

The Haar Wavelet Transform of a Dendrogram: Additional Notes

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Abstract

We consider the wavelet transform of a finite, rooted, node-ranked, p -way tree, focusing on the case of binary ($p = 2$) trees. We study a Haar wavelet transform on this tree. Wavelet transforms allow for multiresolution analysis through translation and dilation of a wavelet function. We explore how this works in our tree context.

Keywords: Haar wavelet transform; binary tree; ultrametric topology; p -adic numbers; hierarchical clustering; data mining; local field; abelian group.

1 Introduction

In a companion paper, which we will refer to as Paper I (“The Haar wavelet transform of a dendrogram”), a new transform is applied to a hierarchical clustering. Various examples are given of uses of this transform, prior to applying the inverse transform.

In this paper, we look at linkages with other ways of understanding the wavelet transform, with the classical démarche described in Appendix 1. Our aim is to understand the wavelet transform when applied to hierarchical clustering dendrograms (where notation and expression as ultrametric topology are summarized in Appendix 2).

After all, both the wavelet transform and hierarchical clustering aspire to multiresolution or multiscale analysis. The natural question is then: how do they differ and are there different aspects that they bring to the data analysis task?

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A good deal of recent work on wavelet transform has been through group theoretic approaches.

Footnote et al. (2000a) point to how group theoretic understanding can lead to a “wealth of new analysis filters” (in the context of multiresolution signal and image analysis). The same point is made by the SMART project (SMART, 2005), including the change to have automatic generation of new transform algorithms. Believing that algorithms should be developed if and only if there is a verifiable user need for them, we would instead point to another reason why group theoretic understanding is crucial to data analysis. A great deal of observed reality can be understood by way of observed symmetries, and groups summarized and encapsulate the properties of these symmetries. For time evolving phenomena, therefore, or spatial coordinate referenced phenomena, it may be possible to replace analysis that is time-referenced or referenced to particular coordinate systems with a more general, more generic, symmetry analysis. This is the vision opened up by the study of group actions on a set of objects.

Parenthetically, one fascinating way as to how this works can be seen in Cendra and Marsden (2003). The authors develop (i) an analytic theory of dynamics as functions of spatial and temporal coordinates; and (ii) group theoretic interpretation of, in parts of the study, return or phase maps.

Our approach can be stated as follows. Let \mathbb{U} be an ultrametric space, associated with an m -dimensional embedding, \mathbb{R}^m . We note that an ultrametric space is necessarily of 0 dimensionality; and that minimal dimensionality real embedding of an ultrametric has been studied by Lemin and others (Lemin, 2001; Bartal et al., 2004).

The partial order of (clopen) set inclusion is denoted by the binary tree or hierarchy, H . Consider the group action comprising rotations or cyclic permutations (these are equivalent) of subnodes of any node in H , and we will denote this group as G_H . Then we study the wavelet transform of $L^2(\mathbb{U})$ resulting from the actions of group G_H .

Having already discussed the new wavelet transform in Paper I, we can give one result relating to it in the context of the group of equivalent representations of H as follows.

Theorem: For all 2^{n-1} equivalent representations of H (here: unlabeled graph isomorphisms), the dendrogram Haar wavelet transform is unique.

The proof follows from the definition of the wavelet coefficients at each level, ν ; whereas the equivalent representations of H are intra level.

It follows from this theorem that we have a unique matrix representation of a dendrogram.

2 Previous Work on Wavelet Transforms of Data Tables

In this section we will review recent work using wavelet transforms on data tables, and show how our work represents a radically new approach to tackling similar objectives.

Approximate query processing arises when data must be kept confidential so that only aggregate or macro-level data can be divulged. Approximate query processing also provides a solution to access of information from massive data tables.

One approach to approximate database querying through aggregates is sampling. However a join operation applied to two uniform random samples results in a non-uniform result, which furthermore is sparse (Chakrabarti, Garofalakis, Rastogi and Shim, 2001). A second approach is to keep histograms on the coordinates. For a multidimensional feature space, one is faced with a “curse of dimensionality” as the dimensionality grows. A third approach is wavelet-based, and is of interest to us in this article.

A form of progressive access to the data is sought, such that aggregated data can be obtained first, followed by greater refinement of the data. The Haar wavelet transform is a favored transform for such purposes, given that reconstructed data at a given resolution level is simply a recursively defined mean of data values. Vitter and Wang (1999) consider the combinatorial aspects of data access using a Haar wavelet transform, and based on a multi-way data hypercube. Such data, containing scores or frequencies, is often found in the commercial data mining context of OLAP, On-Line Analytical Processing.

As pointed out in Chakrabarti et al. (2001), one can treat multidimensional feature hypercubes as a type of high dimensional image, taking the given order of feature dimensions as fixed. As an alternative a uniform “shift and distribute” randomization can be used (Chakrabarti et al., 2001).

There are problems, however, in directly applying a wavelet transform to a data table. Essentially, a relational table (to use database terminology; or matrix) is treated in the same way as a 2-dimensional pixelated image, although the former case is invariant under row and column permutation, whereas the latter case is not (Murtagh, Starck and Berry, 2000). Therefore there are immediate problems related to non-uniqueness, and data order dependence.

What if, however, one organizes the data such that adjacency has a meaning? This implies that similarly-valued objects, and/or similarly-valued fea-

tures, are close together. This is what we do, using any hierarchical clustering algorithm (e.g., the Ward or minimum variance one).

Without loss of generality, as seen in these figures, we assume that a hierarchy is a binary, rooted tree; and equivalently that the series of agglomerations involve precisely two clusters (possibly singleton clusters) at each of the $n - 1$ agglomerations where there are n observations. These n observations are usually represented by n row vectors in our data table.

A significant advantage in regard to hierarchical clustering is that partitions of the data can be read off at a succession of levels, and this obviates the need for fixing the number of clusters in advance. All possible clustering outcomes are considered. (Remark: of course, relative to any one of the commonly used cluster homogeneity criteria, each partition is guaranteed to be sub-optimal at best.)

3 The Haar Wavelet Transform of a Dendrogram: Summary

In this article, we will denote the agglomeration of two clusters, q and q' , as cluster q'' . So (left or right subtree) nodes in the dendrogram are associated with the child (elder or younger) subnodes. We can define the elder cluster as q such that $\nu(q) > \nu(q')$, but we will *not* be concerned with whether or not elder corresponds to left, and younger to right.

For n objects or observation vectors, another notation that we can use is that the hierarchy H is the set of clusters indexed from 1 to n : $H = \{q_1, q_2, \dots, q_{n-1}\}$. We will always assume in this article, for convenience of exposition and with little loss of generality, that for distinct clusters $\nu(q) \neq \nu(q')$.

Whenever the distinction between the following become important, we will clearly distinguish between them: clusters; nodes; sets of objects; sets of indices of objects; and p-adic number representation of indices of objects.

The Haar algorithm, as discussed in Paper I, is as follows:

1. Take each cluster q'' in turn, proceeding in sequence through $q'' \in \{q_1, q_2, \dots, q_{n-1}\}$.
2. Apply the smoothing function, s : $s(q'') = \frac{1}{2}(q + q')$.
3. Thereby apply the detail function, d : $d(q'') = s(q'') - s(q) = -(s(q'') - s(q))$.
4. Return to step 1 until all $n - 1$ clusters are processed.

For details of how the clusters also take terminal nodes (objects) into account, see Paper I.

Now, it is clear from construction that perfect reconstruction of the input data (alternatively expressed, perfect undoing of the foregoing Haar algorithm) is guaranteed, given all of the following: (i) all of the detail function values, (ii) the final smooth, $s(q_{n-1})$, (iii) the definition of the dendrogram, and (iii) a convention of left and right subtree that allows us to traverse down the tree from q'' to both q and q' .

In practice our objectives are to explore the foundations of two distinct approaches. Both seek a Haar wavelet basis. These two approaches are as follows and can express the 2 input data cases considered in section 4.3 (“The Input Data”) of Paper I.

- Wavelet transform in an ultrametric topology: Induce the Haar basis from the hierarchy H that expresses the relationships in a set of ultrametrically related points, I .
- Wavelet transform on embedded subsets: Induce the Haar basis from the hierarchy H defining a set of subsets of I .

In the ultrametric case, each point $i \in I$ defines an m -dimensional vector: $i \in \mathbb{R}^m$. For notational convenience therefore i is either the index, or a vector.

In the set of subsets case, each point $i \in I$ can be defined as an n -dimensional index vector. Thus for example the sequentially second point is defined as $(0, 1, 0, 0, \dots, 0)$.

Both practical cases above can be expressed as follows: we carry out a wavelet transform in $L^2(G)$ where G is the group of alternative representations of a given hierarchy, H . The points $i \in I$ are associated either with vectors in \mathbb{R}^m or with an orthonormal vector set in \mathbb{Z}^n . (Note how \mathbb{Z}^n is n -dimensional, whereas \mathbb{R}^m is m -dimensional. The cardinality of I is n . The dimensionality of a feature or attribute space is m .)

4 Wavelet Transform on Discrete Fields

In this section we look at the wavelet transform on discrete fields, and in particular on $\ell^2(\mathbb{Z}_p)$. This is realized through cyclic representations of the affine group of \mathbb{Z} or of \mathbb{Z}_p .

In wavelets, we are seeking a representation of our data which has “covariant” properties relative to scale: for example, for translation “covariance”, the representation of a shifted signal must be a shifted copy of the

representation of the signal (Torrésani, 1994). In the group theory perspective, such “covariance” properties (i.e. with respect to the action of a symmetry group) are the starting point, and the representation is to be derived from them. The “covariance” group “turns out to be isomorphic (up to a compact factor) to the geometric phase space of the representation” (Torrésani, 1994, p. 6).

Traditionally, the wavelet transform is covariant with respect to a group action applicable to images, signals, time series, etc., viz. the affine group of the real line, which is a continuous group (Antoine et al., 2000a). Thus a first task is to bypass the need for a continuous group.

The group law of the affine group, in generic form $ax + b$, generates translations and dilations. The action of the $ax + b$ group on \mathbb{R} means: $(a, b) : x \longrightarrow ax + b$. We have the following product:

$$(b, a) \cdot (b', a') = (b + ab', aa') \quad (1)$$

Here, the identity is: $(1, 0)$. The inverse of (a, b) is: $(a, b)^{-1} = (a^{-1}, -\frac{b}{a})$. This is a non-commutative Lie (and thus continuous) group.

Flornes et al. (1994) consider a discrete wavelet transform to begin with, specified on the Hilbert space $\ell^2(\mathbb{Z}_p)$ (where \mathbb{Z} are the integers, \mathbb{Z}_p are integers mod p where p is prime for reasons explained below; and ℓ^2 implies finite energy from discrete values, or being square integrable). The Haar measure is defined on locally compact groups, permitting integration over group actions or members; and a locally compact separable group is consistent with the square integrable property. The group at issue here is the cyclic representation of the affine group; or its finite analog, the affine group mod p (Foote et al., 2000a).

A discussion of square integrable group representations in the context of time-frequency transforms, including the continuous wavelet transform, can be found in Torrésani (2000); and Torrésani (1994) discusses the counterexample case of the rotation group, S^2 , on the 2-dimensional sphere, which gives rise to a representation which is not square integrable.

The wavelet transform considered by Flornes et al. (1994) has the following operations:

$$\text{Translation : } T_b f(n) = f(n - b) \quad \text{for } f \in \ell^2(\mathbb{Z}_p) \quad (2)$$

$$\text{Dilation : } D_a f(n) = f(a^{-1}n) \quad (3)$$

The reason why p has to be prime is as follows. Consider the p -adic number representation of \mathbb{Z}_p . For the p -adic representation of \mathbb{Z}_p to be a field, i.e. to have an inverse, p must be prime.

The unitary representation of a group G is a mapping into a (complex) Hilbert space. Flornes et al. (1994) define the following unitary representation, π , on the group, mapping into unitary operations on $\ell^2(\mathbb{Z}_p)$:

$$\pi(g)f(n) = f(a^{-1}(n - b))$$

In terms of the translation and dilation operators, we have:

$$\pi(b, a) = T_b D_a$$

Thus far, a purely discrete wavelet transform is at issue. However if we take our function values f defined on \mathbb{Z}_p as sampled values from a continuous signal, then problems of interpolation arise. It simply is not good enough to transform our discrete data independently of awareness of the underlying continuum. Note that this issue is at the nub of where data analysis differs from signal processing. In data analysis, mostly a data cloud is taken as given (potentially leading to a combinatorial perspective) or as a stochastic realization (leading to a statistical modeling). In signal processing, the observed data are samples of an underlying topological or other continuous structure. It is a prime objective in the signal processing context to keep the processing of the observed, sampled data fully and provably consistent with the underlying continuous structures.

A way to address this issue of interpolation is to use B spline filters (the simplest example of which is the box function used by the Haar wavelet transform) to smooth the data, thereby “filling the holes” between gaps in the sampled values, before dilating. This is termed pseudo-dilation.

Relations (3) can be re-expressed as follows (Torrésani, 1994):

$$\text{Translation : } T_b f(n) = f(n - b) \quad \text{for } f \in \ell^2(\mathbb{Z}_p) \quad (4)$$

$$\begin{aligned} \text{Dilation : } D_a f(n) &= f(a^{-1}n) \quad \text{if } a \text{ divides } n \\ &= 0 \quad \text{otherwise} \end{aligned} \quad (5)$$

Then the affine multiplication law is verified by $\{T_b, D_a\}$ (using relation 1):

$$T_b D_a T_{b'} D_{a'} = T_{b+ab'} D_{aa'} \quad (6)$$

When $f \in \ell^2(\mathbb{Z}_p)$ with p prime, then a always divides n . The affine group on $\ell^2(\mathbb{Z}_p)$ is well-defined; T_b and D_a define a representation of this group; and it turns out that the square integrable property holds.

In general for $f \in \ell^2(\mathbb{Z})$ the à trous algorithm is used incorporating both continuously defined dilation, and discretization of the function.

We could embed each node of a hierarchical clustering, defined as we always do so as binary, rooted tree, in \mathbb{Z}_2 . Lang (1998) develops a wavelet transform approach (including the Haar wavelet transform and others) for such a 2-series local field, or Cantor dyadic group. Taking each cluster $q \in Q$ or node in the tree individually is not satisfactory from our point of view, and so we look further for a more pleasing way to process a hierarchy.

5 Wavelet Bases on Local Fields

Wavelet transform analysis is the determining of a “useful” basis for $L^2(\mathbb{R}^m)$ with the following properties:

- induced from a discrete subgroup of \mathbb{R}^m ,
- using translations on this subgroup, and
- dilations of the basis functions.

Classically (Frazier, 1999; Debnath and Mikusiński, 1999; Strang and Nguyen, 1996) the wavelet transform avails of a wavelet function $\psi(x) \in L^2(\mathbb{R})$, where the latter is the space of all square integrable functions. Wavelet transforms are bases on $L^2(\mathbb{R}^m)$, and the discrete lattice subgroup \mathbb{Z}^m is used to allow discrete groups of dilated translation operators to be induced on \mathbb{R}^m . Discrete lattice subgroups are typical of 2D images (the lattice is a pixelated grid) or 3D images (the lattice is the voxelated grid) or spectra or time series (the lattice is the set of time steps, or wavelength steps).

Sometimes it is appropriate to consider the construction of wavelet bases on $L^2(G)$ where G is some group other than \mathbb{R} . In Foote, Mirchandani, Rockmore, Healy and Olson (2000a, 2000b; see also Foote, 2005) this is done for the group defined by a quadtree, in turn derived from a 2D image. To consider the wavelet transform approach not in a Hilbert space but rather in locally-defined and discrete spaces we have to change the specification of a wavelet function in $L^2(\mathbb{R})$ and instead use $L^2(G)$.

Benedetto (2004) and Benedetto and Benedetto (2004) considered in detail the group G as a locally compact abelian group. Analogous to the integer

grid, \mathbb{Z}^m , a compact subgroup is used to allow a discrete group of operators to be defined on $L^2(G)$. The property of locally compact (essentially: finite and free of edges) abelian (viz., commutative) groups that is most important is the existence of the Haar measure (Ward, 1994). The Haar measure allows integration, and definition of a topology on the algebraic structure of the group.

Benedetto (2004) considers the following cases, among others, of wavelet bases constructed via a sub-structure:

- Wavelet basis on $L^2(\mathbb{R}^m)$ using translation operators defined on the discrete lattice, \mathbb{Z}^m . This is the situation discussed above, which holds for image processing, signal processing, most time series analysis (i.e., with equal length time steps), spectral signal processing, and so on. As pointed out by Foote (2005), this framework allows the multiresolution analysis in $L^2(\mathbb{R}^m)$ to be generalized to $L^p(\mathbb{R}^m)$ for Minkowski metric L^p other than Euclidean L^2 .
- Wavelet basis on $L^2(\mathbb{Q}_p)$, where \mathbb{Q}_p is the p-adic field, using a discrete set of translation operators. This case has been studied by Kozyref, 2002, 2004; Altaisky, 2004, 2005. See also the interesting overview of Khrennikov and Kozyref (2006).
- Wavelet basis on $L^2(\mathbb{Q}_p)$ using translation operators defined on the compact, open subgroup \mathbb{Z}_p . (It is interesting to note that \mathbb{Z}^m is discrete; and that the quotient $\mathbb{R}^m/\mathbb{Z}^m$ is compact. In contrast to this, \mathbb{Z}_p is compact; and the quotient $\mathbb{Q}_p/\mathbb{Z}_p$ is discrete.)
- Discussed is a wavelet basis on $L^2(G)$, for a group G , using translation operators defined on a discrete subgroup, or discrete lattice.
- Finally the central theme of Benedetto (2004) is a wavelet basis on $L^2(G)$ where G is a locally compact abelian group, using translation operators defined on a compact open subgroup (or operators that can be used as such on a compact open subgroup); and with definition of an expansive automorphism replacing the traditional use of dilation.

A motivation for the work of Benedetto (2004) and Benedetto and Benedetto (2004) is laying the groundwork for the wavelet transform on the adelic numbers (see Appendix 3). In this work we are content to be less ambitious in regard to number systems – below we focus on a particular p-adic encoding of dendrograms; and we are also less ambitious in regard to wavelet functions

– staying resolutely with the Haar wavelet in this work. Our motivation is due to our application drivers.

Locally compact abelian groups (LCAG) are a way to take Fourier analysis (hence a particularly important class of harmonic analysis because so versatile) into more general settings than e.g. the reals (although the reals also form a non-compact, but locally compact, abelian group).

The duals of members of a locally compact abelian group, defined as unitary multiplicative characters, $x \longrightarrow e^{-2\pi ix}$, also form a locally compact abelian group (Knapp, 1996). The duality pairing G and \widehat{G} allows for an isometry between $L^2(G)$ and $L^2(\widehat{G})$ (Antoine et al., 2000).

Fourier analysis is the study of real square integrable functions that are invariant under the group of integer translations (see Foote et al., 2000a), while abstract harmonic analysis is the study of functions on more general topological groups that are invariant under a (closed) subgroup.

It is interesting to compare some global properties of our approach relative to the Fourier transform approach applied to decision trees in Kargupta and Park (2004). The Fourier transform lends itself well to a frequency spectrum analysis of binary decision vectors, and the latter can be of importance for supervised classification. On the other hand, our work makes use of binary trees but in the framework of unsupervised classification. The wavelet transform shares with the Fourier transform the property that frequency spectral information is determined from the data; and the wavelet transform additionally determines spatial or resolution scale information from the data. We have found the wavelet transform, as described in this article, to be appropriate for the type of input data that we have considered. In general terms, both we in this work, and Kargupta and Park (2004), have as objectives the filtering and compression of data.

We need affine group action for the wavelet transform, and we have seen above (section 4, “Wavelet transform on discrete fields”) that \mathbb{Z}_p affords us this; but for an arbitrary discrete field, and for an arbitrary locally compact abelian group, it is tricky to find an affine group. Taking further the Flornes et al. (1994) work, Antoine et al. (2000) consider an infinite locally compact abelian group, \mathcal{G} ; the restriction of \mathcal{G} to a lattice $\Gamma \subset \mathcal{G}$; \mathcal{A} , an abelian semigroup; and the actions of \mathcal{A} on $\ell^2(\Gamma)$. Based on a pseudodilation (i.e., the product of a natural dilation by a convolution operator) the case of a continuous underlying signal is studied, i.e. the relation between the semigroup acting on $\ell^2(\Gamma)$, and a continuous affine group acting on $L^2(\mathbb{R})$. Spline functions are again among the wavelet functions used (among which is the Haar wavelet function associated with the B-spline of order 1).

The aspect of greatest interest to us here in the approach of Benedetto (2004), and Benedetto and Benedetto (2004), is to define wavelets on $L^2(G)$, with G taken as the p-adic rationals, \mathbb{Q}_p , and with the p-adic integers, \mathbb{Z}_p , on which we define translation-like operators. Firstly, we are using therefore $L^2(\mathbb{Q}_p)$, i.e. functions defined on the rationals. Secondly, we use a discrete group of operators on $L^2(\mathbb{Q}_p)$ which are not in themselves translation operators, but may be used in an analogous way. The “trick” used is that the quotient $\mathbb{Q}_p/\mathbb{Z}_p$ is discrete, and this will furnish the translation operators. An expansive automorphism is also needed in this context, i.e. what we use in analogy with dilation.

A number of alternatives for the subset of $\mathbb{Q}_p/\mathbb{Z}_p$ (more strictly the quotient of the group dual by the annihilator in the dual of the compact open subgroup) are discussed by Benedetto (2000a). Given our application-driven interest, we will not pursue them further here. What we will do, however, is to look at how the group-based approach of Benedetto (2000a), that for the most part assumes infinite sets, can be tailored for our algorithmic – hence finite – purposes.

We will therefore look at how we can suitably encode any given dendrogram in terms of \mathbb{Q}_p – or indeed, as will be seen, in terms of \mathbb{Z}_p .

Next we will move on to look at how a lattice-proxy is defined on our encoding, and thereby translation operators.

Finally, we will look at how an expansive automorphism can be replaced by expansive mapping in the finite and discrete case.

In all of this, we follow the methodology described by Benedetto (2000a); but we restrict all aspects to the finite, discrete context.

5.1 The Wreath Product Group Corresponding to a Hierarchical Clustering

For the group actions, with respect to which we will seek invariance, we consider independent cyclic shifts of the subnodes of a given node (hence, at each level). Equivalently these actions are adjacency preserving permutations of subnodes of a given node (i.e., for given q , the permutations of $\{q', q''\}$). Due to the binary tree, or strictly pairwise agglomerations represented by the hierarchy, the “adjacency” property is trivial. We have therefore cyclic group actions at each node, where the cyclic group is of order 2.

The symmetries of H are given by structured permutations of the terminals. The terminals will be denoted here by Term H . The full group of symmetries is summarized by the following algorithm:

1. For level $l = n - 1$ down to 1 do:
2. Selected node, $\nu \longleftarrow$ node at level l .
3. And permute subnodes of ν .

Subnode ν is the root of subtree H_ν . We denote H_{n-1} simply by H . For a subnode ν' undergoing a relocation action in step 3, the internal structure of subtree $H_{\nu'}$ is not altered.

The algorithm described defines the automorphism group which is a wreath product of the symmetric group. Denote the permutation at level ν by P_ν . Then the automorphism group is given by:

$$G = P_{n-1} \text{ wr } P_{n-2} \text{ wr } \dots \text{ wr } P_2 \text{ wr } P_1$$

where wr denotes the wreath product.

Call Term H_ν the terminals that descend from the node at level ν . So these are the terminals of the subtree H_ν with its root node at level ν . We can alternatively call Term H_ν the cluster associated with level ν .

We will now look at shift invariance under the group action. This amounts to the requirement for a constant function defined on Term $H_\nu, \forall \nu$. A convenient way to do this is to define such a function on the set Term H_ν via the root node alone, ν . By definition then we have a constant function on the set Term H_ν .

Let us call V_ν a space of functions that are constant on Term H_ν . Possible bases of V_ν that were considered in Paper I are:

1. Basis vector with $|\text{Term}H_\nu|$ components, with 0 values except for value 1 for component i .
2. Set (of cardinality $n = |\text{Term}H_\nu|$) of m -dimensional observation vectors.

The constant function maps

$$L(\text{Term}H) \longrightarrow V_\nu$$

where L is the space of complex valued functions on the set Term H .

Now we consider the resolution scheme arising from moving from $\{\text{Term}H_{\nu'}, \text{Term}H_{\nu''}\}$ to Term H_ν . From the hierarchical clustering point of view it is clear what this represents: simply, an agglomeration of two clusters called Term $H_{\nu'}$ and Term $H_{\nu''}$, replacing them with a new cluster, Term H_ν .

Let the spaces of constant functions corresponding to the two cluster agglomerands be denoted $V_{\nu'}$ and $V_{\nu''}$. These two clusters are disjoint initially, which motivates us taking the two spaces as a couple: $(V_{\nu'}, V_{\nu''})$. In the same way, let the space of constant functions corresponding to node ν be denoted V_{ν} .

The multiresolution scheme uses a space of zero mean denoted $W_{\nu'\nu''}$ with mean defined on the couple of spaces, $(V_{\nu'}, V_{\nu''})$:

$$V_{\nu} = (V_{\nu'}, V_{\nu''}) \oplus W_{\nu'\nu''} = (V_{\nu'} \oplus W_{\nu'\nu''}, V_{\nu''} \oplus W_{\nu'\nu''})$$

In considering spaces of constant functions, $V_{\nu'}$ and $V_{\nu''}$, we know that the support of these spaces are, respectively, Term $H'_{\nu'}$ and Term $H_{\nu''}$. So if, instead of the space of zero mean denoted $W_{\nu'\nu''}$ where mean is defined on the couple of spaces, $(V_{\nu'}, V_{\nu''})$, we considered the mean of the combined support, Term $H'_{\nu'} \cup$ Term $H_{\nu''}$, then the result would be quite different. We would, in fact, have a cluster-weighted mean value.

5.2 Example

Let us exemplify a case that satisfies all that has been defined in the context of the wreath product invariance that we are targeting. It also exemplifies the algorithm discussed in depth in Paper I. Take the constant function on $V_{\nu'}$ to be $f_{\nu'}$. Take the constant function on $V_{\nu''}$ to be $f_{\nu''}$. Then define the constant function on V_{ν} to be $(f_{\nu'} + f_{\nu''})/2$. Next define the zero mean function on $W_{\nu'\nu''}$ to be:

$$w_{\nu'} = (f_{\nu'} + f_{\nu''})/2 - f_{\nu'}$$

in the support interval of $V_{\nu'}$, i.e. Term $H_{\nu'}$, and

$$w_{\nu''} = (f_{\nu'} + f_{\nu''})/2 - f_{\nu''}$$

in the support interval of $V_{\nu''}$, i.e. Term $H_{\nu''}$.

Evidently $w_{\nu'} = -w_{\nu''}$.

5.3 Inverse Transform

Following on from the previous subsection, a demonstration that the algorithm allows for exact reconstruction of the data – the inverse transform – is as follows. The constant function on Term H corresponding to the root node is f_{n-1} . The two subnodes of the root node, at levels ν' and ν'' , are reconstructed from $f_{n-1} \pm w_{\nu'}$ (and, as we have seen, we can either use $w_{\nu'}$ or

$w_{\nu''}$). We next look at the subtrees whose roots are given by the nodes (just considered) at levels ν' and ν'' , and these subtrees are necessarily disjoint. All subnodes of these currently selected nodes are reconstructed using the same algorithm. This procedure is iteratively continued until the terminals have been dealt with.

5.4 Link with Agglomerative Hierarchical Clustering Algorithms

Comparison with traditional clustering criteria is considered next. It is clear why agglomerative levels are very problematic if used for choosing a good partition in a hierarchical clustering tree: they increase with agglomeration, simply because the cluster centers are getting more and more spread out as the sequence of agglomerations proceeds. Directly using these agglomerative levels has been a way to derive a partition for a very long time. An early reference is Mojena (1977). To see how the wavelet transform used by us leads to a very different outcome, see Paper I, where we describe use of the norms of the $w_{\nu'}$ vectors.

When we consider agglomerative hierarchical clustering algorithms it is clear that (i) clusters are often defined in terms of center of gravity, or mean; and (ii) this allows for defining a (vector) difference term between a cluster and its immediate sub-clusters. It is also clear that the Haar wavelet algorithm is close to the method known as median or Gower's or WPGMC – weighted pair group method using centroids: see table, p. 68, of Murtagh (1985).

We could also cater for other agglomerative criteria, subject to storing the cluster cardinality values, and develop an algorithm that is close to the Haar wavelet one. Constant functions on spaces $V_{\nu'}$ and $V_{\nu''}$ remain just as before. The zero mean functions on space $W_{\nu'\nu''}$ would now be generalized to weight mean (with weights given by cluster cardinalities). Viewed in this light, our work has led us to develop a new storage structure for hierarchical clustering trees, which is particularly beneficial for data filtering objectives.

The novelty of our work resides in two areas: (i) we have shown the close association between two classes of multiple resolution data analysis approaches, agglomerative hierarchical clustering algorithms and the wavelet transform; and (ii) our motivation is not at all to construct hierarchical clusterings in a new way but rather to illuminate further inherent ultrametric properties of data (cf. Murtagh, 2004).

6 An Algebraic Representation of a Hierarchy

6.1 Introduction

The dendrogram wavelet transform has been seen to be a set of applications of a function applied to set members, or cluster members, associated with nodes of the hierarchy H . The non-singleton clusters comprise the set $Q = \{q_1, q_2, \dots, q_{n-1}\}$. The wavelet function is applied in turn to q_1, q_2, \dots . In this paper, we are using the Haar wavelet function. But we could well use others (e.g., Altaisky, 2004, uses the Morlet wavelet).

With each $q \in Q$ there is an associated level function, $\nu : q \longrightarrow \mathbb{R}^+$, which induces a total order on Q . We will show that the application of the wavelet function to this sequence of clusters is “dilatory” or “expansive” in two different ways.

We will return to what these two different ways are in a moment. The hierarchy $H_{\nu=0}$ contains an increasingly embedded sequence of subsets (corresponding to increasingly pruning the branches of the tree; Bouchki, 1996). We have: $H_{\nu=0} \supset H_{\nu_1} \supset H_{\nu_2} \supset \dots \supset H_{\nu(n-1)}$. Define Term as the set of terminal nodes of a hierarchy, and Card the cardinality of a set. Then $\text{Term}(H_{\nu=0}) = I$ with $\text{Card}(I) = n$. $\text{Card}(\text{Term}(H_{\nu_1})) = n - 1$. $\text{Card}(\text{Term}(H_{\nu_2})) = n - 2$. \dots $\text{Card}(\text{Term}(H_{\nu(n-1)})) = 1$. Each application of the wavelet function is to the minimal (non-singleton) cluster (again see Bouchki, 1996) in each ν_k . Below, we will see how we *promote* all $q \in H_{\nu_k}$ to the corresponding $q \in H_{\nu(k+1)}$ by multiplying clusters (in a particular algebraic representation) by $1/p$, which is of norm p .

The “dilatory” or “expansive” character of our sequence of operations, viz., application of the wavelet function, comes from (i) the sequence of embedded subsets of H – so we still operate on a cluster, but the data on which we work becomes smaller; or (ii) the sequence of levels at which we work is derived from repeatedly taking the product with $1/p$ of norm p .

In order to introduce this product with $1/p$ of norm p we first describe the p-adic algebraic representation of the hierarchy. In this representation, clusters including singletons, have an operator, denoted \oplus . This operator allows clusters to be defined. Next, we have a null element in Q in this algebraic representation, and a norm of each $q \in Q$. Hence for $q', q'' \in Q$, we can define $q' \oplus q''$. We have: $\exists q \in Q$ s.t. $q = 0$. Finally, $\forall q \in Q$ we have $\|q\|$.

6.2 H Expressed p-Adically: p-Adic Encoding of a Dendrogram

We will introduce now the one-to-one mapping of clusters (including singletons) in H into a set of p-adically expressed integers (a fortiori, rationals, \mathbb{Q}_p). The field of p-adic numbers is the most important example of ultrametric spaces. Addition and multiplication of p-adic integers, \mathbb{Z}_p , are well-defined. Inverses exist and no zero-divisors exist.

A terminal-to-root traversal in a dendrogram or binary rooted tree is defined as follows. We use the path $x \subset q \subset q' \subset q'' \subset \dots \subset q_{n-1}$, where x is a given object specifying a given terminal, and q, q', q'', \dots are the embedded classes along this path, specifying nodes in the dendrogram. The root node is specified by the class q_{n-1} comprising all objects.

A terminal-to-root traversal is the shortest path between the given terminal node and the root node, assuming we preclude repeated traversal (backtrack) of the same path between any two nodes.

By means of terminal-to-root traversals, we define the following p-adic encoding of terminal nodes, and hence objects, in Figure 1.

$$\begin{aligned}
 x_1 : & +1 \cdot p^1 + 1 \cdot p^2 + 1 \cdot p^5 + 1 \cdot p^7 \\
 x_2 : & -1 \cdot p^1 + 1 \cdot p^2 + 1 \cdot p^5 + 1 \cdot p^7 \\
 x_3 : & -1 \cdot p^2 + 1 \cdot p^5 + 1 \cdot p^7 \\
 x_4 : & +1 \cdot p^3 + 1 \cdot p^4 - 1 \cdot p^5 + 1 \cdot p^7 \\
 x_5 : & -1 \cdot p^3 + 1 \cdot p^4 - 1 \cdot p^5 + 1 \cdot p^7 \\
 x_6 : & -1 \cdot p^4 - 1 \cdot p^5 + 1 \cdot p^7 \\
 x_7 : & +1 \cdot p^6 - 1 \cdot p^7 \\
 x_8 : & -1 \cdot p^6 - 1 \cdot p^7
 \end{aligned}$$

If we choose $p = 2$ the resulting decimal equivalents could be identical: cf. contributions based on $+1 \cdot p^1$ and $-1 \cdot p^1 + 1 \cdot p^2$. Given that the coefficients of the p^j terms ($1 \leq j \leq 7$) are in the set $\{-1, 0, +1\}$ (implying for x_1 the additional terms: $+0 \cdot p^3 + 0 \cdot p^4 + 0 \cdot p^6$), the coding based on $p = 3$ is required to avoid ambiguity among decimal equivalents.

A few general remarks on this encoding follow. For the labeled ranked binary trees that we are considering, we require the labels $+1$ and -1 for the two branches at any node. Of course we could interchange these labels, and have these $+1$ and -1 labels reversed at any node. By doing so we will have different p-adic codes for the objects, x_i .

The following properties hold: (i) *Unique encoding*: the decimal codes for each x_i (lexicographically ordered) are unique for $p \geq 3$; and (ii) *Reversibility*: the dendrogram can be uniquely reconstructed from any such set of unique codes.

The p-adic encoding defined for any object set above can be expressed as follows for any object x associated with a terminal node:

$$x = \sum_{j=1}^{n-1} c_j p^j \text{ where } c_j \in \{-1, 0, +1\} \quad (7)$$

In greater detail we have:

$$x_i = \sum_{j=1}^{n-1} c_{ij} p^j \text{ where } c_{ij} \in \{-1, 0, +1\} \quad (8)$$

Here j is the level or rank (root: $n - 1$; terminal: 1), and i is an object index.

In our examples we have used: $a_j = +1$ for a left branch (in the sense of Figure 1), $= -1$ for a right branch, and $= 0$ when the node is not on the path from that particular terminal to the root.

A matrix form of this encoding is as follows, where $\{\cdot\}^t$ denotes the transpose of the vector.

Let \mathbf{x} be the column vector $\{x_1 \ x_2 \ \dots \ x_n\}^t$.

Let \mathbf{p} be the column vector $\{p^1 \ p^2 \ \dots \ p^{n-1}\}^t$.

Define a characteristic matrix C of the branching codes, $+1$ and -1 , and an absent or non-existent branching given by 0 , as a set of values c_{ij} where $i \in I$, the indices of the object set; and $j \in \{1, 2, \dots, n - 1\}$, the indices of the dendrogram levels or nodes ordered increasingly. For Figure 1 we therefore have:

$$C = \{c_{ij}\} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix} \quad (9)$$

For given level j , $\forall i$, the absolute values $|c_{ij}|$ give the membership function either by node, j , which is therefore read off columnwise; or by object

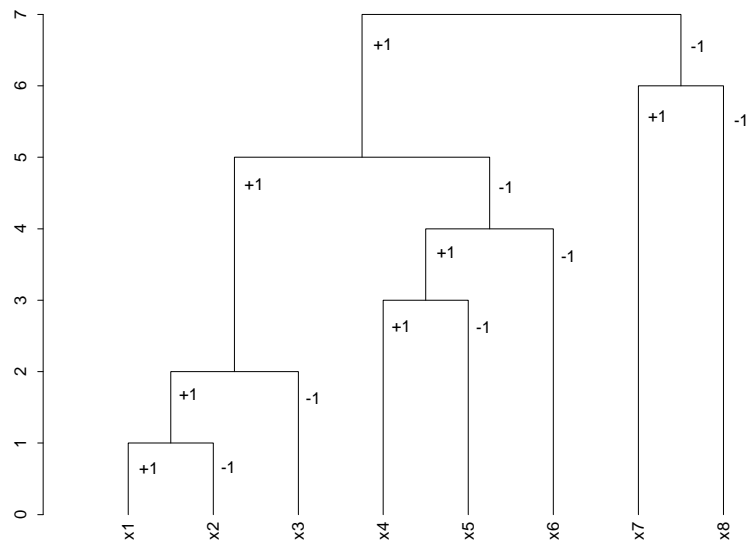


Figure 1: Labeled, ranked dendrogram on 8 terminal nodes, x_1, x_2, \dots, x_8 . Branches are labeled $+1$ and -1 . Clusters are: $q_1 = \{x_1, x_2\}$, $q_2 = \{x_1, x_2, x_3\}$, $q_3 = \{x_4, x_5\}$, $q_4 = \{x_4, x_5, x_6\}$, $q_5 = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $q_6 = \{x_7, x_8\}$, $q_7 = \{x_1, x_2, \dots, x_7, x_8\}$.

index, i which is therefore read off rowwise.

The matrix form of the p-adic encoding is:

$$\mathbf{x} = C\mathbf{p} \tag{10}$$

Here, \mathbf{x} is the decimal encoding, C is the matrix with dendrogram branching codes and \mathbf{p} is the vector of powers of a fixed integer (usually, more restrictively, fixed prime) p .

The tree encoding exemplified in Figure 1, and defined with coefficients in equations (7) or (8), (9) or (10), with labels $+1$ and -1 is not commonly used: zero and one labels are more common. We required the ± 1 labels, however, to fully cater for the ranked nodes (i.e. the total order, as opposed to a partial order, on the nodes).

We can consider the objects that we are dealing with to have equivalent integer values. To show that, all we must do is work out decimal equivalents of the p-adic expressions used above for x_1, x_2, \dots . As noted in Gouvêa (2003), we have equivalence between: a p-adic number; a p-adic expansion; and an element of \mathbb{Z}_p (the p-adic integers). The coefficients used to specify a p-adic number, Gouvêa (2003) notes (p. 69), “must be taken in a set of representatives of the class modulo p . The numbers between 0 and $p - 1$ are only the most obvious choice for these representatives. There are situations, however, where other choices are expedient.”

6.3 P-adic Dendrogram Addition and Multiplication

As noted already the wavelet basis on $L^2(\mathbb{R}^m)$ is often induced from the discrete subgroup, \mathbb{Z}^m . Now for a discrete subgroup we use the dendrogram, H . The addition operation on the group H will now be explored.

In order to define a group structure on the p-adic encoded objects, we require an addition operation. We do not “carry and add” in the traditional way because this does not make sense in this context. Instead we define the following “average and threshold” operation for any coefficients (of values of \mathbf{p} , as used in equations 8 or 10). We define the following compositions for such coefficients.

$$\begin{aligned} + 1 + 1 &\longrightarrow +1 \\ - 1 - 1 &\longrightarrow -1 \\ + 1 - 1 &\longrightarrow 0 \\ - 1 + 1 &\longrightarrow 0 \\ + 1 \pm 0 &\longrightarrow 0 \\ - 1 \pm 0 &\longrightarrow 0 \end{aligned} \tag{11}$$

Examples from the encoding defined above for x_1, x_2, \dots (again with reference to Figure 1, and equations 7 or 8, 9 or 10) follows.

$$\begin{aligned} x_1 \oplus x_2 &= +1 \cdot p^2 + 1 \cdot p^5 + 1 \cdot p^7 \\ x_1 \oplus x_3 &= +1 \cdot p^5 + 1 \cdot p^7 \\ x_1 \oplus x_7 &= 0 \\ x_3 \oplus x_6 &= +1 \cdot p^7 \\ x_5 \oplus x_8 &= 0 \end{aligned}$$

Informally: in the tree, this addition operation only retains non-zero terms for nodes in the tree strictly *above* the first (i.e. lowest level) cluster within which the two objects find themselves. This means that if the two objects only find themselves together for the first time in the same cluster that contains all objects then the result of the addition operation is 0.

Let us use our “average and threshold” operation, which we are using as a customized addition, to define clusters. We will do so by example, taking Figure 1 as our case study. We will call the clusters, ranked by increasing node level, q_1, q_2, \dots as used in the caption of Figure 1.

$$\begin{aligned} q_1 &= x_1 \oplus x_2 = +1 \cdot p^2 + 1 \cdot p^5 + 1 \cdot p^7 \\ q_2 &= q_1 \oplus x_3 = +1 \cdot p^5 + 1 \cdot p^7 \\ q_3 &= x_4 \oplus x_5 = +1 \cdot p^4 - 1 \cdot p^5 + 1 \cdot p^7 \\ q_4 &= q_3 \oplus x_6 = -1 \cdot p^5 + 1 \cdot p^7 \\ q_5 &= q_2 \oplus q_4 = +1 \cdot p^7 \\ q_6 &= x_7 \oplus x_8 = -1 \cdot p^7 \\ q_7 &= 0 \end{aligned}$$

The trivial cluster containing all n objects, q_{n-1} , is of value 0 in this representation.

Definition of Null Element:

On the dendrogram H , the set $q_{n-1} = I$ is the null element when using our p-adic encoding (given in definitions (8) and (10)) and addition operation (11).

Defining p-adic notation for clusters in this way allows us to define norms of clusters; or to define p-adic distances between clusters; or indeed to define p-adic distances between clusters and objects (singletons, terminals). We will look at these in subsection 6.4 below.

For completeness we will provide a definition of p-adic dendrogram multiplication. Take $x = \sum_j c_j p^j$ and let $y = \sum_j c'_j p^j$. The product operation is defined on the formal (Laurent) power series as:

$$xy = \left(\sum_j c_j p^j \right) \left(\sum_{j'} c'_{j'} p^{j'} \right) = \sum_{jj'} c_j c'_{j'} p^{j+j'} \quad (12)$$

with restriction to the term in p^{n-1} . P-adic dendrogram multiplication will be used below in the definition of the expansive operator: this is multiplication by $1/p$.

6.4 P-adic Distance and Norm on a Dendrogram

Thus far, we have been concerned with an analytic framework. Now we will induce a metric topology on H .

To find the p-adic distance, we look for the term p^r in the p-adic codes of the two objects, where r is the lowest level such that the absolute values of the coefficients of p^r are equal.

Let us look at the set of p-adic codes for x_1, x_2, \dots above (Figure 1), to give some examples of this.

For x_1 and x_2 , we find the term we are looking for to be p^1 , and so $r = 1$.
 For x_1 and x_5 , we find the term we are looking for to be p^5 , and so $r = 5$.
 For x_5 and x_8 , we find the term we are looking for to be p^7 , and so $r = 7$.

Having found the value r , the distance is defined as p^{-r} .

See, inter alia, Benzécri (1979), and Gouvêa (2003), for this definition of ultrametric distance.

Examples based on Figure 1:

$$\begin{aligned} |x_1 - x_2|_p &= |x_2 - x_1|_p = p^{-1} \text{ since } r = 1. \\ |x_1 - x_4|_p &= |x_4 - x_1|_p = p^{-5} \text{ since } r = 5. \\ |x_3 - x_6|_p &= |x_6 - x_3|_p = p^{-5} \text{ since } r = 5. \end{aligned}$$

Examples for clusters from Figure 1:

$$\begin{aligned} |q_1 - q_3|_p &= |q_3 - q_1|_p = p^{-5}. \\ |q_2 - q_6|_p &= |q_6 - q_2|_p = p^{-7}. \end{aligned}$$

We take for a singleton object $r = 0$, and so the norm of an object is always 1. We therefore define the p-adic norm, $|\cdot|_p$, of an object corresponding to a terminal node in the following way: for any object, x , $|x|_p = 1$.

The norm of a non-singleton cluster is defined analogously. It is seen to be strictly smaller. We have: $|q_2|_p = p^{-2}$; $|q_4|_p = p^{-4}$.

For the expansive operator that we use for dilation, we will consider product with $1/p$. The norm associated with this operator is seen to be $|1/p|_p = |p^{-1}|_p = p^{-(-1)} = p$.

The operator given by multiplication by $1/p$ therefore has norm or modulus p .

The p -adic norm, or p -adic valuation, satisfies the following properties (Schikhof, 1984):

1. $|x|_p \geq 0$; $|x|_p = 0$ iff $x = 0$
2. $|x + y|_p \leq \max(|x|_p, |y|_p)$
3. $|xy|_p = |x|_p |y|_p$

We also have: $|q|_p \leq 1$ with equality only if q is a singleton.

6.5 Modified Dilation Operation: Multiplication by $1/p$

Consider the set $\{x_i | i \in I\}$ with its p -adic coding considered above. Take $p = 2$. (Non-uniqueness of corresponding decimal codes is not of concern to us now, and taking this value for p is without any loss of generality.) Multiplication of $x_1 = +1 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^5 + 1 \cdot 2^7$ by $1/p = 1/2$ gives: $+1 \cdot 2^1 + 1 \cdot 2^4 + 1 \cdot 2^6$. Each level has decreased by one, and the lowest level has been lost. Subject to the lowest level of the tree being lost, the form of the tree remains the same. By carrying out the multiplication-by- $1/p$ operation on all objects, it is seen that the effect is to rise in the hierarchy by one level.

Let us call product with $1/p$ the operator A . The effect of losing the bottom level of the dendrogram means that either (i) each cluster (possibly singleton) remains the same; or (ii) two clusters are merged. Therefore the application of A to all q implies a subset relationship between the set of clusters $\{q\}$ and the result of applying A , $\{Aq\}$.

Repeated application of the operator A gives Aq, A^2q, A^3q, \dots . Starting with any singleton, $i \in I$, this gives a path from the terminal to the root node in the tree. Each such path ends with the null element, as a result of the Null Element definition (section 3). Therefore the intersection of the paths equals the null element.

Benedetto and Benedetto (2004) discuss A as an expansive automorphism of I , i.e. form-preserving, and locally expansive.

Some implications of Benedetto and Benedetto's (2004) expansive automorphism follow.

For any q , let us take q, Aq, A^2q, \dots as a sequence of open subgroups of I , with $q \subset Aq \subset A^2q \subset \dots$, and $I = \bigcup \{q, Aq, A^2q, \dots\}$. This is termed

an inductive sequence of I , and I itself is the inductive limit (Reiter and Stegeman, 2000).

Each path defined by application of the expansive automorphism defines a spherically complete system (Schikhof, 1984; Gajić, 2001), which is a formalization of well-defined subset embeddedness.

We now return to our starting point, the Haar algorithm given in section 1.4. We apply the averaging and differencing operations to each cluster in sequence. But now, after doing this for cluster q , we apply the operator A , i.e. the $1/p$ product to the p -adic representation of the dendrogram. This causes us to move up a level. This is our enhanced concept of dilation, which we apply to the dendrogram, where we keep the same averaging and differencing operations applied to the cluster in sequence.

7 Wavelet Bases from the Wreath Product Group

In our case we are looking for a new basis for $L^2(G)$ where G is the set of all equivalent representations of a hierarchy, H , on n terminals. Denoting the level index of H as ν (so $\nu : H \rightarrow \mathbb{R}^+$, where \mathbb{R}^+ are the positive reals), and $\nu = 0$ is the level index corresponding to the fine partition of singletons, then this hierarchy will also be denoted as $H_{\nu=0}$. Let I be the set of observations. Let the succession of clusters associated with nodes in H be denoted $Q = \{q_1, q_2, \dots, q_{n-1}\}$. We have $n - 1$ non-singleton nodes in H , associated with the clusters, q . At each node we can interchange left and right subnodes. Hence we have 2^{n-1} equivalent representations of H , or, again, members in the group, G , that we are considering.

So we have the group of equivalent dendrogram representations on $H_{\nu=0}$. We have a series of subgroups, $H_{\nu_k} \supset H_{\nu_{k+1}}$, for $0 \leq k < n - 1$. Symmetries are given by permutations at each level, ν , of hierarchy H . Collecting these furnishes a group of symmetries on the terminal set of any given (non-terminal) node in H .

The practical application arises through identifying the n terminal nodes with (i) m -dimensional vectors, or (ii) n -dimensional hypercube vertices. On the latter sets of vectors we can also consider an associated permutation representation.

Parenthetically, we note that the permutation representation is known as the alternating or zig-zag permutations and are counted by the André or Euler numbers (Murtagh, 1984a; sequence A000111 in Sloane, 2005).

In this work we ignore another form of equivalent representation, i.e. that arising from two or more level values being identical: $\nu_k = \nu_{k+1}$ for

some $0 \leq k \leq n - 1$. This means that successive nodes can be interchanged. This situation happens when we have equilateral triangles in the ultrametric space, as opposed to triangles that are strictly isosceles with small base.

At each non-singleton cluster, q , we define a (trivial) affine group on (q', q'') . The group is defined on $Q = \{q_\nu | \nu = 1, 2, \dots, n - 1\}$.

Footo et al. (2000a) consider group actions on spherically homogeneous rooted trees. The use of the latter is as a quadtree in 2D image processing. (An image is recursively decomposed into spatially homogeneous quadrant covering regions; and this decomposition is represented as a quadtree. For 3D image volumes, the data structure becomes an octree.) Just like for us, the quadtree nodes can “twiddle” around their offspring nodes but, because of the image regions, group action amounts to cyclic shifts or adjacency-preserving permutations of the offspring nodes. The relevant group in this case is referred to as the wreath product group.

8 Matrix Interpretation of the Haar Dendrogram Wavelet Transform

8.1 The Forward Transform

Consider any hierarchical clustering, H , represented as a binary rooted tree. For each cluster q'' with offspring nodes q and q' , we define $s(q'')$ through application of the low-pass filter $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$:

$$s(q'') = \frac{1}{2}(s(q) + s(q')) = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}^t \begin{pmatrix} s(q) \\ s(q') \end{pmatrix} \quad (13)$$

The application of the low-pass filter is carried out in order of increasing node number (i.e., from the smallest non-terminal node, through to the root node).

Next for each cluster q'' with offspring nodes q and q' , we define detail coefficients $d(q'')$ through application of the band-pass filter $\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$:

$$d(q'') = \frac{1}{2}(s(q) - s(q')) = \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix}^t \begin{pmatrix} s(q) \\ s(q') \end{pmatrix} \quad (14)$$

Again, increasing order of node number is used for application of this filter. See Paper I for further details.

8.2 The Ultrametric Case

We now return to the issue of how we start this scheme, i.e. how we define $s(i)$, or the “smooth” of a terminal node. We have distinguished above in section 3 between:

1. H as representing an ultrametric set of relations,
2. H as representing an embedded set of sets.

For case 1 we take $s(i)$ as the m -dimensional observation vector corresponding to i . So, taking all n vectors $s(i)$ we have the initial data matrix X of dimensions $n \times m$.

Then for our set of n points in \mathbb{R}^m given in the form of matrix X we have:

$$X = CD + S_{n-1} \tag{15}$$

where D is the matrix collecting all wavelet projections or detail coefficients, d . The dimensions of C are: $n \times (n - 1)$ (see definition (9)). The dimensions of D are $(n - 1) \times m$.

If s_{n-1} is the final data smooth, in the limit for very large n a constant-valued m -component vector, then let S_{n-1} be the $n \times m$ matrix with s_{n-1} repeated on each of the n rows.

Consider the j th coordinate of the m -dimensional observation vector corresponding to i . For any $d(q_j)$ we have: $\sum_k d(q_j)_k = 0$, i.e. the detail coefficient vectors are each of zero mean.

To recapitulate we have:

X is of dimensions $n \times m$.

C is of dimensions $n \times (n - 1)$.

D is of dimensions $(n - 1) \times m$.

S_{n-1} is of dimensions $n \times m$.

8.3 The Case of Embedded Set of Sets

We have distinguished between

1. H as representing an ultrametric set of relations,
2. H as representing an embedded set of sets.

We now turn attention to the latter.

In this case we take $s(i)$ as an n -dimensional indicator vector corresponding to i . So, taking all n vectors $s(i)$ we have the initial data matrix X which is none other than the $n \times n$ dimensional identity matrix. We will write X_{ind} for this identity matrix.

The wavelet transform in this case is: $X_{\text{ind}} = CD + S_{n-1}$.

X_{ind} is of dimensions $n \times n$.

C , exactly as in case 1 (ultrametric case) is of dimensions $n \times (n - 1)$.

D , of necessity different in values from case 1, is of dimensions $(n - 1) \times n$.

S_{n-1} , of necessity different in values from case 1, is of dimensions $n \times n$.

8.4 The Inverse Transform

In both cases considered (viz., ultrametric, and set of sets) the forward and inverse transforms are performed in the same way. The algorithms are identical – the inputs alone differ.

The inverse transform allows exact reconstruction of the input data. We begin with s_{n-1} . If this root node has subnodes q and q' , we use $d(q)$ and $d(q')$ to form $s(q)$ and $s(q')$.

8.5 Wavelet Filtering

Setting wavelet coefficients to zero and then reconstructing the data is referred to as hard thresholding (in wavelet space) and this is also termed wavelet smoothing or regression. See the companion paper, Paper I, for discussion and examples.

8.6 Hierarchic Wavelet Transform in Matrix Form

We will look at the ultrametric case. The matrix generalization of equation (10) is:

$$X = CP \tag{16}$$

Matrix P is formed from the vectors \mathbf{p} of equation (10) by replicating rows.

Now the wavelet transform gives us: $X = CD + S_{n-1}$. Each (replicated) row of matrix S_{n-1} is a particular measure of central tendency.

Centering X relative to this gives:

$$X - S_{n-1} = CD \tag{17}$$

We conclude from the formal similarity of expressions (16) and (17): the initial p -adic encoding of our data vectors has been mapped into a *wavelet encoding* by the wavelet transform.

With reference to section 7, we note that relation 17 furnishes a unique matrix representation of a dendrogram.

9 Discussion and Conclusions

Generalization to regular p -way trees, for $p > 2$, may also be considered. For $p = 3$ a natural wavelet function is derived from the triangle scaling (Starck et al., 1998) function, which is itself a convolution of a box function (the scaling function defining the Haar transform, used in this article) with itself. The Haar scaling function used above was $(\frac{1}{2}, \frac{1}{2})$. Convolving this with itself gives then the scaling function $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$. Convolving the box function again with the triangle function gives the B_3 spline scaling function, $(\frac{1}{16}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4}, \frac{1}{16})$, which is particularly natural for the analysis of a 5-way, $p = 5$, tree.

A remark on implementation follows: the 3-way tree is unfolded at each node into two 2-way trees. More generally any regular p -way tree is unfolded at each node into $p-1$ two-way branchings. The wavelet transform algorithm described previously is then directly applied.

We now look at other related work.

In Khrennikov and Kozyrev (2004) and Kozyrev (2001) the Haar wavelet transform, defined on binary trees, was also introduced and discussed. Compared to the notation used here, the descriptions are related though a p -adic change of variable (viz., $\sum_0^\infty a_i p^i$ is mapped onto $\sum_0^\infty a_i p^{-i-1}$.)

For the $p = 2$ case, a convenient notational expression is given by the Vladimirov operator (see Avetisov, Bikulov, Kozyrev and Osipov, 2002) which is a modified differentiation operator. The Vladimirov operator is a p -adically expressed derivative for an ultrametric space with *linearly* related hierarchical levels, ν . In Kozyrev (2001, 2003) it is shown how the eigenvalues of the Vladimirov operator are the Haar wavelets. As a consequence, the hierarchical Haar wavelet transform is a spectral analysis of the Vladimirov operator.

Our work differs from the works cited in the following way. Firstly, these other works deal with regular p -way trees. Degeneracies are allowed, which can cater for the irregular p -way trees that we have considered. We have preferred to directly address the dendrogram data structure, given that it models observed data well. Secondly, these other works cater for infinite

trees. We have restricted ourselves to a more curtailed problem, with the aim of having a straightforward implementation, and with the aim of targeting the analysis of practical, constructive data analysis problems.

We have also been more focused in this work compared to the general setting described in comprehensive depth by Benedetto and Benedetto (2004).

An important reason for considering dendrograms rather than infinite regular trees is that the former setting gives rise to (low order) polynomially bound algorithms for all operations; whereas the latter, in the general case, are not polynomially bound.

A final path for future work will be noted. The Haar wavelet transform on a dendrogram (H) gives us information on the rate of change of the clusters (q), with respect to the level index of each cluster (ν). In a sense this Haar wavelet transform is the derivative of H with respect to ν . This perspective may be of benefit when dealing with the dynamics of ultrametric spaces (Avetisov et al., 2002, and references therein; Kuhlmann, 2002).

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Appendix 1. Haar Wavelet Transform Used in Image/Signal Processing

Classically, the Haar wavelet function basis for analysis of $L^2(\mathbb{R}^m)$ is determined by inducing the basis from an m -dimensional pixel (time step, voxel, etc.) grid, \mathbb{Z}^m . Basis functions of a space denoted by V_j are defined from a scaling function ϕ as follows (Starck, Murtagh and Bijaoui, 1998):

$$\phi_{j,i}(x) = \phi(2^{-j}x - i) \quad i = 0, \dots, 2^j - 1 \quad \text{with } \phi(x) = \begin{cases} 1 & \text{for } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases} \quad (18)$$

The functions ϕ are all box functions, defined on the interval $[0, 1)$ and are piecewise constant on 2^j subintervals. We can approximate any function in spaces V_j associated with basis functions ϕ_j , in a very fine manner for V_0 (in this case of V_0 , all values), more crudely for V_{j+1} and so on. We consider the nesting of spaces, $\dots V_{j+1} \subset V_j \subset V_{j-1} \dots \subset V_0$. Equation (1) directly leads to a dyadic analysis.

Next we consider the orthogonal complement of V_{j+1} in V_j , and call it W_{j+1} . The basis functions for W_j are derived from the Haar wavelet. We find

$$\psi_{j,i}(x) = \psi(2^{-j}x - i) \quad i = 0, \dots, 2^j - 1 \quad \text{with } \psi(x) = \begin{cases} 1 & 0 \leq x < \frac{1}{2} \\ -1 & \frac{1}{2} \leq x < 1 \\ 0 & \text{otherwise} \end{cases} \quad (19)$$

This leads to the basis for V_j as being equal to: the basis for V_{j+1} together with the basis for W_{j+1} . In practice we use this finding like this: we write a given function in terms of basis functions in V_j ; then we rewrite in terms of basis functions in V_{j+1} and W_{j+1} ; and then we rewrite the former to yield, overall, an expression in terms of basis functions in V_{j+2} , W_{j+2} and W_{j+1} . The wavelet parts provide the detail part, and the space V_{j+2} provides the smooth part.

For the definitions of scaling function and wavelet function in the case of the Haar wavelet transform, proceeding from the given signal, the spaces V_j are formed by averaging of pairs of adjacent values, and the spaces W_j are formed by differencing of pairs of adjacent values. Proceeding in this direction, from the given signal, we see that application of the scaling or wavelet functions involves downsampling of the data. The low-pass filter is a moving average. The high-pass filter is a moving difference. Other low- and high-pass filters are alternatively used to yield other wavelet transforms.

Appendix 2. Hierarchy, Binary Tree and Ultrametric Topology

A hierarchy, H , is defined as a binary, rooted, unlabeled, node-ranked tree, also termed a dendrogram (Benzécri, 1979; Johnson, 1967; Lerman, 1981; Murtagh, 1985). A hierarchy defines a set of embedded subsets of a given set, I . However these subsets are totally ordered by an index function ν , which is a stronger condition than the partial order required by the subset relation. A bijection exists between a hierarchy and an ultrametric space.

Let us show these equivalences between embedded subsets, hierarchy, and binary tree, through the constructive approach of inducing H on a set I .

Hierarchical agglomeration on n observation vectors, $i \in I$, involves a series of $1, 2, \dots, n - 1$ pairwise agglomerations of observations or clusters, with the following properties. A hierarchy $H = \{q | q \in 2^I\}$ such that (i) $I \in H$, (ii) $i \in H \forall i$, and (iii) for each $q \in H, q' \in H : q \cap q' \neq \emptyset \implies q \subset q'$ or $q' \subset q$. Here we have denoted the power set of set I by 2^I . An indexed hierarchy is the pair (H, ν) where the positive function defined on

H , i.e., $\nu : H \rightarrow \mathbb{R}^+$, satisfies: $\nu(i) = 0$ if $i \in H$ is a singleton; and (ii) $q \subset q' \implies \nu(q) < \nu(q')$. Here we have denoted the positive reals, including 0, by \mathbb{R}^+ . Function ν is the agglomeration level. Take $q \subset q'$, let $q \subset q''$ and $q' \subset q''$, and let q'' be the lowest level cluster for which this is true. Then if we define $D(q, q') = \nu(q'')$, D is an ultrametric. In practice, we start with a Euclidean or other dissimilarity, use some criterion such as minimizing the change in variance resulting from the agglomerations, and then define $\nu(q)$ as the dissimilarity associated with the agglomeration carried out.

Appendix 3: P-Adic Numbers

P-adic numbers were introduced by Kurt Hensel in 1898. The ultrametric topology was introduced by Marc Krasner (1944), the ultrametric inequality having been formulated by Hausdorff in 1934. Essential motivation for the study of this area is provided by Schikhof (1984) as follows. Real and complex fields gave rise to the idea of studying any field K with a complete valuation $|\cdot|$ comparable to the absolute value function. Such fields satisfy the “strong triangle inequality” $|x + y| \leq \max(|x|, |y|)$. Given a valued field, defining a totally ordered Abelian group, an ultrametric space is induced through $|x - y| = d(x, y)$. Various terms are used interchangeably for analysis in and over such fields such as p-adic, ultrametric, non-Archimedean, and isosceles. The natural geometric ordering of metric valuations is on the real line, whereas in the ultrametric case the natural ordering is a hierarchical tree. P-adic numbers, which provide an analytic version of ultrametric topologies, have a crucially important property resulting from Ostrowski’s theorem: Each non-trivial valuation on the field of the rational numbers is equivalent either to the absolute value function or to some p-adic valuation (Schikhof, 1984, p. 22). Essentially this theorem states that the rationals can be expressed in terms of (continuous) reals, or (discrete) p-adic numbers, and no other alternative system.

The p-adic numbers are base p numbers, where p is a prime number. It can be shown that the reals can be expressed as p-adic numbers where p is infinity. The question then arises as to whether any one of $p = 2, 3, 5, 7, 11, \dots, \infty$ can be preferred. For want of justification to limit attention to one or a few values of p, taking them all gives rise to the adelic number system (Brekke and Freund, 1993).

Appendix 4: Some Properties of Ultrametric Spaces

See elsewhere for the basic ultrametric inequality, and the triangle property – isosceles with small base or equilateral. The following is based on Lerman (1981), chapter 0, part IV.

Theorem 1: Every point of a circle in an ultrametric space is a center of the circle.

Proof 1: it suffices to consider the triangle a, b, x , where a is the center of the given circle, b is an element of this circle, and x is an element of the circle with the same radius but with center b . This triangle is isosceles. From the triangle property the result follows.

Corollary 1: Two circles of the same radius, that are not disjoint, are overlapping.

Definition 1: A divisor of the ultrametric space, E , is an equivalence relation D satisfying $\forall a, b, x, y, \in E : aDb$ and $(d(x, y) \leq d(a, b)) \iff xDy$.

Corollary 2: Circles of the same radius form a partition of the ultrametric set. The corresponding equivalence is a divisor of the space.

Definition 2: A valuation of a divisor D of the space E is the number $\nu(D) = \sup_{xDy} d(x, y)$.

Corollary 3: If D and D' are two divisors in E , a finite metric space, verifying $D \leq D'$, then $\nu(D) \leq \nu(D')$ and reciprocally.

Theorem 2: If C and C' are disjoint circles in E , the distance $d(x, y)$ of an $x \in C$ and of an $y \in C'$ depends on C and C' only, and not on x and y .

Proof 2: Consider the triangles x, \bar{x}, y , where $\bar{x} \in C'$ and apply the ultrametric triangle relationship.

Corollary 4: The quotient E/D of an ultrametric space by a divisor is an ultrametric space. The distance between two of its points is strictly greater than $\nu(D)$ in the finite case.

Definition 3: An ultrametric proximity is a positive (possibly infinite) function $p : E \times E \rightarrow \mathbb{R}^+ \cup \{+\infty\}$, verifying (i) $p(y, x) = p(x, y)$, (ii) $p(x, y) = +\infty$ iff $x = y$; and (iii) $p(x, z) \geq \min(p(x, y), p(y, z))$.

Corollary 5: If d is an ultrametric distance, then $-\log d$ is an ultrametric proximity. If p is an ultrametric proximity, then $\exp(-p)$ is an ultrametric distance.

Theorem 3: For an $n \times n$ matrix of positive reals, symmetric with respect to the principal diagonal, to be a matrix of distances associated with an ultrametric distance on E , a sufficient and necessary condition is that a permutation of rows and columns satisfies the following form of the matrix:

1. Above the diagonal term, equal to 0, the elements of the same row are non-decreasing.
2. For every index k , if

$$d(k, k + 1) = d(k, k + 2) = \dots = d(k, k + \ell + 1)$$

then

$$d(k + 1, j) \leq d(k, j) \text{ for } k + 1 < j \leq k + \ell + 1$$

and

$$d(k + 1, j) = d(k, j) \text{ for } j > k + \ell + 1$$

Under these circumstances, $\ell \geq 0$ is the length of the section beginning, beyond the principal diagonal, the interval of columns of equal terms in row k .

Proof 4: Follows from ultrametric triangle inequality. See Lerman (1981), p. 50.

Theorem 5: In an ultrametric topology, every ball is both open and closed (termed clopen).

(The empty set and the universal set are both clopen. The complement of a clopen set is clopen. Finite unions and intersections of clopen sets are clopen.)

From Chakraborty (2004):

A *basic neighborhood* of x , of radius r , is the set $N(x, r) = \{y \in X : d(x, y) < r\}$. An *open set*, $U \subset X$, is a union of basic neighborhoods, i.e. $\forall x \in U, \exists r = r(x) > 0$ s.t. $N(x, r) \subset U$.

Many sets are open and closed at the same time. This property is relative to subspaces. Let (X, d) be a metric space and $Y \subset X$. If $y \in Y$ and $r > 0$, let $N_Y(y, r)$ denote the basic neighborhood of y in Y , and $N_X(y, r)$ the basic neighborhood of y in X . Then $N_Y(y, r) = N_X(y, r) \cap Y$. It follows that a set $U \subset Y$ is open in Y iff \exists an open set V in X s.t. $U = V \cap Y$. An analogous statement holds for closed sets. If $U \subset Y \subset X$ then U can be open (or closed) in Y without being open (or closed) in X .

Appendix 5: Ultrametric Spaces are 0-Dimensional

Informally, a set of points is of necessity 0-dimensional.

From Chakraborty (2004):

A *base* B for the topology T is such that $B \subset T$, and every element of T is a union of elements from B .

A metric space (X, d) is called 0-dimensional if $\forall x \in X, r > 0, \exists U$, a set, which is clopen, and $x \in U \subset N(x, r)$.

Van Rooij (1978): a topology is 0-dimensional if it has a base consisting of clopen sets. I.e., if for every $a \in X$ and for every closed $A \subset X$ that does not contain a , there exists a clopen set U such that $a \in U, A \subset X \setminus U$.