SOME APPLICATIONS

OF

GEOMETRIC MEASURE THEORY

by

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I would like to thank my supervisor , Professor S. J. Taylor , for the kind help and encouragement given throughout.

I am also grateful to the Science Research Council for their support.

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CHAPTER I

1. Introduction.

This dissertation is a general survey of topics in geometric measure theory and diophantine approximation.

The present chapter contains general definitions and notation used throughout and an introduction to the theory of Hausdorff measures. Properties of the Cantor Ternary Set are also derived here, which will be needed for the discussion in Chapter II.

Chapter II is devoted to the study of a problem whose origins are outlined as follows :

Let U be a unitary operator on a Hilbert space $\mathcal{H}$,

$$U : \mathcal{H} \to \mathcal{H}, \qquad U^*U = UU^* = I ,$$

$$U = \int_0^{2\pi} e^{i\theta} dF_\theta ,$$

where

$$F_0 = 0 \leq F_\theta \leq \dots \leq I = F_{2\pi} .$$

U corresponds to a measure μ on $[0, 2\pi)$. (Consult [9], for example).

For any operator $T : \mathcal{H} \to \mathcal{H}$, a subspace $\mathcal{M} \subset \mathcal{H}$ is invariant if $T\mathcal{M} \subset \mathcal{M}$.

$\mathfrak{M}$ is reducing if

$$\left.\begin{array}{l} T\mathfrak{M} \subset \mathfrak{M} \\ T\mathfrak{M}^{\perp} \subset \mathfrak{M}^{\perp} \end{array}\right\}$$

T is singular if every invariant subspace reduces.

Now the measure μ is the direct sum of two measures

$$\mu = \mu_a \oplus \mu_s ,$$

where μ_a is absolutely continuous and μ_s is singular.
μ_s may also be decomposed thus

$$\mu_s = \mu_{as} \oplus \mu_{cs} ,$$

where μ_{cs} is continuous singular.
Similarly,

$$U = U_a \oplus U_s ,$$

$$U_s = U_{as} \oplus U_{cs} .$$

A bilateral shift B is such that for $\{\epsilon_i\}_{-\infty}^{\infty}$ convergent in $\mathcal{H}$, $B\epsilon_i = \epsilon_{i-1}$.
Now U_a is part of a bilateral shift,

$$\|B^n\xi - \xi\| \to \sqrt{2}\|\xi\|, \quad \xi \in \mathcal{H}.$$

The following problem arises:

<u>Problem</u>.

For each $\xi \in \mathcal{H}$, does there exist a sequence $\{\nu_n(\xi)\}_{n=1}^{\infty}$ of positive integers such that

$$U_s^{\nu_n}\xi \to \xi \quad ?$$

It is easily verified that the answer is affirmative for U_{as}. The question of whether the same is true of U_{cs} gives rise to the problem considered in Chapter II; ie. does there exist a sequence $\{\nu_n\}_{n=1}^{\infty} \subset N$ such that

$$e^{i\nu_n\theta}\xi(\theta) \quad \rightarrow \quad \xi(\theta) \quad ,$$

ie.
$$\int_0^{2\pi} X_{cs}|e^{i\nu_n\theta} - 1|^2|\xi|^2 d\mu_{cs} \quad \rightarrow \quad 0 \quad ?$$

Or, writing $|\xi| = 1$, does there exist a sequence of positive integers $\{\nu_n\}_{n=1}^{\infty}$ such that

$$\mu_{cs}\{\ \theta : \ \nu_n\theta \ \rightarrow \ 0, \text{ modulo } 1\ \} \ = \ 1 \quad ?$$

It is proved that the answer to the problem posed is negative, in that it is impossible to find such a sequence $\{\nu_n\}_{n=1}^{\infty}$ of positive integers for certain types of 'regular' set with the given properties.

The first part of the chapter is a general discussion of results related to the problem and contains an account of the properties of the sequence $\{n_k x\}_{k=1}^{\infty}$ modulo 1, where $x \in \mathcal{R}$ and $\{n_k\}_{k=1}^{\infty}$ is any increasing sequence of positive integers.

A problem in geometric measure theory is now considered in Chapter III. An account is given of the origins of the problem, its relationship to other aspects of convexity, of partial solutions, and finally the complete solution.

Chapter IV is a survey of topics in uniform distribution modulo 1 of sequences and related topics. The main interest is in the uniform distribution modulo 1 of sequences of the form $\{n_k x\}_{k=1}^{\infty}$ where $x \in \mathcal{R}$ and $\{n_k\}_{k=1}^{\infty}$ is any increasing sequence of positive integers, and the properties of sets of points x for which $\{n_k x\}_{k=1}^{\infty}$ is uniformly distributed modulo 1. The uniform distribution of the integral parts of such sequences is also discussed. Connections between results here with results in Chapter II are pointed out.

We begin with the basic definitions and notation which will be used freely throughout. Other definitions required for specific results will be given as they arise.

2. Notation and General Definitions.

For any real number x:

$[x]$ denotes the greatest integer not exceeding x; that is, $[x]$ is the unique integer satisfying

$$[x] \leq x < [x] + 1 \quad ;$$

$((x))$ denotes the fractional part of x, namely

$$((x)) = x - [x] \ .$$

The symbol N is used to denote the set of all positive integers.

$\aleph_0$ denotes the cardinal of N and $c = 2^{\aleph_0}$ the cardinal of the continuum.

The definition of a Hausdorff measure applies to any metric space, but problems will only be considered in Euclidean n-space.

A <u>measure function</u> is defined to be a real-valued function $h(t)$ defined for $t > 0$, such that

(i) $h(t)$ is continuous and monotone increasing,

(ii) $\lim_{t \to 0_+} h(t) = 0$, and $h(t) > 0$ for $t > 0$.

Suppose E is a set in a metric space X. For any $\delta > 0$, put

$$h - m_\delta(E) \quad = \quad \inf_{\substack{\cup C_i \supset E \\ d(C_i) \leqslant \delta}} \sum_{i=1}^{\infty} h[d(C_i)] \quad ,$$

where $d(C_i)$ denotes the diameter of C , and the infimum is taken over all coverings of E by sequences $\{C_i\}$ of sets with diameter not greater than δ.

Now define

$$h - m^*(E) \quad = \quad \sup_{\delta > 0} h - m_\delta(E) \tag{1}$$

As the effect of reducing δ is to reduce the class of covers over which the infimum is taken, $h - m_\delta(E)$ does not decrease as δ decreases, and it is the small values of δ that are relevant in taking the supremum. Thus the formula (1) above could be replaced by

$$h - m^*(E) \quad = \quad \lim_{\delta \to 0_+} h - m_\delta(E) \quad ,$$

ie. it is the 'fine' covers - those by sets of small diameter -

that determine $h - m^*(E)$.

Now the set function $h - m^*(E)$ is a Caratheodory outer measure in X : it therefore defines a class of h-measurable subsets of X which includes all Borel sets (see, for example, [19]). When E is measurable with respect to h write

$$h - m(E) = h - m^*(E) ,$$

and call $h - m(E)$ the h-measure of E. All the sets we consider will be obviously measurable.

For the analysis of subsets of E^n of zero Lebesgue measure it is usual to assume that the function $\frac{h(t)}{t^n} \to \infty$ as $t \to 0+$. (h is called a measure function of class n).

In the special case $h(t) = t^\alpha$, $\alpha > 0$, we replace $h - m_\delta(E)$, $h - m^*(E)$ and $h - m(E)$ by the set functions $\Lambda^\alpha_\delta(E)$, $\Lambda^\alpha_*(E)$ and $\Lambda^\alpha(E)$ respectively, and the measure so obtained is called the α-dimensional measure of E. For a given E and $\alpha > 0$, $\Lambda^\alpha(E)$ may be zero, finite and positive, or infinite. E is called an α-set if $\Lambda^\alpha(E)$ is finite and positive.

For example, the classical Cantor Ternary Set C (see section 3) constructed on the real line satisfies

$$\Lambda^\alpha(C) = \begin{cases} 0 , & \alpha > \delta, \\ 1 , & \alpha = \delta, \\ +\infty, & \alpha < \delta, \end{cases}$$

if $h(t) = t^\alpha$, where $\delta = \log 2 / \log 3$.

All subsets E of R^n have a numerical dimension, which is a real number $\gamma \leq n$, and denoted by dim E, given by

$$\dim E = \gamma = \inf \{ \alpha > 0 : \Lambda^{\alpha}(E) = 0 \}.$$

If $\Lambda^{\alpha}(E) = 0$ for all α then we write dim E = 0.
If dim $E = \gamma$, it is possible for $\Lambda^{\gamma}(E)$ to be zero, finite and positive, or infinite, but

$$\alpha > \gamma \Rightarrow \Lambda^{\alpha}(E) = 0, \quad (0 \leq \gamma < \infty),$$
$$0 \leq \alpha < \gamma \Rightarrow \Lambda^{\alpha}(E) \text{ is non } \sigma\text{-finite.}$$

If E is a set in Euclidean n-space, $|E|$ will denote the Lebesgue measure of E. In that case it can be shown that

$$|E| = c_n \Lambda^n(E), \quad n \in N,$$

($\Lambda^n(E)$ is called the (n)-measure of E), where

$$c_n = \frac{\pi^{\frac{1}{2}n}}{2^{n-1} n\Gamma(\frac{1}{2}n)}$$

is the volume of a sphere of unit diameter in n-space.
(See, for example, [19] p. 54).
Thus, while Lebesgue measure assigns unit measure to the cube of unit size, Λ^n assigns unit measure to the sphere of unit diameter.

3. Cantor Ternary Set and Ternary Function.

Denote the open interval $\left(\frac{(3r - 2)}{3^n}, \frac{(3r - 1)}{3^n} \right)$

by $E_{n,r}$ and put

$$G_n = \bigcup_{r=1}^{3^{n-1}} E_{n,r} \quad , \qquad G = \bigcup_{n=1}^{\infty} G_n \quad ;$$

then it is clear that G is an open subset of [0,1] and its complement

$$C = [0,1] \setminus G$$

is called the Cantor Ternary Set. C is obviously closed.

For $x \in [0,1]$, write $x = \sum_{i=1}^{\infty} \frac{b_i}{2^i}$, $(b_i = 0 \text{ or } 1)$,

where the sequence $\{b_i\}$ of 0's and 1's does not satisfy $b_i = 1$ for $i \geqslant N$.

Define

$$g(x) = \sum_{i=1}^{\infty} \frac{a_i}{3^i} \quad , \quad \text{where} \quad a_i = \begin{cases} 0 \ , & \text{if } b_i = 0 \ , \\ 2 \ , & \text{if } b_i = 1 \ . \end{cases}$$

Then $g : [0,1] \to C$ is (1-1) and maps [0,1] on to a proper subset of C.

Since $C \subset [0,1]$, C has cardinal c.

It is clear that

$$x_1 < x_2 \quad \Longrightarrow \quad g(x_1) < g(x_2) \quad ,$$

so that for each $y \in [0,1]$,

$$g^{-1}[0,y] = [0,z] \quad \text{for some } z. \tag{2}$$

If z is defined by (2), then we say that

$$z = f(y).$$

This defines $f : [0,1] \to [0,1]$ as a monotone function which is clearly constant on each of the intervals $E_{n,r}$, for

$$\frac{3r-2}{3^n} \leqslant y \leqslant \frac{3r-1}{3^n} \implies f(y) = \frac{2r-1}{2^n} .$$

The function f is continuous and monotone increasing since

$$0 \leqslant y_1 - y_2 \leqslant 3^{-n-1} \implies 0 \leqslant f(y_1) - f(y_2) < 2^{-n} .$$

The function f (the Cantor Ternary Function) is differentiable with zero derivative at each point of G since f is constant in each of the intervals $E_{n,r}$. f increases at each point of C and

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = +\infty , \quad x \in C.$$

Let ν be the Lebesgue-Stieltjes measure associated with f,

ie. $\nu(a,b] = f(b) - f(a) = \nu[a,b]$

since f is continuous. Then $\nu[0,1] = 1$, $\nu(E_{n,r}) = 0$, $\nu(G_n) = 0$, $\nu(G) = 0$, and $\nu(C) = 1$.

Also, if $\alpha = \dfrac{\log 2}{\log 3}$, then $\Lambda^{\alpha}(C) = 1$. An easy proof of this fact is provided by lemma 4 of [20]. The lemma states:

<u>Lemma 1.1</u>

Suppose that F is a completely additive measure defined on the real Borel sets and that E is a Borel set such that for each $x \in E$,

$$\limsup_{h \to 0} \frac{F[x , x+h]}{\phi(h)} \leqslant k < \infty ,$$

where ϕ is a Hausdorff measure function. Then

$$k\{\phi - m(E)\} \geqslant F(E) .$$

Now the set C is covered by 2^N intervals, each of length 3^{-N}, for N = 1, 2, ... , and so

$$\Lambda^{\alpha}(C) \leqslant 2^N\left\{\frac{1}{3^N}\right\}^{\alpha} = 1 \qquad (3)$$

If $x \in C$ and is of the form

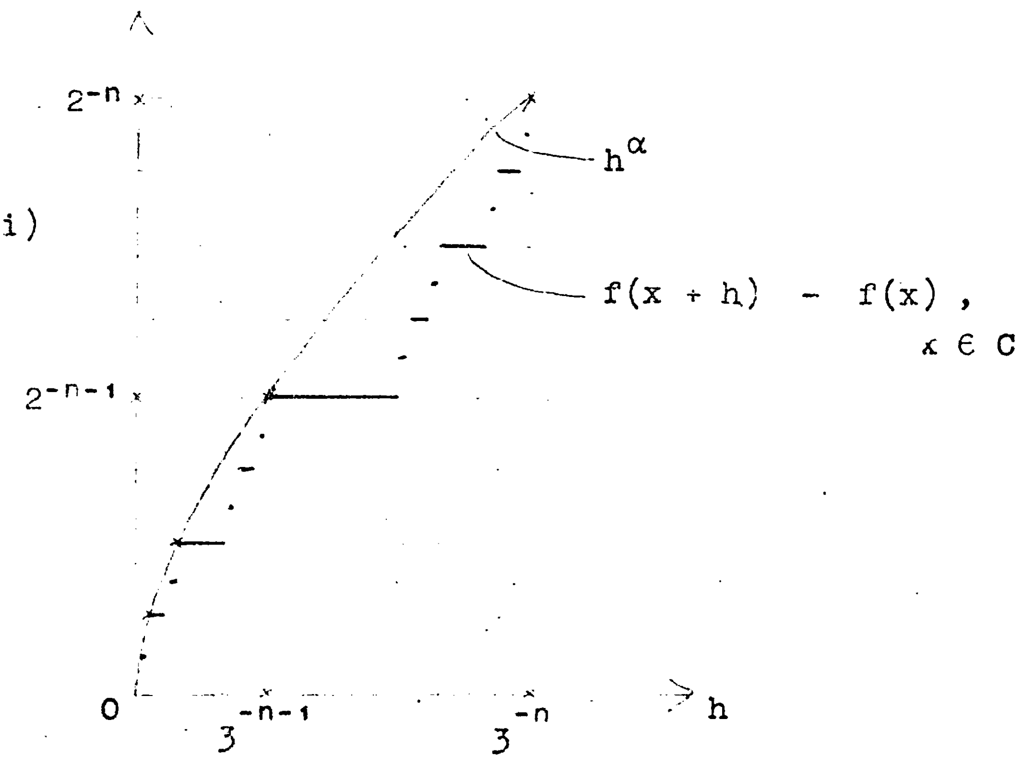
$$x = \frac{3r - 2}{3^N} \quad \text{for some } r , N ,$$

then $\nu[x , x + h] = 0$ for $0 \leqslant h < 3^{-N}$.
Suppose $x \in C$ and is of the form

$$x = \frac{3r - 1}{3^N} \quad \text{for some } r , N ,$$

and consider the relationship between $\nu[x , x + h]$ and h^{α} for $0 < h < 3^{-n}$ for some $n \geqslant N$.

Fig. (i)



Now

$$3^{-(n+1)} \leqslant h < 2.3^{-(n+1)} \Rightarrow f(x+h) - f(x) \leqslant 2^{-(n+1)}$$
$$= \{3^{-(n+1)}\}^{\alpha}$$
$$\leqslant h^{\alpha} ,$$

and

$$2.3^{-(n+1)} \leqslant h < 3.3^{-(n+1)} = 3^{-n}$$

$$\Rightarrow f(x+h) - f(x) \leqslant \{h - 2.3^{-(n+1)}\}^{\alpha} + 2^{-(n+1)}.$$
$$\leqslant h^{\alpha}$$

The argument extends to general points of C by approximation, and So $f(x+h) - f(x) \leqslant h^{\alpha}$ for sufficiently small h for all $x \in C$.

So

$$\limsup_{h \to 0} \frac{f(x+h) - f(x)}{h^{\alpha}} \leqslant 1 \quad \text{for all } x \in C .$$

Applying the lemma with $E = C$, $F = \nu$, $\phi(h) = h^{\alpha}$, $\phi - m = \Lambda^{\alpha}$, we have $k = 1$, and

$$\Lambda^{\alpha}(C) \geqslant \nu(C) = 1 . \qquad (4)$$

Combining (3) with (4),

$$\Lambda^{\alpha}(C) = 1.$$

It follows that $\dim C = \alpha = \dfrac{\log 2}{\log 3}$, for

$$\Lambda^{\beta}(C) = 0 , \beta > \alpha ,$$

and $\Lambda^{\beta}(C)$ is non σ-finite for $\beta < \alpha$.

CHAPTER II

This chapter is concerned with the following problem :

F is a measure on $[0,1]$ concentrated on the subset E_0, $(F(E_0) = 1)$, with $F([0,1]) = 1$, and $|E_0| = 0$.

Does there exist a sequence of integers $\{n_k\}_{k=1}^{\infty}$, increasing to infinity, such that

$$((n_k x)) \to 0 \text{ for almost all } x \in E_0 \text{ , as } k \to \infty \text{ ,}$$

or, more precisely,

$$F \{ x \in [0,1] : ((n_k x)) \to 0 , k \to \infty \} = 1 \quad ?$$

1. The general behaviour of the fractional part of $n_k\theta$.

If θ is an irrational number and α is any number such that $0 \leqslant \alpha < 1$, then it has been known for some time that it is possible to find a sequence of positive integers n_1, n_2, n_3, ... such that

$$((n_k \theta)) \to \alpha \text{ , as } k \to \infty \text{ .}$$

Note : The result, when $\alpha > 0$, asserts that, given any positive number ϵ , there exists an integer ko such that

$$-\epsilon < ((n_k \theta)) - \alpha < \epsilon \text{ , } k \geqslant k_0 \text{ .}$$

The points $((n_k\theta))$, $k = 1, 2, \ldots$, may lie on either side of α. But, since $((n_k\theta))$ is never negative, the formula has a special meaning in the particular case in which $\alpha = 0$:

$$\text{ie.} \quad 0 \leqslant ((n_k\theta)) < \epsilon , \quad k \geqslant k_0 .$$

Any inconvenience arising as a result of this distinction between the value $\alpha = 0$ and other values of α may be avoided by agreeing that, when $\alpha = 0$, the formula $((n_k\theta)) \rightarrow \alpha$ is to be interpreted as meaning 'the set of points $((n_k\theta))$, $k = 1, 2, \ldots$, has, as its sole limiting point or points, one or both of the points 1 and 0 ', ie. for any $k \geqslant k_0$, one or other of the inequalities

$$0 \leqslant ((n_k\theta)) < \epsilon , \qquad 1 - \epsilon < ((n_k\theta)) < 1$$

is satisfied.

This distinction, however, happens to be of no importance in the particular cases considered here.

The following generalisation of this result was first proved by Kronecker (1884) [14] , and a comparatively simpler proof is given by Hardy and Littlewood in [10] :

<u>Theorem 2.1</u>

If $\theta_1, \theta_2, \ldots, \theta_m$ are linearly independent irrationals (ie. if no relation of the type

$$a_1\theta_1 + a_2\theta_2 + \dots + a_m\theta_m + a_{m+1} = 0 ,$$

where $a_1, a_2, \dots, a_{m+1}$ are integers, not all zero , holds between $\theta_1, \theta_2, \dots, \theta_m$) , and $\alpha_1, \alpha_2, \dots, \alpha_m$ are numbers such that $0 \leqslant \alpha_p < 1$, then a sequence $\{n_k\}$ can be found such that

$$((n_k\theta_1)) \to \alpha_1 , \quad ((n_k\theta_2)) \to \alpha_2 , \quad \dots , \quad ((n_k\theta_m)) \to \alpha_m ,$$

as $k \to \infty$.

Further, in the special case when all the α's are zero, it is unnecessary to make any restrictive hypothesis concerning the θ's , or even to suppose them irrational.

The special case when all the α's are zero was known before by Dirichlet - his proof is straightforward and there is virtually no difference between the cases of one and of several variables, [10].

Theorem 2.1 may also be generalised and is proved by induction on k :

Theorem 2.2 [10]

If $\theta_1, \theta_2, \dots, \theta_m$ are linearly independent irrationals, and the α's are any numbers such that $0 \leqslant \alpha < 1$, then a sequence $\{n_k\}$ can be found such that

$$((n_k^i\theta_j)) \to \alpha_{ij} , \quad \begin{cases} i = 1, \dots, p , \\ j = 1, \dots, m . \end{cases}$$

Further, if the α's are all zero, it is unnecessary to suppose the θ's restricted in any way.

For a strictly increasing sequence of positive integers $\{n_k\}$, and an irrational number θ, the set of points $((n_k\theta))$, $k = 1, 2, \ldots$, can exhibit many different kinds of behaviour. The following facts are well-known :

(i) There is no number θ , $0 < \theta < 1$, such that $((k\theta))$ tends to a limit.
(See [18]. If θ is irrational, the points $((k\theta))$, $k = 1, 2, \ldots$, are dense in the unit interval , ie. given any real numbers λ_1, λ_2 satisfying $0 \leq \lambda_1 < \lambda_2 \leq 1$, there is a positive integer k' such that $\lambda_1 < ((k'\theta)) < \lambda_2$).

(ii) Given any arbitrary strictly increasing sequence of integers $\{n_k\}$, the set of real numbers θ for which $0 < \theta < 1$ and $((n_k\theta))$ tends to a limit as k tends to infinity has zero linear Lebesgue measure.
(Hardy and Littlewood prove in [10], p.181, that the set of values of θ for which the set of points $((n_k\theta))$ is not everywhere dense in the interval $(0,1)$ is of measure zero).

It is interesting to examine how the set of θ for which $((n_k\theta))$ tends to α is affected when we consider different sequences of integers $\{n_k\}$. The preceding results show that this set of θ may be empty, and is always of zero linear measure even when it is non-empty.

The following two theorems are due to Eggleston [7] and deal with this problem for two of the commonest types of sequences of integers : when $\frac{n_{k+1}}{n_k}$ is bounded, and when $\frac{n_{k+1}}{n_k} \to \infty$.

Theorem 2.3

If a strictly increasing sequence of positive integers $\{n_k\}$ is such that $\frac{n_{k+1}}{n_k} < K$ where $0 < K < \infty$, for all k, $k = 1, 2, \ldots$, then for any α, $0 \leqslant \alpha < 1$, there are at most an enumerable set of real numbers θ, $0 \leqslant \theta < 1$, for which $((n_k \theta)) \to \alpha$ as $k \to \infty$.

There is an n-dimensional analogue of this result.

Theorem 2.4

If a strictly increasing sequence of positive integers $\{n_k\}$ is such that $\frac{n_{k+1}}{n_k} \to \infty$ as k tends to infinity, then the set of θ for which $((n_k \theta)) \to \alpha$, $0 \leqslant \theta < 1$, as k tends to infinity, has dimension 1, for any given α, $0 \leqslant \alpha < 1$.

There is a corresponding result in n-dimensional space.

A subset of the set concerned in Theorem 2.4 is constructed and an application of another important theorem due to Eggleston [7] shows that this subset has positive

s_1-measure where $0 < s_1 < 1$, thus giving the required result.

The theorem, which will be required later in the chapter, is a useful device for obtaining a lower bound for the dimension of certain types of set :

Theorem 2.5

Suppose I_k is a linear set consisting of N_k closed intervals each of length δ_k. Let each interval of I_k contain n_{k+1} closed intervals of I_{k+1}, $n_{k+1} \geqslant 2$, each of diameter δ_{k+1}, and so distributed that their minimum distance apart is ρ_{k+1}, $\rho_{k+1} > \delta_{k+1}$.

Let

$$P = \bigcap_{k=1}^{\infty} I_k .$$

Then, if

$$\liminf_{k \to \infty} h(\delta_k) N_{k+1} \rho_{k+1} \delta_k^{-1} \geqslant \delta > 0 ,$$

the set P has positive h-measure.

(Note that if the inequality $\rho_{k+1} > \delta_{k+1}$ is not true but $\rho_{j+1} < \delta_j$ for infinitely many j and δ_j, ρ_j both tend to zero, then the result of the theorem still holds).

The most important case of the theorem is when $h(x) = x^s$ so that

$$\liminf_{k \to \infty} N_{k+1} \rho_{k+1} \delta_k^{s-1} > 0 \quad \Rightarrow \quad \Lambda^s(P) > 0 .$$

Erdos and Taylor [8] have obtained a number of results concerning the properties of the set of points x for which the sequence $\{((n_k x))\}$ behaves in certain ways in order to investigate the convergence of the lacunary trigonometric series

$$\sum_{k=1}^{\infty} \sin (n_k x - \mu_k) \tag{5}$$

where $\{\mu_k\}$ $(k = 1, 2, \ldots)$ is a sequence of constants satisfying $0 \leqslant \mu_k \leqslant 2\pi$ and $\{n_k\}$ $(k = 1, 2, \ldots)$ is an increasing sequence of integers satisfying

$$t_k = \frac{n_{k+1}}{n_k} \geqslant \rho > 1 . \tag{6}$$

The classical theory of trigonometric series shows that the series (5) may only converge for values of x in a set of zero Lebesgue measure.

The convergence, or absolute convergence, of the series (5) is closely related to that of the series

$$\sum_{k=1}^{\infty} \{ ((n_k x)) - \alpha_k \} \tag{7}$$

where $\{\alpha_k\}$ $(k = 1, 2, \ldots)$ is a sequence of real numbers satisfying $0 \leqslant \alpha_k \leqslant 1$, and $\{n_k\}$ satisfies (6). The connection is given by

<u>Lemma 2.1</u>

If the series $\sum \left| \left(\left(n_k \frac{x}{2\pi}\right)\right) - \frac{\mu_k}{2\pi} \right|$ converges, then the series $\sum \sin (n_k x - \mu_k)$ converges absolutely.

For

$$|\sin(n_k x - \mu_k)| = \left|\sin\left(2\pi\left\{\left(\left(n_k\frac{x}{2\pi}\right)\right) - \frac{\mu_k}{2\pi}\right\}\right)\right|$$

$$\leqslant 2\pi\left|\left(\left(n_k\frac{x}{2\pi}\right)\right) - \frac{\mu_k}{2\pi}\right| .$$

(The converse is not true but the infinite cardinal or dimension of the sets of absolute convergence of the series (5) and (7) turn out to be the same).

A discussion of the convergence of series (7) leads naturally to the problem of equidistribution of the sequence $\{((n_k x))\}$, $(k = 1, 2, \ldots)$, and this is considered in Chapter IV.

The 'size' of the set of absolute convergence will depend on the rate at which t_k increases.

If t_k is bounded, Theorem 2.3 implies there cannot be more than a countable set of x for which $((n_k x)) \to y$, $(0 \leqslant y \leqslant 1)$, as $k \to \infty$, and so there is at most a countable set of values of x such that

$$\sum_{k=1}^{\infty} ((n_k x)) < \infty .$$

In the case of a sequence $\{n_k\}$ such that $t_k \to \infty$ we have

<u>Theorem 2.6</u> [8]

If $\{n_k\}$ is such that t_k is an integer for large values of k, and $t_k \to \infty$ as $k \to \infty$, then the set of values of x such that $\sum ((n_k x))$ converges (absolutely) has power continuum.

If t_k increases rapidly enough to make $\sum \frac{1}{t_k}$ convergent, we also have

Theorem 2.7 [8]

Suppose $\{n_k\}$ is such that $\sum \frac{1}{t_k}$ converges. Then for any $\{\alpha_k\}$ the series $\sum \{ ((n_k x)) - \alpha_k \}$ converges absolutely for values of x in a set of power continuum.

The dimension in the sense of Besicovitch of the set of x for which $((n_k x))$ converges is of interest only in the case where the set has power continuum since enumerable sets necessarily have dimension 0. The dimension depends on the rate at which $t_k \to \infty$ and among the existing results are :

Theorem 2.8 (Eggleston, [7])

If $\rho > q > 1$ and the sequence of positive increasing integers $\{n_k\}$ satisfies

$$\kappa_1 n_k^{\rho} \leq n_{k+1} \leq \kappa_2 n_k^{\rho} , \quad k = 1, 2, \dots ,$$

κ_1, κ_2 finite positive constants, then the set of real numbers x for which

$$| ((n_k x)) - \alpha | \leq n_k^{1-q}$$

for all sufficiently large k and a fixed α, $0 \leq \alpha < 1$, has dimension $\dfrac{(\rho - q)}{q(\rho - 1)}$;

and

<u>Theorem 2.9</u> (Erdos and Taylor, [8])

Suppose $\lambda > 0$, $\mu > 0$, $\rho > 0$ are constants, and $\{n_k\}$ is an increasing sequence of integers such that

$$\lambda k^{\rho} \leqslant t_k \leqslant \mu k^{\rho}$$

for each integer k, and $\{\alpha_k\}$ is any sequence of constants with $0 \leqslant \alpha_k \leqslant 1$. Then

(i) if $0 < \rho \leqslant 1$, the dimension of the set of x for which $\sum \{ ((n_k x)) - \alpha_k \}$ converges absolutely is zero ;

(ii) if $\rho > 1$, the dimension of the set of x for which $\sum \{ ((n_k x)) - \alpha_k \}$ converges absolutely is $\left(1 - \frac{1}{\rho}\right)$.

Theorem 2.9 is another application of Theorem 2.5 . With ϵ, s satisfying

$$0 < \epsilon < \rho - 1 \ , \quad 0 < s < 1 - \frac{1 + \epsilon}{\rho} \ ,$$

a subset P of the set of x such that

$$| ((n_k x)) - \alpha_k | \leqslant k^{-1-\epsilon}$$

for sufficiently large k is constructed with $\Lambda^s(P) > 0$.

This implies that the set of x for which (7) converges absolutely has positive Λ^s-measure and so has dimension at least $\left(1 - \frac{1}{\rho}\right)$.

More difficult methods show that the set where (5) converges absolutely has dimension at most $\left(1 - \frac{1}{\rho}\right)$ and this proves the theorem.

Similar methods will also yield :

Theorem 2.10

If $\{\alpha_k\}$ is any sequence of constants, $0 \leqslant \alpha_k \leqslant 1$, and h(z) is any measure function of class 1, there is an increasing sequence $\{n_k\}$ of integers such that the set of values of x for which $\sum \{((n_k x)) - \alpha_k\}$ converges absolutely has infinite measure with respect to h(z).

Also, if $t_k \to \infty$, however slowly, and $0 < \delta \leqslant \alpha_k \leqslant 1 - \delta, (k = 1, 2, \ldots)$, then (7) converges for x in a set of dimension 1. Note that this result cannot be true for $\alpha_k = 0$ since the series (7) converges only if it converges absolutely.

2. The solution of the problem.

In section 1 various properties of the set of values of x for which $((n_k x)) \to 0$ as $k \to \infty$ for particular sequences $\{n_k\}$ were stated, and in particular that this set always has Lebesgue measure zero if $\{n_k\}$ increases to infinity.

In this section we will show that it is not possible to find a sequence $\{n_k\}$ to satisfy properties stated in the problem for 'regular' Cantor-like sets. This idea will be made precise later.

We begin by taking E_0 to be the Cantor Ternary Set C constructed in the unit interval as defined in Chapter I, section 3.

<u>Lemma 2.2</u>

There exists a subset $C' \subset C$ and an increasing sequence of positive integers $\{n_k\}$ tending to infinity such that the cardinal of C' is c and

$$\sum_{k=1}^{\infty} ((n_k x)) < \infty , \forall x \in C' .$$

<u>Proof</u> :

We have

$$C = \{ x \in [0,1] : x = \sum_{i=1}^{\infty} \frac{\eta_i}{3^i} , \quad \eta_i \in \{0,2\} \}$$

with cardinal (C) = c , $|C| = 0$, $\dim C = \frac{\log 2}{\log 3}$, (see Chapter 1, section 3).

Let $n_k = 3^{\frac{1}{2}k(k+1)}$, $(k = 1, 2, \ldots)$

Then $t_k = \frac{n_{k+1}}{n_k} = 3^{k+1}$, $(k = 1, 2, \ldots)$

Define

$$C' = \{ x \in [0,1] : x = \sum_{i=1}^{\infty} \frac{\eta_i}{n_i} , \quad \eta_i \in \{0,2\} \} .$$

Clearly $C' \subset C$, and cardinal $(C') = c$, $|C'| = 0$.

Suppose $x \in C'$.

Then

$$n_k x = \sum_{i=1}^{\infty} \frac{\eta_i}{n_i} . n_k ,$$

and so

$$((n_k x)) = \sum_{i>k} \frac{\eta_i}{n_i} . n_k , \quad (k = 1, 2, \ldots)$$

Hence

$$\sum_{k=1}^{\infty} ((n_k x)) = \sum_{k=1}^{\infty} \Big\{ \sum_{i>k} \frac{\eta_i}{n_i} . n_k \Big\}$$

$$= \sum_{i=2}^{\infty} \Big\{ \sum_{k=1}^{i-1} \frac{\eta_i}{n_i} . n_k \Big\} .$$

Now when $k = i - 1$, $\frac{n_k}{n_i} = 3^{-i}$, and for $k \geqslant 1$,

$$\frac{n_{k-1}}{n_k} \leqslant \frac{1}{3} .$$

Hence

$$\sum_{k=1}^{i-1} \frac{\eta_i}{n_i} . n_k \leqslant \frac{2}{3^i} \Big\{ 1 + \frac{1}{3} + \ldots \Big\}$$

$$= \frac{1}{3^{i-1}} ,$$

and so

$$\sum_{k=1}^{\infty} ((n_k x)) \leqslant \sum_{i=2}^{\infty} \frac{1}{3^{i-1}} = \frac{1}{2} .$$

Thus for all $x \in C'$, the series $\sum_{k=1}^{\infty} ((n_k x))$ converges.

Thus $((n_k x)) \to 0$ as $k \to \infty$ for every $x \in C'$.

Suppose ν is the Lebesgue-Stieltjes measure associated with the Cantor Ternary Function f constructed in Chapter I, section 3 :

$$\nu(E) = \Lambda^\alpha(E \cap C) , \quad \alpha = \frac{\log 2}{\log 3} , \quad E \subset [0,1] .$$

From the definition of C' ,

$$C' = \bigcap_{N=1}^{\infty} D_N , \quad \text{where} \quad D_1 \supset D_2 \supset D_3 \supset \ldots$$

and D_N consists of 2^k disjoint intervals in $[0,1]$ each of length 3^{-N} , and ν-measure 2^{-N} , where

$$\tfrac{1}{2}k(k+1) \leqslant N < \tfrac{1}{2}(k+1)(k+2) .$$

So $\nu(D_N) = 2^{k-N} \to 0$ as $N \to \infty$, and

$$\nu(C') = 0 .$$

Also, since C' is covered by 2^k intervals each of length 3^{-N} and $N \sim \frac{1}{2}k^2$, for any $\alpha > 0$,

$$2^k \left\{\frac{1}{3^N}\right\}^\alpha \sim 2^k \left\{\frac{1}{3^{\frac{1}{2}k^2}}\right\}^\alpha$$

$$= \left\{\frac{2}{3^{\frac{1}{2}k\alpha}}\right\}^k$$

$$\to 0 \quad \text{as} \quad k \to \infty ,$$

and so

$$\dim C' = 0 .$$

Note that since $t_k \uparrow \infty$ as $k \uparrow \infty$, it follows in view of Theorem 2.4 that the set of values of x in $[0,1]$

for which $((n_k x)) \to 0$ as $k \to \infty$ has dimension 1.

Lemma 2.2 may be improved in the sense that it is possible to make $\sum_k ((n_k x))$ converge on a larger subset of C.

Lemma 2.3

There exists a subset $D \subset C$ and an increasing sequence of positive integers $\{n_k\}$ tending to infinity such that

$$\sum_{k=1}^{\infty} ((n_k x)) < \infty, \forall x \in D ,$$

and $\dim D = \dim C = \dfrac{\log 2}{\log 3}$.

Proof :

For the sequence $n_k = 3^{\frac{1}{2}k(k+1)}$, $(k = 1, 2, \ldots)$, defined for lemma 2.2, the gaps between the successive terms increase and contain 3^k integers. To define the subset D of C, we make the gaps smaller so that they still increase but now contain only $3^{[\log k]}$ integers, ie. write

$$D = \{ x \in C : x = \sum_{i,k} \frac{\eta_{k_i}}{m_{k_i}} \}$$

for all sequences $\{\eta_{k_i}\}$ where $\eta_{k_i} = 0$ or 2 and

$$m_{k_i} = 3^i n_{k-1} , (k = 1, 2, \ldots) , 0 \leqslant i \leqslant k - [\log k].$$

Then D is the intersection of a descending sequence of intervals $E_1 \supset E_2 \supset E_3 \supset \ldots$ where each E_N consists of 2^J

intervals each of length 3^{-N} and

$$j = \sum_{r=1}^{k-1} \{ r + 1 - [\log r] \} ,$$

$$\tfrac{1}{2}k(k + 1) \leqslant N < \tfrac{1}{2}(k + 1)(k + 2) - [\log k] .$$

As before, $\nu(E_N) = 2^{-g(N)}$ where $g(N)\uparrow \infty$ as $N \to \infty$ so that

$$\nu(D) = 0 .$$

The same methods as in lemma 2.2 give

$$\sum_{k=1}^{\infty} ((n_k x)) \leqslant \sum_{i=2}^{\infty} \left\{ 2.3^{-\{[\log i] + 1\}} (1 + \tfrac{1}{3} + \ldots) \right\}$$

$$= \sum_{i=2}^{\infty} 3^{-[\log i]} , \quad x \in D .$$

Now $3^{[\log i]} > i^{\lambda}$ if $\lambda < \dfrac{[\log i]}{\log i} \log 3$.

Since $\dfrac{[\log i]}{\log i} < 1$ and tends to 1 as $i \to \infty$, choose λ_0

so that $1 < \lambda_0 < \dfrac{1 + \log 3}{2}$.

Then $3^{[\log i]} > i^{\lambda_0}$ for $i \geqslant I$, I a positive integer, and so $\sum_{i=2}^{\infty} 3^{-[\log i]}$ is convergent.

Thus $\sum_{k=1}^{\infty} ((n_k x)) < \infty$ for all $x \in D$ and so $((n_k x)) \to 0$ $\forall\ x \in D$.

Now fix s with $0 < s < \dfrac{\log 2}{\log 3}$.

We apply Theorem 2.5 with

$$N_k = 2^j \quad \text{where} \quad j = \sum_{r=1}^{k-1} (r + 1 - [\log r]) ,$$

$$\delta_k = \rho_{k+1} = 3^{-N}, \quad N \sim \tfrac{1}{2}k^2 .$$

Now $$\sum_{r=1}^{k} \{r + 1 - [\log r]\} = \tfrac{1}{2}k(k+1) + k - \sum_{r=1}^{k} [\log r]$$

$$> \tfrac{1}{2}k(k+3) - ck^{1+\epsilon}$$

for some positive constants c, ϵ .

So

$$N_{k+1}\rho_{k+1}\delta_k^{s-1} \sim 2^{\sum^k \{r+1-[\log r]\}} . 3^{-\frac{1}{2}k^2} . 3^{-\frac{1}{2}k^2(s-1)}$$

$$> 2^{\frac{1}{2}k(k+3) - ck^{1+\epsilon}} . 3^{-\frac{1}{2}k^2 s}$$

$$\sim \left\{\frac{2}{3^s}\right\}^{\frac{1}{2}k^2}$$

$$\to \infty , \quad k \to \infty .$$

Thus D has positive Λ^s -measure for each $s < \dfrac{\log 2}{\log 3}$.

This implies D has Hausdorff dimension $\dfrac{\log 2}{\log 3}$.

<u>Corollary</u>.

There is an increasing sequence of positive integers $\{\lambda_k\}_{k=1}^{\infty} \uparrow \infty$ such that

$$\Lambda^s\{ x \in C : ((\lambda_k x)) \to 0 , k \to \infty \} = + \infty$$

for all $s < \alpha = \dfrac{\log 2}{\log 3}$.

The methods of lemma 2.3 will also give :

Lemma 2.4

If $h(x) = x^{\alpha}\phi(x)$ where $\phi(x)$ is monotone increasing to infinity as $x \downarrow 0$, $\exists \; \{\lambda_k\}_{k=1}^{\infty} \uparrow \infty$ such that

$$h - m\{ x \in C \; : \; ((\lambda_k x)) \to 0 \, , \; k \to \infty \} = +\infty .$$

The sets C', D constructed in lemmas 2.2, 2.3 each have ν-measure equal to zero. This suggests that the same will be true of any subset E of C with the property that $((n_k x)) \to 0$ for all $x \in E$.

Theorem 2.11

Let $\{k_p\}_{p=1}^{\infty}$ be any increasing sequence of integers tending to infinity. If the set $E \subset C$, the Cantor Ternary Set, is such that

$$((k_p x)) \to 0 \quad \text{as} \quad p \to \infty \text{ for all } x \in E ,$$

then

$$\Lambda^{\alpha}(E) = 0 , \qquad \alpha = \frac{\log 2}{\log 3} .$$

Proof :

For $n = 1, 2, \ldots$, $r = 1, 2, \ldots$, define

$$E_{n,r} = \{ x \in C \; : \; ((k_p x)) < \frac{1}{3^r} \; , \; p \geq n \} .$$

Then $E_{1,r} \subset E_{2,r} \subset E_{3,r} \subset \ldots$, $(r = 1, 2, \ldots)$,

$E_{n,1} \supset E_{n,2} \supset E_{n,3} \supset \ldots$, $(n = 1, 2, \ldots)$.

Let

$$F_r = \bigcup_{n=1}^{\infty} E_{n,r} = \lim_{n\to\infty} E_{n,r} , \quad (r = 1, 2, \ldots) .$$

Then $F = \bigcap_{r=1}^{\infty} F_r$ is the set of points x in C for which $((k_p x)) \to 0$ as $p \to \infty$.

We require to show that $\nu(F) = 0$.

It is sufficient to show that for $\epsilon > 0$,

$$\nu(F_r) < \epsilon \text{ for sufficiently large } r ,$$

ie. $\nu(E_{n,r}) < \epsilon$ for $r \geq r(\epsilon)$, $n \geq N$.

Define

$$Q_{p,r} = \{ x \in [0,1] : ((k_p x)) < \frac{1}{3^r} \} ,$$

$(p = 1, 2, \ldots, \quad r = 1, 2, \ldots)$.

Then $Q_{p,r}$ consists of k_p equally-spaced intervals, each of length $\frac{1}{3^r k_p}$, and C is the intersection $\bigcap_{N=1}^{\infty} C_N$ of a descending sequence $\{C_N\}_{N=1}^{\infty}$, where each C_N consists of 2^N intervals, each of length 3^{-N} .

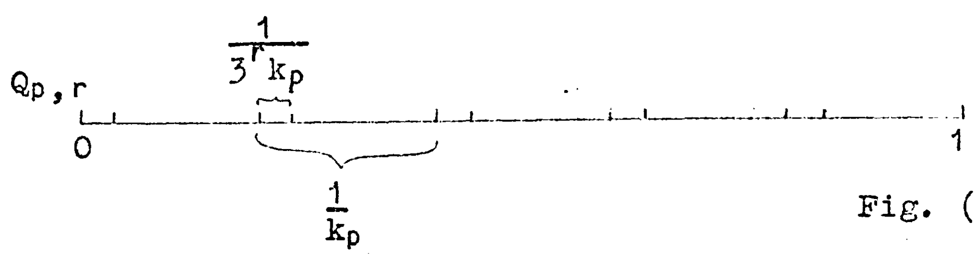


Fig. (ii)

Choose n_p such that

$$\frac{1}{3^{n_p+1}} < \frac{1}{3^r k_p} \leq \frac{1}{3^{n_p}} .$$

Now $(2^s - 1)$ intervals of C_{n_p} do not intersect $Q_{p,r}$ for every one that does, where

$$\frac{1}{3^{r+s}} < \frac{1}{k_p 3^r},$$

$$\text{ie.} \quad 3^s > k_p .$$

So the maximum number of intervals of C_{n_p} intersecting $Q_{p,r}$ is $2^{n_p - s}$ and so

$$\nu(Q_{p,r} \cap C_{n_p}) \leqslant 2^{-s}$$

$< \epsilon$ for sufficiently large s,

ie. for sufficiently large k_p .

Since

$$E_{n,r} = \bigcap_{p \geqslant n} \{ Q_{p,r} \cap C_{n_p} \} ,$$

it follows that $\nu(E_{n,r}) < \epsilon$ for all sufficiently large n.

It follows from this that there is no increasing sequence of integers $\{n_k\}_{k=1}^{\infty}$ such that $((n_k x)) \to 0$ as $k \to \infty$ almost everywhere in C with respect to the measure ν .

Definitions.

The upper and lower symmetric densities of an s-set E at x are given by

$$\limsup_{h \downarrow 0} \frac{\Lambda^s\{ E \cap (x-h, x+h) \}}{h^s} ,$$

and

$$\liminf_{h \downarrow 0} \frac{\Lambda^s\{ E \cap (x-h, x+h) \}}{h^s} .$$

If these densities are equal to each other, their common value is called the circular density.

The following lemmas are known [17] :

Lemma 2.5

The upper circular density of any s-set is less than or equal to 2^s at almost all (with respect to s-measure) its points.

Lemma 2.6

The upper circular density of any s-set is greater than or equal to 1 at almost all its points.

Lemma 2.7

At almost all points outside any s-set, the circular density is equal to zero.

Lemma 2.8

If $0 < s < 1$, the circular density fails to exist at almost every point of any s-set.

Now the Cantor Ternary Set C is an α-set where $0 < \alpha = \dfrac{\log 2}{\log 3} < 1$ and so the circular density fails to exist at almost every point of C and

$$2^\alpha \geqslant \limsup_{h \downarrow 0} \frac{\Lambda^\alpha\{ C \cap (x - h, x + h) \}}{h^\alpha}$$

$$\geq \lim_{n \to \infty} \frac{2^{-n}}{3^{-\alpha n}}$$

$$= \lim_{n \to \infty} \left\{ \frac{3^{\alpha}}{2} \right\}^{n}$$

$$= 1 , \quad \forall x \in C .$$

Also, an argument similar to that of Chapter I shows that for $x \in C$,

$$\nu[x - h, x + h] = f(x + h) - f(x - h) \geq \frac{h^{\alpha}}{2}$$

for sufficiently small positive h and so

$$\liminf_{h \downarrow 0} \frac{\Lambda^{\alpha}\{ C \cap (x - h, x + h) \}}{h^{\alpha}} \geq \tfrac{1}{2} > 0 ,$$

for each $x \in C$.

This suggests the following generalisation of Theorem 2.11 :

<u>Theorem 2.12</u>

Let $\{k_n\}_{n=1}^{\infty}$ be any increasing sequence of integers tending to infinity.

If E is an s-set with $0 < \Lambda^s(E) < 1$, where $0 < s < 1$, and $\exists\, \delta > 0$ such that

$$\liminf_{h \downarrow 0} \frac{\Lambda^{s}\{ E \cap (x - h, x + h) \}}{h^{\alpha}} \geq \delta > 0 ,$$

for each $x \in E$, then

$$\Lambda^{s} \{ x \in E : ((k_n x)) \to 0 \} = 0.$$

Proof :

We may write, where $\eta \to 0$,

$$E = \bigcup_{\eta \in \Xi} E_\eta$$

where Ξ is a countable set and $E_\eta \subset E$ is such that

$$0 < h < \eta \Rightarrow \frac{\Lambda^s\{ E_\eta \cap (x - h, x + h) \}}{(2h)^s} \geqslant \tfrac{1}{2}\delta ,$$

$$\forall x \in E_\eta .$$

Then it suffices to show that

$$\Lambda^s\{ x \in E_\eta : ((k_p x)) \to 0 \} = 0 , \quad \eta \in \Xi .$$

Let $\eta \in \Xi$.

We now use methods similar to those of Theorem 2.11 :

For $n = 1, 2, \ldots$, $r = 1, 2, \ldots$, define

$$E_{n,r} = \{ x \in E_\eta : ((k_p x)) < \frac{1}{r} , p \geqslant n \} .$$

The sequence $\{\{E_{n,r}\}_{n=1}^{\infty}\}_{r=1}^{\infty}$ is increasing in n and decreasing in r .

Let

$$F_r = \bigcup_{n=1}^{\infty} E_{n,r} , \quad (r = 1, 2, \ldots) .$$

Then $F = \bigcap_{r=1}^{\infty} F_r$ is the set of $x \in E_\eta$ for which $((k_p x)) \to 0$.

As before, we require to show

$$\Lambda^s(E_{n,r}) < \epsilon \text{ for } r \geqslant r(\epsilon) , \quad n \geqslant N .$$

Define

$$Q_{p,r} = \{ x \in [0,1] : ((k_p x)) < \frac{1}{r} \} .$$

Choose

$$\frac{1}{k_p r} < 2\eta .$$

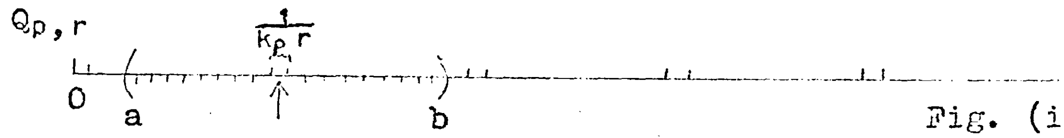


Fig. (iii)

We now argue that if an interval of $Q_{p,r}$ has a point of E_η in it, it cannot have 'a lot' of E_η in it, whereas (a,b) does have a reasonable amount of E_η in it because the lower density at points of E_η is $\geqslant \frac{1}{2}\delta > 0$. The number of smaller intervals which contain points of E_η is therefore large and each contains at least

$$\Lambda^s\{ E_\eta \cap (x - \frac{1}{k_p r}, x + \frac{1}{k_p r}) \} \geqslant \tfrac{1}{2}\delta \left\{ \frac{2}{k_p r} \right\}^s .$$

Thus

$$\Lambda^s\{ Q_{p,r} \cap E_\eta \} \leqslant 2^{1-s}\frac{\Lambda^s(E_\eta)(k_p r)^s}{\delta}$$

$$< \epsilon , \quad r \geqslant R , \quad p \geqslant P .$$

Since

$$E_{n,r} = \bigcap_{p \geqslant n} \{Q_{p,r} \cap E_\eta\} ,$$

the result follows.

Thus, it is impossible for

$$\int_0^1 ((k_n x))\, d\mu(x) \rightarrow 0$$

for some sequence $\{k_n\}_{n=1}^{\infty}$ for the Cantor Ternary Set, because otherwise for a further subsequence, we would have

$$((k_n x)) \rightarrow 0 \quad pp.(\mu) \quad ,$$

and an application of Theorem 2.12 shows that it is also impossible, in general, for any measure μ which is fairly smooth in the sense of Theorem 2.12 (positive lower density with respect to some Hausdorff measure function).

CHAPTER III

In this chapter, we will study a problem concerned with sets known as endsets.

Definition.

For any collection S of line segments in Euclidean n-space E^n, let P(S) denote the set of all endpoints of the members of S. Any set of two or more points has the form P(S) for some S, but a subset M of E^n is here called an endset if and only if $M = P(S)$ for some collection S of pairwise disjoint closed line segments.

1. Statement and origin of the problem.

It is obvious that in the real line E^1 any endset is countable and so has measure zero.

We are thus led to ask :

Must the n-dimensional Lebesgue measure of any compact endset in E^n be zero?

This problem originated in a study of the facial structure of convex bodies [13]. For any convex set C in E^n, denote the interior of C relative to the smallest flat containing C by I(C). Now suppose that B is a convex body in E^n, X is the boundary of B, and X_u is the union of the sets I(C) as C ranges over all maximal convex subsets of X. So, if B is a polytope, X_u consists of the entire boundary except for points on (n - 2)-dimensional faces of B.

Klee and Martin [12] conjecture that, as happens when B is a polytope, X_u is almost all of X in the sense that the (n - 1)-dimensional measure of the set $X \setminus X_u$ is equal to zero,

ie. that the union of the relative interiors of faces of a convex body in E^n covers almost all of the boundary in the sense of (n - 1)-measure.

For $n \leq 3$, Klee and Martin proved this [13, 3.4] by using the fact that in E^1 and E^2, compact endsets have measure zero [12, 13]. They noted that the conjecture could be proved by their methods for general n if the answer to the problem stated at the beginning of this section were affirmative.

Bruckner and Ceder [3], however, produced a counterexample for $n = 4$ (and so for $n > 4$) by using the Axiom of Choice and Nikodym's construction of a Cantor Set X of positive measure in E^2 such that for each point x of X, there is a line in E^2 intersecting X only at x.

(The conjecture has, incidentally, since been established for all n by Larman [16]).

2. Solution of the problem.

There is an endset M_f in E^n associated in a natural way with any real-valued function $f : E^{n-1} \to E^1$.

If the domain of f is a subset D_f of E^{n-1}, the endset is the union of the graph of f and the graph of the function $f + 1$:

$$\text{ie. } M_f = \{ (x, f(x)) : x \in D_f \} \cup \{ (x, f(x) + 1) : x \in D_f \}$$
$$\subset E^{n-1} \times E^1$$
$$= E^n .$$

As the various sets $M_{f+\tau}$, for $0 < \tau < 1$, are pairwise disjoint and all translation equivalent to M_f, it follows that either M_f is non-measurable, or the n-dimensional Lebesgue measure of M_f is zero.

A similar argument shows that the measure of $M = P(S)$ is zero whenever M is measurable and S is a collection of pairwise disjoint parallel segments. As the next paragraph explains, the problem amounts to asking whether almost parallel is as good as parallel in this context.

For each real number $\eta \in (0, \frac{1}{3})$, define a (n, η)-endset as a compact set of the form $P(S)$, where S is a collection of pairwise disjoint segments having one end-point within η of the origin $(0, \ldots, 0, 0)$ in E^n and the other end-point within η of the point $(1, 0, \ldots, 0)$. The segments in such a collection need not be parallel, but are very nearly so, especially when η is small (see Fig. (iv)).

Then, if there exists $\eta_n > 0$ such that any (n, η_n)-endset is of measure zero, it may be derived from

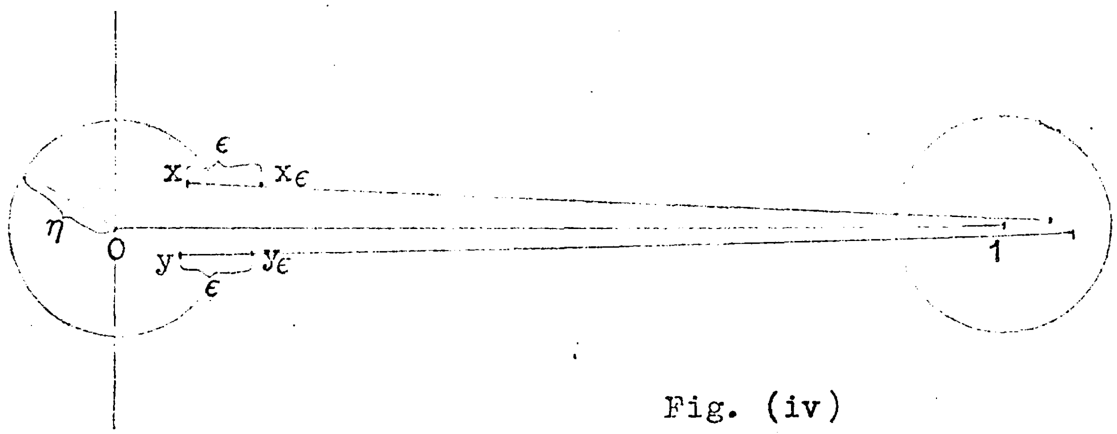


Fig. (iv)

this, using standard and elementary techniques of measure theory, that any measurable endset in E^n is of measure zero.

For $n = 2$, the existence of such a set can be derived from the fact that if $P(S)$ is a $(2, \eta)$-endset for small enough η, and x, y are the left end-points of two members of S, then

$$\|x_\epsilon - y_\epsilon\| \geq \frac{\|x - y\|}{2} \tag{8}$$

for all $\epsilon \in (0, \eta)$, where x_ϵ, y_ϵ are obtained from x, y by moving these points a distance ϵ towards the corresponding right end-points. It follows that the measure of the set of all left end-points is not much reduced by the ϵ-motion, and a similar argument to that above involving an uncountable collection of pairwise disjoint sets shows that $P(S)$ has measure zero.

For, suppose C is a collection of disjoint line segments in E^2 with endset $W = P(C)$, and choose η small and positive, say $0 < \eta < \frac{1}{100}$. For each pair p, q of distinct points of E^2 whose coordinates are all rational, let $C(p, q)$ denote the collection of all members of C that have one end-point within $\eta\|p - q\|$ of p and the other within $\eta\|p - q\|$ of q. Denote the set of former end-points by $W(p, q)$.

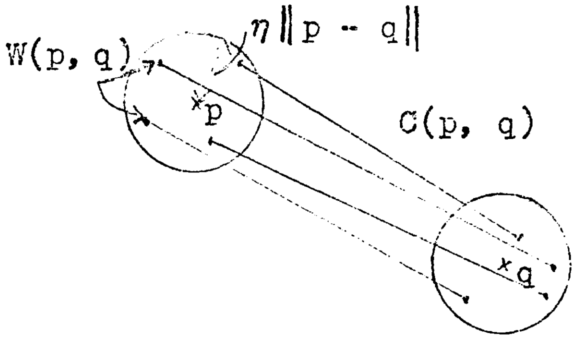


Fig. (v)

Then

$$C = \bigcup_{p,q} C(p, q) \quad \text{and} \quad W = \bigcup_{p,q} W(p, q) .$$

We need only to show that $\Lambda^2 W(p, q) = 0$.

For each point $x \in W(p, q)$, and each ϵ with $0 < \epsilon < \eta$, let $S(x)$ be the member of $C(p, q)$ that has x as one of its end-points and let x_ϵ be the point of $S(x)$ whose distance from x is ϵ. Let

$$W_\epsilon(p, q) = \{ x_\epsilon : x \in W(p, q) \} .$$

Then, assuming $W_\epsilon(p, q)$ to be Λ^2-measurable and that $\|x_\epsilon - y_\epsilon\| \geq \dfrac{\|x - y\|}{2}$ (see [13]) for all $x, y \in W(p, q)$ and $0 < \epsilon < \eta$, it follows that

$$\Lambda^2 W_\epsilon(p, q) \geq \frac{\Lambda^2 W(p, q)}{4} ,$$

and, as the various sets $W_\epsilon(p, q)$ are pairwise disjoint, a contradiction would ensue if $\Lambda^2 W(p, q) > 0$.

The problem is more difficult when $n = 3$. In this case, there is no relation of the form (8) , for the two segments may nearly cross and $\|x_\epsilon - y_\epsilon\|$ may be smaller than $\|x - y\|$, so that the above methods are inapplicable (see Fig. (vi)) .

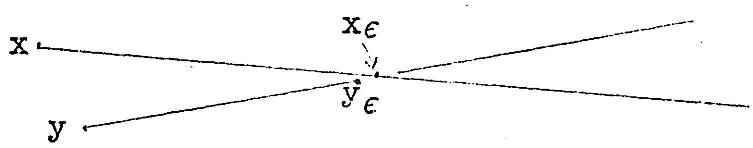


Fig. (vi)

There does exist, however, a compact endset of positive Lebesgue outer measure in E^3 , and there exists a compact set of positive Lebesgue measure in E^3 which is comprised of the end-points of a family of pairwise disjoint arcs.

For the former example, if f is a function from E^2 into E^1 whose graph has positive outer Lebesgue measure, then the graph of f union the graph of the function $f + 1$ yields the desired endset.

For the latter example, suppose C is a nowhere dense perfect subset of E^1 having positive measure. Then there exists a simple closed curve J which contains $C \times C$. So J has positive measure in E^2 . There exists a homeomorphism of the plane which carries J on to the unit circle. Since the unit circle is the set of ends of a family of disjoint arcs, the same is true of J. Extending this example in the obvious way to obtain a homeomorphism of E^3 into E^3 such that the image of the unit sphere contains $C \times C \times C$ which has positive measure in E^3, the result follows as before.(This example is due to Klee and Martin).

Also, the method used by Bruckner and Ceder [3] to construct a compact endset of positive measure in E^n. for $n \geqslant 4$ will work for $n = 3$ by using the set constructed by R. O. Davies [6, corollary to Theorem 7] in place of Nikodym's set.

The answer to the problem is thus affirmative when $n \leqslant 2$ and negative when $n \geqslant 3$.

It is not clear from the construction of Bruckner and Ceder, relying as it does on the Axiom of Choice to extract the line segments, that the set L of line segments can be made Lebesgue measurable. Larman, however, in [15], considers compact sets L of disjoint line segments. Here, a compact set of disjoint, closed, non-degenerate line segments is constructed in E^3 whose endset has positive 3-measure but whose set of 'non-end' points has zero 3-measure. This set has subsequently become known as the 'impossible set' and provides a constructive solution to the problem in 3 and higher dimensions.

3. Impossible Sets.

If L is a set of disjoint closed line segments, let $\mathcal{L}$ denote the point set union of members of L , and $\epsilon(L)$ the point set union of the end-points of L .

Theorem 3.1 (Larman, [15])

If $n \geq 3$ there exists a disjoint set L of closed line segments in E^3 such that $\Lambda^n\{\epsilon(L)\} > 0$ and $\Lambda^n\{\mathcal{L} \setminus \epsilon(L)\} = 0$, where $\mathcal{L}$ (and $\epsilon(L)$) is compact.

Note that it is enough to prove the theorem for E^3 since an example can be obtained in E^n by taking the cartesian product of the 3-dimensional example with a compact nowhere dense set in E^{n-3} of positive $(n - 3)$-measure.

The starting point for the construction is a lemma about plane sets, due to R. O. Davies, which is itself based on a construction of A. S. Besicovitch in connection with the Kakeya Problem. The problem posed by Kakeya (1917) was to find a set of minimum area in which a segment of length 1 could be continuously turned round so as to return to its original position with its ends reversed. The answer was believed to be the deltoid (three-cusped hypocycloid) with area $\pi/8$, ie. half the area of a circle of diameter 1 . Besicovitch, however, constructed (1920) a set with arbitrarily small area which contains segments of length 1 in all directions and realized in [2] that this set could be used to yield the solution of Kakeya's Problem – that there are sets (called Kakeya Sets) of arbitrarily small area in which a segment of length 1 can be turned through 360° by a continuous movement.

(Such examples of Kakeya Sets of small area are highly multiply connected and have large diameters. Further results concerning Kakeya Sets which are simply connected and which eliminate the unboundedness of the Besicovitch examples may be found in [5]).

The construction of Besicovitch forms the basis of a lemma in a paper [6] concerning linear accessibility (a member of a set of points in the plane is linearly accessible if through it there exists a straight line, infinite in both directions, containing no other point of the set).

Lemma 3.1 (Davies, [6])

Let R be a parallelogram $ABB'A'$ and K any closed set contained in R. Then, given a positive number $\epsilon > 0$, we can construct a finite set of parallelograms P_i, $(i = 1, 2, \ldots, n)$, contained in R and with two sides in AB and $A'B'$ such that

$$\text{(i)} \quad K \subset \bigcup_{i=1}^{n} P_i \quad ,$$

$$\text{(ii)} \quad \Lambda^2\{\bigcup_{i=1}^{n} P_i \setminus K\} < \epsilon \ .$$

Then, by considering the 2-dimensional projections of circular cylinders, and using Lemma 3.1 and standard covering theorems, Larman [15] proves the following lemma which is the main result needed for the construction of the 'impossible set'.

Lemma 3.2 (Larman, [15])

Let C be the right circular cylinder

$$C = \{ (x, y, z) : x^2 + y^2 \leqslant a^2, c \leqslant z \leqslant d \} ,$$

and suppose $c \leqslant e < f \leqslant d$. Write

$$C(e, f) = \{ (x, y, z) \in C : e \leqslant z \leqslant f \} .$$

Let $C_1, \ldots, C_k$ be right circular cylinders contained in the cylinder C , whose axes have non-parallel directions $\underline{u}_1, \ldots, \underline{u}_k$ respectively, and each having one end in $C(c, e)$. Let

$$\{\{S_{ij}\}_{j=1}^{t(i)}\}_{i=1}^{k}$$

be disjoint closed convex bodies such that

$$\bigcup_{j=1}^{t(i)} S_{ij} \subset C_i \setminus \bigcup_{\ell<i} C_\ell , \quad i = 1, \ldots, k .$$

Then, given $\epsilon > 0$, there exists a finite collection of right circular cylinders

$$\{\{\{C_{ij\ell}\}_{\ell=1}^{p(i,j)}\}_{j=1}^{t(i)}\}_{i=1}^{k}$$

with $C_{ij\ell} \subset C_i$, $(i = 1, \ldots, k)$, such that

(i) $C_{ijk} \cap C_{i'j'k'} = \phi , \quad i \neq i'$;

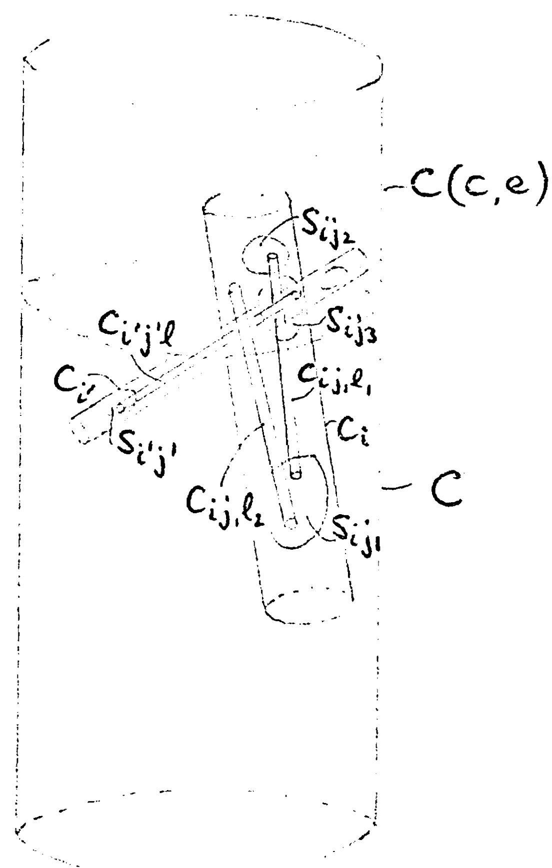
(ii) one end of $C_{ij\ell}$ is contained in $C(c,e)$ and the other end is contained in the interior of S_{ij} ;

(iii) $\Lambda^3\{\bigcup_{i=1}^{k}\bigcup_{j=1}^{t(i)} S_{ij} \setminus \bigcup_{i=1}^{k}\bigcup_{j=1}^{t(i)}\bigcup_{\ell=1}^{p(i,j)} C_{ij\ell}\} < \epsilon$;

(iv) $\Lambda^3\{\bigcup_{i=1}^{k}\bigcup_{j=1}^{t(i)}\bigcup_{\ell=1}^{p(i,j)} C_{ij\ell} \setminus \bigcup_{i=1}^{k}\bigcup_{j=1}^{t(i)} S_{ij}\} < \epsilon$.

Each convex set S_{ij} is thus associated with $p(i,j)$ right circular cylinders C_{ij} ,

Fig. (vii)



each with one end in the interior of S_{ij} , and the other end in $C(c,e)$, so arranged that inequalities of the form (iii) and (iv) above hold, ie. (iii)' $\Lambda^3\{ S_{ij} \setminus \bigcup_{\ell=1}^{p(i,j)} C_{ij\ell} \} < \epsilon'$;

(iv)' $\Lambda^3\{ \bigcup_{\ell=1}^{p(i,j)} C_{ij\ell} \setminus S_{ij} \} < \epsilon'$.

and so that the cylinders $C_{ij\ell}$ 'pass between' the cylinders $C_{i'j\ell}$, $i \neq i'$.

We begin the construction with the cylinder

$$C = \{ (x, y, z) : x^2 + y^2 \leqslant 1 , -2 \leqslant z \leqslant 2 \} ,$$

which we split into three parts :

$$D = \{ (x, y, z) \in C : 1 \leqslant z \leqslant 2 \} ,$$

$$D_0 = \{ (x, y, z) \in C : -1 \leqslant z \leqslant 1 \} ,$$

$$D_{-1} = \{ (x, y, z) \in C : -2 \leqslant z < -1 \} .$$

Define $\{\theta_i\}_{i=1}^{\infty}$ to be a strictly increasing sequence of positive numbers with $\lim \theta_i = \frac{1}{2}\Lambda^3(D)$.

Into D pack disjoint upright (ie. with axes parallel to the z-axis) circular cylinders $D(1), \ldots, D(n_1)$, each with diameter less than 1 , such that

$$\Lambda^3\{ D \setminus \bigcup_{i_1=1}^{n_1} D(i_1) \} < \theta_1 .$$

Into each $D(i_1)$ pack disjoint upright circular cylinders $D(i_1, 1), \ldots, D(i_1, n(i_1))$, each with diameter less than $\frac{1}{2}$, such that

$$\Lambda^3\{ D \setminus \bigcup_{i_1=1}^{n_1} \bigcup_{i_2=1}^{n(i_1)} D(i_1, i_2) \} < \theta_2 .$$

In general, suppose that sets $D(i_1, \ldots, i_k)$ have been defined inductively to satisfy

(i) $D(i_1, \ldots, i_k) \subset D(i_1, \ldots, i_{k-1})$,

$$i_k = 1, \ldots, n(i_1, \ldots, i_{k-1}) \; ;$$

(ii) $\operatorname{diam} D(i_1, \ldots, i_k) < \frac{1}{k}$;

(iii) $\Lambda^3\{ D \setminus \bigcup_{i_1=1}^{n_1} \cdots \bigcup_{i_k=1}^{n(i_1,\ldots,i_{k-1})} D(i_1, \ldots, i_k) \} < \theta_k$;

for $k = 1, 2, \ldots$

Then, if

$$E_k = \bigcup_{i_1=1}^{n_1} \cdots \bigcup_{i_k=1}^{n(i_1,\ldots,i_{k-1})} D(i_1, \ldots, i_k) \; ,$$

$\{E_k\}_{k=1}^{\infty}$ is a nested sequence of compact sets, and from (iii) above,

$$\Lambda^3\{ D \setminus \bigcup_{k=1}^{\infty} E_k\} \leq \lim_{k \to \infty} \theta_k = \tfrac{1}{2}\Lambda^3(D) \; .$$

Consequently,

$$\Lambda^3\{ \bigcup_{k=1}^{\infty} E_k \} \geq \tfrac{1}{2}\Lambda^3(D) > 0 \; . \tag{9}$$

Now let $\{\epsilon_n\}_{n=0}^{\infty}$ be a decreasing sequence of positive numbers such that

$$\sum_{n=0}^{\infty} \epsilon_n < \frac{1}{4}\Lambda^3(D) \; . \tag{10}$$

Applying Lemma 3.2 with $C_i = C$, $S_{ij} = D$, $C(c, e) = D_{-1}$, we construct a finite sequence of right circular cylinders $\{C(j_1)\}_{j_1=1}^{m}$ such that

(i) $C(j_1)$ has one end in the interior of D and the other end is in D_{-1} , $j_1 = 1, \ldots, m$;

(ii) $\Lambda^3\{ D \cap \bigcup_{j_1=1}^{m} C(j_1) \} > \Lambda^3(D) - \epsilon_0$;

(iii) $\Lambda^3\{ (C\backslash D) \cap \bigcup_{j_1=1}^{m} C(j_1) \} < \epsilon_0$.

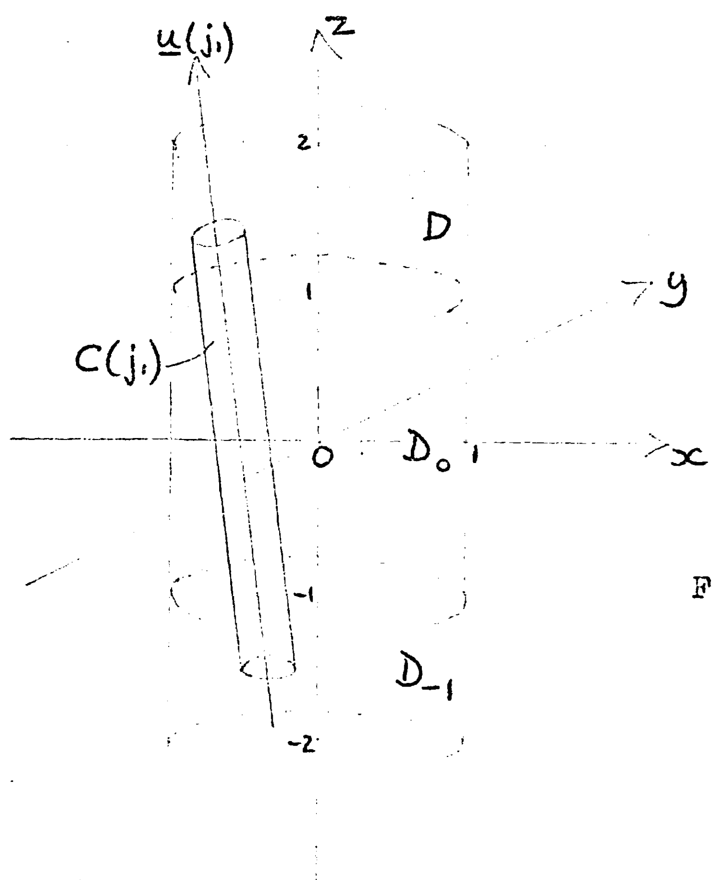


Fig. (viii)

Suppose the axis of $C(j_1)$ has direction $\underline{u}(j_1)$ for $j_1 = 1, \ldots, m$, where, by displacing each cylinder if necessary, we may suppose that the directions $\{\underline{u}(j_1)\}_{j_1=1}^{m}$ are all different.

Define

$$G(j_1) = D \cap \{ C(j_1)\backslash \bigcup_{l_1<j_1} C(l_1) \} .$$

For some positive integer $k_1 > 1$, let

$$\{ D(i_1, \ldots, i_{k_1}) \ , \ (i_1, \ldots, i_{k_1}) \in \Omega_1 \}$$

be the collection of all the sets $D(i_1, \ldots, i_{k_1})$ and let

$$\{ D(i_1, \ldots, i_{k_1}) \ , \ (i_1, \ldots, i_{k_1}) \in \Omega(j_1) \ , \ j_1 = 1, \ldots, m \}$$

be the subcollection such that $(i_1, \ldots, i_{k_1}) \in \Omega(j_1)$ if $D(i_1, \ldots, i_k)$ is in $G(j_1)$. Suppose that k_1 is sufficiently large as to ensure

$$\Lambda^3\{ \bigcup_{\Omega_1} D(i_1, \ldots, i_{k_1}) \setminus \bigcup_{j_1=1}^{m} \bigcup_{\Omega(j_1)} D(i_1, \ldots, i_{k_1}) \} < \epsilon_0 + \epsilon_1 .$$

Applying Lemma 3.2 again, we define a finite collection of right circular cylinders $\{C(j_1, j_2)\}_{j_2=1}^{m(j_1)}$ such that

(i) $C(j_1, j_2) \subset C(j_1)$;

(ii) $C(j_1, j_2) \cap C(j_1', j_2') = \phi \ , \ j_1 \neq j_1'$;

(iii) the directions $\underline{u}(j_1, j_2)$ of the axes of the cylinders $C(j_1, j_2)$ are all different ;

(iv) $C(j_1, j_2)$ has one end in D_{-1} and the other end is in the interior of some $D(i_1, \ldots, i_{k_1})$ with $(i_1, \ldots, i_{k_1}) \in \Omega(j_1)$;

(v) $\Lambda^3\{ \bigcup_{j_1=1}^{m} \bigcup_{\Omega(j_1)} D(i_1, \ldots, i_{k_1}) \setminus \bigcup_{j_1=1}^{m} \bigcup_{j_2=1}^{m(j_1)} C(j_1, j_2) \} < \epsilon_2$;

(vi) $\Lambda^3\{ \bigcup_{j_1=1}^{m} \bigcup_{j_2=1}^{m(j_1)} C(j_1, j_2) \setminus \bigcup_{j_1=1}^{m} \bigcup_{\Omega(j_1)} D(i_1, \ldots, i_{k_1}) \} < \epsilon_2$.

Proceeding inductively, we obtain for n a positive integer, right circular cylinders

$$\{ \ldots \{ C(j_1, \ldots, j_n) \}_{j_n=1}^{m(j_1, \ldots, j_{n-1})} \ldots \}$$

and sets

$$D(i_1, \ldots, i_{k_{n-1}}) \quad , \quad (i_1, \ldots, i_{k_{n-1}}) \in \Omega(j_1, \ldots, j_{n-1})$$

such that, if

$$D(i_1, \ldots, i_{k_{n-1}}) \quad , \quad (i_1, \ldots, i_{k_{n-1}}) \in \Omega_{n-1}$$

is the totality of the sets $D(i_1, \ldots, i_{k_{n-1}})$,

$$\Lambda^3\{ \bigcup_{\Omega_{n-1}} D(i_1, \ldots, i_{k_{n-1}}) \setminus \bigcup_{j_1, \ldots, j_{n-1}} \bigcup_{\Omega(j_1, \ldots, j_{n-1})} D(i_1, \ldots, i_{k_{n-1}}) \} < \sum_{i=0}^{n-1} \epsilon_i \qquad (11)$$

and

(i) $C(j_1, \ldots, j_n) \subset C(j_1, \ldots, j_{n-1})$,

$$\bigcup_{\Omega(j_1, \ldots, j_{n-1})} D(i_1, \ldots, i_{k_{n-1}}) \subset \bigcup_{\Omega(j_1, \ldots, j_{n-2})} D(i_1, \ldots, i_{k_{n-2}}) \ ;$$

(ii) $C(j_1, \ldots, j_{n-1}, j_n) \cap C(j_1', \ldots, j_{n-1}', j_n') = \phi$

if $(j_1, \ldots, j_{n-1}) \neq (j_1', \ldots, j_{n-1}')$;

(iii) the directions $\underline{u}(j_1, \ldots, j_n)$ of the axes of the $C(j_1, \ldots, j_n)$ are all different ;

(iv) $C(j_1, \ldots, j_n)$ has one end in D_{-1} and the other end is in the interior of some $D(i_1, \ldots, i_{k_{n-1}})$,

with $(i_1, \ldots, i_{k_{n-1}}) \in \Omega(j_1, \ldots, j_{n-1})$;

(v) $$\Lambda^3\{ \bigcup_{(j_1, \ldots, j_{n-1})} \bigcup_{\Omega(j_1, \ldots, j_{n-1})} D(i_1, \ldots, i_{k_{n-1}}) \setminus \bigcup_{(j_1, \ldots, j_n)} C(j_1, \ldots, j_n) \} < \epsilon_n ;$$

(vi) $$\Lambda^3\{ \bigcup_{(j_1, \ldots, j_n)} C(j_1, \ldots, j_n) \setminus \bigcup_{(j_1, \ldots, j_{n-1})} \bigcup_{\Omega(j_1, \ldots, j_{n-1})} D(i_1, \ldots, i_{k_{n-1}}) \} < \epsilon_n .$$

We now define the set of lines L .

For each sequence $(j_1, j_2, \ldots)$ of positive integers define

$$l(j_1, j_2, \ldots) = \begin{cases} \bigcap_{n=1}^{\infty} C(j_1, \ldots, j_n) & \text{if } C(j_1, \ldots, j_n) \text{ is defined for } n = 1, 2, \ldots \\ \phi & \text{, otherwise.} \end{cases}$$

Then each non-empty [by (i)] set $l(j_1, j_2, \ldots)$ is a closed line segment, of length at least 2 , which joins D to the closure of D_{-1} .

Also, if $(j_1, j_2, \ldots) \neq (j_1', j_2', \ldots)$, $\exists\, n$ such that $j_i = j_i'$, $i < n - 1$, and $j_{n-1} \neq j_{n-1}'$.

From (ii) above,

$$\begin{aligned} l(j_1, j_2, \ldots) \cap l(j_1', j_2', \ldots) & \subset C(j_1, \ldots, j_n) \cap C(j_1', \ldots, j_n') \\ & = \phi . \end{aligned}$$

So $L = \{ l(j_1, j_2, \ldots) , j_i \in N , i = 1, 2, \ldots \}$ is a collection of disjoint closed non-degenerate line segments.

Also, by (ii) , we have

$$\mathcal{L} = \bigcup_{(j_1, j_2, \ldots)} \bigcap_{n=1}^{\infty} C(j_1, \ldots, j_n) = \bigcap_{n=1}^{\infty} \bigcup_{(j_1, \ldots, j_n)} C(j_1, \ldots, j_n)$$

and so is a closed set.

If $\epsilon^+(L) = \epsilon(L) \cap D$, then $\epsilon^+(L)$ contains the set

$$\bigcap_{n=1}^{\infty} \{ \bigcup_{(j_1, \ldots, j_{n-1})} \bigcup_{\Omega(j_1, \ldots, j_{n-1})} D(i_1, \ldots, i_{k_{n-1}}) \cap \bigcup_{(j_1, \ldots, j_n)} C(j_1, \ldots, j_n) \}$$

$$= \bigcap_{n=1}^{\infty} F_n \ , \quad \text{say} .$$

Thus

$$\Lambda^3(F_n) > \Lambda^3\{ \bigcup_{(j_1, \ldots, j_n)} \bigcup_{\Omega(j_1, \ldots, j_{n-1})} D(i_1, \ldots, i_{k_n}) \} - \epsilon_n \ , \quad \text{by (v)} ,$$

$$> \Lambda^3\{ \bigcup_{\Omega_{n-1}} D(i_1, \ldots, i_{k_{n-1}}) \} - \sum_{i=0}^{n} \epsilon_i \ , \quad \text{by (11)} ,$$

$$> \frac{1}{2} \Lambda^3(D) - \frac{1}{4} \Lambda^3(D) \ , \quad \text{by (9), (10)} ,$$

$$= \frac{1}{4} \Lambda^3(D)$$

$$> 0 .$$

Since $\{F_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets ,

$$\Lambda^3\{ \epsilon^+(L) \} \geq \Lambda^3\{ \bigcap_{n=1}^{\infty} F_n \} \geq \frac{1}{4} \Lambda^3(D) > 0 . \quad (12)$$

For the set of 'non-end' points $\mathcal{L} \setminus \epsilon(L)$ of L , let V_p be the set of points of $\mathcal{L}$ which are at least a distance $\frac{1}{p}$ from either end of the closed line segment on which they lie , so that

$$\mathcal{L} \setminus \mathcal{E}(L) = \bigcup_{p=1}^{\infty} V_p .$$

Then, since $k_n \geqslant n$ and $\operatorname{diam.}\{D(i_1, \ldots, i_n)\} < \frac{1}{n}$, it follows that

$$V_p \subset \bigcup_{(j_1, \ldots, j_{n+1})} C(j_1, \ldots, j_{n+1}) \setminus \bigcup_{(j_1, \ldots, j_n)} \bigcup_{\Omega(j_1, \ldots, j_n)} D(i_1, \ldots, i_{k_n}) \tag{13}$$

for $n = p+1 , p+2 , \ldots$ So, by (vi) ,

$$\Lambda^3(V_p) \leqslant \epsilon_n , \quad n \geqslant p+1 ,$$

and thus $\Lambda^3(V_p) = 0$ and

$$\Lambda^3\{ \mathcal{L} \setminus \mathcal{E}(L) \} = 0 . \tag{14}$$

Since $\mathcal{L}$ is compact , (12) and (14) complete the construction of the 'impossible set' .

A slight modification of the above argument will show that $\Lambda^3\{\mathcal{E}^-(L)\} = 0$ where

$$\mathcal{E}^-(L) = \mathcal{E}(L) \cap D_{-1} = \mathcal{E}(L) \setminus \mathcal{E}^+(L) .$$

For , by defining W_p to be the set of points of $\mathcal{L}$ which are at least a distance $\frac{1}{p}$ from the end which is contained in D of the closed line segment on which they lie , we have

$$\mathcal{E}^-(L) \subset \mathcal{L} \setminus \mathcal{E}^+(L) = \bigcup_{p=1}^{\infty} W_p ,$$

and the inclusion (13) still holds with V_p replaced by W_p, so that $\Lambda^3(W_p) = 0$, $p = 1, 2, \ldots$

In connection with the problem of section 1 however, it is not clear from the above construction that $\epsilon(L)$ is compact. $\epsilon(L)$ is only a G_δ-set in general, rather than compact. The sets F_n above are closed so that $\epsilon^+(L)$ is a compact set. But there is no information about the distribution of the points of $\epsilon^-(L)$ – we only know that each line $l(j_1, j_2, \ldots)$ of L has one end in D_{-1}.

The problem is easily resolved, however, by cutting off the lower ends of the lines of L by the plane $z = -1$, for example. Let L' be the resulting set of lines above the plane $z = -1$. Then, again each line l' in L' has length at least 2. Since $\mathcal{L}$ is compact, the plane $z = -1$ intersects $\mathcal{L}$ in a compact set Z, of measure zero, and this set together with $\epsilon^+(L)$ forms the required compact endset of positive Λ^3-measure of the uncountable and disjoint set of lines L' above the plane $z = -1$.

<u>The impossible set in E^2</u>.

There is an analogue of lemma 3.2 in 2 dimensions provided the disjointness condition (i) is relaxed. Rectangles $R_{ij\ell}$ corresponding to the cylinders $C_{ij\ell}$ may be constructed with analogous properties, but disjointness being a 'stronger' condition in the plane, it will not in general be possible to construct the $R_{ij\ell}$, $l = 1, 2, \ldots, p(i,j)$, without some of the described overlapping occurring.

If R is the rectangle

$$\{ (x, y) : -1 \leqslant x \leqslant 1 , -2 \leqslant y \leqslant 2 \} ,$$

and we define

$$S = \{ (x, y) \in R : 1 \leqslant y \leqslant 2 \} ,$$

$$S_0 = \{ (x, y) \in R : -1 \leqslant y \leqslant 1 \} ,$$

$$S_{-1} = \{ (x, y) \in R : -2 \leqslant y < -1 \} ,$$

the construction leads to a system $S(i_1, \ldots, i_k)$, $k = 1, 2, \ldots$, of rectangles with sides parallel to the sides of R , packed into S with the properties of $D(i_1, \ldots, i_k)$, $k = 1, 2, \ldots$, and a system $R(j_1, \ldots, j_n)$, $n = 1, 2, \ldots$, of rectangles defined for sequences $j_1, j_2, \ldots$, of positive integers , each having one end in S_{-1} and the other end in the interior of some $S(i_1, \ldots, i_k)$, and having all the properties of the $C(j_1, \ldots, j_n)$ described above with the exception of the disjointness condition (ii) .

Then , as before , if for each sequence $(j_1, j_2, \ldots)$ of positive integers we define

$$l(j_1, j_2, \ldots) = \begin{cases} \bigcap_{n=1}^{\infty} R(j_1, \ldots, j_n) & \text{if } R(j_1, \ldots, j_n) \\ & \text{is defined for } n = 1, 2, \ldots \\ \phi , & \text{otherwise,} \end{cases}$$

$$L = \{ l(j_1, j_2, \ldots) , \quad j_i \in N , \quad i = 1, 2, \ldots \}$$

is a collection of closed non-degenerate line segments, each of length at least 2 and joining S to the closure of S_{-1}.

Also, $\Lambda^2\{\epsilon(L)\} > 0$, $\Lambda^2\{\mathcal{L}\setminus\epsilon(L)\} = 0$, and $\epsilon^+(L)$ is compact. Since the rectangles $R(j_1, \ldots, j_n)$ are not in general disjoint, it is not clear whether $\mathcal{L}$ is compact.

It is interesting to ask how much symmetry can be achieved in constructing an impossible set. For example, for the set constructed in E^3, by cutting off the bottom ends of the lines $l(j_1, j_2, \ldots)$ so that

$$|l \cap D| = |l \cap D_0| ,$$

a set of pairwise disjoint lines L' symmetrical in the plane $z = 1$ is obtained with $\Lambda^3\{\epsilon^+(L')\} > 0$, $\Lambda^3\{\mathcal{L}'\setminus\epsilon(L')\} = 0$, but $\Lambda^3\{\epsilon^-(L')\} = 0$.

Also, if σ denotes the operation of reflecting in the plane $z = 0$, the set $\epsilon[L \cup \sigma(L)]$ is symmetrical in the plane $z = 0$ with

$$\Lambda^3\{ \epsilon^+[L \cup \sigma(L)] \} = \Lambda^3\{ \epsilon^-[L \cup \sigma(L)] \} > 0 ,$$

$$\Lambda^3\{ \mathcal{L} \cup \sigma(\mathcal{L}) \setminus \epsilon[L \cup \sigma(L)] \} = 0 ,$$

but since the lines $L \cup \sigma(L)$ are not pairwise disjoint, the set $\epsilon[L \cup \sigma(L)]$ is not strictly an endset.

Probably the main interest in more symmetric endsets is in showing that they cannot exist.

The set constructed above provides an example of sets A , B and a collection L of line segments with the properties that :

(i) every $x \in A$ is an endpoint of at least one $l \in L$;

(ii) every $x \in B$ is an endpoint of at least one $l \in L$;

(iii) each $l \in L$ has one end in A and the other end in B ;

(iv) $|A| > 0$;

(v) $|B| > 0$;

(vi) $|\mathcal{L} \setminus (A \cup B)| = 0$.

Since $0 < |A \cup B| < +\infty$, $A \cup B$ has Hausdorff dimension 3 . It seems highly likely that it is impossible to construct such a set for which

$$\dim \{\mathcal{L} \setminus (A \cup B)\} < \dim \{A \cup B\} .$$

However, this has not been investigated fully and so we conclude by stating it as a conjecture :

<u>Conjecture</u>.

If $\epsilon(L)$ is the endset corresponding to a disjoint set L of line segments in E^3 with the property that $|\epsilon(L)| > 0$, then the lower bound for the Hausdorff dimension achievable by the union $\angle \setminus \epsilon(L)$ of the interiors of these line segments is 3 .

CHAPTER IV

Uniform Distribution

1. In Chapter II the convergence of the series

$$\sum_{k=1}^{\infty} \{ ((n_k x)) - \alpha_k \}$$

where $\{\alpha_k\}$, $(k = 1, 2, \dots)$, is a sequence of real numbers satisfying $0 \leqslant \alpha_k \leqslant 1$, and $\{n_k\}$ satisfies

$$\frac{n_{k+1}}{n_k} \geqslant \rho > 1 ,$$

was considered. This leads to a further discussion on the equidistribution of the sequence $\{ ((n_k x)) \}$, $(k = 1, 2, \dots)$.

Definition.

The sequence $\beta_1, \beta_2, \dots, \beta_r, \dots$ in $(0,1)$ is equidistributed in $(0,1)$ [uniformly distributed in $(0,1)$, or uniformly distributed modulo 1] if , for every l, m satisfying $0 \leqslant l < m \leqslant 1$, the density of integers r for which $l \leqslant \beta_r \leqslant m$ is exactly $(m - l)$; that is , if

$$\epsilon_r = \begin{cases} 1 , \text{ when } l \leqslant \beta_r \leqslant m , \\ 0 , \text{ otherwise} , \end{cases}$$

and $N(\Delta, k)$ is the number of members of the finite sequence $\beta_1, \beta_2, \dots, \beta_k$ in the interval $\Delta = (l,m)$, then

$$\lim_{k \to \infty} \frac{N(\Delta, k)}{k} = \lim_{k \to \infty} \left\{ \frac{1}{k} \sum_{r=1}^{k} \epsilon_r \right\} = |\Delta| ,$$

with a corresponding definition for a sequence of n-dimensional vectors $\underset{\sim}{\beta}^{(r)}, (r = 1, 2, \ldots)$ in R^n , $n > 1$.

In general, any sequence $\beta_1, \beta_2, \ldots, \beta_r, \ldots$ of real numbers is said to be uniformly distributed modulo 1 if the sequence of fractional parts $((\beta_1)), ((\beta_2)), \ldots, ((\beta_r)), \ldots$ (in $[0,1)$) is equidistributed in $(0,1)$.

Theorem 4.1

Let $\underset{\sim}{\beta}^{(r)}, (r = 1, 2, \ldots)$ be any sequence of n-dimensional vectors, not necessarily restricted to lie in the unit cube. The necessary and sufficient condition that it be uniformly distributed modulo 1 is that

$$\lim_{k \to \infty} \left\{ \frac{1}{k} \sum_{r \leqslant k} e(\underset{\sim}{m} \underset{\sim}{\beta}^{(r)}) \right\} = 0$$

for all integral vectors $\underset{\sim}{m} \neq \underset{\sim}{0}$, where

$$e(x) = \exp(2\pi i x) , \quad i^2 = -1 .$$

Note : Since the statements of the theorem are not affected by replacing the vectors $\underset{\sim}{\beta}^{(r)}$ by congruent vectors modulo 1 , we may suppose, if we wish, that they all lie in the unit cube $0 \leqslant \beta_j < 1$, $(1 \leqslant j \leqslant n)$, in which case uniform distribution modulo 1 is simply uniform distribution.

Thus, in the 1-dimensional case, we have the sequence of real numbers $\beta_1, \beta_2, \ldots, \beta_r, \ldots$ is equidistributed in (0,1) if and only if

$$\lim_{k \to \infty} \left\{ \frac{1}{k} \sum_{r=1}^{k} \exp(m\beta_r 2\pi i) \right\} = 0$$

for every positive integer m .

This result is due to Weyl (1916) [22] , Theorem 1 , p.315 .

An application of this theorem yields the following well-known result : (see [4] or [22] , for example)

Theorem 4.2

Let ξ be an irrational number. Then the sequence of multiples of ξ ,

$$\xi \,,\; 2\xi,\; 3\xi,\; \ldots$$

is uniformly distributed modulo 1 , ie. the sequence of fractional parts of the multiples of an irrational number ξ ,

$$((\xi)) \,,\; ((2\xi)) \,,\; ((3\xi)) \,,\; \ldots$$

is uniformly distributed in the unit interval.

Proof :

Let m be a positive integer and let $m\xi = \eta$. Then the result follows if

$$\sum_{r=1}^{k} e(m ((r\xi))) = o(k) .$$

Now

$$\left|\sum_{r=1}^{k} e(\, m((r\xi))\,)\right| = \left|\sum_{r=1}^{k} e(\, r\eta - m[r\xi]\,)\right|$$

$$= \left|\sum_{r=1}^{k} e(\, r\eta\,)\right|$$

$$= \left|\frac{e(\,(k+1)\eta\,) - e(\eta)}{e(\eta) - 1}\right|$$

$$\leqslant \frac{2}{|e(\eta) - 1|}$$

$$= \frac{1}{|\sin \pi\eta|} .$$

Since η is not an integer, the result follows.

For a proof of this result direct from the definition of uniform distribution modulo 1, see [18], p.24 .

In 2-dimensions, if

$$\underset{\sim}{\beta}^{(r)} = (\,((rx)),\ ((ry))\,)\ ,\quad x, y \in R^1 ,$$

$(r = 1, 2, \ldots)$, and $\underset{\sim}{m}$ is a non-zero integer, then

$$\left|\sum_{r\leqslant k} e(\,\underset{\sim}{m}\,\underset{\sim}{\beta}^{(r)})\right| = \left|\sum_{r\leqslant k} e(\, r\,\underset{\sim}{m}\,\underset{\sim}{\theta}\,)\right| ,$$

where $\underset{\sim}{\theta} = (x, y)$

$$= \left|\frac{e(\,(k+1)\underset{\sim}{m}\,\underset{\sim}{\theta}\,) - e(\underset{\sim}{m}\,\underset{\sim}{\theta})}{e(\underset{\sim}{m}\,\underset{\sim}{\theta}) - 1}\right|$$

and so $\{\underset{\sim}{\beta}^{(r)}\}_{r=1}^{\infty}$ is equidistributed in the unit square $(0,1) \times (0,1)$ provided there is no relation $\underset{\sim}{m}\,\underset{\sim}{\theta}$ = integer with integral $\underset{\sim}{m} \neq \underset{\sim}{0}$,

ie. $\Longleftrightarrow$ x , y , 1 are linearly independent over the field of rational numbers ,

$\Longrightarrow$ x , y are both irrational .

In general, in R^n , if $\xi_1, \xi_2, \ldots, \xi_n$ are n real numbers, and there is no relation of the form

$$m_1\xi_1 + m_2\xi_2 + \ldots + m_n\xi_n = m_{n+1} ,$$

where $m_1, m_2, \ldots, m_{n+1}$ are rational numbers (or, equivalently, integers) , then the sequence of points

$$\{ (r\xi_1 , r\xi_2 , \ldots , r\xi_n) \} , \quad (r = 1, 2, \ldots) ,$$

is uniformly distributed modulo 1 ,
ie. the sequence of points

$$\{ (((r\xi_1)), ((r\xi_2)), \ldots , ((r\xi_n))) \} , (r = 1, 2, \ldots),$$

is uniformly distributed (and dense, in the sense of (i) , Chapter II, p.15) in the n-dimensional cube.

A companion result in case $1, \xi_1, \ldots , \xi_n$ are linearly dependent is the following :

<u>Theorem 4.3</u> [18]

Let $\xi_1, \ldots , \xi_n$ be irrational but such that $1, \xi_1, \ldots , \xi_n$ are linearly dependent over the field of rational numbers , say

$$m_1\xi_1 + m_2\xi_2 + \dots + m_n\xi_n = m_{n+1} .$$

Then the points

$$(((k\xi_1)), ((k\xi_2)), \dots , ((k\xi_n))) , \quad (k = 1, 2, \dots) ,$$

whose coordinates are the fractional parts of the multiples of $\xi_1, \dots , \xi_n$ lie on and only on , those portions of the lines

$$m_1x_1 + m_2x_2 + \dots + m_nx_n = t ,$$

where t is any integer , lying within the unit cube.

Furthermore, the points are dense on these segments.

Thus far we have only considered the sequence $n_k = k$, $(k = 1, 2, \dots)$.

Suppose $\{\lambda_k\}_{k=1}^{\infty}$ is an increasing sequence of integers, or real numbers, tending to infinity.

In 1912 Hardy and Littlewood posed the following problems :

Question 1.

For which sequences $\{\lambda_k\}_{k=1}^{\infty}$ is it true that $\{\lambda_k x\}_{k=1}^{\infty}$ is dense modulo 1 for every irrational x ?

(Note : $\{y_k\}$ is dense modulo 1 if $\{ ((y_k)) \}$ is dense in $[0,1]$).

Question 2.

For which sequences $\{\lambda_k\}_{k=1}^{\infty}$ is it true that

$\{((\lambda_k x))\}_{k=1}^{\infty}$ is uniformly distributed modulo 1 for almost all x ?

Theorem 4.4 (Hardy and Littlewood (1914), [10])

Let $\{\lambda_k\}_{k=1}^{\infty}$ be any increasing sequence of real numbers tending to infinity.

Then $\{\lambda_k x\}$ is dense modulo 1 for almost all x .

Now suppose the increasing sequence $\{n_k\}_{k=1}^{\infty}$ of integers is such that

$$\underbrace{n_1 \leqslant n_2}_{h_1 \text{ terms}} \leqslant \underbrace{n_3 \leqslant \cdots}_{h_2 \text{ terms}} \leqslant \underbrace{n_k}_{h_q \text{ terms}} \leqslant \cdots$$

the first h_1 terms are equal, then the next h_2 terms are equal, ... , and the last h_q terms up to n_k are equal , so that in $\{n_1, n_2, \ldots, n_k\}$ there are q distinct terms. Then, if there exist constants c, ϵ such that

$$\max_{1 \leqslant i \leqslant q} h_i \leqslant \frac{ck}{(\log k)^{1+\epsilon}} ,$$

$\{((n_k x))\}$ is uniformly distributed in $[0,1]$ for almost all x (Weyl [22] , § 7) .

To answer question 2 we need a measure for the repetitions in the sequence $\{n_k\}$:

Notation : Let $\{n_k\}$ be any sequence of integers. Suppose that in $\{n_1, \ldots, n_k\}$ there are q distinct elements

occurring with frequencies $h_1, h_2, \dots, h_q$,
$h_1 + h_2 + \dots + h_q = k$.
Put

$$\rho_k = \frac{h_1^2 + h_2^2 + \dots + h_q^2}{k^2} .$$

Definition.

A sequence $\{x_k\}$ of non-negative numbers is called asymptotically small if there exists a divergent series $\sum \alpha_k$ of positive terms such that

$$\sum \alpha_k x_k < +\infty .$$

(Note that this definition is equivalent to $\lim_{k \to \infty} x_k = 0$).

Theorem 4.5 (M. Mehdi)

Let $\{n_k\}_{k=1}^{\infty}$ be any sequence of integers.

Then $\{((n_k x))\}_{k=1}^{\infty}$ is uniformly distributed modulo 1 for almost all x if and only if $\{\rho_k\}_{k=1}^{\infty}$ is asymptotically small.

Proof :

(i) Necessity :

Suppose in $\{n_1, n_2, \dots, n_k\}$ the distinct terms are $n_{\lambda_1}, n_{\lambda_2}, \dots, n_{\lambda_q}$ occurring with frequencies $h_1, h_2, \dots, h_q$ respectively.

Write

$$f_k(x) = \frac{1}{k} \sum_{s=1}^{k} e^{2\pi i m n_s x} ,$$

where $m \neq 0$ is a fixed integer,

$$= \frac{1}{k} \sum_{s=1}^{q} h_s e^{2\pi i m n_{\lambda_s} x} .$$

Then $\int_0^1 |f_k(x)|^2 dx = \frac{1}{k^2} \sum_{s=1}^{q} h_s^2 = \rho_k ,$

and $|f_k(x)| \leqslant 1$, $(k = 1, 2, \ldots)$.

Using Theorem 4.1 , by dominated convergence ,

$$\int_0^1 |f_k(x)|^2 dx \to 0 \text{ as } k \to \infty ,$$

$$\Rightarrow \rho_k \to 0 \text{ as } k \to \infty$$

$$\Rightarrow \{\rho_k\}_{k=1}^{\infty} \text{ is asymptotically small.}$$

(ii) Sufficiency :

Suppose $\{\rho_k\}$ is asynptotically small. Then there exists a divergent series of positive terms $\sum \alpha_k$ such that $\sum \alpha_k \rho_k < +\infty$.

So $$\sum_{k=1}^{\infty} \alpha_k \int_0^1 |f_k(x)|^2 dx < +\infty ,$$

$$\int_0^1 \sum_{k=1}^{\infty} \alpha_k |f_k(x)|^2 dx < +\infty .$$

$$\sum_{k=1}^{\infty} \alpha_k \, |f_k(x)|^2 \quad < \quad +\infty \qquad \text{pp. in } [0,1] \; .$$

This implies $f_k(x) \to 0$ pp. in $[0,1]$ and therefore in R.

Thus if $\{n_k\}$ is a (strictly) increasing sequence of integers, $h_s = 1$ for all s and

$$\rho_k = \frac{h_1^2 + h_2^2 + \ldots + h_q^2}{k^2} \leqslant \frac{1}{k} \max_{1\leqslant i\leqslant q} h_i$$
$$= \frac{1}{k} \; ,$$

and so the set of values of x such that $((n_k x))$, $k = 1, 2, \ldots,$ is not equidistributed in $(0,1)$ has zero Lebesgue measure.

There is an n-dimensional analogue of this result.

A further problem which may be investigated is :

<u>Question 3</u>.

Given two increasing sequences of integers $\{m_k\}$, $\{n_k\}$, tending to infinity, does the set

$$\{ (x, y) \; : \; (((n_k x)), ((m_k y))) \text{ is uniformly distributed} \}$$

have full measure ?

The 'size' of the exceptional set for which $((n_k x))$ is not equidistributed depends on the sequence $\{n_k\}$.

For example, if n_k is given by a polynomial in k with integer coefficients, then the set of x for which $\{((n_k x))\}$ is not equidistributed is enumerable. (We have seen that if $n_k = k$, the exceptional set consists of the rationals). In this case $t_k = \frac{n_{k+1}}{n_k} \to 1$ as $k \to \infty$. However, Erdos and Taylor [8] show that $n_{k+1} - n_k$ bounded is not a sufficient condition to ensure that the exceptional set has power $\aleph_0$.

<u>Theorem 4.6</u> (1957, [8])

There exists a finite constant C, and an increasing sequence of integers $\{n_k\}$ such that

$$n_{k+1} - n_k < C \, , \quad (k = 1, 2, \ldots) \, ,$$

and the set of x such that $((n_k x))$ is not equidistributed is not enumerable.

<u>Proof</u> :

Suppose $\{\lambda_i\}$ is an enumeration of the rationals in $[\frac{1}{8}, \frac{1}{6}]$, and each rational occurs in the sequence infinitely often.

Let

$$k_s = 5^s \, , \quad (s = 0, 1, 2, \ldots) \, .$$

Set $n_1 = 1$ and suppose for some positive integer r, n_k has been defined for $k \leq k_{r-1}$. Suppose n_{k-1} has been defined, and let n_k be the smallest integer greater than n_{k-1} for which

$$\cos(n_k \lambda_r 2\pi) > \tfrac{1}{2} ,$$

so that n_k is defined by induction in the range

$$k_{r-1} < k \leqslant k_r , \quad (r = 1, 2, \ldots) ,$$

Now since $\frac{1}{6} \geqslant \lambda_r \geqslant \frac{1}{8}$, we have

$$n_{k+1} - n_k < \frac{3\pi}{\lambda_r} < 24\pi ,$$

so that $n_{k+1} - n_k < 100$, $(k = 1, 2, \ldots)$.

Also,

$$\sum_{k=k_{r-1}+1}^{k_r} \cos(n_k \lambda_r 2\pi) > \tfrac{1}{2}(k_r - k_{r-1}) = 2k_{r-1} ;$$

and so

$$\frac{1}{k_r} \sum_{k=1}^{k_r} \cos(n_k \lambda_r 2\pi) > \frac{1}{k_r}(2k_0 + 2k_1 + \ldots + 2k_{r-1})$$

$$> \frac{5^{r-1}}{k_r}$$

$$= \frac{1}{5} .$$

Define I_r to be an open interval containing λ_r such that if x is in I_r , then

$$\frac{1}{k_r} \sum_{k=1}^{k_r} \cos(n_k x 2\pi) > \frac{1}{5} . \qquad (15)$$

Let

$$E = \bigcap_{q=1}^{\infty} \bigcup_{r=q}^{\infty} I_r = \limsup I_r .$$

Then E contains all points x which are in infinitely many I_r and so contains every rational in $[\frac{1}{8}, \frac{1}{6}]$. Thus E is everywhere dense in the interval $(\frac{1}{8}, \frac{1}{6})$. Also, E is a G_δ-set, and by the Baire Category Theorem, such a set which is dense in an interval cannot have power $\leqslant \aleph_0$.

If $x \in E$, then given N , x is in I_r for some $r > N$. By (15) , there is an integer $k_r > N$ such that

$$\frac{1}{k_r} \sum_{k=1}^{k_r} \cos\{((n_k x))2\pi\} = \frac{1}{k_r} \sum_{k=1}^{k_r} \cos(n_k x 2\pi) > \frac{1}{5} .$$

Hence, for any x in E ,

$$\limsup_{t \to \infty} \frac{1}{t} \sum_{k=1}^{t} \cos\{((n_k x))\, 2\pi\} \geqslant \limsup_{r \to \infty} \frac{1}{k_r} \sum_{k=1}^{k_r} \cos\{((n_k x))\, 2\pi\}$$

$$\geqslant \frac{1}{5} ,$$

and so, by Theorem 4.1 , $\{((n_k x))\}$ is not equidistributed in (0,1) . So the sequence $\{n_k\}$ satisfies the required conditions with $C = 100$.

Baker [1] , however, has improved on this construction by showing that the constant 100 may be replaced by 2 , and that rather more is true.

Theorem 4.7 (1972, [1])

For a strictly increasing sequence of positive integers $\mathcal{S} = \{n_k\}_{k=1}^{\infty}$, denote by $E(\mathcal{S})$ the set of x for which the sequence $\{((n_k x))\}_{k=1}^{\infty}$ is not equidistributed.

Let N be a given positive integer.

There exists a sequence $\mathcal{S}$ of positive integers such that $E(\mathcal{S})$ is not enumerable, and

(i) $1 \leq n_{k+1} - n_k \leq 2$, $(k \geq 1)$,

(ii) if k_1, k_2 are indices for which $n_{k+1} - n_k = 2$, then $|k_2 - k_1| \geq N$.

Thus the sequence $\{n_k\}_{k=1}^{\infty}$ is just $1, 2, 3, \ldots$ with an integer removed as infrequently as we wish, in a sense. Note that if only $o(n)$ of the first n integers were removed, $E(\mathcal{S})$ would consist of the rationals.

Proof :

Assume $N \geq 2$. Let $\delta = \dfrac{1}{3N+4}$ and $M = 6N+9$. Let $\{\lambda_i\}_{i=1}^{\infty}$ be an enumeration of the rationals in $[2\delta, 3\delta]$ and let each rational occur in the sequence infinitely often.

Let S_i denote the set of integers n such that

$$M^{2i} \leq n < M^{2i+1} , \quad ((n\lambda_i)) \leq \delta , \quad (i = 1, 2, \ldots).$$

Define the sequence $\mathcal{S} = \{n_k\}_{k=1}^{\infty}$ to consist of those $n \geq 1$ which do not belong to any S_i .

To show that $\mathcal{S}$ satisfies (i) and (ii):

Choose $n \in \mathcal{S}$ with

$$M^{2i} \leq n < M^{2i+1} \text{ and } ((n\lambda_i)) > \delta$$

for some $i \geq 1$ and suppose $(n+1) \notin \mathcal{S}$.

Then $(([n+1]\lambda_i)) \leqslant \delta$.

Suppose $n\lambda_i = I_1 + \eta_1$, $I_1 \in N$, $1 > \eta_1 > \delta$.

Then $(n+1)\lambda_i = (I_1 + 1) + \eta_2$, $0 \leqslant \eta_2 \leqslant \delta$.

$$2\delta < \lambda_i < 3\delta \implies I_1 + 1 - 3\delta < n\lambda_i < I_1 + 1 - \delta ,$$

(with possibly equality for the special cases $\lambda_i = 2\delta , 3\delta$) .

Hence

$$I_1 + 1 + \delta < (n+2)\lambda_i < I_1 + 1 + 5\delta$$

$$\implies (([n+2]\lambda_i)) > \delta$$

$$\implies (n+2) \in \mathcal{S} .$$

Thus $\mathcal{S}$ satisfies (i) .

Also,

$$I_1 + 1 + (2r-3)\delta < (n+r)\lambda_i < I_1 + 1 + (3r-1)\delta$$

$$< I_1 + 1 ,$$

for $r = 0, 1, \ldots , N$, so that

$$(([n+r]\lambda_i)) > \delta , \quad 2 \leqslant r \leqslant N ,$$

and so $(n+r) \in \mathcal{S}$, $2 \leqslant r \leqslant N$.

Thus $\mathcal{S}$ satisfies (ii) .

Let J_i denote the set of real x such that

$$((nx)) > \delta ,$$

whenever $M^{2i} \leqslant n < M^{2i+1}$, $n \notin S_i$.

Clearly J_i is open and $\lambda_i \in J_i$.

Let $$E = \bigcap_{q \geq 1} \bigcup_{i \geq q} J_i \, .$$

Then, as in Theorem 4.6, E is uncountable.

Finally, if $x \in E$, and $N_k[0, \delta)$ is the number of terms of $((n_h x))$, $(h \leq k)$, in $[0, \delta)$,

$$\liminf_{k \to \infty} \frac{N_k[0, \delta)}{k} \leq \frac{1}{M(1 - \frac{1}{N})} < \delta ,$$

$$\Rightarrow E \subset E(\delta) .$$

We now consider the Hausdorff dimension of the exceptional set of x for which $\{((n_k x))\}_{k=1}^{\infty}$ is not equidistributed for a sequence $\{n_k\}_{k=1}^{\infty}$ satisfying the conditions of Theorem 4.6 .

<u>Theorem 4.8</u> . [8]

Suppose C is a constant, and $\{n_k\}_{k=1}^{\infty}$ an increasing sequence of integers such that

$$n_{k+1} - n_k < C , \quad (k = 1, 2, \dots) .$$

Then the set of points x for which $\{((n_k x))\}_{k=1}^{\infty}$ is not equidistributed has dimension zero.

The method of proof may also be used to show that the exceptional set of x for which $\{((n_k x))\}_{k=1}^{\infty}$ is not equidistributed has measure zero with respect to the Hausdorff measure function

$$h(z) = \frac{1}{[\log \frac{1}{z}]^{1+\epsilon}} , \quad \text{for every } \epsilon > 0 .$$

Note that since the sequence $\{n_k\}$ constructed in Theorem 4.7 satisfies $n_k = O(k)$, Theorem 4.8 implies that the Hausdorff dimension of $E(\mathfrak{S})$ is zero. Also the set of x for which

$$\limsup_{k \to \infty} \frac{N_k[0, \delta)}{k} < \delta$$

is finite.

The methods of Theorem 4.8 may also be used to yield :

<u>Theorem 4.9</u> [8]

Suppose $C \geq 0$, $\rho \geq 1$ are constants and $\{n_k\}$ is an increasing sequence of integers such that

$$n_k < C\, k^{\rho} , \quad (k = 1, 2, \ldots) .$$

Then the set of points x for which $\{((n_k x))\}_{k=1}^{\infty}$ is not equidistributed has dimension not greater than $(1 - \frac{1}{\rho})$, there being sequences $\{n_k\}_{k=1}^{\infty}$ for which the bound $(1 - \frac{1}{\rho})$ is attained.

The sequences $\{n_k\}$ considered so far do not increase too quickly. The sequences of Theorems 4.8 and 4.9 are not lacunary (ie. they do not satisfy (6) of Chapter II, $t_k = \frac{n_{k+1}}{n_k} \geq \rho > 1$), they satisfy

$$\liminf_{k \to \infty} t_k = 1 .$$

The case $t_k \to \infty$ is already discussed in

Chapter II. For the set E of values of x such that $\sum_{k=1}^{\infty} \{ ((n_k x)) - \alpha \}$ converges , $0 < \alpha < 1$, has dimension 1 . We thus have $((n_k x)) \to \alpha$, as $k \to \infty$, for x in this set E , and so $\{ ((n_k x)) \}_{k=1}^{\infty}$ cannot be equidistributed.

The condition above that the sequence $\{n_k\}$ be lacunary implies that the exceptional set of x for which $\{ ((n_k x)) \}_{k=1}^{\infty}$ is not equidistributed has dimension 1 :

Theorem 4.10 [8]

If $\{n_k\}$ is an increasing sequence of integers such that $t_k \geq \rho > 1$, than the set E of values of x such that $\{ ((n_k x)) \}_{k=1}^{\infty}$ is not equidistributed in $(0,1)$ has dimension 1 .

The proof is another application of Theorem 2.5 to show that E has positive Λ^s-measure for any s satisfying $0 < s < 1$.

2. The uniform distribution of the integral parts of the multiples of an irrational number .

Consider the integral parts of the multiples of an irrational number ξ ,

$$[\xi] , [2\xi] , [3\xi] , \ldots$$

The question arises as to whether, in the limiting sense, there are just as many even numbers as odd, or just as many in each of the three congruence classes modulo 3 , etc.

Consider any sequence of integers $n_1, n_2, n_3, \ldots$ For any modulus $m > 1$, define $N(k, j, m)$ as the number of integers n_h satisfying $h \leqslant k$ and $n \equiv j \pmod m$. Then the sequence $\{n_k\}$ is said to be uniformly distributed modulo m if

$$\lim_{k \to \infty} \frac{N(k, j, m)}{k} = \frac{1}{m} ,$$

for each of the residue classes $j = 1, 2, 3, \ldots , m$. The sequence $\{n_k\}$ is said to be uniformly distributed if it is uniformly distributed modulo m for every positive integer $m \geqslant 2$.

The answer to the above question is then affirmative, for the result of Theorem 4.2 may be used to show that if ξ is an irrational number, then the integral parts of its multiples,

$$[\xi] , [2\xi] , [3\xi] , \ldots$$

form a uniformly distributed sequence of integers [18] .

For, if m is a fixed arbitrary integer, $\frac{\xi}{m}$ is irrational, and the sequence

$$\left\{ \left(\left(\frac{k\xi}{m}\right)\right) \right\} = \left\{ \frac{k\xi}{m} - \left[\frac{k\xi}{m}\right] \right\} , \quad (k = 1, 2, \ldots)$$

is uniformly distributed in the unit interval. Multiplying the terms of this sequence by m , we see that the sequence

$$\left\{ k\xi - m\left[\frac{k\xi}{m}\right] \right\} , \quad (k = 1, 2, \ldots) ,$$

is uniformly distributed (in the obvious sense) over the real line from 0 to m . Hence, by taking the integral parts of this sequence , a sequence of integers which is uniformly distributed modulo m is obtained ,

$$\left\{ \left[k\xi - m\left[\frac{k\xi}{m}\right] \right] \right\} = \left\{ [k\xi] - m\left[\frac{k\xi}{m}\right] \right\}, \quad (k = 1, \ldots)$$

However, since all multiples of m may be ignored modulo m , we have that the sequence $\{[k\xi]\}$ is uniformly distributed modulo m for all $m \geq 2$.

Note that if ξ is irrational and satisfies $-1 < \xi < 1$, then the sequence

$$[\xi] , [2\xi] , [3\xi] , \ldots$$

does not consist of distinct integers, but the definition of a uniformly distributed sequence of integers contains no requirement that the integers be distinct.

For any positive real number α define N_α as the set of integers

$$\{ [\alpha] , [2\alpha] , [3\alpha] , \ldots , [k\alpha] , \ldots \} .$$

Then among other known properties of the sequence of integral parts of the multiples of a real number are the following : (See, for example, [18]) Theorem 4.11

Let α and β be positive real numbers.

Denote the set of all positive integers by N and the empty set by ϕ. Then N_α and N_β are complementary sets of positive integers,

$$\text{ie.} \quad N_\alpha \cup N_\beta = N \quad \text{and} \quad N_\alpha \cap N_\beta = \phi ,$$

if and only if α and β are irrational and

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1 .$$

Theorem 4.12

Given positive real numbers α and β such that 1, $\frac{1}{\alpha}$, $\frac{1}{\beta}$, are linearly independent over the field of rational numbers, ($\Rightarrow \alpha^{-1}$, β^{-1} are irrational), then N_α and N_β have infinitely many common elements.

Theorem 4.13

Let α and β be positive irrational numbers such that

$$\frac{a}{\alpha} + \frac{b}{\beta} = c ,$$

for some integers a, b, c with $ab < 0$ and $c \neq 0$.

Then N_α and N_β have infinitely many common elements.

Theorem 4.14

Let α and β be positive irrational numbers such that

$$\frac{a}{\alpha} + \frac{b}{\beta} = c \quad ,$$

for some positive integers a , b , c , $c > 1$.

Then N_α and N_β have infinitely many common elements.

Theorem 4.15

Let α and β be positive real numbers.

The sets N_α and N_β are disjoint if and only if α and β are irrational and there exist positive integers a and b such that

$$\frac{a}{\alpha} + \frac{b}{\beta} = 1 \quad .$$

Furthermore, if N_α and N_β have one common element they have infinitely many.

Theorem 4.16

Let $\alpha > 1$ and $\beta > 1$ be irrational.

Then $N_\alpha \cup N_\beta = N$, if and only if there are positive integers a and b such that

$$a(1 - \frac{1}{\alpha}) + b(1 - \frac{1}{\beta}) = 1 \quad .$$

Theorem 4.17

Let $\alpha > 1$ and $\beta > 1$ be irrational.

Then $N_\alpha \supset N_\beta$ if and only if there are positive integers a and b such that

$$a(1-\frac{1}{\alpha}) + \frac{b}{\beta} = 1 .$$

Theorem 4.18

There are no positive real numbers α, β, γ such that N_α, N_β, N_γ are pairwise disjoint.

3. The uniform distribution modulo 1 of $\{f(n)\}_{n=1}^{\infty}$.

If $f(t)$, $(t > 0)$, is a differentiable function then the behaviour of $f'(t)$ is an indication of the kind of oscillation (rapid or slow) of $f(t)$ between the bounds 0 and 1. Therefore in certain cases from given properties of $f'(t)$ conclusions can be drawn about the continuous distribution of the values of $f(t)$ (mod 1), and the discrete distribution of the sequence $f(1), f(2), \ldots$ (mod 1).

This section contains some known theorems about the discrete distribution of sequences $f(1), f(2), \ldots$ under given conditions on $f'(t)$. The proofs are generally based on the known behaviour of the corresponding $f(t)$, $(t > 0)$, with respect to the continuous distribution modulo 1.

Definition.

The function $f(t)$ is C-uniformly distributed (mod 1) if for every integral value of $h \neq 0$,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T e^{2\pi i h f(t)} dt = 0 .$$

Theorem 4.19 [11]

Let $f(t)$ be a function, differentiable for $t \geq 0$, and let

(i) $0 \leq f'(t) < \infty$, $t \geq 0$;

(ii) $t^p f'(t) \to$ constant > 0 as $t \to \infty$, (p fixed, $0 < p < 1$).

Then the sequence $f(1), f(2), f(3), \ldots$ is uniformly distributed modulo 1.

Proof :

The condition (ii) implies that $f(t)$ is C-uniformly distributed (mod 1). (See [11], where further references may be found).

Using Euler's summation formula we have

$$\sum_{n=1}^{T} e^{2\pi i h f(n)} = \int_1^T e^{2\pi i h f(t)} dt + \tfrac{1}{2}\{e^{2\pi i h f(T)} + e^{2\pi i h f(1)}\} + 2\pi h i \int_1^T P(t) f'(t) e^{2\pi i h f(t)} dt, \tag{16}$$

$h = \pm 1, \pm 2, \ldots$; $T = 1, 2, \ldots$, where

$$P(t) = t - [t] - \tfrac{1}{2}.$$

If I is the last term on the right-hand side of (16), then

$$\left|\frac{I}{T}\right| \leq \frac{\pi|h|}{T} \int_1^T f'(t) dt \leq \frac{2\pi|h| f(T)}{T}.$$

(ii) implies that $f'(t) \to 0$ as $t \to \infty$, and so

$$\frac{f(t)}{t} \to 0 \quad \text{as} \quad t \to \infty \ .$$

Thus

$$\frac{I}{T} \to 0 \quad \text{as} \quad T \to \infty \ ,$$

and so, from (16), for every integer $h \neq 0$,

$$\lim_{T \to \infty} \frac{1}{T} \sum_{n=1}^{T} e^{2\pi i h f(n)} = 0 \ .$$

The same methods will also give :

<u>Theorem 4.20</u> [11]

Let $f(t)$ be a differentiable function , and let

(iii) $f'(t)$ monotonically decreasing , $(t \geq 0)$;

(iv) $f'(t) \to 0$ as $t \to \infty$;

(v) $tf'(t) \to 0$ as $t \to \infty$.

Then the sequence $f(1)$, $f(2)$, ... is uniformly distributed mod 1 .

For, (iii) , (iv) , (v) imply that

$$\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} e^{2\pi i h f(t)} dt = 0 \ ,$$

and (iv) implies that $\dfrac{f(t)}{t} \to 0$ as $t \to \infty$;

and

Theorem 4.21 [11]

If $f(t)$ is a differentiable function $(t \geq 0)$, and satisfies

(vi) $f(t)$ is C-uniformly distributed mod 1 ;

(vii) $f'(t) > 0 , \quad (t \geq t_0 \geq 0)$;

(viii) $\dfrac{f(t)}{t} \to 0$ if $t \to \infty$;

then the sequence $f(1) , f(2) , \ldots$ is uniformly distributed mod 1 .

Theorem 4.21 also implies that if $f(t)$ is a differentiable function $(t \geq 0)$ and satisfies

$$f'(t) \log t \to C > 0 \quad \text{as} \quad t \to \infty ,$$

then the sequence $f(1) , f(2) , \ldots$ is uniformly distributed mod 1 .

Among other known sufficient conditions for the sequence $f(1) , f(2) , \ldots$ to be uniformly distributed mod 1 are [11] :

(A) $f(t)$ is a differentiable function with
$|tf'(t)| \leq M , \quad (M \geq 0 , t \geq 0)$,

and

(B) $f(t)$ is a function twice differentiable for
$t \geq 1$, and

(ix) $f'(t)$ and $f''(t)$ are bounded for $t \geq 1$;

(x) $f'(t) \to \xi$ (irrational) as $t \to \infty$.

In [11] , the author uses the above theorems to show that the sequences $\sqrt{n} + \sin\frac{1}{n}$, $(n = 1, 2, \ldots)$, and $\sqrt{n} + \sin n$, $(n = 1, 2, \ldots)$, are uniformly distributed mod 1 , whilst the sequence $\cos(n + \log n)$, $(n = 1, 2, \ldots)$, is not .

4. A generalisation of uniform distribution.

Summary of known theorems :

(i) [Weyl]

The necessary and sufficient condition that $\{x_n\}$ be uniformly distributed mod 1 is that for any R-integrable function $f(x)$ in $[0,1]$,

$$\lim_{n\to\infty} \frac{f\{((x_1))\} + \ldots + f\{((x_n))\}}{n} = \int_0^1 f(x)\,dx \,.$$

(ii) [Weyl]

The necessary and sufficient condition that $\{x_n\}$ be uniformly distributed mod 1 is that for $m = 0, \pm 1, \pm 2, \ldots$,

$$\sum_{r=1}^{n} e^{2\pi i m x_r} = o(n) \,.$$

($f(x) = \exp(2\pi i x)$ in (i)) .

(iii) [van der Corput]

Let $g_h(t) = g(t+h) - g(t)$, $(h = 1, 2, \ldots)$. If $\{g_h(n)\}$ is uniformly distributed mod 1 for any h, then $\{g(n)\}$ is uniformly distributed mod 1 .

(iv) [Fejer]

Let $g(t) > 0$ be a continuous increasing function with a continuous derivative $g'(t)$ for $1 \leqslant t < \infty$ and satisfy the following conditions :

(a) $g(t) \to \infty$, as $t \to \infty$,

(b) $g'(t) \to 0$ monotonically, as $t \to \infty$,

(c) $tg'(t) \to \infty$, as $t \to \infty$.

Then $\{g(n)\}$ is uniformly distributed mod 1 .

Thus $\{an^\sigma\}$, $(a > 0, \sigma > 0, \sigma$ not an integer$)$, and $\{a(\log n)^\sigma\}$, $(a > 0, \sigma > 1)$, are uniformly distributed mod 1 . If σ is an integer and a is irrational, then $\{an^\sigma\}$ is uniformly distributed mod 1 .

In [21] , Tsugi generalises the notion of uniform distribution mod 1 as follows :

Let $\lambda_n > 0$ be a sequence which satisfies

(a) $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n > 0$,

(b) $\sum_{n=1}^{\infty} \lambda_n = \infty$.

Let I be an interval in [0,1] and $\phi(x)$ its characteristic function, ie. $\phi(x) = 1$ for $x \in I$, and $\phi(x) = 0$ elsewhere. If for any I ,

$$\lim_{n \to \infty} \frac{\lambda_1 \phi\{((x_1))\} + \dots + \lambda_n \phi\{((x_n))\}}{\lambda_1 + \dots + \lambda_n} = |I| ;$$

then $\{x_n\}$ is said to be $\{\lambda_n\}$-uniformly distributed mod 1 .

The uniform distribution mod 1 is a special case , where $\lambda_n = 1$, $(n = 1, 2, \dots)$.

There are corresponding analogues of the above theorems :

<u>Theorem 4.22</u> [21]

The necessary and sufficient condition that $\{x_n\}$ be $\{\lambda_n\}$-uniformly distributed mod 1 is that , for any R-integrable function $f(x)$ in $[0,1]$,

$$\lim_{n \to \infty} \frac{\lambda_1 f\{((x_1))\} + \dots + \lambda_n f\{((x_n))\}}{\lambda_1 + \dots + \lambda_n} = \int_0^1 f(x)\,dx .$$

<u>Theorem 4.23</u> [21]

The necessary and sufficient condition that $\{x_n\}$ be $\{\lambda_n\}$-uniformly distributed mod 1 is that , for $m = 0, \pm 1, \pm 2, \dots$,

$$\sum_{r=1}^{n} \lambda_r e^{2\pi m i x_r} = o\Big\{ \sum_{r=1}^{n} \lambda_r \Big\}$$

Theorem 4.24 [21]

Let $\lambda_n = \lambda(n)$, where $\lambda(t) > 0$ is a continuous decreasing function with a continuous derivative $\lambda'(t)$ for $1 \leqslant t < \infty$, such that

$$\sum_{r=1}^{n} \lambda_r \sim \int_1^n \lambda(t)\,dt , \quad (n \to \infty) .$$

Let $g(t) > 0$ be a continuous increasing function with a continuous derivative $g'(t)$ for $1 \leqslant t < \infty$, and satisfy the following conditions :

(a) $g(t) \to \infty$, as $t \to \infty$;

(b) $g'(t) \to 0$ monotonically , as $t \to \infty$;

(c) $\dfrac{g'(t)}{\lambda(t)}$ is monotone for $t \geqslant t_0$;

(d) $\dfrac{g'(t)}{\lambda(t)} \displaystyle\int_1^t \lambda(t)\,dt \to \infty$, as $t \to \infty$.

Then $\{g(n)\}$ is $\{\lambda_n\}$-uniformly distributed mod 1 .

Thus, for example, if $g(t) > 0$ is a continuous increasing function with a continuous derivative $g'(t)$ for $1 \leqslant t < \infty$, and satisfies the conditions :

(a) $g(t) \to \infty$, as $t \to \infty$,

(b) $g'(t) \to 0$ monotonically , as $t \to \infty$,

(c) $tg'(t)$ is monotone for $t \geq t_0$,

(d) $t \log t . g'(t) \to \infty$, as $t \to \infty$,

then $\{g(n)\}$ is $\left\{ \frac{1}{n} \right\}$-uniformly distributed mod 1 .

References.

[1] R. C. BAKER
"On a theorem of Erdos and Taylor".
(Bull. Lond. Math. Soc. 4 , (1972) , 373 - 374)

[2] A. S. BESICOVITCH
"On Kakeya's problem and a similar one".
(Math. Zeit. 27 , (1928) , 312 - 320)

[3] A. M. BRUCKNER and J. G. CEDER
"A note on end-sets".
(Amer. Math. Monthly 78 , (1971) , 516 - 518)

[4] J. W. S. CASSELS
"An introduction to Diophantine Approximation".
(Cambridge Tract No. 45 , Cambridge University Press, (1957))

[5] F. CUNNINGHAM
"The Kakeya problem for simply-connected and for star-shaped sets".
(Amer. Math. Monthly 78 , (1971) , 114 - 129)

[6] R. O. DAVIES
"On accessibility of plane sets and differentiation of functions of two real variables".
(Proc. Camb. Phil. Soc. 48 , (1952) , 215 - 232)

[7] H. G. EGGLESTON
"Sets of fractional dimensions which occur in some problems of number theory".
(Proc. Lond. Math. Soc. 54 , (1951) , 42 - 93)

[8] P. ERDOS and S. J. TAYLOR
"On the set of points of convergence of a lacunary trigonometric series and the equidistribution properties of related sequences".
(Proc. Lond. Math. Soc. (3) 7 , (1957) , 598 - 615)

[9] P. HALMOS
"Measure Theory".
(van Nostrand , (1950))

[10] G. H. HARDY and J. E. LITTLEWOOD
"Some problems of diophantine approximation".
(Acta Mathematica 37 , (1914) , 155 - 238)

[11] L. KUIPERS
"Continuous and discrete distribution modulo 1".
(Indagationes Mathematicae 15 , (1953) , 340 - 348)

[12] V. KLEE and M. MARTIN
"Must a compact endset have measure zero?"
(Amer. Math. Monthly 77 , (1970) , 616 - 618)

[13] V. KLEE and M. MARTIN
"Semicontinuity of the face-function of a convex set".
(Commentarii Mathematici Helvetici , (1971) , 1 - 11)

[14] KRONECKER
(Berliner Sitzungsberichte; Werke 3 , (1884) , p. 49)

[15] D. G. LARMAN
"A compact set of disjoint line segments in E^3 whose end-set has positive measure".
(Mathematika 18 , (1971) , 112 - 125)

[16] D. G. LARMAN
"On a conjecture of Klee and Martin for convex bodies".
(Proc. Lond. Math. Soc. (23) 3 , (1971) , 668 - 682)

[17] J. M. MARSTRAND
"Some fundamental geometrical properties of plane sets of fractional dimensions".
(Proc. Lond. Math. Soc. (3) 4 , (1954) , 257 - 302)

[18] I. NIVEN
"Diophantine Approximations".
(Interscience , (1963))

[19] C. A. ROGERS
"Hausdorff Measures".
(Cambridge University Press , (1970))

[20] S. J. TAYLOR and J. G. WENDEL
"The exact Hausdorff measure of the zero set of a stable process".
(Z. Wahrscheinlichkeitstheorie verw. Geb. 6 , (1966) , 170 - 180)

[21] M. TSUGI
"On the uniform distribution of numbers mod. 1".
(Jour. Math. Soc. Japan 4 , Nos. 3 - 4 , (1952) , 313 - 322)

[22] H. WEYL
"Über die Gleichverteilung von Zahlen mod Eins".
(Mathematische Annalen 77 , (1916) , 313 - 352)