A classification of locally semicomplete digraphs

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Abstract

In [19] Huang gave a characterization of local tournaments. His characterization involves arc-reversals and therefore may not be easily used to solve other structural problems on locally semicomplete digraphs (where one deals with a fixed locally semicomplete digraph). In this paper we derive a classification of locally semicomplete digraphs which is very useful for studying structural properties of locally semicomplete digraphs and which does not depend on Huang's characterization. An advantage of this new classification of locally semicomplete digraphs is that it allows one to prove results for locally semicomplete digraphs without reproving the same statement for tournaments.

We use our result to characterize pancyclic and vertex pancyclic locally semicomplete digraphs and to show the existence of a polynomial algorithm to decide whether a given locally semicomplete digraph has a kernel.

1 Introduction

Two classical results on tournaments are the facts that every tournament has a Hamiltonian path and every strongly connected tournament has a Hamiltonian cycle. It is an easy exercise to show that each of these results also hold for *semicomplete digraphs* – a slight generalization of tournaments in which there is at least one arc between each pair of distinct vertices.

In [2] the first author proved that the characterizations for Hamiltonian path and cycle in tournaments extend to *locally semicomplete digraphs* – for every vertex x the set of in-neighbours as well as the set of out-neighbours of x induce a semicomplete digraph. He also showed that several other properties of tournaments hold for locally semicomplete digraphs as well.

Since their introduction in [2], locally semicomplete digraphs have been extensively studied, see e.g. [2, 3, 7, 10, 11, 12, 14, 15, 16, 17, 18, 8, 19, 20, 23]. Locally semicomplete digraphs are interesting, not just because they are a natural generalization of tournaments, but also because of their underlying undirected graphs. These are exactly the proper circular arc graphs (a connected graph is a proper circular arc graph if it is the intersection graph of a family of arcs on a circle, none of which properly contains another) [22]. This fact, together with Huang's structure theorem on locally semicomplete digraphs with no directed cycles of length two [19], was used in [11] to develop an optimal linear algorithm for recognizing proper circular arc graphs, in [8] to develop optimal linear algorithms for chromatic number

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and maximum clique in proper circular arc graphs, and in [7] to develop an optimal linear algorithm to recognize locally semicomplete digraphs.

In [19] Huang characterized local tournaments, i.e. locally semicomplete digraphs without 2-cycles. This is a deep and difficult result. Unfortunately, Huang's characterization, which involves arc-reversals, cannot be easily applied to solve other structural problems on locally semicomplete digraphs. In [4] it was shown that Huang's characterization actually implies another classification of locally semicomplete digraphs which is very useful in the study of structural properties of locally semicomplete digraphs. In this paper we prove a more precise classification theorem without using Huang's result. Our proof is based on ideas from [4] and [12]. The concept of a locally semicomplete digraph was recently used in [5] to obtain a new type sufficient condition for general digraphs to have a Hamiltonian cycle.

In [2] it was shown that there are infinite families of strong locally semicomplete digraphs which are not pancyclic and two sufficient conditions for a locally semicomplete digraph to be pancyclic (see Corollaries 4.7 and 4.8) are given. In [13] and [23] some other sufficient conditions for pancyclicity of locally semicomplete digraphs were obtained (cf. Corollary 4.9). In this paper we show how to use our characterization of locally semicomplete digraphs (Theorem 3.12) to give a characterization of pancyclic and vertex pancyclic locally semicomplete digraphs. We also show that deciding whether a given locally semicomplete digraph has a kernel can be done efficiently.

2 Terminology and preliminaries

We shall assume that the reader is familiar with the standard terminology on graphs and digraphs and refer the reader to [9].

If $X \subseteq V(D)$ then we denote by $D\langle X\rangle$ the subgraph of D induced by X. We also use the notation $D - S$, where $S \subset V(D)$, for the digraph $D\langle V(D) \setminus S\rangle$.

The *underlying graph* $U(D)$ of a digraph D is the graph obtained by ignoring all orientations on the arcs of D and deleting possible multiple arcs arising in this way. We say that a digraph D is *connected* if $U(D)$ is a connected graph.

Let D be a digraph. If there is an arc from a vertex x to a vertex y in D, then we say that x *dominates* y and use the notation $x \rightarrow y$ to denote this. If A and B are disjoint subsets of vertices of D such that there is no arc from B to A and $a{\rightarrow}b$ for every choice of $a \in A$ and $b \in B$, then we say that A completely dominates B and denote this by $A{\Rightarrow}B$. We shall use the same notation when A and B are subdigraphs of D. We let $N^-(x)$ (respectively, $N^+(x)$) denote the set of vertices dominating (respectively, dominated by) x in D. Let $d^-(x) = |N^-(x)|, d^+(x) = |N^+(x)|$. For a subdigraph H of D, $N^+(H) = \cup_{x\in V(H)} N^+(x) - V(H)$ and $N^-(H) = \cup_{x\in V(H)} N^-(x) - V(H)$.

Paths and cycles are always directed. A k-cycle is a cycle of length k. Let $g(D)$ denote the length of a shortest cycle of length at least 3 in D and $g_v(D)$ denote the length of a shortest cycle of length at least 3 in D through a vertex $v \in V(D)$. A digraph D is *pancyclic* if it contains a k-cycle for every $3 \le k \le n$, where n is the number of vertices in D. D is *vertex pancyclic* if it contains a k-cycle through a vertex x for every $3 \le k \le n$ and every $x \in V(D)$.

A digraph D is *strongly connected* (or just *strong*) if there exists a path from x to y and a path from y to x in D for every choice of distinct vertices x, y of D. If a digraph is not strong then we can label its strong components $D_1, \ldots, D_s,\ s \ge 2$, such that there is no arc from D_j to D_i if $j > i$. In general this labelling is not unique, but it is so for locally semicomplete digraphs (see Theorem 3.1 below).

If D is strong and S is a subset of $V(D)$ such that $D - S$ is not strong, then S is a *separating set*. A separating set S is *minimal* if no proper subset of S is a separating set of D.

Let R be a digraph on r vertices and let $L_1, ..., L_r$ be a collection of digraphs. Then $R[L_1, ..., L_r]$ is the new digraph obtained from R by replacing each vertex v_i of R with L_i and adding an arc from every vertex of L_i to every vertex of L_j if and only if $v_i \rightarrow v_j$ is in D $(1 \leq i \neq j \leq r)$. Note that if we have $D = R[L_1, ..., L_r]$, then $R, L_1, ..., L_r$ are subdigraphs of D.

A digraph on n vertices is *round* if we can label its vertices $v_0, v_1, \ldots, v_{n-1}$ so that for each i, $N^+(v_i) = \{v_{i+1}, \ldots, v_{i+d^+(v_i)}\}$ and $N^-(v_i) = \{v_{i-d^-(v_i)}, \ldots, v_{i-1}\}$ (modulo n). Note that every strong round digraph is Hamiltonian.

Theorem 2.1 *[2] A local tournament is round if and only if $N^+(v)$ and $N^-(v)$ induce transitive tournaments for every vertex $v \in V(D)$.*

This result was proved for tournaments in [1].

By Theorem 2.1, if a local tournament is round, then there exists a unique (up to cyclic permutations) round labelling of D. We refer to this as the *round labelling* of D.

A locally semicomplete digraph D is *round decomposable* if there exists a round local tournament R on $r \geq 2$ vertices such that $D = R[S_1, \ldots, S_r]$, where each S_i is a strong semicomplete digraph. We call $R[S_1, \ldots, S_r]$ a *round decomposition* of D.

We shall use the following theorem by Moon.

Theorem 2.2 *[21] Every strongly connected semicomplete digraph is vertex pancyclic.*

3 Structure of locally semicomplete digraphs

In this section we provide a useful classification of locally semicomplete digraphs (see Theorem 3.12). We begin with the structure of non-strong locally semicomplete digraphs.

Theorem 3.1 *[2] Let D be a connected locally semicomplete digraph that is not strong. Then the following holds for D:*

(a) *If A and B are distinct strong components of D then either $A \Rightarrow B$, $B \Rightarrow A$, or there are no arcs between them.*

(b) *If A and B are strong components of D, such that $A \Rightarrow B$, then A and B are semicomplete digraphs.*

(c) *The strong components of D can be ordered in a unique way $D_1, D_2, ..., D_p$ such that there are no arcs from D_j to D_i for $j > i$, and D_i dominates D_{i+1} for $i = 1, 2, ..., p-1$.*

The unique sequence $D_1, D_2, ..., D_p$ of the strong components of D described in Theorem 3.1 (c) is called a *strong decomposition* of D with the *initial component* D_1 and the *terminal component* D_p.

It is easy to derive the following consequence of Theorem 3.1.

Corollary 3.2 *Every connected, but not strongly connected locally semicomplete digraph D has a unique round decomposition $R[D_1, D_2, ..., D_p]$, where $D_1, D_2, ..., D_p$ is the strong decomposition of D and R is a round local tournament containing no cycle.*

Another kind of decomposition theorem for locally semicomplete digraphs was described in [14].

Theorem 3.3 *[14] Let D be a connected locally semicomplete digraph that is not strong and let $D_1, ..., D_p$ be the strong decomposition of D. Then D can be decomposed in $r \geq 2$ subdigraphs $D'_1, D'_2, ..., D'_r$ as follows:*

$$D'_1 = D_p, \quad \lambda_1 = p,$$

$$\lambda_{i+1} = \min\{\ j \mid N^+(D_j) \cap V(D'_i) \neq \emptyset\},$$

$$\text{and} \quad D'_{i+1} = D\langle V(D_{\lambda_{i+1}}) \cup V(D_{\lambda_{i+1}+1}) \cup \cdots \cup V(D_{\lambda_i - 1})\rangle.$$

The subdigraphs $D'_1, D'_2, ..., D'_r$ satisfy the properties below:

(a) D'_i consists of some strong components of D and is semicomplete for $i = 1, 2, ..., r$;

(b) D'_{i+1} dominates the initial component of D'_i and there exists no arc from D'_i to D'_{i+1} for $i = 1, 2, ..., r-1$;

(c) if $r \geq 3$, then there is no arc between D'_i and D'_j for i, j satisfying $|j - i| \geq 2$.

For a connected, but not strongly connected locally semicomplete digraph D, the unique sequence $D'_1, D'_2, ..., D'_r$ defined in Theorem 3.3 is called the *semicomplete decomposition* of D.

In the rest of the section we consider the structure of strong locally semicomplete digraphs. We start with a lemma from [2].

Lemma 3.4 *[2] Let D be a strong locally semicomplete digraph and S a minimal separating set of D. Then $D - S$ is connected.*

Lemma 3.5 *If a strong locally semicomplete digraph D is not semicomplete, then there exists a minimal separating set $S \subset V(D)$ such that $D - S$ is not semicomplete. Furthermore, if $D_1, D_2, \ldots, D_p$ is the strong decomposition and $D'_1, D'_2, \ldots, D'_r$ is the semicomplete decomposition of $D - S$, then $r \geq 3$, $D\langle S\rangle$ is semicomplete and we have $D_p \Rightarrow S \Rightarrow D_1$.*

Proof: Suppose $D - S$ is semicomplete for every minimal separating set S. Then $D - S$ is semicomplete for all separating sets S. Hence D is semicomplete, because any pair of non-adjacent vertices can be separated by some separating set S.

Let S be a minimal separating set such that $D - S$ is not semicomplete. Clearly, if $r = 2$ (in Theorem 3.3), then $D - S$ is semicomplete. Thus, $r \geq 3$. By the minimality of S every vertex $s \in S$ dominates a vertex in D_1 and is dominated by a vertex in D_p. Thus if some $x \in D_p$ was dominated by $s \in S$, then, by Theorem 3.1, we would have $D_1 \Rightarrow D_p$ and $D - S$ would be semicomplete. Hence (using that D_p is strongly connected) we get that $D_p \Rightarrow S$ and similarly $S \Rightarrow D_1$. From the last observation it follows that S is semicomplete. □.

First we shall treat, in more details, the case when D is round decomposable. We shall use the following lemma from [6].

Lemma 3.6 *Suppose that D is a digraph which can be decomposed as $D = F[S_1, S_2, ..., S_f]$, where $f = |V(F)| \geq 2$, and let $D_0 = D - \cup_{i=1}^{f}\{(u, v) :\ u, v \in V(S_i)\}$. Then D is strong if and only if D_0 is strong.*

Proposition 3.7 *Let $R[H_1, H_2, ..., H_\alpha]$ be a round decomposition of a strong locally semicomplete digraph D. Then, for every minimal separating set S, there are two integers i and $k \geq 0$ such that $S = V(H_i) \cup ... \cup V(H_{i+k})$.*

Proof: First, we shall use Lemma 3.6 to prove that

$$\text{if } V(H_i) \cap S \neq \emptyset, \text{ then } V(H_i) \subseteq S. \tag{1}$$

Assume that there exists H_i such that $V(H_i) \cap S \neq \emptyset \neq V(H_i) - S$. Using this assumption we shall prove that $D - S$ is strong, contradicting the definition of S.

Let $s' \in V(H_i) \cap S$. To show that $D - S$ is strong, we consider a pair of different vertices x and y of $D - S$ and prove that $D - S$ has an (x, y)-path. Since S is a minimal separating set, $D' = D - (S - s')$ is strong. By Lemma 3.6, $D'_0 = D' - \{(u, v) : \ u, v \in V(H_i)\}$ is also strong. Consider a shortest (x, y)-path P in D'_0. Since the vertices of H_i in D'_0 have the same in- and out-neighbourhoods, P contains at most one vertex from H_i, unless $x, y \in V(H_i)$ in which case P contains only those two vertices from H_i. If s' is not on P, we are done. Thus, assume that s' is on P. Then, since P is shortest possible, neither x nor y belongs to H_i. Now we can replace s' with a vertex in $V(H_i) - S$. Therefore, $D - S$ has an (x, y)-path, so (1) is proved.

Suppose that S consists of disjoint sets $T_1, ..., T_\ell$ such that

$$T_i = V(H_{j_i} \cup ... \cup H_{j_i+k_i}) \quad \text{and} \quad V(H_{j_i-1} \cup H_{j_i+k_i+1}) \cap S = \emptyset$$

for $i \in \{1, ..., \ell\}$. If $\ell \geq 2$, then $D - T_i$ is strong and hence H_{j_i-1} dominates $H_{j_i+k_i+1}$ for every $i = 1, ..., \ell$. Therefore, $D - S$ is strong; a contradiction. □.

Corollary 3.8 *If a locally semicomplete digraph D is round decomposable, then it has a unique round decomposition $D = R[D_1, D_2, ..., D_\alpha]$.*

Proof: Suppose that D has two different round decompositions: $D = R[D_1, ..., D_\alpha]$ and $D = R'[H_1, ..., H_\beta]$.

By Corollary 3.2, we may assume that D is strong. By the definition of a round decomposition, this implies that $\alpha, \beta \geq 3$. Let S be a minimal separating set of D. By Proposition 3.7, we may assume w.l.o.g that $S = V(D_1 \cup ... \cup D_i) = V(H_1 \cup ... \cup H_j)$ for some i and j. Since $D - S$ is non-strong, by Corollary 3.2, $D_{i+1} = H_{j+1}$,..., $D_\alpha = H_\beta$ (in particular, $\alpha - i = \beta - j$). Now it sufficies to prove that

$$D_1 = H_1, ..., D_i = H_j \text{ (in particular, } i = j) \tag{2}$$

If S is non-strong, then (2) follows by Corollary 3.2. If S is strong, the first consider the case $\alpha = 3$. Clearly, $S = V(D_1)$. Assuming that $j > 1$, we obtain that the subgraph of D induced by S has a strong round decomposition. This contradicts the fact that R' is a local tournament. Therefore, (2) is true for $\alpha = 3$. If $\alpha > 3$, then we can find a separating set in $D\langle S \rangle$ and conclude by induction that (2) holds. □.

Proposition 3.7 allows us to construct a polynomial algorithm for checking whether a locally semicomplete digraph is round decomposable.

Proposition 3.9 *There exists a polynomial algorithm to decide if a given locally semicomplete digraph D has a round decomposition and to find this decomposition if it exists.*

Proof: We only give a sketch of the algorithm. Find a minimal separating set S in D starting with $S' = N^+(x)$ for a vertex $x \in V(D)$ and deleting vertices from S'. Construct the strong components of $D\langle S\rangle$ and $D - S$ and label these $D_1, D_2, ..., D_\alpha$. For every pair D_i and D_j ($1 \le i \ne j \le \alpha$), we check the following: if there exist some arcs between D_i and D_j, then either $D_i \Rightarrow D_j$ or $D_j \Rightarrow D_i$. If we find a pair for which the above condition is false, then D is not round decomposable. Otherwise, we form a digraph $R = D\langle\{x_1, x_2, ..., x_\alpha\}\rangle$, where $x_i \in V(D_i)$ for $i = 1, 2, ..., \alpha$. We check whether R is round by using Theorem 2.1. If R is not round, then D is not round decomposable. Otherwise, D is round decomposable and $D = R[D_1, ..., D_\alpha]$.

It is not difficult to verify that our algorithm is correct and polynomial. □.

Now we consider strongly connected locally semicomplete digraphs which are not semicomplete and not round decomposable. We first show that the semicomplete decomposition of $D - S$ has exactly three components, whenever S is a minimal separating set such that $D - S$ is not semicomplete.

Lemma 3.10 *Let D be a strong locally semicomplete digraph which is not semicomplete. Either D is round decomposable, or D has a minimal separating set S such that the semicomplete decomposition of $D - S$ has exactly three components D'_1, D'_2, D'_3.*

Proof: By Lemma 3.5, D has a minimal separating set S such that the semicomplete decomposition of $D - S$ has at least three components.

Assume now that the semicomplete decomposition of $D - S$ has more than three components $D'_1, ..., D'_r$ ($r \ge 4$). Let $D_1, D_2, ..., D_p$ be the strong decomposition of $D - S$. According to Theorem 3.3 (c), there is no arc between D'_i and D'_j if $|i - j| \ge 2$. It follows from the definition of a locally semicomplete digraph that

$$N^+(D'_i) \cap S = \emptyset \text{ for } i \ge 3 \text{ and } N^-(D'_j) \cap S = \emptyset \text{ for } j \le r - 2. \tag{3}$$

By Lemma 3.5, $D\langle S\rangle$ is semicomplete and $S = N^+(D_p)$. Let $D_{p+1}, ..., D_{p+q}$ be the strong decomposition of $D\langle S\rangle$. Using (3) and the assumption $r \ge 4$, it is easy to check that if there is an arc between D_i and D_j ($1 \le i \ne j \le p + q$), then $D_i \Rightarrow D_j$ or $D_j \Rightarrow D_i$. Let $R = D\langle\{x_1, x_2, ..., x_{p+q}\}\rangle$ with $x_i \in V(D_i)$ for $i = 1, 2, ..., p + q$. Now it suffices to prove that R is a round local tournament.

Since R is a subdigraph of D and no pair D_i, D_j induces a strong digraph, we see that R is a local tournament. By Corollary 3.2 each of the subdigraphs $R' = R - \{x_{p+1}, ..., x_{p+q}\}$, $R'' = R - V(R) \cap V(D'_{r-1})$ and $R''' = R - V(R) \cap V(D'_2)$ is round. Since $N^+(v) \cap V(R)$ (as well as $N^-(v) \cap V(R)$) is completely contained in one of the sets $V(R'), V(R'')$ and $V(R''')$ for every $v \in V(R)$, we see that R is round.

Thus if $r \ge 4$, then D is round decomposable. □.

Our next result is a characterization of locally semicomplete digraphs which are not semicomplete and not round decomposable. This characterization was proved for the first time in [12]. A weaker form was obtained earlier in [4]. Here we give a different proof of this result.

Lemma 3.11 *Let D be a strong locally semicomplete digraph which is not semicomplete. Then D is not round decomposable if and only if the following conditions are satisfied:*

(a) There is a minimal separating set S such that $D - S$ is not semicomplete and for each such S, $D\langle S\rangle$ is semicomplete and the semicomplete decomposition of $D - S$ has exactly three components D'_1, D'_2, D'_3;

(b) *There are integers α, β, μ, ν with $\lambda_2 \leq \alpha \leq \beta \leq p-1$ and $p+1 \leq \mu \leq \nu \leq p+q$ such that*

$$N^-(D_\alpha) \cap V(D_\mu) \neq \emptyset \text{ and } N^+(D_\alpha) \cap V(D_\nu) \neq \emptyset,$$
$$\text{or} \quad N^-(D_\mu) \cap V(D_\alpha) \neq \emptyset \text{ and } N^+(D_\mu) \cap V(D_\beta) \neq \emptyset,$$

where $D_1, D_2, ..., D_p$ and $D_{p+1}, ..., D_{p+q}$ are the strong decompositions of $D - S$ and $D\langle S\rangle$, respectively, and D_{λ_2} is the initial component of D'_2.

Proof: If D is round decomposable and satisfies (a), then $D = R[D_1, D_2, ..., D_{p+q}]$, where R is the digraph obtained from D by contracting each D_i into one vertex. This follows from Corollary 3.2 and the fact that each of the digraphs $D - S$ and $D - V(D'_2)$ has a round decomposition that agrees with this structure. Now it is easy to see that D does not satisfy (b).

Suppose now that D is not round decomposable. By Lemmas 3.5 and 3.10, D satisfies (a), so we only have to prove that it also satisfies (b).

If there are no arcs from S to D'_2, then it is easy to see that D has a round decomposition. If there exist components D_{p+i} and D_j with $V(D_j) \subseteq V(D'_2)$, such that there are arcs in both directions between D_{p+i} and D_j, then D satisfies (b). So we can assume that for every pair of sets from the collection $D_1, D_2, ..., D_{p+q}$, either there are no arcs between these sets, or one set completely dominates the other. Then, by Theorem 2.1, D is round decomposable, with round decomposition $D = R[D_1, D_2, ..., D_{p+q}]$ as above, unless we have three subdigraphs $X, Y, Z \in \{D_1, D_2, ..., D_{p+q}\}$ such that $X \Rightarrow Y \Rightarrow Z \Rightarrow X$ and there exists a subdigraph $W \in \{D_1, D_2, ..., D_{p+q}\} \setminus \{X, Y, Z\}$ such that either $W \Rightarrow X, Y, Z$ or $X, Y, Z \Rightarrow W$.

One of the subdigraphs X, Y, Z, say w.l.o.g. X, is a strong component of $D\langle S\rangle$. If we have $V(Y) \subseteq S$ also, then $V(Z) \subseteq V(D'_2)$ and W is either in $D\langle S\rangle$ or in D'_2 (there are four possible positions for W satisfying that either $W \Rightarrow X, Y, Z$ or $X, Y, Z \Rightarrow W$). In each of these cases it is easy to see that D satisfies (b). For example, if W is in $D\langle S\rangle$ and $W \Rightarrow X, Y, Z$, then any arc from W to Z and from Z to X satisfies the first part of (b). The proof is similar when $V(Y) \subseteq V(D'_3)$. Hence we can assume that $V(Y) \subseteq V(D'_2)$. If $Z = D_p$, then W must be either in $D\langle S\rangle$ and $X, Y, Z \Rightarrow W$, or $V(W) \subseteq V(D'_2)$ and $W \Rightarrow X, Y, Z$ (which means that $W = D_i$ and $Y = D_j$ for some $\lambda_2 \leq i < j < p$). In both cases it is easy to see that D satisfies (b). The last case $V(Y), V(Z) \subseteq V(D'_2)$ can be treated similarly. □.

We can now state a classification of locally semicomplete digraphs.

Theorem 3.12 *Let D be a connected locally semicomplete digraph. Then exactly one of the following possibilities holds.*

(a) *D is round decomposable with a unique round decomposition $R[D_1, D_2, ..., D_\alpha]$, where R is a round local tournament on $\alpha \geq 2$ vertices and D_i is a strong semicomplete digraph for $i = 1, 2, ..., \alpha$;*

(b) *D is not round decomposable and not semicomplete and it has the structure as described in Lemma 3.11;*

(c) *D is a semicomplete digraph which is not round decomposable.*

Below we shall use the following:

Lemma 3.13 *Let D be a strong non-round decomposable locally semicomplete digraph and let S be a minimal separating set of D such that $D - S$ is not semicomplete. Let $D_1, \ldots, D_p$*

be the strong decomposition of $D-S$ *and* $D_{p+1},...,D_{p+q}$ *be the strong decomposition of* $D\langle S\rangle$. *Suppose that there is an arc* $s\to v$ *from* S *to* D_2' *with* $s\in V(D_i)$ *and* $v\in V(D_j)$, *then*

$$D_i\cup D_{i+1}\cup...\cup D_{p+q}\Rightarrow D_3'\Rightarrow D_{\lambda_2}\cup...\cup D_j.$$

Proof: By Lemma 3.5, $D_p\Rightarrow S\Rightarrow D_1$. The fact that $D-S$ is not semicomplete implies that there are no arcs from D_3' to S, since this would imply an arc between D_3' and D_1'. Since $D_i\Rightarrow D_1$ and $s\to v$, we have $D_1\Rightarrow D_j$. By Lemma 3.11 the digraph D^* obtained from D by deleting all arcs between S and D_2' is round decomposable. Hence $D_3'\Rightarrow D_{\lambda_2}\cup...\cup D_j$. The fact $D_i\cup D_{i+1}\cup...\cup D_{p+q}\Rightarrow D_3'$ can be proved analogously. □.

The following result is an easy consequence of [19, Theorem 4.5] and as we shall see also of Theorem 3.12.

Corollary 3.14 *If* D *is a non-round decomposable locally semicomplete digraph, then the independence number of* $U(D)$ *is at most two.*

Proof: If D is semicomplete, then we are done. So we may assume that D is not semicomplete. Thus D has the structure as described in Lemma 3.11. Let S be a minimal separating set of D such that $D-S$ is not semicomplete. We denote by $D_1,D_2,...,D_p$ and $D_{p+1},...,D_{p+q}$ the strong decompositions of $D-S$ and $D\langle S\rangle$, respectively. Let D_1',D_2',D_3' be the semicomplete decomposition of $D-S$.

Suppose to the contrary that D contains three independent vertices x_1,x_2 and x_3. Because $D_2'\Rightarrow D_1'\Rightarrow S$ and D_3' is semicomplete, none of $\{x_1,x_2,x_3\}$ belongs to D_1'. So we may assume w.l.o.g. that $x_1\in V(D_t)\subseteq S, x_2\in V(D_{t'})\subseteq V(D_2')$ and $x_3\in V(D_3')$.

We consider only the case that there are integers α,μ,ν with $\lambda_2\le\alpha\le p-1$ and $p+1\le\mu\le\nu\le p+q$ such that

$$N^-(D_\alpha)\cap V(D_\mu)\neq\emptyset \text{ and } N^+(D_\alpha)\cap V(D_\nu)\neq\emptyset$$

(one can similarly discuss the other case). By Lemma 3.13, we have $t<\mu$ and $t'>\alpha$, furthermore, there is no arc from D_t to D_2'. Since $D_t\Rightarrow D_\nu$ and there is an arc from D_α to D_ν, it follows that $D_\alpha\Rightarrow D_t$. Because $D_\alpha\Rightarrow D_{t'}$, we deduce $D_{t'}\Rightarrow D_t$, in particular, we have $x_2\to x_1$; a contradiction. Therefore, the independence number of $U(D)$ is at most two. □.

4 Pancyclic and vertex pancyclic locally semicomplete digraphs

Lemma 4.1 *Let* R *be a strong round local tournament and let* C *be a shortest cycle of* R *and suppose* C *has* $k\ge 3$ *vertices. Then there exists a round labelling* $v_0,v_1,\dots,v_{n-1}$ *of* R *and indices* $0<a_1<a_2<\dots<a_{k-1}<n$ *so that* $C=v_0v_{a_1}v_{a_2}\dots v_{a_{k-1}}v_0$.

Proof: Let C be a shortest cycle and $v_0,v_1,\dots,v_{n-1}$ a round labelling of R so that $v_0\in V(C)$. If the claim is not true, then there exist k,l so that $C=v_0v_{a_1}v_{a_2}\dots v_{a_{k-1}}v_0$, where $0<a_1<\dots<a_{l-1}$ and $a_l<a_{l-1}$. Now the fact that R is round implies that $v_{l-1}\to v_0$, contradicting the fact that C is a shortest cycle. □.

Lemma 4.2 *A strong round local tournament* R *on* r *vertices has cycles of length* $k,k+1,...,r$, *where* $k=g(R)$.

Proof: By Lemma 4.1 we may assume that R contains a cycle $v_{i_1}v_{i_2}...v_{i_k}v_{i_1}$, where $0=i_1<i_2<...<i_k<r$. Because D is strong, v_{i_m} dominates all the vertices $v_{i_m+1},\dots,v_{i_{m+1}}$

for $m = 1, 2, ..., k$. Now it is easy to see that D has cycles of lengths $k, k+1, ..., r$ through the vertices $v_{i_1}, v_{i_2}, ..., v_{i_k}$. □.

The following lemma is proved in the same way as Lemma 4.2 (by considering a shortest cycle, which by the assumption has length at most k).

Lemma 4.3 *If a strong round local tournament with r vertices has a cycle of length k through a vertex v, then it has cycles of all lengths $k, k+1, \ldots, r$ through v.*

Lemma 4.4 *Let D be a strong round decomposable locally semicomplete digraph with round decomposition $D = R[S_1, \ldots, S_r]$. Then*

1) D is pancyclic if and only if either $g(R) = 3$ or $g(R) \leq \max_{1 \leq i \leq r} |V(S_i)| + 1$.

2) D is vertex pancyclic if and only if for each $i = 1, ..., r$, either $g_{r_i}(R) = 3$ or $g_{r_i}(R) \leq |V(S_i)| + 1$, where r_i is the vertex of R corresponding to S_i.

Proof: As each S_i is semicomplete, it has a Hamiltonian path P_i. Thus, starting from an r-cycle with one vertex from each S_i, we can get cycles of all lengths $r+1, r+2, \ldots, n$, by taking appropriate pieces of Hamiltonian paths $P_1, P_2, \ldots, P_r$ in $S_1, \ldots, S_r$. Thus, if $g(R) = 3$ then D is pancyclic by Lemma 4.2. If $g(R) \leq \max_{1 \leq i \leq r} |V(S_i)| + 1$, then D is pancyclic by Lemma 4.2 and the fact that every S_i has cycles of lengths 3,4,...,$|V(S_i)|$ (by Theorem 2.2). If $g(R) > 3$ and, for every $i = 1, ..., r$, $g(R) > |V(S_i)| + 1$, then D is not pancyclic since it has no $(g(R) - 1)$-cycle. The second part of the lemma can be proved analogously, using Lemma 4.3 and Theorem 2.2. □.

Lemma 4.5 *Let D be a strong locally semicomplete digraph on n vertices which is not round decomposable. Then D is vertex pancyclic.*

Proof: If D is semicomplete, then we are done by Theorem 2.2. So we assume that D is not semicomplete. Thus, D has the structure described in Lemma 3.11.

Let S be a minimal separating set of D such that $D - S$ is not semicomplete and let $D_1, D_2, ..., D_p$ be the strong decomposition of $D - S$. Since the subdigraph $D\langle S\rangle$ is semicomplete, it has also a strong decomposition, denoted by $D_{p+1}, ..., D_{p+q}$ with $q \geq 1$. Recalling Lemma 3.11 (a), the semicomplete decomposition of $D - S$ contains exactly three components D'_1, D'_2, D'_3. Recall that the index of the initial component of D'_2 is λ_2. From Theorem 3.3 and Lemma 3.5, we see that $D'_2 \Rightarrow D'_1 \Rightarrow S \Rightarrow D_1$ and there is no arc between D'_1 and D'_3.

We first consider the spanning subdigraph D^* of D which is obtained by deleting all the arcs between S and D'_2. By Lemma 3.11, D^* is a round decomposable locally semicomplete digraph and $D^* = R^*[D_1, D_2, \ldots, D_{p+q}]$, where R^* is the round locally semicomplete digraph obtained from D^* by contracting each D_i to one vertex (or, equivalently, R^* is the digraph obtained by keeping an arbitrary vertex from each D_i and deleting the rest). It is easy to see that $g(R^*) = 4$. Therefore, D^* is vertex 5-pancyclic by Lemma 4.4 if $n \geq 5$. Thus, it remains to show that every vertex of D lies on a 3-cycle and a 4-cycle.

We define

$$t = \max\{\ i\ |N^+(S) \cap V(D_i) \neq \emptyset, \lambda_2 \leq i < p\},$$

$$A = V(D_{\lambda_2}) \cup ... \cup V(D_t),$$

$$t' = \min\{\ j\ |N^+(D_j) \cap V(D'_2) \neq \emptyset, p+1 \leq j \leq p+q\}$$

$$\text{and } B = V(D_{t'}) \cup ... \cup V(D_{p+q}).$$

By Lemma 3.13 $B \Rightarrow D'_3 \Rightarrow A$.

Because of $S \Rightarrow D_1 \Rightarrow D_{\lambda_2} \Rightarrow D_1' \Rightarrow S$, every vertex of S is in a 4-cycle. Since $B \Rightarrow D_3' \Rightarrow A \Rightarrow D_1' \Rightarrow S$, each vertex of $V(D_3') \cup A \cup V(D_1')$ is contained in a 4-cycle.

¿From the definition of t', there is an arc sa from $D_{t'}$ to A. By Lemma 3.11 (b), it is easy to see that there is an arc $a's'$ from A to B. Let v be a vertex of D_1' and let w be a vertex of D_3'. It is clear that $savs$ and $s'wa's'$ are 3-cycles.

Suppose D_2' contains a vertex x that is not in A, then $A \Rightarrow x$. We also have $x, s' \in N^+(a')$ and this implies that $x \rightarrow s'$. From this we get that $x \Rightarrow D_{t'}$, in particular, $x \rightarrow s$. Hence $xsax$ is a 3-cycle and $xvsax$ is a 4-cycle. Thus, we only need to show that every vertex of $S \cup A$ is contained in a 3-cycle.

Let u be a vertex of S with $u \in V(D_\ell)$. If D_ℓ has at least three vertices, then u lies on a 3-cycle by Theorem 2.2. So we assume $|V(D_\ell)| \leq 2$. If $\ell < t'$, then u and a' are adjacent because D_ℓ dominates the vertex s' of B. If $\ell \geq t'$, then either $u = s$ or $s \rightarrow u$, and hence u, a are adjacent. Therefore, in any case, u is adjacent to one of $\{a, a'\}$. Assume without loss of generality that a and u are adjacent. If $u \rightarrow a$, then $uavu$ is a 3-cycle. If $a \rightarrow u$, then $uwau$ is a 3-cycle because of $D_3' \rightarrow A$. Hence, every vertex of S has the desired property.

Finally, we note that $S' = N^+(D_3')$ is a subset of $V(D_2')$ and it is also a minimal separating set of D. Furthermore, $D - S'$ is not semicomplete. From the proof above, every vertex of S' is also in a 3-cycle. So the proof of the theorem is completed by the fact $A \subseteq S'$. $\square$.

Combining Lemmas 4.4 and 4.5 we have the following characterization of pancyclic and vertex pancyclic locally semicomplete digraphs.

Theorem 4.6 *A strong locally semicomplete digraph D is pancyclic if and only if it is not of the form $D = R[S_1, \ldots, S_r]$, where R is a round local tournament with $g(R) > \max\{2, |V(S_1)|, ..., |V(S_r)|\} + 1$. D is vertex pancyclic if and only if D is not of the form $D = R[S_1, \ldots, S_r]$, where R is a round local tournament with $g_{r_i}(R) > \max\{2, |V(S_i)|\} + 1$ for some $i \in \{1, ..., r\}$, where r_i is the vertex of R corresponding to S_i.*

The following two partial results from [2] on pancyclic locally semicomplete digraphs are immediate consequences of Theorem 4.6 (a digraph D is *chordal* if $U(D)$ is chordal, i.e. it has no induced cycles of length more than 3):

Corollary 4.7 *If a strong locally semicomplete digraph is chordal then it is pancyclic.*

Corollary 4.8 *Let D be a strong locally semicomplete digraph which contains an induced cycle of length 4 such that one arc of the cycle is in a 3-cycle. Then D is pancyclic.*

A vertex v of a digraph D is *locally strongly connected* if $D\langle N^+(v) \cup N^-(v) \cup \{v\}\rangle$ is strong.

Corollary 4.9 *[23] If a locally semicomplete digraph D on n vertices contains a locally strongly connected vertex v, then D is pancyclic and v is contained in cycles of all lengths $3, 4, \ldots, n$.*

Proof: Let v be a locally strongly connected vertex of D. By Theorems 2.2 and 4.6 we may assume that D is not semicomplete and D is round decomposable with round decomposition $D = R[S_1, \ldots, S_r]$, $r \geq 3$. Let S_i be the subgraph containing v. Since v is locally strongly connected and D is not semicomplete, $N^-(v) \setminus V(S_i) \neq \emptyset$ and $N^+(v) \setminus V(S_i) \neq \emptyset$. Let $V(S_j) \cup V(S_{j+1}) \cup \ldots \cup V(S_{i-1})$ be the vertices of $N^-(v) \setminus V(S_i)$ and $V(S_{i+1}) \cup \ldots \cup V(S_k)$ be the vertices of $N^+(v) \setminus V(S_i)$ (with the obvious calculations mod r). Furthermore, the facts that R is round and v is locally strongly connected imply that $S_k \Rightarrow S_j$. Thus R contains a 3-cycle and by Lemma 4.4, D is pancyclic. The fact that v is contained in cycles of all lengths is proved as the last part of Lemma 4.4. $\square$.

Note that Theorem 4.6 provides a polynomial algorithm for checking whether a locally semicomplete digraph is pancyclic or vertex pancyclic. Indeed, using the breadth-first search one can find a shortest cycle of length at least 3 containing a given vertex in linear time. Moreover, by Proposition 3.9 one can verify whether a locally semicomplete digraph has a round decomposition and find this decomposition (if it exists) in polynomial time.

5 Kernels in locally semicomplete digraphs

A *kernel* in a digraph D is a subset $K \subset V(D)$ such that $D\langle K\rangle$ has no arcs and for every $v \in V(D) \setminus K$ there exists a $k \in K$ such that $k \rightarrow v$ is an arc of D.

Thus a semicomplete digraph has a kernel if and only if it has some vertex which dominates all other vertices.

Lemma 5.1 *There exists a polynomial algorithm to decide if a round local tournament has a kernel.*

Proof: Let R be a round local tournament with vertex set $\{v_0, v_1, \ldots, v_{r-1}\}$. Let T_R be a clock with a dial on r hours $v_0, v_1, \ldots, v_{r-1}$ corresponding to the vertices of R, and define for each v_i the time interval $T_i = [v_i, v_{i+d^+(v_i)}]$. We call two time intervals *independent* if they do not overlap. It is easy to see that R has a kernel if and only if the dial of the time clock T_R can be covered by independent time intervals. This can be checked in time $O(r^2)$. Note that if R is not strong and R has a kernel, then it is unique (this corresponds to a unique way to cover the dial of T_R). $\square$.

Theorem 5.2 *There exists a polynomial algorithm to decide if a given locally semicomplete digraph has a kernel.*

Proof: By the remark on semicomplete digraphs above, we may assume that D is a locally semicomplete digraph which is not semicomplete. If $U(D)$ has independence number two, then we can simply check, for each set of two non-adjacent vertices of D, whether they form a kernel. So suppose, by Theorem 3.12 and Corollary 3.14, that D is round decomposable and let $D = R[S_1, \ldots, S_r]$ be the round decomposition of D and recall that each S_i is a strong semicomplete digraph. We may assume, by Lemma 5.1, that some S_i has at least two vertices. Note that, unless S_i has a kernel, no kernel of D can contain a vertex of S_i, because if some vertex of S_j, $j \neq i$ dominates a vertex in S_i, then $S_j \Rightarrow S_i$. Now it is easy to see that D has a kernel if and only if the scale of the time clock T defined with respect to R, but where we have put $T_i = []$ (the empty interval) for each i such that $|S_i| \geq 2$ and S_i has no kernel, can be covered by independent intervals. The complexity of the corresponding algorithm is at most $O(n^3)$, where n is the number of vertices of D. $\square$.

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