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Performance of Monetary Policy with Internal Central Bank Forecasting*

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Abstract

Recent models of monetary policy have analyzed the desirability of different optimal and *ad hoc* interest rules under the restrictive assumption that forecasts of the private sector and the central bank are homogenous. This paper studies the implications of heterogeneity in forecasting by the central bank and private agents for the performance of interest rules in a framework of econometric learning.

Key words: Adaptive learning, stability, heterogeneity, monetary policy.

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1 Introduction

The question whether monetary policy should be forward-looking, i.e. based on forecasts of future inflation and other variables, has raised debates in the recent research into monetary policy making. On one hand, empirical evidence on Germany, Japan and the US since 1979 provided by (Clarida, Gali, and Gertler 1998) suggests that central banks are forward looking in practice. More general discussions also pose the question whether central banks should focus attention to economic fundamentals or "follow the markets", which "sometimes stray far from fundamentals", see pp. 60-61 of (Blinder 1998). Bank of England Inflation Reports, see (Bank of England 2003), discuss private sector forecasts while the June and December Issues of the Monthly Bulletin of the European Central Bank, see (European Central Bank 2003), present both internal macroeconomic projections and forecasts by other institutions. However, the precise role of these forecasts in the decision making of these central banks is not revealed.

On the other hand, theoretical studies have shown that the conduct of optimal monetary policy on the part of the bank can lead to a choice of the instrument, the short-term nominal interest rate, which reacts to the next period forecast of inflation and/or output gap, see (Clarida, Gali, and Gertler 1999) for a survey of the recent literature. This conclusion can nevertheless be problematic as monetary policy rules, both some formulations of optimal setting of the instrument as well as Taylor rules based on forecasts of inflation and/or output gap, are subject to two potentially important difficulties.

First, some interest rate rules lead to indeterminacy of equilibria, see. e.g. (Clarida, Gali, and Gertler 1999), (Bernanke and Woodford 1997), (Bullard and Mitra 2002) and (Evans and Honkapohja 2003b). Under indeterminacy, the economy has multiple stationary rational expectations equilibria (REE), which can include undesirable outcomes.

Second, the problem of instability under learning can also arise depending on the form of the interest rate rule, see (Bullard and Mitra 2002) and (Evans and Honkapohja 2003b). If the economy is not stable under learning, small displacements of expectations away from rational expectations (RE) will lead to volatility as the economy does not return to the REE when agents try to correct their forecast functions. Both of these problems can be avoided by careful design of monetary policy, i.e. the interest rate rule.

In the literature just cited, the forecasts refer to those of the private sector, see e.g. (Hall and Mankiw 1994) for a discussion of targeting of private forecasts. By considering both determinacy and stability under learning, (Evans and Honkapohja 2003b) forcefully make a case for incorporating private forecasts of inflation and output gap into the interest rate rule as the reaction function of the optimal central bank behavior under discretion.[1] Naturally, for such a proposal to make sense it is required that the private sector forecasts are observable. (Evans and Honkapohja 2003b) show that small measurement errors would not lead to large deviations from optimality. However, (Orphanides 2003) and others have argued that there are large errors in private forecasts. While private forecasts by different institutions are regularly published, it is not

[1](Evans and Honkapohja 2003c) extend the results of (Evans and Honkapohja 2003b) to the case of commitment.

self-evident that these published numbers accurately represent the expectations of the private sector that are relevant for the key private economic decisions. Thus the observability problems might in fact be more serious than they appear at first sight. Moreover, if it becomes known that the decisions of the monetary policy maker depend significantly on the forecasts by private institutions, these institutions might alter their forecasts in a strategic way so as to influence the decisions about the conducted monetary policy, so that the central bank may wish to use internal forecasts in its policy making.

The preceding arguments make a case for the use of internal forecasts by the central bank as a proxy for private forecasts in monetary policy decisions. Moreover, it seems likely that internal forecasts, rather than those of other institutions, play the central role in actual monetary policy decisions. The recent literature by (Bernanke and Woodford 1997), (Svensson 1997), (Svensson 1999), (Svensson and Woodford 2003) and (Svensson 2003) incorporates internal forecasts by the central bank in models of monetary policy.

We will focus on the situation in which both the private sector and the central bank use their own forecasts in their decision-making and the forecasts are not available to the other agent. Consequently, the forecasts have no strategic role. This case can be seen as a natural benchmark. Taking the learning viewpoint, we will argue that heterogenous forecasting by the private sector and the central bank poses a new stability concern that needs to be taken into account in policy design.

The learning approach to modelling expectations formation has gained popularity in the recent literature.[2] The economic agents are assumed to use forecast functions that depend on some parameters and, at any moment of time, the economic decisions are made on the basis of expectations/forecasts obtained from these functions. The values of the parameters in the forecast functions and the expectations of the agents are adjusted over time as new data becomes available. Following much of the literature, we will assume that parameter updating is done using standard econometric methods such as recursive least squares (RLS) estimation. We see this approach as a natural first formulation, but it can be noted that other approaches to learning could also be considered in this context.[3]

A key issue of interest is whether this kind of adaptive learning behavior converges to REE over time. If this is the case, the forecast functions of the agents are eventually those associated with the REE. We will analyze the implications of heterogeneity in private sector and central bank forecasts for the performance of forward-looking interest rate rules.[4] Our objective is to study how the conditions for learnability of equilibrium,

[2]There has recently been extensive research into the learning approach to macroeconomics, see (Evans and Honkapohja 2001) for a systematic treatise. Overviews and surveys are provided e.g. by (Evans and Honkapohja 1999) , (Marimon 1997), (Sargent 1993) and (Sargent 1999).

[3]The other approaches include computational intelligence (see e.g. (Arifovic 1998)), models of discrete predictor choice (see e.g. (Brock and Hommes 1997) and (Brock and de Fontnouvelle 2000)) and eductive learning (see (Guesnerie 2002)).

[4]The earlier literature on learning and monetary policy has largely assumed that only private forecasts affect the economy. See (Bullard and Mitra 2002), (Bullard and Mitra 2001), (Evans and Honkapohja 2003b), (Honkapohja and Mitra 2001), (Evans and Honkapohja 2003c) and (Mitra 2003). (Carlstrom and Fuerst 2001) study the standard model of monetary policy under the assumption that

i.e. stability of equilibrium under adaptive learning, are affected by heterogeneities in expectations and learning rules. The model we use is standard in the recent literature on monetary policy conducted with interest rates rules, see the surveys by (Clarida, Gali, and Gertler 1999) and (Woodford 1999).

The heterogeneity in expectations and learning can take some different forms even if attention is restricted to econometric learning. The first and simplest possibility is that both private sector and central bank forecast functions have the same parametric form and the updating of these forecast functions is done using the same learning algorithm. (We specifically assume the RLS algorithm that has been widely used in the literature.) Heterogeneity in expectations is then solely due to differences in initial beliefs.

The second step we consider relaxes the assumption of identical estimation algorithms. One subcase here is that the updating algorithms are in the same class, but the strength of reaction to forecast errors in parameter updating differs between the private sector and the central bank. Another subcase arises when the algorithms used by the private agents and the policy maker are different. For example, the private sector might use the stochastic gradient (SG) algorithm that is simpler to implement than RLS.[5]

The analysis of learning dynamics in the context of monetary policies provides a very natural setting in which adaptive learning can take place under *structural heterogeneity.* We say that a model exhibits structural heterogeneity when different agents respond differently to expectations; compare p. 42 of (Evans and Honkapohja 2001). In the model of monetary policy the private sector expectations influence the economy directly through aggregate demand and the new Phillips curves, while the central bank forecasts enter through the interest rate rule. Our analysis make use of the general theoretical results for forward-looking multivariate linear models with structural heterogeneity derived in the companion paper (Honkapohja and Mitra 2003).

We will show that the learnability restrictions for interest rate rules under the assumption of homogenous expectations continue to be important. They are a necessary condition for convergence of adaptive learning with heterogenous forecasts and learning rules. However, these conditions need not be sufficient for learnability. We look systematically at additional conditions that might lead to stability or instability. Interestingly, these results have natural interpretations as suggestions concerning the forecasting activity of the central bank.

private sector has rational expectations and the Central Bank tries to learn. Learning and monetary policy has been recently surveyed in (Evans and Honkapohja 2003a).

[5]These forms of heterogeneity in learning are studied in (Honkapohja and Mitra 2003) for a general framework. In independent work (Giannitsarou 2003) considers similar cases under the more restrictive assumption that the economy depends on the average expectations of the agents.

2 The Model and First Results

2.1 Analytical Framework

The analysis will be conducted using a standard model with a representative consumer and monopolistic competition in markets for differentiated products. It is assumed that firms face restrictions on price changes, so that only a fraction of firms can change its price in any given period. Real balances enter the utility function of the consumer, who can also make savings in the form of government bonds. We employ directly the log-linearized model and thus adopt the framework that is formally as outlined in Section 2 of the survey paper by (Clarida, Gali, and Gertler 1999). (See e.g. (Woodford 1996) for the nonlinear model and its log-linearized version.) We clearly need a specific model and we emphasize that our approach is applicable to the very similar frameworks that have been used in the recent literature.

The structural model consists of two equations:

$$\begin{aligned} z_t &= -\varphi(i_t - \hat{E}_t^P \pi_{t+1}) + \hat{E}_t^P z_{t+1} + g_t, \quad (1) \\ \pi_t &= \lambda z_t + \beta \hat{E}_t^P \pi_{t+1} + u_t, \quad (2) \end{aligned}$$

where z_t is the "output gap", i.e. the difference between actual and potential output, π_t is the inflation rate, i.e. the proportional rate of change in the price level from $t-1$ to t, and i_t is the nominal interest rate. $\hat{E}_t^P \pi_{t+1}$ and $\hat{E}_t^P z_{t+1}$ denote *private sector expectations* of inflation and output gap next period. We will use the same notation without the "^" and superscript P to denote RE of the private sector. All the parameters in (1) and (2) are positive. $0 < \beta < 1$ is the discount rate of the representative firm.

(1) is a dynamic "IS" curve that can be derived from the Euler equation associated with the household's savings decision. (2) is a "new Phillips curve" that can be derived from optimal pricing decisions of monopolistically competitive firms facing constraints on the frequency of future price changes. (1) and (2) are the key behavioral relationships indicating that consumers and firms need to predict future inflation and output gap in order to make their current consumption and pricing decisions. Thus consumers and firms will try to learn the stochastic process of π_t and z_t.

u_t and g_t denote observable shocks following first order autoregressive processes:

$$\begin{pmatrix} u_t \\ g_t \end{pmatrix} = F \begin{pmatrix} u_{t-1} \\ g_{t-1} \end{pmatrix} + \begin{pmatrix} \tilde{u}_t \\ \tilde{g}_t \end{pmatrix}, \; F = \begin{pmatrix} \rho & 0 \\ 0 & \mu \end{pmatrix}, \quad (3)$$

where $0 < \mu < 1, 0 < \rho < 1$ and $\tilde{g}_t \sim iid(0, \sigma_g^2), \tilde{u}_t \sim iid(0, \sigma_u^2)$. The demand shock g_t may be rationalized as a preference shock or as expected changes in government purchases relative to expected changes in potential output. The "cost push" shock u_t captures features that affect expected marginal costs other than those through z_t.

We supplement equations (1) and (2) with monetary policy that is conducted by means of control of the nominal interest rate i_t.[6] We focus on rules where the interest

[6]It should be noted that we have left out explicit consideration of the intertemporal government

rate is adjusted in accordance with the central bank expectations $\hat{E}_t^{CB}(.)$ of output gap and inflation next period and possibly the exogenous shocks. Then

$$i_t = \chi_0 + \chi_\pi \hat{E}_t^{CB}\pi_{t+1} + \chi_z \hat{E}_t^{CB} z_{t+1} + \chi_g g_t + \chi_u u_t. \tag{4}$$

Again the same notation without the "^" and superscript CB will denote RE (of the central bank). We assume $\chi_z \geq 0$ and $\chi_\pi \geq 0$ throughout the paper.

The rule (4) can be interpreted as a *Taylor rule* considered, for instance, in (Bullard and Mitra 2002). (The term "Taylor rule" is used to commonly refer to non-optimal rules.) Formally similar rules can also arise from *optimal policy* on the part of the central bank.[7] As there can be large errors in measuring private expectations or these might be subject to strategic behavior, a plausible procedure would use the internal forecasts by the central bank in place of the private sector expectations. This suggests the rule

$$\begin{aligned} i_t \quad = \quad & [1 + (\lambda^2 + \alpha)^{-1}\varphi^{-1}\lambda\beta]\hat{E}_t^{CB}\pi_{t+1} + \varphi^{-1}\hat{E}_t^{CB} z_{t+1} \\ & + \varphi^{-1} g_t + (\lambda^2 + \alpha)^{-1}\varphi^{-1}\lambda u_t, \end{aligned} \tag{5}$$

where α is the relative weight for output deviations in the central bank's objective function, see footnote 7. Following (Evans and Honkapohja 2003b), we will refer to (5) as the *expectations based (EB-) optimal rule.* The optimal interest rate under discretion can be characterized in different ways, as pointed out in (Clarida, Gali, and Gertler 1999). Their suggestion, coupled with internal forecasting, is

$$i_t = (1 + \frac{(1-\rho)\lambda}{\rho\alpha\varphi})\hat{E}_t^{CB}\pi_{t+1} + \varphi^{-1} g_t. \tag{6}$$

We will refer to (6) as the *rational expectations (RE-) optimal rule* as in (Evans and Honkapohja 2003b). Clearly, (5) and (6) are special cases of (4). Note that (5) and (6) presuppose that the central bank knows the structural parameters φ and λ. We make this assumption for simplicity; see (Evans and Honkapohja 2003b) for learning of structural parameters by the central bank when private expectations are observable.

As mentioned before, (Bullard and Mitra 2002) assume identical forecasts to derive conditions for (local) stability of fundamental or minimal state variable (MSV) solutions under learning dynamics. Bullard and Mitra found that the MSV solution is learnable if and only if it satisfies the *Taylor principle*, which is formally defined as

$$\chi_\pi + \frac{1-\beta}{\lambda}\chi_z > 1 \tag{7}$$

in our model. (See (Woodford 2003) and (Bullard and Mitra 2002), p.1116 for details on the Taylor principle.) Regarding optimal rules, (Evans and Honkapohja 2003b) found

budget constraint. This is appropriate only if fiscal policy in the form of lump-sum taxes is passively adjusted in the sense of (Leeper 1991), so that taxes are set to ensure fulfillment of the intertemporal government budget constraint.

[7](Evans and Honkapohja 2003b) consider optimal discretionary policy by minimizing a quadratic objective function, $\alpha z_t^2 + \pi_t^2$, subject to general private sector expectations and (2), which approach leads to a specific interest rate rule that depend on private expectations.

that, with private expectations, the rule (5) yields both stability under learning and determinacy whereas rule (6) is stable but leads to indeterminacy in some parameter domains.

Our purpose in this paper is to analyze the robustness of these results to the heterogeneity in forecasts and/or learning algorithms for the private sector and the central bank. Our model, therefore, comprises of equations (1), (2), and (3) supplemented with the interest rate rule (4). We thus substitute equation (4) into (1) and write the above system in the general form

$$\begin{aligned} y_t &= D + A^P \hat{E}_t^P y_{t+1} + A^{CB} \hat{E}_t^{CB} y_{t+1} + B w_t, \\ w_t &= F w_{t-1} + v_t, \end{aligned} \tag{8}$$

where

$$\begin{aligned} A^P &= \begin{pmatrix} 1 & \varphi \\ \lambda & \beta + \lambda\varphi \end{pmatrix}, A^{CB} = \begin{pmatrix} -\varphi\chi_z & -\varphi\chi_\pi \\ -\lambda\varphi\chi_z & -\lambda\varphi\chi_\pi \end{pmatrix} \\ B &= \begin{pmatrix} -\varphi\chi_u & 1 - \varphi\chi_g \\ 1 - \lambda\varphi\chi_u & \lambda(1 - \varphi\chi_g) \end{pmatrix}, D = \begin{pmatrix} -\varphi\chi_0 \\ -\lambda\varphi\chi_0 \end{pmatrix} \end{aligned}$$

and $y_t = (z_t, \pi_t)'$, $w_t = (u_t, g_t)'$, $v_t = (\tilde{u}_t, \tilde{g}_t)'$.

We will consider learnability of the minimal state variable (MSV) solution for the model (8). It takes the form

$$y_t = a + b w_t, \tag{9}$$

where a, b are to be computed in terms of the structural parameters of the model.[8] Proposition 1 of (Honkapohja and Mitra 2003) shows that the MSV solution is generically unique:

Proposition 1 *The model of monetary policy (8), has a unique MSV solution of the form (9) if the matrices $I - (A^P + A^{CB})$ and $I - F' \otimes (A^P + A^{CB})$ are invertible.*

2.2 Heterogenous Forecasts Under Homogenous Learning Rules

As a first case we assume that the central bank and the private sector have different forecasts although their forecast functions take the same general form and they use asymptotically identical learning algorithms in updating the parameters of their forecast functions. Note that the forecasts of the private sector and the central bank will differ along the actual time path since the learning rules can start with different initial beliefs and the gain sequences can differ for finite time. It is well known for a wide variety of different models (with homogenous forecasts and learning) that convergence to the REE obtains if and only if certain stability conditions, known as expectational stability (or

[8] There are other stationary REE in addition to the MSV solution if the model is indeterminate. See (Honkapohja and Mitra 2001) for analysis of indeterminacy in the model of monetary policy.

E-stability) conditions, are satisfied, see e.g. (Evans and Honkapohja 2001). We now obtain the extended E-stability conditions that govern convergence of the economy to the REE under real time learning as long as the bank and private agents use asymptotically identical versions of RLS. (This result is discussed in more detail in the companion paper (Honkapohja and Mitra 2003).)

The private agents and the central bank are assumed to have their own *perceived laws of motion* (PLM) that take the same parametric form (9) as the REE of interest. For given values of the parameters of the PLM and the current values of the exogenous variables, the agents use the estimated PLM to make forecasts about the values of the endogenous variables next period. Given the forecasts a temporary equilibrium of the economy, also called the *actual law of motion* (ALM) obtains. We note that this formulation of forecasting by the private agents and the central bank is a natural first approach, since the forecast functions correspond, under specific parameter values, to the equilibrium forecast functions. However, we do acknowledge that they represent a greatly simplified view of actual forecasting practices.

Formally, the private sector and the central bank, respectively, have PLMs of the form that corresponds to the MSV solutions (9) but they have different parameter values:

$$y_t = a^P + b^P w_t = (\phi^P)' x_t \tag{10}$$
$$y_t = a^{CB} + b^{CB} w_t = (\phi^{CB})' x_t \tag{11}$$

with corresponding forecast functions[9]

$$\hat{E}_t^P y_{t+1} = a^P + b^P F w_t, \tag{12}$$
$$\hat{E}_t^{CB} y_{t+1} = a^{CB} + b^{CB} F w_t, \tag{13}$$

where $x_t = (1, w_t)'$ is a vector of variables relevant in forecasting and $(\phi^i)' = (a^i, b^i)$, with a^i a $2-$dimensional vector and b^i an 2×2 matrix for $i = P, CB$. We remark that constant terms are incorporated in (10)-(11) since it is natural to assume that agents also need to estimate intercepts; see (Bullard and Mitra 2002), p. 1118, for further discussion.

Inserting these forecasts into the model (8), one obtains the ALM followed by inflation and output as

$$\begin{aligned} y_t &= D + A^P a^P + A^{CB} a^{CB} + [(A^P b^P + A^{CB} b^{CB})F + B] w_t \\ &= [D + A^P a^P + A^{CB} a^{CB}, (A^P b^P + A^{CB} b^{CB})F + B] \begin{bmatrix} 1 \\ w_t \end{bmatrix} \\ &\equiv T(\varphi_1', \varphi_2') x_t. \end{aligned}$$

From this ALM it is easy to construct the explicit form of the mapping $T(\varphi_1', \varphi_2')$ from the PLM to the ALM. We look at E-stability of the REE in which the bank and the

[9] Parameter estimates are assumed to depend on data up to $t-1$ but the current observation on exogenous variables is used in the forecasts. This is commonly done in the literature. For simplicity, F is assumed to be known (if not, it could be estimated).

private sector have identical forecasts, that is when $a^P = a^{CB} = \bar{a}$ and $b^P = b^{CB} = \bar{b}$. The REE is said to be *E-stable* if it is a locally asymptotically stable fixed point under differential equation

$$\frac{d}{d\tau}(\varphi_1', \varphi_2') = T(\varphi_1', \varphi_2') - (\varphi_1', \varphi_2'),$$

which explicitly is (here I denotes the identity matrix)

$$\begin{aligned}
da^P/d\tau &= D + (A^P - I)a^P + A^{CB}a^{CB}, \\
db^P/d\tau &= A^P b^P F - b^P + A^{CB} b^{CB} F + B, \\
da^{CB}/d\tau &= D + A^P a^P + (A^{CB} - I)a^{CB}, \\
db^{CB}/d\tau &= A^P b^P F + A^{CB} b^{CB} F - b^{CB} + B.
\end{aligned}$$

This system is linear and the equations for (a^P, a^{CB}) and (b^P, b^{CB}) are independent from each other. Using the results in (Honkapohja and Mitra 2003), stability for the (a^P, a^{CB}) and (b^P, b^{CB}) subsystems are determined, respectively, by the matrices

$$\begin{pmatrix} A^P - I & A^{CB} \\ A^P & A^{CB} - I \end{pmatrix}, \begin{pmatrix} F \otimes A^P - I & F \otimes A^{CB} \\ F \otimes A^P & F \otimes A^{CB} - I \end{pmatrix},$$

where $\otimes$ is the Kronecker product. Consequently, the necessary and sufficient condition for E-stability are that the eigenvalues of

$$A^P + A^{CB} - I \text{ and } F \otimes (A^P + A^{CB}) - I$$

have negative real parts. (We assume that the case of zero real parts does not arise.)

These requirements for stability are exactly the E-stability condition when the bank and the private sector have identical forecasts. In this case $\hat{E}_t^P y_{t+1} = \hat{E}_t^{CB} y_{t+1}$ and the matrix in front of the common expectations $\hat{E}_t^P y_{t+1}$ in (8) becomes $A^P + A^{CB}$. Since the matrix F in (3) is diagonal and has positive elements (i.e. $\mu > 0$ and $\rho > 0$), the necessary and sufficient conditions for E-stability in fact reduce to the condition that the eigenvalues of $A^P + A^{CB} - I$ have negative real parts, as shown in (Bullard and Mitra 2002). Finally, their Proposition 5 shows that the E-stability condition is equivalent to the Taylor principle (formally, their model has only one shock but this does not affect the results). We have thus verified:

Proposition 2 *The MSV solution to the model of monetary policy (8) is E-stable under heterogenous forecasts if and only if the corresponding model with homogenous expectations is E-stable. The E-stability condition is that the eigenvalues of the matrix $A^P + A^{CB} - I$ have negative real parts, which is equivalent to the Taylor principle.*

We note that under some (mild) regularity conditions, the RLS algorithm will converge to an E-unstable symmetric (MSV) solution with probability zero. See (Evans and Honkapohja 2001) for these details.

Proposition 2 shows that the stability conditions obtained in the homogenous case in (Bullard and Mitra 2002) and (Evans and Honkapohja 2003b) are not as restrictive as they might seem. The assumption that the bank and private sector have the same type of forecast functions and same learning algorithms serves as a good first approximation.

3 Heterogenous Learning Rules

The preceding stability result allowed for different forecasts for the central bank and private agents but assumed that the bank and private sector use learning rules that were asymptotically identical (even though these rules differed along the transition to REE). A greater degree of heterogeneity would allow for learning rules that differ even asymptotically or for altogether different learning algorithms. Here we take up these two further forms of heterogeneity.

The first subcase assumes that both the bank and the private agents use versions of RLS in their updating schemes but they differ in the degree of adaption to forecast errors. This allows for inertia in the formation of expectations as well as various weighting schemes for data in later periods relative to early ones.

The second subcase considers a scenario where the agents use different learning rules. The rules we consider are RLS and SG type algorithms. These algorithms involve a trade-off between simplicity and efficiency (see Section 3.2 for further discussion).

3.1 RLS Learning with Different Gain Sequences

We continue to assume that the private sector and the bank use forecast functions (12)-(13) in forming their forecasts of inflation and output. Consequently, the analysis of E-stability is identical to that in Section 2.2. However, the analysis of real time learning is different since we now allow the bank and the private sector to display different speeds of adaption in their updating of estimates of parameters.

Versions of the RLS algorithm take the form

$$\begin{aligned} (\phi_t^i)' &= (\phi_{t-1}^i)' + \gamma_{i,t}(R_t^i)^{-1}x_{t-1}(y_{t-1} - \phi_{t-1}^i x_{t-1})' \\ R_t^i &= R_{t-1}^i + \gamma_{i,t}[x_{t-1}(x_{t-1})' - R_{t-1}^i], \end{aligned} \tag{14}$$

for $i = P, CB$, where we have used the notation in equation (10) from the preceding section. The first equation describes the updating of the parameters of the PLM of the private sector and the central bank, while the second equation updates the matrix of second moments of x_t that is needed in the updating of the PLM parameters. As already noted, the estimation of parameters for time t is based on information available in $t-1$ but forecasts use current data.

The sequence $\gamma_{i,t}, i = P, CB$ is known as the sequence of gains. The gain sequences can differ even asymptotically, and our interest is in the implications of heterogeneity in the form of different gain sequences of the central bank and private agents. The gain sequence indicates how much weight, say, the private agent puts on forecast errors $y_{t-1} - \phi_{t-1}^P x_{t-1}$ when he updates the parameters of the PLM. For standard RLS it is given by $\gamma_{i,t} = t^{-1}$. Modifications to standard RLS can be obtained by permitting greater or smaller response than t^{-1} to the forecast errors, adjusted for by the matrix of second moments and the state of exogenous variables. It is possible to include various weighting schemes, inertia in updating of forecast rules and even independent random fluctuations in adaption speeds, see e.g. (Ljung and Söderström 1983), (Marcet and

Sargent 1989) and (Evans, Honkapohja, and Marimon 2001) for different possibilities and the companion paper (Honkapohja and Mitra 2003) for further discussion.[10] We assume throughout the paper that agents perceive their environment to be stationary as the gain sequences are taken to be decreasing, i.e. $\lim_{t\to\infty}\gamma_{i,t}=0$. Reformulation of the learning to allow e.g. for structural shifts raises many further issues that are not taken into account for reasons of brevity.[11]

We write the (possibly random) gain sequences in the form

$$\gamma_{P,t}=\gamma_t(\gamma_{P,t}\gamma_t^{-1}) \text{ and } \gamma_{CB,t}=\gamma_t(\gamma_{CB,t}\gamma_t^{-1}),$$

where γ_t is an exogenously given, nonincreasing deterministic sequence satisfying certain properties. The different asymptotics are captured by assuming that

$$E(\gamma_{P,t}\gamma_t^{-1})\to\delta_P \text{ and } E(\gamma_{CB,t}\gamma_t^{-1})\to\delta_{CB}, \text{ as } t\to\infty \text{ with } \delta_P\neq\delta_{CB}.$$

Here the mathematical expectations are taken over the possible independent randomness in the individual gains.

The results in (Honkapohja and Mitra 2003) imply that local convergence of learning under these algorithms is determined by the following two matrices

$$\begin{pmatrix} \delta_P(A^P-I) & \delta_P A^{CB} \\ \delta_{CB}A^P & \delta_{CB}(A^{CB}-I) \end{pmatrix} = \begin{pmatrix} \delta_P I & 0 \\ 0 & \delta_{CB}I \end{pmatrix}\begin{pmatrix} A^P-I & A^{CB} \\ A^P & A^{CB}-I \end{pmatrix} \quad (15)$$

and

$$\begin{pmatrix} \delta_P(F\otimes A^P-I) & \delta_P F\otimes A^{CB} \\ \delta_{CB}F\otimes A^P & \delta_{CB}(F\otimes A^{CB}-I) \end{pmatrix} = \begin{pmatrix} \delta_P I & 0 \\ 0 & \delta_{CB}I \end{pmatrix}\begin{pmatrix} F\otimes A^P-I & F\otimes A^{CB} \\ F\otimes A^P & F\otimes A^{CB}-I \end{pmatrix}, \quad (16)$$

and we state the formal convergence result:

Proposition 3 *If the private sector and the central bank use modified RLS learning algorithms with different gain sequences, then learning converges locally if the matrices (15) and (16) have eigenvalues with negative real parts.*

We note that if the gain sequences are the same asymptotically, i.e. $\delta_P=\delta_{CB}$, the necessary and sufficient condition for convergence to the MSV solutions is given by the E-stability conditions. However, the situation is quite different if $\delta_P\neq\delta_{CB}$. The stability conditions are in general affected by the relative size of the gain parameters, though the earlier E-stability condition is still relevant, as we now show.

[10] See also Section 15.2 of (Evans and Honkapohja 2001) for discussion of gain sequences. Inertia in the formation of expectations is observed in experimental data, see for instance (Marimon and Sunder 1993) and (Evans, Honkapohja, and Marimon 2001).

[11] Constant gain algorithms, for which $\gamma_{i,t}$ is a small positive constant, are a natural way for allowing for the possibility of structural shifts; see e.g. Chapter 14 of (Evans and Honkapohja 2001) and (Cho, Williams, and Sargent 2002).

3.1.1 Stability and Instability Conditions for Interest Rate Rules

The first substantive result that follows from Proposition 3 concerns the general form of interest rate policies (4). We first provide some necessary conditions that must be satisfied for an equilibrium to be locally stable under learning. The next corollary is proved in Appendix A.1.

Corollary 4 *Consider model (8) and assume that the private sector and the central bank use the algorithms (14) with different gain sequences. The two conditions*

$$(1-\beta)\chi_z + \lambda(\chi_\pi - 1) \;>\; 0, \tag{17}$$
$$\chi_z + \lambda\chi_\pi - \delta_P\delta_{CB}^{-1}[\lambda - \varphi^{-1}(1-\beta+\delta_P^{-1}\delta_{CB})] \;>\; 0 \tag{18}$$

are necessary for local stability of the symmetric equilibrium under learning.

Note that (17) is precisely the Taylor principle (7) that completely characterized stability under learning for the homogenous case considered in (Bullard and Mitra 2002). Corollary 4 shows the continued importance of the Taylor principle in the presence of heterogenous rules. In particular, it shows that rules violating this principle continue to be unstable as is the case under homogenous forecasts.

However, the Taylor principle is in general not sufficient for stability under learning since condition (18) depends on $\delta_P^{-1}\delta_{CB}$. The interest rule may require a stronger response to inflation and/or output (via larger χ_π or χ_z) than what is dictated by the Taylor principle, especially for small values of $\delta_P^{-1}\delta_{CB}$. To illustrate this, assume that $\beta+\lambda\varphi > 1$.[12] Then, if $\chi_z = 0$, the necessary condition (18) requires $\chi_\pi > \delta_P\delta_{CB}^{-1}[1-\lambda^{-1}\varphi^{-1}(1-\beta+\delta_P^{-1}\delta_{CB})]$, which is strictly more than 1, for any $\delta_P^{-1}\delta_{CB} < (\beta+\lambda\varphi-1)(1+\lambda\varphi)^{-1}$.

It turns out that the Taylor principle is necessary and sufficient for stability when $\delta_P^{-1}\delta_{CB} \geq 1$. Intuitively, this last requirement means that the central bank should put at least as much weight on incoming information about the economy when revising its parameter estimates as does the private sector. The next corollary is also proved in Appendix A.1.

Corollary 5 *Consider model (8) when the private sector and the central bank use modified RLS learning algorithms with different gain sequences and assume that*

$$\delta_P^{-1}\delta_{CB} \geq 1.$$

The dynamics of the economy is then locally stable under learning if and only if the Taylor principle (7) holds.

[12] This parameter restriction will typically be satisfied since the discount factor β is assumed to be close to 1. For example, the restriction is satisfied for the calibrated values in both (Woodford 1999) and (Clarida, Gali, and Gertler 2000).

A common theme that emerged from (Bullard and Mitra 2002) for a variety of interest rate rules is that a strong enough response towards inflation or output always resulted in stability under learning dynamics. Corollary 5 shows this to be true when $\delta_P^{-1}\delta_{CB} \geq 1$.[13]

On the other hand, Corollary 4 suggests that policies satisfying the Taylor principle can sometimes lead to instability. This can indeed happen if the bank puts much less weight on incoming information about the economy than the private sector (i.e. if δ_{CB} is much smaller than δ_P). This can be seen informally by examining the left-hand side of (15). We can write

$$\begin{pmatrix} \delta_P(A^P - I) & \delta_P A^{CB} \\ \delta_{CB}A^P & \delta_{CB}(A^{CB} - I) \end{pmatrix} = \delta_P \begin{pmatrix} A^P - I & A^{CB} \\ \delta_P^{-1}\delta_{CB}A^P & \delta_P^{-1}\delta_{CB}(A^{CB} - I) \end{pmatrix}, \tag{19}$$

so that, if $\delta_P^{-1}\delta_{CB}$ is sufficiently small, half of the eigenvalues of the matrix (19) are approximately equal to zero while the other half are approximately the eigenvalues of $A^P - I$. The latter set contains an eigenvalue with positive real part. This intuition is made rigorous below in the result:

Corollary 6 *Consider model (8) when the private sector and the central bank use modified RLS learning algorithms with different gain sequences and assume that $\beta + \lambda\varphi > 1$. If*

$$\delta_P^{-1}\delta_{CB} < \frac{\beta + \lambda\varphi - 1}{1 + \varphi(\chi_z + \lambda\chi_\pi)}, \tag{20}$$

the dynamics of the economy is locally unstable under learning.

The proof is immediate by making a strict reversal of condition (18). As mentioned before, the parameter restriction $\beta + \lambda\varphi > 1$ is very often satisfied since β is close to 1. Corollary 6 points to the danger of instability even for interest rules satisfying the Taylor principle when the central bank does not put enough weight on the forecast errors while revising its parameter estimates. The general intuition for the result is as follows.

One observes from the model (8) that, while the central bank has a stabilizing effect, the private sector has a de-stabilizing influence on the economy. Inspection of the matrices A^P and A^{CB} makes it clear that if private sector expectations of inflation (or output) deviate upward from the RE value, then actual inflation (and output) increase, which leads, *ceteris paribus*, to upward revisions of both $\hat{E}_t^P\pi_{t+1}$ and $\hat{E}_t^P z_{t+1}$ (note that all the entries of A^P are positive). On the other hand, if the central bank's expectations $\hat{E}_t^{CB}\pi_{t+1}$ or $\hat{E}_t^{CB}z_{t+1}$ deviate upwards from the RE value, π_t and z_t fall, which tends to guide the bank's non-rational expectations towards the RE values as all the entries of A^{CB} are negative. More formally, one observes that the eigenvalues of A^{CB} are non-positive, while the eigenvalues of A^P are positive and one of them exceeds 1. The

[13]It can be shown that, for any $\delta_P^{-1}\delta_{CB}$ (and values of the structural parameters), the symmetric equilibrium is stable if the central bank is aggressive enough by choosing large values of χ_z and χ_π. For brevity, we do not formally develop this result.

eigenvalue exceeding one is the key to understanding our instability results under heterogenous forecasts and learning. (This intuition continues to be true when we examine different types of learning rules in Section 3.2.)

Under homogenous forecasts, it is the sum of the matrices A^P and A^{CB} that determined stability under learning dynamics. Pursuit of the Taylor principle by the bank is then able to guide non-rational expectations of the private sector towards RE. However, under heterogenous forecasts, this is no longer sufficient because of the differential effects of the different forecasts via A^P and A^{CB} and the different weights in parameter updating on π_t and z_t. It now becomes very important for the bank to put sufficient weight on new data about the exogenous observables while revising its forecasts of inflation and output, so that, in conjuction with the Taylor principle, its stabilizing influence outweighs the de-stabilizing influence of the private sector to render the REE stable. This makes intuitive sense since, after all, these shocks are indicative of inflationary pressures in the economy.

The results indicate the degree to which the results in (Bullard and Mitra 2002) and (Evans and Honkapohja 2003b) are affected by the use of differential gains by the bank and the private sector. If $\delta_P^{-1}\delta_{CB}$ satisfies the condition in Corollary 6, then the Taylor principle no longer suffices to guarantee convergence. Moreover, even if the central bank behaves optimally by following the rules (5) or (6), convergence to the equilibrium may not take place unless the bank makes $\delta_P^{-1}\delta_{CB}$ sufficiently big (say, ≥ 1 as in Corollary 5) in the updating of the PLM parameters. We illustrate this further in the next subsection.

3.1.2 Robustness of the Rules to Gain Differentials

Corollary 6 provides only sufficient conditions for the learning dynamics to be unstable. We now use numerical techniques to study to what extent the differences in learning actually influence stability for plausible values of structural parameters. As a useful by-product, we also consider the desirability of different optimal interest rate rules (5) and (6) advocated under the assumption of homogenous forecasts in the previous literature.

We first look at variants of optimal policies considered in Section 2. The two variants are the EB-optimal rule, equation (5), and the RE-optimal rule, equation (6). (Evans and Honkapohja 2003b) recommend the EB-optimal rule in part on grounds of determinacy: rule (5) is always determinate under RE, rule (6) can become indeterminate for values of ρ close to zero. Regarding learnability, (Evans and Honkapohja 2003b) show that under homogenous forecasts and learning, both the rules (5) and (6) are stable under learning. We now consider the implications of heterogeneity in learning rules for these results by means of numerical analysis.

Throughout the paper the results reported by Figures 1-7 show domains of specified parameters for which numerical computation of the eigenvalues of the relevant matrices indicates stability with heterogenous learning. The calibrated parameters values in (Woodford 1999) are used in this discussion:

Calibrated Example: $\varphi = (.157)^{-1}, \lambda = .024$, and $\beta = .99$.[14]

[14]We have found that our results are in fact robust to the calibrated values in (Clarida, Gali, and

We allow α to range in the interval $(0, 1]$, which captures the scenarios ranging from strict inflation targeting (α close to 0) to that of flexible inflation targeting. The ratio $\delta_P^{-1}\delta_{CB}$ is allowed to range in the interval $(0, 2]$. (Both rules are always stable for all $\alpha \in (0, 1)$ when $\delta_P^{-1}\delta_{CB} > 2$.)

FIGURES 1 AND 2 HERE

Figures 1 and 2 illustrate the stability region for the EB-optimal rule (5) and the RE-optimal rule (6), respectively, with the value $\rho = .9$ for the persistence parameter of the u_t shock, which was used in (Clarida, Gali, and Gertler 2000). (The value for μ is not needed for Figures 1 and 2 as the g_t shock is neutralized in these rules.) The height of the vertical lines indicate stability for the relevant range of $\delta_P^{-1}\delta_{CB}$. The figures are drawn for a grid search of 0.01 for α. For the EB-optimal rule, we find that for $\alpha \in (0, 1]$, the symmetric equilibrium is stable under learning dynamics whenever $\delta_P^{-1}\delta_{CB} \geq 0.2$. Instability can only arise for $\delta_P^{-1}\delta_{CB} < 0.2$. However, for the RE-optimal rule stability is guaranteed only when $\delta_P^{-1}\delta_{CB} \geq 1$ whereas most values of $\delta_P^{-1}\delta_{CB} < 1$ lead to instability for $\alpha \in (0, 1]$. We, therefore, find that the EB-optimal rule performs better than the RE-optimal rule as it yields more robustly stability with differences in the gain parameters of the learning rules. (Recall that, with identical forecasting, EB and RE optimal rules both yield learnability so that the entire region in Figures 1 and 2 is stable in this case.)

FIGURE 3 HERE

Differences in gain sequences also affect the stability of *ad hoc* Taylor type rules like (4) considered in (Bullard and Mitra 2002). Figure 3 plots the stability region of the rule (4) with $\chi_z = 0$ and the above calibration. We assume that there is no cost-push shock u_t and that $\mu = 0.35$ as in (Woodford 1999). The horizontal axis indicates values for χ_π while the vertical axis indicates the values for $\delta_P^{-1}\delta_{CB} \in (0, 2]$. We recall that, in (Bullard and Mitra 2002), all rules with $\chi_\pi > 1$ are stable under learning. However, for values of $\delta_P^{-1}\delta_{CB} < 1$ even rules with $\chi_\pi > 1$ can now lead to instability. Stability is guaranteed for rules satisfying the Taylor principle (i.e. $\chi_\pi > 1$) only when $\delta_P^{-1}\delta_{CB} \geq 1$, as shown in Corollary 5. Of course, rules with $\chi_\pi < 1$ continue to deliver unstable dynamics, as shown in Corollary 4. The general message is that, with a Taylor rule in place, the central bank should put enough weight on incoming information (i.e. make $\delta_P^{-1}\delta_{CB}$ sufficiently big) to ensure stability of the economy.

3.2 RLS Learning and SG Learning

In this section we consider a different form of heterogeneity in the learning rules of the private agents and the bank. The broad aim is to consider a setting where the bank is using a learning algorithm that is either more or less sophisticated than the algorithm of the private sector. Central banks usually devote a large amount of resources in forming

Gertler 2000) who use $\varphi = 1$, $\lambda = .3$ and the same β.

forecasts of economy-wide variables, see (Romer and Romer 2001). Our analysis in this section provides an analytical answer to the question whether such actions on the part of the central bank are justified.

Specifically, we assume that there are two possible types of learning algorithms, the RLS and the stochastic gradient (SG) algorithms that the private agents or the central bank might use. The RLS and SG algorithms are very commonly used algorithms in adaptive control and filtering. The RLS algorithm is somewhat more common than SG in the economics literature. For parameter estimation of fixed exogenous stochastic processes, both the RLS and SG algorithms yield consistent estimates of parameters but the former in addition possesses some optimality properties.[15] However, the SG algorithm is computationally much simpler than the RLS algorithm and it has an interpretation as a robust optimal prediction rule; see (Evans, Honkapohja, and Williams 2003) for further discussion. We re-emphasize that we stay within the framework of econometric learning. Further ways of modeling the degree of sophistication could naturally be considered if other types of learning were introduced.

When the central bank uses the SG algorithm, parameter updating takes the form

$$(\phi_t^{CB})' = (\phi_{t-1}^{CB})' + \gamma_{CB,t} x_{t-1}(y_{t-1} - \phi_{t-1}^{CB} x_{t-1})', \tag{21}$$

where we have used the notation in equation (11) from Section 2.2. The main difference from the RLS algorithm (14) is that (21) does not involve the matrix of second moments. The private sector is assumed to use RLS. For simplicity, it is assumed that the gain sequences are identical, i.e. $\delta_P = \delta_{CB} = 1$ in the notation of Section 3.1.

The results in the companion paper (Honkapohja and Mitra 2003) show that local convergence under learning is determined by the following two matrices

$$\begin{pmatrix} A^P - I & A^{CB} \\ A^P & A^{CB} - I \end{pmatrix}, \tag{22}$$

$$\begin{pmatrix} F \otimes A^P - I & F \otimes A^{CB} \\ M_w F \otimes A^P & M_w F \otimes A^{CB} - M_w \otimes I \end{pmatrix}, \tag{23}$$

when the private sector uses RLS and the central bank uses the SG algorithm. Here

$$M_w = \lim_{t\to\infty} E(w_t w_t') = \begin{pmatrix} (1-\rho^2)^{-1}\sigma_u^2 & 0 \\ 0 & (1-\mu^2)^{-1}\sigma_g^2 \end{pmatrix}$$

with $\sigma_u^2 = var(\tilde{u}_t)$ and $\sigma_g^2 = var(\tilde{g}_t)$. Formally, we state:

Proposition 7 *If the private sector uses RLS and the central bank the SG algorithm, then learning converges locally if the matrices (22) and (23) have eigenvalues with negative real parts.*

[15]For instance, if the underlying shock process is *iid* normal, then the RLS estimator is minimum variance unbiased. These properties refer to the usual statistical analysis that involves parameter estimation for exogenous processes.

Analogous conditions may be obtained when the private agents use SG learning and the bank uses RLS by inter-changing the roles of A^P and A^{CB} in (22) and (23).

We observe from Proposition 7 that the E-stability conditions continue to be necessary for stability. An application of the results in (Bullard and Mitra 2002) immediately yields the following necessary condition:

Corollary 8 *Consider model (8). If the private sector uses RLS and the central bank the SG algorithm in their learning rules (or vice versa), the dynamics of the economy is stable under learning only if* $\lambda(\chi_\pi - 1) + (1 - \beta)\chi_z > 0$.

We, therefore, see that the conclusion that interest rules violating the Taylor principle are undesirable is quite robust. In particular, such "passive" rules lead to instability irrespective of whether the central bank uses a sophisticated algorithm like RLS or a simple algorithm like SG.

However, as is clear from Proposition 7, the E-stability conditions are no longer sufficient for convergence of learning. The learnability conditions are now influenced by the persistence in the shocks (ρ and μ) and their variances (σ_u^2 and σ_g^2) via the matrix (23). We also note one result, which is easily seen from (23): If ρ and μ are small enough, the E-stability conditions are sufficient for learning stability irrespective of whether the private agents use RLS or the SG algorithm. Next, we study in more detail, first, the situation when the central bank uses the SG algorithm in its updating equations while the private sector uses RLS and, second, the converse situation when the bank uses RLS and the private agents SG in the following section.

3.2.1 Bank Uses SG and Private Agents Use RLS

We now consider the situation when the central bank uses the SG algorithm in its updating equations and the private sector uses RLS. As observed above, the matrices (22) and (23) need to have eigenvalues with negative real parts for stability.

Since stability of (22) is equivalent to the Taylor principle, we concentrate on the matrix (23). It can be shown (e.g. using Mathematica) that the characteristic polynomial of the matrix (23) is symmetric in the quantities (ρ, σ_u^2) and (μ, σ_g^2) of the two shocks u_t and g_t, so that w.l.o.g. the relevant stability or instability conditions can be obtained by considering the case of a single shock, say, g_t. Thus, we may formally assume that F is a scalar μ when examining the eigenvalues of (23). When we obtain stability or instability conditions in terms of μ and σ_g^2, it should be kept in mind that the same conditions are required also for ρ and σ_u^2.

The next thing to note is that (23) is exactly the same matrix (16) (or (30) in Appendix A.1) which appears in Section 3.1, if we replace $\delta_P^{-1}\delta_{CB}$ by (now the scalar) M_w (compare (23) and (30)). In other words, with this identification, (23) will have eigenvalues with negative real parts if and only if (16) has so, given that both $\delta_P^{-1}\delta_{CB}$ and M_w are positive. This observation is useful since we are able to directly apply most of the results of Section 3.1. Henceforth, for the general interest rate rules (4), we confine ourselves to the one shock case, g_t, although, as noted, the symmetric conditions in u_t are also needed in the case of two shocks.

By the above arguments, Corollary 5 implies the following:

Corollary 9 *Consider model (8). Assume that* $(1-\mu^2)^{-1}\sigma_g^2 \geq 1$ *and that the central bank uses the SG algorithm while the private sector uses the RLS algorithm in their learning rules. The dynamics of the economy is then locally stable under learning if and only if* $\lambda(\chi_\pi - 1) + (1-\beta)\chi_z > 0$.

We note that the condition $\sigma_g^2 + \mu^2 \geq 1$ in Corollary 9 is likely to be easily satisfied for plausible values of these parameters. For example, the calibrated values in (Woodford 1999) satisfy this condition since $\mu = 0.35$ and $\sigma_g^2 = 3.72$. In addition, large enough values of χ_z and χ_π continue to make the symmetric equilibrium necessarily stable.

As in Corollary 6, the Taylor principle does not guarantee stability under learning and we consider this further. The following Corollary is proved in Appendix A.2.

Corollary 10 *Consider model (8). Assume that the central bank uses the SG algorithm and the private sector uses the RLS algorithm in their learning rules. Then for all* $\mu > \bar{\mu} \equiv 2(1+\beta+\lambda\varphi)^{-1}$,[16] *the dynamics of the economy is unstable under learning if*

$$(1-\mu^2)^{-1}\sigma_g^2 < \frac{\mu(1+\beta+\lambda\varphi) - 2}{1+\mu\varphi(\chi_z + \lambda\chi_\pi)}. \tag{24}$$

Roughly, the intuition for Corollaries 9 and 10 is as follows. While the Taylor principle suffices for stability of (22), it does not for (23). From (23) one observes that a large σ_g^2 enhances the stabilizing influence of the bank (recall that A^{CB} has negative entries only), which provides intuition for Corollary 9. On the other hand, a small σ_g^2 effectively works towards eliminating the stabilizing influence of the bank and a large μ (in conjunction with this) enhances the de-stabilizing influence of the private sector via (23) since A^P has an eigenvalue more than 1. This provides some intuition for Corollary 10. More formally, the latter is also evident from the fact that in this case half the eigenvalues of (23) are approximately zero and the other half are approximately the eigenvalues of $F \otimes A^P - I$ and that A^P has an eigenvalue more than 1.

Even though Corollary 10 gives theoretical conditions for instability, these conditions will in general be hard to satisfy for plausible values of parameters. It, therefore, seems that stability under learning is very likely to obtain when the bank uses the SG algorithm and subscribes to the Taylor principle in view of Corollary 9.

These results are also borne out by the numerical results for the optimal and Taylor rules. (We do not provide details, which are available on request from the second author.) Generally, we find that both the RE-optimal and the EB-optimal rules lead to stability under learning provided that σ_g^2 is not very small. However, instability arises for small enough values of σ_g^2 and relatively large values of μ, as expected from the discussion

[16] We note that $\bar{\mu} \approx 0.93$ with the Woodford values and is approximately 0.87 with the Clarida *et al* values.

above. Nevertheless, the EB-optimal rule continues to be more robust than the RE-optimal rule in the sense that it delivers a stable economy for a larger domain of values of μ and σ_g^2. In addition, rules fulfilling the Taylor principle yield stability as long as σ_g^2 (or σ_u^2, by symmetry) are not too small.

3.2.2 Bank Uses RLS and Private Agents Use SG

We now consider the converse situation when the central bank uses RLS and the private sector the SG algorithm. In this case we need (22) and

$$\begin{pmatrix} F \otimes A^{CB} - I & F \otimes A^{P} \\ M_w F \otimes A^{CB} & M_w F \otimes A^{P} - M_w \otimes I \end{pmatrix} \tag{25}$$

to have eigenvalues with negative real parts for stability.

We first show that the Taylor principle continues to completely characterize stability for interest rules under certain conditions as shown in the next corollary, which is proved in Appendix A.3.

Corollary 11 *Consider model (8). Let the central bank use RLS and the private sector the SG algorithm in their learning rules. Assume the following two conditions*

$$\begin{aligned} 1 &\leq (1-\mu^2)^{-1}\sigma_g^2, \\ \mu &\leq 2^{-1}\bar{\mu} = (1+\beta+\lambda\varphi)^{-1}, \end{aligned}$$

with $\bar{\mu}$ as in Corollary 10. Then the symmetric equilibrium is locally stable under learning if and only if $\lambda(\chi_\pi - 1) + (1-\beta)\chi_z > 0$ for the interest rule (4).

Thus, stability obtains if μ (and ρ) is small enough and σ_g^2 (and σ_u^2) is not too small in the sense made precise in the Corollary. Woodford's calibrated values satisfy the conditions of Corollary 11 since $\mu = 0.35$ and $\sigma_g^2 = 3.72$ so that, with these values, the Taylor principle completely characterizes stability.

However, instability may arise when μ is large. The next corollary, which is also proved in Appendix A.3, shows that the symmetric equilibrium may be rendered unstable in this case. Before stating the result we define the following expressions

$$\upsilon_1 \equiv \frac{2-\mu(1+\beta+\lambda\varphi)+\mu\varphi[(2-\beta\mu)\chi_z+2\lambda\chi_\pi]}{\mu\lambda\varphi-(1-\mu)(1-\beta\mu)}, \tag{26}$$

$$\upsilon_2 \equiv \frac{1+\mu\varphi(\chi_z+\lambda\chi_\pi)}{\mu(1+\beta+\lambda\varphi)-2}. \tag{27}$$

Corollary 12 *Consider the model (8) and assume that the central bank uses RLS and the private sector uses the SG algorithm in their learning rules. Let $\bar{\mu}$ be as in Corollary 10 and define*

$$\tilde{\mu} \equiv (2\beta)^{-1}[1+\beta+\lambda\varphi-\sqrt{(1+\beta+\lambda\varphi)^2-4\beta}].$$

We have $\bar{\mu} > \tilde{\mu}$.[17] *The dynamics of the economy is locally unstable under learning if*
(a) $(1-\mu^2)^{-1}\sigma_g^2 > \upsilon_1$ *when* $\mu > \tilde{\mu}$
or
(b) $(1-\mu^2)^{-1}\sigma_g^2 > \min[\upsilon_1, \upsilon_2]$ *when* $\mu > \bar{\mu}$,
where υ_1, υ_2 *are defined in (26)-(27).*

Some intuition for Corollaries 11 and 12 follows from our discussion in the previous section. A small enough value of μ always contributes to stability in (25), which explains Corollary 11. On the other hand, a large value of μ (together with a large σ_g^2) increases the possibility of instability arising from the behavior of the private sector (since A^P has an eigenvalue more than 1) when it uses the SG algorithm, which provides some intuition for Corollary 12, see (25).

Regarding policy response, we note that υ_1, υ_2 in Corollary 12 are increasing in χ_π (and χ_z) so that if the central bank reacts aggressively enough to inflation, the inequalities (a) and (b) will be violated. In general, the central bank can continue to contribute towards stability under learning by choosing large values of χ_π (and χ_z).

For further analysis we revert to numerics for the calibrated example. When considering the performance of the two optimal rules, we note that the rules fully neutralize the g_t shock, and so we must state the conditions in terms of ρ and σ_u^2. We have stability for small values of ρ with both versions of the optimal rules, as might be expected from Corollary 11. However, instability arises with either rule when ρ and σ_u^2 are large enough (as expected from Corollary 12). As in Section 3.1.2, the EB-optimal rule continues to yield stability under learning for a larger range of values of ρ and σ_u^2 than the RE-optimal rule. Figures 4-5 illustrate this phenomenon in the (ρ, σ_u^2) space with $\alpha = .1$, $\rho \in [0.7, 1)$ and $\sigma_u^2 \in (0, 5]$. (For values of $\rho \in (0, 0.7)$ and $\sigma_u^2 \in (0, 5]$, we have stability with either version of the optimal rule.) Recall that the entire region is stable in Figures 4 and 5 in the homogenous case.

FIGURES 4 AND 5 HERE

A similar picture emerges with the Taylor rules. Rules fulfilling the Taylor principle lead to instability for large enough values of ρ or μ (as expected from the corollary above). This is illustrated in Figure 6, where we have assumed that there is no cost push shock as in (Woodford 1999). In Figure 6 we set $\sigma_g^2 = 3.72$ (the calibrated value in (Woodford 1999)) and $\chi_z = 0$. Note that rules with $\chi_\pi > 1$ imply instability for values of μ close to 1. Figure 7 also (re)emphasizes this instability. It plots Taylor rules in the (χ_π, σ_g^2) space with $\chi_z = 0$ and $\mu = .9$. Most of the space associated with rules satisfying $\chi_\pi > 1$ yield instability.[18] (Recall that the entire region to the right of $\chi_\pi = 1$ is stable in the homogenous case in Figures 6 and 7.)

FIGURES 6 AND 7 HERE

[17] $\tilde{\mu} \approx 0.68$ with the Woodford values and is approximately 0.58 with the Clarida *et al* values.

[18] On the other hand, with $\mu = .35$ as in (Woodford 1999) rules fulfilling the Taylor principle yield stability under learning illustrating that the problem of instability arises only for μ close to 1.

4 Discussion and Concluding Remarks

In this paper we have considered the argument that the use of central bank internal forecasts in monetary policy making might be a source of instability of the economy. We studied the consequences of the use of internal forecasts for the stability of the economy by means of the learning approach to expectations formation, in which agents may at least temporarily have non-rational forecast functions that are corrected over time. For the analysis we employed a standard forward looking model that is currently the workhorse for studies of monetary policy. For modeling adaptive learning we have focused attention of the benchmark case, where agents use standard econometric learning procedures and they perceive the environment as stationary.

The limitations of our analysis suggest many possibilities for further research. The model of monetary policy could be enriched and the implications of other types of learning procedures could be studied. Moreover, our benchmark assumptions allowed us to leave out the strategic aspects of expectations formation, which can become important under different informational assumptions.

While our results pertain to the specific model and specific ways of learning, they do suggest a general lesson. In these kinds of models the learning behavior of private agents can be a source of instability which needs to be offset by appropriate policy. We have shown that this task is more difficult when policy cannot react directly to private expectations and instead uses internal forecasts as a proxy and additional conditions have to be met to ensure stability of the economy under learning.

The paper has looked at both some optimal policies and Taylor rules for some typical cases of heterogenous learning. Looking at specific policies, the forecast based rule with internal central bank forecasts, recently suggested by (Evans and Honkapohja 2003b), performed well more robustly than the other formulations of optimal discretionary policy. However, that policy - as well as learnable Taylor rules - may not be stable under heterogenous learning for some parameter configurations.

Based on our analysis, we can make the following general suggestions for the conduct of good monetary policy on the part of the central bank.

First, the interest rate rule should satisfy the Taylor principle. Our analysis supports this suggestion since, with forward looking rules, the Taylor principle is equivalent to E-stability of the equilibrium and it is always a necessary condition for convergence of the economy under heterogenous learning.

Second, the central bank should take incoming information about the economy seriously and put sufficient weight on these indicators while setting its interest rate rule. This suggestion is supported by our analysis of the differences in the degree of responsiveness when forecast functions are updated (Section 3.1). We emphasize that our focus has been on the use of incoming information in the updating of the internal forecasting. Naturally, there are further aspects of fine tuning in practice, e.g. getting good information on the exogenous shocks, which we have not covered. Observation errors need not affect stability under learning, see (Evans and Honkapohja 2003b). The implications of observation errors in the data for learning could be analyzed but this must be left for another occasion.

Third, the bank should spend sufficient resources in obtaining large amount of information about the exogenous shocks. This suggestion is supported by results in Section 3.2. It is also supported by our work in progress for cases of asymmetric information.

Informal discussions of monetary policy do tend to support these suggestions. Our contribution has been to lend weight to these informal discussions in an analytical treatment of monetary policy. We conclude by re-emphasizing the importance of the learning approach for monetary policy design.

A Appendix

A.1 Proof of Corollaries 4 and 5

To shorten notation, define $\delta \equiv \delta_P^{-1}\delta_{CB}$. Consider the matrix on the right hand side of (19). Ignoring the scalar $\delta_P > 0$ (which does not affect the sign of the real parts of the eigenvalues of (19)), it can be checked that one of the eigenvalues of the matrix within parentheses on the right hand side of (19) is $-\delta$ and the remaining 3 eigenvalues are given by the characteristic polynomial

$$\begin{aligned} p(m) &= m^3 + a_1 m^2 + a_2 m + a_3, \text{ where} \qquad (28)\\ a_1 &= 1-\beta+\delta+\delta\varphi[\chi_z+\lambda(\chi_\pi-\delta^{-1})] \equiv 1-\beta+\delta+\zeta_1,\\ a_2 &= \delta(1-\beta)+\delta\varphi[(2-\beta)\chi_z+\lambda(2\chi_\pi-\delta^{-1}-1)] \equiv \delta(1-\beta)+\zeta_2,\\ a_3 &= \delta\varphi[(1-\beta)\chi_z+\lambda(\chi_\pi-1)]. \end{aligned}$$

where the definitions of ζ_1, ζ_2 are introduced to shorten notation and should be obvious from above. The necessary and sufficient conditions for the eigenvalues of $p(m)$ to have negative real parts (the Routh conditions) are $a_1 > 0$, $a_3 > 0$, and $a_1 a_2 > a_3$. These conditions imply that $a_2 > 0$ also.

Note that $a_3 > 0$ if and only if $(1-\beta)\chi_z + \lambda(\chi_\pi - 1) > 0$, which provides the necessary condition (17) in Corollary 4. Second, $a_1 > 0$ if and only if condition (18) in Corollary 4 is satisfied. This completes the proof of Corollary 4.

To prove Corollary 5 we first note that, when $\delta > 1$, the Taylor principle suffices to make $a_1 > 0$. We still need to show that $a_1 a_2 > a_3$.

Note that $\zeta_2 = \zeta_1 + a_3$, which we use below. In addition, ζ_1 and ζ_2 are positive when $\delta > 1$. Now

$$\begin{aligned} a_1a_2 - a_3 &= \delta(1-\beta)(1-\beta+\delta)+\delta(1-\beta)\zeta_1+(1-\beta+\delta)\zeta_2+\zeta_1\zeta_2-a_3 \qquad (29)\\ &= \delta(1-\beta)(1-\beta+\delta)+\delta(1-\beta)\zeta_1+(1-\beta+\delta)(\zeta_1+a_3)+\zeta_1(\zeta_1+a_3)-a_3\\ &= \delta(1-\beta)(1-\beta+\delta)+[(\delta+1)(1-\beta)+\delta]\zeta_1+\zeta_1^2+[(1-\beta+\delta)+\zeta_1-1]a_3\\ &= \delta(1-\beta)(1-\beta+\delta)+[(\delta+1)(1-\beta)+\delta]\zeta_1+\zeta_1^2+[\delta-\beta+\zeta_1]a_3. \end{aligned}$$

$\delta > 1$ suffices to make the coefficient of a_3 in the final line above positive since $\zeta_1 > 0$, which in turn implies $a_1 a_2 - a_3 > 0$.

We still need to check the matrix (16) for stability. As in (19), rewrite matrix (16) as (by pulling out δ_P)

$$\delta_P \begin{pmatrix} F \otimes A^P - I & F \otimes A^{CB} \\ \delta_P^{-1}\delta_{CB} F \otimes A^P & \delta_P^{-1}\delta_{CB}(F \otimes A^{CB} - I) \end{pmatrix} \tag{30}$$

and we examine the eigenvalues of the matrix within parentheses in (30). By e.g. using Mathematica, we first note that the characteristic polynomial of this (8×8) matrix is symmetric in the shocks ρ and μ, so that we may consider only the shock ρ and the resulting 4th degree polynomial. One eigenvalue of this polynomial is $-\delta$ and the remaining 3 eigenvalues are given by the polynomial

$$\begin{aligned} q(m) &= m^3 + b_1 m^2 + b_2 m + b_3, \text{ where} \qquad (31) \\ b_1 &= 2 - \rho(1+\beta) + \delta + \delta\rho\varphi[\chi_z + \lambda(\chi_\pi - \delta^{-1})] \equiv 2 - \rho(1+\beta) + \delta + \tau_1, \\ b_2 &= (1-\rho)(1-\beta\rho) + \delta[2 - \rho(1+\beta)] + \delta\rho\varphi[(2-\beta\rho)\chi_z + \lambda(2\chi_\pi - \delta^{-1} - 1)] \\ &\equiv (1-\rho)(1-\beta\rho) + \delta[2-\rho(1+\beta)] + \tau_2, \\ b_3 &= \delta(1-\rho)(1-\beta\rho) + \delta\rho\varphi[(1-\beta\rho)\chi_z + \lambda(\chi_\pi - 1)] \\ &\equiv \delta(1-\rho)(1-\beta\rho) + \tau_3, \end{aligned}$$

where the definitions of τ_i $(i = 1, 2, 3)$ should again be obvious from above.

Note that $\tau_2 = (\tau_1 + \tau_3)$, and that $\tau_i > 0$ by the Taylor principle and the assumption that $\delta > 1$. Next we observe that $b_1 > 0$ and $b_3 > 0$ also by the Taylor principle and the assumptions $0 < \rho < 1$ and $0 < \beta < 1$. We now need to determine the sign of $b_1 b_2 - b_3$ for which we use $\tau_2 = (\tau_1 + \tau_3)$ below.

$$\begin{aligned} b_1 b_2 - b_3 &= [2 - \rho(1+\beta) + \delta + \tau_1][(1-\rho)(1-\beta\rho) + \delta\{2 - \rho(1+\beta)\} + \tau_2] \\ &\quad -\delta(1-\rho)(1-\beta\rho) - \tau_3 \\ &= [2 - \rho(1+\beta) + \delta][(1-\rho)(1-\beta\rho) + \delta\{2-\rho(1+\beta)\}] + \\ &\quad \tau_1[(1-\rho)(1-\beta\rho) + \delta\{2-\rho(1+\beta)\}] + \tau_2[2 - \rho(1+\beta) + \delta] \\ &\quad +\tau_1\tau_2 - \delta(1-\rho)(1-\beta\rho) - \tau_3 \\ &= [2-\rho(1+\beta)][(1-\rho)(1-\beta\rho) + \delta\{2-\rho(1+\beta)\}] + \delta(1-\rho)(1-\beta\rho) \\ &\quad +\delta^2[2-\rho(1+\beta)] + \tau_1[(1-\rho)(1-\beta\rho) + \delta\{2-\rho(1+\beta)\}] \\ &\quad +\tau_2[2-\rho(1+\beta)+\delta] + \tau_1\tau_2 - \delta(1-\rho)(1-\beta\rho) - \tau_3 \\ &= [2-\rho(1+\beta)][(1-\rho)(1-\beta\rho) + \delta\{2-\rho(1+\beta)\}] + \delta^2[2-\rho(1+\beta)] \\ &\quad +\tau_1[(1-\rho)(1-\beta\rho) + \delta\{2-\rho(1+\beta)\}] + (\tau_1+\tau_3)[2-\rho(1+\beta)+\delta] \\ &\quad +\tau_1(\tau_1+\tau_3) - \tau_3 \\ &= [2-\rho(1+\beta)][(1-\rho)(1-\beta\rho) + \delta\{2-\rho(1+\beta)\}] + \delta^2[2-\rho(1+\beta)] \\ &\quad +\tau_1[(1-\rho)(1-\beta\rho) + \delta\{2-\rho(1+\beta)\}] + \tau_1[2-\rho(1+\beta)+\delta+\tau_1] \\ &\quad +\tau_3[1+\delta-\rho(1+\beta)+\tau_1]. \end{aligned}$$

Note that in the final line above all terms are positive since $0 < \rho < 1$, $0 < \beta < 1$ and $\delta > 1$. Hence, $b_1 b_2 - b_3 > 0$ when $\delta > 1$.

Finally, when $\delta = 1$ the stability matrices are the same as in the homogenous case so that E-stability follows from the Taylor principle.

A.2 Proof of Corollary 10

As mentioned in the text, the conditions for stability of (23) when $F \equiv \mu$ are identical to that of (16) or (30). Therefore, the characteristic polynomial (31), in Appendix A.1, determines stability in this case after making the substitution $\delta \equiv (1-\mu^2)^{-1}\sigma_g^2$, so that it takes the form

$$\begin{aligned}
\hat{p}(m) &= m^3 + \hat{b}_1 m^2 + \hat{b}_2 m + \hat{b}_3, \text{ where} \\
\hat{b}_1 &= 2 - \mu(1+\beta+\lambda\varphi) + (1-\mu^2)^{-1}\sigma_g^2[1+\mu\varphi(\chi_z+\lambda\chi_\pi)], \\
\hat{b}_2 &= (1-\mu)(1-\beta\mu) - \mu\lambda\varphi \\
&\quad +(1-\mu^2)^{-1}\sigma_g^2[2-\mu(1+\beta)+\mu\varphi\{(2-\beta\mu)\chi_z+\lambda(2\chi_\pi-1)\}], \\
\hat{b}_3 &= (1-\mu^2)^{-1}\sigma_g^2[(1-\mu)(1-\beta\mu)+\mu\varphi\{(1-\beta\mu)\chi_z+\lambda(\chi_\pi-1)\}].
\end{aligned}$$

in (31). The necessary and sufficient conditions for the eigenvalues of $\hat{p}(m)$ to have negative real parts are given by $\hat{b}_1 > 0, \hat{b}_3 > 0$ and $\hat{b}_1\hat{b}_2 > \hat{b}_3$.

The instability condition in Corollary 10 is simply $\hat{b}_1 < 0$. We note that for $\hat{b}_1 < 0$ it is necessary that $\mu > \bar{\mu} \equiv 2(1+\beta+\lambda\varphi)^{-1}$.

A.3 Proof of Corollaries 11 and 12

We first prove the inequalities

$$\bar{\mu} > \tilde{\mu} > 2^{-1}\bar{\mu}. \tag{32}$$

Here $\bar{\mu}$ and $\tilde{\mu}$ are defined in Corollaries 10 and 12, respectively.

Now,

$$\bar{\mu} > \tilde{\mu} \Leftrightarrow \frac{2}{(1+\beta+\lambda\varphi)} > \frac{1+\beta+\lambda\varphi - \sqrt{(1+\beta+\lambda\varphi)^2 - 4\beta}}{2\beta}.$$

Rearranging and squaring, we require

$$(1+\beta+\lambda\varphi)^2 - 4\beta > (1+\beta+\lambda\varphi)^2 + \frac{16\beta^2}{(1+\beta+\lambda\varphi)^2} - 8\beta$$

or

$$(1+\beta+\lambda\varphi)^2 > 4\beta$$

or

$$(1-\beta)^2 + \lambda^2\varphi^2 + 2\beta\lambda\varphi + 2\lambda\varphi > 0,$$

which is true, given that the parameters are all positive.

In a similar way we show that

$$2^{-1}\bar{\mu} < \tilde{\mu} \Leftrightarrow \frac{1}{(1+\beta+\lambda\varphi)} < \frac{1+\beta+\lambda\varphi - \sqrt{(1+\beta+\lambda\varphi)^2 - 4\beta}}{2\beta}.$$

Rearranging and squaring, we require

$$(1+\beta+\lambda\varphi)^2 - 4\beta < (1+\beta+\lambda\varphi)^2 + \frac{4\beta^2}{(1+\beta+\lambda\varphi)^2} - 4\beta$$

or

$$4\beta^2(1+\beta+\lambda\varphi)^{-2} > 0,$$

which is true.

To prove Corollary 11, we first note that necessity of the Taylor principle $(1-\beta)\chi_z + \lambda(\chi_\pi - 1) > 0$ follows from the necessity of E-stability through the matrix (22). Thus assume that the Taylor principle holds and consider the characteristic polynomial of the (8×8) matrix (25). It is again symmetric in the shocks ρ and μ, so that we may consider only the shock μ and the resultant 4th degree polynomial. One eigenvalue of this polynomial is -1 and the remaining 3 eigenvalues are given by the polynomial

$$\begin{aligned}
r(m) &= m^3 + c_1 m^2 + c_2 m + c_3, \qquad (33)\\
c_1 &= 1 + \mu\varphi(\chi_z + \lambda\chi_\pi) + \{2 - \mu(1+\beta+\lambda\varphi)\}(1-\mu^2)^{-1}\sigma_g^2,\\
c_2 &= (1-\mu^2)^{-1}\sigma_g^2[2 - \mu(1+\beta+\lambda\varphi) + \mu\varphi\{(2-\beta\mu)\chi_z + 2\lambda\chi_\pi\} +\\
&\quad (1-\mu^2)^{-1}\sigma_g^2\{(1-\mu)(1-\beta\mu) - \mu\varphi\lambda\}],\\
c_3 &= (1-\mu^2)^{-2}(\sigma_g^2)^2[\mu\varphi\{(1-\beta\mu)\chi_z + \lambda\chi_\pi\} + \{(1-\mu)(1-\beta\mu) - \mu\varphi\lambda\}].
\end{aligned}$$

Clearly, $c_1 > 0$ and $c_3 > 0$ since $\mu < 2^{-1}\bar{\mu}$, $0 < \beta$, $\mu < 1$ and the Taylor principle holds. We still need to show that $c_1c_2 - c_3 > 0$. We introduce the notation $\delta \equiv (1-\mu^2)^{-1}\sigma_g^2$ below. We first write

$$\begin{aligned}
c_1c_2 &= \delta\{1 + \mu\varphi(\chi_z + \lambda\chi_\pi)\}\{2 - (1+\beta+\lambda\varphi)\mu\} + \delta^2\{2 - \mu(1+\beta+\lambda\varphi)\}^2 +\\
&\quad \delta\mu\varphi\{1 + \mu\varphi(\chi_z + \lambda\chi_\pi)\}\{(2-\beta\mu)\chi_z + 2\lambda\chi_\pi\} +\\
&\quad \delta^2\mu\varphi\{2 - \mu(1+\beta+\lambda\varphi)\}\{(2-\beta\mu)\chi_z + 2\lambda\chi_\pi\} +\\
&\quad \delta^2\{1 + \mu\varphi(\chi_z + \lambda\chi_\pi)\}\{(1-\mu)(1-\beta\mu) - \mu\varphi\lambda\} +\\
&\quad \delta^3\{2 - \mu(1+\beta+\lambda\varphi)\}\{(1-\mu)(1-\beta\mu) - \mu\varphi\lambda\}
\end{aligned}$$

In this expression $(1-\mu)(1-\beta\mu) - \mu\varphi\lambda$ is positive for all $\mu < \tilde{\mu}$ and hence for all $\mu < 2^{-1}\bar{\mu}$ (using (32)) since the expression $\mu\lambda\varphi - (1-\mu)(1-\beta\mu)$ is increasing in μ and is zero when $\mu = \tilde{\mu}$ so that it is negative for all $\mu < \tilde{\mu}$.

In computing $c_1c_2 - c_3$ we ignore first, second, third and fifth terms of the preceding expression for c_1c_2 and (to economize on space) denote them only by .. while keeping in mind that these terms are all positive. We retain only the fourth and final (sixth) term and obtain

$$\begin{aligned}
c_1c_2 - c_3 &= \delta^2\mu\varphi\{2-\mu(1+\beta+\lambda\varphi)\}\{(2-\beta\mu)\chi_z + 2\lambda\chi_\pi\} + \\
&\quad \delta^3\{2-\mu(1+\beta+\lambda\varphi)\}\{(1-\mu)(1-\beta\mu)-\mu\varphi\lambda\} \\
&\quad -\delta^2[\mu\varphi\{(1-\beta\mu)\chi_z + \lambda\chi_\pi\} + \{(1-\mu)(1-\beta\mu)-\mu\varphi\lambda\}] + .. \\
&= \delta^2\mu\varphi\{2-\mu(1+\beta+\lambda\varphi)\}[\{(1-\beta\mu)\chi_z + \lambda\chi_\pi\} + \{\chi_z + \lambda\chi_\pi\}] + \\
&\quad \delta^3\{2-\mu(1+\beta+\lambda\varphi)\}\{(1-\mu)(1-\beta\mu)-\mu\varphi\lambda\} - \\
&\quad \delta^2[\mu\varphi\{(1-\beta\mu)\chi_z + \lambda\chi_\pi\} + \{(1-\mu)(1-\beta\mu)-\mu\varphi\lambda\}] + .. \\
&= \delta^2\mu\varphi\{(1-\beta\mu)\chi_z + \lambda\chi_\pi\}\{1-\mu(1+\beta+\lambda\varphi)\} + \\
&\quad \delta^2\mu\varphi\{2-\mu(1+\beta+\lambda\varphi)\}\{\chi_z + \lambda\chi_\pi\} + \\
&\quad \delta^2\{(1-\mu)(1-\beta\mu)-\mu\varphi\lambda\}[\delta\{2-\mu(1+\beta+\lambda\varphi)\} - 1] + ..
\end{aligned}$$

The first two terms in the final expression are positive since $\mu < 2^{-1}\bar{\mu}$. Regarding the final explicit term, it was shown above that $(1-\mu)(1-\beta\mu)-\mu\varphi\lambda > 0$ for all $\mu < 2^{-1}\bar{\mu}$. Moreover, since $\delta \geq 1$ and $\mu < 2^{-1}\bar{\mu}$,

$$\delta\{2-\mu(1+\beta+\lambda\varphi)\} - 1 \geq 2-\mu(1+\beta+\lambda\varphi) - 1 = 1-\mu(1+\beta+\lambda\varphi) > 0,$$

which proves that the final term is positive. Thus $c_1c_2 - c_3 > 0$ (recall that the .. terms in the expression for $c_1c_2 - c_3$ above are all positive).

To prove Corollary 12, consider the characteristic polynomial (33). Assume that the Taylor principle holds since otherwise we immediately have instability. We note that for $c_1 < 0$, it is necessary that $\mu > \bar{\mu}$ and for $c_2 < 0$, it is necessary that $\mu\lambda\varphi - (1-\mu)(1-\beta\mu) > 0$. The rest follows, since $c_1 < 0$ or $c_2 < 0$ suffices for instability.

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