PROCESSES OF EVOLUTION IN ASTROPHYSICS

THESIS

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ABSTRACT

This thesis consists of two parts; part I deals with the origin of the terrestrial planets while part II is concerned with evolution of larger objects, stellar clusters.

PART I

Chapter I:

A general outline of the problems involved in the formation of planets and a brief indication of our proposed solution is given.

Chapter 2:

The resistance offered by a gas to an object moving through it with different speeds is calculated and a comparison between the expressions found is given. The equation of growth for an object accreting material is also found.

Chapter 3:

The time taken by an object to fall to the centre of a gas cloud under the action of the gravitational attraction of the cloud and the resistance of the gas is found. The resistance and growth laws found in Chapter 2 are used.

Chapter 4:

The possibility of forming very large grains, or clumps, by accretion and the probability that these

escape from the cloud on collision is investigated.

Chapter 5:

An estimate for the dispersal time of a condensation violating the Roche limit is given.

Chapter 6:

This gives a brief outline of the proposed theory together with the conclusions about the whole part.

PART II

Chapter 1:

This gives an explanation to some of the terms used in connection with stellar evolution and indicates some of the problems involved.

Chapter 2:

The problem of stars contracting onto the main sequence is considered, showing the importance of the initial conditions chosen.

Chapter 3:

A discussion about the blue stragglers found in some clusters is given and an investigation into the possibility that this phenomena is caused by continuous creation of stars in the cluster is carried out.

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P A R T I

THE ORIGIN OF THE TERRESTRIAL PLANETS

CHAPTER I

Introduction

One of the most surprising aspects about the solar system is the sharp division that exists between the two types of planets present, namely the Major, or Jovian, planets and the terrestrial planets. They differ widely, both in mass and in chemical composition. The major planets have a mass of the order of 10^{30} grammes and according to present-day calculations, probably consist mainly of hydrogen, presumably with the usual sprinkling of the heavier elements. They are thus similar to most other astronomical objects as regards chemical composition. The terrestrial planets on the other hand are only about 1% as massive as the major planets, consisting mainly of the heavier elements. It is now commonly thought that the terrestrial planets originated after the heavier elements were separated in some manner from the hydrogen. Various theories have been proposed in order to achieve this result, among them the formation of a disc of material surrounding the sun as suggested by Hoyle [1] and the theories presented by Russian astronomers such as Schmidt [2]. Most of these theories concentrate on giving an explanation of the difference in composition and take no account of the fact that if we returned hydrogen to the terrestrial planets such that their chemical composition would be similar to the major planets, then their masses would also be comparable to the major planets. A theory by Professor McCrea [3] is one of the few theories that take account of this second feature.

According to this theory, formation of stars takes place, in clusters of a hundred or more at a time, from an initial massive cloud of hydrogen gas in molecular form, with the usual small amount of other elements present. This cloud is not assumed to have uniform density, but to consist of cloudlets of higher density called floccules, these floccules moving at random amongst themselves. A star is formed at a point where the path of a number of these floccules converge. By such an arrangement clearly many stars are formed in a cluster, none of them possessing a phenomenally high amount of angular momentum since the floccules approach from all directions. Hence, this suggestion agrees with present-day beliefs that stars are formed in clusters, not individually, and at the same time overcomes the angular momentum problem, which has been the downfall of many a theory in the past and has led to many a drastic proposal in order that it could be resolved.

All the floccules in a cloud could not possibly be moving directly towards these points and so, after an interval of time, we find that while most of the floccules have formed into stellar condensations, a few will still be moving, gravitationally captured by one of these condensations but not having actually joined any of them. As time passes, these trapped floccules will tend to settle into the invariable plane defined by the angular momentum of the system. As this process is taking place the floccules will tend to collide together and grow, in a similar way to the formation of the parent star, though on a smaller scale of course. As a result of this we would expect the stars to be eventually surrounded by a few condensations, all of these condensations being roughly similar to each other. (In his paper, Professor McCrea shows that the common mass would be about equivalent to the mass of a major planet.) The major planets can thus be formed simply by further condensing these condensations.

McCrea points out that for any region closer to the sun than Jupiter's orbit, a condensation having a density similar to the floccule density assumed could not hold together for long due to the tidal action of the sun. He suggests that this is the reason for the difference between the two types of planets, but does not give any indication as to how either the separation of the elements, or indeed the formation of a condensation in this region, could possibly take place.

It is intended that this work should give some indication of how this formation of the terrestrial planets could come about. It is fairly obvious that if we could, in some way, cause the heavier elements to form a core at the centre of the planetary condensation, with a density high enough to hold together on its own, then the tidal action of the sun could only remove the regions of lower density, namely the outer hydrogen layers, leaving us with a stable condensation of heavier material or just the ingredients required for a terrestrial planet. It is our intention to investigate the possibility of forming such a core. Clearly the only practical way of forming this core is by the heavier elements falling to the centre of the condensation due to the gravitational attraction of the cloud. We shall thus investigate various ways by which this falling process can take place, and in what form the heavier material must be before the time of fall becomes realistic.

We have also made estimates of the time

required by the tidal action of the sun to disperse such a condensation as the one above. It is important that this time be fairly short, otherwise the time taken for the outer hydrogen layers to disperse might be so long as to make the occurrence impossible during the sun's life. Ideally we would like this time to be longer than the time required for the formation of the heavy core, as the formation of a terrestrial planet then follows a very simple pattern. The condensations form by collisions between floccules; the heavy material begins to fall and the tidal action disperses, but as the time of fall is less than the dispersal time formation of the core takes place before any appreciable dispersal. This is an ideal situation and we shall investigate under what conditions this comes about. If these requirements are unreasonable we can still obtain the desired result by having the condensation formed and orbiting in a region where it does not disperse, and occasionally dipping into the other region, whence the outer layers are removed.

In chapter 2, all the equations concerned with the resistance of a gas cloud to motions through it are found. Equations are also obtained giving the rate of growth of a moving particle assuming various methods of accretion.

Chapter 3 deals with the actual motion of the heavy material through a spherical cloud of hydrogen gas. Solutions of the equations of motion are obtained in all cases likely to be of interest, together with numerical values for the time of fall of the heavy material.

In some of these motions we consider very large grains. The formation of these grains is considered in chapter 4, together with an investigation into the probability of these objects escaping from the surrounding gas cloud when the cloud is involved in a collision. It is to be noted that these large grains are comparable with the planetesimals postulated by various authors on the origin of the planets. We compare our theory with these planetesimal theories in chapter 5, where all the conclusions are given together with a complete outline of the theory, without mathematical detail.

CHAPTER 2

Equations concerned with the motion of the heavy material

In order to investigate the formation of the heavy core we require the equation of motion of this material as it moves through the surrounding gas condensation, the only forces operating being the gravitational attraction of the condensation itself and the resistance offered by this gas to any motion through it.

We shall first investigate this resisting force, finding various expressions depending on the speed and size of the object moving through the gas. For obtaining some of these we can make use of published work if we regard the falling heavy material as being another gas.

Throughout this discussion we shall retain the following notation.

m_1 = mass of a molecule of the resisting gas.

m_2 = mass of a molecule of the moving gas, or of the particle, as the case may be.

σ_1 = radius of a molecule of the resisting gas.

σ_2 = radius of a molecule of the moving gas or particle.

n_1 = number density of the resisting gas.

n_2 = number density of the other gas.

$\rho_1 = m_1 n_1$ = density of the resisting gas.

T = temperature in degrees Kelvin (assumed constant).

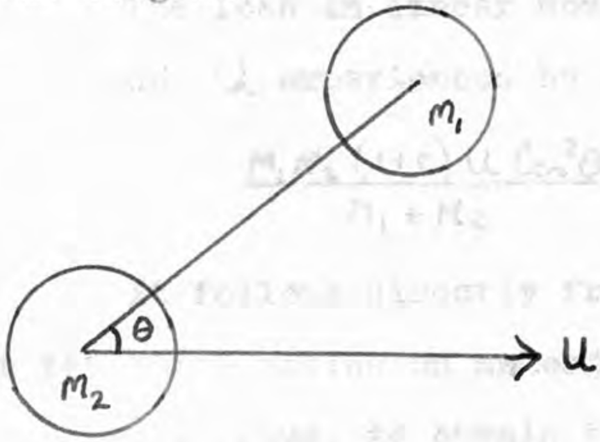
k = Boltzman's gas constant.

Any other notation will be explained as the need for it arises.

Throughout this discussion we assume that all the molecules and particles can be considered as spheres, the coefficient of elasticity being e for any collisions between them.

In our first derivation, we follow a method given by McCrea [4] in his paper on "Gas motions in prominences, Wolf-Rayet stars and Novae" with the small generalization that the coefficient of elasticity is e instead of unity as assumed by him.

Consider a molecule of the moving gas approaching a molecule of the resisting gas with relative speed u. This is the same as taking a sphere of mass m_1 at rest, with a sphere of mass m_2 approaching it with speed u. Let the straight line joining the centres of these two spheres subtend an angle θ with the direction of the speed u. The situation is illustrated in the diagram.



If v_1 and v_2 are the speeds of molecules (1) and (2) after the collision, along the line of centres, then conservation of linear momentum along this line of centres gives

$$m_2 U \cos\theta = m_1 v_1 + m_2 v_2$$

The coefficient of elasticity is e, and thus

$$e U \cos\theta = v_1 - v_2$$

Hence, on substituting for v_2

$$v_1 = \frac{m_2 U \cos\theta (1+e)}{m_1 + m_2}$$

The gain in linear momentum experienced by molecule (1) is thus

$$m_1 v_1 = \frac{m_1 m_2 U \cos\theta (1+e)}{m_1 + m_2}$$

along the line of centres, with the corresponding loss for molecule (2).

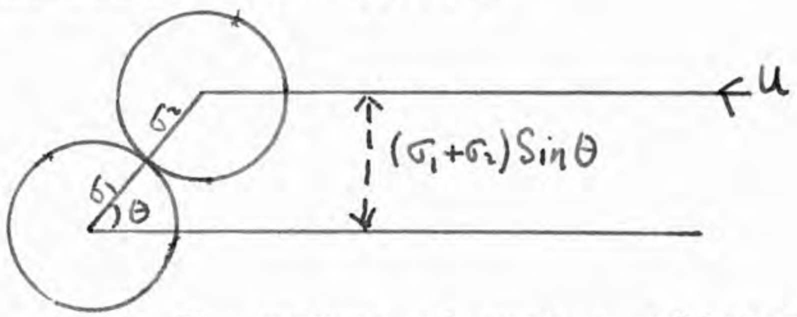
The loss in linear momentum along the direction of the speed U experienced by molecule (2) is thus

$$\frac{M_1 M_2 (1+e) U \cos^2\theta}{M_1 + M_2} \qquad (1)$$

It follows directly from Newton's laws of motion that the force acting on an object is its rate of change of momentum. Thus, to obtain the resistance we have to find the number of such exchanges per unit time, that is clearly the number of such collisions per unit time.

If the thermal velocity of the gas is small compared with the translation speed U then we can obtain the number of collisions as follows.

If two molecules collide, their line of centres subtending an angle between 0 and θ with the speed U, then their centres must lie in a cylinder of length Ut and radius $(\sigma_1+\sigma_2)\sin\theta$.



The volume of such a cylinder is

$$\pi(\sigma_1+\sigma_2)^2 Ut \sin^2\theta$$

The volume of the corresponding cylinder, if the angle can lie between θ and $\theta + d\theta$, is

$$\pi(\sigma_1+\sigma_2)^2 ut \, \mathrm{Sin}^2(\theta+d\theta)$$

Hence if two molecules collide with their centres at an angle between θ and $\theta+d\theta$ with the direction of the speed u, then their centres must lie in a volume

$$\pi(\sigma_1+\sigma_2)^2 ut \left\{ \mathrm{Sin}^2(\theta+d\theta) - \mathrm{Sin}^2\theta \right\}$$

$$= 2\pi(\sigma_1+\sigma_2)^2 ut \, \mathrm{Sin}\,\theta \, \mathrm{Cos}\,\theta \, d\theta$$

If we have n_1 molecules of mass M_1 and n_2 molecules of mass M_2, then the number of collisions per unit time is

$$2\pi(\sigma_1+\sigma_2)^2 n_1 n_2 \, \mathrm{Sin}\,\theta \, \mathrm{Cos}\,\theta \; u \, d\theta \tag{2}$$

So the resistance acting on a unit volume of gas (2) would be, due to collision in the range θ to $\theta+d\theta$

$$\frac{2\pi(\sigma_1+\sigma_2)^2 n_1 n_2 M_1 M_2 (1+e) \, \mathrm{Sin}\,\theta \, \mathrm{Cos}^3\theta \, u^2 d\theta}{M_1+M_2} \tag{3}$$

On integrating over the total range of θ, we obtain

$$\frac{2\pi(\sigma_1+\sigma_2)^2 n_1 n_2 M_1 M_2 (1+e)}{M_1+M_2} \int_0^{\pi/2} \mathrm{Sin}\,\theta \, \mathrm{Cos}^3\theta \, d\theta \cdot u^2$$

$$= \frac{\pi(1+e) M_1 M_2 n_1 n_2 (\sigma_1+\sigma_2)^2}{2(M_1+M_2)} \cdot u^2 \tag{4}$$

This is the resisting force on unit volume of the gas.

The force on one molecule is clearly

$$\frac{\pi(1+e)M_1M_2\,n_1(\sigma_1+\sigma_2)^2\,u^2}{2(M_1+M_2)} \qquad (5)$$

and this will hold whether the object M_2 is a molecule or a grain (as we assumed molecules were spheres anyway to obtain the formula).

Clearly only minor alterations to this method are required to give the resistance ~~of~~ the speed of the moving material is small compared with the thermal speed of the gas, which is why all the details were reproduced from McCrea's paper. The total linear momentum transferred from one gas to the other remains unaltered by the above assumption; thus equation (1) is still operative.

The number of collisions will be modified by replacing $u\cos\theta$ by W , where W is the main thermal velocity of the gas, and is

$$2\pi(\sigma_1+\sigma_2)^2 n_1 n_2 \sin\theta\, d\theta\, W$$

Thus giving a total force due to all collisions in the range θ to $\theta+d\theta$ as

$$\frac{2\pi\, M_1M_2\,(1+e)\,n_1n_2\,(\sigma_1+\sigma_2)^2\,u\,W\sin\theta\cos^2\theta\,d\theta}{M_1+M_2}$$

or, upon integrating over all possible values of θ , the resistance to the motion of the second gas is

$$\frac{2\pi M_1 M_2 (1+e) n_1 n_2 (\sigma_1+\sigma_2)^2 U.W}{3(M_1+M_2)}$$

Now for a gas at a temperature T whose molecules have a mass M_1 , the mean thermal velocity is defined to be

$$W_1 = 2\sqrt{\frac{2kT}{\pi M_1}}$$

If we have another gas whose molecules weigh M_2 , then

$$W_2 = 2\sqrt{\frac{2kT}{\pi M_2}}$$

Thus a convenient mean thermal velocity would be

$$W = 2\sqrt{\frac{2kT(M_1+M_2)}{\pi M_1 M_2}}$$

When we introduce this for W in the resistance, it becomes

$$\frac{4}{3}(1+e) n_1 n_2 (\sigma_1+\sigma_2)^2 \sqrt{\frac{2\pi kT M_1 M_2}{M_1+M_2}} \cdot U \qquad (6)$$

If we are interested in one molecule only, then we have $n_2 = 1$ and use the expression W_1 for the thermal velocity, giving

$$R = \frac{4}{3}\frac{(1+e) M_2 n_1 (\sigma_1+\sigma_2)^2}{M_1+M_2}\sqrt{2\pi kT M_1} \cdot U \qquad (7)$$

Expression (6) was also found by McCrea [4] but proceeding from Chapman's [5] general theory of diffusion.

When the temperature and pressure are assumed uniform the force is

$$\frac{n_1 n_2 kT\, u}{(n_1+n_2)\, D_{12}} \tag{8}$$

where D_{12} is the coefficient of diffusion.

Chapman's first approximation for the value of this coefficient gives

$$\frac{3(M_1+M_2)}{2\pi(n_1+n_2)M_1M_2\, \kappa_{12}'(0)}$$

where

$$\kappa_{12}'(0) = 4(\sigma_1+\sigma_2)^2\left\{\frac{M_1+M_2}{M_1M_2}\cdot\frac{2kT}{\pi}\right\}^{1/2}$$

if the molecules are assumed to be elastic spheres. Thus the total resistance turns out to be

$$\frac{8}{3}\, n_1 n_2 (\sigma_1+\sigma_2)^2 \sqrt{\frac{2\pi kT M_1 M_2}{M_1+M_2}}\cdot u \tag{9}$$

This is the same as expression (6) obtained by us, apart from the added assumption in (9) that the spheres are elastic, thus $e = 1$.

In future work, we shall be interested in material other than gas molecules falling through the gas. For this reason, we require an expression for the force on these larger bodies. There is nothing in the work done by us to obtain expressions (6) and (7) that actually specifies an application to molecules only. Thus we can deduce that the resistance would be

$$\frac{4}{3}(1+e)\, n_1 \sigma_2^2 \sqrt{2\pi k T m_1}\ .\ u \qquad (10)$$

since, for a grain, $\sigma_2 >> \sigma_1$, $m_2 >> m_1$.

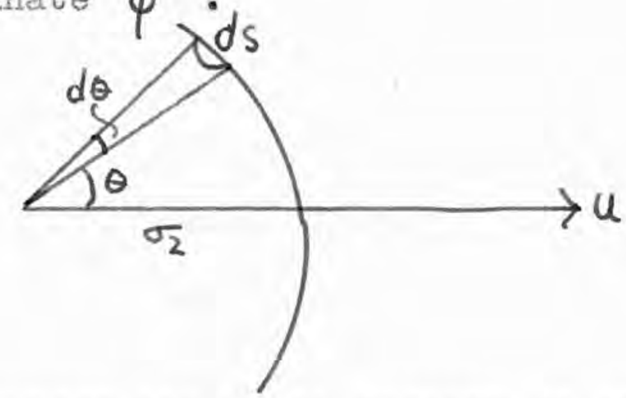
Though there seems to be no objection to obtaining the resistance in this way, there is no particular reason why the resistance should be the same for gas and grains. As we can calculate the resistance on a grain directly, we shall do this and not rely on deductions from the gas case.

We assume that the molecules of the resisting gas have a Maxwellian distribution of velocities, that is to say we assume that the probability of any molecule having a speed between v and $v+dv$ along a given direction is

$$\left(\frac{h m_1}{\pi}\right)^{1/2} \exp\left\{-h m_1 v^2\right\} dv$$

where h is a constant of value $1/2kT$.

Let us then take a particle, considered as a sphere of radius σ_2, moving with speed u through a stationary gas with the properties σ_1, m_1, n_1 and T as already defined by us. We take the centre of this sphere as the pole of a spherical polar system of coordinates. The direction of the speed u is taken as the axis of the θ coordinate and another arbitary direction, perpendicular to u, as the axis of the third coordinate ϕ.



Consider the small element of area dS on the surface of the sphere and subtending angles between θ and $\theta + d\theta$, ϕ and $\phi + d\phi$ at the centre of the sphere. Then by elementary mathematics we obtain

$$dS = \sigma_2^2 \sin\theta \, d\theta \, d\phi$$

If we now take a new set of rectangular right-handed coordinate axis stationary in the gas, the axis of x parallel to the normal to dS, the y and z axis thus being parallel to the plane of dS.

The probability that any molecule has speed between u and $u + du$ along the x-axis is thus

$$\left(\frac{hM_1}{\pi}\right)^{1/2} \exp\{-hM_1 u^2\} \, du$$

The linear momentum transferred when a molecule in such a velocity range collides with the particle is

$$M_1(1+e)(u + U\cos\theta)$$

along the normal, with no contribution perpendicular to this normal. In obtaining this expression we have assumed that the mass of the particle greatly exceeds the mass of the colliding molecule. v and ω are perpendicular to the normal, so have no contribution.

The number of collisions per unit time between this area dS and molecules having speed in the range u to $u+du$ along the prescribed axis is

$$n(u + U\cos\theta)\,dS$$

where n is the number of molecules per unit volume with the prescribed speed.

But

$$n = n_1\left(\frac{hM_1}{\pi}\right)^{1/2} \exp\left\{-hM_1u^2\right\} du$$

by virtue of the Maxwellian distribution, and thus the number of collisions per unit time is

$$n_1(u + U\cos\theta)\left(\frac{hM_1}{\pi}\right)^{1/2} \exp\left\{-hM_1u^2\right\} du\,dS$$

Hence the force acting, which is equal to the rate of change of momentum, due to the molecules in the above speed range is

$$M_1(1+e)(u+U\cos\theta)^2 n_1\left(\frac{hM_1}{\pi}\right)^{1/2} \exp\{-hM_1u^2\}\,du\,dS$$

Now, since we are investigating the case of the thermal velocity exceeding the translation velocity, $u \gg U$, and so we have the force as

$$M_1(1+e)(u^2+2uU\cos\theta)n_1\left(\frac{hM_1}{\pi}\right)^{1/2} \exp\{-hM_1u^2\}\,du\,dS$$

Integrating over all possible values of u, the force acting on dS along its normal is

$$F_1 = M_1(1+e)\left(\frac{hM_1}{\pi}\right)^{1/2} n_1\,dS\int_0^{\infty}(u^2+2uU\cos\theta)\exp\{-hM_1u^2\}\,du$$

$$F_1 = M_1(1+e)\left(\frac{hM_1}{\pi}\right)^{1/2} n_1\,dS\left[\frac{1}{4}\sqrt{\frac{\pi}{h^3M_1^3}} + \frac{U\cos\theta}{hM_1}\right]$$

This is the force along the normal to dS.

From elementary mathematics it is obvious that the total force in the direction of U is $F_1\cos\theta$ with $F_1\sin\theta$ perpendicular to this.

From symmetry it is obvious that when integrated over the whole sphere, the sum of $F_1\sin\theta$ is zero.

The total resistance to motion is thus given by

$$(1+e)M_1\left(\frac{hM_1}{\pi}\right)^{1/2} n_1\int_0^{2\pi}\sigma_2^2\,d\phi\, 2\int_0^{\pi/2}\left[\frac{1}{4}\sqrt{\frac{\pi}{h^3M_1^3}} + \frac{U\cos\theta}{hM_1}\right]\sin\theta\cos\theta\,d\theta$$

$$= \frac{-4\pi}{3hM_1}U(1+e)M_1n_1\sigma_2^2\left(\frac{hM_1}{\pi}\right)^{1/2}$$

or, since $h = 1/2kT$

the total resistance to the motion is

$$\frac{4}{3}(1+e)\,n_1\sigma_2^2\,(2\pi kTM_1)^{1/2}\,U \tag{11}$$

This expression (11) is exactly the same as what we deduced the resistance for a particle to be from the corresponding resistance for a gas, namely expression (10). Thus we can safely conclude that if the translation velocity is smaller than the thermal velocity of the resisting medium, the resistance is given by the expression derived by us, namely

$$\frac{4}{3}(1+e)\,n_1\sigma_2^2\sqrt{2\pi kTM_1}\;U \tag{12}$$

There exists a well known formula for the resistance of a fluid to the motion of a sphere, called Stokes's law. According to this law the resistance is

$$R = 6\pi\eta\sigma_2 U \tag{13}$$

where η is called the coefficient of viscosity, about which more information is given below, all the other symbols being as already defined. Proofs and derivations of this formula are given in various books, for example, Basset "Treatise in Hydrodynamics" [6], Lamb "Hydrodynamics" [7] and Green "Hydro and Aerodynamics" [8].

For this reason we do not include a detailed proof, we only give a brief outline of the method, which is enough to show the basic principles involved and the assumptions made. Initially the Stokes current function is found, which is a solution of Laplace's equation

$$\nabla^2 \psi = 0$$

which satisfies the boundary conditions prescribed.

This allows us to find the motion of the fluid. From this we derive the tangential and normal components of the force (resistance) due to the liquid on a small element of area dS on the surface of the sphere. Sumation of all such forces clearly gives the total resistance acting. This turns out to be

$$R = 6\pi\eta\sigma_2 U \frac{1 + \frac{2\eta}{\lambda\sigma_2}}{1 + \frac{3\eta}{\lambda\sigma_2}}$$

where λ is a proportionality constant denoting the amount of slip between the liquid and the sphere. Stokes in his formula assumes no slip and hence $\lambda \to \infty$, giving the result

$$R = 6\pi\eta\sigma_2 U$$

as given above. Two points are to be noted from this derivation of the resistance.

1) The resisting medium is considered as a viscous fluid throughout.
2) The layer of fluid immediately in contact with the sphere is assumed to move with the sphere.

In deriving equation (12) giving the resistance to a particle, we assumed:

1) That the resisting medium is considered as a collection of molecules;
2) The layer immediately in contact with the sphere is exactly similar to the layer far removed from the sphere.

Thus we see clearly the difference between the Stokes law of resistance and the one derived by us.

The first set of conditions (Stokes's law) would clearly be better satisfied when the sphere is large compared with the mean free path, while the other conditions apply if the sphere is small compared with the mean free path. In other words Stokes's law applies for large spheres while the other resistance law applies for small spheres.

Clearly before any intelligent comparison can be carried out between the two expressions (12) and (13) for the resistance we require some equation for the coefficient of viscosity η .

The simple derivation of such an expression which we are about to give is essentially the same as one given by Meyer in his book "The Kinetic Theory of Gases" [9] apart from changes of notation. We have included it in the discussion because it is very simple, shows clearly what is meant by viscosity, and gives an expression that is very similar to what is given by more refined and complicated methods.

We consider a gas moving in such a way that, on choice of a suitable set of axis, the velocity in a given plane is constant and numerically equal to the distance of this plane from a fixed plane parallel to it, which we shall call the base plane. We now take two adjacent layers in the fluid, their plane of separation being parallel to the base plane and at a distance x from it. Now the number of molecules which pass in a unit time through a unit area of the separation plane from the layer nearest the base plane to the other layer is

$$1/6 \; n_1 W$$

where, as before, n_1 is the number of molecules per unit volume and W is the mean thermal velocity of these molecules. The coefficient $1/6$ is obtained by assuming, as first suggested by Joube, that only $1/3$ of the molecules have speed along any given direction. Half of this number would thus be moving in a positive sense.

The molecules crossing this separation plane must, on average, have come from a distance equal to the mean free path, L , of the gas away from this separation plane, that is from a layer distance $(x-L)$ from the base plane for crossing one way and $(x+L)$ the other way.

By our assumption the mean forward velocity of these molecules is equal to the distance from the base plane, and thus the mean linear momentum of a molecule passing from the slower layer to the other is

$$m_1 (x-L)$$

The total linear momentum passing from the slower layer through a unit area of the separation plane in a unit time is thus

$$\frac{1}{6} n_1 W m_1 (x-L)$$

By an obviously similar argument the amount of linear momentum transferred from the fast moving layer to the other is

$$\frac{1}{6} n_1 W m_1 (x+L)$$

Hence the layer nearest the base plane is gaining linear momentum through a unit area of the separation plane at a rate

$$\frac{1}{6} n_1 W m_1 (x+L) - \frac{1}{6} n_1 W m_1 (x-L) = \frac{1}{3} m_1 n_1 W L$$

while the other layer loses the same amount.

But the coefficient of viscosity is defined to be the force acting on a unit surface and is thus equal to the rate of change of linear momentum through a unit area, and is thus

$$\eta = \frac{1}{3} m_1 n_1 W L \qquad (14)$$

This expression for the coefficient of viscosity was first obtained by Maxwell in 1860. More refined methods for obtaining this coefficient have since been discovered. These can be found in "Dynamical Theory of gases" by Jeans [10] or "Kinetic Theory of Gases" [11] by the same author, and Kennard's "Kinetic Theory of Gases" [12] .

Kennard also gives a result obtained by Boltzman in "Gas Theory" [13] which is

$$\eta = 0{\cdot}350\, W m_1 n_1 L$$

This result is modified by Kennard to give a final value of

$$\eta = 0.310\, W m_1 n_1 L$$

Jeans gives reference to a more rigorous derivation carried out by Chapman in 1911 [14], who obtains the value

$$\eta = \frac{5\pi}{32} \cdot \frac{m_1 W}{4\sqrt{2}\,\pi\,\sigma_1^2}$$

If we use Chapman's value for the mean free path of the gas, namely

$$L = \frac{1.38}{4\sqrt{2}\,\pi\, n_1 \sigma_1^2}$$

then the expression quoted above for the viscosity becomes

$$\eta = 0.36\, m_1 n_1 W L$$

More recently (1962) Deslodge in "The American Journal of Physics" [15] finds

$$\eta = \frac{5\,(m_1 \pi k T)^{1/2}}{64\pi\,\sigma_1^2}$$

Using the expression for L given above and the mean thermal velocity W as $2\sqrt{\frac{2kT}{\pi m_1}}$ this expression becomes

$$\eta = \frac{5\, m_1 W}{32 \cdot 4\sqrt{2}\,\sigma_1^2}$$

which is the same as given above by Chapman, thus giving

$$\eta = 0.36\, M_1 n_1 W L$$

Thus we see that all the above refinements do not appreciably change the basic expression obtained by Maxwell and any differences occur in the numerical coefficient only. As Maxwell's result is much simpler than the others we shall find it more convenient to use this expression for comparison purposes. We thus retain

$$\eta = \tfrac{1}{3} W M_1 n_1 L$$

As already pointed out $W = 2\sqrt{\frac{2kT}{\pi M_1}}$ and so

$$\eta = \frac{2}{3} \cdot \frac{n_1}{\pi} \sqrt{2\pi k T M_1}$$

The resistance due to a gas when a large sphere moves through it is thus given by substituting this value of η into equation (13), giving

$$R = 4\sigma_2\, n_1 L \sqrt{2\pi k T M_1} \cdot u \qquad (15)$$

We have thus obtained various expressions for the resistance experienced by a particle (or sphere) as it moves through a gas cloud. These were -

1. <u>If the velocity of translation is small</u>

   (a) radius larger than the mean free path of the gas

$$R = 4 n_1 \sigma_2 L \sqrt{2\pi k T M_1} \cdot u \qquad (15)$$

(b) radius smaller than the mean free path of the gas but greater than the radius of a molecule

$$R = \frac{4}{3}(1+e)\, n_1\, \sigma_2^{\,2} \sqrt{2\pi k T M_1} \cdot u \qquad (12)$$

(c) radius similar in magnitude to a molecular radius

$$R = \frac{4}{3}(1+e)\, n_1 (\sigma_1+\sigma_2)^2 \sqrt{\frac{2\pi k T M_1 M_2}{M_1+M_2}} \cdot u \qquad (6)$$

2. <u>If the velocity of translation is large compared with the thermal velocity of the gas</u>

(a) radius larger than a molecular radius

$$R = \frac{\pi}{2}(1+e)\, M_1 n_1\, \sigma_2^{\,2}\; u^2$$

(b) radius comparable with a molecular radius

$$R = \frac{\pi(1+e) M_1 M_2\, n_1 (\sigma_1+\sigma_2)^2}{2(M_1+M_2)} \cdot u^2 \qquad (5)$$

We are now in possession of the above expressions for the resistance offered by a gas to motions through it. It would be both interesting and instructive to compare these expressions.

Initially let us compare the expressions taking account of the variation in the resistance with speed. The expressions to be compared now are (5) and (6), (5) being for high speeds while (6) is for a slower moving object. These two resistances have been plotted in a diagram, which we have called figure 1. So as to

Fig 1. – Relation between resistance and speed

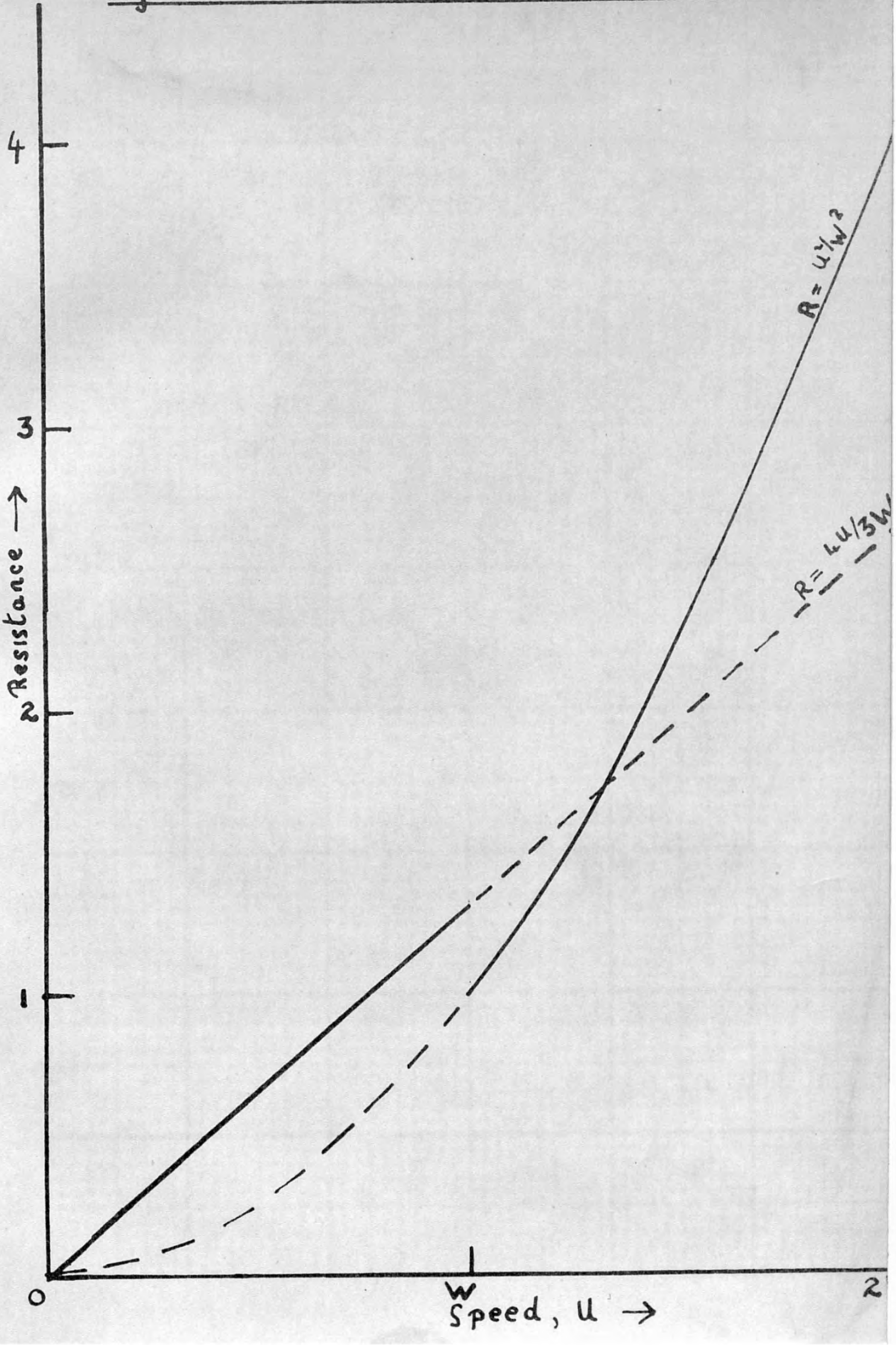
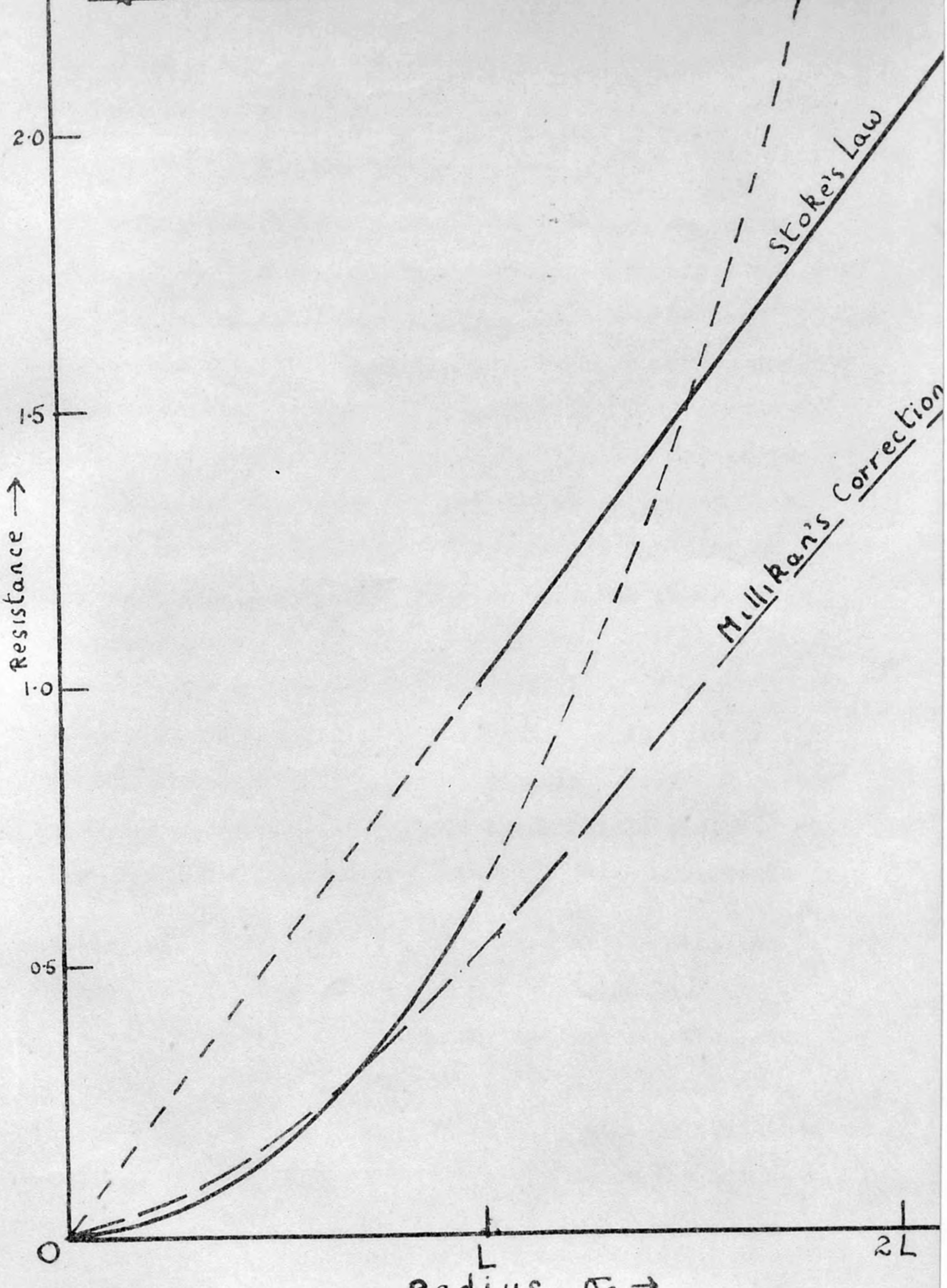


Fig 2 - Relation between resistance and radius



be able to use a simple scale for the resistance we have taken the quantity $\frac{\pi(1+e)M_1 M_2(\sigma_1+\sigma_2)^2 W^2}{2(M_1+M_2)}$ as a scale factor, W being the thermal velocity of the gas, as usual.

In this diagram, the straight line gives the resistance for small speeds while the curve is the applicable one for higher speeds. We have used a continuous marking to show the region where the curve or line is the correct resistance and broken markings for the other region. From the diagram it is immediately evident that the resistance law actually applying at any given speed is always greater than the other resistance law.

Let us now turn to the variation of the resistance with the particle radius. Again this is best illustrated by means of a diagram. Figure 2 gives the resistance plotted against the particle radius. We have now taken the quantity $4\pi M_1 n_1\sqrt{\frac{2kT}{\pi M_1}}\,uL$ as a scale factor, L being the mean free path of the gas, the reason again being just to simplify the scale.

As before a continuous line denotes the actual resistance while a broken line gives the resistance using the other law. The dotted line will be explained later. We note that the diagram now shows the actual resistance to be smaller than the resistance calculated from the incorrect law. We see that the difference between these laws for radii less than about 2 mean free paths is small.

Comparing the two resistance laws involved qualitatively we see that one is proportional to σ_2^2 while the other is proportional to $\sigma_2 L$, all other variable terms being similar in both. From an intuitive point of view this is what is to be expected. In the first derivation, the force arose from the impact of molecules against the particle and would thus depend on the area available for impact, that is σ_2^2. In Stokes's law the resistance is due to the drag of the fluid on the edges, and is thus proportional to the edge length, or σ_2. To keep the dimensions similar another length must be involved, and the only term of such dimensions that could conceivably be involved is the mean free path of the gas, L; hence for Stokes's law we obtain $\sigma_2 L$.

A well known experiment, first conducted by Millikan, to measure the charge of an electron makes use of the resistance of air to the motion of an oil drop. (For a full account of the experimental details see Tolansky [16] or Millikan [17].) For fairly large oil drops Millikan found that using Stokes's law as given above gave accurate and consistent results, hence indicating that the law is correct. For smaller oil drops, however, he found that the Stokes resistance does not give very accurate results. A resistance

$$R = \frac{4\sigma_2 n_1 L\sqrt{2\pi kTM_1}}{1 + AL/\sigma_2} \cdot u \qquad (16)$$

was found to give very satisfactory and accurate results. This law is the same as Stokes's, apart from $(1 + AL/\sigma_2)$ in the denominator, A being a constant.

For very small values of σ_2, AL/σ_2 becomes much the dominant term and so for very small radii the Millikan correction gives the Stoke resistance as

$$R = \frac{4\sigma_2^2 n_1}{A}\sqrt{2\pi kTM_1} \cdot u$$

which is exactly the same as the force found by us for small radii, apart from numerical constants.

This Millikan law, equation (16), has been plotted into figure 2, which was the diagram showing the variation of resistance with radius. This is the dashed line. At first sight this does not agree at all well with the two curves already present, and as the experimental evidence shows that this curve cannot be far wrong for the region studied by Millikan, it could suggest that our theoretical results do not apply. It is to be noted, however, that for very small radii, where we are very certain that our theoretical curve is correct, agreement is good, and the experimental data do not extend to this region. For large radii, the ratio of the two resistances, Stokes's and Millikan's, approach closer and closer to unity, and so agreement is also good for this region.

It is thus only for a region near the mean free path of the gas that Millikan's corrected resistance, which is in good agreement with experimental data, differs from both our resistance laws. But this is the region where one law becomes applicable instead of the other. Nevertheless the method by which this comes about is surprising. We would expect from physical considerations that at any radius in this region the actual resistance would be fairly close to both the resistance laws, and possibly lying in between the two of them. We find however a fairly large difference, the actual resistance lying below both of the theoretical ones. It is to be noted, however, that this curve found by Millikan is very similar to the curve one would draw to join the curve found by us for small radii and Stokes's law for very large radii.

We can thus conclude that Millikan's experiment gives definite proof that the expressions found by us are correct when the radius is not comparable with the mean free path of the gas and also shows how to change from one law to the other.

It is perhaps interesting to note that the actual separation between the Stokes line and the Millikan corrected resistance is independent of radius and so of course (as we expect) the difference between the two expressed as a percentage of one of them gets smaller and smaller.

In a paper "On graphite particles as interstellar grains" Hoyle and Wickramasinghe [18] use an expression for the resistance of a gas to a small object moving through it. They use Stokes's law, but modify the viscosity by replacing the mean free path with the object radius. They are thus in effect using essentially the same resistance as was found by us apart from small numerical factors.

There thus seems to be abundant evidence, both theoretical and experimental, for concluding that the resistances we have obtained and given above as equations (5), (6), (12) and (15) are essentially correct and that the region of applicability of these formulae are roughly as described by us. We have thus completed our investigation into the resisting forces acting in a gas cloud.

Possible growth of the moving object

As we have stated, in this chapter we are interested in the equations governing the motion of particles through a gas cloud. Under certain conditions these moving particles could accrete material as they moved. We thus have to investigate the rate of growth of such particles assuming different methods of accretion. We must point out that some of the methods of accretion outlined below are very unlikely to have any physical application. They have been included only for interest and for completeness. The particle that is accreting material can either be moving faster or slower than the thermal velocity of the accreted material. This accreted material could also be in the form of either gas molecules or other particles like interstellar grains. There are thus four possibilities to be investigated. We shall consider these in turn.

1. Accretion of gas molecules by a fast moving particle.

Let the probability of a molecule being picked up on collision with the particle be λ , all other notation being as already defined.

When the particle moves through a distance dx , the number of molecules colliding with it is clearly

$$\pi \sigma_2^2 n_1 \, dx$$

The amount of material accreted in this distance is thus

$$dM_2 = \lambda \pi \sigma_2^2 M_1 n_1 \, dx$$

But $dM_2 = 4\pi \sigma_2^2 d\sigma_2 \rho_2$, where ρ_2 is the density of a particle and is assumed constant. Thus

$$d\sigma_2 = \frac{\lambda M_1 n_1 dx}{4 \rho_2}$$

Integrating we obtain

$$\sigma = \sigma_0 + \frac{\lambda M_1 n_1 x}{4 \rho_2} \qquad (17)$$

where σ_0 is the initial radius of the particle at the point $x = 0$.

Equation (17) gives the growth of the particle under the stated conditions.

2. Accretion of particles by fast moving particles.

Let the probability of being picked up after a collision be λ again. Let the ratio of particles to gas molecules, by mass, be μ , then

$$M n = \mu M_1 n_1$$

where M and n are the mass and number density of the particles. As before, clearly

$$dM_2 = \lambda \pi \sigma_2^2 M n \, dx$$

$$= \lambda \mu \pi \sigma_2^2 M_1 n_1 \, dx$$

so

$$d\sigma_2 = \frac{\lambda \mu M_1 n_1 \, dx}{4 \rho_2}$$

ρ_2 being as defined before.

Thus

$$\sigma_2 = \sigma_0 + \frac{\lambda \mu M_1 n_1 x}{4 \rho_2}$$

where σ_0 is the radius of the moving particle at some initial stage where $x = 0$.

3. <u>Accretion of gas molecules by particles moving slower than the thermal speed of the gas</u>.

Let λ have the same meaning as before. As the particle is moving slower than the thermal speed of the gas, more molecules are involved in collision due to this speed than due to the motion of the particle, and so the number of collisions depends on the time rather than the distance moved. Assume again that the velocity distribution of the gas molecules is Maxwellian. Then the number of collisions between molecules having speed in the range u to $u + du$ and an area dS of the particle in time dt is

$$n_1 \left(\frac{h M_1}{\pi} \right)^{1/2} \exp\{-h M_1 u^2\} \, u \, du \, dS \, dt$$

The number of collisions, allowing for all possible velocities, is thus

$$n_1 \int_0^\infty \left(\frac{hM_1}{\pi}\right)^{1/2} exp\{-hM_1u^2\} u\, du\, ds\, dt$$

$$= n_1 \sqrt{\frac{1}{\pi h M_1}}\; dt\; dS$$

Integrating over the whole surface of the particle, the number of collisions in time dt is

$$\frac{n_1}{2}\sqrt{\frac{1}{\pi h M_1}} \cdot 4\pi\sigma_2^2\, dt$$

The mass gained by the particle in this time is thus

$$dM_2 = \frac{\lambda M_1 n_1}{2}\sqrt{\frac{1}{\pi h M_1}} \cdot 4\pi\sigma_2^2\, dt$$

hence

$$d\sigma_2 = \frac{\lambda M_1 n_1}{2\rho_2}\sqrt{\frac{1}{\pi h M_1}} \cdot dt$$

so

$$\sigma_2 = \sigma_0 + \frac{\lambda M_1 n_1}{2\rho_2}\sqrt{\frac{1}{\pi h M_1}}\, t$$

where σ_0 is the radius of the particle when $t = 0$. But $h = 1/2kT$ and $W = 2\sqrt{\frac{2kT}{\pi M_1}}$ so we can write the above result as

$$\sigma_2 = \sigma_0 + \frac{\lambda M_1 n_1 W t}{4\rho_2} \qquad (19)$$

which gives the size of the particle under the given growth conditions.

4. Accretion of particles by particles moving slower than the thermal velocity of the accreted material.

Let λ and μ have the meaning already defined. Then as before

$$Mn = \mu M_1 n_1$$

Clearly the same analysis can be applied to this case as to the preceding one. Thus the mass gained by the moving particle in time dt is

$$dM_2 = \frac{\lambda M n}{2}\sqrt{\frac{1}{\pi h M}} \cdot 4\pi\sigma_2^2\, dt$$

$$d\sigma_2 = \frac{\lambda M n}{2\rho_2}\sqrt{\frac{1}{\pi h M}}\, dt$$

$$\sigma_2 = \sigma_0 + \frac{\lambda M n}{2\rho_2}\sqrt{\frac{1}{\pi h M}}\, t$$

But $h = 1/2kT$ and $2\sqrt{\frac{2kT}{\pi M}} = W_2$ (say) the thermal velocity of the particles accreted. Thus

$$\sigma_2 = \sigma_0 + \frac{\lambda M n W_2 t}{4\rho_2}$$

or

$$\sigma_2 = \sigma_0 + \frac{\lambda \mu M_1 n_1 W_2 t}{4\rho_2} \qquad (20)$$

giving the particle size for this case.

This completes the investigation into the growth of particles by accretion. There now remains only one other factor to discuss about the motion of the heavy material, the accelerating force. We are primarily interested in motions through a gas cloud, the only accelerating force present is thus the gravitational attraction of this cloud. This force will be acting towards the centre of the cloud, and by Newton's law of gravitation its magnitude will be

$$\frac{GmM}{r^2}$$

where m is the mass of the particle r the position of the particle measured from the cloud centre, M being the mass of cloud within this radius and G the universal gravitation constant.

If we assume the cloud to possess a uniform density ρ then the mass inside a radius r is

$$M = \frac{4}{3}\pi \rho r^3$$

But we have already defined the number density of this cloud to be n_1, so $\rho = M_1 n_1$ and

$$M = \frac{4}{3}\pi n_1 M_1 r^3$$

which leads to an accelerating force of

$$F = \frac{4}{3}\pi G m M_1 n_1 r \qquad (21)$$

The resistance of the cloud will be in direct opposition to this force and its magnitude will clearly be one of the types already discussed by us above. Any changes in the size of the moving particle are also governed by the equations of growth due to accretion we have obtained. We are thus able to derive the equations of motion for any type of motion likely to arise. No useful purpose will be served by writing them all down here as they can be obtained very simply if required from the information given above. Practical applications of these equations are left until a later chapter. Hence all the information required at the beginning of the chapter has been found.

CHAPTER 3

Motion of the heavy material, with numerical values

In this chapter we shall initially fix the numerical values of most of the parameters involved in the formation of the terrestrial planets. By making use of these values and of the expressions for the forces acting on a moving particle found in the last chapter we shall attempt to solve the equations of motion for the heavy material and by doing this obtain an estimate for the time required to form the heavy core, that is the time taken by the heavy material to fall.

Throughout this discussion the prime consideration is the formation of a terrestrial planet. We are considering the transportation of heavy material originally spread about in a condensation, to the centre of this gas condensation, the object being to form a terrestrial planet out of this heavy material. Hence the mass of this heavy material must be roughly equivalent to a terrestrial planet mass, but possibly slightly greater, to allow for any inefficiency in the process. The mass of the Earth is about 6×10^{27} gms with

Venus approximately the same, while Mars and Mercury are slightly smaller. The mass of heavy material in a condensation must thus be of this order of magnitude.

Now according to Allen in 'Astrophysical Quantities' [19], the average proportion, by weight, of heavy material to gas in an interstellar cloud is about 1 : 100. This proportion is roughly the same in the Sun and other stars. The floccules considered by McCrea [3] and hence our clouds and condensations (that are assumed to be formed from the floccules) are assumed to be perfectly normal cloudlets, so this proportion of heavy material to gas must be the same in these. Thus in order to have a final planet with a mass of 6×10^{27} gms the total mass of the condensation would be 6×10^{29} gms if the mechanism was completely efficient. If we allow for small inefficiencies in this process, while at the same time obtaining a more convenient mass, numerically speaking, we obtain 10^{30} gms for the initial mass of the condensation.

Now, the mass of Jupiter is 2×10^{30} gms while Saturn is 6×10^{29} gms and thus these major planets also have a mass of about 10^{30} gms. We thus have the very satisfactory situation where all planetary condensations are initially of the same mass and composition. The difference in the present day appearance being due only to the terrestrial planets losing most of their initial hydrogen at some stage. From now on, in all numerical work we shall assume that both the mass and composition of the condensations are fixed, their actual values being as given above.

The initial density of our problem must be the density of a floccule, the value chosen by McCrea for this quantity being 6.6×10^{-9} gm/cc. Throughout the period of interest to us, the density of any condensation is unlikely to have changed much from this value but any tendency in the condensations to condense would increase this slightly. Consequently, we shall take the round number 10^{-8} gm/cc for our condensation density for any general argument and use the less convenient value 6.6×10^{-9} gm/cc only if this causes a great difference in the result. We will, of course, attempt to solve all the problems using algebraic symbols first and will only introduce these numerical values to obtain a numerical physical solution, so that other values for these parameters could be used if the need arose.

There are other known constants whose values are required. These have all been taken from Allen's book [19]. We assume that the temperature remains constant throughout the time interval we are considering and at all times this will be taken equal to the floccule temperature. McCrea has given reasons for assuming this temperature to be $50^{\circ}K$ and so we adopt this value for our condensation.

The complete list of the numerical values for the constants so far determined by us is as follows.

Mass of the condensation = $M = 10^{30}$ gms

Density of the condensation = $\rho = M_1 n_1 = 10^{-8}$ gm/cc

Proportion of heavy material to gas = $\mu = 10^{-2}$

Temperature of the condensation = $T = 50^{\circ}K$

Mass of hydrogen molecule = $M_1 = 3.3 \times 10^{-24}$ gms

Radius of hydrogen molecule = $\sigma_1 = 1.25 \times 10^{-8}$ cms

Mass of interstellar grain = M_2, M_g = 10^{-13} gms

Radius of interstellar grain = 2.8×10^{-5} cms

Density of interstellar grain = ρ_2 = 1 gm/cc [Allen value 1.1]

Universal gravitational constant = $G = 6.67 \times 10^{-8}$ dyne cm^2/gm

Boltzman's gas constant = $k = 1.38 \times 10^{-16}$ erg deg^{-1}

Motion of the heavy material

With this information we now have to investigate the motion of the heavy material, assuming various initial configurations for this material and finding the time required for the formation of a heavy core.

No means of transportation is likely to exist that could move the heavy material to the centre faster than allowing the particles to fall freely. It would thus appear advisable to investigate this free fall time and so deduce whether this minimum time is a short and sensible time before proceeding to more complicated transportation mechanisms.

If we have a particle of mass m in a spherical cloud of gas with a uniform density ρ at a distance r from the centre of the cloud, then the force of attraction towards the centre of this cloud is, by the usual gravitational laws, given by

$$\frac{GMm}{r^2} = \frac{4}{3}\pi G\rho r m$$

where M is the mass included in a radius r .

As we are considering free-fall, no resistance exists and so, by Newton's second law of motion, the equation of motion of the particle is

$$-\ddot{r} = \frac{4}{3}\pi G\rho r \qquad (1)$$

We note that the mass of the particle, m , is not involved and so this equation of motion holds independent of what form we have the heavy material in, be it molecules, grains or any other particle.

If we denote the speed of the falling particle at any instant by v , then equation (1) becomes

$$-v\frac{dv}{dr} = \frac{4}{3}\pi G\rho r$$

Integrating this equation, we obtain

$$v^2 = \frac{4}{3}\pi G\rho\left[r_0^2 - r^2\right] \qquad (2)$$

where r_0 is the initial position of the particle, where the speed was assumed to be zero.

Then

$$v = \pm\sqrt{\frac{4}{3}\pi G\rho\,(r_0^2 - r^2)}$$

Now the speed v is positive but if we denote the velocity by $\dot{r}$ this is now negative so the above equation can be written as

$$-\frac{dr}{dt} = \sqrt{\frac{4}{3}\pi G\rho\,(r_0^2 - r^2)}$$

Upon integration this gives

$$-\sin^{-1}\left(\frac{r}{r_0}\right) = \sqrt{\frac{4}{3}\pi G\rho}\;t + \text{Constant}$$

If we begin our time measurement at the point $r = r_0$ when the particle is beginning to move, then clearly the value of the constant is $-\pi/2$.

Hence

$$\sin^{-1}\left(\frac{r}{r_0}\right) = \frac{\pi}{2} - \sqrt{\frac{4\pi G\rho}{3}}\, t$$

At the centre of the cloud, $r = 0$ and so the time taken by the particle to fall to this point is clearly

$$t = \frac{\pi}{2}\sqrt{\frac{3}{4\pi G\rho}} = \sqrt{\frac{3\pi}{16 G\rho}} \qquad (3)$$

With the value for the density of 10^{-8} gm/cc this equation gives a time of fall of 2.96×10^{7} seconds or 0.94 years.

If we were to use the exact floccule density given by McCrea of 6.6×10^{-9} gm/cc, this gives a time of fall of 1.15 years.

Hence, for any density likely to be of interest to us the time of free-fall for a particle is very small and is very close to one year. It thus seems quite in order to proceed to investigate the time taken in the more realistic cases when the resistance of the gas is taken into account. We must note that the above time of one year is the minimum time possible and so any of the following methods must give a longer period than this.

It will obviously be beneficial if we investigate the simpler configurations first since clearly if these give a satisfactory solution and an acceptable time there is no need to investigate more complicated mechanisms.

The simplest possible situation we can have after the free-fall is obviously when a molecule of the heavier material falls towards the centre of the gas cloud. Clearly, as the time of fall cannot be less than the free fall time, the speed of the molecule cannot be greater than the thermal velocity of the gas for any significant period. Also the molecules start from rest so their initial translation speed is less than the mean thermal velocity. The radius of this molecule must, for obvious reasons, be less than the mean free path of the gas. The resistance offered by the gas to such a motion as this is that given by equation (6) of the previous chapter.

$$R = \frac{4}{3}(1+e)n_1(\sigma_1+\sigma_2)^2\sqrt{\frac{2\pi kTM_1M_2}{M_1+M_2}}\cdot u$$

We assume that this molecule does not accrete any material, so its radius is constant through the motion and the coefficient of elasticity for each collision will be unity.

The accelerating force towards the centre of the cloud has already been noted and is

$$F = \frac{4}{3}\pi G \rho M_2 r$$

M_2 being used to denote the mass of the heavy molecule.

The equation of motion can now obviously be written down, and is

$$- M_2 \ddot{r} = F - R$$

or

$$\ddot{r} + \frac{4}{3}\pi G \rho r + \frac{8}{3} n_1 \frac{(\sigma_1+\sigma_2)^2}{M_2} \sqrt{\frac{2\pi k T M_1 M_2}{M_1+M_2}}\ \dot{r} \qquad (4)$$

where we have used the position vector r as the variable parameter.

The above equation (4) is clearly of the form

$$\frac{d^2 r}{dt^2} + \alpha \frac{dr}{dt} + \beta r = 0 \qquad (5)$$

where α and β are both constants and are given by

$$\left.\begin{aligned} \alpha &= \frac{8}{3} n_1 \frac{(\sigma_1+\sigma_2)^2}{M_2} \sqrt{\frac{2\pi k T M_1 M_2}{M_1+M_2}} \\ \beta &= \frac{4}{3}\pi_1 G M_1 n_1 \end{aligned}\right\} \qquad (6)$$

Equation (5) is a standard second order differential equation with constant coefficients so the solution is given in the usual manner by

$$r = A \exp\left\{\frac{(-\alpha - \sqrt{\alpha^2 - 4\beta})t}{2}\right\} + B\exp\left\{\frac{(-\alpha - \sqrt{\alpha^2 - 4\beta})t}{2}\right\} \qquad (7)$$

where A and B are constants of integration that can be determined from initial conditions. We shall be interested in the solution when $\alpha^2 \gg \beta$, in which case

$$\sqrt{\alpha^2 - 4\beta} \doteq \alpha\left(1 - \frac{2\beta}{\alpha^2}\right)$$

and hence the solution is

$$r = A\exp\left\{-\alpha t + \frac{\beta t}{\alpha}\right\} + B\exp\left\{-\frac{\beta t}{\alpha}\right\}$$

The initial conditions defined by us are $r = r_0$ and $\dot{r} = 0$ at $t = 0$, therefore

$$A = -\frac{\beta r_0}{\alpha^2} \qquad B = \frac{r_0(\alpha^2 - \beta)}{\alpha^2}$$

giving us a solution as

$$r = \frac{r_0(\alpha^2 - \beta)}{\alpha^2}\exp\left\{-\frac{\beta t}{\alpha}\right\} - \frac{\beta r_0}{\alpha^2}\exp\left\{-\alpha t + \frac{\beta t}{\alpha}\right\}$$

On expanding the small term $\exp\left\{\pm\frac{\beta t}{\alpha}\right\}$ and taking the first order terms only we obtain

$$r = r_0 - \frac{r_0\beta t}{\alpha} - \frac{r_0\beta}{\alpha^2}\exp\{-\alpha t\}$$

As αt is likely to be fairly large this gives

$$t \doteq \frac{(r_0 - r)\alpha}{\beta r_0} \qquad (8)$$

and so the time taken to fall to the centre of the cloud is given when $r = 0$, and is

$$t = \frac{\alpha}{\beta}$$

From equation (5) this is the sort of expression we would intuitively expect for the time of fall. Inserting into this expression the values of the constants α and β from equation (6) the time of fall is then given by

$$t = \frac{2}{\pi}\frac{(\sigma_1+\sigma_2)^2}{GM_1}\sqrt{\frac{2\pi kTM_1}{M_2(M_1+M_2)}} \qquad (9)$$

where M_2 and σ_2 are the mass and radius of the falling molecule. From Allen's book [19] the average radius of the heavier molecules would appear to be about 2×10^{-8} cm while on average the mass would be about $10M_1$ or 3.3×10^{-23} gms . Inserting these values into expression (9) together with the values for all the other constants we obtain a time of fall of 9.5×10^{19} seconds or 3.0×10^{12} years. We note that this time is independent of the cloud density. The physical reason for this is that both the resistance and the accelerating force are proportional to the density and so they cancel each other out.

This time is a phenomenally long time and means that if the core was formed by molecules falling in this way, then the falling process must have started before any of the known astronomical bodies, including galaxies, were formed. Clearly this is very unlikely and so this mode of transportation is of little use in the formation of terrestrial planets.

The above solution is valid provided the two constants α and β satisfy the relation

$$\alpha^2 \gg \beta$$

Now on using equation (6)

$$\frac{\alpha^2}{\beta} = \frac{32 m_1 (\sigma_1+\sigma_2)^4 kT}{3 G M_2 (M_1+M_2)}$$

Inserting the numerical values we have chosen, this gives

$$\frac{\alpha^2}{\beta} \sim 10^{25}$$

and hence $\alpha^2 \gg \beta$ and the solution we have found above is valid.

Clearly before a solution becomes of use to us, the time of fall must be greatly reduced. Expression (9) shows that this time of fall is proportional to σ_2^2/M_2 and hence if we can reduce this we reduce the time. Thus we have to use a larger body if we require a smaller time. It is normal to assume that some of the heavy material in interstellar space will be present in the form of grains. It would thus be sensible to investigate the time taken by such objects to fall. The same conditions as before apply on the resistance law, but now as the falling object is larger we can use the simple expression (12) of the previous chapter, namely

$$R = \frac{8}{3} n_1 \sigma_2^2 \sqrt{2\pi k T M_1} \,.\, u$$

All the other expressions are the same as in the case of the falling molecule and so the equation of motion becomes

$$\ddot{r} + \frac{8}{3} \frac{n_1 \sigma_2^2}{M_2} \sqrt{2\pi k T M_1}\, \dot{r} + \frac{4}{3}\pi G M_1 n_1 r = 0$$

which is again of the form

$$\frac{d^2 r}{dt^2} + \alpha \frac{dr}{dt} + \beta r = 0 \qquad (10)$$

with now

$$\left.\begin{aligned} \alpha &= \frac{8}{3} \frac{n_1 \sigma_2^2}{M_2} \sqrt{2\pi k T M_1} \\ \beta &= \frac{4}{3}\pi G M_1 n_1 \end{aligned}\right\} \qquad (11)$$

Clearly the solution of (10) for the case $\alpha^2 \gg \beta$ will once again be

$$t = \frac{\alpha}{\beta} = \frac{2\sigma_2 \sqrt{2\pi k T M_1}}{\pi G M_1 M_2} \qquad (12)$$

Once again the time is independent of the cloud density (for the same physical reason). Inserting numerical values for all the known symbols we obtain a time of fall of 8.6×10^{15} seconds or 2.7×10^{8} years.

This solution is also valid only if $\alpha^2 >> \beta$.

Again

$$\frac{\alpha^2}{\beta} = \frac{32}{3} \cdot \frac{\sigma_2 4 n_1 k T}{G M_2^2} \sim 10^{18}$$

Thus $\alpha^2 >> \beta$ and the solution is valid.

This time again is rather long and is about the same as the age of an average galactic cluster and hence clearly of little use for the formation of planets. It is however significantly shorter than the time required by the molecules to fall and does suggest that if we had a large enough body the time of fall could be as close as we please to the free-fall time.

For the motion of these large bodies however we cannot use the given expression (12) for the resistance as the radius of the body might exceed the mean free path of the gas and the proper resistance law is now Stokes's law. For an investigation of the motion of large bodies we thus have to start the investigation again.

The resistance due to Stokes's law is given by equation (15) of the last chapter and is

$$R = 4 n_1 \sigma_2 L \sqrt{2\pi k T M_1} \, . u$$

where L is the mean free path of the gas, arising from the viscosity, and is given as in the last chapter by

$$L = \frac{1.38}{4\sqrt{2}\pi n_1 \sigma_1^2}$$

Hence the resistance becomes

$$R = \frac{1.38}{\pi} \cdot \frac{\sigma_2}{\sigma_1} \sqrt{\pi k T M_1} \cdot U \tag{13}$$

The accelerating force due to gravity is the same as usual and is

$$\vec{F} = \frac{4}{3}\pi G M_1 n_1 M_2 r$$

The equation of motion for these large bodies thus is

$$\ddot{r} + \frac{1.38}{\pi} \cdot \frac{\sigma_2}{M_2 \sigma_1^2} \sqrt{\pi k T M_1}\, \dot{r} + \frac{4}{3}\pi G M_1 n_1 r = 0$$

which is again of the same form

$$\frac{d^2r}{dt^2} + \alpha \frac{dr}{dt} + \beta r = 0 \tag{14}$$

with now the new values for α and β of

$$\left.\begin{aligned} \alpha &= \frac{1.38}{\pi} \cdot \frac{\sigma_2}{M_2 \sigma_1^2} \sqrt{\pi k T M_1} \\ \beta &= \frac{4}{3}\pi G M_1 n_1 \end{aligned}\right\} \tag{15}$$

The solution of equation (14) will now be the same as in the previous cases if $\alpha^2 >> \beta$, namely

$$t = \frac{\alpha}{\beta} = \frac{3 \times 1.38}{4\pi^2} \times \frac{\sigma_2 \sqrt{\pi k T M_1}}{G M_1 n_1 M_2 \sigma_1^2}$$

Inserting all the numerical constants into this expression we obtain

$$t = \frac{6.4 \times 10^{2}}{\sigma_2^{2}\,\rho}$$

and thus, with the cloud density of $\rho = 10^{-8}$ gm/cc we are interested in, the time of fall becomes of the same order as the free-fall time, namely one year when the radius of the falling body becomes about 70 cm.

This above solution is valid only as long as $\alpha^2 \gg \beta$.

Now

$$\frac{\alpha^2}{\beta} = \frac{27}{64\pi^4}\,\frac{(1.38)^2 kT}{\sigma_1^{4}\sigma_2^{4} G M_1} \doteq 10$$

and hence this gives $\alpha^2 \gg \beta$ while at the same time indicates that we cannot increase β/α further to reduce the time or $\alpha^2 \not\gg \beta$ and the time is smaller than the free-fall time.

Hence if the heavy material in a gas cloud is concentrated into large bodies with a radius of about 70 cm then these objects can fall to the centre of the cloud of density 10^{-8} gm/cc in about one year, which is a very suitable time for the process we have in mind. We do not wish to enter at this stage into any discussion about the formation of these large objects, or of how tightly bound in the gas condensations they are. These questions will be investigated in chapter 4. Here we are only interested in the formation of the heavy core in a reasonable time. This we have succeeded in doing.

The motion of growing grains

So far we have only considered objects with a constant size and mass falling through the gas cloud. If a falling object grows as it falls then clearly, in view of what we have already found, its time of fall must be less than the time it would take to fall if it moved with the constant initial small size. From this point of view, having a growing particle is very desirable in our theory. The only conceivable way in which any falling particle could grow is by accretion of some of the surrounding material, namely gas and dust grains. We do not enter into any discussion about this mechanism of accretion. We postulate that if a falling grain is involved in collision with another grain, accretion can take place. Though no actual proof is given, in the next chapter we shall discuss briefly the possibility of this occurrence. It is also possible (theoretically at least) for the growing particle to accrete gas molecules. As we only require the transportation of the heavy elements we must postulate some mechanism by which molecules of the heavy elements are accreted while hydrogen ones are not. This second method of accretion is included for academic interest and for completeness. It is unlikely to be one which occurs in nature as we can find no means by which only the heavy element molecules would be accreted.

We shall consider briefly first the unlikely method of accreting the heavy molecules. We have already remarked that no interest exists for us if the speed of translation greatly exceeds the thermal velocity of the gas. The particle accretes the heavy molecules under these conditions, and so the equation of growth is equation (19) of the previous chapter, namely

$$\sigma_2 = \sigma_0 + \frac{\mu M_1 n_1 W t}{4 \rho_2} \tag{16}$$

This proportionality constant μ is the ratio of heavy elements to gas molecules in the cloud. From the values we have adopted this becomes 10^{-2}.

The accelerating force on this growing particle is the same as we have always used, and is

$$F = \frac{4}{3}\pi G M_1 n_1 M_2 r$$

where M_2 is the mass of the moving particle.

The resistance however presents us with some difficulty as we do not yet know whether to use Stokes's law or one of the other resistance laws found by us for smaller objects.

We know that the change over from Stokes's law occurs at a radius of the order of the mean free path of the gas. We also know from the comparison of these laws which has already been carried out that if we use the incorrect law we shall be overestimating the resistance and so overestimating the time of fall.

With the cloud density of 10^{-8} gm/cc which we are interested in, the mean free path of the gas is roughly 0.15 cm. Hence if the radius of the body exceeds this amount we should use Stokes's law.

Using the above equation (16), the equation of growth, we see that the radius of the growing body exceeds any given amount σ after a time t given by

$$t = \frac{4\rho_2 \sigma}{\mu M_1 n_1 W}$$

Inserting numerical values for the density and radius we are interested to see that after a time of about 6×10^4 seconds the radius exceeds the mean free path and we should use Stokes's law. Now the free-fall time, which is the minimum time of fall for such a cloud, has been found and is about one year or $3/\times10^{7}$ seconds. Hence we are outside the region of applicability of Stokes's law for $6\times10^{4}/3\times10^{7}$, or 0.2% of the total time of fall. We shall thus use Stokes's law for the resistance throughout the motion, noting that for 0.2% of the time we are overestimating the resistance.

We thus have the resistance due to Stokes's law as

$$R = 6\pi\eta\sigma_2 u$$

where η is the coefficient of viscosity and is given by one of the expressions of chapter 2. The most accurate of these gives

$$\eta = \frac{5\sqrt{\pi k T m_1}}{64\pi\sigma_1^2} = 4.4\times10^{-5}$$

The equation of motion thus becomes

$$\frac{d}{dt}(m_2\dot{r}) + 6\pi\eta\sigma_2\dot{r} + \frac{4}{3}\pi G m_1 n_1 m_2 r = 0 \qquad (17)$$

where σ_2 and m_2 are the radius and mass of the moving particle at the given instant and are connected by

$$m_2 = \frac{4}{3}\pi\rho_2\sigma_2^3$$

and of course

$$\sigma_2 = \sigma_0 + \frac{\mu m_1 n_1 W t}{4\rho_2} \qquad (16)$$

σ_0 is the initial value of the radius of the moving object and clearly for most of the time $\sigma_2 \gg \sigma_0$ and so we take

$$\sigma_2 = \frac{\mu M_1 n_1 W t}{4 \rho_2}$$

Inserting this into equation (17) it becomes

$$\frac{d}{dt}(M_2 \dot{r}) + \frac{3\pi \eta \mu M_1 n_1 W t}{2 \rho_2} \dot{r} + \frac{16\pi^2}{9} G M_1^4 n_1^4 \rho_2 \frac{\mu^3 W^3 t^3}{64 \rho_2^3} r = 0$$

which can clearly be written as

$$\frac{d}{dt}(M_2 \dot{r}) + \gamma t \dot{r} + \delta t^3 r = 0 \qquad (18)$$

where again γ and δ are constants whose values are

$$\left.\begin{aligned} \gamma &= \frac{3\pi \eta \mu M_1 n_1 W}{2 \rho_2} \\ \delta &= \frac{\pi^2 G M_1^4 n_1^4 \mu^3 W^3}{36 \rho_2^2} \end{aligned}\right\} \qquad (19)$$

In order to obtain the time of fall we require a solution of equation (13). A completely rigorous solution is clearly rather difficult to obtain. However, from physical considerations, we would expect the speed of the falling body to approximate to the terminal velocity of the motion after a short interval of time. We thus take as a solution the limiting velocity of the body at any instant to be its actual speed. Hence we take $\gamma t \dot{r} + \delta t^3 r = 0$ (20)

as a solution.

Integrating this we obtain

$$\gamma \operatorname{Log}_e\left(\frac{r_0}{r}\right) = \frac{\delta t^3}{3}$$

with the obvious initial conditions $r = r_0$ at $t = 0$.

The time of fall is thus given by

$$t = \left[\frac{3\gamma}{\delta} \operatorname{Log}_e\left(\frac{r_0}{r}\right)\right]^{1/3} \tag{21}$$

Now inserting numerical values into this equation for all the terms and finding the time of fall down to a radius of $r_0/10$, with the density of 10^{-8} gm/cc we find that this time of fall is 3.3×10^7 seconds, or 1.1 years.

In obtaining the above solution we have assumed that the particle moves with its local limiting velocity at any instant and so the rate of change of linear momentum should be small. We can check to see if the approximation is self consistent.

Using (20) $m_2 \dot{r}$ becomes

$$-\frac{\pi \mu^3 m_1^3 n_1^3 W^3 t^3}{48 \rho_2^2} \cdot \frac{\delta t^2 r}{\gamma}$$

And so the rate of change of linear momentum is

$$\frac{d}{dt}(m_2 \dot{r}) = -\frac{\pi \mu^3 m_1^3 n_1^3 W^3 \delta}{48 \rho_2^2 \gamma}\left(5 t^4 r + t^5 \dot{r}\right)$$

Substitute for $\dot{r}$ again from (20)

$$\frac{d}{dt}(M_2\dot{r}) = \frac{\pi\mu^3 M_1^3 n_1^3 W^3 \delta}{48\,\rho_2^2\gamma}\left(t^5 \cdot \frac{\delta t^2 r}{\gamma} - 5t^4 r\right)$$

Our approximation is self consistent if this term is small compared with the other terms of equation (13) or if

$$\frac{d}{dt}(M_2\dot{r}) \Big/ \delta t^3 r$$

is small.

That is, if

$$\frac{\pi\mu^3 M_1^3 n_1^3 W^3}{48\,\rho_2^3\gamma}\left(5t - \frac{\delta t^4}{\gamma}\right)$$

is small compared with unity.

This has a maximum positive value when $t^3 = \frac{5\gamma}{4\delta}$ and a maximum negative value at the greatest permissible value of t, this second value being greater in absolute terms for the t we have given above, and this maximum value is about 0.50 .

Thus for all the range

$$\left|\frac{d}{dt}(M_2\dot{r}) \Big/ \delta t^3 r\right| < 1$$

and so our solution is justified. We note, however, that our solution is not amply justified and if we increase the density this would not be justified.

Clearly we are on the border of justification as the time is also very near the free-fall time and an increase in density would also decrease this.

After falling for this period of 3.3×10^7 seconds, the radius of the growing particle is given by

$$\sigma = \frac{\mu M_1 n_1 W t}{4 \rho_2} = 110 \text{ cm}$$

Hence if a grain falls through a cloud of gas similar to the one we have in mind for a planetary condensation, picking up heavy molecules as it collides with them, then it will reach the centre of the cloud in just over one year, the radius of the body by this time being about 100 centimetres. As we have already stated, this method of transportation is included for completeness only as we require rather unusual conditions before it works. We require the vast majority of the heavy elements in molecular form, with just enough grains present to pick these up and transport them to the centre. No obvious method exists by which the heavy molecules could adhere to the grains while normal gas molecules do not. We thus move on to the next method of accretion, which is a method that could occur in practice.

Grains growing by accreting other grains

In this method grains accrete other grains and we have already postulated that all grains involved in a collision adhere together afterwards. We do not enter into any discussion about the validity of the above postulate. We do, however, point out that, as far as is known, these interstellar grains are to be considered as soft and wispy like snow flakes rather than hard pellets like lead shot. It is not impossible therefore for the postulate to be valid.

We first show that the time for any growth due to thermal collisions is fairly long so that any growth of interest must be caused by the movement of the grain through other grains.

The equation of growth due to thermal motions is equation (20) of the previous chapter, namely

$$\sigma_2 = \sigma_0 + \frac{\lambda \mu M_1 n_1 W_2 t}{4 \rho_2}$$

Since all the grains, in view of the postulate, adhere together, $\lambda = 1$, and the size after any given time is given by

$$\sigma_2 = \frac{\mu M_1 n_1 W_2 t}{4 \rho_2}$$

where W_2 is the thermal velocity of the grains and is given by $W_2 = 2\sqrt{\frac{2kT}{\pi M_g}}$ where M_g is the mass of a grain.

Using numerical values, the time taken to grow to 1000 times its original size, that is 10^{-2}cm, would be 2.7×10^9 seconds or nearly 100 years. If the thermal velocity of the grains decreases as the grain size increases [increase in M_g] the above time of 100 years would be further increased. We hope to show that in about one year the grains will have reached the condensation centre, and thus the above method of growth can be ignored.

Hence if a grain originally, for some reason, is larger than average, then it will fall through the other grains and grow by this process. It can do this for some time before growth of the thermal types can be compared with it. The same is clearly true for a fast moving grain.

We have shown that the time of fall for an average grain is in excess of 10^8 years and so in the present work these grains can be considered at rest. Hence if we have one of these runaway grains we have in effect a grain moving through stationary grains and so the equation of growth is

$$\sigma_2 = \frac{\mu M_1 n_1 x}{4 \rho_2} \qquad (23)$$

as given by equation (18) of the last chapter.

As before, if the radius of the grain exceeds about 10^{-1}cm the resistance law will be Stokes's law. This will be so when

$$10^{-1} < \frac{\mu M_1 n_1 x}{4\rho_2}$$

With the density we have in mind, when x exceeds 5×10^{9}cm the resistance law becomes Stokes's law. But the total distance to be covered in the cloud radius, which is about 3×10^{12}cm and thus Stokes's law does not strictly apply for only about 0.17% of the path. We can thus take the resistance to be given by Stokes's law, thus overestimating the time for 0.2% of its path.

Both the resistance and the accelerating force are the same as in the previous method and so the equation of motion becomes

$$\frac{d}{dt}(m_2\dot{x}) = \frac{4}{3}\pi G M_1 n_1 m_2 (r_0 - x) - 6\pi\eta\sigma_2\dot{x}$$

Now, as before, m_2 and σ_2 are the mass and radius of the falling particle, so $m_2 = \frac{4}{3}\pi\rho_2\sigma_2^3$, while we are using x to denote its position from the outside of the cloud and

$$\sigma_2 = \frac{\mu M_1 n_1 x}{4\rho_2} \tag{23}$$

The equation of motion thus becomes

$$\frac{d}{dt}(M_2\dot{x}) = \frac{\pi^2 G M_1{}^4 n_1{}^4}{36\rho_2{}^2}\mu^3 x^3(r_0-x) - \frac{3\pi}{2\rho_2}\eta\mu M_1 n_1 x\dot{x}$$

which can clearly be written as

$$\frac{d}{dt}(M_2\dot{x}) = \delta x^3(r_0-x) - \gamma x\dot{x} \qquad (24)$$

if required, where γ and δ are constants given by

$$\gamma = \frac{3\pi\eta M_1 n_1\mu}{2\rho_2}$$
$$\delta = \frac{\pi^2 G M_1{}^4 n_1{}^4}{36\rho_2{}^2}\mu^3 \qquad (25)$$

We again require a solution of equation (24) and again a rigorous solution of the equation as it stands is rather difficult. We can however try taking the speed of the particle at any instant to be approximately its limiting velocity and so we try as a solution

$$\delta x^3(r_0-x) = \gamma x\dot{x}$$

or

$$\dot{x} = \delta\frac{x^2(r_0-x)}{\gamma} \qquad (26)$$

This is

$$\frac{dx}{x^2(r_0-x)} = \frac{\delta\, dt}{\gamma}$$

Integrating

$$\delta . t = \frac{\gamma}{r_0{}^2}\left[\log_e\left(\frac{x}{r_0-x}\right) - \frac{r_0}{x}\right]_{x_1}^{x_2}$$

We cannot take the limits of integration as $x=0$ and $x=r_0$ since both these points are points where the approximation has singularities; at $x=0$ the body radius is zero and at $x=r_0$ the gravitational field is zero. We thus take the end points $x=r_0/10$ and $x=9r_0/10$. This gives

$$\delta . t = \frac{\gamma}{r_0^2}\left[-\log_e 81 + \frac{80}{9}\right]$$

And so our approximation gives a time of

$$t \doteqdot \frac{13\gamma}{8 r_0^2} \qquad (27)$$

With the density of 10^{-8} gm/cc specified by us and the other numerical values inserted we obtain a time of fall of 7 years.

The size of the object after this time is given by equation (23) as

$$\sigma_2 = \frac{\mu M_1 \eta_1 x}{4 \rho_2} = 75 \text{ cms}$$

Hence all that we have to do now is to justify using the limiting velocity approximation.

For this purpose we shall use the full expressions, not those derived on substituting γ and δ. The limiting velocity is thus given by

$$\dot{x} = \frac{\pi \mu^2 \rho^3 G x^2 (r_0 - x)}{54 \eta \rho_2} \qquad (28)$$

We are attempting to show that the term $\frac{d}{dt}(M_2\dot{x})$ is small.

Now

$$M_2\dot{x} = \frac{\pi^2\mu^5\rho^6 G x^5(r_0-x)}{48.54\,\eta\,\rho_2^3}$$

on substituting for M_2 and $\dot{x}$. Thus

$$\frac{d}{dt}(M_2\dot{x}) = \frac{\pi^2\mu^5\rho^6 G}{48.54\,\eta\,\rho_2^3}\,x^4(5r_0-6x)\,\dot{x}$$

Substitute for $\dot{x}$ from (28)

$$\frac{d}{dt}(M_2\dot{x}) = \frac{\pi^3\mu^7\rho^9 G^2}{48.54^2\,\eta^2\,\rho_2^4}\,x^6(5r_0-6x)(r_0-x)$$

This term has to be small compared with the other terms of equation (24) before we can justify our solution. Hence $\frac{d}{dt}(M_2\dot{x})$ less than $\delta x^3(r_0-x)$ or

$$\frac{\pi\mu^4\rho^5 G\,x^3(5r_0-6x)}{72.54\,\eta^2\,\rho_2^2} < 1 \qquad (29)$$

The maximum positive value of this quantity, i.e. for the region $x < 5r_0/6$, is given when $x = 15r_0/24$. The value of the above expression (29) is then given as about 0·75. We have used the formula given for η to give a value of $4{\cdot}4\times10^{5}$, which agrees well with tabulated values.

The maximum absolute value of this quantity $\frac{d}{dt}(M\dot{x})/\delta x^3(r_0-x)$ occurs at the maximum value permissible for x , which in this case is $9r_0/10$ as that is where we have terminated our investigation. With this value for x , the value of the expression is 1.9 and is thus close to unity. x occurs to the fourth power in the expression so we can conclude that the approximation holds good up to very nearly $9r_0/10$ and so the time given by us in expression (27) is a fairly accurate estimate of the time of fall.

The term we have been neglecting is the term $\frac{d}{dt}(m_2 \dot{x})$. It is now interesting to compare the two components of this term, $\dot{m}_2 \dot{x}$ and $m_2 \ddot{x}$. Now

$$\dot{m}_2 \dot{x} = \frac{\pi \mu^3 \rho^3 x^2 \dot{x}^2}{16 \rho_2^2}$$

while

$$m_2 \ddot{x} = \frac{\pi^2 \mu^5 \rho^6 G x^4 (2r_0 - 3x) \dot{x}}{48 \cdot 54 \eta \rho_2^3}$$

and $\dot{x}$ is given by equation (28). So

$$\frac{m_2 \ddot{x}}{\dot{m}_2 \dot{x}} = \frac{\pi \mu^2 \rho^3 G x^2 (2r_0 - 3x)}{3 \cdot 54 \eta \rho_2 \dot{x}}$$

$$= \frac{(2r_0 - 3x)}{3(r_0 - x)} < 1 \qquad (30)$$

which is a dimensionless quantity as we expect. For small x we see that $\dot{m}_2 \dot{x}$ is slightly greater than $m_2 \ddot{x}$

at $x = 2r_0/3$ the ratio is 0 and then the ratio is negative, but with $\dot{M_2}\dot{x}$ grater than $M_2\ddot{x}$ and increasing until the ratio becomes infinite at $x = r_0$.

Thus through most of the range the change in mass exceeds the change in speed.

We can also give one other indication that for most of the range the terminal velocity approximation is good.

If we allow the velocity to differ by an amount ϵ from the terminal velocity, then

$$\dot{x} = \frac{\delta x^2 (r_0 - x)}{\gamma} + \epsilon$$

Substitute this into the equation (24)

$$\frac{d}{dt}(M_2 \dot{x}) = \delta x^3 (r_0 - x) - \gamma x \left[\frac{\delta x^2 (r_0 - x)}{\gamma} + \epsilon \right]$$

$$\frac{d}{dt}(M_2 \dot{x}) = -\gamma x \epsilon$$

and so a negative change in the rate of change of linear momentum is introduced to compensate for the too high velocity. The reverse will be true if the velocity is too low. Hence there will always be this compensation unless the velocity is close to the terminal velocity.

We can thus conclude that if a grain falls through one of our given condensations, accreting all other grains in its path, then it can reach the centre of this condensation in about 7 years, its size then being about 100 cm. It is thus possible to form a core of heavy material at the centre of a condensation just by allowing the grains to fall in the above way.

We can thus summarize the chapter as follows. The heavy material in a cloud must be in a suitable form before it can be transported to the centre of a condensation in a suitable time. It can be as large grains, or clumps of material, with a radius of about 70 cm; this then just falls under gravity. Alternatively the grains can grow as they fall, the final size of these grains being about 100 cm. The method of growth being accretion of other grains, or in theory at least heavy molecules.

In all the above cases the time of fall is of the order of one year, certainly less than 10 years. The acceleration remains fairly small throughout and so the velocity of these grains reaching the centre of the condensation is fairly close to the average velocity for the motion. This average velocity could not be higher than $3 \times 10^{12} / 3 \times 10^{7} = 10^{5}$ cm/sec allowing a time of fall of one year. Using the time of 7 years found above, this speed is less than 2×10^{4} cm/sec and is thus not so high as to cause overheating of the core, or to cause fragmentation of the core, with possible escape. We must note however that this does not remove all the difficulties about the temperature of the earth as this core still has to contract by an appreciable amount before a planet is formed.

Hence, if condensations are formed out of floccules as postulated by McCrea [3] of mass 10^{30} gms and density 10^{-8} gm/cc, then a core of heavy material can be formed in a fairly short time using one of the methods described above.

If at some subsequent epoch one of these condensations dips inside what is usually called the Roche limit, (see Jeans (20)) of which more will be said later, then the outer hydrogen layers would be swept away by the sun's tidal action. As a result we have a small mass of about 10^{28} gms composed mainly of the heavier elements. This can now contract to give a planet. For the density of 10^{-8} gm/cc this Roche limit comes at about the Asteroidal belt. Hence the observed difference between major and terrestrial planets agrees with what we obtain from this theory.

CHAPTER 4

About the large sized grains

The methods we have described for the generation of the heavy core at the centre of the gas cloud would appear to be the only two possible ways of forming this core in a reasonable time. That is to say, the core could be formed either by bodies falling and growing as they fall, or the condensation could contain a few large bodies, or clumps, which would then fall inwards. The problem of the growing body was discussed in the last chapter, where we actually showed as well that a clump must be about 100 centimetres across before it would fall inwards in a time of about one year. In this chapter we shall be interested in these large bodies, or clumps. There are two problems concerning these which must be discussed. One is obvious, and is the formation of such bodies. Just to postulate their existence would be unreasonable and so we must be able to produce these clumps before the method of generating a core using these becomes justified. Not only must we form these clumps, they must also be formed in a condensation, and carried with the condensation after formation. It is no good to form them if immediately they leave the gas condensation, or indeed if they leave the condensation at any subsequent time until the heavy core is formed. We must thus investigate the probability of these clumps escaping from the condensation after formation as well as show that formation can take place.

These two problems will be tackled in turn, taking the formation problem first since clearly the other problem has no meaning if the clumps have not been formed.

The formation problem

In his paper "The origin of the solar system" Professor McCrea [3] points out that when the floccules are initially captured by the sun they will be distributed in a roughly spherical volume with the sun as centre. About ten thousand years later these floccules will have grown by colliding with each other until they become condensations with a mass similar to the major planets and will have flattened into the invariable plane defined by the angular momentum of the system. Thus at the end of this period the individual condensations, out of which the planets are to be formed, are available; hence the large grains, or clumps, should also now be available.

Before the floccules were captured by the sun they were essentially part of a diffuse cloud of gas and this does not appear to be an ideal place for the formation of the required objects. Thus there remains only this period, of 10^4 years duration, during which the floccules are growing into condensations and settling into the invariable plane of the angular momentum in which the formation of the clumps could conceivably take place.

The only way in which these clumps could be formed under the above conditions is by accretion. This process could come about in several ways. A grain could accrete molecules of the heavy elements, collisions being caused by the thermal motions of the molecules. A grain could also grow by accretion of other grains. In this case the required collisions could be caused either by the thermal motions of the grains or by the growing grain moving through the other grains due to it possessing some velocity.

All of these possibilities will be investigated in turn. Again we do not enter into any discussion about the process by which the grains adhere together on collision. As in the previous chapter we postulate that they do.

I. Accretion due to collisions between a grain and molecules of the heavy elements.

It was pointed out in the last chapter how unlikely it was for a grain to move with a speed greater than the thermal velocity of the gas molecules. It is even more unlikely an occurrence in the present context since the accelerating forces due to gravity must be less for the floccules than for the condensations. We thus consider only accretion caused by the thermal collisions of the heavy element molecules with grains. (Collision with all types of molecules obviously occur but we only allow the heavy element molecules to adhere and so are only interested in these.) We have already worked out the equation of growth under such conditions in chapter 2, equation (20), namely

$$\sigma = \frac{\mu M_1 n_1 W t}{4 \rho_2} \qquad (1)$$

where W is the thermal velocity of the molecules involved and $\mu M_1 n_1$ is the density of these molecules, μ clearly being the proportion by weight of heavy elements to gas which has already been chosen as $1:100$.

McCrea has chosen a density of 6.6×10^{-9} gm/cc for the density of the floccules and hence

$$\mu M_1 n_1 = 6.6 \times 10^{-11} \text{ gm/cc}$$

The thermal velocity is $2\sqrt{\frac{2kT}{\pi M}}$ as has been indicated before.

Introducing all these numerical values into equation (1) we obtain

$$\sigma = 3.9 \times 10^{-7}\, t \text{ cm}$$

which gives us the radius of the clump so formed after a time t seconds.

Now the actual maximum time available for growth is 10^4 years and thus after this period of time the grain can grow into a clump with a radius of

$$1.25 \times 10^5 \quad \text{centimetres}$$

This is rather larger than the size required before the time of fall to the centre of a condensation becomes about one year and hence there appears to be no difficulty in obtaining clumps with the required radius of 100 cm or so.

In practice the clumps would probably not grow to the above size of 10^5 cm since molecules might not adhere so easily at such sizes and material could also be running short, but it certainly appears possible to produce clumps as required for the formation of a core in the last chapter.

If we assume this adhering process to be active all the time, then growth of all grains in interstellar space would occur. It would be satisfying if the size of the grains so produced did not conflict with the observed size of grains. From Allen [19] we find that an average interstellar cloud has a density around 1.5×10^{-23} gm/cc. Introducing this value into the above equation (1) and allowing a time of 10^9 years for the growth we obtain a radius of 2.5×10^{-5} cm for the resulting object. This is about the estimated radius of interstellar grains and hence even if the adhering process had been active since the formation of interstellar clouds the result would not be grains larger than what is observed.

Hence the process of growth by collisions of heavy molecules with grains can give us the required size for our clumps, while at the same time not producing too large objects in interstellar space if the process was active for the whole of the past history of a cloud.

II. <u>Accretion of other grains due to thermal collisions.</u>

The equation of growth for this type of collision will clearly be the same as equation (1) above

$$\sigma = \frac{\mu M_1 n_1 W t}{4 \rho_2}$$

where in this case W is the thermal velocity of the grains and $\mu m_1 n_1$ the mass of grains in a unit volume of cloud, μ once again being the ratio of grain to gas, which is $1:100$.

With the new values for the thermal velocity given by $2\sqrt{\frac{2kT}{\pi M_g}}$ where M_g is the mass of a grain, the equation of growth becomes on inserting numerical values,

$$\sigma = 7 \times 10^{-12}\, t \text{ cm} \qquad (2)$$

As before the total time available is 10^4 years and so the maximum size which a clump can grow to using this method is 2.2 centimetres in radius.

Thus this method does not give clumps of the required radius, though the radius found is sufficiently close to the required value for this method not to be completely useless. A slightly higher floccule density or a floccule containing more grains than average could render this method useful. If the floccule, for some reason, was formed before this period of 10^4 years then the increased time could also make this method satisfactory.

If this method of growth had been in operation throughout the history of the galaxy, then introducing the interstellar cloud density and the new value for the time into the above equation gives a body with a radius of 10^{-9} cm. Thus this process would certainly not produce interstellar grains with a radius greater than what is observed today and so in this respect is very satisfactory.

III. <u>Accretion of other grains by a grain moving through them.</u>

The equation of growth for this type of accretion is equation (19) of chapter 2, namely

$$\sigma = \frac{\lambda \mu m_1 n_1 x}{4 \rho_2}$$

Here μ as usual is the proportion of grains to gas. λ is the probability that grains adhere on collision and as we have postulated that they all adhere this is 1. With the usual value for the floccule density we obtain

$$\sigma = \frac{10^{-10} x}{6} \qquad (3)$$

where x is the distance covered by this grain.

Hence in order to produce a clump comparable in size to what we require, x must be about 4×10^{12} cm.

Thus if our grain is to grow into a clump having a radius large enough to fall as described in the last chapter then it must move through the other grains for the above distance during the period of time available.

During the period now under discussion the floccules are continually colliding together and growing. Their final mass is about 10^{30} gms, being a condensation mass, while initially the mass is that of one floccule, namely 2×10^{28} gms. Thus a convenient mean mass for this stage of the proceedings is about 2×10^{29} gms. The radius of a cloudlet having this mass and the prescribed floccule density is 2×10^{12} cm in round numbers.

Let us then first find the time required by a grain to fall to the centre of such a mean condensation in much the same way as investigated for the final condensation in the last chapter. The time of fall, taken from equation (27) of the last chapter, is

$$t = \frac{13\gamma}{\delta r_0^2} \qquad (4)$$

where

$$\gamma = \frac{3\pi \eta_2 M_1 n_1 \mu}{2\rho_2}$$

and

$$\delta = \frac{\pi^2 G M_1^4 n_1^4 \mu^3}{36 \rho_2^2}$$

all the notation being as previously defined and r_o being the radius of the condensation.

Inserting all the numerical values into equation (4) with the usual density of 6.6×10^{-9} gm/cc and $r_0 = 2 \times 10^{12}$ cm the time of moving is given as 3×10^8 seconds, or 10 years. Thus in 10^4 years a clump can move through many such mean condensations and can quite easily reach the given size of about 100 cm.

With this type of motion there is no connection with conditions existing in interstellar space and hence this method could not possibly produce any effects that would be in disagreement with the size of interstellar grains.

We see that of the three methods available for the growth of clumps by accretion in floccules, one of them, the one involving growth due to the thermal motions of the grain, does not give the required size for a clump. The other two, accretion by the thermal collision of the heavy element molecules and accretion due to a grain moving through other grains, both give satisfactory results. In both cases the required size is obtained without any difficulty and indeed it appears that larger sizes could be produced if required. The first method also shows that if this process had been operating for all time, the size of the resulting interstellar grain would be no larger than present day interstellar grains. The main weakness of this method is that no apparent way exists in which the heavy element molecules could adhere to the grain on collision.

The other method requires the adhering of grains as they collide. Nothing has been said so far about the possibility of this happening, beyond stating that these grains are more similar to snow flakes than lead shot, and so accretion is not impossible. In a discussion at the Royal Astronomical Society [21] on the origin and constitution of the planets, during which a brief outline of the above work was presented by Professor McCrea, Professor Bernal expressed the opinion that certain types of grain could indeed act in the way postulated by us. Hence the assumption that grains adhere to other grains on collision, which is one of the major assumptions of the theory, does not seem unrealistic.

Having thus concluded that it is quite possible to produce the clumps required for the core formation we now turn our attention to the second problem concerning these objects.

Possible escape of clumps.

If two clouds of gas containing such clumps as described above are involved in a collision, as they must if condensations are to be formed, there is a possibility that these clumps will retain their previous velocities and in this way travel through the gas, thus escaping from the condensation. If this occurs then these clumps are lost from the condensation and cannot take any part in the formation of the core and so nothing is achieved in forming them. Hence before the method of core formation involving clumps becomes useful we must show that these do not escape from the surrounding gas as the condensations collide.

Let the clumps be given a speed u relative to the gas by a collision. Clearly the bigger the value of u, the easier escape will be. We thus take the initial value of the speed u to be greater than the thermal velocity of the resisting gas.

Now the resistance offered by a gas to an object moving through it under the above conditions has been found by us in chapter 2 and is given by equation (5) of that chapter as

$$R = \frac{\pi}{2}(1+e)\frac{M_1 M_2}{M_1+M_2}\, n_1(\sigma_1+\sigma_2)^2\, v^2 \qquad (5)$$

If, as in this case, the object is much larger than a gas molecule and all collisions are assumed elastic then the above equation reduces to

$$R = \pi M_1 n_1 \sigma_2^2 v^2 \tag{6}$$

and the deceleration caused by this force in the clump is thus

$$\frac{\pi M_1 n_1}{M_2} \sigma_2^2 v^2$$

Now in practice the gravitational field of a condensation will tend to prevent escape as well. We shall however ignore gravity and hence our conclusion will be stronger if we can show that no escape is possible. Thus the equation of motion of a clump can be written as

$$\frac{dv}{dt} = -\frac{\pi M_1 n_1}{M_2} \sigma_2^2 v^2$$

or

$$v\frac{dv}{dr} = -\alpha v^2 \tag{7}$$

where α is the constant given by

$$\alpha = \frac{\pi M_1 n_1 \sigma_2^2}{M_2}$$

or, as $M_2 = \frac{4}{3}\pi \rho_2 \sigma_2^3$

$$\alpha = \frac{3 M_1 n_1}{4 \rho_2 \sigma_2} \tag{8}$$

Integrating equation (7) we obtain

$$\log_e v = -\alpha r + \text{constant}$$

The initial conditions are that at $r = 0$ the speed is and hence the constant is $\log_e u$ and

$$\log_e\left(\frac{v}{u}\right) = -\alpha r$$

or

$$v = u \exp\{-\alpha r\} \qquad (9)$$

The equation for the resistance we have used is only correct for large velocities. At some velocity, V say, (which as we shall see we do not require a numerical value for) the resistance law will change to the well known Stokes law which is of the form βv where β is a constant whose value can be found from chapter 2 if required. If we assume a smooth change over from one law to the other, then at V,

$$\alpha V^2 = \beta V \qquad (10)$$

For motion below this speed V, again ignoring gravity

$$v \frac{dv}{dr} = -\beta v$$

Integrating

$$v = V - \beta r \qquad (11)$$

where now the initial speed is V .

Thus the speed of the clump becomes zero after travelling a distance

$$r_1 = \frac{V}{\beta}$$

or, in view of equation (10)

$$r_1 = \frac{1}{\alpha}$$

Now the distance travelled before the speed became less than V , from equation (9), is r_2 , given by

$$r_2 = \frac{1}{\alpha} \log_e \frac{u}{V}$$

Hence the total distance which this clump must travel before coming to rest is $r_1 + r_2$ which is

$$\frac{1}{\alpha}\left\{ 1 + \log_e \frac{u}{V} \right\} \tag{12}$$

If this quantity is less than the radius of a floccule, then the clumps cannot possibly escape either from a floccule, or a condensation graater than a floccule.

Let us now insert numerical values for the case that is of interest to us.

The random speed of a floccule as given by McCrea is about a kilometre a second, or 10^5 cm/sec . The collision speed of two floccules will not thus be greater than twice this. The maximum initial speed that the clumps can have relative to the gas is thus 2×10^5 cm/sec. (This is when one floccule is completely reversed in motion by the collision and is obviously an overestimate.) As the proof will be more convincing if we overestimate the speed rather than underestimate we shall take the initial speed of the clumps relative to the gas to be 10^6 cm/sec.

We are interested in the possibility of escape during the period when the floccules are growing by collisions into condensations and, as already pointed out, the mean mass for this period is 2×10^{29} gms. Again for the reason that we prefer to use unfavourable conditions rather than favourable ones, we shall take the condensation to be as small as possible, namely one floccule mass and possessing the usual floccule density of 6.6×10^{-9} gm/cc. Hence the radius of a floccule is 9×10^{11} cm and $r_1 + r_2$ as given above must be less than this.

The actual size of the clumps required by us is slightly less than 100 cm, but again as large ones can escape easier we shall use 10^2 cm as the clump radius. This allows our results to apply for the objects produced in the other methods of forming a core, though it seems unlikely that these would escape as they are not formed until after the planetary condensation is formed and thus there would be no collisions between condensations.

There now only remains V to be specified. In reality this is likely to be in the region of the mean thermal velocity, hence about 10^5 cm/sec. However we shall take it to be only 1 cm/sec, a very unfavourable speed as far as equation (12) is concerned.

Inserting all the numerical values into equation (12) we find that the clump must travel a distance

$$\frac{1}{5\times10^{-11}}\left\{1 + \log_e 10^6\right\} \doteqdot 3\times10^{11}\text{ cm}$$

before it is brought to rest.

But this is only one-third of the radius of a floccule and so the clump, under very unfavourable conditions, cannot escape from a condensation. Hence there is strong evidence to conclude that they do not actually escape.

Throughout this discussion we have ignored gravitational effects. If we take account of these then we do not have to reduce the speed of the clump to zero, only reduce it until it is less than the escape velocity from the condensation. The escape velocity from a spherical body is given roughly by

$$\frac{1}{2} V_e^2 = \frac{GM}{R}$$

where M is the mass and R the radius of the body. For a floccule, this gives an escape velocity as 7.7×10^4 cm/sec. If the clump has speed below this then it cannot escape.

Hence, as we have actually succeeded in reducing the speed to zero we can conclude that the clumps mentioned above will not escape from the surrounding gas when two condensations collide.

On the face of it this is a very surprising result, that tenuous gas like we have in our floccules can actually bring to rest a body like a clump which weighs about a hundredweight. We can however see very simply that this is a sensible result by considering the mass of the material which a clump has to push out of the way to escape. This is the mass inside a cylinder of radius equal to the clump radius, and length equal to the floccule length, which is

$$M = \pi (10^2)^2 \times 2 \times 9 \times 10^{11} \times 6.6 \times 10^{-9}$$

$$M \doteqdot 4 \times 10^8 \text{ gms}$$

The mass of a clump is $\frac{4}{3}\pi 10^6$ gms.

And hence the clump has to remove a mass equal to about

100 times its own mass before any escape is possible. Again in this argument we are ignoring the gravitational effects.

Hence, with absolute conviction, we can conclude that clumps do not escape from the condensations in any collision.

A speculative note on comets and meteorites

It would be interesting to obtain an estimate of the size which a clump must have before any possibility of escape exists. Again using adverse conditions we take the mass of the condensation to be 10^{30} gms and the radius is thus 3.3×10^{12} cm. If escape is possible, equation (12) must not give a value less than 3.3×10^{12} cm and so

$$3.3 \times 10^{12} < \frac{1}{\alpha}\left\{1 + \log_e \frac{u}{V}\right\}$$

Let us now take a more rational value for V, namely 10^5 cm/sec then

$$\alpha \lessdot 10^{-12}$$

if escape is to be possible. But

$$\alpha = \frac{3M_1 n_1}{4 \rho_2 \sigma_2}$$

and hence we have $\sigma_2 > 5 \times 10^3 cm$.

Thus before a clump can possibly escape from a condensation it must have a radius of about $10^4 cm$. If it is to escape, its velocity must exceed the escape velocity given as $10^5 cm/sec$ when it reaches the outer layers, hence equation (9) gives

$$v = u \exp\{-\alpha r\}$$

as $10^5 < 10^6 \exp\{-3.3 \times 10^{12} \alpha\}$.

This means that

$$\alpha < \frac{2.3}{3.3 \times 10^{12}}$$

which again requires a radius of about $10^4 cms$. A clump of such radius would have a mass of about $4 \times 10^{12} gms$. These clumps would thus be greater than the meteorites. The average mass of comets, calculated from

$$\log M = 19 - 0.4 M_0$$

where M is its mass and M_0 its absolute magnitude appear to be about $10^{16} gms$ and thus is slightly larger than these escaped clumps.

It is tempting however to suggest that these clumps, when they manage to escape, are the origin of comets and meteorites, as these escaped clumps would presumably have rather eccentric orbits about the sun. Comets could be formed when a few of them collide together and adhere, while meteorites (in showers) could be formed when collision results in fragmentation.

Thus, in this chapter we have demonstrated how large grains or clumps could be formed in the gas floccules in the period when they are growing into condensations. It was further shown that these methods would not give rise to abnormally large grains in interstellar space, even if the growing process had been operative for the derivation of the galaxy. We have also concluded that it was impossible for these clumps to escape from the condensation when this condensation is involved in a collision. A suggestion is made that somewhat larger clumps could escape and form the comets and meteorites of the solar system.

CHAPTER 5

The dispersal of a condensation

As is well known there is a critical distance L from the sun within which a condensation with a given density ρ cannot be in equilibrium due to the tidal action of the sun. This is often called Roche's limit and has been given by Jeans in 'Astronomy and Cosmogony' [20] as approximately

$$L = 1.523 \left(\frac{M_\odot}{\rho} \right)^{1/3}. \qquad (1)$$

where $M_\odot$ is the mass of the sun.

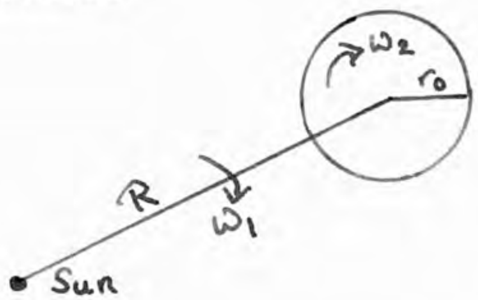
Jeans further shows that if this fluid condensation, assumed incompressible, is in equilibrium it must be ellipsoidal in shape. If a, b and c denote the semi-axis of the critical ellipsoid, that is the ellipsoid when the fluid is at the critical Roche limit, then we have that

$$a = 1.65\,r \qquad \text{and} \qquad b = 0.81\,r$$

where $r = abc$.

What we require is some expression for the dispersal time of a condensation if it cannot hold together. This we could do fairly easily if we could write down the gravitational attraction of such an ellipsoid on a particle on its outer surface. Unfortunately this attraction is given by a very complicated expression for an ellipsoid, and involves elliptic integrals and the associated elliptic functions as can be seen in McMillan's 'Theory of the Potential' [22]. A sphere is the only regular volume that has a simple expression for the gravitational attraction and consequently we will estimate the dispersal time for the sphere indicating that a similar result holds for an ellipse if we chose the sphere properly.

Consider a spherical cloud of gas, with radius r_0 and density ρ_1 moving in a circular orbit around the sun, the radius of the orbit being R. Let the angular velocity of the cloud about the sun be ω_1 and the angular velocity of the cloud about its centre be ω_2, both being in the same plane and in the same direction.



The condition that the whole of the cloud (that is the centre of gravity) moves in a circular orbit about the sun is clearly

$$R\omega_1^2 = \frac{GM_\odot}{R^2} \tag{2}$$

where $GM_\odot/R^2$ is the attraction per unit mass caused by the sun.

Consider the outermost particle of the system. If the condensation is not to disperse then this must keep its position relative to the cloud centre and hence it must move, at this instant, around the sun, getting neither closer nor further away.

Now the centrifugal force on this particle is $mv^2/R+r_0$ where v is its speed.

But clearly $v = R\omega_1 + r_0\omega_2$ and so the centrifugal force is

$$\frac{m(R\omega_1 + r_0\omega_2)^2}{(R+r_0)}$$

The attractive force towards the sun is now

$$\frac{GmM_\odot}{(R+r_0)^2} + \frac{GmM}{r_0^2}$$

where M is the mass of the condensation and is clearly given by

$$M = \frac{4}{3}\pi r_0^3 \rho_1$$

Hence

$$\frac{(R\omega_1 + r_0\omega_2)^2}{R+r_0} = \frac{GM_\odot}{(R+r_0)^2} + \frac{GM}{r_0^2} \tag{3}$$

In the conditions investigated by Roche, the condensation always is 'face on' to the sun and hence if we apply the same conditions, $\omega_1 = \omega_2 = \omega$ (say) and (5) becomes

$$\omega^2 (R + r_0) = \frac{G M_\odot}{(R + r_0)^2} + \frac{G M}{r_0^2}$$

But, from equation (2)

$$\omega^2 = \frac{G M_\odot}{R^3}$$

and hence, if the outermost particle always keeps its position, then the following condition must be satisfied

$$\frac{G M_\odot (R + r_0)}{R^3} = \frac{G M_\odot}{(R + r_0)^2} + \frac{G M}{r_0^2}$$

giving

$$\frac{M_\odot \left[(R + r_0)^3 - R^3 \right]}{R^3 \; (R + r_0)^2} = \frac{M}{r_0^2} \qquad (4)$$

If we now make the usual assumption, also made by Roche, that the distance from the condensation to the sun exceeds the radius of the condensation by a large factor, or $R >> r_0$ then equation (4) simplifies to

$$3 M_\odot r_0^2 = M R^3 \qquad (5)$$

We have already noted that $M = \frac{4}{3}\pi r_0^3 \rho_1$, and so (5) gives

$$\rho_1 = \frac{9M_\odot}{4\pi R^3} = \frac{.716M_\odot}{R^3} \qquad (6)$$

If we consider a particle advanced by $\pi/2$ from the one just considered, that is the one following the same path around the sun as the centre of the cloud, then the forces along the line from sun to particle are the same as for the centre, thus

$$R\omega^2 = \frac{GM_\odot}{R^2}$$

In the direction of the centre of the cloud we have a gravitational force of $\frac{GmM}{r_0^2}$ and a centrifugal force in opposition to this of

$$mr_0\omega^2 = \frac{mr_0 GM_\odot}{R^3}$$

Hence, the force towards the centre of the condensation is

$$\frac{GmM}{r_0^2} - \frac{GmM_\odot r_0}{R^3}$$

But, for this cloud equation (5) holds and so

$$3M_\odot r_0^3 = MR^3$$

The above equation thus becomes

$$\text{Total force towards centre} = \frac{2GmM}{3r_0^3}$$

This force is always positive and will always exist if the conditions are as described.

If the condensation is incompressible, then this force will set up pressures inside the condensation which will disturb the equilibrium of the outer particle and so this will be pushed out and the condensation in fact dispersed. This is what we expect to happen since if we investigate the density given by equqtion (6) for a condensation at a distance R from the sun, we see that they violate the condition of Roche given by equation (1).

If however the condensation is compressible, then the parts on which the force is acting can move inwards causing an increase in the density of the cloud. As this density increases, it becomes more incompressible. Let us now find what density this cloud would have if we allow it to condense until it is the same shape as the critical ellipsoid found by Roche. Clearly as the outermost particle (and a similar argument holds for the innermost) keep their position (we have defined conditions so that they do) then the major axis of the ellipsoid is equal to the diameter of the sphere, or $a = r_0$.

Now the volume of the ellipsoid is $\frac{4}{3}\pi abc = \frac{4}{3}\pi r^3$

But for the critical ellipsoid $a = 1.65 r$ and so the volume of the ellipsoid is $\frac{4}{3}\pi \frac{a^3}{(1.65)^3}$.

The volume of the spherical cloud is $\frac{4}{3}\pi a^3$ $\left(\text{or } \frac{4}{3}\pi r_0^3\right)$. Hence, the ratio of the densities of the two volumes is

$$\frac{\rho_1}{\rho} = \frac{1}{(1.65)^3} \qquad (7)$$

But we have found the density of the sphere, ρ_1 in equation (6), as $\frac{.716\, M_\odot}{R^3}$.

Hence, the density of the ellipsoid is

$$\rho = (.716)(1.65)^3 \frac{M_\odot}{R^3}$$

or the critical distance at which this ellipsoid should be is

$$R = (.716)^{1/3}(1.65)\left(\frac{M_\odot}{\rho}\right)^{1/3} = 1.48\left(\frac{M_\odot}{\rho}\right)^{1/3} \qquad (8)$$

Thus the agreement is remarkably good, with the critical distance found in equation (1).

Hence, we conclude that if the spherical cloud with density and position satisfying equation (5) is incompressible it will disperse as it does not satisfy Roche's limit, but if it is compressible (and thus not covered by Roche's case) it will condense until it has a similar density to Roche's critical case when it has the same shape. If by now the gas has become incompressible we have equilibrium as given by Roche, if not it will proceed to condense.

It thus appears that if we investigate the dispersal of such a gas cloud the results so obtained will give a fair indication of what happens in the more complex ellipsoidal case. We thus proceed to investigate the dispersal of this spherical cloud.

Consider then such a spherical cloud of radius r_0 with a density that does not satisfy the condition of equation (6) at a distance R. Then it must disperse. Let the centre of this cloud move in a circular orbit with angular velocity ω_1 about the sun. Now, as the condensation cannot be in equilibrium, if the outermost particle is <u>not</u> to be thrown outwards it must move slower than the centre of the cloud. Let its angular velocity be ω_2 about the sun. (NOT about the cloud centre as before.)

The cloud centre follows a circular path around the sun and so

$$\frac{GM_\odot}{R^2} = R\omega_1^2 \qquad (9)$$

For the outermost particle

$$(R+r_0)\omega_2^2 = \frac{GM_\odot}{(R+r_0)^2} + \frac{GM}{r_0^2} \qquad (10)$$

as we also have this particle moving in a smaller circle.

After an interval of time τ (say) the angle swept out by the radius from the sun to the centre of the cloud is $\omega_1 \tau$ while the corresponding quantity for the particle is $\omega_2 \tau$.

Thus, after this interval of time τ the angle of separation (at the sun) of these two particles is given by θ , where

$$|\theta| = \tau(\omega_2 - \omega_1)$$

Where by $|\theta|$ we mean that the signs are chosen so that a positive angle is given.

The time taken for a displacement θ is thus

$$\tau = \frac{|\theta|}{\omega_2 - \omega_1} \tag{11}$$

again with the signs chosen so that we have a positive time. This will be true as long as θ is small enough for the cloud to be still fairly close to a sphere, and the attraction equations thus being unaltered.

Substituting the values for ω_1 and ω_2 from equations (9) and (10) into equation (11) we obtain

$$\tau = \frac{|\theta| G^{-1/2} (R + r_0)^{3/2} R^{3/2} r_0}{\left\{ R^{3/2} \left[r_0^2 M_\odot + (R+r_0)^2 M \right]^{1/2} - r_0 (R+r_0)^{3/2} M_\odot^{1/2} \right\}}$$

Rationalizing the denominator in the usual manner,

$$\tau = \frac{|\theta| G^{-1/2}(R+r_0)^{3/2} R^{3/2} r_0 \left\{ R^{3/2}\left[r_0^2 M_\odot + (R+r_0)^2 M\right]^{1/2} + r_0 (R+r_0)^{3/2} M_\odot^{1/2} \right\}}{R^3\left[r_0^2 M_\odot + (R+r_0)^2 M\right] - r_0^2 (R+r_0)^3 M_\odot}$$

If again we assume that $R \gg r_0$ or that the distance of the condensation from the sun greatly exceeds the condensation radius, the above equation simplifies to

$$\tau = \frac{|\theta| G^{-1/2} R r_0 \left\{ R^{3/2}\left[r_0^2 M_\odot + R^2 M\right]^{1/2} + r_0 R^{3/2} M_\odot^{1/2} \right\}}{R^3 M - 3 r_0^3 M_\odot} \quad (12)$$

As we expect, we see that the dispersal time becomes infinite when the condition

$$R^3 M - 3 r_0^3 M_\odot = 0$$

is satisfied, which was the condition of non-dispersal we found for a compressible sphere.

If dispersal occurs, then $R^3 M < 3 r_0^3 M_\odot$ since the gravitational effects of the cloud are then smaller than the sun's. Thus, if we take account of the signs,

$$\tau = \frac{\theta G^{-1/2} R r_0 \left\{ R^{3/2}\left[r_0^2 M_\odot + R^2 M\right]^{1/2} + r_0 R^{3/2} M_\odot^{1/2} \right\}}{3 r_0^3 M_\odot - R^3 M} \quad (13)$$

In general this equation (13) will be the best we can obtain, but in certain cases we can simplify this expression further.

<u>If $R^2M \ll r_0^2 M_\odot$</u>

then

$$R^{3/2}\left[r_0^2 M_\odot + R^2 M\right]^{1/2} + r_0 R^{3/2} M_\odot^{1/2}$$

clearly reduces to $2R^{3/2} r_0 M_\odot^{1/2}$.

The expression for the dispersal time then becomes

$$\tau = \frac{2\theta G^{-1/2} r_0^2 R^{5/2} M_\odot^{1/2}}{3r_0^3 M_\odot - R^3 M} \qquad (14)$$

<u>If $R^2M \gg r_0^2 M_\odot$</u>

then

$$R^{3/2}\left[r_0^2 M_\odot + R^2 M\right]^{1/2} + r_0 R^{3/2} M_\odot^{1/2}$$

reduces to $R^{5/2} M^{1/2}$.

and the time in equation (13) becomes

$$\tau = \frac{\theta G^{-1/2} r_0 R^{7/2} M^{1/2}}{3r_0^3 M_\odot - R^3 M} \qquad (15)$$

In all these expressions for the dispersal time we observe that τ varies linearly with the displacement θ. There are no governing factors on the value of θ beyond the constraint that it must be small. A satisfactory choice of value for this angle would be r_0/R since then the displacement in position of the particle is proportional to the position of the particle relative to the centre. We must however stress that at this point we have introduced a completely arbitrary quantity into the theory; we could just as well take $\Theta = 2r_0/R$, or $5r_0/R$. Hence from now on the time can only be an order of magnitude estimate only. With this value for Θ equations (13), (14) and (15) for the time become

$$\tau = \frac{r_0^2 G^{-\frac{1}{2}} R^{3/2} \left\{ [r_0^2 M_\odot + R^2 M]^{1/2} + r_0 M_\odot^{1/2} \right\}}{3 r_0^3 M_\odot - R^3 M} \qquad (16)$$

$$\tau = \frac{2 G^{-1/2} r_0^3 R^{3/2} M_\odot^{1/2}}{3 r_0^3 M_\odot - R^3 M} \qquad (17)$$

$$\tau = \frac{G^{-1/2} r_0^2 R^{5/2} M^{1/2}}{3 r_0^3 M_\odot - R^3 M} \qquad (18)$$

We now have to apply these equations to physical situations, with numerical values. We shall do this for two situations. We will take a cloud similar to the one already discussed with a mass 10^{30} gms and a density of 10^{-8} gm/cc. We shall find the dispersal time for various distances from the sun, or varying R. We shall also take a cloud with the given mass of 10^{30} gms at a fixed distance from the sun and compute the dispersal time for various densities (or various r_0 since $M = \frac{4}{3}\pi \rho r_0^3$).

Let us initially take the first case

$M = 10^{30}$ gms, $\rho = 10^{-8}$ gm/cc, $r_0 = 2.9 \times 10^{12}$ cm, $M_\odot = 2 \times 10^{33}$ gms

The condition $\mathcal{R}^2 M \ll r_0^2 M_\odot$ is satisfied for $\mathcal{R} < 8\times10^{13}$ cm or about 6 AU.
Hence the inequality is satisfied for any cloud within the orbit of Jupiter, and as this is the only region of interest to us we can use the simplified equation (17) throughout, giving

$$\tau = \frac{2G^{-1/2} r_0^3 \mathcal{R}^{3/2} M_\odot^{1/2}}{3r_0^3 M_\odot - \mathcal{R}^3 M}$$

Introducing all the necessary parameters, this gives

$$\tau = \frac{8\cdot25\times10^{57}\mathcal{R}^{3/2}}{1\cdot43\times10^{71} - 10^{30}\mathcal{R}^3} \text{ secs.}$$

or

$$\tau = \frac{2\cdot62\times10^{20}\mathcal{R}^{3/2}}{1\cdot43\times10^{41} - \mathcal{R}^3} \text{ years.} \qquad (19)$$

The results of the numerical calculations for different values of $\mathcal{R}$ are most easily expressed in tabular form. The actual values for $\mathcal{R}$ we have taken are fairly close to the mean orbital radii of the terrestrial planets, plus one or two other convenient points. In the second column we indicate which planet the value of $\mathcal{R}$ taken corresponds to.

DISTANCE	PLANET	DISPERSAL TIME
6×10^{12} cm	MERCURY	.027 years
10^{13} cm	VENUS	.058 years
1.5×10^{13} cm	EARTH	.11 years
2.5×10^{13} cm	MARS	.23 years
4×10^{13} cm	ASTEROIDS	.84 years
5×10^{13} cm	---	5.1 years
5.23×10^{13} cm	---	∞

TABLE 1. The variation of dispersal time with orbital radius R .

We shall leave any discussion of the above results until after we have completed the second investigation.

In this case we take a cloud of fixed mass 10^{30} gms to be at a constant distance from the sun, the dispersal time is then found for various densities. Since most of us are more attached to the earth than any other planet we shall take the constant distance to be that of the earth's orbit, thus $R = 1.5 \times 10^{13}$ cm .

Now, the period of rotation of the earth about the sun is, by definition, one year. Hence the angular velocity of any body rotating along the sun's orbit must be 2π radians/year. Hence,

$$\frac{GM_\odot}{R^3} = (2\pi)^2$$

$$\text{or} \quad G^{-\frac{1}{2}} R^{\frac{3}{2}} = \frac{M_\odot^{\frac{1}{2}}}{2\pi} \tag{20}$$

Thus we can introduce this into any of the expressions for the time, and the units will then be years. For the case under consideration

$$R = 1.5 \times 10^{13}\ \mathrm{cm}, \quad M_\odot = 2 \times 10^{33}\ \mathrm{gms}, \quad M = 10^{30}\ \mathrm{gms}.$$

Hence $R^2 M \ll r_0^2 M_\odot$ for all $r_0 > 8 \times 10^{11}\ \mathrm{cm}$

Now, for any density that is of any interest to us, must be greater than the above value, and so once again the dispersal time is given by equation (17), now simplified further by the introduction of

$$G^{-\frac{1}{2}} R^{\frac{3}{2}} = M_\odot^{\frac{1}{2}} / 2\pi$$

giving

$$\tau = \frac{r_0^3 M_\odot}{\pi (3 r_0^3 M_\odot - R^3 M)} \tag{21}$$

Introducing the numerical values chosen by us for the quantities into this expression, we obtain

$$\tau = \frac{6.36\, r_0^3}{60\, r_0^3 - 3.37 \times 10^{37}} \text{ years.}$$

The results of this numerical calculation are given in Table 2. In columns 2 and 3 we show the value of the density and number density of a cloud corresponding to the radius r_0 given in column 1. The variation in time with r_0 becomes rather rapid as tends to the solution of

$$3 r_0^3 M_\odot - R^3 M = 0$$

For this reason values for more frequent intervals in have been considered near this solution than in a range further away.

Radius r_0 (cm)	Density ρ (gm/cc)	No.Density n (mol/cc)	time τ (years)
2×10^{12}	3×10^{-8}	9×10^{15}	.11
10^{12}	2.4×10^{-7}	7.2×10^{16}	.11
9×10^{11}	3.3×10^{-7}	10^{17}	.46
8.5×10^{11}	3.9×10^{-7}	1.17×10^{17}	1.28
8.4×10^{11}	4.05×10^{-7}	1.23×10^{17}	2.07
8.3×10^{11}	4.17×10^{-7}	1.265×10^{17}	6.3
8.26×10^{11}	4.24×10^{-7}	1.285×10^{17}	54.4
8.257×10^{11}	4.242×10^{-7}	1.29×10^{17}	150
8.255×10^{11}	4.244×10^{-7}	1.29×10^{17}	∞

TABLE 2. Showing the variation of the dispersal time with cloud density.

We observe that the first two values given for the dispersal time in the above table are exactly the same. From consideration of the formula we are using, the reason for this is obvious.

We have

$$\tau = \frac{2G^{-\frac{1}{2}} r_0^3 R^{3/2} M_\odot^{1/2}}{3r_0^3 M_\odot - R^3 M}$$

For small densities $3r_0^3 M_\odot >> R^3 M$ and so this expression reduces to

$$\tau = \frac{2G^{-\frac{1}{2}} R^{3/2}}{3 M_\odot^{1/2}}$$

which is independent of the parameter r_0, hence the constant time.

Physically the reason is even more obvious. When the density of the cloud is small the gravitational field of the cloud is too weak to have any effect on the dispersal time. The dispersal time in this case is just the time taken by a collection of particles, with no interaction, orbiting about the sun to separate out.

In both the above tables, the last row with a dispersal time of ∞ is obviously the conditions when the critical density or distance has been reached, and this corresponds to a solution of

$$3r_0^3 M_\odot - R^3 M = 0$$

in the first table R being the unknown while in the second case r_0 is the unknown.

From the two tables given we can clearly draw some general conclusions. If the condition connecting the density and position of a condensation is violated then the time taken by such a condensation to disperse is extremely small and is smaller than the orbital period of the condensation, even if we only violate the given condition by a small amount. We can thus in general say that a condensation will either hold together, not dispersing at all, or disperse in a very short period of time.

What we have actually found above is the time required for one particle at a distance r_0 from the centre of a condensation to be left behind by a further distance r_0 . What we require is the time taken for a significant amount of material to be displaced. From the formula for the dispersal time, (17)

$$\tau = \frac{2G^{-\frac{1}{2}} r_0^3 R^{3/2} M_\odot^{\frac{1}{2}}}{3r_0^3 M_\odot - R^3 M}$$

$$= \frac{2G^{-\frac{1}{2}} R^{3/2} M_\odot^{\frac{1}{2}}}{3M_\odot - \frac{4}{3}\pi R^3 \rho}$$

on using $M = \frac{4}{3}\pi \rho r_0^3$.

We see that the dispersal time is independent of the value of r_0 taken and hence, once the outer particle has been removed a small amount, the next one in will also be removed. Hence in the given time we actually have a stream of particles being removed and thus it can be taken as an estimate of the time required for some of the material to be removed.

Thus if a condensation was considered to move in a circular orbit, violating the given density-distance condition, a significant amount of material would be removed in the times indicated in the tables. These are all less than one year for a density in the region of 10^{-8} gm/cc and so for a circular orbit dispersal takes place in a relatively short time.

Clearly the problem of a condensation moving in a circular orbit has no real physical significance as, if any tendency towards dispersal existed then the condensation could not be formed in such a position in the first place. What we have to consider is a condensation orbiting about the sun on some orbit, with this orbit dipping inside the critical distance from time to time. Now the dispersal is rather more complex. The tendency to disperse must still be present when the condensation is inside the critical distance, but now when the condensation returns outside, any dispersal effects may be repaired before the condensation dips inside the critical distance again.

We are unable to give a rigorous proof that this is not so, but will endeavour to show this qualitatively. Let us, again for the sake of choosing some value, consider a condensation moving in such a way that when it is inside the critical distance its mean separation from the sun is roughly similar to the earth's orbital distance. Then from the tables we see that if its density is roughly similar to 10^{-8} gm/cc dispersal of amount equivalent to the condensation radius will be set up in about 1/10 year. Hence the furthest particle is now farther away from the centre by a factor $\sqrt{2}$ with a corresponding weakening of the gravitational field of the condensation. Hence repairing this damage could only take place where the sun's tidal action was weakened by more than the corresponding amount, or if the orbit went outside the critical distance by a distance $\sqrt{2}$ times this distance. Hence part of this orbit must lie outside Jupiter's orbit before any repair can take place. Such an orbit is likely to spend a time greater than 1/10 year inside the critical distance, say about ½ year (Jupiter's orbital period is 10 years). But in this time five times the damage we have allowed for would be caused and hence to repair it, by the same argument as above, we need an orbit with a part outside $\sqrt{26}$ times the critical distance, which now takes us outside Uranus' orbit, which has an orbital period of 84 years.

Hence, though complete dispersal might not take place after the condensation has dipped inside the critical distance once, we appear to have reasons for concluding that any dispersal so caused will not be repaired while the condensation is outside the critical distance, and so more dispersal takes place when the condensation is next inside the critical distance, with complete dispersion being the final result.

From a common sense point of view, total dispersion might be said to have taken place when the outermost particle has been displaced by about 10 condensation radii. Our equations for the dispersal clearly do not hold for such conditions as the shape differs significantly from a sphere, but clearly the time will be less than 10 times the time required for one radii dispersal as the gravitational field of the condensation gets weaker and weaker. From what we have already said, dispersal would thus come about in about 10 orbits and hence the total time required would be ten times the orbital period which will not exceed about 10^2 years.

It thus seems reasonable to conclude that the outer layers of hydrogen surrounding the heavy core we have formed at the centre of the condensation will be swept away in a time not exceeding 10^2 years, the actual time being more like one year. Whatever the actual value is, we can be certain that it is a short time astronomically speaking. The heavy core is not dispersed because of its higher density.

As the reader will no doubt be familiar, Jeans has spent considerable time investigating the effects of the sun's tidal action on various bodies (e.g. Jeans [20] and [23], 'Astronomy & Cosmogony' and 'The motions of tidally distorted masses'). His investigations however only appear to cover the equilibrium and stability, or otherwise, of bodies acted upon by a tidal force. He does not give any estimates of the time taken by the tidal force to cause appreciable effects. He only concludes, as we have done, that if the density-distance conditions are violated, instability leading to dispersal sets in. Jeans' investigations into these matters are much more detailed and rigorous than ours, but there does not appear to be any way of applying his results so as to give an estimate of the dispersal time. It might be added that this dispersal time was of no real interest to Jeans, all he required was that the tidal action of a passing star could cause the ejection of a small amount of matter from the sun, this material then being used as the origin of the planets. The problem investigated by Jeans and the one investigated by us are related closely enough to each other for Jeans' rigorous results to add further weight to the conclusions we have drawn from less rigorous methods, that dispersal does actually take place in a short time.

In 'Astronomy and Cosmogony' [20] Jeans shows that the effect of a resisting medium on a planet's orbit is to reduce the eccentricity and bring the orbit closer to the sun. Hence, if some resisting material existed, we have no difficulty at all in reducing the initially eccentric orbit of the condensation, and possibly of the heavy core, to the nearly circular. orbits we have today for the terrestrial planets, and indeed for the major planets. Indeed it seems very likely that the material actually removed from the condensation could act as this resisting medium.

There is also the possibility that the actual act of dispersing a condensation might have the effect of reducing the eccentricity of the remainder. This possibility is one that needs investigating in the future, but whether this action does reduce eccentricity or not does not alter or affect our conclusions about the dispersal.

These are that a condensation dipping inside the critical distance as defined by the density-distance relation will be dispersed in a short time, astronomically speaking. If the orbit is circular, with the condensation continually inside the critical distance, the same dispersal takes place in a time slightly shorter than the orbital period, while total dispersal occurs in a time one order of magnitude greater. Thus no grave difficulty exists in connection with the removal of the excess outer layers of hydrogen which surround the heavy core that is going to form a terrestrial planet.

CHAPTER 6

Conclusions about the origin of the terrestrial planets

Professor McCrea [3] has given reasons for concluding that interstellar material in which stars are about to be formed consists of fragments, or floccules, moving at random amongst themselves and probably composed mainly of molecular hydrogen, the temperature being about $50^{\circ}K$ and the random speed of the floccules about 1 Km/sec. Many stars would then be formed simultaneously, each by aggregation of portions of the material that happen to be moving towards each other. The process producing condensations that grow into stars would also produce minor condensations in material that became trapped in the growing gravitational field of these stars. These minor condensations could obviously form a planetary system. Each of these minor condensations would initially be basically similar to any other minor condensation, consisting mainly of molecular hydrogen and weighing about 10^{30} gms. We have demonstrated in the preceding work some methods by which the heavy material in these condensations, about 1% of the total material by mass, could fall to the centre of a condensation in the very short time, astronomically speaking, of about one year. All of these methods depend on the formation of very large grains, or clumps as we have called them, with a radius of roughly 100 cm . In two of these methods the growth of the clumps is caused by accretion during the actual falling process, while the third method requires the clumps to be formed before the falling to the centre of the condensation begins. We have shown that such clumps could be formed during the period when the initial floccules are colliding together, forming minor condensations and settling into the invariable plane defined by the angular momentum of the system. In these methods we have assumed that the heavy material can adhere to the grains upon collision with them. This is the only basic postulate in the whole theory and no real investigation has been carried out into a mechanism by which this could come about, though reasons have been given to show that the postulate is not impossible to satisfy.

We have thus concluded that formation of a heavy core (that is a core consisting of the heavier elements) at the centre of a condensation can take place in about one year simply by allowing the clumps to fall under the gravitational attraction of the condensation. The speed with which each clump arrives at the condensation centre has been estimated and found to be of the order of a kilometre a second. It is thus not so high as to cause any anxiety about the effect of the energy released in the resulting collision at the centre of the condensation. It has also been shown that if the condensations are involved in collisions before the core is formed, the clumps will not be lost from the condensations, but will be carried along with them. Hence we conclude that the formation of the heavy core can take place without any difficulty.

If any such condensation, with a given density, approaches within a certain critical distance from the sun then it cannot hold together for long, due to the effect of the tidal action of the sun. We have estimated the time that must ellapse before a noticeable amount of dispersal occurs, and found this to be less than one year. Further, if the condensation approaches too close to the sun and then receeds again, we have indicated that the damage caused by the approach will not be repaired and thus, under such conditions, dispersal occurs in an astronomically short time. Due to the simplifications we have used in investigating this dispersal time, we cannot be too certain of the exact numerical values obtained. We would appear to be safe however in concluding that dispersal can occur in an astronomically short time and is likely to be of the same order as the time required for the formation of the heavy core. We note that this tendency for dispersing will not be present in the heavy core as it has a much higher density. Hence the net result of this tidal action is to remove the outer layers of hydrogen and leave us with a smaller body composed of the heavy elements. As the two times (core formation and dispersal) are roughly of the same order, it is possible for the core to be formed even if dispersal begins before the formation is completed.

This remaining core of heavy material is exactly what is required for the formation of a terrestrial planet. It will have a mass of approximately 1% of the initial mass and hence a mass comparable with the terrestrial planets. It also consists mainly of the heavier elements, again what is required. Hence this is in effect a diffuse planet. This material must now condense under its own gravitational attraction and a terrestrial planet, as we know it, has been formed. It is to be noted that we do not state that the heavy core is the terrestrial planet, this core is only the planetary material, which must condense further before a planet is formed.

We have also suggested that other objects in the solar system could be formed as a bi-product of some of the above events. This requires that one of the clump-forming mechanisms forms clumps with a radius in excess of 10^4 cm. These clumps would now escape from the surrounding condensations, but not from the solar system. These clumps would thus have rather eccentric orbits and could thus form meteorites and comets.

As the reader will have realized by now, this theory is in many ways similar to the group of theories that are commonly known as 'The Planetesimal Theories'; 'planetesimal' because the planets are assumed to be formed from an agglomeration of small bodies, planetesimals. Such theories were first proposed by Professors Chamberlin [24] and Moulton [25] around 1900. Theirs was a simple theory, where material was ejected from the sun and condensed into planetesimals, and so by agglomeration to planets. This was of course during the period when the sun was thought to be composed of heavy material and so no separation from the hydrogen was necessary. Since then many variations of this general theory have been proposed; the material could be ejected from a companion star to the sun as this exploded as a super novae, or it could be material left over from the formation of the sun. The more recent theories, proposed by Hoyle [1], [26] also involves ejection from the sun, the material now forming a disc about the sun, ejection being caused by too rapid rotation of the sun. This disc is assumed to be magnetically coupled to the sun, allowing for a transfer of momentum between the two bodies. As the sun contracts angular momentum is transferred to the disc causing it to move outwards. Hoyle has shown that planetesimals, having condensed in the cooling disc, with a radius greater than 100 cm will be left behind by this moving disc. These planetesimals will, of course, consist of the non-volatile elements, which are essentially the same as our heavy elements, and so separation of these from the hydrogen has been accomplished. This theory now required some means by which the planetesimals can collect together and form a planet.

The clumps formed in our theory are very similar in many ways to Hoyle's planetesimals. The planetesimals consist of the non-volatile material while the clumps are made of heavy material. Our knowledge of the early composition of the clouds and of planetary composition is too restricted for us to be able to draw any conclusions about the validity of either theory from a comparison of these two facts. Both theories give better agreement than most previous theories.

There exists however one fundamental difference between our theory and the planetesimal theories. After separation of the hydrogen from the planetesimals has taken place the planetesimals are scattered in a disc about the vicinity of the sun, and before any planet can be formed these must all be brought together and fused into one body. In our theory on the other hand, once separation takes place our clumps are virtually all in the same spot, forming the heavy core. It must be a much simpler task to form a planet out of the clumps under such conditions.

Hence, though in many ways our theory is similar to the planetesimal theories, it presents a great advancement on most of them in that no juggling is required to bring the clumps together to form a planet.

Thus, in brief conclusion, the picture we present for the formation of the terrestrial planets is as follows. Floccules are captured by the young sun. These collide and form condensations with a mass of about 10^{30} gms, which is roughly equivalent to a major planet mass. By one of the methods described a core consisting of the heavy material is formed at the centre of each condensation in a comparatively short time. This condensation, by now orbiting in some fashion in the invariable plane defined by the angular momentum, approaches within the critical distance defined for the particular condensation density and the outer layer of hydrogen will probably be swept away by the tidal action of the sun in a short time. We are thus left with a smaller object composed mainly of the heavy elements which can now condense further to form a terrestrial planet. If the initial condensation does not approach too close to the sun then the sweeping away of the hydrogen does not take place and condensation of the whole body occurs, resulting in a major planet.

One of the major attractions of this theory is that initial conditions can be the same for all the planets. They all originate from similar condensations consisting of the same materials. The difference in the final planets comes about only because some approach closer to the sun than others. The one disadvantage is that we have to postulate the mechanism for the growth of the clumps rather than being able to offer a definite proof that such a process of grain adhering to grain can take place.

We can however conclude that a satisfactory theory for the formation of the terrestrial planets has been outlined.

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P A R T II

CHAPTER 1

Introduction

Throughout this part we shall be interested in stellar clusters, or to be more precise in the evolution and appearance of clusters of stars. We do not wish to enter into any discussion as to what constitutes a stellar cluster, we just take the words to have the usual meaning attached to them in astronomy and astrophysics, namely a collection of stars occupying essentially the same region in space and moving together through the remaining star field as one complete unit.

As is well known, a diagram, named after them, was constructed by Hertzsprung and Russell. This plotted some measure of the luminosity of given stars against their colour. Later it was realized that the colour of a star was an indication of its effective, or black body, temperature. As, for theoretical purposes, the temperature is much more useful a tool than the colour, the diagram now became a plot of some measure of the luminosity against some measure of the temperature. Unfortunately, due to the way Hertzsprung and Russell had chosen their colour axis with the blue on the left and red towards the right it turns out that the temperature increases from the right to the left instead of the more usual arrangement. This axis has been kept on as the diagram had become well known before the temperature connection became known.

When large-scale observations of stars started, it was found more convenient to measure the star's magnitude rather than its luminosity. This magnitude is simply some constant minus the logarithm of its luminosity,

$$M = \text{Constant} - A \log L$$

Clearly we could also write

$$M = \text{Constant(other)} - A \log L/L_\odot$$

where $L_\odot$ denotes the sun's luminosity.

It was also found that the apparent magnitude of a star's blue light minus the apparent magnitude in the violet region gave a good indication of the temperature, and so nowadays the observational data are given in terms of the magnitude and $B-V$ where $B-V$ is the quantity just defined.

We will describe the conversion from one set of axis to the other [M_v, $B-V$ to $\log L$, $\log T$] with some detail in a later chapter.

If a star is in equilibrium then its luminosity and temperature are constant and so it occupies one well defined point in the Hertzsprung-Russell diagram, independent of the epoch at which it is observed. If all stars of a cluster were similar, apart from their mass, and each one in equilibrium, then this distribution of stars would have a well defined pattern, independent of epoch. Furthermore, any collection of stars with a similar mass range should give a similar pattern. Such a unique pattern has been found in the Hertzsprung-Russell diagram of most clusters, and has been called the main sequence.

Common-sense tells us that stars cannot be in equilibrium for all time, as some evolution must take place to provide the energy required to give the observed luminosity. Hence some stars will not confer orm to the above pattern. The theory of stellar interior and stellar evolution informs us that this stage is reached when a given proportion of hydrogen is converted into helium. When this point is passed rapid expansion takes place, accompanied by cooling down of the star. In the Hertzsprung-Russell diagram this corresponds to a rapid motion to the right. As stars that would occupy this region are much larger in radii than the main sequence stars, this branch is sometimes called the giant branch.

The evolution of one star can then be described as follows. For most of its life it will be a star on the main sequence, occupying one defined point in this sequence. It then moves off to the right after a given proportion of its hydrogen has been consumed. Hence, the brighter the star, the quicker it moves off.

If we have a random distribution of stars, both in mass and age, then we could possibly still detect the main sequence, but stars would be evolving away at all points as both young bright stars and old not so bright ones would move off. This is not what is found when we observe stellar clusters which suggests that the stars in a cluster are not completely at random in both mass and age.

If we have a random distribution in mass, but not in age, then stars would leave the main sequence in an orderly fashion, the bright ones leaving first. The pattern would thus be a main sequence with stars that are just burning the required proportion on the point of moving off, brighter stars being to the right of this point. This is what is observed in clusters which suggests strongly that stars in a stellar cluster are all of the same age. The point at which the stars turn away from the main sequence onto the giant branch has been called the main sequence turn-off point, or sometimes just the turn-off point.

If all the stars of a stellar cluster were indeed formed at the same epoch, then very useful information concerning stellar ages, luminosity and distances can be obtained just by comparing the Hertzsprung-Russell diagrams of various clusters. It is thus of paramount importance that we make sure that the assumption about stellar ages is correct.

In this part we shall investigate two points that at first sight appear to contradict the hypothesis that all stars in a cluster were formed at one and the same epoch.

The first of these points, which we deal with in chapter 2, is concerned with very young stars. Ifwe have a very young star contracting from some initial state then it will not have had enough time to reach a state of equilibrium on the main sequence. Calculations about the position of such a star at any given time seem to be in conflict with the observed position. In this chapter 2 we give calculations showing the great importance of what initial conditions we assume.

In chapter 3 we discuss another point that does not appear to agree with the assumption of a unique age for a stellar cluster. In a few clusters, stars are found lying in a region beyond the turn-off point from the main sequence, forming a rough continuation to the main sequence. Clearly if everything is strictly as we have assumed it, these stars should not be present. We have constructed a diagram in which stars are assumed to have non-unique ages to see if such conditions give a better agreement with observations or not.

By means of these two investigations we hope either to disprove the assumption that all stars in a cluster are of the same age, or give additional evidence for assuming it to be true.

At the risk of some repetition we have attempted to make each chapter intelligible in its own right so we might have to describe some methods twice.

CHAPTER 2

On the contraction of pre-main sequence stars

Some astronomers, especially Walker [1,2,3] and Whiteoak [4] have observed what, to judge by their turn-off point from the main sequence, must be very young clusters. These, being observational results, are given in terms of the apparent visual magnitude V and $B-V$, which is an indication of the colour of the star. From the point of view of any theoretical work that is to be done, these units are not very useful. The usual units for theoretical work are the logarithm of the luminosity and the logarithm of the effective or black body temperature, or $\log L$ against $\log T$ with obvious meanings for L and T. Thus before we proceed with any theoretical discussion it would be helpful if we could convert the observational data to the more useful $\log L$, $\log T$ system.

Before any progress can be made, the apparent visual magnitude, V, must be given in terms of the absolute magnitude M_V. These are connected by the following relation

$$M_V = V - D.M \tag{1}$$

$D.M$ denotes the distance modulus of the cluster. The value of this will be given when the results of work on any cluster is published. Now before we can convert to luminosities account of emission in all frequency ranges must be taken, so we require the bolometric magnitude rather than the visual magnitude M_V. Now

$$M_{bol} = M_V + B.C \tag{2}$$

where M_{bol} is the bolometric magnitude and $B.C$ stands for the bolometric correction. This bolometric correction depends on the colour of the star and tables are published connecting the two quantities.

Finally

$$M_{bol} = 4.62 - 2.5 \log L/L_{\odot} \tag{3}$$

where L is the luminosity of the star and $L_{\odot}$ that of the sun. We shall use ℓ to denote the quantity $L/L_{\odot}$, or the luminosity in solar units. By means of equations (1), (2) and (3) above we are thus able to convert from the apparent visual magnitude to the luminosity of a star.

In his book "Structure and evolution of the stars" [5] Schwarzschild gives a table of the relation between the colour, $B-V$, and the bolometric correction $B.C$ and also the variation in temperature with colour.

Hence we can convert from the observational V against $B-V$ diagram to the theoretical $\log L$, $\log T$ Hertzsprung-Russell diagram.

By using this method we have converted the outline of the region occupied by stars from the observed data to a theoretical diagram in $\log L$ against $\log T$, for two clusters NGC 2264 and NGC 6530. The results are given as figures 1 and 2. These are two of the clusters, mentioned at the beginning of the chapter, that have been observed by Walker. The third, IC 5146 is very similar to the others but contains very few stars. For this reason no conversion was carried out. The cluster IC 2602 observed by Whiteoak is also rather sparsely populated and so was not converted.

The unconverted diagrams in V against $B-V$, which are very similar apart from size, to the diagram, can of course be seen for all four of these clusters in references [1], [2], [3] and [4], the original publications on the subject.

These diagrams show that below a certain luminosity L_c all the stars lie to the right of the main sequence. As they lie to the right, their luminosity must be greater than that of a main sequence star of equivalent temperature and as these stars

Fig 1. Observational results for NGC 2264

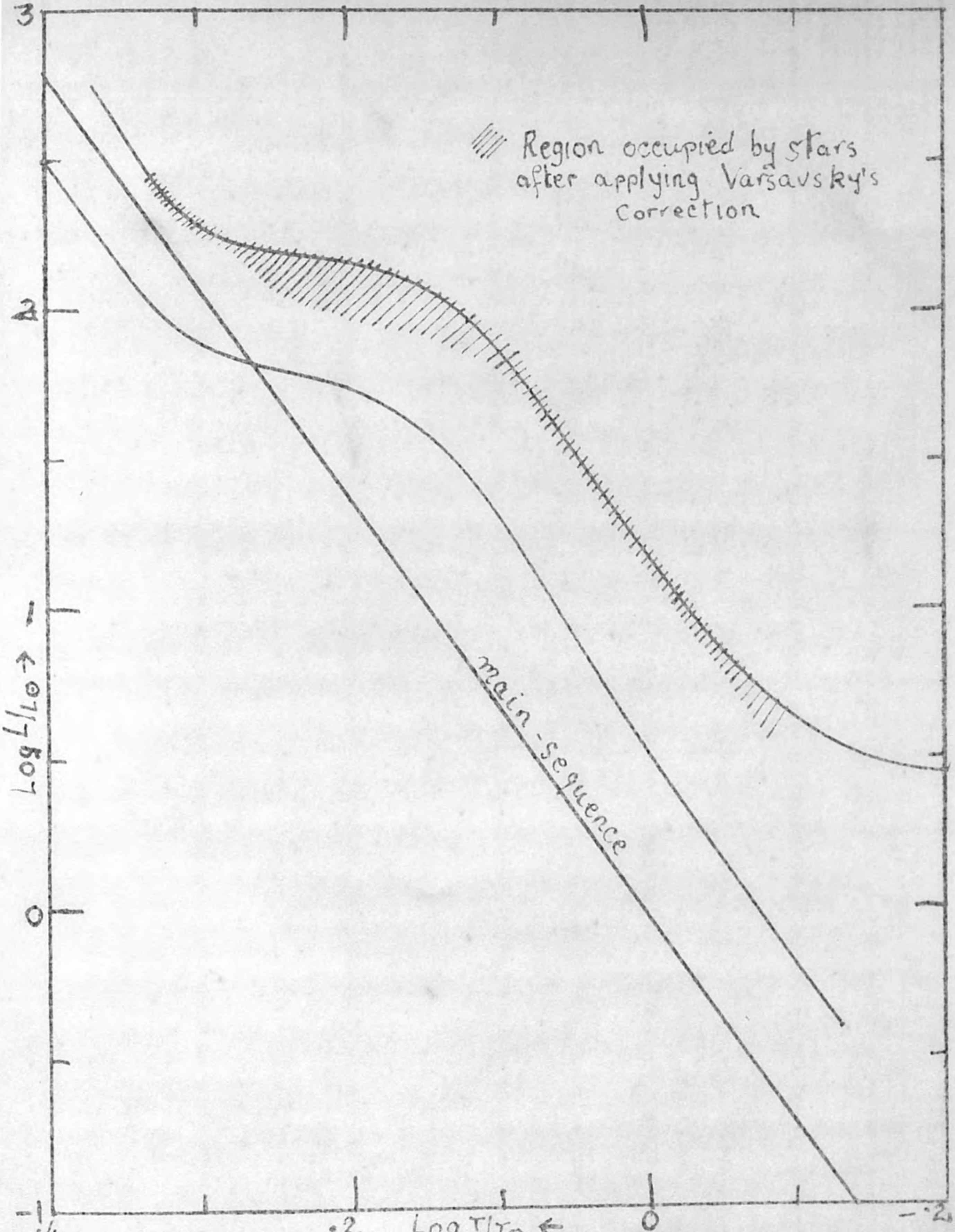
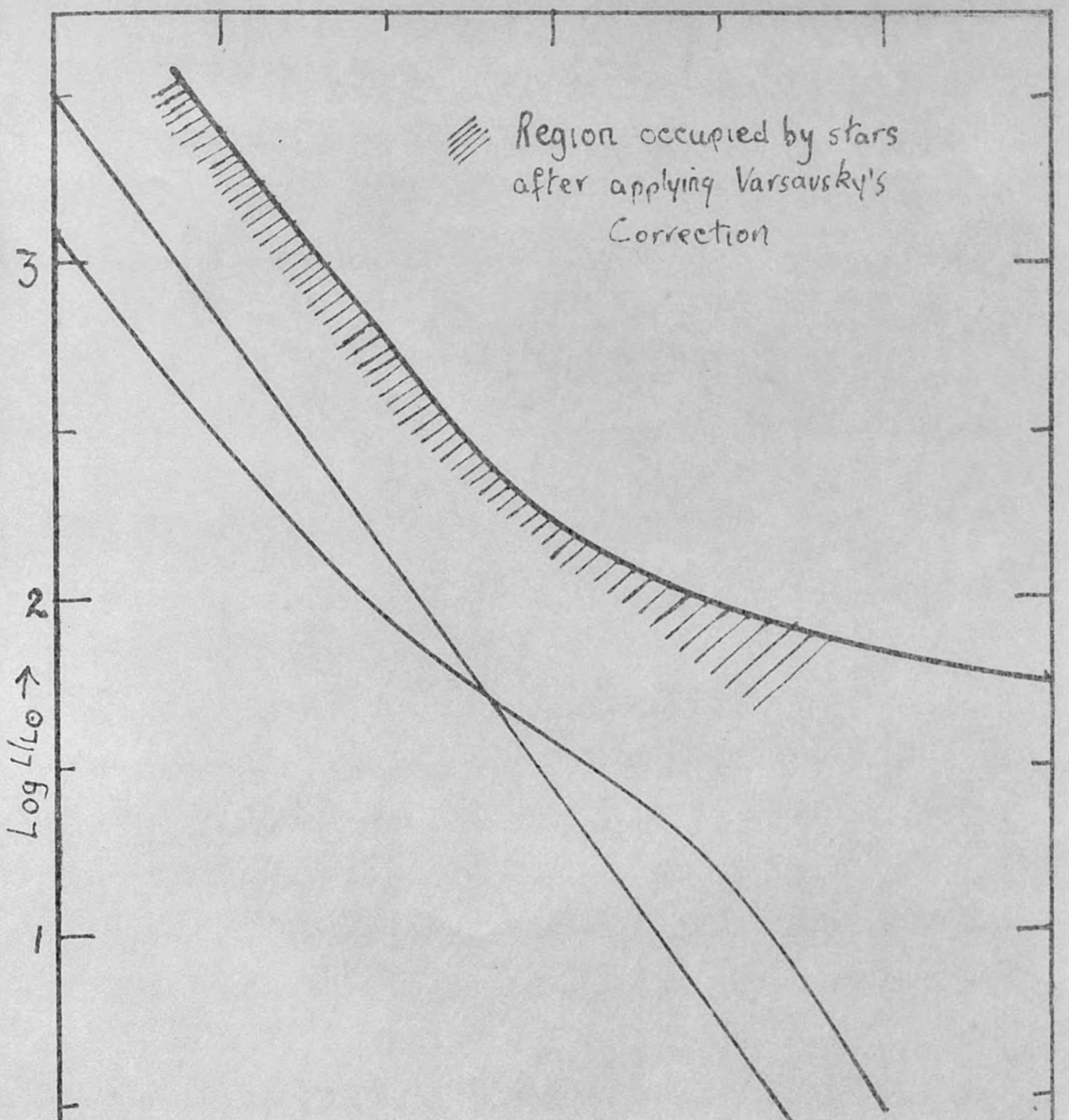


Fig 2 Observational results for NGC 6530



behave roughly like black bodies,

$$L \sim R^2 T^4$$

R being the radius of the star. Thus the radii of these stars under consideration must be greater than a main sequence star. This, together with the information from the turn-off point from the main sequence that the cluster is very young, leads to the natural interpretation that these stars are still in the process of gravitational contraction, not having as yet had enough time to reach the main sequence. Indeed estimates of the time taken for this gravitational contraction from vanishingly small densities are in fair agreement with the age estimates derived by other methods.

Had all members of a stellar cluster started at one and the same epoch to condense from vanishingly small density, then those now having a luminosity fainter than L_c would be expected to lie upon a certain calculable locus in the Hertzsprung-Russell diagram. However, in cases where age determination can be carried out fairly accurately, it is found that the stars concerned lie, not on this locus, but much nearer to the main sequence. That is to say, the gravitational contraction of the fainter stars in the cluster appears to have progressed much further than is possible within the lifetime of the cluster.

All who have written on the subject regard this as presenting a very serious problem and some very drastic solutions have been proposed. We wish to point out however that it is not possible to say whether there is a real difficulty without knowing more than we do about the early stages of the process of formation of the stars concerned. We have to contrast the present problem of stars condensing onto the main sequence at its lower end with that of stars evolving off the main sequence at its upper end. In the latter the problem is well defined. In the case of the condensing stars on the other hand we do not as yet know what has to be treated as initial conditions.

We hope to illustrate this conclusion by the example we are about to give. This is intended to support the conclusions to be stated; it is not intended, or even suggested, that it arises from any particular theory of star formation. In working this example we follow Su-Shu Huang in his paper "Distribution of pre-main-sequence stars in the Hertzsprung-Russell diagram" [6] and merely for the sake of argument we shall exhibit the consequences of supposing that at some epoch all the stars concerned have the same mean density.

Assume that a star of mass M undergoing homologous gravitational contraction towards the main sequence has luminosity L when its radius is R, where

$$L = A M^{\alpha} R^{-\beta} \tag{4}$$

A, α and β are all constants.

For this type of star, undergoing contraction, the only source of energy present is assumed to be the gravitational energy of the star itself, and the change in this as the star contracts by an amount dR is clearly proportional to

$$\frac{G M^2 dR}{R^2} \tag{5}$$

If this collapse takes place in a time dt, then

$$L\,dt = -C G \frac{M^2}{R^2} dR \tag{6}$$

where C is a constant taking account of the amount of energy actually being released as luminosity rather than just released. G is obviously the gravitational constant. Combining equations (4) and (6) gives us

$$A\,dt = -C G \frac{M^{2-\alpha}}{R^{2-\beta}} dR \tag{7}$$

Now, if no gain or loss of mass takes place during this contraction stage in any star, M of equation (7) can be considered a constant and so the equation can be integrated to give

$$\left[t\right]_{t_1}^{t_2} = \left[-\frac{C}{A}\cdot\frac{GM^{2-\alpha}}{(1-\beta)R^{1-\beta}}\right]_{r_1}^{r_2}$$

Now the time taken by this star to contract from vanishingly small density (infinitely large radius) is clearly

$$t = \frac{C}{A}\cdot\frac{GM^{2-\alpha}}{(1-\beta)R^{1-\beta}} \tag{8}$$

In this form the expression is not very useful as it consists of both the constants C and A amongst other quantities. A great simplification can be brought about if we use solar units as we will show.

The Helmholtz-Kelvin time scale for the sun (hereafter denoted by HK units) is defined by Chandrasekhar in 'An introduction to the study of stellar structure' [7] to be the length of time for which the suncould go on emitting radiation at its present rate, if the only sources of energy it had available was the gravitational potential energy released as the sun contracted from infinity. This is clearly

$$1\,HK = C \cdot \frac{G M_\odot^2}{R_\odot L_\odot} \qquad (9)$$

where again C gives the proportion actually released as radiation.

Chandrasekhar has a value $\frac{3}{5-n} \cdot \frac{3\gamma-4}{3(\gamma-1)}$ for the constant C where n is the power at which the density decreases with radius inside the star and γ is the usual ratio of specific heats.

With $n = 3$ and $\gamma = 5/3$ and the usual numerical values for the other quantities $G, M_\odot, R_\odot$ and $L_\odot$, Chandrasekhar obtains a value of 2.4×10^7 years for one Helmholtz-Kelvin unit.

We have assumed homologous contraction, and so equation (4) applies to the sun as well as any other star, so

$$L_\odot = A \frac{M_\odot^\alpha}{R_\odot^\beta}$$

(As usual we use $\odot$ to denote any quantity belonging to the sun.)

Substituting this into equation (9) gives

$$1\,HK = \frac{C}{A} \cdot G \frac{M_\odot^{2-\alpha}}{R_\odot^{1-\beta}} \qquad (10)$$

Now, if we express the time in expression (8) in units of the Helmholtz-Kelvin scale, using τ to denote this

$$\tau \frac{CG}{A} \cdot \frac{M_\odot^{2-\alpha}}{R_\odot^{1-\beta}} = \frac{C}{A} \frac{GM^{2-\alpha}}{(1-\beta)R^{1-\beta}}$$

If we now use solar units to measure the mass and radius of a star, using script notation for this units $\mathcal{M} = M/M_\odot$, $\mathcal{R} = R/R_\odot$ etc. then

$$\tau = \frac{\mathcal{M}^{2-\alpha}}{(1-\beta)\mathcal{R}^{1-\beta}} \tag{11}$$

Hence if the units are taken to be solar units, the time of contraction, given in the Helmholtz-Kelvin scale is given by the simple expression (11).

In these units equation (4) reduces to

$$\mathcal{L} = \mathcal{M}^{\alpha}/\mathcal{R}^{\beta}$$

If we define the density and temperature to be ρ and T in solar units then we have the following relations

$$\mathcal{L} = \mathcal{R}^2 T^4$$

and (12)

$$\mathcal{M} = \rho\,\mathcal{R}^3$$

The first being the equation of black body radiation while the second is the fundamental definition of mean density. By making full use of equations (4), (11) and (12) substituting where necessary we can quite simply obtain the following relations.

$$\log L = (\alpha - \beta/3)\log M + \frac{\beta}{3}\log \rho \qquad (13)$$

$$\log T = \frac{1}{4}\left\{\alpha - \frac{2+\beta}{3}\right\}\log M + \frac{2+\beta}{12}\log \rho \qquad (14)$$

$$\log \tau = \left\{2-\alpha - \frac{1-\beta}{3}\right\}\log M - \log(1-\beta) + \frac{(1-\beta)}{3}\log \rho \qquad (15)$$

$$\log L = \frac{\beta}{(1-\beta)}\log \tau + \frac{\beta}{(1-\beta)}\log(1-\beta) - \left\{\frac{(2-\alpha)\beta}{1-\beta} - \alpha\right\}\log M \qquad (16)$$

$$\log T = \frac{(2+\beta)}{4(1-\beta)}\log(1-\beta) + \frac{(2+\beta)}{4(1-\beta)}\log \tau - \frac{1}{4}\left\{\frac{(2-\alpha)(2+\beta)}{1-\beta} - \alpha\right\}\log M \qquad (17)$$

$$\left(\frac{3\alpha}{2} - 2 - \beta\right)\log L = 2(\alpha - 2\beta)\log T - \alpha\log \tau - \alpha\log(1-\beta) \qquad (18)$$

Su-Shu Huang in his paper has written down only this last equation (18) which clearly is sufficient for one to be able to construct the Hertzsprung-Russell diagram for a distribution of stars with any given age, τ . The actual method used by us in deriving these equations is also different from Su-Shu Huang's, though the principle and the treatment are the same, since we have started working in conventional units and introduced solar units at a later stage only because they obviously bring about a great simplification, while Su-Shu Huang uses solar units throughout his derivation.

By making a study of the transfer of energy through a star, taking account of the equilibrium (relative, as changes take place slowly) of the star, Henyey Lelevier and Levee [8] have computed a series of evolutionary tracks for stars in the gravitationally contracting stage. From their results, it appears that for each star the relation between luminosity and effective temperature may be represented as

$$\mathcal{L} \propto T^{1.13} \tag{19}$$

But we already have

$$\mathcal{L} = \mathcal{R}^2 T^4 \qquad \text{and} \qquad \mathcal{L} = \mathcal{M}^{\alpha}/\mathcal{R}^{\beta}$$

which leads to

$$\mathcal{L} \propto T^{\frac{4\beta}{2+\beta}}$$

and so by using equation (19) we have that

$$\beta = 0.79 \tag{20}$$

If the opacity of the outer areas of the collapsing star is assumed to be given by modified Kramer's law as done by Schwarzschild in his book [5] then the value of α corresponding to this value of β

given in (20) is 5.8 .

If instead of relying on this law of opacity we use the results of direct calculation by Henyey, Lelevier and Levee [8] the value of α turns out to be 5.4 , provided that the relation between the radii and masses of main sequence stars given by Russell and Moore [9] is assumed.

On the other hand if we calibrate α by the aid of the empirical mass-luminosity relation and the temperature-luminosity relation of the main sequence stars, we obtain

$$\alpha = 4.7$$

From now on we shall use the same values as Su-Shu Huang adopted in his paper, a value which is about the mean of the three given above, so we take

$$\alpha = 5.4 \qquad \text{and} \qquad \beta = 0.79 \tag{21}$$

Actually it is very simple to show that the above equations are very insensitive to the value of α .

With the values of α and β given by expression (21), equations (13) to (18) can be expressed in numerical form as the following

$$\log L = 5.14 \log M + 0.26 \log \rho \tag{22}$$

$$\log T = 1.12 \log M + 0.23 \log \rho \tag{23}$$

$$\log \tau = -3.47 \log M + 0.07 \log \rho + 0.68 \tag{24}$$

$$\log L = 3.76 \log \tau + 18.2 \log M - 2.55 \tag{25}$$

$$\log T = 3.32 \log \tau + 12.6 \log M - 2.25 \tag{26}$$

$$\log L = 1.44 \log T - 1.01 \log \tau + 0.69 \tag{27}$$

In equations (13) to (18) the logarithm could be taken to any base we preferred, but in these equations it has been fixed as base 10.

In the resulting $\log L - \log T$ diagram, a line of fixed density, a line of fixed age, τ and a line of constant mass M, which is the evolutionary track of the star with this mass, are all straight lines. Such lines have been plotted by Su-Shu Huang in his paper and some are shown in figure 3.

For the purpose of this diagram we have taken the relevant part of the main sequence to be a straight line given by

$$L = T^{\gamma} \tag{28}$$

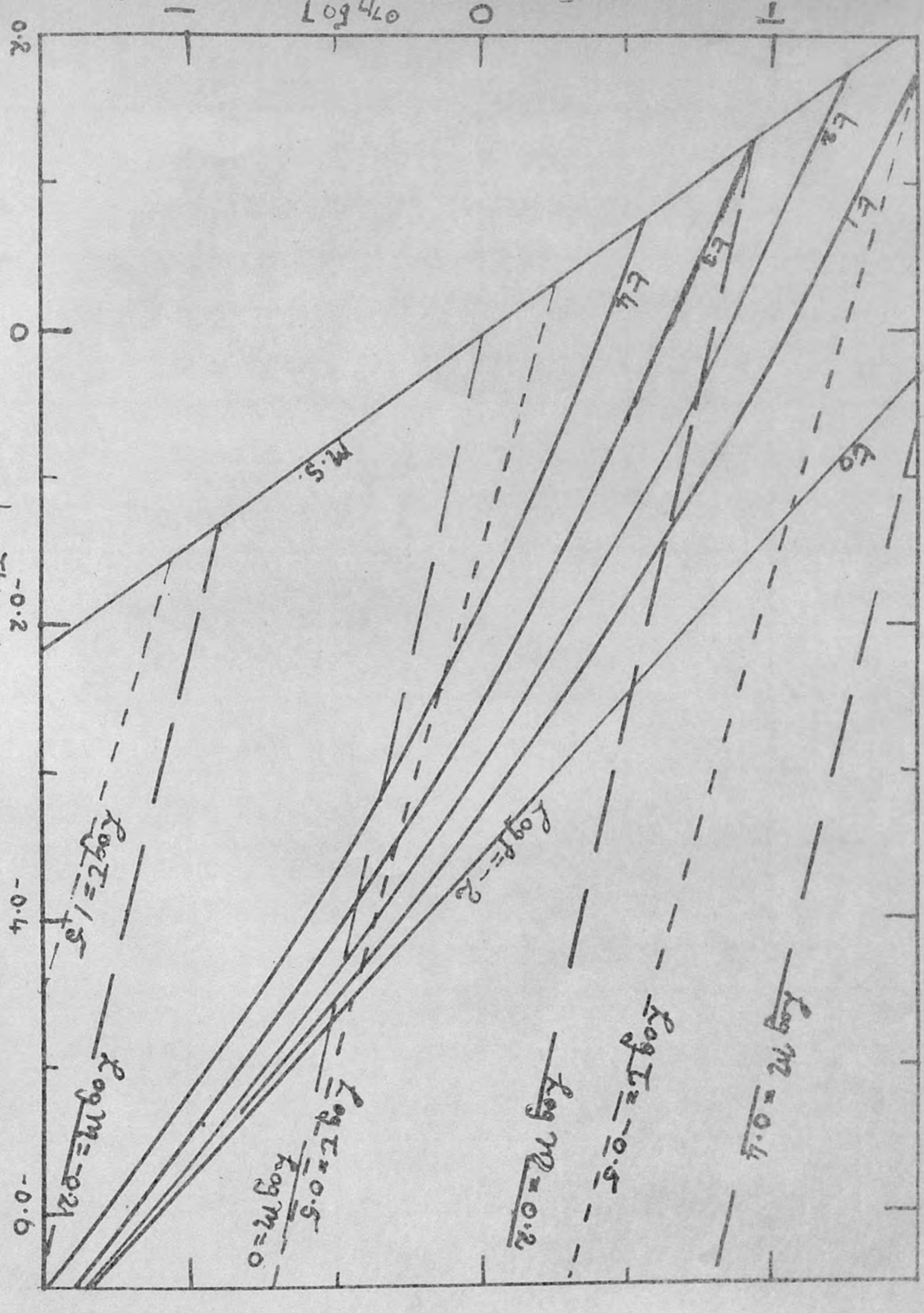
with $\gamma = 7$ as given by Su-Shu Huang. The position of this line in the diagram is fixed by assuming that the sun is a main sequence star, thus lying on this line, and so if the units are solar the main sequence passes through the origin in the $\log L - \log T$ diagram.

For the purpose of our illustration we consider a set of pre-main-sequence stars of the type under consideration that at some epoch $t = t_0$ have the property that all the stars have the same mean density $\rho = 0.01$, or one percent of the solar mean density. This value of ρ has been chosen simply because we had to choose some value before we could proceed with the illustration and 0.01 is a fairly convenient value, numerically speaking, while at the same time not being an impossible value for a star to possess. There are, as far as we are aware, no physical peculiarities attached to this value of ρ which gives it preference over any other value.

In figure 3 we exhibit the loci on which these stars would lie at epoch t_1, t_2, t_3 and t_4, where we have taken

$$t_1 - t_0 = 0.05, \quad t_2 - t_0 = 0.1, \quad t_3 - t_0 = 0.2, \quad t_4 - t_0 = 0.4$$

Fig 3 Theoretical Evolutionary tracks



The units in all cases being the Helmholtz-Kelvin contraction time for the sun, and the value of such an unit we have already stated is about 3×10^7 years.

However, for our discussion it is more instructive to note that the locus for t_2 meets the main sequence at a point which has the rough properties,
$\log \mathcal{M} = 0.28$, $\log L = 1.3$, $\log T = 0.18$, $\tau = 0.48$
and $\rho = 0.28$; while the corresponding point on the locus for t_0 (that is point of intersection with the main sequence) with the same value of $\log \mathcal{M}$ has
$\tau = 0.38$.

The homologous family is meant to consist of stars in which the luminosity is produced by the release of gravitational potential energy due to contraction alone. Were account taken of the release of nuclear energy in the final stages of contraction we should have the evolutionary tracks joining the main sequence in a somewhat different way, similar to the final stages computed by Henyey, Lelevier and Levee [8], but this is of no importance for the present discussion.

It is to be noted that our locus for, say, $t = t_2$ is roughly like those plotted from observations of young clusters, for example NGC 2264 and NGC 6530 which are our figures 1 and 2, taken from observational results by Walker [1,2]. The value $\tau \doteq 0.5$ just mentioned above is of the order of the age usually ascribed to such young clusters. Thus we, in our example, are actually dealing with the sort of situation that is of interest in practice, even though we are not advocating any particular theory of evolution.

The loci τ = constant have, of course, the property that if a set of stars of the present type are represented at some epoch by points on one of these loci then at any later epoch they must be represented by points on another of the loci, unless they move on to the main sequence in which case their locus clearly becomes the main sequence. Loci of this type are the only ones normally considered hitherto.

The loci t = constant also however have by construction the property that if the stars lie on one of them at one epoch then they lie on another of them at any other epoch.

There are three features about these loci which are immediately noticeable from a study of figure 3, namely,

(a) The loci of t_1, t_2, t_3 and t_4 are all quite different from the usual τ-loci.

(b) These loci are all qualitatively similar to each other.

(c) They span a time interval comparable with the estimated ages of the known young stellar clusters, namely something of the order of 10^7 years, or 0.5 Helmholtz-Kelvin units.

The physical reasons for these features are evident.

The age of the cluster within the meaning we have in mind, (contraction time) is the time required for a star at the lower end of the main sequence just to reach the main sequence from whatever has to be treated as the initial state. Now the rate of contraction is highly sensitive to mass, as can be seen from equation (24)

$$\log \tau = -3.47 \log M + 0.07 \log \rho + 0.68$$

which shows that the time required is proportional to 3.5^{th} power of the mass.

Thus stars of mass much less than the one just reaching the main sequence cannot have moved far from the initial state in the available time. Therefore if the initial conditions are such that at any epoch the pre-main-sequence stars lie on a locus appreciably different from that representing gravitational contraction starting at the same epoch from vanishingly small densities, then this will remain the case throughout the time interval of interest.

On finding a distribution of pre-main-sequence stars such as those already mentioned by us, it consequently seems natural to look for an explanation in terms of initial conditions (e.g. the observed effect would be produced were there a tendency for stars of a smaller mass to be formed before stars of a greater mass), rather than postulate new phenomena such as large scale mass loss during the gravitational contraction of a star.

As has already been pointed out, we can see from equations (22) to (27) that the result is insensitive to the value of α . Clearly no substantial changes in the diagram will occur if we change the slope of the main sequence slightly and hence the result is not sensitive to the value of γ either.

The calculations given above were carried out before a very important paper on the subject by C. Hayashi [10] came to our notice. Taking account of the presence of a hydrogen convective zone in stars of late spectral type, Hayashi concludes that the early part of the evolutionary track of a star of given mass is very different from the straight line tracks shown in figure 3. This affects the age calculations and on making allowances for this Hayashi gets good agreement with the observational results for NGC 2264 on making the usual assumption that all the member stars of the cluster originated with an arbitrary low density at the same epoch.

Hayashi's explanation will probably be accepted in this case, and because of its importance a section at the end of this chapter has been devoted to a discussion of Hayashi's work.

These results do however give another example of what we have been drawing attention to in the preceding discussion; they show that by altering the initial condition of a star (in Hayashi's case, the presence of a hydrogen convection zone is taken) we obtain a Hertzsprung-Russell diagram that need not be similar to the straight lines found by Su-Shu Huang (and also shown by us in figure 3). Hence, as regards homologous contraction, Hayashi's work substantiates the general conclusions about the importance of the initial conditions to which we seek to call attention.

If we assume any one of the initial conditions mentioned above we see that the predicted Hertzsprung-Russell diagram for any young stellar cluster would be a single track. This track could be curved or straight depending on which model for this contraction stage we are using, but in all cases it would be a single track. The loci $\log \mathcal{M}$ = constant, $\log \tau$ = constant and $\log t$ = constant are all single line tracks and hence any theoretical model will always give us the loci of these young stars as a single track.

From our figures 1 and 2, or from any of the diagrams given by Walker [1,2,3] and Whiteoak [4], it is clear that the stars observed do not lie upon a single line in the Hertzsprung-Russell diagram, but rather occupy a region between two curves, this band being much wider than the region occupied by the main sequence stars (which also should be a single line theoretically). Some divergence from a single track due to the uncertainties involved in measuring stellar luminosities and effective temperature and the effects of interstellar absorption, but these effects would be the same for all cluster members, main sequence or pre-main sequence, and so we would not expect the pre-main sequence band to be wider than the main sequence band and so apparently we have a discrepancy here between theory (any model) and observation.

Varsavsky [11] has observed a group of stars in Taurus, presumably in the gravitational contracting stage, photoelectrically and has compared his results with earlier observations of the spectrum of the same group of stars conducted by Joy [12].

Comparison of the two results can only have a meaning if we have the two following assumptions:
1) The spectral types, as deduced from the absorption lines give the correct values for the effective temperature. That is, Joy's results enable us to find the correct value for the effective temperature.
2) These effective temperatures have not altered significantly during the years that have elapsed between Joy's spectroscopic observations in 1949 and Varsavsky's photoelectric observations of 1960.

Clearly these assumptions are quite general and are very likely to be satisfied. On comparing the two sets of results Varsavsky finds that the photo-electric colour observations $(B-V)$ do not give a unique definition of the effective temperature. He shows that for any given effective temperature, T, measured spectroscopically, the colour $(B-V)$ varies within a range of approximately $0 \cdot 8$ magnitude from this and always (with the exception of Ge stars) towards the blue side of the normal relation.

There is no reason for doubting Varsavsky's work or for rejecting the two general assumptions used above. No conceivable reason exists either for these results applying only to the stars in the Taurus cloud, and so it seems natural to conclude that all the pre-main sequence stars in the clusters mentioned by us should have this correction of up to 0·8 magnitude applied to them. We note here that the width of the pre-main sequence band, measured parallel to the effective temperature axis, is just about 0·8 magnitude. Thus if this correction were applied when we transform to the temperature results from the colour $(B-V)$ measurement, the wide area which the stars appear to occupy would be reduced to a single track.

This correction has been applied by Varsavsky to NGC 2264; we have done the same to NGC 6530. (Actually we have only corrected the stars with the greatest displacement from the red border of the band. If these stars can be corrected onto a single track, clearly any star requiring less correction can be corrected). The tracks obtained for both of the above clusters are shown in figures 1 and 2, superimposed on the diagram obtained by assuming the normal relation between $B-V$ and the effective temperature.

Hence we can conclude that the above mentioned discrepancy between observation and theory concerning the width of the region containing stars in the Hertzsprung-Russell diagram can be resolved, but the solution raises the problem of determining the cause of this irregular relation between the colour (B-V) and the effective temperature. Varsavsky suggests that this may be due to emission lines in the star's spectrum together with the presence of a blue continuum, the strength of these being the factors chiefly determining the colour of the star. If these are weak the star would have normal colours, otherwise the star would be too blue for its spectral type.

<u>The distribution of pre-main-sequence stars in the Hertzsprung-Russell diagram</u>

We now turn to a completely different aspect of the diagram for these pre-main sequence stars. This is the relative distribution of these stars in comparison with the main sequence distribution. In his paper on this type of star, Su-Shu Huang [6] has found the distribution of these stars assuming that stars are formed at a constant rate, but has not found this distribution for stars in a cluster, all formed at the same epoch. As this distribution can be a very important tool in testing the validity of a theory, since the theoretical and observational distribution must agree, and as we have already obtained most of the equations necessary to determine this distribution we shall now include a determination of this distribution despite the fact that it has no real connection with the conclusions given above.

We can clearly obtain this distribution in either of two ways, graphically or theoretically. Graphically the position of a pre-main sequence star is given by the point of intersection of the straight line locus $\log \tau$ = constant with the evolutionary track of the star with the given mass.

Theoretically we have to take a distribution of masses such as one finds in a stellar cluster and calculate by means of equations (22) to (28) the position of these masses at any given epoch in the Hertzsprung-Russell diagram.

Let us take the masses of the stars concerned as $\log \mathcal{M} = 0.35$, 0.4, by increments of $\cdot 0.5$ to $\log \mathcal{M} = 1.1$.

The two equations out of the six mentioned above that are applicable to the problem under discussion are

$$\log L = 3.76 \log \tau + 18.2 \log M - 2.55 \qquad (25)$$

and

$$\log T = 3.32 \log \tau + 12.6 \log M - 2.25 \qquad (26)$$

The values of $\log L$ and $\log T$ can now be found for all the values of $\log M$ given above if we determine a value for $\log \tau$.

If the value of $\log \tau$ we choose is too large, then all the stars in the above mass range will have evolved onto the main-sequence and so the required distribution will not be given. On the other hand if we choose too small a value for $\log \tau$ the stars might still have, in effect, an infinite radius and so would be of no use for any sensible diagram. A value for $\log \tau$ that avoids both these difficulties while at the same time being a convenient number, numerically speaking, is $\log \tau = -1$. The age of the distribution is thus one tenth of the Helmholtz-Kelvin time scale or about 3×10^6 years. This time is thus close to the estimated age of young clusters and so we are in fact dealing with a case that is likely to exist in practice.

For the purpose of this discussion the main-sequence is again taken to be given by

$$\log L = 7 \log T$$

the units again being solar.

If, as a result of our computation, we find that a star has luminosity and effective temperature such as to place it below (to the left) of the main sequence, then clearly this star has reached the main sequence and its position should be given by solving the equation of the main sequence with the locus of a star of given mass $\mathfrak{M}$, namely the equation obtained by eliminating ρ from equations (13) and (14) above

$$\log L = \left(\alpha - \frac{\beta}{3}\right) \log \mathfrak{M} + \frac{\beta}{3} \log \rho \qquad (13)$$

$$\log T = \frac{1}{4}\left(\alpha - \frac{2+\beta}{3}\right) \log \mathfrak{M} + \frac{2+\beta}{12} \log \rho \qquad (14)$$

On eliminating ρ we obtain

$$\log L = \frac{4\beta}{2+\beta} \log T + \frac{2\alpha}{2+\beta} \log \mathfrak{M} \qquad (29)$$

Introducing the values we have adopted for the constants α and β, namely $\alpha = 5.4$, $\beta = 0.79$ into the above equation gives

$$\log L = 1.13 \log T + 3.87 \log \mathfrak{M} \qquad (30)$$

Solution of this equation with the equation of the main sequence for each value of the mass $\mathcal{M}$ gives the position of a star with mass $\mathcal{M}$ on the main sequence and hence giving us the distribution of the stars as well.

These points we obtain theoretically using the described method should clearly be the same as those given by the intersections of the locus of the given masses $\mathcal{M}$ with either the locus $\log \tau = -1$ or the main sequence depending on which of the two intersections occur nearest to the right of the diagram.

The distribution we obtained using the chosen values of $\log \mathcal{M}$ and $\log \tau$ is drawn as figure 4. We note immediately that this diagram shows a much more dense population of the main sequence region than the pre-main sequence region. With the mass distribution we have taken, namely $\log \mathcal{M}$ = constant interval, we have introduced a very strong tendency towards the lighter masses being present in greater numbers in the cluster. With a stellar distribution in which the masses were more evenly spread out the population of the main sequence would be even thicker while the pre-main sequence region became even more thickly populated

Clearly no great change in the distribution

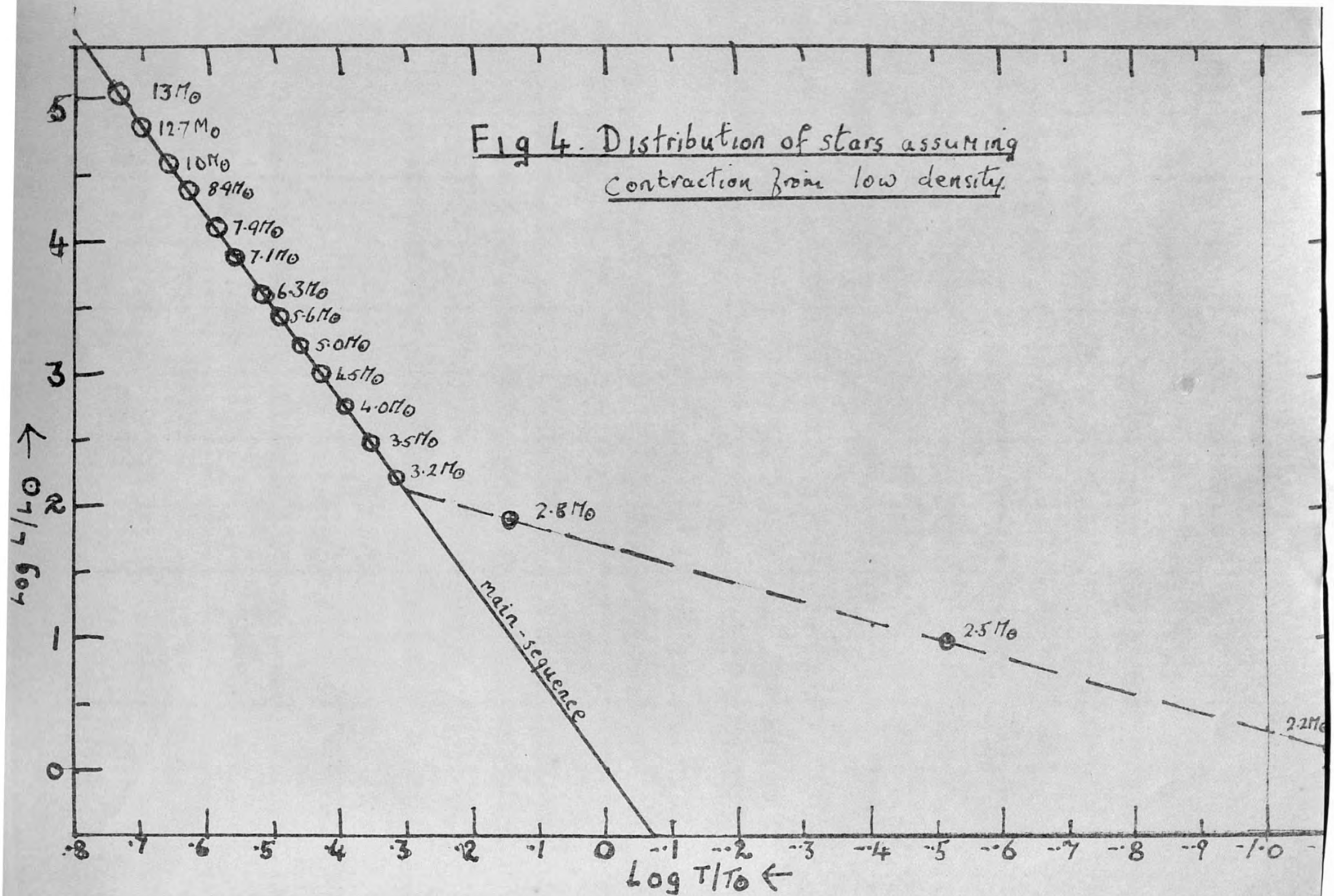


Fig 4. Distribution of stars assuming contraction from low density.
13M⊙
12.7M⊙
10M⊙
8.4M⊙
7.9M⊙
7.1M⊙
6.3M⊙
5.6M⊙
5.0M⊙
4.5M⊙
4.0M⊙
3.5M⊙
3.2M⊙
2.8M⊙
2.5M⊙
2.2M⊙
main-sequence
Log L/L⊙ →
Log T/T⊙ ←
5
4
3
2
1
0
·8
·7
·6
·5
·4
·3
·2
·1
0
-·1
-·2
-·3
-·4
-·5
-·6
-·7
-·8
-·9
-1·0

will take place if we change the value of $\log \tau$. Indeed, all this does is to change the position of the locus $\log \tau$ = constant, not its direction and so the pattern of intersections with a set of parallel lines, which are the loci for the different masses, is unaltered.

We can thus conclude that, irrespective of the particular values of $\log m$ and $\log \tau$ we choose, the number of stars in the pre-main sequence stage is very much smaller than the number on the main sequence if we assume that the model for contraction described by Su-Shu Huang is correct.

The actual range of masses in the pre-main sequence contraction stage must also be very small. With the values we have chosen we see that from $\log L = 0$ to $\log L = 2.25$ (a range of seven magnitudes in stellar brightness) the mass increases from 2.2 solar masses to 3.16 solar masses in the pre-main sequence region, while an increase from 3.16 to 8.9 solar masses is observed for a range of seven magnitudes in brightness along the main sequence. Hence, if we were to assume a one-to-one correspondence between mass and number, the ratio of the number of stars in main sequence and pre-main sequence stages over an equivalent brightness interval (7 magnitudes say) is about 6 : 1.

In a normal cluster the main sequence would cover about three times as large a brightness range as the pre-main sequence section, hence the ratio of main sequence stars to pre-main sequence stars becomes 18 : 1.

Hence if we had about sixty stars in the pre-main sequence stage in a cluster, this would mean that the total cluster membership was about 1000 stars if the found distribution is correct.

Now in all cases of young clusters observed the pre-main sequence branch is as thickly populated as the main sequence branch and certainly more than about one star in twenty are observed to be in this contracting stage. Hence we can say that over this point of the stellar distribution a serious discrepancy exists between the observed facts and the predictions of the simple theory, which assumes creation at very low density at a unique epoch for all stars, the stars then evolving along a path found by Henyey and others, as described by Su-Shu Huang.

Hence, quite apart from any differences that exist between the predicted and observed tracks or between the estimates of the age of these clusters using different methods for measuring it, we can conclude that the actual number of stars in the contracting stage is in serious disagreement with the predictions of the simple theory described (but not advocated) by Su-Shu Huang and so it appears that this theory must be abandoned in favour of some theory which takes more account of the initial stages of the stars as suggested by us.

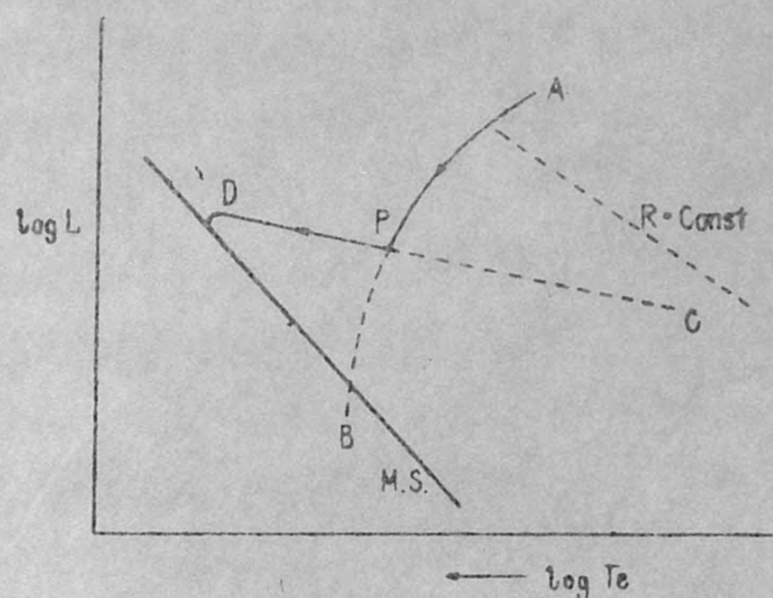
An account of Hayashi's theory

Hayashi has obtained very different results from the straight line tracks shown by Su-Shu Huang in his paper and by us in figure 3 by taking account of a hydrogen convection zone in stars of late spectral type.

In a previous paper, Hayashi [13], together with Hoshi, has investigated the outer envelopes of late type giant stars and calculated the locus E = constant in the Hertzsprung-Russell diagram. E is the characteristic value which determines the degree of central condensation of the solution with polytropic index $3/2$, and is given by

$$E = 4\pi G^{3/2} \left(\frac{\mu H}{k}\right)^{5/2} M^{1/2} R^{3/2} P / T^{5/2}$$

For the meaning of the symbols the reader should consult Hayashi and Hoshi [13]; the exact meaning of E is not important for an understanding of this paper by Hayashi. The maximum value which this function can take is 45, beyond which no quasi-state solutions exist. In figure 5, which is reproduced from Hayashi's paper [10], APB denotes this curve $E=45$ for a given mass and chemical composition. (Hayashi does not state what they are and again this is not important for a general understanding.) No quasi-state solutions can exist to the right of this curve. CPD is the usual evolutionary track for a star of the given mass found by Henyey, Lelevier and Levee [8]. When a star is formed in the forbidden region to the right of APB Hayashi states that it will adjust its internal structure in a relatively short period of time in such a way that it takes up a position on the curve APB. A quasi-state solution is now possible and so the star will stay on this track APB until it reaches a point where it can follow the normal evolution found by Henyey, Lelevier and Levee, that is the point P, and then evolves along CPD. This new track would only be useful in giving a new Hertzsprung-Russell diagram if the time taken by the star to reach P along the new track is appreciably shorter than the time the star would need to evolve along the conven-



Schematic track for contracting stars with given mass and chemical composition. The curve CPD shows the track calculated by LEVEE, SALPETER and HENYEY et. al. and APB shows a curve with $E=45$, the right region of which is forbidden for the existence of the quasi-static solutions.

Fig 5 Reproduced from Hayashi's paper

tional track to P along CPD. Hayashi assumes that the time taken by a star to evolve from whatever state it was formed in to some point on APB is small compared with the time taken to evolve to P and can thus be ignored. No apparent way exists of estimating this and so we can only accept Hayashi's assumption.

The time of evolution when the only source of energy available is the gravitational potential energy of the star is given by a solution of the equations

$$\frac{dE}{dt} = -L \qquad \text{and} \qquad E = -cG\frac{M^2}{R}$$

which are essentially the same as those already used by us in the previous work as equation (6), namely

$$L\,dt = -cG\frac{M^2}{R^2}\,dR \tag{6}$$

As Hayashi is interested only in the evolution of one particular star at any instant, the variation of any quantity with mass is not important and so he takes

$$L \sim R^{-\beta}$$

which is again essentially the same as what we have assumed in equation (4)

$$L = \frac{AM^{\alpha}}{R^{\beta}} \tag{4}$$

(It is to be noted that in his paper Hayashi uses somewhat different notation to what we have used above, but we have converted his notation to the one already defined by us to avoid confusion.)

Using the above equations Hayashi obtains a value for the contraction time of a star starting from an arbitrary large radius as

$$t = \frac{10^{7.2} \mathcal{M}^2}{(1-\beta)\mathcal{R}\mathcal{L}} \quad \text{years}$$

where $\mathcal{M}$, $\mathcal{R}$ and $\mathcal{L}$ denote the mass, radius and luminosity of the star in solar units. The expression found by us as equation (11) was

$$\tau = \frac{\mathcal{M}^{2-\alpha}}{(1-\beta)\mathcal{R}^{1-\beta}} = \frac{\mathcal{M}^2}{(1-\beta)\mathcal{R}\mathcal{L}} \quad \text{H.K units}$$

Hayashi assumes that n , the power of the radius with which the density varies, is two while the value used by us to give one H.K. unit as 3×10^7 years is three. Using this new value of $n = 2$, one H.K. unit becomes 1.6×10^7 years or $10^{7.2}$ years. Thus Hayashi's results are exactly what we have already found, and so no new theory has been inserted so far.

However, for the stellar model proposed by Hayashi evolving along the track APB the value of the constant β is $-3/2$, thus giving a contraction time to the point P as $2D/5$ where

$$D = \frac{10^{7.20} m^2}{RL}$$

Along the conventional evolutionary track found initially by Henyey and others, the value of this constant β is 0.79 (Hayashi has used the very approximate value of 0.5 , but we shall use this value already given of 0.79 .) The contraction time using this value for β becomes $\frac{D}{.21} \doteq 5D$, D being the same as defined above.

Thus the time required by a star contracting in the manner postulated by Hayashi to reach the point P is very much shorter than the time required by a star of equal mass evolving along the more conventional track given by Henyey, Su-Shu Huang and us.

In his paper Hayashi gives the Hertzsprung-Russell diagram he obtained after allowances had been made for this shorter evolutionary time and obtains good agreement with the diagram drawn from observations of NGC 2264.

Hayashi however does not give details of how he constructed his diagram and so we shall now attempt to construct one employing similar methods to what we have already used in the previous example. So as to reproduce results with the same condition as Hayashi we shall use $n = 2$ (already defined) and thus a new value for the Helmholtz Kelvin time scale of $1{\cdot}6 \times 10^7$ years.

The equations Hayashi obtains in his paper are exactly the same as those we have already obtained in the first part of the chapter. The only differences are in the value of the constants α and β . Along AP Hayashi takes β to be $-3/2$ while along PD the value is $0{\cdot}79$, while for the conventional model the value is $0{\cdot}79$ all the time.

We have already found the equation connecting $\log L$ and $\log T$ at any given epoch, this was equation (19), namely,

$$\left(\frac{3\alpha}{2} - 2 - \beta\right) \log L = 2(\alpha - 2\beta) \log T - \alpha \log C - \alpha \log(1-\beta) \qquad (19)$$

The locus in the Hertzsprung-Russell diagram of stars that have not yet reached the point P on their evolutionary track is thus given just by substituting the values of α and β into equation (19). The value of β we know is $-3/2$. Unfortunately, as we have already mentioned, Hayashi is interested primarily in one star at a time and so no relation is given between luminosity and mass, that is, no value for α . From the paper by Hayashi and Hoshi [13] , however, we can deduce that $L \sim M$ and thus the value of α is 1 .

Thus the locus of stars evolving in the manner postulated by Hayashi up to the point P is given by

$$\log L = 8\log T - \log \tau - \log 5/2 \qquad (31)$$

This equation only holds up to the point P which has a lready been defined as the point where the conventional track of a star, CD, crosses the Hayashi track APB. We have already found the ratio of the time of evolution to the point P along these two tracks as $2\times.21/5 \sim 1/12$. Thus a star with any given mass would reach P along APB in a time τ_1 while it would take a time $12\tau_1$ along CPD, and hence in the Hertzsprung-Russell diagram the point P can be given by the intersection of locus (31)

$$\log L = 8\log T - \log \tau_1 - \log 5/2$$

with the locus of star evolving along CPD with a time $12\tau_1$, that is equation (27) with $\tau = 12\tau_1$

$$\log L = 1{\cdot}44\log T - 1{\cdot}01\log 12\tau_1 + 0{\cdot}69 \qquad (32)$$

For stars that have evolved far beyond P, the mode of evolution up to the point P is immaterial as the time along both tracks AP and CP are short compared with the time taken by the star to evolve from P to its present position. Hence for these stars the locus is equation (27)

$$\log L = 1.44 \log T - 1.01 \log \tau + 0.69 \qquad (27)$$

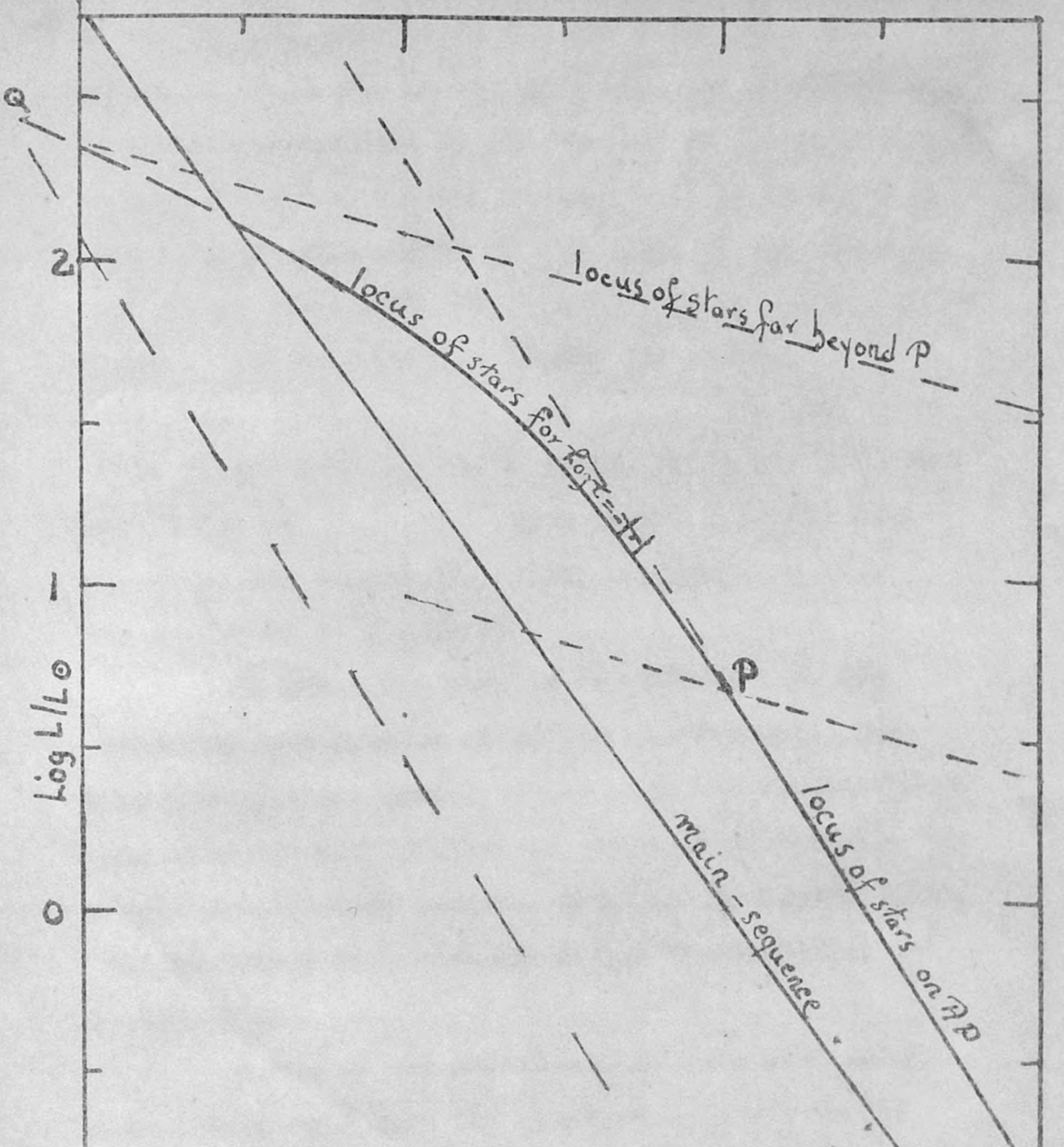
The time to reach P is short compared with the age of the star when it is about $1/6$ of its age when travelling along CP, thus about $1/72$ of its age travelling along APB. The point Q where the locus becomes equation (27) is thus given by the intersection of this (27) with equation (31) with $\tau = 72\,\tau_1$.

The locus of the stars will thus be the two lines given by (27) and (31) beyond P and Q and the smooth curve that has these lines as its tangents at P and Q between P and Q.

These lines, curves and points have been drawn as figure 6. The age we have taken for the distribution is 1.26×10^6 years ($\log t = 6.1$) or $\log \tau_1 = -1.1$ using Helmholtz Kelvin units. This diagram we see compares very favourably with the diagram obtained from observation of NGC 2264 or NGC 6530, and the track is also very similar to the one given by Hayashi in his diagram.

We have already stated the importance of obtaining the correct stellar distribution from any theory concerning the gravitationally contracting stars. We shall thus now find the distribution working

Fig 6 Evolutionary tracks using the Hayashi theory

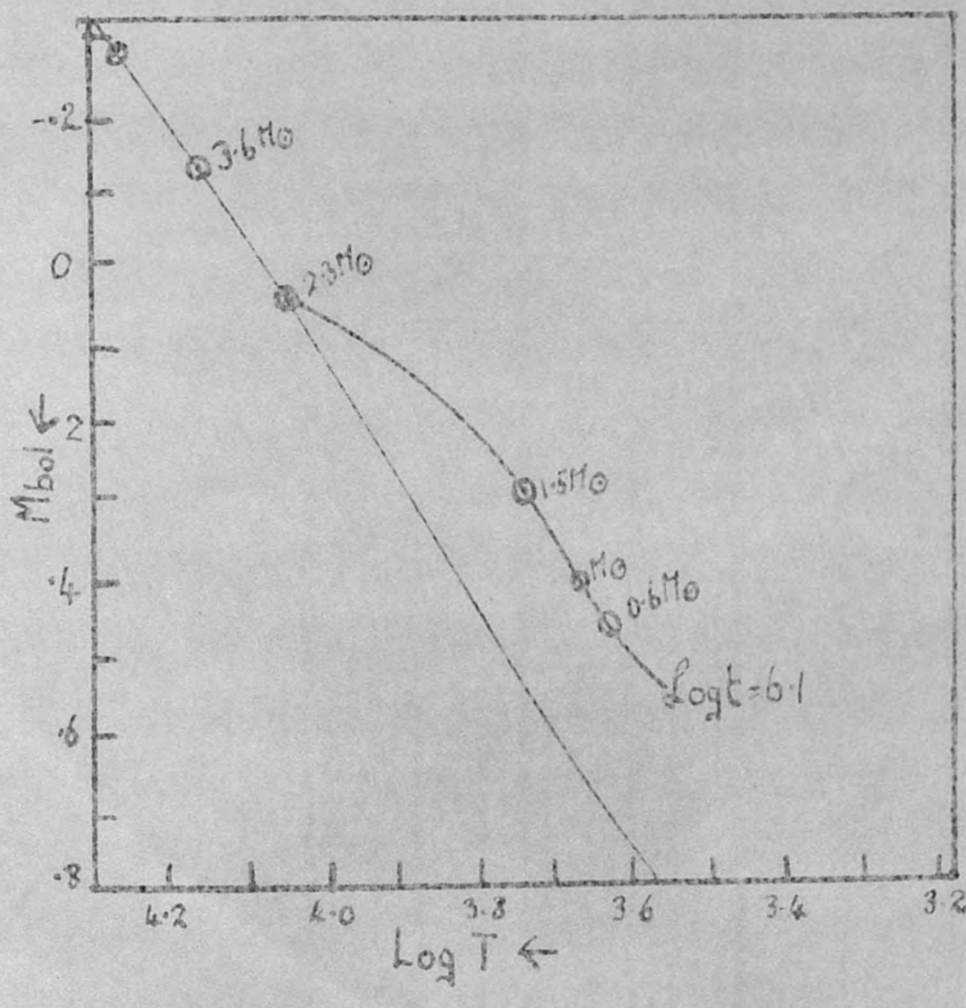


on the Hayashi hypothesis for the evolution. The method we shall use is the graphical one described in an earlier discussion in this chapter on distribution. We have taken the stellar distribution to be given by the interception points of the locus of the stars at any given epoch with the evolutionary tracks for given masses. The evolutionary tracks for masses, $0.6 M_{\odot}$, $1.0 M_{\odot}$, $1.5 M_{\odot}$, $2.3 M_{\odot}$ and $3.6 M_{\odot}$ we have taken from figure 2 of Hayashi's paper, while the locus has been taken at $\log t = 6.1$ from figure 3 of the same paper (or our figure 6). These interception points are exhibited in figure 7.

We can again see, by comparison with the observational results of Walker and Whiteoak, that this distribution agrees better with the observational one than the distribution we obtained by assuming the simple evolutionary pattern described by Su-Shu Huang, and is indeed very similar to the observational distribution.

Hence we can conclude that Hayashi's theory agrees very well with the observed data and so his theory is a great advancement on all previous work on this phase of stellar evolution.

Fig 7 Distribution of stars assuming the Hayashi model for contraction obtained by superimposing Hayashi's figure 2 on his figure 3.



In this chapter we have shown the importance of choosing proper initial conditions for a stellar cluster by means of an illustrative example. Hayashi's theory bears this out, he having altered the initial evolutionary path of a star, and indeed it appears that Hayashi's results give a fairly good approximation to the actual observed evolution. We have also found the distribution of these pre-main sequence stars working on both Hayashi's theory and on the older theories described by Su-Shu Huang. It appears that Hayashi's gives a fair approximation to what is observed while the other does not.

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CHAPTER 3

The blue stars beyond the main sequence turn-off point in the Hertzsprung-Russell diagram

Many astronomers and writers of literature on the Hertzsprung-Russell diagram (one of the most commonly presented forms for this diagram being the colour-magnitude diagram) as related to stellar clusters have commented on the presence of stars in a region of this diagram where, according to present-day beliefs, there should be none. This is the region to the left, or blue side, of the turn-off point from the main sequence, the actual stars forming a pattern corresponding roughly to a continuation of the main sequence in this region, though not so thickly populated as the actual main sequence. The presence of these stars are very easily detected in the globular cluster M3 and in the galactic clusters M67 and NGC 7789. The first two of these clusters were observed by Johnson and Sandage [1,2] while NGC 7789 has been observed by Burbidge and Sandage [3]. Stars such as those under discussion have also been observed in the following galactic clusters, though not as clearly as in three clusters mentioned initially, Coma Berenices [4,5] , Praesepe [6,7] and possibly in h and χ Persei [8].

Various suggestions have been proposed from time to time as to the cause of this phenomena but, as far as we can find out, none of these suggestions have been investigated with any detail. Consequently nothing definite is known about the extent of the agreement between theoretical predictions based on these suggestions and the observed facts. As things stand at the moment we are not in a position either to accept or reject any of the proposed theories.

One of these suggestions is that the formation of stars in a stellar cluster is a continuous process and not confined to any particular epoch in the history of the cluster. As a result, stars in such a cluster have a range of different ages. A direct consequence of this is that stars of similar mass and composition have consumed varying amounts of hydrogen and are thus in varying stages of evolution. Hence some stars with a given mass can occupy a position on the main sequence while others with a similar mass have evolved to the right of this main sequence. There could thus be a turn-off point defined by the evolution of the older stars together with a less heavily populated continuation to the main sequence beyond this point consisting only of the younger stars. It is the purpose of this chapter to investigate whether a cluster formed in a continuous process as suggested above will have a Hertzsprung-Russell diagram similar to what is observed in the above mentioned clusters, or not.

Clearly what we have to do is to produce a theoretical Hertzsprung-Russell diagram for a stellar cluster in which the stars were formed at various epochs, and hence now possess a wide range of ages. This diagram must then be compared with the one we can draw from observations of one of the clusters under discussion. We shall assume that while the formation of stars in a cluster is taking place, stars are formed regularly. That is, a new star will appear regularly after a fixed period of time has elapsed. The problem of constructing a theoretical Hertzsprung-Russell diagram is now much simpler than if formation of the stars was a random process in time. We now have to find the position in the Hertzsprung-Russell diagram of stars, within a given range of masses, whose ages are determined by the regularity of their formation. Our problem is thus to calculate the luminosity and effective temperature of any star with a given mass, its age also being known as we have fixed the epoch of its formation.

Professor Hoyle has recently published two papers "The ages of type I and type II sub-giants" [9] and "On the main sequence band and the Hertzsprung gap" [10]. Both of these papers contain valuable information about the variation of the luminosity and effective temperature of stars with given masses as they evolve. We shall make use of the information contained in both of these papers when we calculate the position of stars in the Hertzsprung-Russell diagram at different epochs. No attempts have been made by us to recalculate or modify any of these results; they have been taken as published by Hoyle.

In tables 1, 2 and 3, which show the evolution of three different types of stars, the values of the luminosity and effective temperature at any given time are taken from Hoyle's papers while the other quantities involved have been calculated by us, making use of Hoyle's results if necessary. According to Hoyle the first row in each of these tables gives the value of the quantity concerned when helium burning begins in the stellar interior. In the second column we give the age of the star at any stage using the stated time for helium burning to begin as a unit. All logarithms in these tables have been taken to base 10.

Using these tables we can clearly plot the logarithm of the luminosity against the logarithm of the temperature for all the values given in these tables. The resulting diagram would be the evolutionary track in the Hertzsprung-Russell diagram of the particular star concerned. Each point plotted on this track will have associated with it the time (whose value can be found from the tables) that must elapse before a star with the given properties can evolve to this point. These evolutionary tracks have been drawn and the points associated with the times given in the tables have been indicated. They are exhibited as figures 1, 2 and 3. For convenience, in all of these diagrams we have taken the luminosity of the sun as a scale factor and have thus plotted $\log(L/L_\odot)$ against $\log T$ (with obvious notation), hence reducing the luminosity scale to more convenient numerical values. The main sequence and a continuation of the evolutionary track beyond the helium burning stage are also included in these diagrams. More will be said about these later.

We see immediately from these three diagrams that the evolutionary track in figure 1 is very different from the tracks in figures 2 and 3. For

Fig 1 - Evolution of a star of Mass $3 \cdot 89 M_{\odot}$

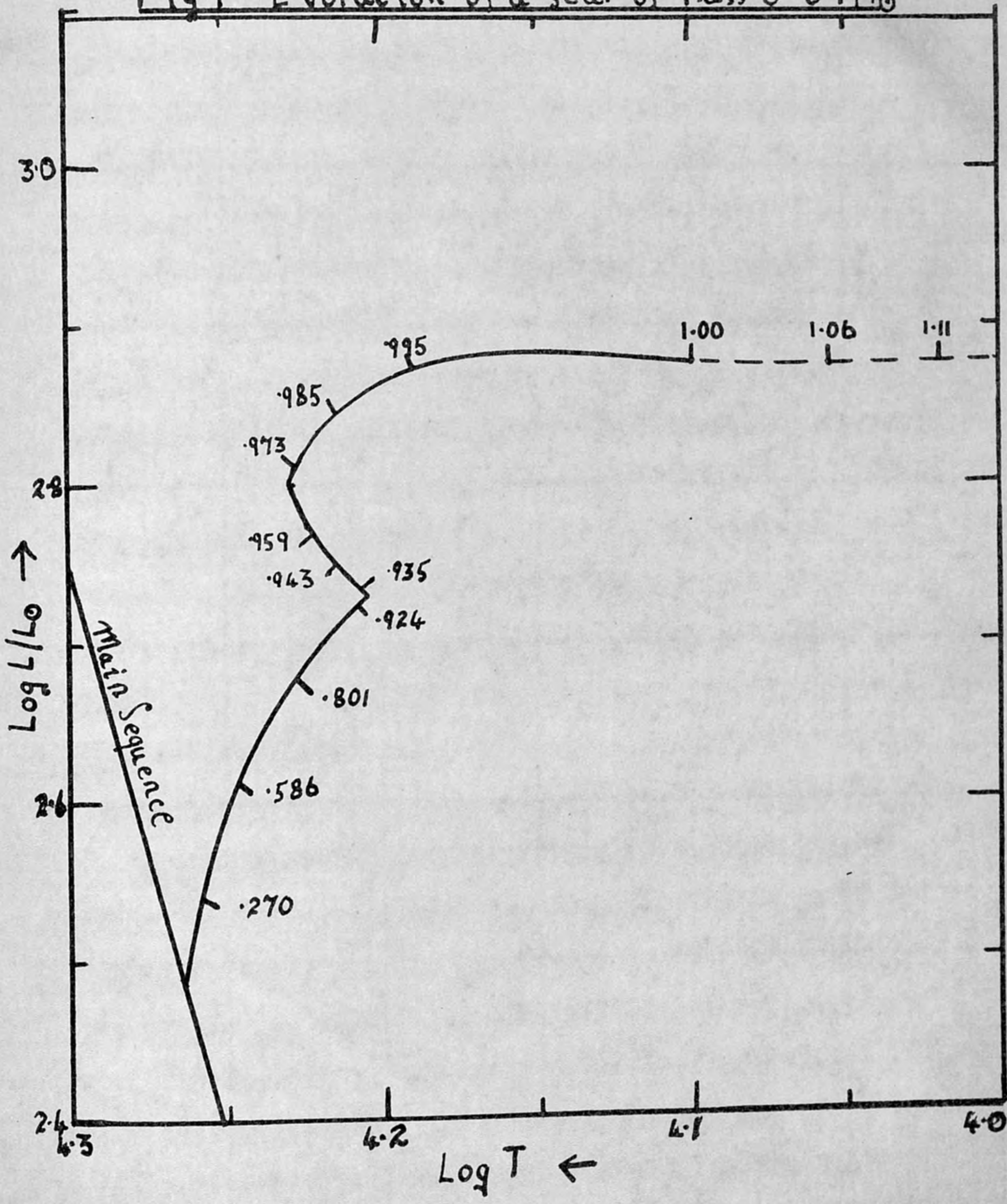


Fig 2 - Evolution of a star of Mass $1 \cdot 09 M_\odot$

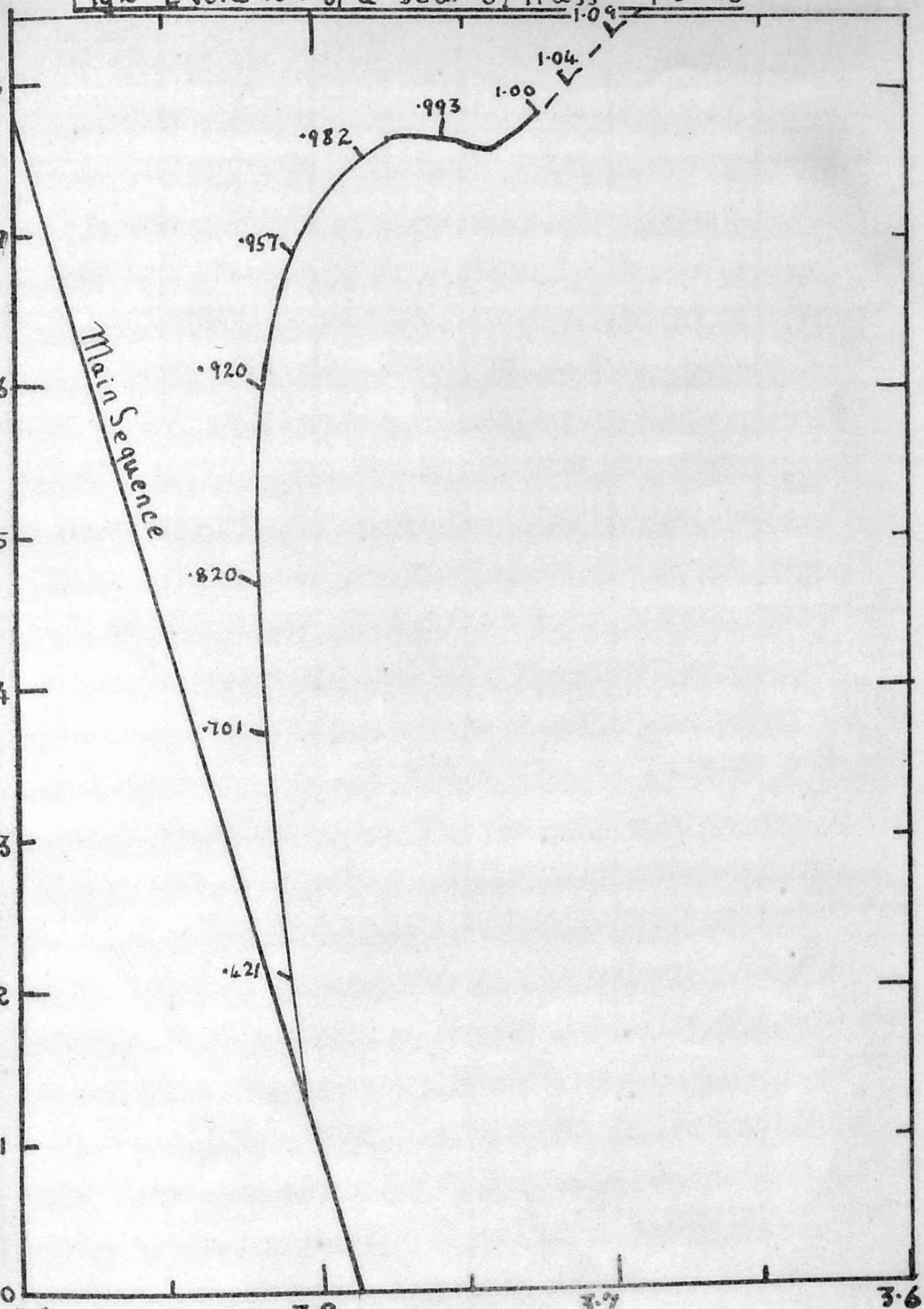
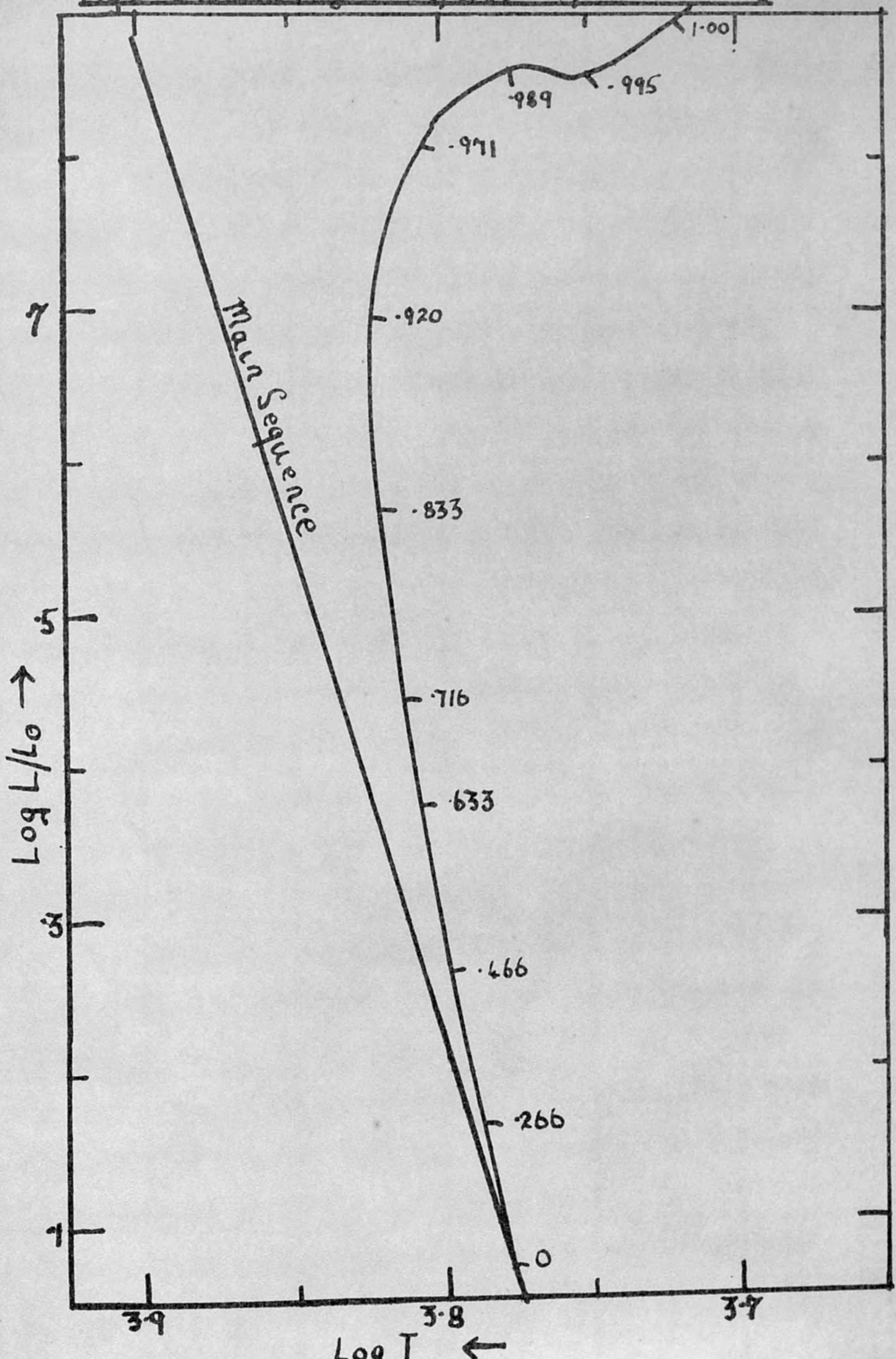


Fig 3 - Evolutionary track of a star of mass $1{\cdot}35 M_{\odot}$



this reason we shall now divide the investigation into two parts, one for tracks such as shown in figure 1 which is for masses greater than one solar mass, and one for the combined figures 2 and 3 which apply for masses in the neighbourhood of one solar mass.

In the second of the two papers mentioned above, Professor Hoyle has compared, by means of a diagram, the evolutionary tracks of stars with various different masses. Hoyle has used the bolometric magnitude as ordinate instead of the logarithm of the luminosity as used by us. The connection between the two quantities is

$$M_{bol} = 4.62 - 2.5 \log\left(\frac{L}{L_{\odot}}\right) \qquad (1)$$

as can be seen in Allen's 'Astrophysical Quantities' [11] for example, where M_{bol} is the bolometric magnitude and $L_{\odot}$ is the solar luminosity.

Hence the bolometric magnitude used by Hoyle is essentially the same, apart from added constants, as the logarithm of the luminosity.

So as to be able to use the same scale for all the tracks to be compared, Hoyle has taken the luminosity at zero age and effective temperature at zero age of the stars in question as a scale factor for the tracks. For this reason all the tracks originate from the same point (the origin) in the diagram.

This diagram has been reproduced by us and is included as figure 4. It underlines the differences, mentioned above, between the track of stars comparable in mass with the sun and the track of stars heavier than this. It also shows that the evolutionary tracks for all masses greater than the solar mass are exceedingly similar to each other, irrespective of mass, and differ only in their zero age position on the main sequence. This suggests that for this type of star all we have to do to move from one track to the track of another star is to slide the initial track along the main sequence until the starting point is in the proper place.

In a paper entitled "The transition from hydrogen burning to helium burning in a star of mass $5 M_\odot$", Emil J. Polak [12] has calculated the values of the luminosity and effective temperature of a star of five solar masses for given ages as evolution progresses. The evolutionary track so obtained compares favourably with the track for a mass $3{\cdot}89 M_\odot$ calculated by Hoyle and exhibited as figure 1 by us.

There thus appears to exist strong evidence for assuming that the evolutionary tracks for any mass

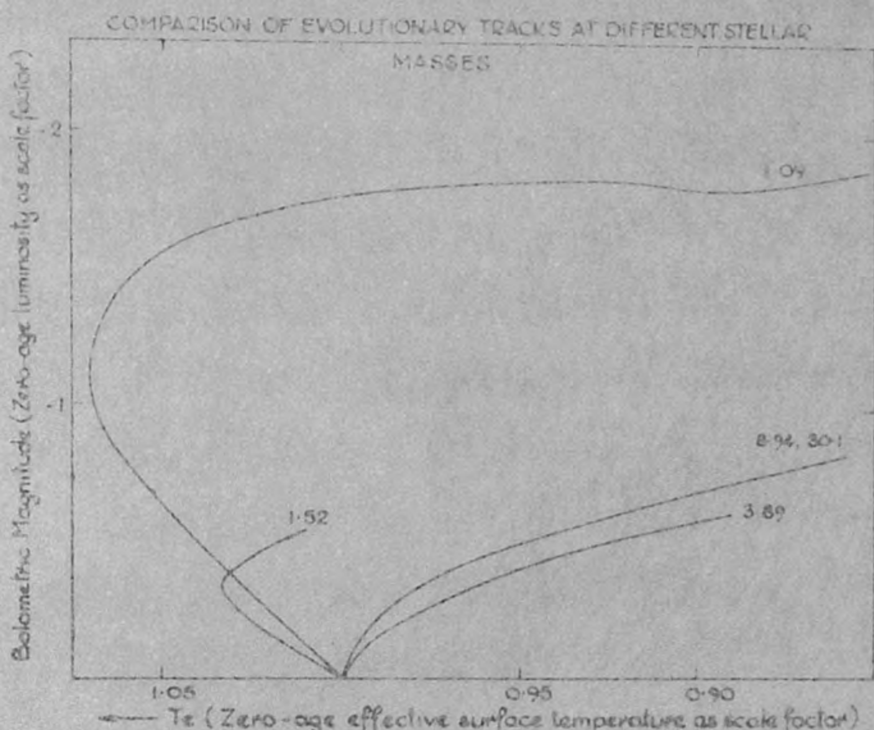


Fig 4 taken from
Hoyle's paper

significantly greater than one solar mass will be qualitatively similar to the evolutionary track for any other mass in this range, but differing appreciably from the evolutionary tracks for a star with a mass similar to the sun.

As already mentioned we shall now consider these two mass ranges separately and attempt to construct a theoretical Hertzsprung-Russell diagram for each of them in turn.

Masses significantly greater than one solar mass.

In order to construct a theoretical Hertzsprung-Russell diagram we require information about the luminosity and effective temperature of stars for a range of different masses. So far, we are only in possession of this information for one mass, 3.89 solar masses as given in table 1. We thus require some means of generalizing this information so that we have knowledge about other masses as well.

We have already pointed out that, to the best of our knowledge, the evolutionary tracks for all masses in the range now under consideration are qualitatively similar to each other. Thus for any smaller mass range completely included in this mass range (such as we would have for the stars in the region of the main sequence turn-off point) the evolutionary tracks are bound to be similar to each other, the only difference between them being their actual position in the diagram, the greater the mass the further along the main sequence the track would join the main sequence. The time required by a star to evolve along any given part of the track would also obviously depend on the mass (among other things possibly). Thus, before we can obtain the information about the luminosity and effective temperature, we have to find the variation in the time scale and also the position at which the track leaves the main sequence for all the masses concerned.

In a paper entitled "Main sequence stars" by Hazelgrove and Hoyle [13] values are given for the mass, luminosity and effective temperature of main sequence stars. For the mass range we are interested in, namely masses in the region of $3.89\, M_{\odot}$ (which is the mass for which the evolutionary track is known) we find the following quantities given.

Mass (10^{33} gms)	Luminosity (10^{35} erg/sec)	Effective temperature (°K)
5.7407	3.8363	15090
6.8342	7.4701	16960
7.7384	1.2027 x 10	18410
11.841	4.4020 x 10	21850
17.761	1.6594 x 10^2	26950

This information allows us to find two of the facts we require for the construction of a Hertzsprung-Russell diagram. The two columns relating luminosity and effective temperature obviously allow us to plot the main sequence into any diagram we wish to construct for this range of masses. As we expect, we find that the relation between luminosity and effective temperature is fairly close to the rough relation usually used for this part of the main sequence of

$$\log L \doteqdot 7 \log T + \text{Constant}$$

From the above information we can also find a relation between mass and luminosity in the region of 4 solar masses, that is in the region of 8×10^{33} gms.

We can write

$$\log L = A \log M + B$$

for this small region, where A and B do not depend on the luminosity or the mass and can thus be treated as constants for this purpose. Using the information given above we can clearly evaluate these constants for the required range. These turn out to be

$$A = 3.05 \ , \quad B = -67.25$$

Hence in the region of 4 solar masses, which is the region we are interested in, the relation between luminosity and mass for stars on the main sequence is

$$\operatorname{Log} L = 3.05 \operatorname{Log} M - 67.25 \tag{2}$$

We note that this agrees very well with the rough relation given by Allen [11] for this part of the main sequence with masses greater than the solar mass of

$$L \propto M^3$$

As we have already concluded that the evolutionary tracks for all the stars in this mass range are similar to each other in shape, differing only in position, a relation similar to the above equation (2) must hold for any set of equivalent points on all the tracks, the only difference being in the constant B . This is so because on moving along a track we keep the mass fixed and to move to any equivalent point on any track we just add the appropriate amount of luminosity and effective temperature, this amount being the same for all tracks as all the tracks are similar.

Thus for any set of equivalent points (for example the helium burning points or the main sequence turn-off point) a relation of the type

$$\log L = 3.05 \log M + B \qquad (3)$$

exists. For the special case of the join with the main sequence the value of B becomes -67.25 .

By the use of equation (2) we can clearly obtain the point at which the evolutionary track of any given mass leaves the main sequence (since the equation gives the star luminosity and the main sequence is known, thus giving a unique point). Hence the shape and position of the evolutionary track for any given mass in the range now under consideration is now known to us. There only remains for us to determine the position of a star on one of these fixed evolutionary tracks after a given period of time has elapsed since the formation of the star.

In the evolutionary track already determined by us for a star with a mass $3.89 M_{\odot}$, points have been inserted indicating where a star has advanced after the passing of the period of time shown, the unit for this measurement of the time being taken as the time required before helium burning begins. In view of our assumption about the similarity of all the evolutionary tracks concerned it follows that the ratio of the time taken to reach any specific point on a track to the time taken to reach any other specified point will be the same for all tracks. In particular, the time taken to reach a given point on one track, using the time for helium burning as a unit, must be the same as that required to reach a corresponding point on another track (now using the time for helium burning on this track as a unit of course). In other words, if we take the time for helium burning as an unit of time for a track, the relative position of astar on any track (in particular the track for the mass $3.89M_{\odot}$) would be the same after any interval of time x , where x is expressed in the units of time defined above. Hence the position of a star on a given evolutionary track after a given interval of time can be determined if we can obtain the unit of time we have defined, that is the time required before helium burning begins, for this particular track.

Clearly the time required by a star to reach any given point on its evolutionary track, which really means the time required by a star to burn, or transform, a certain part of its interior, is inversely proportional to the amount of material to be burnt, that is its mass, and directly proportional to the rate at which burning takes place, that is its luminosity, hence

$$t = C . \frac{M}{L} \tag{4}$$

where obviously t is the time required, M is the stellar mass and L its luminosity, C being a proportionality constant.

We have already obtained equation (3) which connects the luminosity of a star at any point with its mass, which was

$$\log L = 3.05 \log M + B .$$

which can clearly be written as

$$L = D M^{3.05}$$

where D is another constant whose value is 10^B . Substituting this into equation (4)

$$t = \frac{CM}{DM^{3.05}} = E . M^{-2.05} \tag{5}$$

where E is another constant with the value C/D .

Now for the mass $3.89\,M_\odot$ the time taken by a star to reach the helium burning point is given in table 1 as 1.28×10^8 years. Inserting this information into equation (5) gives us the value of the constant E as

$$1.28 \times 10^8 \{3.89\,M_\odot\}^{2.05} \tag{6}$$

and thus equation (5) may be written as

$$t = \frac{1.28 \times 10^8 \{3.89\,M_\odot\}^{2.05}}{M^{2.05}} = \frac{2.07 \times 10^9}{\mathcal{M}^{2.05}} \text{ years.} \tag{7}$$

where $\mathcal{M}$ denotes the mass of the star in solar units as in the prèvious chapter, the time now being given in years.

Thus, to find the position of a star with a given mass in the Hertzsprung-Russell diagram at any given epoch after its formation we proceed as follows.

By means of equation (2) we determine the point on the main sequence from which the evolutionary track of the star originates. The shape of this evolutionary track is the same as the shape of the track for a particular mass given as figure 1 and so the evolutionary track of a star is completely determined. The actual position of the star is somewhere along this track. Using equation (7) the unit of time as defined by us for this particular track, which is the time required by a star to evolve to the helium burning stage, can be found. The age of the star in question can now be expressed as a fraction of this unit of time. The position of this star will now correspond exactly to the position of a star of mass $3.89\,M_\odot$ on its own track after the same fraction of the unit time has elapsed (the position of a star with mass $3.89\,M_\odot$ being known from figure 1 and table 1). Hence we can plot the position of any star with a given mass at any epoch after its formation. We are thus in a position to construct a theoretical Hertzsprung-Russell diagram if we can choose suitable masses and ages for the stars.

What we require is the Hertzsprung-Russell diagram for a stellar cluster in which the stars are formed at regular intervals of time. We must choose some value for this set interval if we are to construct a diagram. We have chosen an interval of 4×10^6 years which appears to be a reasonable time, about the same length of time as is estimated a star of mass comparable to what we are now considering requires to condense from a very low density to its position on the main sequence. Hence we have a new star formed directly after the already formed star has attained stability on the main sequence. We have considered ten stars to be formed in this manner, the total period of time during which formation was taking place, that is the spread in age of the cluster, is thus 3.6×10^7 years (there being 9 periods in between 10 stars).

We have more information about the mass $3.89 M_\odot$ than any other stellar mass and so it is reasonable to take the mass distribution for the stars in the neighbourhood of $4 M_\odot$. We have taken a set of twelve masses, $3 M_\odot$, $3.5 M_\odot$, $3.7 M_\odot$, $3.8 M_\odot$, $3.89 M_\odot$, $4 M_\odot$, $4.1 M_\odot$, $4.2 M_\odot$, $4.3 M_\odot$, $4.4 M_\odot$, $4.5 M_\odot$ and $4.6 M_\odot$.

The time required before a star of mass evolves to the helium burning stage is about 1.28×10^8 years and the time required by all the above masses to evolve to the same stage must be of the same order. As the region we are interested in is near the main sequence turn-off point, just before the helium burning stage is reached we take the ages of the ten sets of stars formed to be 1.28×10^8 yrs, 1.24×10^8, — — — 1.00×10^8, — — — 9.2×10^7 years

By calculating the luminosity and effective temperature of each star under the above conditions we obtain a set of 120 points for the construction of the Hertzsprung-Russell diagram.

Table 4 gives the results of the calculation for the unit of time, as defined by us, for each individual mass. The equation used for this purpose is equation (7)

$$t = \frac{2.07 \times 10^9}{m^{2.05}} \text{ years}$$

Table 5 gives the value of the main sequence luminosity for all the values for the mass of the stars listed above. The equation required for this is equation (2),

$$\log L = 3.05 \log M - 67.25$$

Finally table 6 gives all the stellar ages mentioned above as fractions of the unit of time defined by us for each individual mass and given in table 4.

With this information we should be able to construct the Hertzsprung-Russell diagram along the lines already mentioned (that is by plotting each star individually using the described methods).

When we attempt to do this however we find, as is reasonable to expect, that some of the above stars have evolved beyond the helium burning point and thus have an age, expressed in the units described, greater than unity. No information has been given about this stage in the evolution of a star and we were unable to find any results about this part of the evolutionary track. This region is not the one we are primarily interested in during this discussion as we are mainly concerned with the observed 'continuation' to the main sequence. It would however give the constructed diagram a more 'completed' look if we could include a few stars in this region past the helium burning stage. Consequently crude efforts have been made by us to evaluate some of these points. This method depends on knowing what the Hertzsprung-Russell diagram for stars all of the same age in this region looks like.

Arguing from very general grounds it is obvious that no violent changes will occur as helium burning begins as this burning will be very slow at first. Consequently we would not expect any discontinuities or sudden changes in direction in the evolutionary track. We shall thus assume that the track after helium burning is a smooth continuation of the pre-helium burning stage. (This is not too improbable an assumption as no reason exists for supposing a decrease in luminosity when helium burning begins and as this evolutionary track must clearly intersect the observed locus of stars having a constant age it cannot move upwards too rapidly; thus broadly speaking the track must be a continuation of the pre-helium burning part.)

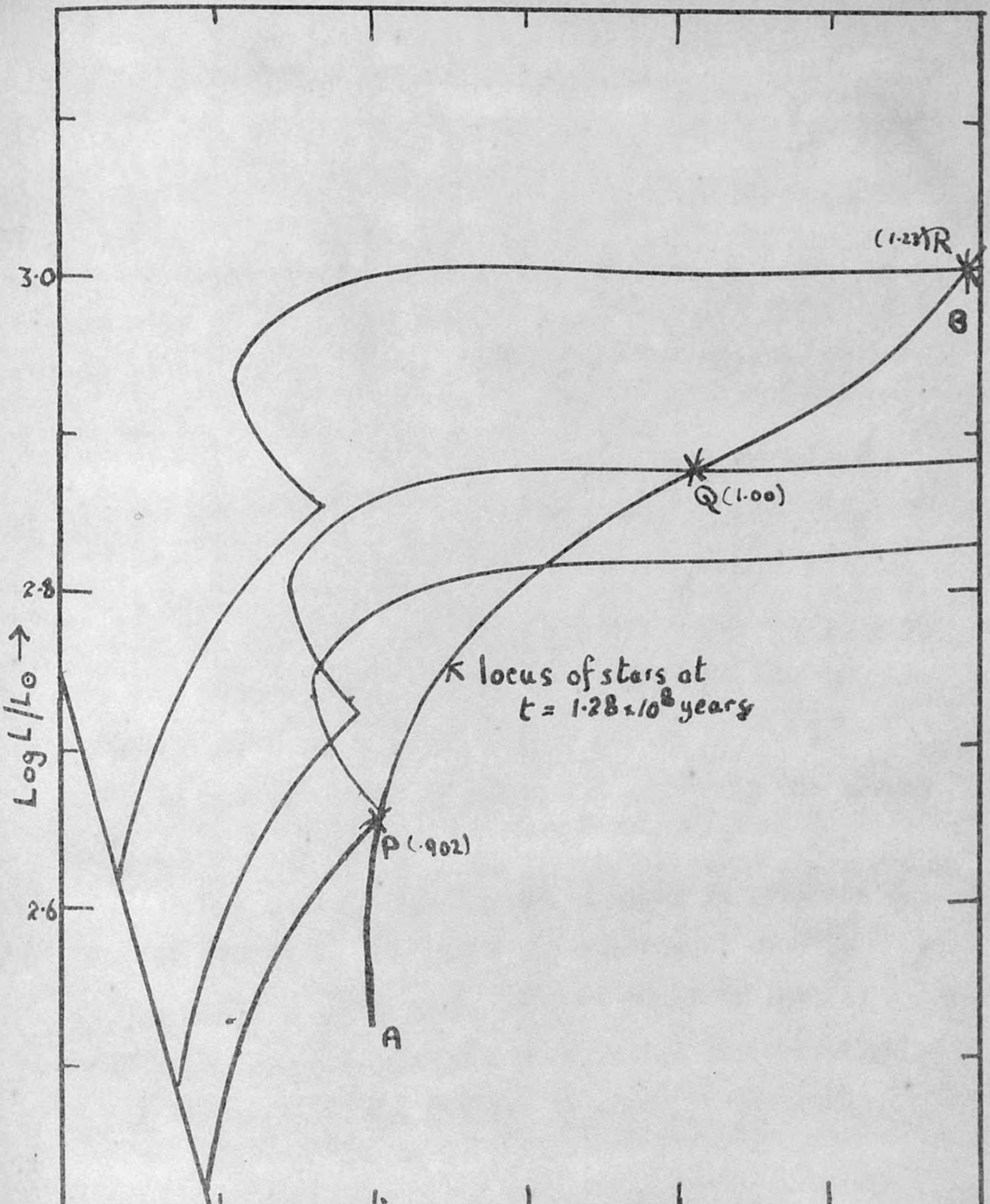
What we now have to do is to determine how far along this track a star has evolved in a given time.

Now the locus of stars of a given unique age but different masses is well known from observations of stellar clusters. As the age changes so does the position of this locus, but not its shape. The position of a star on the evolutionary track is clearly given by the intersection of its own evolutionary track with the constant age locus mentioned above.

Thus if we can determine the position of this constant age locus, we have solved the problem. But the position of stars for any age up to the helium burning is given, and the lower end of the locus is thus fixed by just joining the position of the masses at the required time. So we obtain the position of the stars beyond helium burning point.

We have included figure 5 in order to clarify this method. AB is the locus of stars with a unique

Fig 5 - Points beyond the Helium burning stage



age. The three other curves are the evolutionary tracks for three stellar masses. P and Q are the positions of two of these stars at a given epoch, AB must pass through these points. The interception of AB with the third curve, R , gives the position of a star of this mass at the given epoch. Repeated application of this method gives us the position of any star beyond the helium burning point at any epoch.

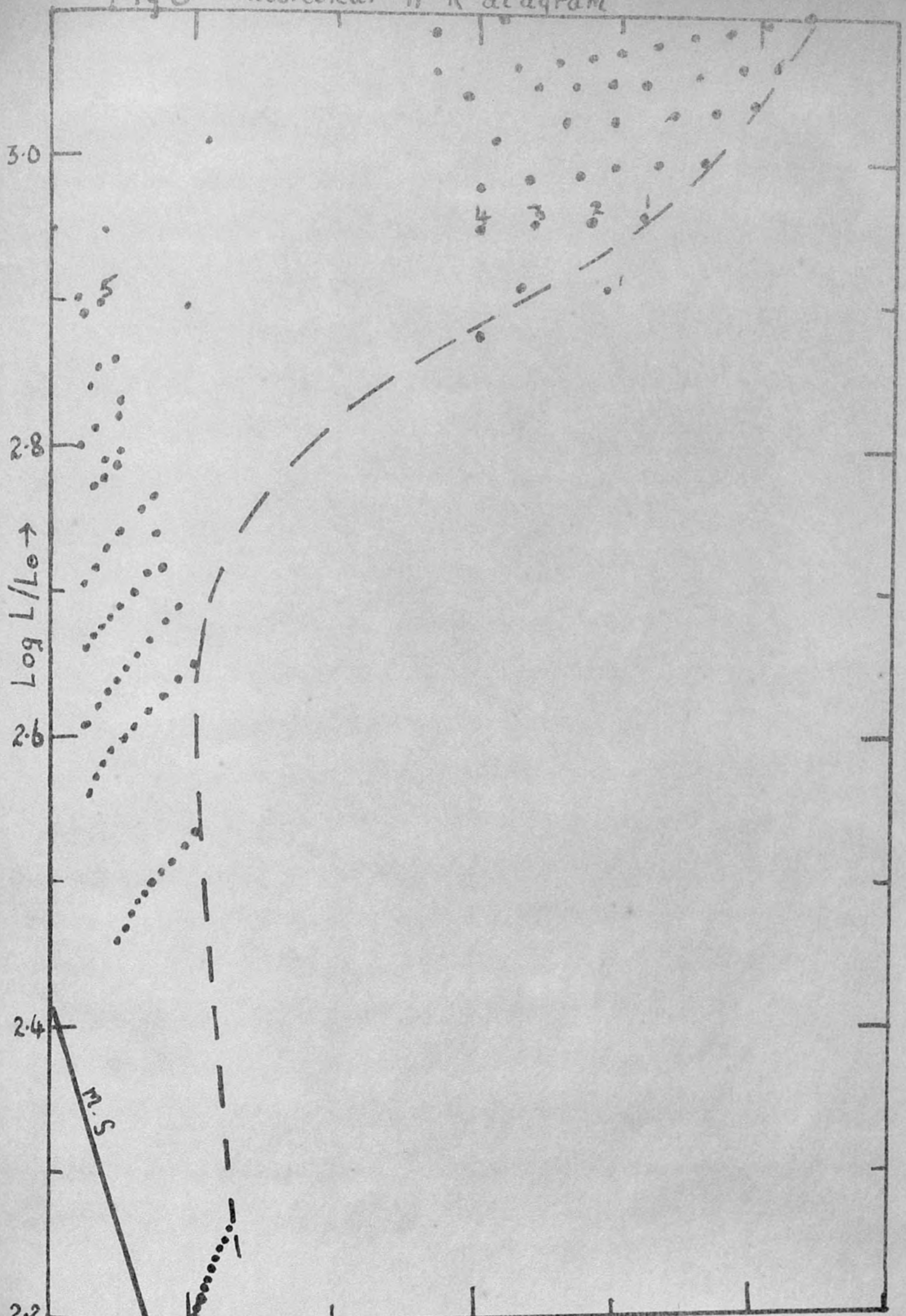
We can now plot the theoretical Hertzsprung-Russell diagram for all the stars. The evolutionary track for each individual mass is plotted using table 5 to give its position and its similarity with figure 1 to give its shape. The position of a star with the above mentioned ages are plotted making use of the correspondence between their position and age using the helium burning age as a unit.

The Hertzsprung-Russell diagram so obtained is given as figure 6. We leave any discussion of this diagram until a later part, when we shall be able to compare it with the diagrams constructed from observational facts about the stellar clusters in question.

Masses in the neighbourhood of one solar mass

We have noted that the evolutionary tracks for

Fig 6 Theoretical H-R diagram



stars in this mass range are different from the tracks in the other mass range just discussed. We see however that figures 2 and 3 are fairly similar to each other and we shall thus only consider stars evolving along one of these tracks, figure 2. The results obtained if we were to use figure 3 would evidently be very similar to those we shall now obtain.

As before we can assume that the evolutionary tracks are homologous to each other for a small mass range on either side of the mass of $1.09\ M_\odot$ (whose evolutionary track is given as figure 2). Hence we can employ the same methods to obtain the theoretical Hertzsprung-Russell diagram for this mass range as was described in detail for the last mass range.

From the paper by Hazelgrove and Hoyle [13] the information about the main sequence for stars in this mass range is as follows

Mass (10^{33} gms)	Luminosity (10^{33} ergs/sec)	Effective Temperature (°K)
2.5592	9.8896	7030
2.3545	6.3535	6565
2.1699	5.1202	6390
1.9963	2.6958	5880

As the whole time scale is now changed, the time that elapses before each generation of new stars is formed (the interval between the ages of the stars) must also be changed. If we were to use the same interval of time as we had before, the whole time span of the formation of twelve stars would now only be about 1/300 of the time required by any one of the stars to evolve to the main sequence turn-off point. For this type of star we have thus taken the interval of time between each generation of stars in the cluster to be 3×10^8 years, and their actual ages in the region of interest now becomes 9.24×10^9, 8.94×10^9, --- 6.24×10^9 and 5.94×10^9 years. The total span in time while generation of stars is taking place in the cluster is now about 1/3 of the age of the eldest stars in the sets we have taken.

For this case we have taken the stellar masses to be 0.8 , 0.9 , 1.0 , 1.09 , 1.1 , 1.2 and 1.3 solar masses.

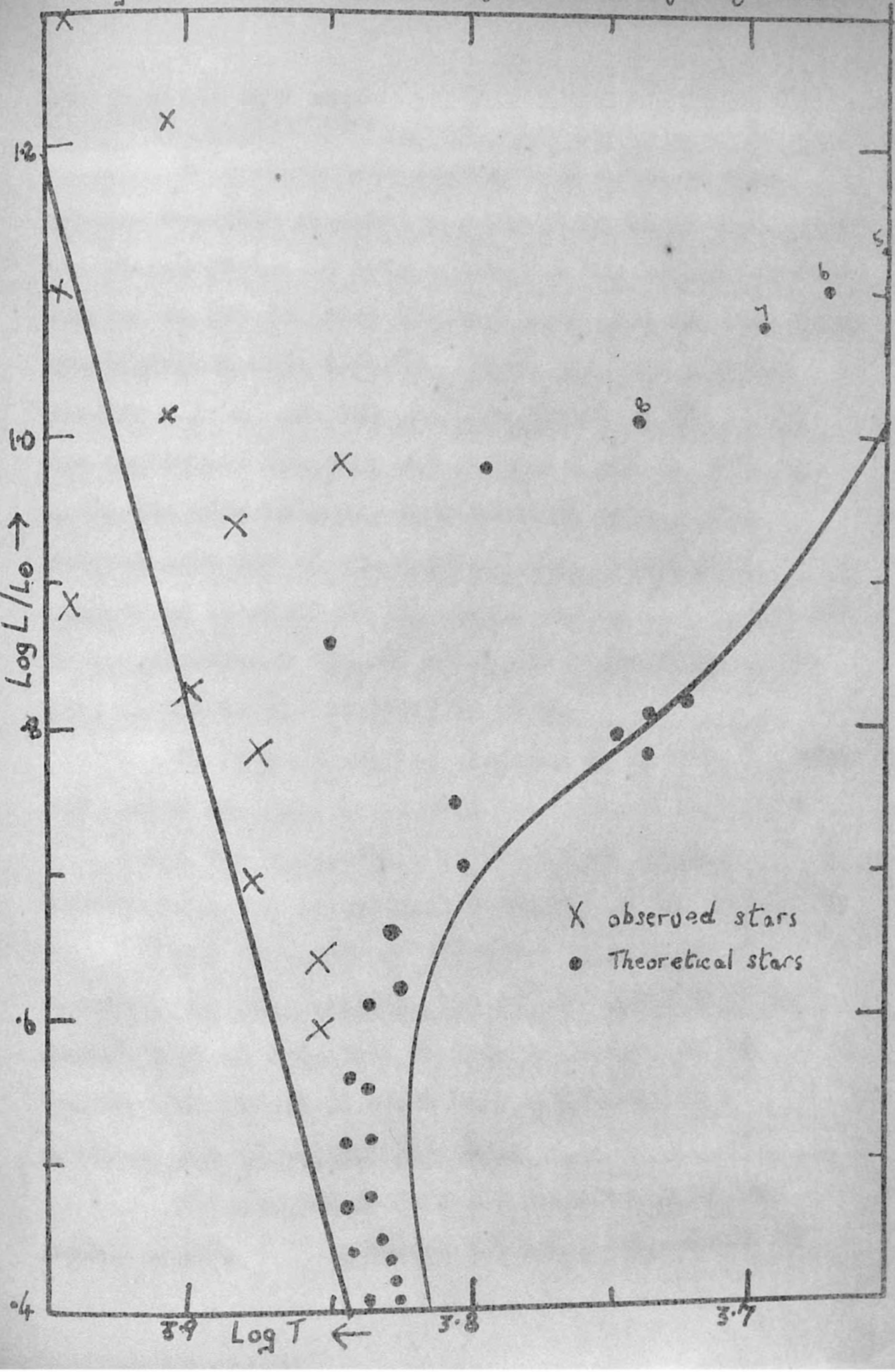
Table 7 gives the information connecting mass, luminosity and the unit of time. It is similar to a combination of tables 4 and 5 for the previous mass range and its derivation was completely analogous to the derivation of tables 4 and 5 of the previous case.

Table 8 was obtained in a similar manner to table 6 and contains information about the ages of the 84 stars mentioned above, the unit of time in each case being the time required before helium burning begins, as given in table 7.

Proceeding in a similar manner to what has already been described in connection with the previous case, we can now construct a theoretical Hertzsprung-Russell diagram for the set of stars given above. This diagram is given as figure 7. A discussion of this diagram is also left until after we have given the observational data, when comparison with these becomes possible.

In both this case and the previous one, if we had constructed the diagram around any other masses but the ones we have taken, the diagrams we would have obtained would clearly be similar in appearance to the ones we have actually obtained, the only difference would be in the absolute position of the stars, not in their position relative to each other. Hence we are justified in comparing the diagrams we have obtained with diagrams constructed from observational data if we take no account of any positional differences that might exist.

Fig 7 - Theoretical H-R diagram in the region of 1·09M⊙



The observational data

References have already been given to some stellar clusters in which the phenomena under discussion can be observed. We have reproduced the colour-magnitude diagram of two of these clusters that show the star under consideration very clearly. These are, the globular cluster M 3 [2] and the galactic cluster M 67 [1] and the reproduced diagrams are figures 8 and 9. However, as is the case with all observational data, these observations are of the apparent visual magnitude, V , against an estimate of the colour index, $B-V$, and not of the luminosity against effective temperature as the plot theoretically produced by us is.

In another chapter (chapter 2) we have described the method employed to convert the apparent magnitude (V) and the measurement of the colour index ($B-V$) results into the theoretical logarithm of the luminosity ($\log L$) and logarithm of effective temperature ($\log T$) results. We will however give a brief account of the method here as this will be more convenient to the reader than having to refer back constantly to a previous set of independent work.

The conversion from the measurement of the colour index, $B-V$, to the effective temperature is

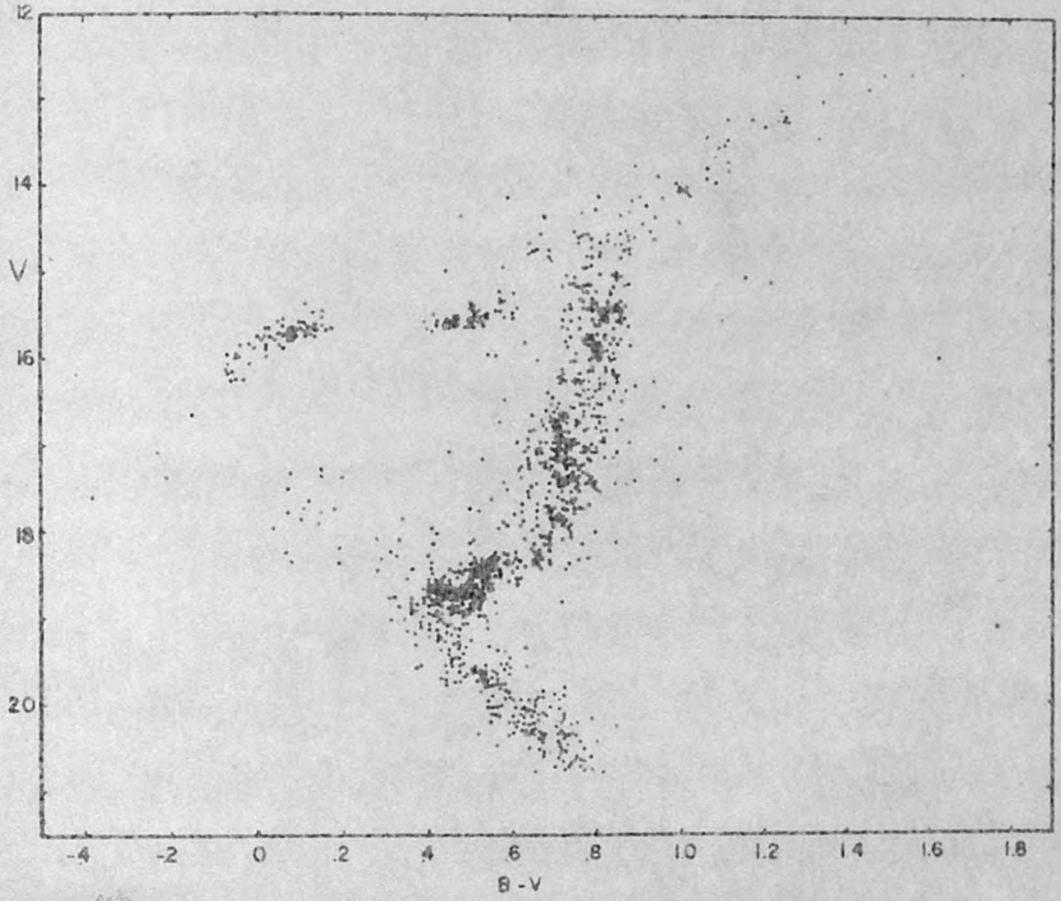


FIG 8 Color-magnitude diagram for M3 stars in the arguments V and $B - V$

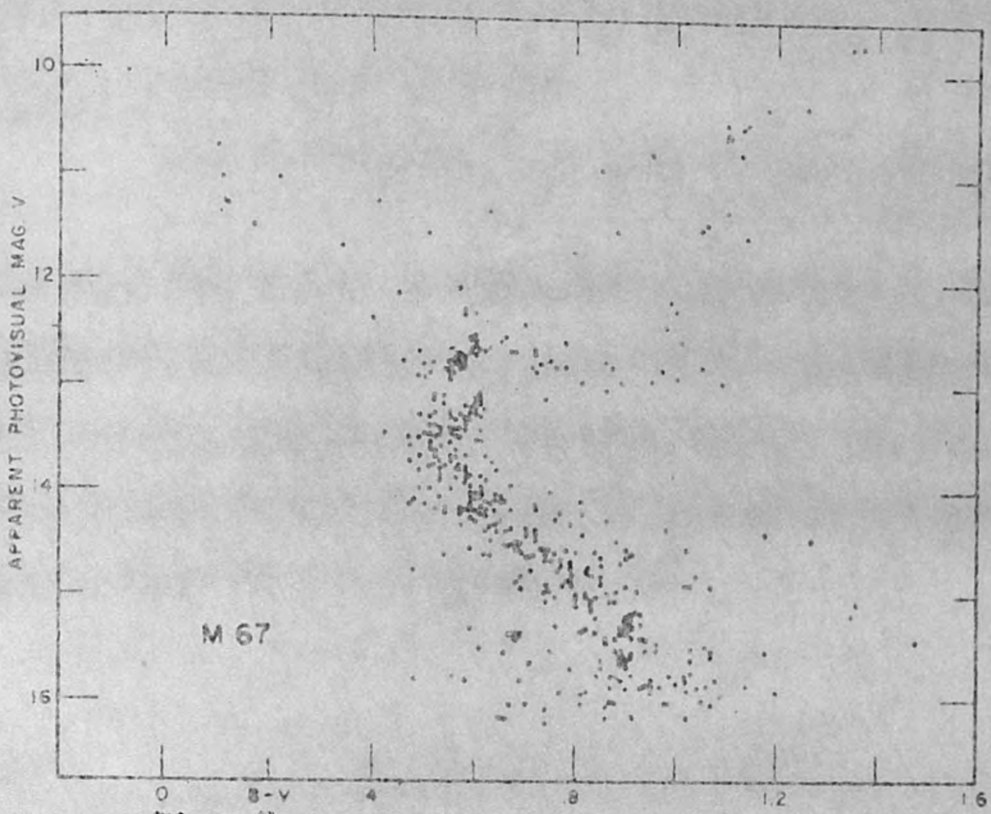


Fig 9 The color-magnitude diagram for M67

carried out by means of tables given in Schwarzschild's book "The structure and evolution of the stars" [14]. These tables also give us the bolometric correction, B.C at any effective temperature.

The bolometric magnitude is then given by

$$m_{bol} = V + B.C \tag{8}$$

where m_{bol} is the apparent bolometric magnitude and V the apparent visual magnitude.

The absolute bolometric magnitude is now given by

$$M_{bol} = m_{bol} - D.M \tag{9}$$

where D.M. denotes the distance modulus.

This distance modulus will be given as part of the data with every set of published results on observations of stellar clusters.

The relation

$$M_{bol} = 4.62 - 2.5 \log L/L_{\odot} \tag{10}$$

where $L_{\odot}$ denotes the solar luminosity, clearly enables us to determine the logarithm of the star luminosity.

As we are not actually interested in the absolute position in the Hertzsprung-Russell diagram of the observed stars, only in the shape of the pattern they form, we need not, for the present purpose, reduce the values of the stellar magnitude that have been observed to the absolute magnitude scale, we can just take

$$M = m_{bol} - \text{Constant} \qquad (11)$$

choosing the value of the constant in such a way that the numerical values of M fit on a suitable and convenient numerical scale when we plot the Hertzsprung-Russell diagram. The logarithm of the luminosity (again non-absolute) is given by

$$M = 2.5 \, \text{Log} \, L/L_\odot \qquad (12)$$

In converting the observed diagram into the diagram using $\text{Log}\, L$ and $\text{Log}\, T$ we have not actually carried out the above conversion for all the stars in the cluster. We have taken selected points on the boundary of the region occupied by stars, plus any star that appears to be of interest and converted them in the described manner to the $\text{Log}\, L$, $\text{Log}\, T$ diagram. This we have done for both the above mentioned stellar clusters, M 3 and M 67. The results are given as figures 10 and 11. Some of the more interesting points have also been inserted into figure 7.

Actually in these figures we have not

Fig 10 Observed data for M3

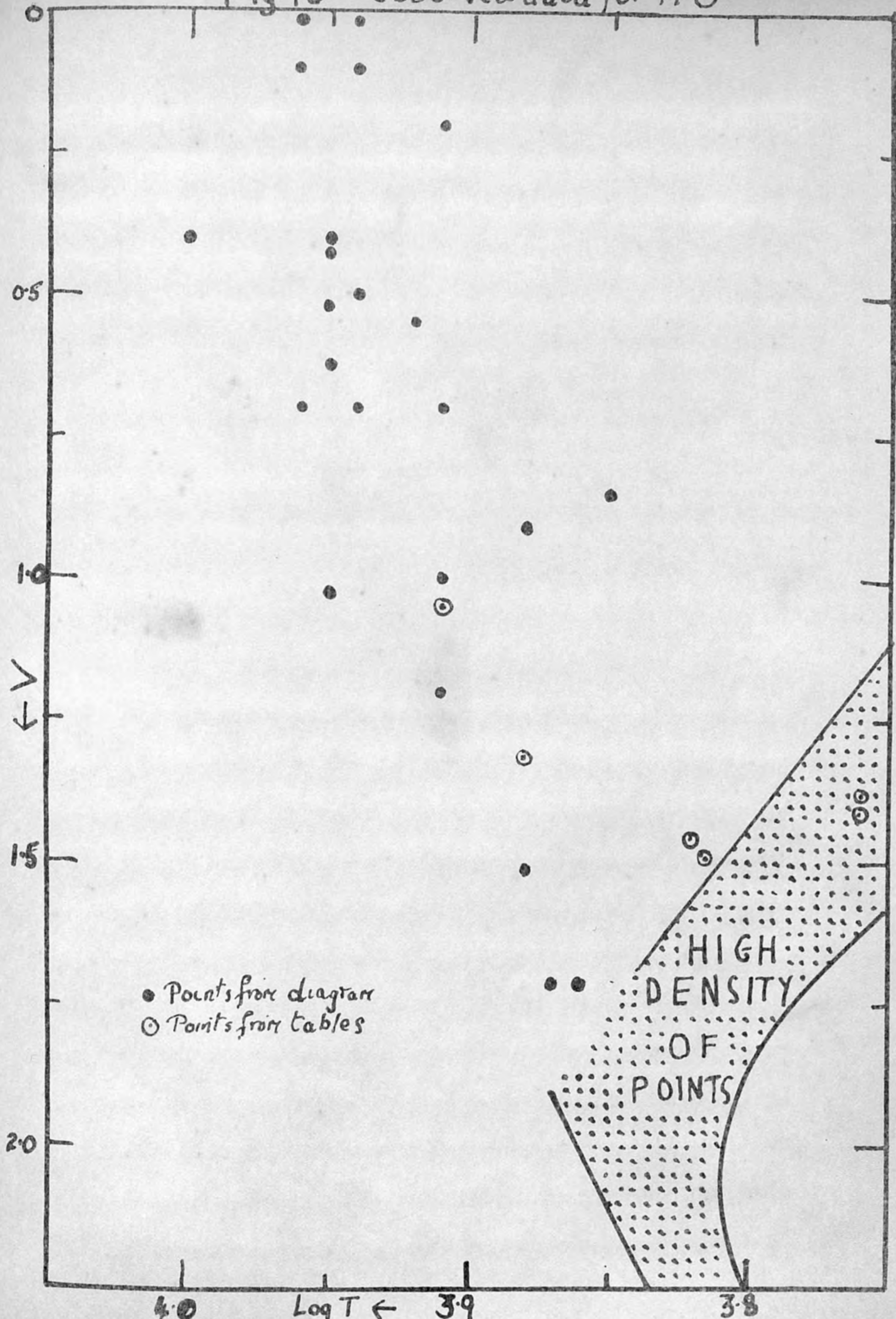
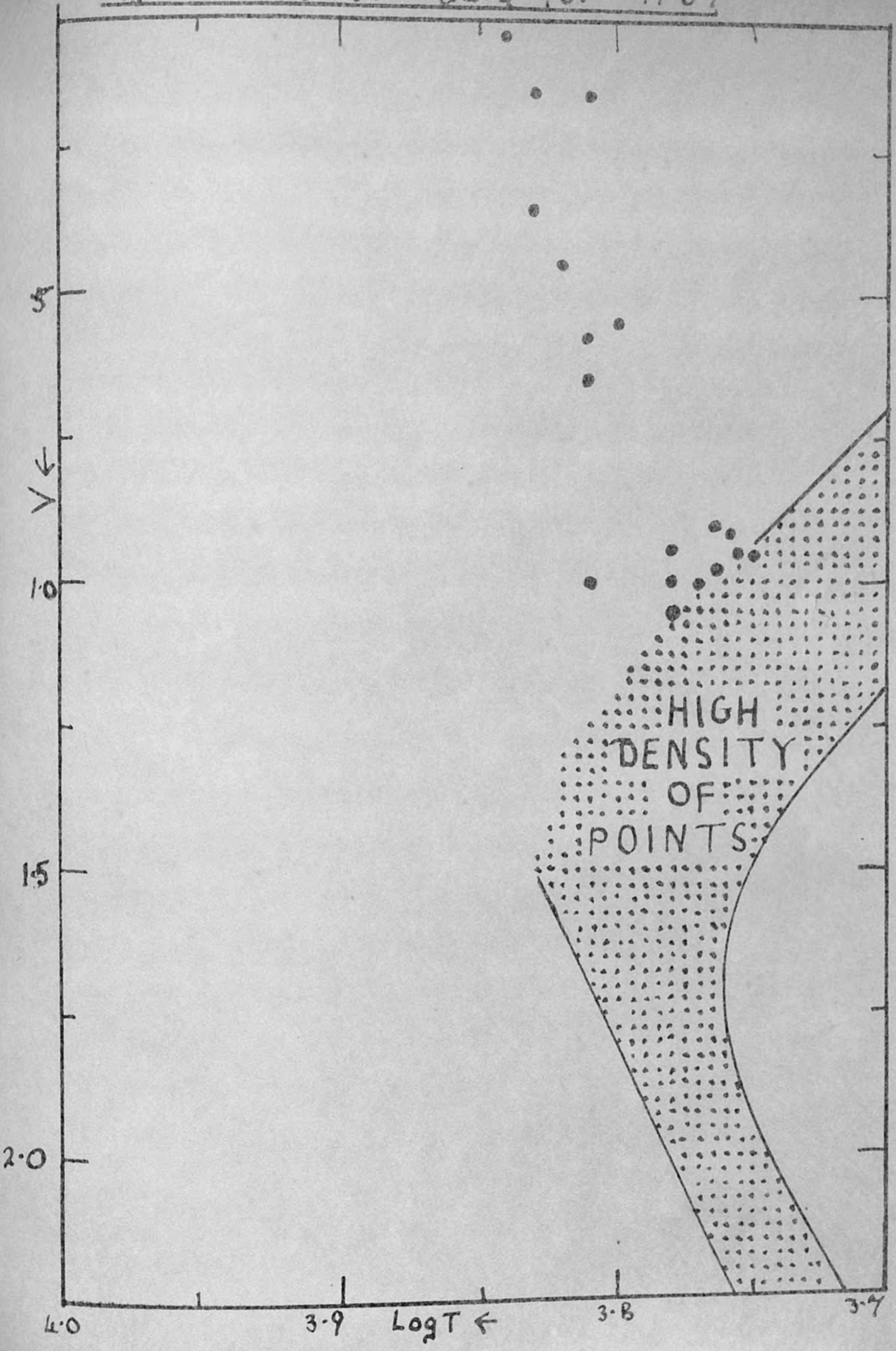


Fig 11. Observed data for M67



converted the apparent visual magnitude, M_v to $Log\ L$, only to the quantity M defined in equation (11) above. We have then chosen the scale for this quantity M to be 2.5 times the scale we have used in constructing the theoretical diagram in $Log\ L/L_{\odot}$. By doing this we have avoided the necessity of dividing each value for the stars by 2.5 while at the same time producing a diagram that has the same ordinate scale as the theoretical one.

Comments on the diagrams and comparison of results.

The Hertzsprung-Russell diagrams constructed from observational data about the stars in the two stellar clusters M 3 and M 67 (figures 10, 11) show a distinct turn-off point from the main sequence. There is no indication of the existence of a Hertzsprung gap and the width of the star bands in all the regions of the diagram are roughly the same. That is, the main sequence, giant branch and the main sequence 'continuation', (which we are interested in) all have roughly the same spread. On the other hand we note that the main sequence and giant branch are both very thickly populated in comparison with the main sequence 'continuation'. These are the main points that are evident in the diagram constructed from observational data and these are thus the points which any theoretically constructed diagram should possess if it is to give a good representation of the facts.

The diagram we have constructed for stars with a mass greater than one solar mass, figure 6, does exhibit a continuation to the main sequence similar to what is required. This continuation is not as extended as the one found in the observational diagram, but this could be because we have allowed creation to continue for too short an interval of time. On this count we must conclude that our theoretical diagram is in fair agreement with the diagram constructed from observations.

In this theoretical diagram however there exists a very distinct Hertzsprung gap, extending roughly from $Log\,T = 4.2$ to $Log\,T = 4.1$. This is in complete disagreement with the observed facts.

The width of the stellar band along the main sequence appears to be fairly uniform and compares well with the observed main sequence, but there is much more spread in the giant branch and so this again is in disagreement with the diagram produced from observational data. The 'continuation' to the main sequence is comparable in width to the observed continuation.

In the theoretical diagram all three branches mentioned above are roughly equally populated while in the observational diagram the 'continuation' is very sparsely populated compared with the other two branches. Thus on yet another point the theoretical is in complete disagreement with the observational data.

Hence the only sensible conclusion we can come to is that the theoretical Hertzsprung-Russell diagram constructed by us for stars with a mass greater than the solar mass, and exhibited as figure 6, does not agree with the observed facts.

We now turn to the second theoretical diagram constructed by us, the one for stars with a mass in the neighbourhood of one solar mass; this was exhibited as figure 7. No Hertzsprung gap appears to exist in this diagram, so on this point we appear to have agreement with the observational diagram.

Again the spread of the star along the giant branch is much wider than spread along the main sequence and so we have a disagreement here between observations and theory.

The most important failing in this second theoretical diagram however is that the 'continuation' to the main sequence which is the phenomena we have been trying to explain, does not exist. Indeed, there is no trace of even a tendency in the diagram for the formation of such a continuation.

There can thus be no doubt at all about our conclusions regarding this diagram for masses in the neighbourhood of one solar mass. The theoretical diagram does not meet with the requirements of the diagram constructed from observational data and so the observed facts could not be caused in the way proposed by the theory.

Thus neither of the two possible theoretical Hertzsprung-Russell diagrams appear to agree with the diagram produced from the observed facts and hence it is true to conclude that the 'continuation' to the main sequence observed in some clusters is not formed by the existence of a spread in the creation epoch of the stars in the cluster.

The allowable spread in the ages
<u>of the stars</u>

We have concluded above that the Hertzsprung-Russell diagrams we obtain by allowing a spread in the formation epoch of the stars in a cluster do not agree with the observed diagrams of such clusters. However it is clearly ridiculous to suggest that all the stars in any particular cluster were formed at exactly the same instant and hence, inany normal cluster, there must be some spread in the ages of the stars. It will thus be interesting to investigate how large a spread can be allowed before discrepancies appear between the observed cluster diagram and the one obtained allowing for the spread in ages.

Clearly we can deduce this information from the diagrams we have already constructed. All we have to do is to find which of the stars lie outside the boundary of the regions occupied by stars in the observational diagram and find which of these stars is the closest in age to the oldest stars in the set. The 'allowable time spread' would then be the difference in ages between these stars. In figures 6 and 7 we have numbered one set of stars to show which one is the first to leave the region occupied by the observed stars.

For figure 6 this turns out to be the fifth star, and as the time interval between each star is 4×10^6 years, the 'allowable time spread' is thus 1.6×10^7 years. The age of the eldest star in this set is about 1.3×10^8 years and so the 'allowable spread' is about 12% of the age of the eldest cluster members.

For figure 7, (stars with mass in the region of one solar mass) again the star closest to the boundary of the region occupied by observed stars turns out to be the fifth star formed. In this case the interval of time that elapses before each star is formed is 3×10^8 years and so we now have an 'allowable time' of 1.2×10^9 years. The age of the eldest star now however is 9.2×10^9 years and so in this case the greatest spread that can be allowed is 13% of the age of the eldest stars, a very similar number to that obtained for the previous case.

We can thus conclude that a spread in the epoch of creation of the stars in a stellar cluster of about 10% of the age of the eldest stars in this cluster can be allowed before any discrepancies from the usual diagrams observed for clusters become apparent.

Conclusions

From the comparison between the theoretical Hertzsprung-Russell diagrams we have produced and the diagram obtained from observational data we have concluded that it is not possible to produce the observed continuation to the main sequence by allowing a spread in the formation epoch of the stars, and hence the presence of this continuation does not indicate the breaking down of the hypothesis that all stars in a cluster are approximately of the same age, a far more satisfactory state of affairs than if we had been able to solve the origin of these blue stars at the expense of dropping the hypothesis.

We have also shown that a spread in age of about 10% of the age of the eldest stars can be allowed in any cluster before any discrepancies with the normal observed cluster diagrams become evident.

As for the cause of the blue stars forming the continuation we have been discussing, we note that our conclusions do not exclude the possibility that the great majority of stars in a cluster are all of the same age while a few stars of a much lesser age could form the continuation. Other suggestions that have been offered as the cause of these stars are that much more mixing than normal takes place in their interior and so the hydrogen that is being consumed at the centre keeps on being replenished by supplies carried in from the outer region. This allows the star to stay on the main sequence for a much longer period than usual. It might also be possible that these stars, in some manner, are accreting material and are thus increasing in mass. They have not consumed the required amount of hydrogen to evolve off the main sequence because the major part of the star is a comparatively late arrival, as it was accreted.

This still allows an uncertainty about the cause of these stars that cause the continuation to the main sequence in the Hertzsprung-Russell diagram but by means of this work we hope at least to have disposed of one possibility, while at the same time strengthening the belief that every stellar cluster possesses a unique age that can be determined by studying the turn-off point from the main sequence.

TABLE 1

Evolutionary track of star of Mass 3.89 $M_\odot$ $X = .75$ $Y = .23$ initially

Age 10^{15} secs	Age using H_e-burning age as unit	Luminosity $L, 10^{36}$ erg/sec	Surface Temp Te ^oK	Log $^L/_{L_\odot}$	Log Te
4.033	1	2.916	12540	2.877	4.098
4.028	.999	2.922	14390	2.879	4.157
4.022	.997	2.893	14910	2.875	4.173
4.013	.995	2.845	15480	2.867	4.190
4.001	.992	2.784	15930	2.858	4.202
3.988	.989	2.726	16240	2.849	4.211
3.973	.985	2.663	16490	2.839	4.217
3.957	.981	2.593	16680	2.828	4.222
3.941	.977	2.532	16800	2.817	4.225
3.924	.973	2.467	16860	2.805	4.227
3.906	.969	2.401	16860	2.794	4.227
3.887	.964	2.336	16810	2.782	4.226
3.867	.959	2.272	16700	2.770	4.223

(cont'd.)

TABLE 1 (cont'd.)

3.846	.954	2.213	16550	2.758	4.219
3.825	.949	2.170	16390	2.750	4.215
3.803	.943	2.135	16220	2.743	4.210
3.769	.935	2.095	16030	2.736	4.205
3.725	.924	2.073	16100	2.730	4.207
3.507	.870	1.974	16510	2.709	4.218
3.230	.801	1.867	16930	2.684	4.229
2.743	.680	1.707	17440	2.645	4.241
2.362	.586	1.600	17710	2.617	4.248
1.835	.455	1.479	17940	2.583	4.254
1.088	.270	1.337	18190	2.540	4.260
0	0	1.177	18340	2.484	4.263

TABLE 2

Mass 1.09 $M_\odot$ Type I $X = .75$ $Y = .24$

Age 10^{17} secs	Age, H_e-burning age as unit	Luminosity $L(10^{36}$ erg/sec)	Surface Temp Te °K	Log $L/L_\odot$	Log Te
2.915	1.0	22.78	5335	.771	3.727
2.913	.999	22.41	5375	.764	3.730
2.910	.998	22.14	5425	.759	3.734
2.906	.997	21.97	5495	.756	3.740
2.902	.996	22.05	5580	.757	3.747
2.898	.994	22.23	5660	.760	3.753
2.893	.993	22.40	5740	.764	3.759
2.889	.991	22.52	5815	.766	3.765
2.877	.987	22.41	5945	.764	3.774
2.863	.982	21.87	6095	.753	3.785
2.845	.976	20.99	6230	.735	3.794
2.821	.968	19.88	6350	.712	3.803
2.790	.957	18.62	6445	.683	3.809

(cont'd.)

TABLE 2 (cont'd.)

2.746	.942	17.17	6535	.648	3.815
2.683	.920	15.52	6605	.604	3.820
2.585	.887	13.63	6645	.548	3.822
2.414	.828	11.41	6655	.471	3.823
2.043	.701	9.058	6620	.370	3.821
1.226	.421	6.200	6455	.206	3.810
0	0	5.120	6390	.123	3.805

TABLE 3

Type II star   Mass 2.6856×10^{33} gms   X = .99   Y = .009

Age 10^{17} secs	Age H_e-burning Age as unit	Luminosity $L(10^{36}$ erg/sec)	Surface Temp Te ^{o}K	Log $^{L}/_{L_\odot}$	Log Te
3.894	1	30.62	5490	.899	3.740
3.875	.995	27.66	5630	.855	3.750
3.853	.989	27.72	5975	.856	3.776
3.822	.981	26.36	6250	.834	3.796
3.780	.971	24.52	6425	.803	3.808
3.722	.956	22.47	6550	.765	3.816
3.581	.920	19.07	6655	.694	3.823
3.503	.900	17.19	6665	.649	3.824
3.243	.833	14.28	6650	.568	3.823
3.049	.783	12.47	6600	.509	3.819
2.789	.716	10.72	6535	.444	3.815
2.659	.683	10.00	6505	.413	3.813
2.465	.633	9.095	6460	.372	3.810

(cont'd.)

TABLE 3 (cont'd.)

2.205	.566	8.154	6410	.325	3.807
1.816	.466	7.069	6335	.263	3.802
1.557	.400	6.507	6290	.227	3.799
1.038	.266	5.654	6210	.166	3.793
.5189	.133	5.009	6135	.113	3.788
0	0	4.492	6065	.066	3.783

TABLE 4

The relation between mass and time for helium burning for masses in the region of 4 solar masses.

Mass (solar units)	Time x 10^8 (years)	Mass (solar units)	Time x 10^8 (years)
		4.1	1.15
3.0	2.22	4.2	1.09
3.5	1.59	4.3	1.04
3.7	1.42	4.4	.992
3.8	1.34	4.5	.95
3.89	1.28	4.6	.91
4.0	1.21		

TABLE 5

The relation between mass and luminosity in the region of 4 solar masses.

Mass (solar units)	Log $\frac{L}{L_\odot}$	Mass (solar units)	Log $\frac{L}{L_\odot}$
		4.1	2.56
3.0	2.14	4.2	2.59
3.5	2.34	4.3	2.62
3.7	2.42	4.4	2.65
3.8	2.45	4.5	2.69
3.89	2.48	4.6	2.71
4.0	2.52		

TABLE 6

Specified ages of stars as proportion of Helium burning time for each star

TIME	10^8 yrs	1.28	1.24	1.20	1.16	1.12	1.08	1.04	1.00	..96	.92
PROPORTION FOR STATED MASS	3.0	.576	.558	.540	.522	.504	.486	.468	.450	.432	.414
	3.5	.805	.780	.755	.730	.704	.679	.654	.629	.604	.579
	3.7	.902	.873	.845	.817	.789	.760	.732	.704	.676	.648
	3.8	.952	.923	.893	.863	.833	.803	.774	.744	.714	.684
	3.89	1	.968	.937	.905	.874	.844	.812	.781	.750	.719
	4.0	1.06	1.02	.992	.957	.926	.892	.859	.826	.793	.760
	4.1	1.11	1.03	1.05	1.01	.976	.939	.904	.870	.835	.800
	4.2	1.175	1.14	1.10	1.06	1.02	.988	.952	.915	.879	.842
	4.3	1.23	1.19	1.15	1.11	1.07	1.03	.997	.959	.920	.882
	4.4	1.29	1.25	1.21	1.17	1.13	1.09	1.05	1.01	.967	.927
	4.5	—	1.29	1.26	1.22	1.18	1.14	1.09	1.05	1.01	.968
	4.6	—	—	—	1.26	1.22	1.18	1.14	1.10	1.05	1.01
	5.0	—	—	—	—	—	1.41	1.36	1.31	1.26	1.20

TABLE 7

Relation between the mass, luminosity on main sequence, and time for helium burning for a star in the neighbourhood of 1 solar mass.

Mass (solar units)	Log ($^L/_{L_\odot}$)	Time (years x 10^9)
0.8	−.548	31.8
0.9	−.293	19.9
1.0	−.064	13.0
1.09	0.123	9.23
1.1	0.143	8.91
1.2	0.332	6.29
1.3	0.505	4.57

TABLE 8

Specified ages of stars as proportion of helium burning time

AGE	PROPORTION FOR STATED MASS (SOLAR UNITS)						
10^9yrs	0.8	0.9	1.0	1.09	1.1	1.2	1.3
9.24	.290	.464	.711	1	1.04	1.47	--
8.94	.281	.449	.688	.967	1.00	1.42	--
8.64	.272	.434	.665	.935	.970	1.37	--
8.34	.262	.419	.641	.903	.936	1.33	--
8.04	.253	.404	.618	.870	.902	1.28	--
7.74	.243	.389	.595	.838	.869	1.23	--
7.44	.233	.374	.572	.805	.835	1.18	--
7.14	.224	.359	.549	.773	.801	1.13	--
6.84	.215	.344	.526	.740	.768	1.09	1.47
6.54	.206	.329	.503	.708	.734	1.04	1.43
6.24	.196	.314	.480	.675	.700	.992	1.37
5.94	.187	.298	.457	.643	.667	.944	1.30

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PRE-MAIN-SEQUENCE STARS

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Various observers have discovered some apparently very young galactic clusters. In many of these, stars fainter than some particular luminosity L_c, say, lie to the right of the main sequence in the HR diagram. Naturally, this is interpreted as showing that these stars have not yet had time to condense on to the main sequence. Indeed, if t_{gc} is the calculated time for gravitational contraction of a star from vanishingly small density to luminosity L_c on the main sequence, the value of t_{gc} is found to be in fair agreement with other estimates of the age of the cluster concerned.

Had all members of a cluster started, at one and the same epoch, to condense from vanishingly small density, then those now having luminosity fainter than L_c would be expected to lie upon a certain calculable locus in the HR diagram. However, in the cases studied, the stars concerned are found to lie, not on this locus, but much nearer to the main sequence. That is to say, the gravitational contraction of the fainter stars in the cluster appears to have progressed much further than is possible within the lifetime of the cluster. All who have written on the subject regard this as presenting a very serious problem, and some very drastic solutions have been proposed. Su-Shu Huang (S.S.H.)[1] has usefully reviewed the situation; he gives references to the literature of the subject.

We wish to point out that it is not possible to say whether there is a real difficulty without knowing more than we do about the early stages of the process of formation of the stars concerned. We have to contrast the present problem of stars condensing on to the main sequence at its lower end with that of stars evolving off the main sequence at its upper end. In the latter problem the initial state is well-defined. In the case of condensing stars, on the other hand, we do not yet know what have to be treated as initial conditions.

The situation may be illustrated by the example we are about to give. This is intended to support the conclusions to be stated below; it is not suggested that it arises from any particular theory of star-formation. In fact, we adopt the treatment given by S.S.H. and, merely for the sake of argument, we shall exhibit the consequences of supposing that at some epoch all the stars concerned have the same mean density.

Following S.S.H., we assume that a star of mass $\mathscr{M}$ undergoing homologous gravitational contraction towards the main sequence has luminosity L when its radius is R, where

$$L=\mathscr{M}^{\alpha}/R^{\beta}. \tag{1}$$

Here α, β are constants; $\mathscr{M}$, L, R, as well as the effective temperature T and the mean density ρ are all measured in solar units. It then follows from (1) that the time τ required for the star to contract from infinite radius to radius R is

$$\tau=(1-\beta)^{-1}\mathscr{M}^{2-\alpha}/R^{1-\beta} \qquad (0<\beta<1). \tag{2}$$

This measure of τ is in the unit of the Helmholtz-Kelvin (HK) time-scale for the Sun. The definitions of ρ, T in the present units are given by

$$L=R^2T^4, \quad \mathscr{M}=\rho R^3. \tag{3}$$

S.S.H. adopts the values

$$\alpha = 5\cdot4, \quad \beta = 0\cdot79. \tag{4}$$

Using (4) in (1)–(3) we derive

$$\log L = 5\cdot14 \log \mathcal{M} + 0\cdot263 \log \rho,$$
$$\log T = 1\cdot12 \log \mathcal{M} + 0\cdot233 \log \rho; \tag{5}$$

$$\log \tau = -3\cdot47 \log \mathcal{M} + 0\cdot07 \log \rho + 0\cdot678; \tag{6}$$

$$\log L = 18\cdot2 \log \mathcal{M} + 3\cdot76 \log \tau - 2\cdot55,$$
$$\log T = 12\cdot6 \log \mathcal{M} + 3\cdot32 \log \tau - 2\cdot25. \tag{7}$$

These equations are not given by S.S.H. although, of course, he must have used them in the construction of his diagram.

In the resulting $\log L$–$\log T$ diagram, a line of fixed density, a line of fixed τ and a line of constant $\mathcal{M}$, which is the evolutionary track for the star concerned, are all straight lines. Such lines are plotted by S.S.H. and some are shown in Fig. 1.

S.S.H. takes the relevant part of the main sequence to be given by

$$L = T^{\gamma} \quad \text{with} \quad \gamma = 7 \tag{8}$$

in the units used here, it being assumed that the Sun lies on the main sequence.

For the purpose of our illustration, we consider a set of pre-main-sequence stars of the present sort that, at epoch $t = t_0$, all have mean density $\rho = 0\cdot01$, or one per cent of the solar mean density. In Fig. 1 we exhibit the loci on which these stars lie at epochs $t_0, \ldots, t_4$ where, in HK units,

$$t_1 - t_0 = 0\cdot05, \; t_2 - t_0 = 0\cdot1, \; t_3 - t_0 = 0\cdot2, \; t_4 - t_0 = 0\cdot4.$$

One HK unit is about 3×10^7 years. However, for our discussion it is more instructive to note that the locus for t_2 meets the main sequence at a point for which roughly $\log \mathcal{M} = 0\cdot28$, $\log L = 1\cdot3$, $\log T = 0\cdot18$, $\tau = 0\cdot48$, $\rho = 0\cdot28$, while the corresponding point in the locus for t_0 for the same $\mathcal{M}$-value has $\tau = 0\cdot38$.

The homologous family is meant to consist of stars of which the luminosity is produced by gravitational contraction alone. Were account to be taken of the release of nuclear energy in the final approach to the main sequence, we should have evolutionary tracks joining on to the main sequence in a somewhat different way. But this is of no importance for present purposes.

It is now to be noted that our locus for, say, $t = t_2$ is roughly like those plotted from observations of young clusters; the value $\tau \doteq 0\cdot5$ just mentioned is of order of the age usually ascribed to such clusters. Thus we are dealing with the sort of situation that is of interest in practice (even though we are not advocating any particular theory of evolution).

The loci τ = constant have, of course, the property that if a set of stars of the present type are represented at some epoch by points on one of these loci, then at any epoch they are represented by points of such a locus (unless they move on to the main sequence). Loci of this type are the only ones normally considered hitherto.

The loci t = constant also, however, have by construction the property that if stars lie on one of them at one epoch then they lie on another of them at any other epoch. The features to notice are: (*a*) The loci for $t_1, \ldots, t_4$

are all quite different from the τ-loci; (*b*) the loci are qualitatively similar to each other; (*c*) they span a time-interval comparable with the estimated ages of the known young clusters.

The physical reasons for these features are evident. The age of a cluster is the time for a star at the lower end of the main sequence just to reach the main sequence from whatever has to be treated as the initial state. Now the rate of contraction is highly sensitive to the mass. Thus, stars of mass much less than the one just mentioned cannot have moved far from the initial state in the time available.

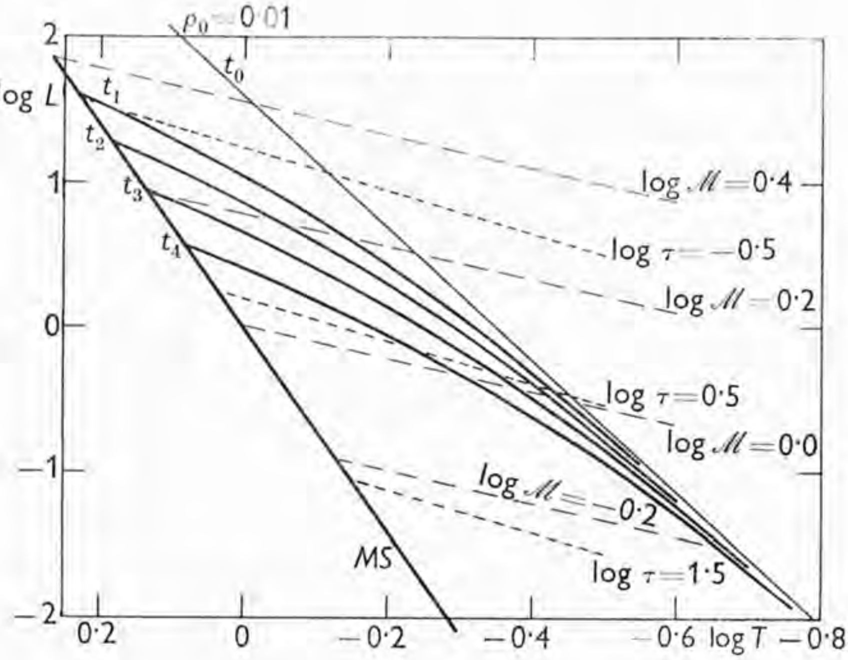

FIG. 1

According to the hypotheses of this paper, MS is the main sequence, the broken lines $\mathcal{M}$ = constant are the evolutionary tracks of stars having the indicated masses, the dotted lines τ = constant are the loci of stars at epoch τ after starting to condense from vanishingly small density at epoch $\tau = 0$.

If stars start at epoch t_0 to condense from mean density ρ_0 then the loci at epochs $t_1, \ldots, t_4$ are as shown, the loci being drawn for $\rho_0 = 0{\cdot}01$, $t_1 - t_0 = 0{\cdot}05$, $t_2 - t_0 = 0{\cdot}1$, $t_3 - t_0 = 0{\cdot}2$, $t_4 - t_0 = 0{\cdot}4$.

$\mathcal{M}$, L, T, ρ_0 are measured in solar units, and τ, t in the Helmholtz-Kelvin time-scale for the Sun.

Therefore, if initial conditions are such that at any epoch the pre-main-sequence stars lie on a locus appreciably different from that representing gravitational contraction starting at the same epoch from vanishingly small initial density, then this will remain the case throughout the time intervals of interest.

On finding a distribution of pre-main-sequence stars as in NGC 2264, it consequently seems natural to look for an explanation in terms of initial conditions (*e.g.*, the observed effect would be produced if there were a tendency for stars of smaller mass to be formed before stars of greater mass), rather than to postulate new phenomena such as large mass-loss during the gravitational contraction of a star.

It must be pointed out that the conclusion does not depend very sensitively upon the values of α, β, γ.

The calculations described above had been carried out before the authors saw the very important paper on the same problem by C. Hayashi[2]. Taking account of the hydrogen convection zone in stars of late spectral type, Hayashi concludes that the early part of the evolutionary track of a star of given mass is very different from the straight-line tracks shown in Fig. 1. This affects the age-calculation. Allowing for this, Hayashi gets good agreement for NGC 2264, if he makes the usual assumption that the member-stars all originated at the same epoch at very low density.

Hayashi's explanation will probably be accepted in this case. Nevertheless, it seems useful to present our results as a demonstration of the vital importance of initial conditions in such problems. Indeed, so far as the homologous contraction is concerned, Hayashi discloses a mechanism by which fainter stars get into the homologous series more quickly than had previously been thought possible. As regards the homologous contraction, his work substantiates the general conclusion to which we seek to call attention.

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