AN ANALYTICAL APPROACH TO SOME DIOPHANTINE INEQUALITIES

by

J. GAJRAJ

(Royal Holloway College, University of London).

Thesis submitted for the Ph.D of the University of London.

CONTENTS

ABSTRACT 2

PREFACE 4

STATEMENT OF COLLABORATION WITH DR.BAKER 5

NOTATION 6

INTRODUCTION 7

PART 1

CHAPTER 1 15

CHAPTER 2 25

CHAPTER 3 39

CHAPTER 4 57

CHAPTER 5 72

CHAPTER 6 79

PART 2

CHAPTER 7 98

BIBLIOGRAPHY 141

ABSTRACT

Heilbronn proved that for any $\varepsilon > 0$ there exists a number $C(\varepsilon)$ such that for any real numbers θ and $N (\geq 1)$

$$\min_{1 \leq n \leq N} ||n^2 \theta|| < C(\varepsilon) N^{-\frac{1}{2}+\varepsilon} .$$

In the first part of this thesis we prove various extensions of this result. We find values of $g(r, k, s)$ so that the inequality

$$\min_{1 \leq |\underline{X}| \leq N} \max_{1 \leq i \leq r} ||f_i(\underline{X})|| < C(r, k, s, \varepsilon) N^{-\frac{1}{g(r,k,s)} + \varepsilon}$$

is soluble, where $\underline{X}$ is an integral s-dimensional vector and the f's are either polynomials (without constant term) or forms, both of degree $\leq k$.

The method used depends upon estimates for certain exponential sums. Using Weyl's estimates, we look, in Chapter 2, at monomials of different degree, and, in Chapter 3, at additive forms of degree k in s variables and quadratic polynomials. Using Hua's improvement of Vinogradov's estimates, we improve, for large values of k, the results of Chapter 2 and the results of Chapter 3 on additive forms. Also using Hua's estimates, we look, in Chapter 6 at polynomials of any degree ($\leq k$). In the course of this work we improve some results of Liu and Cook.

Birch, Davenport and Ridout proved that if Q is an indefinite quadratic form in $n (\geq 21)$ variables, the inequality

$$|Q(\underline{X})| < \varepsilon ,$$

has an integral solution $\underline{X}$ with $|\underline{X}| \geq 1$.

In the second part of the thesis we investigate the inequality

$$\min_{1 \leq |\underline{X}| \leq N} |Q(\underline{X})| < C(n, k, \varepsilon) N^{-f(n, k)} ,$$

where Q is an indefinite quadratic form in $n (\geq 21)$ variables of rank r, and $k = \min (r, n-r)$. We find values for $f(n, k)$ and show

that (i) for $k \leqslant 6$, $\lim_{n\to\infty} f(n, k) = \frac{1}{2}$, (ii) for $k \geqslant 7$, $10k < 3n$, $\lim_{k\to\infty} f(n, k) = \frac{1}{2}$, and (iii) for $10k \geqslant 3n$, $\lim_{n\to\infty} f(n, k) = \frac{1}{2}$.

PREFACE

I would like to express my thanks to Dr.A.J.Jones, who was my supervisor from January to November 1974, and who introduced me to the topic upon which the first part of this thesis is based, and to Dr.R.C.Baker, my supervisor from November 1974, whose help has been invaluable.

The results of this thesis are, to the best of my knowledge, original and my own, except for some joint work with Dr.R.C.Baker, which is specified elsewhere.

Theorem 17 is to be published in the Mathematical Proceedings of the Cambridge Philosophical Society as a joint paper with Dr.Baker, and a substantial portion of Chapters 1, 2 and 4 is to be published in Acta Arithmetica as a joint paper with Dr.Baker.

Finally, I would like to thank the University of London for awarding me a studentship which supported me during my two years of research.

STATEMENT OF COLLABORATION WITH DR.BAKER

In Chapter 1, the proof of Lemma 1.5 was given by Dr. Baker in the case of equal λ_j's and he showed its simplest applications. The idea for the proof of Theorem 17 came from his working out the case $r = 2$, $s = 2$, and he suggested that Lemma 4.1 was possible, thus opening up the work of Chapters 5 and 6.

Dr.Baker also steered me through some rather subtle arguments in the work of Davenport and Ridout [21] which were crucial to the work of §2 of Chapter 7.

He constantly checked my work and encouraged me.

NOTATION

Throughout the thesis, unless otherwise specified, k denotes a positive integer ≥ 2, ε an arbitrary positive number (usually very small) and $C(a_1, a_2, \ldots)$ a positive number depending at most upon $a_1, a_2, \ldots$, not necessarily the same at each occurrence.

In the proof of any theorems, by $F \ll G$ we mean $G > 0$, $|F| < C_0 G$, where C_0 is a constant depending at most upon the same variables as the constant in the statement of the theorem or as specified in the statement of the lemma. We also assume that N is an integer and, in any proof, that $N > N_0$ where N_0 depends at most upon the same variables as C_0 above, and that $\varepsilon < \varepsilon_0$ where ε_0 depends upon the same variables as C_0, except ε. At all times A will denote a positive integer, not necessarily the same at each occurrence, depending on the same variables as ε_0.

Finally, we write $K = 2^{k-1}$, $e(z)$ denotes $e^{2\pi iz}$ and $||\alpha||$ denotes the distance between α and the nearest integer. Also, when $\underline{x}$ is a vector, $\underline{x} = (x_1, \ldots, x_n)$, then

$$|\underline{x}| = \max_{1 \leq i \leq n} |x_i|.$$

INTRODUCTION

This thesis is divided into two parts, the first part of which is concerned with extensions to a certain theorem of Heilbronn. The second part is concerned with quantifying certain results of Davenport, Birch and Ridout concerning quadratic forms.

The starting point for the first part of this thesis is a paper of Heilbronn [23], published in 1948, in which he proved the following theorem:

Theorem 1: For any $N \geq 1$ and any real θ, there is an integer x satisfying

$$1 \leq x \leq N \quad \text{and} \quad ||\theta x^2|| < C(\varepsilon) N^{-\frac{1}{2}+\varepsilon} .$$

This theorem improved a result of Vinogradov [32] who had obtained as exponent $-2/5+\varepsilon$ instead of $-\frac{1}{2}+\varepsilon$. We will give a proof of Theorem 1 in Chapter 1 since it demonstrates the essential argument, which is used throughout the first part of the thesis, in its simplest form. We note that Theorem 1 is a one-dimensional analogue to the famous theorem of Dirichlet on diophantine approximation (Lemma 1.1) in that the inequality is soluble for every $N \geq 1$ and the constant and the degree of approximation are independent of θ.

We may restate Theorem 1 thus:

$$\min_{1 \leq x \leq N} ||\theta x^2|| < C(\varepsilon) N^{-\frac{1}{2}+\varepsilon} \quad (N \geq 1, \theta \text{ real}).$$

Immediately, the following three questions arise:

1) Can one improve the exponent in Theorem 1 without losing the uniformity of the result ?

2) What type of upper bound can we obtain for

$$\min_{1 \leqslant x \leqslant N} \|\theta x^k\| \qquad (N \geqslant 1, \theta \text{ real}) ?$$

3) In view of the multi-dimensionality of Lemma 1.1, can one generalise Theorem 1 to r dimensions ?

According to a conjecture of Hardy and Littlewood [22], made in 1914, the exponent $-\frac{1}{2}+\varepsilon$ in Theorem 1 could be replaced by $-1+\varepsilon$. The exponent certainly cannot be improved beyond -1, for, as Professor Heilbronn remarked, if p is an odd prime and a is not divisible by p then

$$\left\|\frac{a\,n^2}{p}\right\| \geqslant \frac{1}{p} \qquad \text{for } 1 \leqslant n \leqslant p-1.$$

Professor Heilbronn has also remarked that if one could improve the exponent to $-1+\varepsilon$ then it would follow that the absolutely least quadratic non-residue (mod p) is less than $C(\varepsilon)\,p^{\varepsilon}$. For if a is a quadratic non-residue (mod p) then so is each of the numbers $a\,n^2$ $(1 \leqslant n \leqslant p-1)$ and

$$\left\|\frac{a\,n^2}{p}\right\| < C(\varepsilon)\,p^{-1+\varepsilon}$$

implies that $a\,n^2$ is congruent (mod p) to a number of absolute value less than $C(\varepsilon)\,p^{\varepsilon}$. However, no improvement, other than a slight sharpening by Liu [27] in which he replaces ε by $\varepsilon(N) = \frac{1}{\log\log N}$, has been made to date.

In answer to question 2, Danicic [13] proved

Theorem 2: $\min_{1 \leqslant x \leqslant N} ||\theta x^k|| < C(\varepsilon, k) N^{-\frac{1}{K}+\varepsilon}$ (N $\geqslant$ 1, θ real).

The first attempt to generalise Theorem 1 to several dimensions was made by Danicic [13] who proved

$$\min_{1 \leqslant x \leqslant N} \max (||\theta_1 x^2||, ||\theta_2 x^2||) < C(\varepsilon) N^{-1/9+\varepsilon}$$

($N \geqslant 1$, θ_1, θ_2 real).

He improved the exponent to $-1/8 + \varepsilon$ [15], and this was later improved by Liu [28, 29] who proved the more general theorem:

Theorem 3: Suppose $f_1(\underline{x})$, $f_2(\underline{x})$ are additive forms in s variables of degree k with real coefficients. Then, for $N \geqslant 1$,

$$\min_{1 \leqslant |\underline{x}| \leqslant N} \max (||f_1(\underline{x})||, ||f_2(\underline{x})||) < C(\varepsilon, k, s) N^{-\frac{1}{g(k,s)}+\varepsilon}.$$

where

$$g(k, s) = \begin{cases} 7 & k = 2, s = 1 \\ 3K + \frac{1}{k} & k \geqslant 3, s = 1 \\ 2K + 1 + \frac{K}{ks} & s \geqslant 2 \end{cases}$$

Cook [5] using the method of Danicic [13] proved

$$\min_{1 \leq x \leq N} \max_{1 \leq i \leq r} (||\theta_i x^k||) < C(\varepsilon, k, r) N^{-\frac{1}{g(r,k)}+\varepsilon},$$

$(N \geq 1, \theta_1, \ldots \theta_r$ real), where $g(1, k) = K$,

and

$$g(r, k) = 2\, g(r-1, k) + Kr + 1 \qquad (r \geq 2).$$

Liu [30] improved Cook's result in again proving a more general result:

Theorem 4: Suppose $f_i(\underline{x})$ $(i = 1, \ldots r)$ are additive forms in s variables of degree k with real coefficients. Then, for $N \geq 1$,

$$\min_{1 \leq |\underline{x}| \leq N} \max_{1 \leq i \leq r} (||f_i(\underline{x})||) < C(\varepsilon, k, s, r)\, N^{-\frac{1}{g(k,s,r)}+\varepsilon}$$

where $g(k, s, 2) = g(k, s)$ in Theorem 3,

and $$g(k, s, r) = 2\, g(k, s, r-1) + \frac{(r-1)K}{k} + 1.$$

The method of [23] has also been used in proving several other results. Davenport [17] proved

Theorem 5: For any polynomial f of degree k, with real coefficients and without constant term, and $N \geq 1$,

$$\min_{1 \leq x \leq N} ||f(x)|| < C(k, \varepsilon)\, N^{-\frac{1}{2K-1}+\varepsilon}$$

Cook [6] generalised this result of Davenport for polynomials of degree 2 proving

<u>Theorem 6</u>: <u>Suppose $f_i(x)$ $(i = 1, \ldots r)$ are quadratic polynomials with real variables and no constant term. Then, for $N \geq 1$,</u>

$$\min_{1 \leq x \leq N} \max_{1 \leq i \leq r} ||f_i(x)|| < C(\varepsilon, r)\, N^{-\frac{1}{g(r)} + \varepsilon}$$

<u>where $g(1) = 3$, $g(r) = 4g(r-1) + 4r + 2$ for $r \geq 2$.</u>

All the above results have been proved by making use of Weyl's estimates for certain exponential sums (Lemma 1.3). By using the estimates of Hua (Lemma 4.1) for these sums, one is able to improve some of the above results for large values of k. Thus Cook [5] improved Theorem 2 for large values of k, proving

<u>Theorem 7</u>: <u>For $N \geq 1$, θ real, $k \geq 12$</u>

$$\min_{1 \leq x \leq N} ||\theta x^k|| < C(k, \varepsilon)\, N^{-\frac{1}{\rho(k)} + \varepsilon}$$

<u>where $\rho^{-1}(k) = 4\,k(k-1)\,\log(12(k-1)^2)$.</u>

Similarly, Cook [5] improved Theorem 5, proving

<u>Theorem 8</u>: <u>Suppose $F(x)$ is a polynomial of degree $k (\geq 12)$, with real coefficients and no constant term. Then, for $N \geq 1$,</u>

$$\min_{1 \leq x \leq N} ||F(x)|| < C(k, \varepsilon)\, N^{-\frac{1}{\rho_1(k)} + \varepsilon}$$

<u>where $\rho_1^{-1}(k) = 4\,k(k-1)^2\,\log(12(k-1)k^2)$.</u>

Cook [7] generalised Theorem 8 proving

Theorem 9: $$\min_{1 \leqslant x \leqslant N} \max_{1 \leqslant i \leqslant r} (\|f_i(x)\|) < C(k,r,\varepsilon)\ N^{-\frac{1}{g(r,k)}+\varepsilon}$$

where $g(1, k) = 16\ k^3 \log k$

and $g(r, k) = 8\ g(r-1, k) + 4 \qquad \text{for } k \geqslant 2.$

Danicic in two very elegant papers [14, 16], using some very delicate results from the Geometry of Numbers proved the following two theorems:

Theorem 10: Let $Q(x_1, \ldots x_n)$ be a real quadratic form in n variables. Then, for $N \geqslant 1$, there are integers $x_1, \ldots x_n$, not all zero, satisfying

$$|x_j| \leqslant N \qquad (j = 1, \ldots n)$$

and $$\|Q(x_1, \ldots x_n)\| < C(\varepsilon, n)\ N^{-\frac{n}{n+1}+\varepsilon}.$$

Theorem 11: Let Q_1, Q_2 be two real quadratic forms as in Theorem 10. Then, for $N \geqslant 1$, there are integers $x_1, \ldots x_n$, not all zero, satisfying

$$|x_j| \leqslant N \qquad (j = 1, \ldots n)$$

and $$\|Q_i(x_1, \ldots x_n)\| < C(n, \varepsilon)\ N^{-\delta_n+\varepsilon} \qquad (i = 1, 2)$$

where $\delta_n = (3 + 4\ n^{-1} + 2\ \theta\ n^{-1})^{-1}$ with $\theta = \sum_{r=1}^{n} r^{-1}.$

These results use Weyl's estimates for exponential sums but the method of proof is rather different from that of Heilbronn.

Recently Cook [10, 11] has been able to prove analogous results for cubic forms by using some results of Jane Pitman. See also [8, 9] for other results of Cook; also [12].

In Chapter 1, I will set up the mathematical structure needed to prove Heilbronn - type theorems using Weyl's estimates for exponential sums, and I will give a proof of Theorem 1. In Chapter 2, I will consider simultaneous approximations to monomials of different degrees

$$\min_{1 \leqslant x \leqslant N} \max_{1 \leqslant i \leqslant r} (||\theta_i x^{a_i}||) \qquad (\text{integers } a_i \geqslant 1)$$

and in Chapter 3, I will give improvements of Theorems 4 and 6.

In Chapters 4, 5, 6, I will use Hua's estimates for exponential sums to improve the results of Chapter 2 for large values of k and also to prove a result corresponding to Theorem 4 on additive forms and to prove results which improve Theorems 7, 8 and 9.

In Part 2 of this thesis, we study certain diophantine inequalities concerned with quadratic forms. In a series of papers in the late 1950's, Davenport, Birch and Ridout [1, 18, 19, 20, 21, 31] proved

Theorem 12: For any real indefinite quadratic form $Q(x_1, \dots x_n)$ in 21 or more variables, the inequality

$$|Q(x_1, \dots x_n)| < \varepsilon \tag{1}$$

is soluble for any $\varepsilon > 0$ in integers $x_1, \dots x_n$ not all zero.

In [19], Davenport suggested that by modifying the proof of his theorem it might be possible to show that (1) is soluble with $|\underline{x}| < P$ and with $\varepsilon = P^{-\delta}$ for any sufficiently large P with a suitable δ (depending upon n and r, the rank of Q). In Chapter 7 we do this using the methods of [1, 19, 21, 31].

P A R T 1

CHAPTER 1

In this chapter we will state the basic lemmas of analytic number theory which are necessary to prove the results of the next two chapters, and we will give a proof of Theorem 1.

§1. Lemma 1.1 (Dirichlet). Suppose $\theta_1, \dots \theta_r$ to be real numbers and $Q \geq 1$. There are integers $a_1, \dots a_r, q$ satisfying

$$1 \leq q \leq Q \quad , \quad |\theta_i - a_i q^{-1}| \leq q^{-1} Q^{-\frac{1}{r}} \qquad (1 \leq i \leq r)$$

and $\quad (a_1, \dots a_r, q) = 1.$

Proof: The lemma is well-known and a proof is easily accessible. See, for example [4].

Lemma 1.2 (Vinogradov). Let $0 < \Delta < \frac{1}{2}$ and let a be a positive integer. Then there exists a real function $\psi(z)$, of period 1, such that

$$\psi(z) = 0 \qquad \text{for} \quad ||z|| > \Delta ,$$

$$\psi(z) = \sum_{m=-\infty}^{\infty} \alpha_m \, e(m\,z) \qquad ,$$

$$\alpha_0 = \Delta \quad , \quad \alpha_{-m} = \alpha_m \qquad \text{and}$$

(1.1) $$|\alpha_m| < C(a) \min (\Delta, m^{-a-1} \Delta^{-a}) , m \geq 1.$$

Proof: This is a special case of Lemma 12 of Chapter 1 of [33].

Lemma 1.3 (Weyl): Let θ be real and suppose there are integers $a, q \geq 1$ satisfying

$$|\theta - a q^{-1}| \leq q^{-2} , \quad (a, q) = 1.$$

Then, for any positive integers P, N,

$$\sum_{j=P+1}^{P+N} \min (N, \frac{1}{||j \theta||}) \ll N + q \log q,$$

where the implied constant is absolute.

Proof: This is a special case of Lemma 8a of Chapter 1 of [33].

Lemma 1.4: Let $f(x) = \theta x^k + \theta_1 x^{k-1} + \ldots + \theta_{k-1} x$ be a polynomial with real coefficients. Let $m (\geq 1)$ be an integer, and let, for $N \geq 1$,

$$S(m) = \sum_{x=1}^{N} e(m f(x)).$$

Suppose $|\theta - a q^{-1}| \leq q^{-2}$ for some integers $q (\geq 1)$, a, where $(a, q) = 1$. Then for $H \geq 1$,

$$(1.2)\qquad \sum_{m=1}^{H} |S(m)|^{K} \ll (H\,N)^{\varepsilon}\, N^{K-k} \left(\frac{H\,N^{k-1}}{q} + 1\right)(N + q \log q)\,.$$

<u>If $k = 1$, $q \geqslant 2H$, then</u>

$$(1.3)\qquad \sum_{m=1}^{H} |S(m)| \ll q \log q.$$

<u>The constants implied in (1.2), (1.3) depend at most upon ε and k.</u>

<u>Proof</u>: By Satz 266 of [26]

$$|S(m)|^{K} \ll N^{K-1+\varepsilon/2} + N^{K-k+\varepsilon/2} \sum_{v=1}^{k!N^{k-1}} \min\left(N, \frac{1}{||m\,v\,\theta||}\right).$$

But
$$\sum_{m=1}^{H} \sum_{v=1}^{k!N^{k-1}} \min\left(N, \frac{1}{||m\,v\,\theta||}\right) \ll (HN)^{\varepsilon/2} \sum_{x=1}^{k!HN^{k-1}} \min\left(N, \frac{1}{||x\,\theta||}\right)$$

because $x = mv$ has $\ll (HN)^{\varepsilon/2}$ solutions m, v for any x. Now we can complete the proof of (1.2) using Lemma 1.3.

For the second part,

$$\sum_{m=1}^{H} |S(m)| \leqslant \sum_{m=1}^{H} ||m\,\theta||^{-1} \leqslant \sum_{1 \leqslant m \leqslant \frac{1}{2}q} ||m\,\theta||^{-1}\,.$$

Since $(a, q) = 1$, the integers $k \equiv am \pmod q$, $1 \leqslant m \leqslant q$, are distinct and not zero, and so for $1 \leqslant m \leqslant \frac{1}{2} q$

$$||m\,\theta|| = ||m\,aq^{-1} + \tfrac{1}{2}\,\phi\, q^{-1}||\,, \quad (|\phi| < 1)$$

$$= ||k\, q^{-1} + \tfrac{1}{2}\,\phi\, q^{-1}|| \gg \frac{\min\,(k, q-k)}{q}$$

and thus

$$\sum_{1<m<\frac{1}{2}q} ||m\,\theta||^{-1} \ll \sum_{1\leqslant k\leqslant \frac{1}{2}q} q\,k^{-1} \ll q \log q.$$

The result (1.2) is due to Weyl.

§2. Before giving a proof of Theorem 1, we prove a lemma which contains the basic idea of Heilbronn's method. An 'improvement on Dirichlet's theorem' (see (1.5), (1.6)) is purchased by a 'Heilbronn hypothesis', as we call the hypothesis of Lemma 1.5 about the absence of an integral solution to certain inequalities.

Lemma 1.5 Let $\lambda_1, \dots \lambda_r$ be positive real numbers and $\theta_1, \dots \theta_r$ any real numbers. Suppose there is no integral solution of the inequalities

$$1 \leqslant x \leqslant N\,,\quad ||x^{a_j}\,\theta_j|| \leqslant N^{-\lambda_j} \qquad (j = 1, \dots r)\,,$$

where the a_j $(1 \leqslant j \leqslant r)$ are positive integers, $1 \leqslant a_1 < a_2 < \dots < a_r$, and $N \geqslant 1$.

Then there is a j, $1 \leqslant j \leqslant r$, such that either

(1.4) $K_j\,(\lambda_1 + \dots + \lambda_j) \geqslant 1 - \varepsilon(j\,K_j + 3)$

or

there is an integer $q \geqslant 1$ satisfying

$$q \ll N^{K_j(\lambda_1 + \dots + \lambda_j) + \varepsilon(j\,K_j + 2)} \qquad (1.5)$$

and

(1.6) $\underline{||q\,\theta_j|| \ll N^{-a_j - \lambda_j + K_j(\lambda_1 + \dots + \lambda_j) + \varepsilon(j K_j + 2)}}$

where $K_j = 2^{a_j - 1}$ $(j = 1, \dots r)$ and the implied constants in (1.5), (1.6) depend at most upon $a_1, \dots a_r$, r and ε.

Proof: Let $\psi_i(z)$ be the function obtained in Lemma 1.2 on letting $\Delta = \frac{1}{2} N^{-\lambda_i}$, $a = [\varepsilon^{-1}] + 1$ for $1 \leqslant i \leqslant r$. Then, by the hypothesis,

$$\sum_{x=1}^{N} \prod_{i=1}^{r} \psi_i(x^{a_i} \theta_i) = 0$$

or

$$(1.7) \qquad 2^{-r} N^{1-(\lambda_1 + \dots + \lambda_r)} + \sum{}^{*} \alpha_{m_1} \dots \alpha_{m_r} T(\underline{m}^r) = 0 \quad ,$$

where $\sum^*$ denotes a summation over all non-zero integral vectors $\underline{m}^r = (m_1, \dots m_r)$, and

$$T(\underline{m}^j) = \sum_{x=1}^{N} e(m_1 \theta_1 x^{a_1} + \dots + m_j \theta_j x^{a_j}) \quad \text{for } 1 \leqslant j \leqslant r.$$

From (1.7) we deduce

$$N^{1 - (\lambda_1 + \dots + \lambda_r)} \ll \sum{}^{*} |\alpha_{m_1} \dots \alpha_{m_r} T(\underline{m}^r)| \, .$$

On summing over the $\underline{m}^r$ with $|m_1| > N^{\lambda_1 + \varepsilon}$ we have, by (1.1),

$$\sum |\alpha_{m_1} \dots \alpha_{m_r} T(\underline{m}^r)| \ll N \sum_{m_1 > N^{\lambda_1 + \varepsilon}} N^{a \lambda_1} m_1^{-a-1}$$

$$\ll N^{1-a\varepsilon} \ll 1$$

and similarly for $|m_2| > N^{\lambda_2+\varepsilon}$, etc. Thus (unless $1 - (\lambda_1 + \dots + \lambda_r) < \varepsilon$, which we may ignore as it would prove (1.4)),

$$N^{1 - (\lambda_1+\dots+\lambda_r)} << \sum^{(r)} |\alpha_{m_1} \dots \alpha_{m_r} T(\underline{m}^r)|,$$

where $\sum^{(r)}$ is a sum over $|m_i| \leqslant N^{\lambda_i+\varepsilon}$ $(i=1, \dots r)$, $\underline{m}^r \neq 0$, and using (1.1), we obtain

$$N << \sum^{(r)} |T(\underline{m}^r)| .$$

Clearly there is a j, $1 \leqslant j \leqslant r$ such that

(1.8) $$N << \sum_{m_j \neq 0}^{(j)} |T(\underline{m}^j)| .$$

This completes the first stage of the lemma. We now distinguish between the two cases: $a_j = 1$, $a_j > 1$. First consider the latter.

Suppose $Q = N^{\mu}$ where $\mu = a_j - K_j(\lambda_1 + \dots + \lambda_j) + \lambda_j - \varepsilon(j K_j + 2)$. Then, by Lemma 1.1 there exist integers q $(\geqslant 1)$, a, satisfying

(1.9) $$(a, q) = 1,\ 1 \leqslant q \leqslant Q,\ |\theta_j - aq^{-1}| \leqslant q^{-1} Q^{-1} \leqslant q^{-2}.$$

Using Hölder's inequality in (1.8) we obtain

$$N^{K_j} << N^{(\lambda_1+\dots+\lambda_j + j\varepsilon)(K_j-1)} \sum_{m_j \neq 0}^{(j)} |T(\underline{m}^j)|^{K_j} ,$$

and on using (1.2),

$$(1.10)\qquad N^{K_j} \ll N^{K_j(\lambda_1+\ldots\lambda_j+j\ \varepsilon)-\lambda_j+\varepsilon+K_j-a_j}\left(\frac{N^{a_j-1+\lambda_j+\varepsilon}}{q}+1\right)(N + q \log q)$$

and (1.10) yields

$$N^{a_j-K_j(\lambda_1+\ldots+\lambda_j)+\lambda_j-(j\ K_j+1)\varepsilon} \ll N^{a_j+\lambda_j+\varepsilon}\, q^{-1} + N^{a_j-1+\lambda_j+\varepsilon} \log q$$

$$+ q \log q$$

Since $q \leqslant Q$, the term $q \log q$ is negligible, and since we may assume (1.4) is false, the term $N^{a_j-1+\lambda_j+\varepsilon} \log q$ is negligible. We are thus able to deduce

$$q \ll N^{K_j(\lambda_1+\ldots+\lambda_j)+\varepsilon(j\ K_j+2)}$$

which is (1.5), and finally (1.9) implies (1.6). We now need only consider the case $a_j = 1$. Then $j = 1$, and from (1.8) we obtain

$$N \ll \sum_{m_1=1}^{N^{\lambda_1+\varepsilon}} |T(\underline{m}^1)|.$$

We suppose q to be an integer, $1 \leqslant q \leqslant Q = N^{1-\varepsilon}$, satisfying (1.9) with $j = 1$. If $q \geqslant 2N^{\lambda_1+\varepsilon}$, by Lemma 1.4 we obtain

$$N \ll q \log q,$$

which is absurd. So $q \leqslant 2N^{\lambda_1+\varepsilon}$, $||q\ \theta_1|| \leqslant N^{-1+\varepsilon}$ which implies (1.6) and (1.5). This complete the proof of the lemma.

We now give a proof of Theorem 1. The proof given here is due to Davenport [17].

Proof of Theorem 1: We assume that for some positive number λ,

(1.11) $$||n^2 \theta|| > N^{-\lambda} \quad \text{for } 1 \leqslant n \leqslant N.$$

We will show

(1.12) $$\lambda > \frac{1}{2} - 5\varepsilon \ ,$$

and clearly this proves the theorem.

Now, by Lemma 1.5, on letting $r = 1$, whence $j = 1$ and $a_j = 2$,

either

(1.13) $$2\lambda \geqslant 1 - 5\varepsilon$$

or

there is an integer q ($\geqslant 1$) satisfying

(1.14) $$q \ll N^{2\lambda+4\varepsilon} \ ,$$

(1.15) $$||q\theta|| \ll N^{-2+\lambda+4\varepsilon} \ .$$

We may suppose (1.13) to be false, for otherwise we would have proved (1.12).

Now $$q \ll N^{2\lambda+4\varepsilon} \leqslant N^{1-\varepsilon} \ ,$$

for otherwise $\lambda > \frac{1}{2} - 5\varepsilon$ which would prove (1.12). Also,

$$||q^2\theta|| \leqslant q\ ||q\ \theta|| \ll N^{2\lambda+4\varepsilon} . N^{-2+\lambda+5\varepsilon} = N^{-2+3\lambda+9\varepsilon} \ ,$$

and thus, by (1.11),

$$2 + 3\lambda + 9\varepsilon > -\lambda - \varepsilon ,$$

which implies (1.12) and thus proves the theorem.

CHAPTER 2

In this chapter we consider simultaneous approximations to monomials of different degrees:

$$\min_{1 \leqslant x \leqslant N} \max_{1 \leqslant i \leqslant r} (\|\theta_i x^{a_i}\|) \qquad \text{(integers } a_i\ (\geqslant 1)),$$

and in Theorem 14 we prove an analogue to the following theorem of Minkowski:

Theorem 13: Let $0 < \eta_i < 1$, $(i = 1, \dots k)$, $\eta_1 \dots \eta_k \geqslant N^{-1}$. Let $\theta_1, \dots \theta_k$ be real. There is an integer, x, satisfying

$$1 \leqslant x \leqslant N, \qquad \|\theta_i x\| \leqslant \eta_i \qquad (1 \leqslant i \leqslant k).$$

This theorem is a special case of Minkowski's convex body theorem, a proof of which may be found in [4].

§1. Theorem 14: Let $0 < \eta_i < 1$, $(i = 1, \dots k)$, and

$$\eta_1 \dots \eta_k \geqslant C(k, \varepsilon)\, N^{-\frac{1}{K} + \varepsilon} \qquad (2.1)$$

Let θ be real. Then there is an integer, x, satisfying

$$1 \leqslant x \leqslant N, \qquad \|\theta x^i\| \leqslant \eta_i \qquad (i = 1, \dots k). \qquad (2.2)$$

Proof: Suppose that there is no solution of the inequalities (2.2). We apply Lemma 1.5 with $r = k$, $a_j = j$, $N^{-\lambda_j} = \eta_j$ and $\theta_j = \theta$ $(1 \leqslant j \leqslant k)$. We shall show that

(2.3) $$K_k (\lambda_1 + \ldots + \lambda_k) \geqq 1 - A\varepsilon \ .$$

This implies

$$\eta_1 \ldots \eta_k \leqslant N^{-\frac{1}{K_k} + \frac{A_\varepsilon}{K_k}} ,$$

and therefore proves the theorem. Thus it is sufficient to show (2.3). By Lemma 1.5, since we have made a 'Heilbronn hypothesis', there is a j, $1 \leqslant j \leqslant k$, such that

either

(2.4) $$K_j (\lambda_1 + \ldots + \lambda_j) \geqslant 1 - \varepsilon(j\, K_j + 3)$$

or

there is an integer q $(\geqslant 1)$ satisfying

(2.5) $$q \ll N^{K_j(\lambda_1 + \ldots + \lambda_j) + \varepsilon(j\, K_j + 2)}$$

and

(2.6) $$||q\, \theta|| \ll N^{-j - \lambda_j + K_j(\lambda_1 + \ldots + \lambda_j) + \varepsilon(j\, K_j + 2)}$$

We may suppose that (2.4) is false since this would automatically imply (2.3). So we assume (2.5), (2.6) hold. Clearly (2.5) implies

(2.7) $$q \ll N^{1-\varepsilon} ,$$

for otherwise (2.3) is true.

Now, for $1 \leqslant i \leqslant k$,

$$||q^i \theta|| \leqslant q^{i-1} ||q\theta|| << N^{i K_j(\lambda_1 + \dots + \lambda_j) - j - \lambda_j + b\varepsilon}$$

where $b = i\,(j K_j + 2)$. By the 'Heilbronn hypothesis' and (2.7), for some i, $1 \leqslant i \leqslant k$

$$N^{i K_j (\lambda_1 + \dots + \lambda_j) - j - \lambda_j + (b + 1)\varepsilon} > N^{-\lambda_i}$$

and therefore

$$\frac{i K_j}{j} (\lambda_1 + \dots + \lambda_j) - \frac{\lambda_j}{j} + \frac{\lambda_i}{j} \geqslant 1 - A\varepsilon \quad . \tag{2.8}$$

If $i = j$, clearly (2.8) implies (2.3) .

If $i > j$, then

$$\frac{i K_j}{j} (\lambda_1 + \dots + \lambda_j) - \frac{\lambda_j}{j} + \frac{\lambda_i}{j} \leqslant \frac{iK_j}{j} (\lambda_1 + \dots + \lambda_j + \lambda_i)$$

$$\leqslant \frac{i K_k}{k} (\lambda_1 + \dots + \lambda_k) \leqslant K_k (\lambda_1 + \dots + \lambda_k)$$

and thus (2.8) implies (2.3).

If $i < j$, then

$$\frac{i K_j}{j} (\lambda_1 + \dots + \lambda_j) - \frac{\lambda_j}{j} + \frac{\lambda_i}{j} \leqslant \frac{i K_j + 1}{j} (\lambda_1 + \dots + \lambda_j)$$

$$\leqslant \frac{k K_j}{j} (\lambda_1 + \dots + \lambda_k) \leqslant K_k (\lambda_1 + \dots + \lambda_k)$$

and thus (2.8) again implies (2.3). Since we have covered all possible values of i, this completes the proof of the theorem.

We observe in particular

Corollary to Theorem 14: Let θ be real and $N \geq 1$.

Then $\min_{1 \leq x \leq N} \max_{1 \leq i \leq k} ||\theta x^i|| < C(k, \varepsilon) N^{-\frac{1}{Kk} + \varepsilon}$.

Proof: We let $\eta_1 = \eta_2 = \ldots = \eta_k = N^{-\frac{1}{Kk} + \frac{\varepsilon}{k}}$ and the result follows.

Recently Cook [9] using a slightly more elementary argument proved the following weaker result:

Theorem: For any $\varepsilon > 0$, there exists a constant $C(\varepsilon)$, such that for any real θ, $N \geq 1$, there is an integer x satisfying

$$1 \leq x \leq N \text{ and } \max(||x\theta||, ||x^2\theta||) < C(\varepsilon) N^{-\frac{1}{5} + \varepsilon}$$

Proof: By Lemma 1.1, there is an integer y satisfying

$$1 \leq y \leq N^{\frac{3}{5}}, \quad ||y\theta|| \leq N^{-\frac{3}{5}}.$$

By Theorem 1, there is an integer z satisfying

$$1 \leq z \leq N^{\frac{2}{5}}, \quad ||z^2 y^2 \theta|| < C(\varepsilon) N^{-\frac{1}{5} + \varepsilon}.$$

Clearly $||zy\theta|| \leq N^{-\frac{1}{5}}$, and on noting $zy \leq N$, the result follows.

§2. In this section we investigate what happens when we have different θ's. We prove a result corresponding to Theorem 14 and we improve this result in a certain special case - when we try to obtain a uniform degree of approximation - as in the Corollary to Theorem 14.

Theorem 15: Let $\lambda_1, \ldots \lambda_r$ be real positive numbers, with $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_r > 0$, and $\theta_1, \ldots \theta_r$ be any real numbers. Let $1 \leqslant a_1 < a_2 < \ldots < a_r = k$ be integers. Then there is an integer x satisfying

$$(2.9) \quad 1 \leqslant x \leqslant N, \quad \max_{1 \leqslant j \leqslant r} \|\theta_j x^{a_j}\| < C(k, r, \varepsilon) N^{-\lambda_j}$$

$$\text{if} \quad \sum_{i=1}^{r} v_i (k, r) \lambda_i \leqslant 1 - \varepsilon$$

where

$$(2.10) \quad v_i (k, r) = \max (v_i (k - 1, r - 1) + 2^{k-1}, v_i(k, r-1) + 2^{k-2}),$$

for $i = 1, \ldots r - 1$,

$$\text{and} \quad v_r (k, r) = 2^{k-1}, \quad v_1 (k, 1) = 2^{k-1}.$$

Proof: We call the theorem 'case (k, r)'. Case (k, 1) is true, by Theorem 2 and Lemma 1.1. Assume Case (j, h) is known for $1 \leqslant h \leqslant j < k$, and that cases (k, 1), ... (k, r-1) are known (r > 1). We deduce case (k, r); this will obviously prove the theorem.

Suppose there are no integral solutions of the inequalities

(2.11) $$1 \leqslant x \leqslant N, \quad \max_{1\leqslant j\leqslant r} ||x^{a_j} \theta_j|| \leqslant N^{-\lambda_j} .$$

We shall deduce

(2.12) $$\sum_{i=1}^{r} v_i\,(k, r)\,\lambda_i \geqslant 1 - A\varepsilon ,$$

which will complete the proof.

By Lemma 1.5, there is a j, $1 \leqslant j \leqslant r$ such that

either

(2.13) $$K_j\,(\lambda_1 + \ldots + \lambda_j) \geqslant 1 - \varepsilon\,(j\,K_j + 3)$$

or

there is an integer $q \geqslant 1$ satisfying

(2.14) $$q << N^{K_j(\lambda_1 + \ldots + \lambda_j) + \varepsilon(j\,K_j + 2)}$$

with $||q\,\theta_j|| << N^{-a_j - \lambda_j + K_j(\lambda_1 + \ldots + \lambda_j) + \varepsilon(j\,K_j + 2)}$.

We may suppose that (2.13) is false for otherwise, since $v_i\,(k, r) \geqslant 2^{k-1} \geqslant K_j$, for $i = 1, \ldots r$, we would have proved (2.12). There are two cases to consider.

(a) $\underline{j = r}$. For every z, $1 \leqslant z \leqslant N^{1 - K_r(\lambda_1 + \ldots + \lambda_r) - \varepsilon(r\,K_r + 3)}$,

$zq << N^{1-\varepsilon} \leqslant N$,

and

$$||z^{a_r} q^{a_r} \theta_r|| \leqslant z^{a_r} q^{a_r - 1} ||q\,\theta_r|| << N^{\mu_1} < N^{-\lambda_r}$$

$$\text{since}\quad \mu_1 = a_r\,(1 - K_r(\lambda_1 + \dots + \lambda_r)) + (a_r - 1)(K_r(\lambda_1 + \dots + \lambda_r))$$

$$- a_r - \lambda_r + K_r\,(\lambda_1 + \dots + \lambda_r) - A\varepsilon$$

$$= - \lambda_r - A\varepsilon\ .$$

Thus, by the insolubility of (2.11) and the case (k-1, r-1),

$$\sum_{i=1}^{r-1} v_i\ (k - 1,\ r - 1)\ \lambda_i \geqslant 1 - K_r\,(\lambda_1 + \dots + \lambda_r) - A\varepsilon$$

for otherwise we would have a contradiction. Thus

$$\sum_{i=1}^{r-1} v_i\ (k - 1,\ r - 1)\ \lambda_i + K_r\,(\lambda_1 + \dots + \lambda_r) \geqslant 1 - A\varepsilon\ . \tag{2.15}$$

(b) $\underline{j \leqslant r - 1}$. For every z, $1 \leqslant z \leqslant N^{1 - K_j(\lambda_1 + \dots + \lambda_j) - \varepsilon(j K_j + 3)}$

$$z\, q \ll N^{1-\varepsilon} \leqslant N,$$

and

$$||z^{a_j} q^{a_j} \theta_j|| \leqslant z^{a_j} q^{a_j - 1} ||q\, \theta_j|| \ll N^{\mu_2} < N^{-\lambda_j}$$

$$\text{since}\quad \mu_2 = - \lambda_j - A\varepsilon\ .$$

Thus, by the insolubility of (2.11) and the case (k, r-1),

$$\sum_{i=1}^{j-1} v_i\ (k,\ r\text{-}1)\ \lambda_i + \sum_{i=j+1}^{r} v_{i-1}\ (k,\ r\text{-}1)\ \lambda_i \geqslant 1 - K_j\,(\lambda_1 + \dots + \lambda_j) - A\varepsilon$$

for otherwise we would have a contradiction.

Since $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_r > 0$, we obtain

$$\sum_{i=1}^{r-1} v_i\ (k,\ r - 1)\ \lambda_i + K_j\ (\lambda_1 + \dots + \lambda_j) \geqslant 1 - A\varepsilon\ .$$

and since $j \leqq r\text{-}1$, and $a_{r-1} \leqq k\text{-}1$, it follows that

$$\sum_{i=1}^{r-1} v_i(k,\ r\text{-}1)\ \lambda_i + 2^{k-2}\ (\lambda_1 + \dots + \lambda_{r-1}) \geqslant 1\text{-}A\varepsilon. \tag{2.16}$$

Now (2.15) or (2.16), along with (2.10) implies (2.12) and this completes the proof of the theorem.

The following lemma gives an exact value for $v_i\ (k, r)$.

<u>Lemma 2.1:</u> <u>Let $v_i\ (k, r)$ be defined as in (2.10). Then $v_i\ (k, r) = (r - i + 2)\ .\ 2^{k-2}$</u>

<u>Proof:</u> We prove the result by induction on r. Suppose $r = 1$. Then $v_1\ (k, 1) = 2^{k-1} = (1 - 1 + 2)\ 2^{k-2}$, which proves the result for this case.

So suppose $r > 1$. Then $v_r\ (k, r) = (r - r + 2)\ 2^{k-2} = 2^{k-1}$ which proves the result for $i = r$.

For $1 \leqslant i < r$,

$$v_i\ (k, r) = \max\ (v_i\ (k - 1, r - 1) + 2^{k-1},\ v_i\ (k, r - 1) + 2^{k-2})$$

$$= \max\ \{(r - i + 1)\ 2^{k-3} + 2^{k-1},\ (r - i + 1)\ 2^{k-2} + 2^{k-2}\}$$

$$= \max\ \{(r - i + 5)\ 2^{k-3},\ (2r - 2i + 4)\ 2^{k-3}\}$$

$$= (r - i + 2)\ 2^{k-2}, \quad \text{since } r - i \geqslant 1,$$

and the lemma is proved.

The following theorem is an improvement of Theorem 15 in the special case $\lambda_1 = \lambda_2 = \ldots = \lambda_r$.

<u>Theorem 16:</u> <u>Let $1 \leqslant a_1 < \ldots < a_r = k$ be integers, and $\theta_1, \ldots \theta_r$ be real numbers. Then, for $N \geqslant 1$, there is an integer x satisfying</u>

$$1 \leq x \leq N,\ \max_{1 \leqslant i \leqslant r} ||\theta_i\ x^{a_i}|| < C(k, r, \varepsilon)\ N^{-\frac{1}{U_{k,r}} + \varepsilon},$$

where

$$(2.17)\qquad U_{k,r} = \begin{cases} 2^{k-r}\,((r-1)2^r + 1) & r = 1, 2, 3, 4 \\ 2^{k-4}\,(65 + 2\,(r-5)(r+4)) & r \geq 5 \end{cases} .$$

Before we prove the theorem, we prove a recursive lemma.

Lemma 2.2: Let $U_{k,r}$ $(k \geq 1,\ r = 1, \ldots k)$ be positive integers defined recursively by

$$(2.18)\qquad U_{k,1} = 2^{k-1},\ U_{k,r} = \max\,(U_{k,r-1} + (r-1)2^{k-2},\ U_{k-1,r-1} + r.2^{k-1})$$

if $r > 1$.

Then $U_{k,r}$ satisfies (2.17).

Proof: The cases $r \leq 4$ are clearly true. So we suppose $r \geq 5$. Now

$$U_{k,5} = \max\,(U_{k,4} + 4.2^{k-2},\ U_{k-1,4} + 5.2^{k-1})$$

$$= \max\,(2^{k-4}\,(3.2^4 + 1) + 16.2^{k-4},\ 2^{k-5}.\,49 + 80.2^{k-5})$$

$$= \max\,(130.2^{k-5},\ 129.2^{k-5}) = 65.2^{k-4}.$$

Now suppose $h \geq 6$ and the result known for all $r < h$, for all k. Then

$$U_{k,h} = \max\,(2^{k-4}\,(65 + 2(h-6)\,(h+3)) + (h-1)2^{k-2},$$

$$2^{k-5}\,(65 + 2(h-6)\,(h+3)) + h\,.\,2^{k-1}).$$

Clearly the first of this pair is larger and is

$$2^{k-4}\ (65 + 2(h - 5)\ (h + 4)),$$

and this proves the lemma.

Proof of Theorem 16: Again we call the theorem 'Case (k, r)', and the proof is similar to that of Theorem 15.

Suppose there are no integer solutions of the inequalities

(2.19) $$1 \leqslant x \leqslant N, \max_{1\leqslant j\leqslant r} ||x^{a_j}\theta_j|| \leqslant N^{-\lambda}.$$

We shall prove

(2.20) $$\lambda \geqslant \frac{1}{U_{k,r}} - A\,\varepsilon\ ,$$

and this will prove the theorem.

In Theorem 15, let $\lambda_1 = \ldots = \lambda_r = \lambda$ and replace $\sum_{i=1}^{\ell} v_i(m, \ell)$ by $U_{m,\ell}$. We follow the proof of Theorem 15 until (2.15) in (a), and (2.16) of (b). Then, instead of (2.15), (2.16) we get

(2.21) $$(U_{k-1,\ r-1} + r.2^{k-1})\ \lambda \geqslant 1 - A\,\varepsilon\ ,$$

(2.22) $$(U_{k,\ r-1} + (r - 1)2^{k-2})\ \lambda \geqslant 1 - A\,\varepsilon\ .$$

Then (2.21) or (2.22) together with Lemma 2.2 imply (2.20) which proves the theorem.

Corollary to Theorem 16: For θ_i $(i = 1, \dots k)$ real, $N \geq 1$,

$$\min_{1 \leq x \leq N} \max_{1 \leq i \leq k} ||x^i \theta_i|| < C(k, \varepsilon) N^{-\frac{1}{U_k} + \varepsilon}$$

where

$$U_k = \begin{cases} (k - 1)2^k + 1 & k = 1, 2, 3, 4 \\ 2^{k-4} (65 + 2(k - 5)(k + 4)) & k \geq 5. \end{cases}$$

In Theorems 15 and 16, there are restrictions upon the relative sizes of the λ_j's. In certain specific cases it is possible to relax this restriction and we prove the following

Corollary to Lemma 1.5: Suppose $1 \leq k_1 < k_2$ where k_1, k_2 are integers and suppose θ_1, θ_2 to be real numbers. Then for $N \geq 1$, there is an integer x satisfying $1 \leq x \leq N$,

$$||\theta_1 x^{k_1}|| < C(k_1, k_2, \varepsilon) N^{-\lambda_1 + \varepsilon}$$

$$||\theta_2 x^{k_2}|| < C(k_1, k_2, \varepsilon) N^{-\lambda_2 + \varepsilon}$$

for any λ_1, λ_2 satisfying

$$\lambda_1 (K_1 + K_2) + \lambda_2 K_2 \leq 1,$$

where $K_1 = 2^{k_1 - 1}$, $K_2 = 2^{k_2 - 1}$.

We note that this result is at least as good as Theorem 15 in the case $r = 2$, since from Theorem 15, $v_1 (k, 2) = 3/2\ K_2$,

$v_2\ (k, 2) = K_2$ (on putting $k = k_2$), while the above Corollary gives $v_1\ (k, 2) \leqslant 3/2\ K_2$, $v_2\ (k, 2) = K_2$, since $K_1 \leqslant \frac{1}{2} K_2$. This anomaly arises because of the generality for which Theorem 15 was proved.

Proof of the Corollary to Lemma 1.5: We suppose that for any positive λ_1, λ_2

$$(2.22) \qquad ||\theta_1 x^{k_1}|| > N^{-\lambda_1}, \quad ||\theta_2 x^{k_2}|| > N^{-\lambda_2}$$

for all integers x, $1 \leqslant x \leqslant N$.

Clearly, it is enough to show

$$(2.23) \qquad \lambda_1 (K_1 + K_2) + \lambda_2 (K_2) \geqslant 1 - A\varepsilon .$$

By (2.22) and Lemma 1.5, there is a j, $1 \leqslant j \leqslant 2$, such that

either

$$(2.24) \qquad K_j (\lambda_1 + \dots + \lambda_j) \geqslant 1 - A\varepsilon$$

or

there is an integer $q\ (\geqslant 1)$ satisfying

$$(2.25) \qquad \begin{gathered} q \ll N^{K_j(\lambda_1 + \dots + \lambda_j) + A\varepsilon} \\ ||q\,\theta_j|| \ll N^{-k_j - \lambda_j + K_j(\lambda_1 + \dots + \lambda_j) + A\varepsilon} \end{gathered}$$

We consider the possible cases:

$j = 1$: Then (2.24), (2.25) become

either

$$(2.26) \qquad K_1 \lambda_1 \geqslant 1 - A\varepsilon$$

or

there is an integer q satisfying

$$q \ll N^{K_1 \lambda_1 + A\varepsilon}$$

(2.27)

$$||q\,\theta_1|| \ll N^{-k_1 + (K_1-1)\lambda_1 + A\varepsilon}$$

We may suppose (2.26) false for otherwise (2.23) is proved.

Now, by Theorem 2, there is an integer t satisfying

(2.28) $$1 \leq t \leq N^{1-K_1\lambda_1 - A\varepsilon}, \quad ||t^{k_2} q^{k_2} \theta_2|| \ll N^{-(1-K_1\lambda_1-A\varepsilon)\frac{1}{K_2} + \varepsilon}.$$

Also $||q^{k_1} t^{k_1} \theta_1|| \ll q^{k_1-1} t^{k_1} ||q\,\theta_1|| \ll N^{-\lambda_1 - A\varepsilon}$.

Thus, since $q\,t \ll N^{1-A\varepsilon}$, it follows from (2.23), (2.28) that

$$-(1 - K_1\lambda_1 - A\varepsilon)\frac{1}{K_2} \geq -\lambda_2 - A\varepsilon$$

i.e. $\qquad -1 + K_1\lambda_1 \geq -K_2\lambda_2 - A\varepsilon$

i.e. $\qquad K_1\lambda_1 + K_2\lambda_2 \geq 1 - A\varepsilon$, which proves (2.23) in this case.

j = 2: Then (2.24), (2.25) become

either

(2.29) $$K_2(\lambda_1 + \lambda_2) \geq 1 - A\varepsilon$$

or

there is an integer q satisfying

$$q \ll N^{K_2(\lambda_1 + \lambda_2) + A\varepsilon}$$

(2.30) $$||q\,\theta_2|| \ll N^{-k_2 - \lambda_2 + K_2(\lambda_1 + \lambda_2) + A\varepsilon}.$$

We may suppose (2.29) false, for otherwise (2.23) is proved. Again, by Theorem 2, there is an integer t satisfying

(2.31) $$1 \leqslant t \leqslant N^{1 - K_2(\lambda_1+\lambda_2)-A\varepsilon}, \quad ||t^{k_1} q^{k_1} \theta_1|| \ll N^{-(1-K_2(\lambda_1+\lambda_2)-A\varepsilon)\frac{1}{K_1}+\varepsilon}$$

Also $||q^{k_2} t^{k_2} \theta_2|| \ll q^{k_2-1} t^{k_2} ||q\,\theta_2|| \ll N^{-\lambda_2 - A\varepsilon}$.

Thus, since $q\,t \ll N^{1-A\varepsilon}$, it follows from (2.23), (2.31) that

$$-(1 - K_2(\lambda_1 + \lambda_2) - A\varepsilon)\frac{1}{K_1} \geqslant -\lambda_1 - \varepsilon$$

i.e. $$\lambda_1 (K_1 + K_2) + \lambda_2 K_2 \geqslant 1 - A\varepsilon$$

which proves (2.23) in this case.

This completes the proof of the theorem.

CHAPTER 3

In this chapter we consider simultaneous approximations to certain types of polynomials without constant term. We prove a theorem on additive forms which improves Theorem 4 and one on quadratic polynomials which improves Theorem 6.

The method of proof of Theorem 17 is that of Liu [28, 29, 30] with some improvements in the arrangement of the proof. The method of proof of Theorem 18 is that of Cook [6] where the improvement is obtained by incorporating an idea of Liu. (See pp.79-80 [29]).

§1: In this section we prove

Theorem 17: For $i = 1, \ldots r$, let $f_i(\underline{X}) = \sum_{j=1}^{s} \theta_{ij} x_j^k$ be additive forms in s variables. Then

$$(3.1) \qquad \min_{1 \leqslant |\underline{X}| \leqslant N} \ \max_{1 \leqslant i \leqslant r} \ ||f_i(\underline{X})|| < C(\varepsilon, k, r, s) N^{-\frac{1}{g(k,r,s)} + \varepsilon}$$

where

$$(3.2) \qquad g(k, r, s) = \begin{cases} 2K + 1 & r = 2, s = 2 \\ 2^{r+1} - 1 & k = 2, r \geqslant 2, s = 1 \\ 2^{r-2}(3K + \frac{1}{k} + \cdot 1) - 1 & k \geqslant 3, r \geqslant 2, s = 1 . \end{cases}$$

The case $k \geqslant 2$, $r = 2$, $s = 1$ is proved in Theorem 3. We improve the result of Theorem 3 if $r = 2$, $s \geqslant 2$. We also improve the result of Theorem 4 for $r \geqslant 3$, $s = 1$ and hence for $s \geqslant 1$. The results can more easily be compared by noting that in (3.2)

(3.3) $$g(k, r, 1) = 2g(k, r-1, 1) + 1 \qquad (r \geqslant 3).$$

By a straightforward extension of Theorem 9 it is possible to obtain the following result:

Let $Q_i(x_1, \dots x_s), (i = 1, \dots r)$, be real quadratic forms in s variables. Then there exist integers $x_1, \dots x_s$, not all zero, satisfying, for $N \geqslant 1$,

$$\max_{1 \leqslant i \leqslant r} \| Q_i(x_1, \dots x_s) \| < C(s, r, \varepsilon) \; N^{-\frac{1}{g(s,r)} + \varepsilon},$$

$$|x_j| \leqslant N \qquad (j = 1, \dots s)$$

where $g(s, r) = 2r\, s^{-1} + 1 + 2\rho s^{-1}$

with $\rho = \sum_{n=1}^{s} g(n, r-1)$

and $g(s, 1) = (s+1).s^{-1}$.

(See Cook [8]).

Suppose $s = 1$. Then $g(1, 1) = 2$, $g(1, r) = 2r + 1 + 2\rho$ where $\rho = 2g(1, r-1)$ and this gives

$$g(1, r) = 9.2^{r-1} - 2r - 5.$$

Now the result obtained in Theorem 17 for $g(2, r, 1)$ is a special case of this result, and we note $g(2, r, 1) = 2^{r+1} - 1$. Clearly this is an improvement on $g(1, r)$ for $r \geqslant 6$, which is what one would expect.

Proof of Theorem 17: The proof of the theorem is by induction on r. In the case $r \geq 2$, $s = 1$ Theorem 3 starts the induction. For the case $r = 2$, $s \geq 2$ Theorem 2 starts the induction. We will prove the theorem in the form of three lemmas.

Lemma 3.1: Let λ $(< \frac{1}{r})$ be a positive number such that

(3.4) $$\min_{1 \leq |\underline{X}| \leq N} \max_{1 \leq i \leq r} ||f_i(\underline{X})|| \geq N^{-\lambda}.$$

Then there is a ρ, $0 \leq \rho \leq \lambda + \varepsilon$, and a j, $1 \leq j \leq s$, such that

(3.5) $$|S(\underline{U})| \gg N^{1-r\rho/s}$$

for at least $[N^{r\rho-\varepsilon}] + 1$ integer points $\underline{U} = (U_1, \ldots U_r)$ with

(3.6) $$1 \leq |\underline{U}| \leq N^{\lambda+\varepsilon}$$

where $S(\underline{U}) = \sum_{x=1}^{N} e((\underline{U}.T)\, x^k)$,

$T = (\theta_{ij}, \ldots \theta_{rj})$, $\underline{U}.T = U_1 \theta_{ij} + \ldots + U_r \theta_{rj}$

Proof: By Lemma 1.2, with $\Delta = \frac{1}{2} N^{-\lambda}$ and $a = [s\varepsilon^{-1}] + r$,

$$\sum_{1\leqslant|\underline{X}|\leqslant N} \psi\left\{\prod_{i=1}^{r} f_i\,(\underline{X})\right\} = 0 \;,$$

i.e. $$\sum_{U_1=-\infty}^{\infty} \cdots \sum_{U_r=-\infty}^{\infty} \alpha_{U_1} \cdots \alpha_{U_r} \prod_{j=1}^{s} S_j(\underline{U}) = 0$$

where $$S_j\,(\underline{U}) = \sum_{|x|=1}^{N} e((\underline{U}.T)x^k) \;.$$

On separating out the term with $\underline{U} = \underline{0}$, we obtain

$$2^{-r} N^{s-r\lambda} + \sum_{U_1=-\infty}^{\infty} \cdots \sum_{U_r=-\infty}^{\infty} \alpha_{U_1} \cdots \alpha_{U_r} \prod_{j=1}^{s} S_j(\underline{U}) = 0$$

(where $\underline{U} \neq \underline{0}$ in the summation), which implies

(3.7) $$\sum_{U_1=-\infty}^{\infty} \cdots \sum_{U_r=-\infty}^{\infty} |\alpha_{U_1} \cdots \alpha_{U_r}| \prod_{j=1}^{s} |S_j(\underline{U})| \geqslant 2^{-r} N^{s-r\lambda}$$

(where $\underline{U} \neq \underline{0}$ in the summation).

For $0 \leqslant \ell < r$,

$$\sum_{|U_1|<N^{\lambda+\varepsilon}} \cdots \sum_{|U_\ell|<N^{\lambda+\varepsilon}} \sum_{|U_{\ell+1}|\geqslant N^{\lambda+\varepsilon}} \cdots \sum_{|U_r|\geqslant N^{\lambda+\varepsilon}} |\alpha_{U_1} \cdots \alpha_{U_r}| \prod_{j=1}^{s} |S_j(\underline{U})|$$

$$\ll N^{s-\ell\lambda} N^{\ell(\lambda+\varepsilon)} \sum_{|U_{\ell+1}|\geqslant N^{\lambda+\varepsilon}} \cdots \sum_{|U_r|\geqslant N^{\lambda+\varepsilon}} |\alpha_{U_1} \cdots \alpha_{U_r}|$$

$$\ll N^{s+\ell\varepsilon} N^{\lambda a(r-\ell)} N^{-(\lambda+\varepsilon)(r-\ell)a} \;, \text{ by (1.1)}$$

$$= N^{s+\ell\varepsilon} N^{\varepsilon a\ell-\varepsilon ar} \ll 1, \text{ by the choice of } a.$$

Thus (3.7), along with the fact that $\alpha_{m_i} \ll \Delta(i = 1, .. r)$, implies

$$\sum_{U_1} \cdots \sum_{U_r} \prod_{j=1}^{s} |S_j(\underline{U})| \gg N^s$$

where $0 < |\underline{U}| \leqslant N^{\lambda+\varepsilon}$,

and therefore

$$\sum_{j=1}^{s} \sum_{U_1} \cdots \sum_{U_r} |S_j(\underline{U})|^s \gg N^s$$

where $0 < |(\underline{U})| \leqslant N^{\lambda+\varepsilon}$.

Thus, there is some j, $1 \leqslant j \leqslant s$, for which we write $S_j(\underline{U}) = S(\underline{U})$, satisfying

(3.8) $$S = \sum_{U_1} \cdots \sum_{U_r} |S(\underline{U})|^s \gg C\, N^s$$

where $0 < |\underline{U}| \leqslant N^{\lambda+\varepsilon}$.

So there exists a ρ, $0 \leqslant \rho \leqslant \lambda + \varepsilon$ such that

(3.9) $$|S(\underline{U})| > C_1^{1/s} N^{1-r\rho/s}$$

for at least $[N^{r\rho-\varepsilon}] + 1$ vectors $\underline{U}$ in $1 \leqslant |\underline{U}| \leqslant N^{\lambda+\varepsilon}$,

where $C_1 = C_1\ (C, r, s)$.

For suppose that such a ρ does not exist. For integer m, with $\varepsilon m > 2r\ (\lambda + \varepsilon)$, write

$$S = \sum |S(\underline{U})|^s + \sum_{j=0}^{m-1} T_j,$$

where the first summation on the right is over $1 \leqslant |\underline{U}| \leqslant N^{\lambda+\varepsilon}$ and $|S(\underline{U})|^s < C_1 N^{s-r(\lambda+\varepsilon)}$, while $T_j = \sum |S(\underline{U})|^s$, where the summation is taken over

$$C_1 N^{s-r(j+1)(\lambda+\varepsilon)m^{-1}} \leqslant |S(\underline{U})|^s < C_1 N^{s-rj(\lambda+\varepsilon)m^{-1}}.$$

It then follows that, by a suitable choice of C_1,

$$S < CN^s,$$

for every large N, which contradicts (3.8). Thus (3.9) holds and together with (3.8) proves the lemma.

Lemma 3.2: Let $\lambda > 0$ and suppose that (3.4) holds for $r \geqslant 2$. Then there is a ρ, $0 \leqslant \rho \leqslant \lambda + \varepsilon$, such that at least one of the following inequalities hold:

$$\lambda \geqslant \frac{s}{rK} - A\varepsilon \tag{3.10}$$

$$\lambda \geqslant \left(\frac{r^2K}{ks} + \frac{rK}{ks} + \frac{r}{k}\right)^{-1} - A\varepsilon \tag{3.11}$$

$$\lambda\,(2g(k,\, r-1,\, 1) + 1) + \rho\left(\frac{rK}{s} + \frac{1}{k} - 1 - g\,(k,\, r-1,\, 1)\right) \geqslant 1 - A\varepsilon\,. \tag{3.12}$$

Proof: Let ρ and T be as in Lemma 3.1 and we write $(\theta_1, \ldots \theta_r)$ in place of $(\theta_{1j}, \ldots \theta_{rj})$. Let $\underline{U}$ be any of the vectors satisfying (3.5) and (3.6).

We may assume $r\,K\rho/s \leqslant 1 - 4\varepsilon$, for otherwise (3.10) holds and the lemma is proved. By Lemma 1.1 there are integers $a = a(\underline{U})$, $q = q(\underline{U})$ satisfying

$$(a,\, q) = 1,\; 1 \leqslant q \leqslant N^{k-Kr\rho/s\, -\, 2\varepsilon} \tag{3.13}$$

and

$$(3.14) \qquad |q(\underline{U}.T) - a| \leqslant N^{-k + Kr\rho/s + 2\varepsilon} .$$

By Lemma 1.4,

$$|S(\underline{U})|^K << N^{K+\varepsilon} q^{-1} + N^{K-1+\varepsilon} + N^{K-r\, K\rho/s - \varepsilon}$$

and so by (3.5)

$$N^{K-rK\rho/s} << N^{K+\varepsilon} q^{-1} + N^{K-1+\varepsilon} + N^{K-r\, K\rho/s-\varepsilon} .$$

The last two terms are negligible, and so

$$(3.15) \qquad q << N^{r\, K\rho/s + \varepsilon} .$$

Suppose now $\rho \geqslant 2\varepsilon$. By Lemma 1.1, there are integers $W, b_1, \ldots b_r$ such that

$$(3.16) \qquad 1 \leqslant W \leqslant N^{k-r\, K\rho/s-3\varepsilon} , \quad (W, b_1, \ldots b_r) = 1,$$

and

$$(3.17) \qquad \max_{1\leqslant i\leqslant r} |W\theta_i - b_i| \leqslant N^{-\frac{1}{r}(k-r\, K\rho/s - 3\varepsilon)} .$$

It now follows from (3.14), (3.15), (3.16), (3.17) that

$$|q(U_1b_1 + \ldots + U_rb_r) - a\, W| \leqslant \sum_{i=1}^{r} q|U_i|\; |b_i - W\,\theta_i| + W|q(U.T) - a|$$

$$<< N^{r\, K\rho/s + \lambda + 2\varepsilon - \frac{1}{r}(k - r\, K\rho/s - 3\varepsilon)} + N^{-\varepsilon} .$$

Since $\rho \leqslant \lambda + \varepsilon$, we may assume that

$$rK\rho/s + \lambda + 2\varepsilon - \frac{1}{r}(k - rK\rho/s - 3\varepsilon) \leqslant -\varepsilon, \text{ for}$$

otherwise (3.11) holds and the lemma is proved.

Thus

$$|q(U_1 b_1 + \ldots + U_r b_r) - a W| << N^{-\varepsilon}$$

and therefore we may conclude that $q = q(\underline{U})$ divides W. But W has fewer than N^{ε} divisors. Thus there are at least $N^{r\rho-2\varepsilon}$ vectors $\underline{U}$ satisfying (3.5) and (3.6) for which $q(\underline{U})$ takes the same value q; say for $\underline{U} \in H$.

Consider those $\underline{U} \in H$ which fall into the various regions

$$k_j b N^{\lambda-\rho+2\varepsilon} \leqslant U_j < (k_j + 1) b N^{\lambda-\rho+2\varepsilon} \quad (j = 1, \ldots r),$$

where $k_1, \ldots k_r$ are integers and b is a positive number. There are $<< b^{-r} N^{r(\rho-\varepsilon)} + 1$ of these regions that meet H, by (3.6). If $b = b(\varepsilon, k, r)$ is suitably chosen, one of the regions contains two members of H, $\underline{U}_1$ and $\underline{U}_2$ say. If $\underline{U} = \underline{U}_1 - \underline{U}_2$, we have

(3.18) $$\underline{U} \neq 0, \quad ||q(\underline{U}.T)|| \leqslant 2N^{-k+rK\rho/s+2\varepsilon}$$

(3.19) $$|\underline{U}| << N^{\lambda-\rho+3\varepsilon}$$

If $\rho \leqslant 2\varepsilon$, we simply take U to be any of the vectors satisfying (3.5) and (3.6). Without loss of generality we may assume $U_1 > 0$.

We now apply the induction hypothesis (with Theorem 3 starting the induction if $r \geqslant 2$, $s = 1$, and Theorem 2 starting if $r = 2$, $s \geqslant 2$.) We write $g = g\,(k, r - 1, 1)$. There is an integer n such that

$$(3.20) \qquad 1 \leqslant n \leqslant N^{(2\lambda + 5\varepsilon - \rho)\, g},$$

$$(3.21) \qquad ||\theta_i\, U_1^{k-1}\, q^k\, n^k|| << N^{-2\lambda - 4\varepsilon + \rho} \qquad (2 \leqslant i \leqslant r).$$

Let $x = n\, q\, U_1$. Then, by (3.19), (3.21)

$$(3.22) \qquad ||\theta_i\, x^k|| << U_1||\,\theta_i\, U_1^{k-1}\, q^k\, n^k|| << N^{-\lambda-\varepsilon} \quad (2 \leqslant i \leqslant r).$$

Also, for θ_1, we have by (3.15), (3.18), (3.19), (3.20) and (3.21),

$$(3.23) \qquad ||\theta_1\, x^k|| \leqslant n^k\, q^{k-1}\, U_1^{k-1}||q(U_1\theta_1 + \dots + U_r\,\theta_r)|| + \sum_{i=2}^{r} |U_i|\; ||\theta_i\, U_1^{k-1}\, q^k\, n^k|| << N^{\sigma} + N^{-\lambda-\epsilon}$$

where

$$\sigma = kg\,(2\lambda + 5\varepsilon - \rho) + (k - 1)\left(\frac{rK\rho}{s} + \varepsilon\right) + (k - 1)(\lambda + 3\varepsilon - \rho) - k + \frac{rK\rho}{s} + 2\varepsilon$$

Suppose

(3.24) $$\sigma \leqslant - \lambda - \varepsilon$$

Then $$||\theta_i x^k|| < N^{-\lambda} \qquad (1 \leqslant i \leqslant r).$$

We also have $x << N^{\tau}$

where $\tau = (2\lambda + 5\varepsilon - \rho)\ g + \frac{rK\rho}{s} + \varepsilon + \lambda - \rho + 3\varepsilon$.

By (3.24) $\tau < 1$. Thus $\underline{X} = (0, 0, \ldots 0, x, 0, \ldots 0)$ (x in j^{th} place) satisfies $|\underline{X}| \leqslant N$, and $||f_i(\underline{X})|| = ||\theta_i x^k|| < N^{-\lambda}$ $(1 \leqslant i \leqslant r)$, which contradicts (3.4). Thus (3.24) does not hold, which implies that (3.12) holds and thus proves the lemma.

Lemma 3.3: Let $\lambda > 0$ and suppose (3.4) holds for $r \geqslant 2$. Then,

if $r = 2, s = 2$

(3.25) $$\lambda \geqslant \frac{1}{2K + 1} - A\varepsilon \quad ,$$

and for $r \geqslant 2, k = 2, s = 1$

(3.26) $$\lambda \geqslant \frac{1}{2^{r+1} - 1} - A\varepsilon \quad ,$$

and for $r \geqslant 2, k \geqslant 2, s = 1$,

(3.27) $$\lambda \geqslant \frac{1}{2^{r-2}(3K + \frac{1}{k} + 1) - 1} - A\varepsilon \quad ,$$

Clearly this proves the theorem.

Proof: Suppose $r = 2$, $s = 2$. Then $g(k, r-1, 1) = g(k, 1, 1) = K$. Thus the coefficient of ρ in (3.12) is $K + \frac{1}{k} - 1 - K = \frac{1}{k} - 1 < 0$. Then, by Lemma 3.2,

$$\lambda \geqslant \min\left(\frac{1}{K}, \frac{k}{3K + 2}, \frac{1}{2K + 1}\right) - A\varepsilon$$

$$= \frac{1}{2K + 1} - A\varepsilon$$

and this proves (3.25).

Now suppose $r \geqslant 3$, $k = 2$, $s = 1$ (since by Theorem 3, (3.26) is true for $r = 2$). Then the coefficient of ρ in (3.12) is

$$2r + \tfrac{1}{2} - 1 - (2^r - 1) = 2r + \tfrac{1}{2} - 2^r,$$

which is negative for $r \geqslant 3$.

Then, by Lemma 3.2,

$$\lambda \geqslant \min\left(\frac{1}{rK}, \frac{k}{r^2K + rK + r}, \frac{1}{2^{r+1} - 1}\right) - A\varepsilon ,$$

and by a simple calculation we obtain

$$\lambda \geqslant \frac{1}{2^{r+1} - 1} - A\varepsilon ,$$

which proves (3.26).

We now suppose $r \geqslant 3$, $k \geqslant 3$, $s = 1$ (since by Theorem 3, (3.27) is true for $r = 2$).

Then the coefficient of ρ in (3.12) is

$$r\,K + \frac{1}{k} - 2^{r-3}\,(3K + \frac{1}{k} + 1),$$

which is negative for $k \geqslant 3$, $r \geqslant 3$. Thus by Lemma 3.2

$$\lambda \geqslant \min\left(\frac{1}{rK}, \frac{k}{r^2K + rK + r}, \frac{1}{2^{r-2}(3K + \frac{1}{k} + 1) - 1}\right) - A\varepsilon$$

and a simple calculation shows

$$\lambda \geqslant \frac{1}{2^{r-2}(3K + \frac{1}{k} + 1) - 1} - A\varepsilon ,$$

which proves (3.27) and hence the lemma.

We notice that other than in the special case $r = 2$, $s = 2$, the result does not improve as the value of s increases. This is because our knowledge of the value of ρ is very limited. All we know is that $0 \leqslant \rho \leqslant \lambda + \varepsilon$, and so when its coefficient is negative we must implicitly suppose $\rho = 0$. If somehow we could restrict ρ even further, i.e. allow ρ to lie in a range like $[\delta, \lambda + \varepsilon]$ for some positive δ, we would obtain a result dependent upon s, which would improve as s increases. This, however, I have been unable to do.

§2: In this section we prove

Theorem 18: Suppose that $f_1, \ldots f_r$ are real quadratic polynomials having no constant term. Then there is an integer x satisfying

$$1 \leqslant x \leqslant N, \quad \max_{1 \leqslant i \leqslant r} ||f_i(x)|| < C(\varepsilon, r)\, N^{-\frac{1}{g(r)} + \varepsilon} \tag{3.28}$$

where

$$g(r) = 4g(r-1) + 2 \qquad (r \geqslant 3) \tag{3.29}$$

with $g(1) = 3$ and $g(2) = 15\frac{1}{2}$.

It is easily seen that an explicit formula for $g(r)$ is

$$\underline{g(r) = 2/3\ (4^{r-2} - 1) + 62.4^{r-3} \qquad (r \geq 2).}$$

Proof: We prove the theorem by induction on r. Theorem 5 starts the induction and so we may assume $r \geq 2$. Suppose

$$f_i(n) = \theta_i n^2 + \phi_i n \qquad (i = 1, \ldots r). \tag{3.30}$$

Suppose there is some number $\lambda > 0$ such that the following inequalities have no integral solution:

$$1 \leq x \leq N, \qquad ||f_i(x)|| \leq N^{-\lambda} \qquad (i = 1, \ldots r). \tag{3.31}$$

We will show

$$\lambda > (g(r))^{-1} - \varepsilon \quad . \tag{3.32}$$

By a slight adaptation of the argument of Lemma 3.1 we can show that there is a ρ, $0 \leq \rho \leq \lambda + \varepsilon$ such that

$$|T(\underline{m})| >> N^{1-r\rho} \tag{3.33}$$

for at least $[N^{r\rho-\varepsilon}] + 1$ r-tuples $\underline{m} = (m_1, \ldots m_r)$ with

$$1 \leq |\underline{m}| \leq N^{\lambda+\varepsilon} \quad , \tag{3.34}$$

where $$T(\underline{m}) = \sum_{n=1}^{N} e(\underline{m} \cdot \underline{\theta}\, n^2 + \underline{m} \cdot \underline{\phi}\, n)$$

and $\underline{m}.\theta = \sum_{i=1}^{r} m_i \theta_i$, $\underline{m}.\underline{\phi} = \sum_{i=1}^{r} m_i \phi_i$.

We now confine ourselves to r-tuples $\underline{m}$ satisfying (3.33), (3.34).

Suppose

(3.35) $$Q = N^{2-2r\rho-2\varepsilon}, \quad T = N^{D},$$

where $D > 0$ will be chosen later. Since we know $0 \leqslant \rho \leqslant \lambda + \varepsilon$, we may assume $Q \geqslant 1$ for otherwise (3.32) follows. Then, by Lemma 1.1, for each $\underline{m}$, there exist integers $a = a(\underline{m})$, $b = b(\underline{m})$, $q = q(\underline{m})$, $t = t(m)$, such that

(3.36) $$\underline{m} \cdot \underline{\theta} = a\, q^{-1} + \alpha, \ (a, q) = 1, \ 1 \leqslant q \leqslant Q, \ q|\alpha| \leqslant Q^{-1},$$

(3.37) $$\underline{m} \cdot \underline{\phi} = b\, t^{-1} + \beta, \ (b, t) = 1, \ 1 \leqslant t \leqslant T, \ t|\beta| \leqslant T^{-1}.$$

Now, by Lemma 1.4,

$$|T(\underline{m})|^2 << q^{-1} N^{2+\varepsilon} + q N^{\varepsilon} + N^{1+\varepsilon}$$

and, thus, by (3.33)

(3.38) $$N^{2-2r\rho} << q^{-1} N^{2+\varepsilon} + q N^{\varepsilon} + N^{1+\varepsilon}.$$

Clearly $q N^{\varepsilon} = o\,(N^{2-2r\rho})$ and therefore (3.38) implies

<u>either</u>

(3.39) $$\lambda > \frac{1}{2r} - A\varepsilon$$

<u>or</u>

(3.40) $$q << N^{2r\rho+\varepsilon}.$$

We may suppose (3.39) to be false for otherwise (3.32) would have been proved.

Suppose $\rho \geqslant 2\varepsilon$. By Lemma 1.1 there exist integers $W, z_1, \ldots z_r$ satisfying

$$1 \leqslant W \leqslant N^{2-2r\rho-3\varepsilon} \quad , \quad (W, z_1, \ldots z_r) = 1,$$

and $$|W\,\theta_i - z_i| \leqslant N^{(-2+2r\rho+3\varepsilon)\frac{1}{r}} \qquad (i = 1, \ldots r).$$

Then

$$|q(m_1\, z_1 + \ldots + m_r\, z_r) - a\,W| \ll W|q(\underline{m}.\underline{\theta}-a)| + \sum_{i=1}^{r} q|\underline{m}|\;|W\,\theta_i - z_i|$$

$$\ll N^{-\varepsilon} + N^{2r\rho+\varepsilon+\lambda+\varepsilon - \frac{1}{r}(2-2r\rho-3\varepsilon)}$$

Now, <u>either</u>

(3.41) $$\lambda\;(2r + 3) > \frac{2}{r} - A\varepsilon$$

<u>or</u>

(3.42) $$|q\;(m_1 z_1 + \ldots + m_r z_r) - a\,W| \ll N^{-\varepsilon}\;.$$

We may suppose that (3.41) is false, for otherwise (3.32) is proved. Since $(a, q) = 1$, $q|W$ and thus there are $O(N^{\varepsilon})$ choices for $q = q(\underline{m})$. Thus there are at least $N^{r\rho-2\varepsilon}$ r-tuples $\underline{m}$ satisfying (3.33), (3.34) for which $q(\underline{m})$ takes the same value q; say for $\underline{m} \in H$.

Consider those $\underline{m} \in H$ which fall into the various regions

$$k_j\; b\; N^{\lambda-\rho+2\varepsilon} \leqslant m_j \leqslant (k_j + 1)\; b\; N^{\lambda-\rho+2\varepsilon} \quad (j = 1, \ldots r),$$

where $k_1, \dots k_r$ are integers and b is a positive number. There are $\ll b^{-r} N^{r(\rho-\varepsilon)} + 1$ of these regions that meet H, by (3.34). If $b = b(\varepsilon, r)$ is suitably chosen, one of the regions contains two members of H, $\underline{m}_1, \underline{m}_2$ say. If $\underline{m} = \underline{m}_1 - \underline{m}_2$, we have

$$(3.43) \qquad ||q(\underline{m}.\underline{\theta})|| < 2 Q^{-1}$$

$$(3.44) \qquad 0 < |\underline{m}| \ll N^{\lambda-\rho+2\varepsilon} .$$

If $\rho \leqslant 2\varepsilon$, we simply suppose $\underline{m}$ to be any of the r-tuples satisfying (3.33) and (3.34). We may suppose $m_1 > 0$.

We now apply the inductive hypothesis, and we write $g = g(r - 1)$. Let

$$(3.45) \qquad \sigma = 2g\lambda + 4g\varepsilon - g\rho.$$

Then there is an integer x satisfying

$$(3.46) \qquad 1 \leqslant x \leqslant N^{\sigma}, \quad ||f_i^*(x)|| \ll N^{-2\lambda+\rho-3\varepsilon} \qquad (i = 2, \dots r)$$

where $f_i^*(n) = m_1 q^2 t^2 \theta_i n^2 + qt \phi_i n \qquad (i = 2, \dots r).$

Suppose

$$(3.47) \qquad y = m_1 q t x, \; D = 1 + \rho(\tfrac{1}{2} - 2r) - \lambda - \sigma - 4\varepsilon$$

Clearly $D > 0$, for otherwise (3.32) is true. Then

$$y = m_1 q t x \ll N^{1-\rho/2-\varepsilon}, \text{ and therefore}$$

$$(3.48) \qquad y \leqslant N .$$

By (3.44), (3.46), for $i = 2, \ldots r$,

$$(3.49) \qquad ||f_i(y)|| = ||m_1{}^2q^2t^2\theta_i x^2 + m_1 qt\phi_i x|| < m_1||f_i{}^*(x)|| \ll N^{-\lambda-\varepsilon}.$$

Also $||f_1(y)|| = ||m_1{}^2q^2t^2\theta_1 x^2 + m_1 qt\phi_1 x||$

$$\leqslant ||m_1{}^2q^2t^2x^2\underline{m}.\underline{\theta}|| + ||qtx\underline{m}.\underline{\phi}||$$

$$+ ||\sum_{i=2}^{r} m_i(m_1 q^2t^2\theta_i x^2 + qt\phi_i x)||$$

By (3.49),

$$(3.50) \qquad ||\sum_{i=2}^{r} m_i(m_1 q^2t^2\theta_i x^2 + qt\phi_i x)|| \leqslant \sum_{i=2}^{r} m_i||f_i{}^*(x)|| \leqslant N^{-\lambda-\varepsilon}.$$

Also,

$$(3.51) \qquad ||m_1 q^2t^2x^2\underline{m}.\underline{\theta}|| \leqslant |m_1 qt^2x^2|\ ||q\underline{m}.\underline{\theta}|| \ll N^{-\lambda-\varepsilon}$$

Thus by the Heilbronn hypothesis that (3.31) is insoluble and (3.49), (3.50), (3.51),

$$||q\ t\ x\ \underline{m}.\underline{\phi}|| \geqslant N^{-\lambda-\varepsilon}.$$

Now $||q\ t\ x\ \underline{m}.\underline{\phi}|| \leqslant |qx|\ ||t\ \underline{m}.\underline{\phi}|| \ll N^{\sigma_1}$

where $\sigma_1 = \sigma + 2r\rho + \varepsilon - 1 - \rho(\tfrac{1}{2} - 2r) + \lambda + \sigma + 4\varepsilon$.

Thus

$$\lambda + 2\sigma + \rho(4r - \tfrac{1}{2}) + 5\varepsilon - 1 \geqslant -\lambda - \varepsilon,$$

and so by (3.45),

(3.52) $$\lambda\ (2 + 4g) + \rho(4r - \tfrac{1}{2} - 2g) > 1 - A\varepsilon\ .$$

We first consider the case $r = 2$. Then $g(r - 1) = g(1) = 3$, and so the coefficient of ρ in (3.52) is 3/2. Thus (3.52) implies

$$\lambda > \frac{1}{15\tfrac{1}{2}} - A\varepsilon\ ,$$

which proves (3.32) in this case.

We now suppose $r > 3$. By the proof of the case $r = 2$ and the inductive hypothesis, it is clear that $4r - \tfrac{1}{2} - 2g < 0$, and so (3.52) gives

$$\lambda > \frac{1}{2 + 4g\ (r-1)} - A\varepsilon\ ,$$

which proves (3.32) in this case and hence completes the proof of the theorem.

CHAPTER 4

In this chapter we use Hua's estimates for certain types of exponential sums to improve the results of Chapter 2 for large values of k.

§1: We start by proving a result corresponding to Lemma 1.4.

Lemma 4.1: Let $f(x) = \theta_k x^k + \dots + \theta_1 x + \theta_o$ be a polynomial with real coefficients. Let $m\ (\geq 1)$ be an integer, and for $P \geq 1$, suppose

$$S(m) = \sum_{x=1}^{P} e\,(m\,f(x))\,.$$

Suppose further that the number of solutions in integers of

(4.1) $x_1^h + \dots + x_{2t}^h = y_1^h + \dots + y_{2t}^h$ is $\ll P^{4t-\frac{1}{2}k(k-1)+\delta^1}$ $(1 \leq h \leq k-1,\ 1 \leq x_j, y_j \leq P)$, for some $\delta^1 > 0$, where t is a positive integer. Suppose further that there are integers $q\ (\geq 1)$, a, satisfying

$$(a, q) = 1, \qquad |\theta_k - a\,q^{-1}| \leq q^{-2},$$

and that p is an integer, $1 \leq p \leq P$. Then

(4.2)
$$\sum_{m=1}^{H} |S(m)|^{4t}$$
$$\ll (HP)^{\varepsilon}\ (Hp^{4t} + p^{\delta^1}\, P^{4t-1}\, ((\frac{HP}{q} + 1)(1 + p^{1-k}q\log q) + H)).$$

In particular, the estimates (4.1) and hence (4.2) hold if
$\delta^1 = \frac{1}{2}k(k-1)(1 - \frac{1}{k-1})^{\ell} + \varepsilon,\quad t = \left[\frac{1}{4}k(k-1) + \frac{\ell(k-1)}{2}\right] + 1,$

where ℓ is any positive integer.

Proof: The second assertion of the lemma is Hua's 'mean value theorem'. (Theorem 1 of [24] with s replaced by 2t and k replaced by k-1 ≥ 1). We prove the first part of the lemma by slightly adapting the proof of Theorem 4 of [24].

Let $S_m(y) = \sum_{x=1}^{p} e\,(m\,f(x+y))$.

Then,

$$S(m) = \frac{1}{p} \sum_{x=1}^{p} \sum_{z=1}^{p} e\,(m\,f(z))$$

$$= \frac{1}{p} \sum_{x=1}^{p} \sum_{y=1-x}^{P-x} e\,(m\,f(x+y))$$

i.e.

(4.3) $$S(m) = \frac{1}{p} \sum_{y=1}^{p} S_m(y) + \varrho_1 p \quad \text{where } |\varrho_1| \leq 2.$$

Write $f(x+y) = A_k x^k + A_{k-1} x^{k-1} + \ldots + A_o$.

Then

(4.4) $$A_k = \theta_k,\ A_{k-1} = \theta_{k-1} + k\,\theta_k y,\ \ldots,\ A_o = \theta_k y^k + \theta_{k-1} y^{k-1} + \ldots + \theta_o.$$

By Hölder's inequality,

$$\Big|\sum_{y=1}^{p} S_m(y)\Big|^{2t} \leq p^{2t-1} \sum_{y=1}^{p} |S_m(y)|^{2t}$$

$$= p^{2t-1} \sum_{y=1}^{p} \{S_m(y)\}^t \{\overline{S_m(y)}\}^t$$

i.e.

(4.5) $$\Big|\sum_{y=1}^{p} S_m(y)\Big|^{2t} \leq p^{2t-1} \sum_{y=1}^{p} \Big(\sum_{x_1=1}^{p} \cdots \sum_{x_t=1}^{p} \sum_{x_1'=1}^{p} \cdots \sum_{x_t'=1}^{p} e(\phi)\Big)$$

where $\emptyset = m\,(f(x_1 + y) + \ldots + f(x_t + y) - f(x_1' + y) - \ldots - f(x_t' + y))$

$$= m\,\left(\sum_{j=0}^{k} A_j \left(\sum_{i=1}^{t} (x_i^{\,j} - x_i'^{\,j})\right)\right)$$

Let $\psi\,(N_{k-1}, \ldots N_1)$ be the number of solutions of

$$x_1^h + \ldots + x_t^h - x_1'^h - \ldots - x_t'^h = N_h \quad (1 \leqslant h \leqslant k-1,\ 1 \leqslant x, x' \leqslant p).$$

Then we obtain

$$\sum_{y=1}^{p} |S_m(y)|^{2t} \leqslant \sum_{x_1=1}^{p} \cdots \sum_{x_t=1}^{p} \sum_{x_1'=1}^{p} \cdots \sum_{x_t'=1}^{p} \left|\sum_{y=1}^{p} e(\emptyset)\right|$$

$$\leqslant \sum_{|N_1| \leqslant t\,p} \cdots \sum_{|N_{k-1}| \leqslant t\,p^{k-1}} \psi(N_{k-1}, \ldots N_1) \left|\sum_{y=1}^{p} e(m(A_{k-1}N_{k-1} + \ldots + A_1 N_1))\right|$$

$$(4.6) \leqslant \sqrt{\left\{ \sum_{|N_1| \leqslant t\,p} \cdots \sum_{|N_{k-1}| \leqslant t\,p^{k-1}} \psi^2\,(N_{k-1} \ldots N_1) \sum_{|N_1| \leqslant t\,p} \cdots \sum_{|N_{k-1}| \leqslant t\,p^{k-1}} \left|\sum_{y=1}^{p} e(m(A_{k-1}\,N_{k-1} + \ldots + A_1\,N_1))\right|^2 \right\}}$$

by Schwarz's inequality.

First, the expression

$$\sum_{|N_1| \leqslant t\,p} \cdots \sum_{|N_{k-1}| \leqslant t\,p^{k-1}} \psi^2\,(N_{k-1}, \ldots N_1)$$

$$= \sum_{|N_1| \leqslant t\,p} \cdots \sum_{|N_{k-1}| \leqslant t\,p^{k-1}} \left| \int_0^1 \cdots \int_0^1 \left|\sum_{x=1}^{p} e(\theta_{k-1} x^{k-1} + \ldots + \theta_1\,x)\right|^{2t} \times \right.$$

$$\times\, e(-N_{k-1}\theta_k - \dots - N_1\theta_1)\, d\theta_1 \dots d\theta_{k-1}\Big|^2$$

$$< \int_0^1 \dots \int_0^1 \Big| \sum_{x=1}^{p} e(\theta_{k-1} x^{k-1} + \dots + \theta_1 x)\Big|^{4t} d\theta_1 \dots d\theta_{k-1}$$

by the Parseval relation. By (4.1) we thus obtain

$$(4.7)\quad \sum_{|N_1|\leq t\,p} \dots \sum_{|N_{k-1}|\leq t\,p^{k-1}} \psi^2 (N_{k-1} \dots N_1) \ll p^{4t-\frac{1}{2}k(k-1)+\delta'}$$

Next, by (4.4)

$$\sum_{|N_1|\leq t\,p} \dots \sum_{|N_{k-1}|\leq t\,p^{k-1}} \Big| \sum_{y=1}^{p} e(m(A_{k-1}N_{k-1} + \dots + A_1N_1))\Big|^2$$

$$\leq t^{k-2}\, p^{\frac{1}{2}(k-1)(k-2)} \sum_{y_1=1}^{p} \sum_{y_2=1}^{p} \Big| \sum_{|N_{k-1}|\leq t\,p^{k-1}} e(m\,k\,\theta_k(y_1-y_2)N_{k-1})\Big|$$

$$(4.8)\quad \leq t^{k-2}\, p^{\frac{1}{2}(k-1)(k-2)}\; p \sum_{y=0}^{p} \Big| \sum_{|N_{k-1}|\leq t\,p^{k-1}} e(m\,y\,\theta_k N_{k-1})\Big| ,$$

since the number of solutions of $k(y_1 - y_2) = y$ does not exceed P. Combining (4.7), (4.8) we obtain

$$\sum_{y=1}^{p} |S_m(y)|^{2t} \ll \sqrt{p^{4t+1-k+\delta'}}\; p \sum_{y=0}^{p} \Big| \sum_{|N_{k-1}|\leq t\,p^{k-1}} e(m\,y\,\theta_k N_{k-1})\Big|$$

and on using Hölder's inequality we obtain

$$\sum_{y=1}^{p} |S_m(y)| \ll p^{1+(\frac{\delta'+1-k}{4t})}\; p^{1-\frac{1}{4t}} \Big(\sum_{y=0}^{p} \Big| \sum_{|N_{k-1}|\leq t\,p^{k-1}} e(m\,y\,\theta_k N_{k-1})\Big|\Big)^{\frac{1}{4t}} .$$

Thus, by (4.3),

$$|S(m)|^{4t} \ll p^{4t} + p^{\delta'-(k-1)} P^{4t-1} \sum_{y=0}^{P} \Big| \sum_{|N_{k-1}| \leq t\, p^{k-1}} e(m\, y\, \theta_k\, N_{k-1}) \Big|$$

$$\ll p^{4t} + p^{\delta'-(k-1)} P^{4t-1+\varepsilon/2} \Big(1 + \sum_{y=0}^{P} \min\Big(p^{k-1}, \frac{1}{\|m\, y\, \theta_k\|}\Big)\Big)$$

by Satz 266 of [26]. Thus,

$$\sum_{m=1}^{H} |S(m)|^{4t} \ll H\, p^{4t} + p^{\delta'-(k-1)} P^{4t-1+\varepsilon/2} \Big(H + \sum_{m=1}^{H} \sum_{y=0}^{P} \min\Big(p^{k-1}, \frac{1}{\|m\, y\, \theta_k\|}\Big)\Big)$$

$$\ll H\, p^{4t} + p^{\delta'-(k-1)} P^{4t-1+\varepsilon} H^{\varepsilon/2} \Big(H\, p^{k-1} + \sum_{x=1}^{HP} \min\Big(p^{k-1}, \frac{1}{\|x\, \theta_k\|}\Big)\Big)$$

because $x = m\, y$ has $\ll (H\, N)^{\varepsilon/2}$ solutions m, y for any x other than in the special case $y = 0$, which gives the term $H\, p^{k-1}$.

Now Lemma 1.3 gives

$$\sum_{m=1}^{H} |S(m)|^{4t} \ll H\, P^{4t} + p^{\delta'-(k-1)} P^{4t-1+\varepsilon} H^{\varepsilon/2} \Big(H\, p^{k-1} + \Big(\frac{H\, P}{q} + 1\Big) (p^{k-1} + q \log q)\Big)$$

$$\ll (H\, P)^{\varepsilon} \big(H\, p^{4t} + p^{\delta'} P^{4t-1} \big(H + \Big(\frac{H\, P}{q} + 1\Big) (1 + p^{1-k} q \log q)\big)$$

which proves the lemma.

We now prove an analogue to Lemma 1.5, using (4.2) instead of (1.2). We will assume that with every $j \leqslant r$ such that $a_j > 1$, there is associated an integer $\ell_j \geqslant 1$ and we write

$$\delta_j = \tfrac{1}{2} a_j \,(a_j - 1)\, (1 - \frac{1}{a_j - 1})^{\ell_j} + \varepsilon,$$

$$t_j = \left[\frac{1}{8} a_j \,(a_j - 1) + \frac{\ell_j \,(a_j-1)}{2}\right] + 1$$

but $\delta_1 = 0$, $4t_1 = 1$ if $a_1 = 1$. We shall always have $\delta_j < 1$, and when there is no ambiguity we drop the subscripts from ℓ_j, δ_j and t_j. This also holds for Chapters 5 and 6.

Lemma 4.2: Let $\theta_1, \ldots \theta_r$ be real and suppose there are no integral solutions of the inequalities

$$1 \leqslant x \leqslant N, \quad \max_{1\leqslant i\leqslant r} ||x^{a_i} \theta_i|| < N^{-\lambda},$$

where $1\leqslant a_1 < \ldots < a_r = k$ are integers. Then there exists a j, $1 \leqslant j \leqslant r$ such that

either

$$\lambda > \frac{1 - \delta_j}{j\,(4t_j + k - 1 - \delta_j)} - A\varepsilon \tag{4.9}$$

or

there is an integer q $(\geqslant 1)$ satisfying

$$q \ll N^{\delta_j+\lambda(4j\, t_j-\delta_j) + A\varepsilon} \tag{4.10}$$

and

$$||q\, \theta_j|| \ll N^{-a_j + \delta_j + j\lambda(4t_j + a_j - 1 - \delta_j - \frac{1}{j}) + A\varepsilon}. \tag{4.11}$$

Proof: By following the argument of Lemma 1.5 until (1.8), and using the same notation (with $\lambda_1 = \lambda_2 = \ldots = \lambda_r = \lambda$), we obtain for some j, $1 \leqslant j \leqslant r$

$$N \ll \sum_{m_j \neq 0}^{(j)} |T(\underline{m}^j)| \quad . \tag{4.12}$$

If $a_j = 1$, by using the same argument as in Lemma 1.5, we obtain either $\lambda \geqslant 1 - 4\varepsilon$, which implies (4.9) or that (4.10), (4.11) are true. We may thus assume $a_j > 1$. By Hölder's inequality,

$$N^{4t_j} \ll N^{j(\lambda+\varepsilon)(4t_j-1)} \sum_{m_j \neq 0}^{(j)} |T(\underline{m}^j)|^{4t_j}$$

Suppose $Q = N^U$ where

$$U = a_j - \delta_j - j\lambda(4t_j + a_j - 1 - \delta_j - \tfrac{1}{j}) - A\varepsilon. \tag{4.13}$$

Clearly $U > 0$ (for otherwise (4.9) is true), and so we choose a, q, integers, satisfying

$$1 \leqslant q \leqslant Q, \ (a, q) = 1, \ |\theta_j - aq^{-1}| \leqslant q^{-1} Q^{-1}.$$

Then, by Lemma 4.1, for some integer p, $1 \leqslant p \leqslant N$,

$$N^{4t_j} \ll N^{(4j t_j - 1)(\lambda+\varepsilon)} (p^{4t_j} N^{\lambda+2\varepsilon} + \tag{4.14}$$

$$p^{\delta_j+\varepsilon} N^{4t_j-1} (N^{\lambda+\varepsilon} (\frac{N^{1+\lambda+\varepsilon}}{q} + 1) (1 + p^{1-a_j} q \log q))).$$

We choose $p = \left[N^{1-j\lambda-(j+1)\varepsilon}\right]$; then $1 \leqslant p \leqslant N$, unless $\lambda \geqslant \frac{1}{j} - 3\varepsilon$, which we may exclude, since it implies (4.9). Then

$$p^{4t_j} N^{(4j t_j-1)(\lambda+\varepsilon)+\lambda+2\varepsilon} = o(N^{4t_j}),$$

and thus (4.14) implies

$$(4.15) \qquad N^{1-(4j\,t_j-1)(\lambda+\varepsilon)}\,p^{-\delta_j-\varepsilon} << \frac{N^{1+\lambda+\varepsilon}}{q} + N^{\lambda+\varepsilon} + \frac{q \log q}{p^{a_j-1}} + \frac{N^{1+\lambda+\varepsilon}\log q}{p^{a_j-1}}$$

We may assume (4.9) is false. Then, with a suitable choice of A in (4.9) the terms $N^{\lambda+\varepsilon}$, $N^{1+\lambda+\varepsilon} \log q.p^{1-a_j}$ are negligible. Similarly, by a suitable choice of A in (4.13), the term $q \log q.p^{1-a_j}$ is negligible.

Thus (4.15) implies

$$q << N^{4j\,t_j\,\lambda+A\varepsilon}\,p^{\delta_j+\varepsilon}$$

which is (4.10), while (4.13) implies (4.11). This completes the proof of the lemma.

§2. In this section, we look at approximation to monomials, similar to the work of Chapter 2. We first obtain an improvement of Theorem 7.

<u>Theorem 19</u>: <u>Let θ be real. Then for $k \geqslant 3$, and any $N \geqslant 1$,</u>

$$\min_{1 \leqslant x \leqslant N} ||\theta x^k|| < C(k)\, N^{-\sigma_k},$$

<u>where $\sigma_k^{-1} = \frac{\log k}{\log k-1} \cdot 2k^2 (\log k^2 + \log\log k) \sim 4k^2 \log k$.</u>

<u>Proof</u>: We suppose that there are no integer solutions of the following inequalities, for some positive λ,

(4.16) $$1 \leqslant x \leqslant N, \ ||\theta x^k|| \leqslant N^{-\lambda}$$

We will show

(4.17) $$\lambda > \sigma_k \ .$$

We apply Lemma 4.2 with $r = 1$, $a_1 = k$, $\Theta_1 = \theta$ and since there is no ambiguity, $\ell_j = \ell$, $\delta_j = \delta$, $t_j = t$. We choose

$$\ell = \left\lceil \frac{\log \tfrac{1}{2} k (k-1) + \log \log k}{-\log \left(1 - \frac{1}{k-1}\right)} \right\rceil + 1$$

so that $\delta < \frac{1}{\log k}$, and

$$4t \leqslant (k-1)(2\ell + \tfrac{1}{2}k) + 4 \ .$$

$$\leqslant (k-1)^2 (2 \log \tfrac{1}{2} k^2 + 2 \log \log k) + \tfrac{1}{2} k(k+3) + 2.$$

We assume first that (4.9) is false and that (4.10) and (4.11) hold. Then

$$||q^k \theta|| \leqslant q^{k-1}||q \theta|| \ll N^{-k+k\delta+\lambda(k(4t+1-\delta)-2)+A\varepsilon} \ ,$$

and $q \leqslant N$, (since (4.9) is false.) Therefore, by the insolubility of (4.16),

$$- k + k\delta + \lambda(k(4t + 1 - \delta) - 2) + A\varepsilon \geqslant - \lambda - \varepsilon,$$

so that $\lambda \geqslant \dfrac{1 - \delta}{4t + k - 1} - A\varepsilon$.

Even if (4.9) holds instead, we can always conclude that

$$\lambda \geqslant \frac{1 - \delta}{4t + k - 1} - A\varepsilon \ ,$$

i.e. $$\lambda \geq \frac{1 - (\log k)^{-1}}{(k-1)^2(2 \log \tfrac{1}{2} k^2 + 2 \log\log k) + \tfrac{1}{2} k(k+5) + 1}$$

$$\geq \frac{1 - (\log k)^{-1}}{2k^2(\log k^2 + \log\log k)}$$

which proves (4.17) and hence the theorem.

We now use Lemma 4.2 to deduce improvements of the Corollary to Theorem 14, and Theorem 16 for large values of k.

Theorem 20: For $k \geq 3$, θ real, $N \geq 1$

$$\min_{1 \leq x \leq N} \max_{1 \leq j \leq k} \|\theta x^j\| < C(k, \varepsilon)\, N^{-\tau_k}$$

where $$\tau_k^{-1} = \frac{\log k . 2k^3}{\log k - 1} (\log k^2 + \log\log k)$$

Proof: Suppose there are no integral solutions of

(4.18) $$1 \leq x \leq N, \quad \max_{1 \leq j \leq k} \|\theta x^j\| \leq N^{-\lambda},$$

where $\lambda > 0$. We will show

(4.19) $$\lambda \geq \tau_k$$

We apply Lemma 4.2 with $r = k$, $a_j = j$, $\theta_j = \theta$ for $1 \leq j \leq k$, and

$$\ell_j = \left[\frac{\log \tfrac{1}{2} k(j-1) + \log\log k}{-\log\left(1 - \frac{1}{j-1}\right)} \right] + 1$$

Then $\delta_j < \frac{j}{k \log k}$,

and $4t_j \leqslant (j-1)(2\ell + \frac{1}{2} j) + 4$

$$\leqslant (j-1)^2 \, (2 \log \tfrac{1}{2}(j-1)\, k + 2 \log \log k)$$

$$+ \tfrac{1}{2} j\,(j+3) + 2$$

$$\leqslant 2\, j^2 \, (\log j\, k + \log \log k)$$

We first suppose that (4.9) is false and therefore (4.10) and (4.11) hold. Then for $1 \leqslant i \leqslant k$,

$$||q^i \,\theta|| \leqslant q^{k-1} ||q\, \theta||$$

$$\ll N^{-j+k\delta_j+\lambda\,((4t_j-\delta_j)j\,k+j^2-j)}$$

and since (4.9) is false $q \leqslant N$. Thus, by the insolubility of (4.18),

$$- j + k\delta_j + \lambda((4t_j - \delta_j)\, j\, k + j^2 - j) + \varepsilon \geqslant - \lambda$$

i.e. $$\lambda \geqslant \frac{1 - k\, j^{-1}\, \delta_j}{4kt_j - k\delta_j + j - 1} - \varepsilon \, .$$

Even if (4.9) holds instead, we can always include that

$$\lambda > \frac{1 - k\, j^{-1}\, \delta_j}{k(4t_j + k - 1)}$$

$$\geqslant \frac{1 - (\log k)^{-1}}{2k^3\, (\log k^2 + \log \log k)} \quad \text{(since } j \leqslant k\text{)}$$

$$= \tau_k \, ,$$

which proves (4.19) and hence the theorem.

We now prove a result corresponding to Theorem 16.

Theorem 21: For $k \geqslant 3$, $\theta_1, \ldots \theta_r$ real and $N \geqslant 1$,

$$\min_{1 \leqslant x \leqslant N} \max_{1 \leqslant i \leqslant r} ||x^{a_i} \theta_i|| < C(k, r, \varepsilon) N^{-\frac{1}{W_{k,r}} + \varepsilon} \tag{4.20}$$

where the a_i are integers satisfying $1 \leqslant a_1 < a_2 \ldots < a_r = k$, and $W_{k,r} = \frac{2}{\log_e k} (\sigma_k^{-1} + (r + 2)(r - 1) k^2 (\log k^3 + \log \log k))$, where σ_k is defined as in Theorem 19.

Proof: Define $\nu_{k,1} = \sigma_k^{-1}$. Let $r \geqslant 2$ and we assume that for $\nu_{k,r-1} > 0$ we have

$$\min_{1 \leqslant x \leqslant M} \max_{1 \leqslant i \leqslant r-1} ||x^{b_i} \phi_i|| < C(k, r, \varepsilon) M^{-\frac{1}{\nu_{k,r-1}}} \tag{4.21}$$

($\phi_1, \ldots \phi_r$ real, $M \geqslant 1$, $1 \leqslant b_1 < b_2 \ldots < b_{r-1} \leqslant k$ integers).

Clearly, by Theorem 19, we may make this assumption. Now, suppose there is no integral solution of

$$1 \leqslant x \leqslant N, \max_{1 \leqslant i \leqslant r} ||x^{a_i} \theta_i|| \leqslant N^{-\lambda} ,$$

for some $\lambda > 0$. We apply Lemma 4.2 with

$$\ell_j = \left[\frac{\log (\tfrac{1}{2} k\, a_j\, (a_j - 1) \log k)}{- \log (1 - \frac{1}{a_{j-1}})} \right] + 1$$

$$\leqslant \quad (k - 1)(\log \tfrac{1}{2} k^3 + \log \log k) + 1,$$

so that $\delta_j < \frac{1}{k \log k}$,

and $4t_j \leqslant 2(k - 1)^2 \log \tfrac{1}{2} k^3 + 2k^2 \log \log k + \tfrac{1}{2} k(k + 3) + 2$ for $j = 1, \ldots r$.

We first assume that (4.9) is false, and that (4.10) and (4.11) hold. Suppose

$$\mu = 1 - \delta_j - j\lambda(4t_j - \delta_j + 1) - A\varepsilon \tag{4.22}$$

(Where j is as in (4.10), (4.11)); since (4.9) is false, $\mu \geqslant 0$. Thus, by (4.21), there is an integer z, $1 \leqslant z \leqslant N^{\mu}$, satisfying

$$\max_{\substack{1 \leqslant i \leqslant r \\ i \neq j}} ||z^{a_i} q^{a_i} \theta_i|| \ll N^{-\frac{\mu}{k,r-1}} ,$$

Now $||z^{a_j} q^{a_j} \theta_j|| \leqslant z^{a_j} q^{a_j - 1} ||q\, \theta_j|| \ll N^{\nu}$,

$$\text{where } \nu = \mu\, a_j + (a_j - 1)(\delta_j + \lambda(4j\, t_j - \delta_j)) - a_j + \delta_j$$

$$+ j\lambda(4t_j + a_j - 1 - \delta_j - \tfrac{1}{j}) + A\varepsilon \leqslant - \lambda$$

since $\delta_j < \frac{1}{k \log k}$ and $a_j \leqslant k$.

Now $zq \leqslant N$, and so by the 'Heilbronn hypothesis',

$$\frac{\mu}{\nu_{k,r-1}} \leqslant \lambda \ ,$$

i.e.
$$\lambda \geqslant \frac{1 - \delta_j}{(4t_j - \delta_j + 1)\ j + \nu_{k,r-1}} - A\varepsilon \ .$$

Even if (4.9) holds instead we can always conclude that

$$\lambda \geqslant \frac{1 - \delta_j}{(4t_j + k - 1)\ j + \nu_{k,r-1})}$$

and so

$$(4.23) \qquad \min_{1 \leqslant x \leqslant N} \ \max_{1 \leqslant i \leqslant r} ||x^{a_i}\ \theta_i|| < C(k,\ r,\ \varepsilon)\ N^{-\frac{1}{\nu_{k,r}}}$$

where $\nu_{k,r} = (\nu_{k,r-1} + (4t_k + k - 1)\ r)\ (1 + \frac{2}{k \log k})$.

Thus (4.23) is true for $r = 1, \ldots, k$. Now

$$\nu_{k,r} \leqslant \nu_r \quad \text{where} \quad \nu_1 = \sigma_k^{-1}, \text{ and for } r \geqslant 2,$$

$$\nu_r = (\nu_{r-1} + \alpha r)\ (1 + \beta),$$

where $\alpha = 2\ k^2\ (\log k^3 + \log \log k)$, $\beta = \frac{2}{k \log k}$.

Now, by induction,

$$\nu_r = \nu_1\ (1 + \beta)^{r-1} + \sum_{j=2}^{r} j\ \alpha (1 + \beta)^{r+1-j}$$

$$< (1 + \beta)^k \, (\nu_1 + \alpha(\tfrac{1}{2} r \, (r + 1) - 1))$$

$$< \exp \left(\frac{2}{\log k}\right) \, (\sigma_k^{-1} + (r + 2) \, (r - 1) \, k^2 (\log k^3 + \log \log k)).$$

Since $\nu_{k,r} \leqslant \nu_r \leqslant W_{k,r}$ the theorem is proved.

Corollary: $\min_{1 \leqslant x \leqslant N} \; \max_{1 \leqslant i \leqslant k} \; ||x^i \, \theta_i|| < C(k, \varepsilon) \, N^{-\frac{1 - \varepsilon_k}{3 k^4 \log k}}$,

for $N \geqslant 1$, $\theta_1, \ldots \theta_k$ real, where $\varepsilon_k \to 0$ as $k \to \infty$.

CHAPTER 5

In this chapter, we use Lemma 4.1 to obtain an improvement of Theorem 17 for large values of k. As in Theorem 17 we prove the results for additive forms in one variable since the same results hold good for more than one variable. We prove

Theorem 22: Make the hypotheses of Theorem 17 and further suppose $k \geq 6$. Then

$$(5.1) \qquad \min_{1 \leq |\underline{X}| \leq N} \; \max_{1 \leq i \leq r} \; ||f_i(\underline{X})|| < C(\varepsilon, k, r)\, N^{-\frac{1}{g(k,r)} + \varepsilon}$$

where

$$(5.2) \qquad g(k, r) = \begin{cases} k^2 (12 \log 2k + 6 \log \log 3k) + k, & r = 1 \\ \exp(\frac{4}{\log 3k})\,(2^{r-1}(g(k, 1) + 1)), & r \geq 2. \end{cases}$$

We drop s from the notation of Theorem 17, since we are only concerned with the case $s = 1$; thus $\underline{X} = x$.

Proof: We prove the theorem by induction on r. We will use the same notation as in the proof of Theorem 17, except for dropping j, since $s = 1$. By Theorem 19, since $k^2 (12 \log 2k + 6 \log \log 3k) + k \geq \frac{\log k}{\log k - 1} \cdot 2k^2 (\log k^2 + \log \log k)$ for $k \geq 6$, the theorem is true for $r = 1$. So we suppose $r \geq 2$. We write $f_i(x) = \theta_i x^k$ $(i = 1, \ldots r)$.

Suppose that for some positive λ, there are no integral solutions of the inequalities.

(5.3) $$1 \leqslant x \leqslant N, \quad \max_{1 \leqslant i \leqslant r} ||\theta_i x^k|| \leqslant N^{-\lambda}.$$

We will show

(5.4) $$\lambda \geqslant \frac{1}{g(k,r)} - A\varepsilon \quad .$$

By (5.3) and Lemma 3.1, there is a ρ, $0 \leqslant \rho \leqslant \lambda + \varepsilon$ such that

(5.5) $$|S(\underline{U})| >> N^{1-r\rho}$$

for at least $[N^{r\rho-\varepsilon}] + 1$ integer points $\underline{U} = (U_1, \ldots U_r)$ with

(5.6) $$1 \leqslant |\underline{U}| \leqslant N^{\lambda+\varepsilon} \quad .$$

We confine ourselves to those points satisfying the above. We will use Lemma 4.1 with

$$\ell_r = \left[\frac{\log \frac{1}{2} r^2 k(k-1) + \log \log 3k}{- \log (1 - \frac{1}{k-1})} \right] + 1.$$

Then

(5.7) $$\delta_r < \frac{1}{r^2 \log 3k} \quad \text{and} \quad 4t_r \leqslant 2k^2 (\log r^2 k^2 + \log \log 3k).$$

Now, by Lemma 1.1 there are integers $b = b(\underline{U})$, $q = q(\underline{U})$ such that

(5.8) $$(b, q) = 1, \qquad 1 \leqslant q \leqslant N^a$$

and

(5.9) $$|q\,(\underline{U}.T) - b| < N^{-a},$$

where $a = k - \delta_r - r\rho\,(4t_r + k - 1 - \delta_r) - k\varepsilon$.

By Lemma 4.1, and (5.7), for integer p, $1 \leqslant p \leqslant N$,

$$|S(\underline{U})|^{4t_r} \ll p^{\delta_r+\varepsilon} N^{4t_r-1} (1 + (Nq^{-1} + 1)\,(1 + p^{1-k} q \log q))$$

$$+ p^{4t_r+\varepsilon}$$

and if we substitute from (5.5), we obtain

(5.10) $$N^{4t_r-4rt_r\rho} \ll p^{\delta_r+\varepsilon} N^{4t_r-1} (Nq^{-1} + 1)(1 + p^{1-k} q \log q)$$

$$+ p^{4t_r+\varepsilon}$$

Let $p = [N^{1-r\rho-\varepsilon}]$, and since we may suppose that $0 \leqslant r\rho \leqslant 1 - 2\varepsilon$, $1 \leqslant p \leqslant N$. Then $p^{4t_r+\varepsilon}$ is negligible, and (5.10) then implies

(5.11) $$N^{1-\delta_r-r\rho(4t_r-\delta_r)-\varepsilon(1-\delta_r)} \ll Nq^{-1} + 1$$

$$+ N^{2-k+r\rho(k-1)+k\varepsilon} + N^{1-\delta_r-r\rho(4t_r-\delta_r)-\varepsilon}$$

Clearly the last term on the right is negligible. Now, either

$$\lambda \geqslant \frac{1 - \delta_r}{r(4t_r - \delta_r)} - A\varepsilon$$

$$\geqslant \frac{1}{(r+1)(2\,k^2 \log r^2 k^2 + \log\log 3k)} - A\varepsilon$$

$$\geqslant \frac{1}{g(k,r)} - A\varepsilon.$$

or 1 is negligible. We may suppose that the first possibility does not hold, for, if so, we would have proved (5.4); hence 1 is negligible. Similarly the third term on the right in (5.11) is negligible, for otherwise

$$\lambda \geqslant \frac{k - 1 - \delta_r}{r(4t_r - \delta_r - k + 1)} - A\varepsilon$$

$$\geqslant \frac{1}{r(2k^2(\log r^2 k^2 + \log\log 3k))} - A\varepsilon$$

$$\geqslant \frac{1}{g(k, r)} - A\varepsilon ,$$

and (5.4) would follow. Thus, from (5.11), we deduce

(5.12) $$q \ll N^{\delta_r + r\rho(4t_r - \delta_r) + \varepsilon(1-\delta_r)}$$

Now, by Lemma 1.1 there are integers $W, z_1, \ldots z_r$ such that

$$1 \leqslant W \leqslant N^{a-\varepsilon}, \ |W\,\theta_i - z_i| \leqslant N^{-(a-\varepsilon)1/r} \quad \text{for } i = 1, 2, \ldots r.$$

Then

$$|q(U_1 z_1 + U_2 z_2 + \ldots + U_r z_r) - bW| \leqslant q \sum_{i=1}^{r} |U_i|\ |W\,\theta_i - z_i|$$

$$+ \ W|q(\underline{U}.T) - b|$$

$$\ll N^{\delta_r + r\rho(4t_r - \delta_r) + \varepsilon(1-\delta_r) + \lambda + \varepsilon - (a-\varepsilon)1/r} + N^{a-\varepsilon-a}$$

$$\ll N^{-\varepsilon} ,$$

for otherwise

$$\lambda \geq \frac{k - (r+1)\,\delta_r}{r(4t_r\,(r+1) + k - \delta_r\,(r+1))} - A\varepsilon$$

$$\geq \frac{k-1}{r((r+1)\,(2k^2 \log r^2\,k^2 + \log\log 3k) + k)} - A\varepsilon$$

$$\geq \frac{1}{g\,(k,\,r)} - A\varepsilon \;.$$

Thus, as in the argument of Lemma 3.2, $q \mid W$, and therefore there is an r-tuple $\underline{U}$ satisfying (5.5) and

(5.13) $$||q\,(\underline{U}.T)|| \ll N^{-a},$$

and

(5.14) $$1 \leq |\underline{U}| \ll N^{\lambda+2\varepsilon-\rho} \;.$$

We may suppose $U_1 \geq 1$. We now confine ourselves to the above r-tuple $\underline{U}$. By our inductive hypothesis, for the real numbers $\theta_i\,U_1^{k-1}\,q^k$ (i = 2, ... r) and the integer $[N^{(2\lambda+4\varepsilon-\rho)g}]$, (where $g = g\,(k,\,r-1)$), there is an integer n satisfying

(5.15) $$1 \leq n \leq N^{(2\lambda+4\varepsilon-\rho)g},\; ||\theta_i\,U_1^{k-1}\,q^k\,n^k|| \ll N^{-2\lambda-3\varepsilon+\rho},$$

for i = 2, ... r. Suppose $x = n\,q\,U_1$. Then for i = 2, ... r

$$||\theta_i\,x^k|| \leq |U_1|\;||\theta_i\,U_1^{k-1}\,q^k\,n^k|| \ll N^{-\lambda-\varepsilon} \;.$$

Now,

$$||\theta_1 x^k|| \leqslant \sum_{i=2}^{r} |U_i|\; ||\theta_i U_1^{k-1} q^k n^k|| + n^k q^{k-1} U_1^{k-1} \times$$

$$\times \; ||q(U_1 \theta_1 + \ldots + U_r \theta_r)||$$

$$<< N^{-\lambda-\varepsilon} + N^{\sigma_1} \quad \text{, by (5.15),}$$

where $\sigma_1 = k(2\lambda + 4\varepsilon - \rho)\, g + (k - 1)(\delta_r + r\rho(4t_r - \delta_r) + \varepsilon(1 - \delta_r))$

$$+ (k - 1)\,(\lambda + 2\varepsilon - \rho) - k + \delta_r + r\rho(4t_r + k - 1 - \delta_r) + k\varepsilon.$$

Suppose

(5.16) $$\sigma_1 \leqslant - \lambda - \varepsilon \quad .$$

Then $||\theta_i x^k|| < N^{-\lambda}$ $(1 \leqslant i \leqslant r)$.

We also have $x << N^{\sigma_2}$,

where $\sigma_2 = (2\lambda + 4\varepsilon - \rho)\, g + \delta_r + r\rho(4t_r - \delta_r) + \varepsilon(1 - \delta_r)$

$$+ \lambda + 2\varepsilon - \rho.$$

By (5.16), $\sigma_2 < 1 - \varepsilon$. Thus $x \leqslant N$ satisfies

$$||\theta_i x^k|| < N^{-\lambda} \quad (1 \leqslant i \leqslant r),$$

which contradicts (5.3). Hence (5.16) does not hold, and therefore

(5.17) $$\lambda\,(2g + 1) + \rho(4rt_r + r - r\delta_r - g - 1 + (1 - r)\, k^{-1})$$

$$> 1 - \delta_r - A\varepsilon \quad .$$

When $r = 2$, $2(4t_2 + 1) \leqslant 4k^2 (\log 4k^2 + \log\log 3k) + 2$

$$\leqslant g(k, 1),$$

and for $r > 2$, $r(4t_r+1) \leqslant r\,(2k^2 (\log r^2 k^2 + \log\log 3k))$

$$\leqslant g(k, r - 1).$$

Thus, the coefficient of ρ in (5.17) is negative for $r \geqslant 2$. Therefore, (5.17) implies

$$\lambda > \frac{1 - \delta_r}{2g(k, r - 1) + 1} - A\varepsilon \tag{5.18}$$

Now write $F(r) = \{2\, g(k, r - 1) + 1\} (1 - \delta_r)^{-1}$.

$$F(2) = (2\, g(k, 1) + 1) \left(1 - \frac{1}{4 \log 3k}\right)^{-1}$$

$$< \exp\left(\frac{4}{\log 3k}\right)(2g(k, 1) + 1) = g(k, 2).$$

A simple inductive argument shows that, for $r \geqslant 3$,

$$F(r) = (2\, g(k, r - 1) + 1) (1 - \delta_r)^{-1}$$

$$= 2^{r-1} g(k, 1) \prod_{j=2}^{r} (1 - \delta_j)^{-1} + \sum_{j=2}^{r} 2^{r-j} \prod_{i=1}^{j} (1 - \delta_i)^{-1}$$

$$< 2^{r-1} (g(k, 1) + 1) \exp\left(\frac{4}{\log 3k}\right) = g(k, r),$$

and thus, from (5.18) we deduce

$$\lambda > \frac{1}{g(k, r)} - A\varepsilon,$$

which is (5.4). This completes the proof of the theorem.

CHAPTER 6

In this Chapter we look at approximations to polynomials without constant term, and we prove Theorems 23 and 24 which are improvements of Theorems 8 and 9. Note that in the proof of Theorem 24, we do not use Lemma 4.1 but, instead, a similar result of Hua which is slightly more convenient for our purposes.

§1: We first prove a lemma which is applicable to the two theorems of this chapter.

Lemma 6.1: Let $f_1, \ldots f_r$ be real polynomials of degree k, $f_i(x) = \sum_{j=1}^{k} \theta_{ij} x^j$. Suppose there is no integral solution of the inequalities

$$(6.1) \qquad 1 \leq x \leq N, \quad \max_{1 \leq i \leq r} ||f_i(x)|| < N^{-\lambda},$$

for some positive $\lambda < \frac{1}{r}$. Then,

$$(6.2) \qquad \sum{}' |T(\underline{m})| >> N,$$

where $\sum{}'$ denotes a summation over $\underline{m}$ satisfying

$$1 \leq |\underline{m}| \leq N^{\lambda+\varepsilon},$$

and $T(\underline{m}) = \sum_{x=1}^{N} e\left(\sum_{j=1}^{k} \underline{m}.\underline{\theta}^j x^j\right)$

and $\underline{m}.\underline{\theta}^j = \sum_{i=1}^{r} m_i \theta_{ij}$

Proof: By Lemma 1.2, with $\Delta = \frac{1}{2} N^{-\lambda}$ and $a = [2\varepsilon^{-1}] + 1$, we obtain

$$\sum_{x=1}^{N} \prod_{i=1}^{r} \psi(f_i(x)) = 0,$$

and therefore

$$\sum_{m_1=-\infty}^{\infty} \cdots\cdots \sum_{m_r=-\infty}^{\infty} \alpha_{m_1} \cdots \alpha_{m_r} \; T(\underline{m}) = 0.$$

On separating out the term with $\underline{m} = 0$, we obtain

$$(6.3) \qquad 2^{-r} N^{1-r\lambda} + \sum_{\substack{m_1=-\infty \\ \underline{m} \neq 0}}^{\infty} \cdots \sum_{m_r=-\infty}^{\infty} \alpha_{m_1} \cdots \alpha_{m_r} \, T(\underline{m}) = 0$$

Summing $\alpha_{m_1} \cdots \alpha_{m_r} T(\underline{m})$ over those values of $\underline{m}$ for which $|m_1| > N^{\lambda+\varepsilon}$, we obtain, by Lemma 1.2,

$$\sum_{|m_1| > N^{\lambda+\varepsilon}} |\alpha_{m_1} \cdots \alpha_{m_r} T(\underline{m})| \ll N \sum N^{a\lambda} m^{-a-1}$$

$$\ll N^{1-a\varepsilon} \ll 1,$$

and similarly for the other regions $|m_i| > N^{\lambda+\varepsilon}$ for $i = 2, \ldots r$. Thus, from (6.3), we obtain

$$N^{1-r\lambda} \ll \sum{}' \, |\alpha_{m_1} \cdots \alpha_{m_r}| \; |T(\underline{m})| + 1,$$

whence by Lemma 1.2, and since $\lambda < \frac{1}{r}$,

$$N \ll \sum{}' \, |T(\underline{m})| \,,$$

which is (6.2).

We will use the same notation throughout this chapter. We now restrict ourselves to one polynomial (thus we can drop one suffix from the notation of Lemma 6.1).

Theorem 23: Let $F(x)$ be a polynomial of degree $k \geq 2$ with real coefficients and no constant term. Then, for $N \geq 1$, there is an integer x satisfying

$$(6.4) \qquad 1 \leq x \leq N, \ ||F(x)|| < C(k, \varepsilon)\, N^{-\rho_1^{-1}(k)+\varepsilon}$$

where $\rho_1(k) = (3k^2 \log k)(k + 4)$.

Proof: We prove the theorem by induction on k, and we split the proof up into three parts, as in Chapter 3.

Lemma 6.2: Suppose there is no integer x, such that for some positive $\lambda_k < 1$, the following inequalities are satisfied:

$$(6.5) \qquad 1 < x < N, \ ||F(x)|| < N^{-\lambda_k}.$$

Then, either

$$(6.6) \qquad \lambda_k > \frac{1 - \delta_k}{4t_k + k-1 - \delta_k} - A\varepsilon$$

or there is an integer q (≥ 1) satisfying

$$(6.7) \qquad q \ll N^{\delta_k+\lambda_k(4t_k-\delta_k)+\varepsilon(4t_k+2-2\delta_k)}$$

and

$$(6.8) \qquad ||q\,\theta_k|| \ll N^{-(k-\delta_k-\lambda_k(4t_k+k-2-\delta_k)-\varepsilon(4t_k+2k+1-\delta_k))}$$

where, for any positive integer ℓ_k,

$$t_k = \left[\tfrac{1}{8}\, k(k-1) + \ell_k \frac{(k-1)}{2}\right] + 1, \quad \delta_k = \tfrac{1}{2}k(k-1)\left(1 - \frac{1}{k-1}\right)^{\ell_k}.$$

Proof: By (6.5) and Lemma 6.1, we have

$$\sum{}' \, |T(\underline{m})| >> N,$$

where $\underline{m}$ is a one-dimensional vector. Thus, by Hölder's inequality

$$(6.9) \qquad \sum{}' \, |T(\underline{m})|^{4t_k} >> N^{4t_k - (4t_k - 1)(\lambda_k + \varepsilon)}$$

Now, by Lemma 1.1, there are integers b, q satisfying

$$(6.10) \qquad 1 \leqslant q \leqslant N^a, \ (b, q) = 1, \ |q\,\theta_k - b| \leqslant N^{-a},$$

where $a = k - \delta_k - \lambda_k\,(4t_k + k - 2 - \delta_k) - \varepsilon(4t_k + 2k + 1 - \delta_k)$.

Then, by Lemma 4.1,

$$(6.11) \qquad \sum{}' \, |T(\underline{m})|^{4t_k} << N^{2\varepsilon}\,(N^{\lambda_k+\varepsilon}\, p^{4t_k} + p^{\delta_k}\, N^{4t_k - 1}\,(N^{\lambda_k+\varepsilon} + \left(\frac{N^{1+\lambda_k+\varepsilon}}{q} + 1\right)(1 + p^{1-k}\, q \log q))),$$

where p is an integer, $1 \leqslant p \leqslant N$. Thus, since we may suppose that $\lambda_k < 1 - 2\varepsilon$ (for otherwise (6.6) would be trivially true), we may let

$$(6.12) \qquad p = \left[N^{1-\lambda_k-2\varepsilon}\right].$$

On substituting from (6.9) into (6.11), we obtain

$$(6.13)\qquad N^{4t_k-(4t_k-1)(\lambda_k+\varepsilon)} \ll N^{2\varepsilon}(N^{\lambda_k+\varepsilon}p^{4t_k}+p^{\delta_k}N^{4t_k-1}$$

$$(N^{\lambda_k+\varepsilon}+(\frac{N^{1+\lambda_k+\varepsilon}}{q}+1)(1+p^{1-k}q\log q))).$$

By (6.12), $p^{4t_k}N^{\lambda_k+3\varepsilon}$ is negligible, and on substituting from (6.10), (6.12), (6.13) implies

$$(6.14)\qquad N^{1-\lambda_k(4t_k-1)-\varepsilon(4t_k+1)} \ll N^{1+\delta_k+\lambda_k(1-\delta_k)+\varepsilon(1-2\delta_k)}q^{-1}$$

$$+N^{\delta_k-\lambda_k(\delta_k-1)+\varepsilon(1-2\delta_k)}+N^{1-\lambda_k(4t_k-1)-\varepsilon(4t_k+2)}$$

$$+N^{2+\delta_k-k-\lambda_k(\delta_k-k)-2\varepsilon(\delta_k-k)}.$$

Clearly the third term on the right-hand side is negligible. Now either

$$\lambda_k > \frac{1-\delta_k}{4t_k-\delta_k} - A\varepsilon$$

or $N^{\delta_k-\lambda_k(\delta_k-1)+\varepsilon(1-2\delta_k)}$ is negligible. We may suppose the former to be false, for otherwise (6.6) would be true. Similarly, either

$$\lambda_k > \frac{k-1-\delta_k}{4t_k+k-1-\delta_k} - A\varepsilon$$

or $N^{2+\delta_k-k-\lambda_k(\delta_k-k)-2\varepsilon(\delta_k-k)}$ is negligible. Again we may suppose the former to be false, for otherwise (6.6) would be true. Thus, since the second, third and fourth terms on the right of (6.14) are negligible, we deduce

$$q \ll N^{\delta_k+\lambda_k(4t_k-\delta_k)+\varepsilon(4t_k+2-2\delta_k)}$$

which is (6.7) and together with (6.10) completes the proof of the lemma.

Lemma 6.3: There is an integer x satisfying

(6.15) $$1 \leqslant x \leqslant N \quad \text{and} \quad ||F(x)|| < C(k, \varepsilon)\, N^{-\alpha_1^{-1}(k)+\varepsilon}$$

where

(6.16) $$\alpha_1(k) = \{\alpha_1(k - 1) + 4t_k + k\}\,(1 - \delta_k)^{-1}, \quad (k > 2)$$

and $\alpha_1(2) = 5\frac{1}{2}$,

and t_k, δ_k, ℓ_k are as in Lemma 6.2.

Proof: We prove the lemma by induction on k. Let $k = 2$. Then the result holds by Theorem 5. So we suppose that $k > 2$ and that the result has been proved for all $\ell < k$. We make the usual 'Heilbronn hypothesis'; suppose that there is no integral solution of the inequalities

(6.17) $$1 \leqslant x \leqslant N, \qquad ||F(x)|| < N^{-\lambda_k},$$

for some $\lambda_k > 0$. We will show

$$\lambda_k > \alpha_1^{-1}(k) - \varepsilon$$

By Lemma 6.2, either

(6.18) $$\lambda_k > \frac{1 - \delta_k}{4t_k + k - 1 - \delta_k} - A\varepsilon \quad ,$$

or there is an integer q ($\geqslant 1$) satisfying

(6.19) $$q << N^{\delta_k+\lambda_k\ (4t_k-\delta_k)+\ \varepsilon(4t_k+2-2\delta_k)}$$

and

$$||q\ \theta_k|| << N^{-k+\delta_k+\lambda_k(4t_k+k-2-\delta_k)+\varepsilon(4t_k+2k+1-\delta_k)}.$$

We may suppose that (6.18) is false for otherwise (6.16) is true and the lemma is proved. We now suppose $n = q\ t_1$ where $1 \leqslant t_1 \leqslant T$. Then

$$||F(n)|| \leqslant q^{k-1}\ t_1^{\ k}||q\ \theta_k|| + ||\theta_{k-1}\ q^{k-1}\ t_1^{\ k-1} + \ldots + \theta_1 q\ t_1|| .$$

By the inductive hypothesis, there exists t, $1 \leqslant t \leqslant T$, such that

$$||\theta_{k-1}\ q^{k-1}\ t^{k-1} + \ldots + \theta_1\ q\ t|| << T^{-\alpha\ +\varepsilon}$$

where $\alpha^{-1} = \alpha_1^{\ -1}(k-1)$. Thus, for this value of t_1,

(6.20) $$||F(n)|| << N^{\sigma}\ T^{k} + T^{-\alpha^{-1}+\varepsilon} ,$$

where $\sigma = -\ k + k\ \delta_k + \lambda_k\ (k\ (4t_k - \delta_k + 1)\ -2) + A\varepsilon$.

Suppose

(6.21) $$T = N^{\sigma_1} ,$$

where $$\sigma_1 = \frac{-\ \sigma}{k + \alpha^{-1} + \varepsilon} .$$

Then $q\ t << N^{\sigma_2}$,

where $$\sigma_2 = \frac{k - \lambda_k(k - 2) + \alpha^{-1}(\delta_k + \lambda_k\ (4t_k - \delta_k))}{k + \alpha^{-1}} - A\varepsilon .$$

Now $k \geqslant 3$ and since (6.24) is false

$$\delta_k + \lambda_k (4t_k - \delta_k) < 1 - \varepsilon \quad .$$

Hence $\sigma_2 < 1 - A\varepsilon$, whence

(6.22) $$q\, t << N^{1-\varepsilon} \quad .$$

Now, by (6.21),

$$||F(n)|| << T^{-\alpha^{-1} + \varepsilon} ,$$

and therefore, by (6.17), (6.22),

$$\sigma_1 (- \alpha^{-1} + \varepsilon) > - \lambda_k - \varepsilon$$

i.e. $$\lambda_k > \frac{1 - \delta_k}{\alpha_1(k-1) + 4t_k - \delta_k + 1} - A\varepsilon$$

$$> \alpha_1^{-1} (k) - \varepsilon$$

which proves the lemma.

<u>Completion of the proof of Theorem 23</u>: By Theorem 5 the theorem holds for $k = 2$. We may therefore suppose $k \geqslant 3$. Now, by Lemma 6.3, if we show

$$\alpha_1 (k) \leqslant \rho_1(k) \qquad (k \geqslant 3),$$

we will have proved the theorem. In the notation of Lemma 6.2, let

$$\ell_i = \left[\frac{\log \frac{1}{2} i^2 (i-1)}{-\log (1 - \frac{1}{i-1})} \right] + 1 \qquad (i = 3, \ldots k).$$

Then, for $i = 3, \ldots k$,

$$\delta_i < \frac{1}{i}, \text{ and}$$

$$\begin{aligned} 4t_i &\lesssim (i-1)\ (\tfrac{1}{2} i + 2\ell_i) + 4 \\ &\lesssim (i-1)\ (2(i-1) \log \tfrac{1}{2} i^2 (i-1) + \tfrac{1}{2} i + 2) + 4 \\ &\lesssim 6\, i^2 \log i + 4 . \end{aligned}$$

Now, $\alpha_1 (k) = (4t_k + k + \alpha_1 (k-1))\ (1 - \delta_k)^{-1}$,

and a simple calculation shows

$$\begin{aligned} \alpha_1(k) &= \sum_{i=3}^{k} (4t_i + i) \prod_{j=i}^{k} (1 - \delta_j)^{-1} + \alpha_1(2) \prod_{j=3}^{k} (1 - \delta_j)^{-1} \\ &< \sum_{i=3}^{k} (4t_i + i) \prod_{j=i}^{k} (\frac{j}{j-1}) + \frac{11}{2} \prod_{j=3}^{k} (\frac{j}{j-1}) \\ &\lesssim \frac{11\,k}{4} + k \sum_{i=3}^{k} \frac{6\, i^2 \log i + i + 4}{i-1} \\ &< \frac{11\,k}{4} + \frac{7\,k^2}{2} + 6\,k \log k\ (\tfrac{1}{2} k(k-1) + \frac{3k}{2}) \\ &< (3k^2 \log k)\ (k+4) = \rho_1(k) \end{aligned}$$

which proves the theorem.

§2. In this section we prove

Theorem 24: Let $f_1(x), \ldots f_r(x)$ be polynomials with real coefficients and no constant term of degree k. Then, there is, for $N \geqslant 1$, an integer x satisfying

$$(6.24) \qquad 1 \leqslant x \leqslant N, \quad \max_{1 \leqslant i \leqslant r} ||f_i(x)|| < C(\varepsilon, k, r)\, N^{-\rho_2^{-1}(k,r)+\varepsilon}$$

where

$$(6.25) \qquad \rho_2\,(k, r) = (2 + 4\rho_2(k, r-1)) \cdot \frac{r^2}{r^2 - 1} \qquad (r \geqslant 2),$$

and $\rho_2\,(k, 1) = 8\,k^3 \log 4\,k^3$.

Some simple calculations show

$$(6.26) \qquad 4^r\,(2\,k^3 \log 4\,k^3) \leqslant \rho_2\,(k, r) \leqslant 4^{r-1} \cdot e^4\,(8\,k^3 \log 4\,k^3 + 1).$$

Before proving the theorem, we need a lemma.

Lemma 6.4: Let $f(x) = \alpha_k x^k + \ldots + \alpha_1 x$ be a polynomial with real coefficients. Suppose that the number of solutions in integers of, for $N \geqslant 1$,

$$x_1^h + \ldots + x_t^h = y_1^h + \ldots y_t^h \quad (1 \leqslant h \leqslant k-1,\ 1 \leqslant x_j, y_j \leqslant N)$$

is

$$(6.27) \qquad \ll N^{2t-\frac{1}{2}k(k-1) + \delta}$$

for some $\delta > 0$ where t is a positive integer. Suppose also that there exist integers a, q, satisfying

$$(a, q) = 1, \qquad |\alpha_r - a/q| \leqslant \frac{1}{q^2}, \qquad 1 \leqslant q \leqslant N^r$$

for some r, $2 \leqslant r \leqslant k$. Further suppose p is an integer $1 \leqslant p \leqslant N$. Then

(6.28) $$\left|\sum_{x=1}^{N} e\,(f(x))\right|^{2t} \ll N^{2t+k-1+\delta}\, p^{-k}\,(1 + \frac{p}{q} + \frac{pq}{N^r})$$

$$+ p^{2t}.$$

In particular, the estimates (6.27) and hence (6.28) hold if

$$\delta = \tfrac{1}{2} k\,(k - 1)(1 - \frac{1}{k - 1})^{\ell} + \varepsilon, \quad t = \left[\tfrac{1}{4} k\,(k - 1) + \frac{\ell(k-1)}{2}\right] + 1$$

and ℓ is any positive integer.

Proof: This is Lemma 5.10 of Hua [14] where we stop at ℓ. - 9 of p.62.

Proof of the Theorem: We use induction on r, and clearly by Theorem 23, the theorem is true for $r = 1$. We now prove a lemma which does the work, in this case, of Lemma 4.2.

Lemma 6.5: Make the hypotheses of Lemma 6.1. Then, either

(6.29) $$\lambda > \min\left\{\frac{2 - (r + 1)\,\delta_r}{r(r(2t_r + k - 1) + 2t_r + k)} - A\varepsilon,\ \frac{1 - \delta_r}{r(2t_r + k)} - A\varepsilon\right\}$$

or there is a ρ, $0 \leqslant \rho \leqslant \lambda + \varepsilon$ having the following properties: for each j, $2 \leqslant j \leqslant k$ there exists an integer q_j satisfying

(6.30) $$q_j \ll N^{\delta_r + r\rho(2t_r + k-1) + (k-1)\varepsilon}$$

and

$$||q_j\,(\underline{m}.\theta^j)|| \ll N^{-a_j}$$

where $a_j = j - \delta_r - r\rho\ (2t_r + k - 1) - k\varepsilon$

with

(6.31) $$1 \leqslant |\underline{m}| \ll N^{\lambda+(k+1)\varepsilon-\rho}, \quad m_r \geqslant 1,$$

and $t_r = [\tfrac{1}{4}k\ (k-1) + \ell_r\ (k-1)] + 1,$

$$\delta_r = \tfrac{1}{2}k\ (k-1)\ (1 - \frac{1}{k-1})^{\ell_r} + \varepsilon,$$

for anypositive integer ℓ_r.

Proof: By Lemma 6.1,

$$\sum{}' |T(\underline{m})| \gg N,$$

and therefore by the argument of Lemma 3.1, there exists ρ, $0 \leqslant \rho \leqslant \lambda + \varepsilon$, such that

(6.32) $$|T(\underline{m})| \gg N^{1-r\rho}$$

for at least $[N^{r\rho-\varepsilon}] + 1$ distinct r-tuples $\underline{m}$, with

(6.33) $$1 \leqslant |\underline{m}| \leqslant N^{\lambda+\varepsilon}.$$

We now confine ourselves to the r-tuples $\underline{m}$ satisfying (6.32) and (6.33). Suppose $\rho \geqslant k\varepsilon$. The following argument holds for each j, $2 \leqslant j \leqslant k$.

By Lemma 1.1, there are integers b_j, q_j satisfying

(6.34) $$1 \leqslant q_j \leqslant N^{a_j}, \quad (q_j, b_j) = 1, \quad |q_j \cdot \underline{m} \cdot \underline{\theta}^j - b_j| \leqslant N^{-a_j}.$$

Thus, by Lemma 6.4, for integer p, $1 \leqslant p \leqslant N$,

$$|T(\underline{m})|^{2t_r} \ll N^{2t_r+k-1+\delta_r} p^{-k} (1 + pq_j^{-1} + pq_j N^{-j}) + p^{2t_r}.$$

On substituting from (6.32) we obtain

$$N^{2t_r-2t_r r\rho} \ll p^{2t_r} + N^{2t_r+k-1+\delta_r} p^{-k} (1 + pq_j^{-1} + pq_j N^{-j}).$$

Choose $p = \left[N^{1-r\rho-\varepsilon}\right]$. Then, since we have supposed $\lambda < \frac{1}{r}$, $1 \leqslant p \leqslant N$. Hence p^{2t_r} is negligible in the above inequality. We thus deduce

$$N^{1-2t_r r\rho} \ll N^{\delta_r+kr\rho+k\varepsilon} (1 + N^{1-r\rho-\varepsilon} q_j^{-1} + N^{1-j-r\rho-\varepsilon} q_j)$$

Since $1 \leqslant q_j \leqslant N^{a_j}$, the term $N^{1-j-r\rho-\varepsilon+\delta_r+kr\rho+k\varepsilon} . q_j$ is negligible. Thus, either

$$\lambda > \frac{1 - \delta_r}{r(2t_r + k)} - A\varepsilon \tag{6.35}$$

or

$$q_j \ll N^{\delta_r+r\rho(2t_r+k-1) + (k-1)\varepsilon} \tag{6.36}$$

We may assume that (6.35) is false for otherwise (6.29) would be true. Now, by Lemma 1.1, there are integers $z_{1j}, \ldots z_{rj}$, W_j, satisfying, for $i = 1, \ldots r$,

$$1 \leqslant W_j \leqslant N^{a_j-\varepsilon}, \ (z_{1j}, \ldots z_{rj}, W_j) = 1, \ |W \theta_{ij} - z_{ij}| \leqslant N^{-(a_j-\varepsilon)\frac{1}{r}}. \tag{6.37}$$

Thus, by (6.34), (6.36), (6.37)

$$|q_j(m_1 z_{1j} + \dots + m_r z_{rj}) - b_j W_j| \ll \sum_{i=1}^{r} q_j |\underline{m}| \, |W_j \theta_{ij} - z_{ij}|$$

$$+ W_j |q_j \underline{m} \cdot \underline{\theta}^j - b_j|$$

$$\ll N^{-\varepsilon} + N^{\delta_r + r\rho(2t_r + k-1) + (k-1)\varepsilon + \lambda + \varepsilon - (a_j - \varepsilon)\frac{1}{r}}$$

and therefore either

$$\lambda > \frac{j - (r+1)\,\delta_r}{r(r(2t_r + k-1) + 2t_r + k)} - A\varepsilon \tag{6.38}$$

or

$$|q_j(m_1 z_{1j} + \dots + m_r z_{rj}) - b_j W_j| \ll N^{-\varepsilon} \tag{6.39}$$

We may suppose (6.38) false for, otherwise, (6.29) would be true. Hence since $(q_j, b_j) = 1$, $q_j | W_j$. There are thus $\ll N^{\varepsilon}$ choices for q_j and hence $\ll N^{(k-1)\varepsilon}$ choices of the vectors $(q_2, \dots q_k)$. Thus, since $[N^{r\rho-\varepsilon}] + 1$ (> 1) r-tuples $\underline{m}$ satisfy (6.32), (6.33), $\gg N^{r\rho-k\varepsilon}$ of them have the same $\underline{q} = (q_2, \dots q_k)$. We now use the argument of Lemma 3.2 to obtain an r-tuple $\underline{m}$ satisfying

$$m_r \geqslant 1, \quad |m_i| \ll N^{\lambda+(k+1)\varepsilon-\rho}, \quad i = 1, \dots r \tag{6.40}$$

and

$$\| q_j (\underline{m} \cdot \underline{\theta}^j) \| \ll N^{-a_j} .$$

If $\rho \leqslant k\varepsilon$, we take any of the r-tuples $\underline{m}$ satisfying (6.32), (6.33). This, together with (6.35), (6.36) and (6.38) proves the lemma.

<u>Completion of the proof of Theorem 24</u>: We now suppose that (6.1) has no integral solutions. We will show that

(6.41) $$\lambda > \rho_2^{-1}(k, r) - A\varepsilon \ .$$

Since (6.1) has no integral solution, by Lemma 6.5 either (6.29) holds or (6.30) and (6.31) do. We choose

$$\ell_r = \left[\frac{\log r^2 k (k-1)^2}{-\log\left(1 - \frac{1}{k-1}\right)}\right] + 1 \ .$$

Then $\delta_r < \dfrac{1}{2r^2 (k-1)}$, and

$$2t_r \leqslant 2k^2 (\log k^3 + \log 2r^2).$$

We may suppose that (6.29) is false, for otherwise, since by the choice of ℓ_r the first value is the smaller, and

$$\frac{2 - (r+1)\delta_r}{r(r(2t_r + k-1) + 2t_r + k)} > \frac{1}{r(r(2k^2(\log 2r^2k^3) + k-1) + 2k^2(\log 2r^2k^3) + k)}$$

$$\geqslant \frac{1}{4r^2 k^2 \log 2k^3 r^2} \geqslant \frac{1}{4^r 2k^3 \log 4k^3}$$

$$\geqslant \frac{1}{\rho_2(k, r)} - A\varepsilon \ ,$$

(6.41) would be proved.

We write $\rho_2 = \rho_2(k, r-1)$ and suppose

(6.42) $$\sigma = 2\rho_2 \lambda + (k+3)\rho_2 \varepsilon - \rho_2\rho \ .$$

By the inductive hypothesis, there is an integer x satisfying

$$1 \leqslant x \leqslant N^\sigma \ , \quad ||f_i^*(x)|| << N^{-2\lambda-(k+2)\varepsilon+\rho} \quad \text{for } i = 1, \ldots r-1$$

where $f_i^*(n) = \sum_{j=1}^{k} m_r^{-1} \theta_{ij} (m_r Qn)^j$ for $i = 1, \ldots r-1$

and $Q = \prod_{j=1}^{k} q_j$ where q_1 is an integer satisfying

$$(b_1, q_1) = 1,\ 1 \leqslant q_1 \leqslant N^{a_1},\ |q_1\ \underline{m}.\underline{\theta}^1 - b_1| \leqslant N^{-a_1}$$

for some integer b_1, where a_1 will be determined later.

Let $y = m_r Q x$. Then

(6.43) $$||f_i(y)|| \ll |m_r|\ ||f_i^*(x)|| \ll N^{-\lambda-\varepsilon} \quad \text{for } i = 1, \ldots r-1.$$

Also

$$||f_r(y)|| \leqslant \sum_{j=1}^{k} |m_r Qx|^j\ |m_r q_j|^{-1}\ ||q_j\ \underline{m}.\theta^j|| + \sum_{i=1}^{r-1} |m_i|\ ||f_i^*(x)||.$$

The contribution of the second sum on the right is clearly $\ll N^{-\lambda-\varepsilon}$. We consider the first sum. Choose

$$a_1 = \sigma + (k-1)(\delta_r + r\rho(2t_r + k-1) + (k+1)\varepsilon) + \lambda + \varepsilon.$$

Then the contribution of the first term is $\ll N^{-\lambda-\varepsilon}$.

For $j \geqslant 2$, the contribution of the j-th term is

(6.44) $$m_r^{j-1} Q^j x^j q_j^{-1} ||q_j\ \underline{m}.\underline{\theta}^j|| \ll N^{\sigma_j},$$

where $\sigma_j = \lambda(4j\ \rho_2 + 2j-1) + \rho(rj(2k-2)(2t_r+k-1) + 1 - j - 2j\ \rho_2)$
$+ \delta_r\ (j\ (2k-2)) - j + A\varepsilon$.

Suppose that for each j, $2 \leqslant j \leqslant k$

(6.45) $$\sigma_j < -\lambda - \varepsilon\ .$$

Now,

(6.46) $$y = m_r Q x \ll N^{\sigma_1}$$

where $\sigma_1 = \lambda(2 + 4\rho_2) + \rho(2r(k-1)(2t_r + k-1) - 1 - 2\rho_2) + 2\delta_r(k-1) + A\varepsilon$, and so by (6.45) $\sigma_1 < 1-\varepsilon$ whence $y \leqslant N$. This, together with (6.43), (6.45) contradicts the assumption that (6.1) had no integral solutions and therefore (6.45) must be false, i.e. for some j, $2 \leqslant j \leqslant k$,

$$\lambda(4\rho_2 + 2)+\rho(r(2k-2)(2t_r+k-1)+\frac{1}{j} -1-2\rho_2)+2\delta_r(k-1) > 1-A\varepsilon.$$

Now by (6.26) and our inductive hypothesis

$$\begin{aligned}\rho_2 &\geqslant 4^{r-1} (2k^3 \log 4k^3)\\ &\geqslant r(k-1)\ (2k^2 \log 2r^2k^3)\\ &\geqslant r(k-1)\ (2t_r + k-1)\ ,\end{aligned}$$

and therefore the coefficient of ρ above is negative. It therefore follows that

$$\begin{aligned}\lambda &> \frac{1 - \delta_r(2k-2)}{2 + 4\rho_2} - A\varepsilon\\ &> \frac{r^2 - 1}{r^2(2 + 4\rho_2(k, r-1))} - A\varepsilon\end{aligned}$$

which proves (6.41). It only remains to verify (6.26). Now,

$$\rho_2(k, r) = (2 + 4\rho_2(k, r-1)) \frac{r^2}{r^2 - 1}$$, and it is easily seen that

$$\begin{aligned}\rho_2(k, r) &= \sum_{i=2}^{r} 4^{r-i}\ 2.\prod_{j=1}^{r} \frac{j^2}{j^2 - 1} + 4^{r-1} \rho_2(1, k) \prod_{i=2}^{r} \frac{i^2}{i^2 - 1}\\ &< \sum_{i=2}^{r} 4^{r-i}.2.\prod_{j=i}^{r} (1 + \frac{2}{j^2}) + 4^{r-1} \rho_2(1, k) \prod_{i=2}^{r} (1 + \frac{2}{i^2})\end{aligned}$$

$$< 4^{r-1} e^4 (\rho_2 (1, k) + 1),$$

and clearly $\rho_2(r, k) > 4^{r-1} \rho_2(1, k)$

$$= 4^r . 2k^3 \log 4k^3.$$

This completes the proof of Theorem 24.

The proof of Theorem 24 uses the method of Cook [7] where the improvement is obtained by incorporating an idea of Liu as was done in the proof of Theorem 18.

PART 2

CHAPTER 7

In this chapter we investigate a diophantine inequality concerned with quadratic forms. We actually prove a quantitative form of Theorem 12. In addition to the notation specified earlier we use the following notation in this chapter. $Q = Q(x_1, \dots x_n)$ will always represent a real indefinite quadratic form in n variable of rank r, and we will write

$$Q(x_1, \dots x_n) = \sum_{j=1}^{n} \sum_{k=1}^{n} \lambda_{jk} x_j x_k$$

$$= \sum_{j=1}^{n} x_j L_j (x_1, \dots x_n)$$

where $L_j (x_1, \dots x_n) = \sum_{k=1}^{n} \lambda_{jk} x_k$.

Also, for any real $\alpha > 0$, we write

$$S(\alpha) = \sum_{x_1=P_1}^{2P_1} \dots \sum_{x_n=P_n}^{2P_n} e(\alpha Q(x_1, \dots x_n)) \tag{7.1}$$

where $P_1, \dots P_n$ are positive integers.

We will spend the rest of this chapter, and thus the rest of this thesis proving the following:

Theorem 25: For any Q, let $k = \min (r, n-r)$. Suppose $n \geq 21$. Then for every $P \geq 1$, there are integers $x_1, \dots x_n$, not all 0, satisfying

(7.2) $\underline{|\underline{x}| \leqslant P, \quad |Q(x_1, \ldots x_n)| \leqslant C(Q, \varepsilon)\, P^{-f(n,k)}}$

where

(a) for $10k \geqslant 3n$, $n \geqslant 21$

(7.3) $$f(n, k) = \frac{n - 20}{2(n + 5)} - \varepsilon \; ;$$

(b) for $10k < 3n$, $n \geqslant 21$, $k \geqslant 7$

(7.4) $$f(n, k) = \frac{k - 6}{2k + 3} - \varepsilon \; ;$$

(c) for $n \geqslant 21$, $1 \leqslant k \leqslant 4$

(7.5) $$f(n, k) = \frac{n - 20}{2(n + 5)} - \varepsilon \; ;$$

(d) for $n \geqslant 21$, $k = 5$

(7.6) $$f(n, k) = \frac{n^2 - 21n + 4}{2n^2 + 9n - 2} - \varepsilon \; ;$$

(e) for $n \geqslant 22$, $k = 6$

(7.7) $$f(n, k) = \frac{n^3 - 22n^2 + 7n + 4}{2n^3 + 8n^2 - 5n - 2} - \varepsilon \; ;$$

(f) for $n = 21$, $k = 6$

(7.8) $$f(n, k) = \frac{8}{5987} - \varepsilon.$$

We will prove (a), (b) and (f) in §2, (c) and (d) in §3 and (e) in §4.

§1: In this section we state the lemmas which we will use in the proof of the theorem and we introduce a few ideas from the geometry of numbers which we will find useful.

Lemma 7.1: There exists a real function $K(\alpha)$ of the real variable α, satisfying

$$K(-\alpha) = K(\alpha),$$

$$|K(\alpha)| \ll 1, \quad |K(\alpha)| \ll |\alpha|^{-N-1},$$

for all α, where N is any large positive integer, with the following property. Let

$$\psi(\theta) = \int_{-\infty}^{\infty} e(\theta\alpha)\, K(\alpha)\, d\alpha .$$

Then

$$0 \leqslant \psi(\theta) \leqslant 1 \qquad \text{for all real } \theta,$$

$$\psi(\theta) = 0 \qquad \text{for } |\theta| \geqslant 1,$$

$$\psi(\theta) = 1 \qquad \text{for } |\theta| \leqslant 1/3.$$

Proof: This follows from Lemma 1 of [19] on replacing n by N.

Lemma 7.2: There exist real positive constants $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ such that for all real $\underline{x}$

$$Q(x_1, \dots x_n) \leqslant \gamma_1 \sum_{j=1}^{r} x_j^2 - \gamma_2 \sum_{j=r+1}^{n} x_j^2,$$

and

$$Q(x_1, \dots x_n) \geqslant \gamma_3 \sum_{j=1}^{r} x_j^2 - \gamma_4 \sum_{j=r+1}^{n} x_j^2 .$$

Proof: This is Lemma 7 of [20].

Lemma 7.3: Let $\underline{x}^{(1)}, \dots \underline{x}^{(5)}$ be linearly independent points in $2n$ dimensions with $\underline{x}^{(k)} = (x_1^{(k)}, \dots x_{2n}^{(k)})$

for $k = 1, \dots 5$. Suppose that, for any real α, they satisfy

$$|x_j^{(k)}| < \eta_{1k} \qquad (j = 1, \dots n)$$

and $|2\alpha L_j(x_1^{(k)} \dots, x_n^{(k)}) - x_{n+j}^{(k)}| \leq \eta_{2k} \quad (j = 1, \dots n)$

where $\eta_{1k} \geq 1$, $0 < \eta_{2k} < 1$, for $k = 1, 2, \dots, 5$. Then, provided

$$\eta_{21} \; \eta_{22} \; \eta_{23} \dots \eta_{25} \; (\max_k \; \eta_{1k} \; \eta_{2k}^{-1})^4 < C$$

for a suitable constant $C = C(n)$, the points $\underline{X}^{(1)}, \dots \underline{X}^{(5)}$ defined by $\underline{X}^{(k)} = (x_1^{(k)}, \dots x_n^{(k)})$ for $k = 1, \dots 5$ are linearly independent in n dimensions

Proof: This is Lemma 1 of [21].

Lemma 7.4: Let $f(T_1, \dots T_5) = \sum_{u=1}^{5} \sum_{v=1}^{5} f_{uv} T_u T_v$ be a quadratic form in 5 variables with integral coefficients which is either indefinite or represents zero. Let $g_1, \dots g_5$ be any five positive numbers. Then, there exist integers $T_1, \dots T_5$, not all 0, such that

$$f(T_1, \dots T_5) = 0,$$

and $g_u T_u^2 \ll \left(\sum_{u=1}^{5} \sum_{v=1}^{5} f_{uv}^2 \; g_u^{-1} \; g_v^{-1} \right)^2 g_1 \dots g_5 \quad (u = 1, \dots 5).$

Proof: This is a special case of Theorem B of [2].

We now define the successive minima of a convex body. Suppose M is a symmetric convex set in n-dimensional Euclidean space, where by symmetric we mean that if $\underline{x} \in M$, then also $-\underline{x} \in M$. We assume that M is compact and has a non-empty interior. For $\lambda > 0$, let λM be the set consisting of the points $\lambda\underline{x}$ with $\underline{x} \in M$. The first minimum, λ_1, is defined to be the least positive value of λ such that λM contains an integer point $\underline{x} \neq \underline{0}$. More generally, for $1 \leqslant j \leqslant n$, the j-th minimum λ_j is the least positive value of λ such that λM contains j linearly independent integer points. It is clear that $0 < \lambda_1 \leqslant \lambda_2 \cdots \leqslant \lambda_n < \infty$ and that γM contains j linearly independent integer points if and only if $\gamma > \lambda_j$.

We now suppose $M_1, \ldots M_{2n}$ are the $2n$ successive minima of $B(P, \alpha)$, the parallelopiped in $2n$ dimensions defined by

$$|x_j| \leqslant P \qquad (j = 1, \ldots n), \tag{7.9}$$

$$|2\alpha\, L_j(x_1, \ldots x_n) - x_{n+j}| \leqslant P^{-1} \qquad (j = 1, \ldots n) \tag{7.10}$$

where $P \geqslant 1$ and α is some positive real number.

Lemma 7.5: Suppose that in the definition of $S(\alpha)$, for some $P \geqslant 2$, $1 \leqslant P_i \leqslant \frac{1}{2}P$ $(i = 1, \ldots n)$. Then

$$|S(\alpha)|^2 < C(Q)P^n (\log P)^n \quad (M_1 \cdots M_n)^{-1} \tag{7.11}$$

where $M_1, \ldots M_n$ are the first n successive minima of the body $B(P, \alpha)$.

Proof: This follows from the proof of Lemma 5 of [20].

Lemma 7.6: Let $\phi_1, \ldots \phi_5$ be real numbers, not all of the same sign, and all of absolute value 1 at least. Then, for any $\varepsilon > 0$, there exist integers $t_1, \ldots t_5$, not all 0, satisfying

$$|\phi_1 t_1^2 + \ldots + \phi_5 t_5^2| < 1$$

and

$$|\phi_\nu t_\nu^2| < C(\varepsilon)|\phi_1 \ldots \phi_5|^{1+\varepsilon} \qquad (\nu = 1, \ldots 5).$$

Proof: This is the theorem of [3].

Lemma 7.7: Suppose that $m < n$ and let $J_1, \ldots J_m$ be m real linear forms in $x_1, \ldots x_n$; say

$$J_i = \sum_{j=1}^{n} \gamma_{ij} x_j \qquad (1 \leqslant i \leqslant m).$$

Let $\theta, \theta_1, \ldots \theta_m$ be fixed positive numbers satisfying

$$\sum_{i=1}^{m} \theta_i = n\theta; \quad \theta > 1$$

Then, for any $P \geqslant 1$, there exist integers $x_1, \ldots x_n$, not all 0, such that

$$|x_j| \leqslant P^{\theta} \qquad (1 \leqslant j \leqslant n),$$

$$|J_i| \ll P^{\theta-\theta_i} \sum_{j=1}^{n} |\gamma_{ij}| \quad (1 \leqslant i \leqslant m).$$

where '<<' is at most dependent upon $n, \theta_1, \ldots \theta_m$.

Proof: This is the Lemma of [31], and we give a proof here. The result is clearly true for $P < 2$. So we suppose $P \geq 2$. Let $\underline{x}$ run through the integer points in the cube,

$$|x_j| \leq [\tfrac{1}{2} P^{\theta}],$$

the number of such points being $(2[\tfrac{1}{2} P^{\theta}] + 1)^n$. For each $\underline{x}$ we have a m-dimensional point $\underline{J} = (J_1, \dots J_m)$ in the box

$$|J_i| \leq [\tfrac{1}{2} P^{\theta}] \sum_{j=1}^{n} |\gamma_{ij}| .$$

If this box is divided into $\ll (P-1)^{n\theta}$ parts by dividing the range for each co-ordinate of J into $[(P-1)^{\theta_i}]$ equal segments, there will be two distinct points $\underline{x}'$, $\underline{x}''$ such that the corresponding points $\underline{J}'$, $\underline{J}''$ lie in the same range. Suppose $\underline{x} = \underline{x}' - \underline{x}''$. Then $\underline{x}$ satisfies

$$|x_j| \leq 2[\tfrac{1}{2} P^{\theta}] \leq P^{\theta} ,$$

and

$$|J_i| \leq \frac{[\tfrac{1}{2} P^{\theta}] \sum_{j=1}^{n} |\gamma_{ij}|}{[(P-1)^{\theta_i}]}$$

$$\leq \frac{2^{\theta_i - 1} P^{\theta} \sum_{j=1}^{n} |\gamma_{ij}|}{2^{\theta_i}[(P-1)^{\theta_i}]} \ll P^{\theta-\theta_i} \sum_{j=1}^{n} |\gamma_{ij}|$$

We will use Lemmas 7.1 - 7.5 in the proof of parts (a), (b), (f) of Theorem 25 and Lemmas 7.6 7.7 in the proof of parts (c), (d), (e).

§2: Proof of parts (a), (b), (f) of Theorem 25: Until otherwise specified we suppose $n \geq 21$ and $k \geq 6$. Without loss of generality we may suppose $k = r$. We further suppose that for all vectors $\underline{x}$ satisfying $0 < |\underline{x}| \leqslant P$, and some positive λ,

(7.12) $$|Q(x_1, \ldots x_n)| > P^{-\lambda} .$$

We will show that, for all $P > P_o\ (Q, \varepsilon)$, in order for (7.12) to hold we must have that

(7.13) $$\lambda \geqslant f\ (n, k),$$

and clearly this proves the theorem, in these cases.

We suppose $Q = X_1^{\ 2} + \ldots + X_r^{\ 2} - (X_{r+1}^{\ 2} + \ldots + X_n^{\ 2})$ where $X_1, \ldots X_n$ are real linear forms in $x_1, \ldots x_n$ of non-zero determinant. We choose any real numbers $b_1, \ldots b_n$ satisfying

(7.14) $$Q\ (b_1, \ldots b_n) = 0,$$

and without loss of generality we may also suppose

(7.15)
(i) $0 < b_i < \frac{1}{2}$ $\qquad (1 \leqslant i \leqslant n)$
(ii) for $x_1 = b_1, \ldots x_n = b_n$ none of the values of $X_1, \ldots X_n$ is zero.

In the definition (7.1) of $S(\alpha)$ we suppose

(7.16) $$P_i = [2/3\ b_j\ P] .$$

Then $1 \leqslant P_i < P/2 \quad (i = 1, \ldots n)$.

Now, by Lemma 7.1, $\psi(P^{\lambda}\theta) = 0$ for $|\theta| \geqslant P^{-\lambda}$, and thus by (7.12) we obtain $\mathrm{Re} \int_0^{\infty} e(\alpha P^{\lambda} Q(x_1, \ldots x_n))\ K(\alpha)\ d\alpha = 0,$

for all $x_1, \ldots x_n$ satisfying $P_i \leqslant x_i \leqslant 2P_i \ (i = 1, \ldots n)$.

Summing over these ranges we obtain

(7.17) $$\mathrm{Re} \int_0^{\infty} S\ (P^{\lambda}\alpha)\ K(\alpha)\ d\alpha = 0.$$

We divide the range of integration into four parts, viz,

$$\text{(i)}\quad 0 < \alpha < P^{-3/2-\lambda}, \qquad \text{(ii)}\quad P^{-3/2-\lambda} < \alpha < P^{-\frac{1}{2}-\lambda},$$

$$\text{(iii)}\quad P^{-\frac{1}{2}-\lambda} < \alpha < P^{\varepsilon}, \qquad \text{(iv)}\quad \alpha > P^{\varepsilon},$$

and find values of the integral over each part of the range. From these values we will obtain a contradiction of (7.17) under the condition that (7.13) is not satisfied. This will imply that (7.13) is true and hence will prove the theorem for the cases (a), (b), (f).

Part 1: $0 < \alpha < P^{-3/2-\lambda}$

Define

$$I(\alpha) = \int_{P_1-\frac{1}{2}}^{2P_1+\frac{1}{2}} \cdots \int_{P_n-\frac{1}{2}}^{2P_n+\frac{1}{2}} e(\alpha\, Q(x_1, \ldots x_n))\, d\,x_1 \ldots d\,x_n.$$

Lemma 7.8: For $0 < \alpha < P^{-1-\lambda}$

$$|S(P^{\lambda}\alpha) - I(P^{\lambda}\alpha)| \ll P^{n+2+2\lambda}\alpha^2 + P^{n+\lambda}\alpha.$$

Proof: This follows from Lemma 7 of [19].

Lemma 7.9: For $\alpha > P^{-2-\lambda}$, we have

$$|I(P^{\lambda}\alpha)| \ll P^{-n\lambda/2}\,\alpha^{-n/2}\,(\log 2P^{2+\lambda}\alpha)^{n/2}.$$

Proof: This follows from Lemma 8 of [19].

Lemma 7.10:

$$\mathrm{Re}\int_0^{\infty} I(P^{\lambda}\alpha)\, K(\alpha)\, d\alpha \gg P^{n-2-\lambda}.$$

Proof: By Lemma 7.1 and the definition of $I(\alpha)$, the left hand side is

$$J = \int_{P_1-\frac{1}{2}}^{2P_1+\frac{1}{2}} \cdots \int_{P_n-\frac{1}{2}}^{2P_n+\frac{1}{2}} \psi(P^\lambda\, Q(\underline{x}))\, d\,\underline{x}\, .$$

Again by Lemma 7.1, the integrand is non-negative and is 1 if $|Q(\underline{x})| \leq 1/3\, P^{-\lambda}$. Using the representation of Q, $Q = X_1^2 + \ldots + X_r^2 - X_{r+1}^2 - \ldots - X_n^2$, we transform the variable from $x_1, \ldots x_n$ to $X_1, \ldots X_n$, obtaining

$$J \gg \int \cdots \int_{\Pi} \psi(P^\lambda(X_1^2 + \ldots + X_r^2 - X_{r+1}^2 - \ldots - X_n^2)$$

$$d\, X_1 \ldots d\, X_n$$

where Π is a parallelopiped of fixed shape, the lengths of whose edges are all $\gg P$. The centre of Π is at the point in X-space corresponding to

$$x_1 = 3/2\, P_1, \ldots x_n = 3/2\, P_n\ .$$

By (7.16), the centre is at a bounded distance from the point in X-space corresponding to the point $x_1 = b_1\, P_1, \ldots$ $x_n = b_n\, P_n$, a point at which $Q(x_1, \ldots x_n) = 0$. We denote this point in X-space by $X_1 = d_1\, P, \ldots X_n = d_n\, P$ where the $d_1, \ldots d_n$ are constants, none of which is zero by our original construction, and

(7.18) $$d_1^2 + \ldots + d_r^2 - (d_{r+1}^2 + \ldots + d_n^2) = 0.$$

Clearly this point is in Π for large enough values of P and thus Π contains the cube

$$(d_j - b) < X_j < (d_j + b) \qquad (j = 1, \dots n)$$

for some small positive constant b. Thus

$$J \gg \int_{(d_1-b)P}^{(d_1+b)P} \dots \int_{(d_n-b)P}^{(d_n+b)P} \psi(P^{\lambda}(X_1^2 + \dots + X_r^2 - X_{r+1}^2 - \dots - X_n^2)$$

$$d X_1 \dots d X_n.$$

By the change of variable $X_j = P Y_j^{\frac{1}{2}}$, $(j = 1, \dots n)$, we obtain

$$J \gg P^n \int_{e_1}^{f_1} \dots \int_{e_n}^{f_n} \psi(P^{2+\lambda}(Y_1 + \dots + Y_r - Y_{r+1} - \dots - Y_n)$$

$$(Y_1 \dots Y_n)^{-\frac{1}{2}} d Y_1 \dots d Y_n$$

where $f_j = (|d_j| + b)^2$, $e_j = (|d_j| - b)^2$, $(j = 1, \dots n)$, so that

$f_j > d_j^2 > e_j > 0$, $(j = 1, \dots n)$.

In view of (7.18), if h is a suitable small positive constant, $(h \ll b)$, the inequalities

$$|Y_2 - d_2^2| < h, \dots |Y_n - d_n^2| < h,$$

$$|Y_1 + \dots + Y_r - Y_{r+1} - \dots - Y_n| < 1/3 \, P^{-2-\lambda},$$

imply $|Y_1 - d_1^2| < n\,h$ and so define a region within the range of integration. The volume of this region is $\gg P^{-2-\lambda}$ and the integrand is $\gg 1$ there. Thus

$$J \gg P^{n-2-\lambda}$$

which proves the lemma.

Lemma 7.11: $\mathrm{Re} \int_0^{P^{-3/2-\lambda}} S(P^\lambda\alpha)\, K(\alpha)\, d\alpha \gg P^{n-2-\lambda}$

Proof: By Lemmas 7.8 and 7.1,

$$\int_0^{P^{-3/2-\lambda}} |S(P^\lambda\alpha) - I(P^\lambda\alpha)|\, |K(\alpha)|\, d\alpha \ll \int_0^{P^{-3/2-\lambda}} (P^{n+2+2\lambda}\alpha^2 + P^{n+\lambda}\alpha)\, d\alpha$$

$$\ll P^{n-5/2-\lambda} .$$

By Lemma 7.9 and 7.1.

$$\int_{P^{-3/2-\lambda}}^{\infty} |I(P^\lambda\alpha)\, K(\alpha)|\, d\alpha \ll \int_{P^{-3/2-\lambda}}^{\infty} \alpha^{-n/2} P^{-n\lambda/2} (\log 2P^{2+\lambda}\alpha)^{n/2}\, d\alpha$$

$$\ll P^{-n\lambda/2} \int_{P^{-3/2-\lambda}}^{\infty} (\alpha^{-n/2}(\log 2P^{2+\lambda})^{n/2} + \alpha^{-n/2}(\log \alpha)^{n/2})\, d\alpha$$

$$\ll P^{3/4(n-2)-\lambda+\varepsilon} + P^{-n\lambda/2} \int_{P^{-3/2-\lambda}}^{1} \alpha^{-n/2} (\log \alpha)^{n/2}\, d\alpha$$

$$+ P^{-n\lambda/2} \int_1^{\infty} \alpha^{-n/2} (\log \alpha)^{n/2}\, d\alpha$$

$$\ll P^{3/4(n-2)-\lambda+\varepsilon} .$$

Since $n > 2$, the result now follows by Lemma 7.10.

We have thus found a lower bound for the integral over this part of the range. We will now calculate upper bounds for the integral over the second and fourth parts of the range. Clearly this will give us some information about the value of the integral in the third part of the range.

Part 2: $P^{-3/2-\lambda} < \alpha < P^{-\frac{1}{2}-\lambda}$.

Lemma 7.12: For $P^{-2-\lambda} < \alpha < P^{-\lambda}$, we have

$$|S(P^{\lambda}\alpha)| \ll \alpha^{-n/2} P^{-n\lambda/2} (\log P)^{n/2} + P^{n/2} (\log P)^{n/2} + \alpha^{n/2} P^{\frac{n+n\lambda}{2}} .$$

Proof: This follows from Lemma 11 of [19].

Lemma 7.13: $$\int_{P^{-3/2-\lambda}}^{P^{-1/2-\lambda}} |S(P^{\lambda}\alpha)\ K(\alpha)|\ d\alpha \ll P^{n-2-1/4-\lambda} .$$

Proof: By Lemma 7.12 and Lemma 7.1

$$\int_{P^{-3/2-\lambda}}^{P^{-1/2-\lambda}} |S(P^{\lambda}\alpha)\ K(\alpha)|\, d\alpha \ll \int_{P^{-3/2-\lambda}}^{P^{-1/2-\lambda}} (\alpha^{-n/2} P^{-n\lambda/2+\varepsilon})\ d\alpha$$

$$+ \int_{P^{-3/2-\lambda}}^{P^{-1/2-\lambda}} P^{n/2+\varepsilon}\ d\alpha + \int_{P^{-3/2-\lambda}}^{P^{-1/2-\lambda}} P^{\frac{n+n\lambda}{2}} \alpha^{n/2}\ d\alpha$$

$$\ll P^{3n/4-3/2-\lambda+\varepsilon} + P^{n/2-1/2-\lambda+\varepsilon} + P^{3n/4-1/2-\lambda}$$

$$\ll P^{n-2-1/4-\lambda} \qquad \text{since } n \geqslant 7.$$

We have now found an upper bound for the value of the integral in the second part of the range. By straightforward methods we now obtain the following lemma which gives us our result for the fourth range.

Lemma 7.14: $$\int_{P^{\varepsilon}}^{\infty} |S(P^{\lambda}\alpha)\ K(\alpha)|\, d\alpha \ll P^{n-3} .$$

Proof: In Lemma 7.1 suppose $N = [3\varepsilon^{-1}] + 1$. Then

$$\int_{P^{\varepsilon}}^{\infty} |S(P^{\lambda}\alpha)\ K(\alpha)|\, d\alpha << P^{n} \int_{P^{\varepsilon}}^{\infty} \alpha^{-N-1}\, d\alpha << P^{n-N\varepsilon} << P^{n-3}.$$

It now follows from (7.17), Lemmas 7.11, 7.13 and 7.14 that either

(7.19) $$\int_{P^{-1/2-\lambda}}^{P^{\varepsilon}} |S(P^{\lambda}\alpha)\ K(P^{\lambda}\alpha)|\, d\alpha >> P^{n-2-\lambda}$$

or $\lambda \geq 1-\varepsilon$.

We may suppose the second assertion to be false for otherwise (7.13) is true. So we assume (7.19) to be true and investigate the third part of the range.

Part 3: $P^{-1/2-\lambda} < \alpha < P^{\varepsilon}$. The contribution made to the integral on the left hand side of (7.19) by those α for which $|S(P^{\lambda}\alpha)| \leq P^{n-2-\lambda-\varepsilon}$ is, by Lemma 7.1 with N as in Lemma 7.14,

$$P^{n-2-\lambda-\varepsilon} \int_{P^{-1/2-\lambda}}^{P^{\varepsilon}} \min(1, \alpha^{-N-1})\, d\alpha << P^{n-2-\lambda-\varepsilon}$$

Hence, if $\mathcal{J}$ is the set of those α in

(7.20) $$P^{-1/2-\lambda} < \alpha < P^{\varepsilon},$$

for which

(7.21) $$|S(P^{\lambda}\alpha)| > P^{n-2-\lambda-\varepsilon}$$

we have

$$(7.22) \qquad \int_{\mathfrak{g}} |S(P^{\lambda}\alpha)\ K(\alpha)|\, d\,\alpha >> P^{n-2-\lambda},$$

by (7.19).

We now suppose α belongs to $\mathfrak{g}$. Suppose $M_1, \ldots M_{2n}$ are the $2n$ successive minima of $B(P, P^{\lambda}\alpha)$. Then, by Lemma 7.5,

$$|S(P^{\lambda}\alpha)|^2 << P^n (\log P)^n\ (M_1 \ldots M_n)^{-1},$$

and thus, by (7.21),

$$(7.23) \qquad M_1 \ldots M_n << P^{-n+4+2\lambda+A\varepsilon}.$$

We also note that, by the proof of Lemma 3 of [20],

$$M_n << 1 << M_{n+1}.$$

We now suppose that $\underline{x}^{(1)}, \ldots \underline{x}^{(n)}$ are the integer points at which the first n successive minima of the body $B(P, P^{\lambda}\alpha)$ are attained. These points are linearly independent in 2n-dimensional space and satisfy

$$(7.24) \qquad |x_j^{(k)}| \leqslant M_k\, P \qquad (j = 1, \ldots n),$$

$$(7.25) \qquad |2P^{\lambda}\alpha\, L_j(x_1^{(k)}, \ldots x_n^{(k)}) - x_{n+j}^{(k)}| \leqslant M_k\, P^{-1} \quad (j=1, \ldots n),$$

for $k = 1, \ldots n$. Let $\underline{X}^{(1)}, \ldots \underline{X}^{(5)}$ be the points in n dimensions derived from $\underline{x}^{(1)}, \ldots \underline{x}^{(5)}$ as in Lemma 7.3.

<u>Lemma 7.15</u>: <u>The integer points $\underline{X}^{(1)}, \ldots \underline{X}^{(5)}$ corresponding to any α in $\mathfrak{g}$ are such that either the five numbers $Q(\underline{X}^{(1)})$, $Q(\underline{X}^{(2)}), \ldots Q(\underline{X}^{(5)})$ are all of the same sign or else (7.13) holds.</u>

Proof: By (7.24), (7.25), the points $\underline{X}^{(1)}, \ldots \underline{X}^{(5)}$ satisfy

(7.26) $$|x_j^{(k)}| \leqslant M_k P \qquad (j = 1, \ldots n),$$

(7.27) $$||2 P^{\lambda}\alpha\, L_j\, (x_1^{(k)}, \ldots x_n^{(k)})|| \leqslant M_k P^{-1} \quad (j = 1, \ldots n),$$

for $k = 1, \ldots 5$. By Lemma 7.3 the points $\underline{X}^{(1)}, \ldots \underline{X}^{(5)}$ are linearly independent in n dimensions provided that

$$(M_1 P^{-1}) \ldots (M_5 P^{-1})\, (P^2)^4 < C,$$

i.e.

(7.28) $$M_1 \ldots M_5 < C\, P^{-3},$$

for some constant C depending at most upon n. We suppose (7.28) to be true. Consider the quadratic form in 5 variables given by

$$\Phi(T_1, \ldots T_5) = 2\, P^{\lambda}\alpha\, Q(T_1 \underline{X}^{(1)} + \ldots + T_5 \underline{X}^{(5)}).$$

By (7.26) and the supposition that (7.28) is true, for all integers $T_1, \ldots T_5$, not all 0, and satisfying

(7.29) $$|M_k T_k| \leqslant 1/5 \qquad (k = 1, \ldots 5),$$

we have

(7.30) $$|\Phi(T_1, \ldots T_5)| \geqslant 2P^{\lambda}\alpha . P^{-\lambda} = 2\alpha.$$

Writing $\Phi(T_1, \ldots T_5) = \sum_{u=1}^{5} \sum_{v=1}^{5} \phi_{uv}\, T_u\, T_v$, we have

$$\phi_{uv} = \sum_{j=1}^{n} x_j^{(u)}\, 2P^{\lambda}\alpha\, L_j\, (\underline{X}^{(v)}), \text{ and thus}$$

$$\phi_{uv} << \alpha\, M_u\, M_v\, P^{2+\lambda}, \quad ||\phi_{uv}|| << M_u\, M_v .$$

We now write $\phi_{uv} = f_{uv} + \psi_{uv}$ where f_{uv} is the nearest integer to ϕ_{uv}. Then

$$|f_{uv}| << \alpha\, M_u\, M_v\, P^{2+\lambda}, \quad |\psi_{uv}| << M_u\, M_v.$$

Suppose the numbers $Q(\underline{X}^{(k)})$, $(k = 1, \ldots 5)$, are not all of the same sign. If $Q(\underline{X}^{(k)}) > 0$, then $\phi_{kk} = 2\, P^{\lambda}\alpha\, Q(\underline{X}^{(k)}) > 0$, whence $f_{kk} \geqslant 0$. Similarly $Q(\underline{X}^{(\ell)}) < 0 \Longrightarrow f_{\ell\ell} \leqslant 0$. The form $f(T_1, \ldots T_5) = \sum_{u=1}^{5} \sum_{v=1}^{5} f_{uv}\, T_u\, T_v$ is thus either indefinite or represents zero. We now apply Lemma 7.4 with $g_u = M_u^{\,2}$, $(u = 1, \ldots 5)$. Then there exist integers $T_1, \ldots T_5$, not all 0, satisfying $f(T_1, \ldots T_5) = 0$, and, for $u = 1, \ldots 5$,

$$M_u^{\,2}\, T_u^{\,2} << \Big(\sum_{u=1}^{5} \sum_{v=1}^{5} f_{uv}^{\,2}\, M_u^{-2}\, M_v^{-2}\Big)^2\, M_1^{\,2} \ldots M_5^{\,2}$$

$$<< \alpha^4\, P^{8+4\lambda}\, M_1^{\,2} \ldots M_5^{\,2},$$

and hence $M_u\, T_u << \alpha^2\, P^{4+2\lambda}\, M_1 \ldots M_5$.

Now, on using (7.23), and since $(M_1 \ldots M_n)^5 >> (M_1 \ldots M_5)^n$,

$$\alpha^2\, P^{4+2\lambda}\, M_1 \ldots M_5 << P^{4+2\lambda-5+20/n\, +\, \frac{10\lambda}{n}\, +\, A\varepsilon}$$
$$= P^{-1+20/n+\lambda(\frac{10}{n}+2)+A\varepsilon}$$

We can now deduce that either

(7.31) $$\lambda > \frac{n-20}{2(n+5)} - A\varepsilon$$

or

(7.32) $$M_u\, T_u \leqslant \frac{1}{5} \qquad (u = 1, \ldots 5).$$

We may suppose (7.31) to be false for otherwise we would have proved (7.13). Thus (7.29) is satisfied. Therefore, by (7.30) and since $f(T_1, \ldots T_5) = 0$,

$$2\alpha \leqslant |\Phi(T_1, \dots T_5)| = |\sum_{u=1}^{5} \sum_{v=1}^{5} \psi_{uv} \; T_u T_v|$$

$$\ll \alpha^4 P^{8+4\lambda} M_1^{\;2} \dots M_5^{\;2}.$$

Hence $\quad M_1 \dots M_5 \gg \alpha^{-3/2} \; P^{-4-2\lambda} \gg P^{-4-2\lambda-3\varepsilon/2}$.

This has been proved on the assumption that (7.28) holds but it is clearly also true if (7.28) is false. Thus, by (7.23)

$$(P^{-4-2\lambda-A\varepsilon})^n \ll (M_1 \dots M_5)^n \ll (M_1 \dots M_n)^5 \ll P^{-5n+20+10\lambda+A\varepsilon}$$

and hence $\lambda > \dfrac{n - 20}{2(n+5)} - A\varepsilon$.

If this is true we have proved (7.13) and hence the lemma. If this does not hold we have a contradiction of the supposition that the numbers $Q(\underline{x}^{(k)})$ $(k = 1, \dots 5)$ are neither all positive or negative and this proves the lemma.

We may now suppose that the first alternative of Lemma 7.15 holds, for otherwise we would have proved the theorem. Thus the set of values of α in $\mathcal{J}$ fall into 2 parts $\mathcal{J}+$, $\mathcal{J}-$, say, depending on whether the values of $Q(\underline{x}^{(1)}), \dots Q(\underline{x}^{(5)})$ are positive or negative. We consider $\mathcal{J}-$ only since the following argument also holds for $\mathcal{J}+$ on interchanging the roles of r and $n - r$. We introduce a parameter U satisfying

(7.33) $$1 \leqslant U \leqslant P .$$

The inequalities

(7.34) $$|x_j| \leqslant P \qquad (1 \leqslant j \leqslant r) ,$$

(7.35) $$|x_j| \leqslant U \qquad (r < j \leqslant n) ,$$

(7.36) $$|2P^{\lambda}\alpha \, L_j \, (x_1, \dots x_n) - x_{n+j}| \leqslant P^{-1} \qquad (1 \leqslant j \leqslant n),$$

define a parallelopiped $B(U, P, P^\lambda\alpha)$ in 2n dimensions which reduces to $B(P, P^\lambda\alpha)$ when $U = P$. Let $M_1(U), \ldots M_{2n}(U)$ denote the 2n successive minima of $B(U, P, P^\lambda\alpha)$. If $U_1 > U_2$ then $B(U_1, P, P^\lambda\alpha)$ contains $B(U_2, P, P^\lambda\alpha)$ and is contained in $(U_1/U_2)\, B(U_2, P, P^\lambda\alpha)$. Hence

$$M_k(U_1) < M_k(U_2) \leqslant (U_1/U_2)\, M_k(U_1) \qquad \text{for } k = 1, \ldots 2n.$$

In particular each $M_k(U)$ is a continuous function of U which increases as U decreases.

For each U there exist 2n minimal points $\underline{x}^{(1)}(U)$, $\ldots$ $\underline{x}^{(2n)}(U)$ satisfying

$$|x_j^{(k)}| \leqslant M_k(U)\, P \qquad (1 \leqslant j \leqslant r) \tag{7.37}$$

$$|x_j^{(k)}| \leqslant M_k(U)\, U \qquad (r < j \leqslant n) \tag{7.38}$$

$$|2P^\lambda\alpha\, L_j\,(x_1^{(k)}, \ldots x_n^{(k)}) - x_{n+j}^{(k)}| \leqslant M_k(U)P^{-1} \quad (1 \leqslant j \leqslant n) \tag{7.39}$$

for $k = 1, \ldots 2n$.

We now consider the effect of diminishing U from the initial value $U = P$. Only a finite number of integer points can qualify as minimal points for any U. Hence the interval $P \geqslant U \geqslant 1$ splits up into a finite number of intervals

$$P = U_o > U_1 > \ldots > U_t = 1,$$

such that the same minimal points $\underline{x}^{(1)}(U), \ldots \underline{x}^{(2n)}(U)$, can be chosen throughout each interval $U_p > U > U_{p+1}$. These points can also be chosen when $U = U_p$ or U_{p+1}. Initially, when $U = P$, by the note following (7.23) we have $M_n \ll 1 \ll M_{n+1}$. Since $M_k(U)$ increases as U decreases, it is plain that for some time at least in the process there exists some $s = s(U)$ with $1 \leqslant s \leqslant n$ such that

(7.40) $$M_s(U) \ll 1 \ll M_{s+1}(U)$$

We also note that if any integer point $\underline{X}$ satisfies $Q(\underline{X}) < 0$, then, by Lemma 7.2, it satisfies

(7.41) $$\max_{j \leqslant r} |x_j| \ll \max_{j > r} |x_j| \quad .$$

Lemma 7.16: For any α in $\mathcal{J}-$, and any U, we have

(7.42) $$|S(P^{\lambda}\alpha)|^2 \ll P^{2n-r} U^{-(n-r)} (\log P)^n \{M_1(U) \ldots M_s(U)\}^{-1}$$

where s is defined as in (7.40).

Proof: This follows from Lemma 4 of [21].

Lemma 7.17: For any α in $\mathcal{J}-$, and any U satisfying

(7.43) $$U^{n-r} > P^{8-r+2\lambda+A\varepsilon}$$

there exists $s \geqslant 5$ satisfying (7.40) and we have

(7.44) $$\{M_1(U) \ldots M_5(U)\} \ll P^{5(4-r+2\lambda)} U^{-5(n-r)} (\log P)^{5n} \quad .$$

Proof: This follows from the proof of Lemma 5 of [21].

Lemma 7.18: If α is in $\mathcal{J}-$, and U satisfies both (7.43) and

(7.45) $$U^r \leqslant P^{r-4-2\lambda-A\varepsilon} \quad ,$$

then the 5 numbers $Q(\underline{X}^{(1)}(U)), \ldots Q(\underline{X}^{(5)}(U))$ cannot all be negative where, for k = 1, ... 5, $\underline{X}^{(k)}(U)$ is derived from the integer point $\underline{x}^{(k)}(U)$ defined by (7.37), (7.38) and (7.39).

Proof: This follows from the proof of Lemma 6 of [21].

Lemma 7.19: For each α in $\mathcal{J}$ -, either (7.13) holds, or there exists a value of U satisfying

(7.46) $$U^r \geq P^{r-4-2\lambda-A\varepsilon},$$

with the following properties. First U satisfies (7.43) so that Lemma 7.17 applies. Secondly there are two alternative choices of the first 5 minimal points of $B(U, P, P^{\lambda}\alpha)$; for one choice, say $\underline{x}^{(1)}(U), \ldots \underline{x}^{(5)}(U)$,

(7.47) $$Q(\underline{X}^{(k)}(U)) < 0, \qquad k = 1, \ldots 5.$$

For the other choice $\underline{y}^{(1)}(U), \ldots \underline{y}^{(5)}(U)$, say, we have

(7.48) $$Q(\underline{Y}^{(k)}(U)) > 0 \quad \text{for some } k,\ 1 \leq k \leq 5.$$

Proof: We first show that (7.46) implies (7.43). This will be so if $(n - r)(r - 4 - 2\lambda - A\varepsilon) > r(8 - r + 2\lambda + A\varepsilon)$, i.e.

$$\text{if } (n - r)(r - 4) - r(8 - r) > 2n\lambda + A\varepsilon .$$

Now either (7.13) holds or $2n\lambda < n-20$ and so it suffices to show $rn - 4n - 4r > n-20 + A\varepsilon$

i.e. $n(r - 5) > 4(r - 5) + A\varepsilon$,

which is true for $n > 4$ since $r \geq 6$. This proves the first assertion about U. The second follows from the proof of Lemma 7 of [21].

Lemma 7.20: Either (7.13) holds, or, for the value of U determined by Lemma 7.19 we have

$$M_1(U) \ldots M_5(U) \gg P^{-2\lambda-\frac{1}{2}-A\varepsilon}\, U^{-7/2}\,. \tag{7.49}$$

Proof: We use the same notation as in Lemma 7.19 and we write $M_k(U) = M_k$, $\underline{x}^{(k)}(U) = \underline{x}^{(k)}$ $(k = 1, \ldots 5)$. The points $\underline{x}^{(1)}, \ldots \underline{x}^{(5)}$ satisfy, by (7.37), (7.41),

$$|x_j^{(k)}| \ll M_k U \qquad (j = 1, \ldots n) \tag{7.50}$$

$$\|2P^{\lambda}\alpha\, L_j(x_1^{(k)}, \ldots x_n^{(k)})\| \ll M_k P^{-1} \quad (j = 1, \ldots n) \tag{7.51}$$

We suppose that

$$M_1 \ldots M_5 < C\, P\, U^{-4}, \tag{7.52}$$

for some constant C depending at most upon n.

Then the five points are linearly independent. Let

$$\Phi\,(T_1, \ldots T_5) = 2P^{\lambda}\alpha\, Q(T_1\, \underline{x}^{(1)} + \ldots + T_5\, \underline{x}^{(5)}).$$

We now consider two cases.

Case 1: Suppose Φ is indefinite; by (7.50), (7.52), it satisfies

(7.53) $$|\Phi(T_1, \ldots T_5)| \geqslant 2\alpha$$

for all integers $T_1, \ldots T_5$, not all 0, satisfying

(7.54) $$M_k T_k \leqslant \frac{1}{5} U^{-1} P \qquad (k = 1, \ldots 5).$$

If we write $\Phi(T_1, \ldots T_5) = \sum_{u=1}^{5} \sum_{v=1}^{5} \phi_{uv} T_u T_v$, we obtain

$$|\phi_{uv}| << P^{\lambda}\alpha M_u M_v U^2, \quad ||\phi_{uv}|| << M_u M_v U P^{-1}.$$

Putting $\phi_{uv} = f_{uv} + \psi_{uv}$ where f_{uv} is the integer nearest to ϕ_{uv} we obtain

(7.55) $$|f_{uv}| << P^{\lambda}\alpha M_u M_v U^2, |\psi_{uv}| << M_u M_v U P^{-1}.$$

Now the question is whether the form $f = \sum_{u=1}^{5} \sum_{v=1}^{5} f_{uv} T_u T_v$ is indefinite or represents zero. This will be so if det $\{f_{uv}\}$ is of the same sign as det $\{\phi_{uv}\}$ and if the analogous result holds for the principal sub-determinants of $\{f_{uv}\}$. Clearly, from (7.55),

$$|\det\{f_{uv}\} - \det\{\phi_{uv}\}| << (U P^{-1}) (P^{\lambda} U^2 \alpha)^4 M_1^2 \ldots M_5^2.$$

Similar inequalities hold for each principal sub-determinant but with one or more factors $P^{\lambda} U^2 M_u^2 \alpha$ missing. We may suppose $P^{\lambda} U^2 M_u^2 \alpha >> 1$ for each u, since otherwise $f_{uu} = 0$, whence the form represents zero. So it suffices to show that $(U P^{-1}) (P^{\lambda} U^2 \alpha)^4 M_1^2 \ldots M_5^2$ is less than a suitable positive constant. This will be so if

(7.56) $$M_1 \ldots M_5 << P^{\frac{1}{2}-2\lambda-A\varepsilon} U^{-9/2},$$

and we suppose this is true. Clearly (7.56) supersedes (7.52).

We now apply Lemma 7.4 with $g_u = M_u^2$. Then there are integers $T_1, \ldots T_5$, not all zero, satisfying $f = 0$, and for $u = 1, \ldots 5$, $M_u^2 T_u^2 \ll \{\sum_{u=1}^{5} \sum_{v=1}^{5} P^{2\lambda}\alpha^2 M_u^2 M_v^2 U^4 M_u^{-2} M_v^{-2}\}^2 \times$

$$\times M_1^2 \ldots M_5^2 .$$

We thus obtain for $u = 1, \ldots 5$,

$$M_u T_u \ll P^{2\lambda}\alpha^2 U^4 M_1, \ldots M_5.$$

If we show that $P^{2\lambda}\alpha^2 U^4 M_1, \ldots M_5 = o(U^{-1} P)$ we will have shown that $T_1, \ldots T_5$ satisfy (7.54). On substituting from (7.44) and noting that $U \leqslant P$, it is enough to show that $5/n\ (4-r + 2\lambda) + 2\lambda + 5r/n - 1 + A\varepsilon < -\varepsilon$, and either this is true or

$$\lambda > \frac{n - 20}{2(n + 5)} - A\varepsilon . \tag{7.57}$$

We may suppose (7.57) false for otherwise the lemma is proved. Thus $T_1, \ldots T_5$ satisfy (7.54), and therefore by (7.53) and since $f = 0$,

$$2\alpha \leqslant \sum_{u=1}^{5} \sum_{v=1}^{5} \psi_{uv}|T_u T_v| \ll U^9 P^{-1+4\lambda} \alpha^4 M_1^2 \ldots M_5^2$$

i.e. $M_1 \ldots M_5 \gg P^{\frac{1}{2}-2\lambda} \alpha^{-3/2} U^{-9/2} \gg P^{\frac{1}{2}-2\lambda-A\varepsilon} U^{-9/2}$.

Clearly this holds whether (7.56) is true or not and since $U \leqslant P$, $M_1 \ldots M_5 \gg P^{-\frac{1}{2}-2\lambda-A\varepsilon} U^{-7/2}$, which proves (7.49) and hence the lemma in this case.

Case 2: Suppose the form Φ is definite, and thus negative definite, since its diagonal coefficients are negative. The form is non-singular since $\underline{X}^{(1)}, \dots \underline{X}^{(5)}$ are linearly independent.

We may assume in (7.48), without loss of generality, that $k = 5$. The points $\underline{X}^{(1)}, \dots \underline{X}^{(4)}, \underline{Y}^{(5)}$ are linearly independent, for it not, $\underline{Y}^{(5)}$ would be a linear combination of $\underline{X}^{(1)}, \dots \underline{X}^{(5)}$ with rational coefficients, and so $2\alpha P^{\lambda} Q(\underline{Y}^{(5)})$ would be a value of $\Phi(T_1, \dots T_5)$ and so would be negative. Now consider the form.

$$\Phi'(T_1, \dots T_5) = 2 P^{\lambda} \alpha Q(T_1 \underline{X}^{(1)} + \dots + T_4 \underline{X}^{(4)} + T_5 \underline{Y}^{(5)}),$$

which is clearly indefinite. Now $\underline{X}^{(1)} \dots \underline{X}^{(4)}$ satisfy

(7.58) $$|X_j^{(k)}| \ll M_k U \qquad (j = 1, \dots n),$$

(7.59) $$||2P^{\lambda} \alpha L_j(X_1^{(k)}, \dots X_n^{(k)})|| \ll M_k P^{-1} \quad (j = 1, \dots n),$$

and $\underline{Y}^{(5)}$ satisfies

(7.60) $$|Y_j^{(5)}| \ll M_5 P \qquad (j = 1, \dots n),$$

(7.61) $$||2P^{\lambda}\alpha \; L_j(Y_1^{(5)}, \dots Y_n^{(5)})|| \ll M_5 P^{-1} \quad (j = 1, \dots n).$$

Thus, for all integers $T_1, \dots T_5$, not all 0, satisfying

(7.62) $$| M_k T_k| < \tfrac{1}{5} U^{-1} P \quad (k = 1, \dots 4),$$

(7.63) $$| M_k T_k| < \tfrac{1}{5},$$

$$|\Phi'(T_1, \dots T_5)| \geqslant 2 \alpha.$$

Clearly the coefficients ϕ'_{uv} of Φ' satisfy

$$|\phi'_{uv}| \ll \begin{cases} P^{\lambda}\alpha M_u M_v U^2 & u < 5, v < 5, \\ P^{1+\lambda}\alpha M_u M_v U & u < 5, v = 5, \\ P^{2+\lambda}\alpha M_5^2 & u = 5 = v, \end{cases}$$

$$||\phi'_{uv}|| \ll \begin{cases} M_u M_v U P^{-1} & u < 5, v \leqslant 5, \\ M_5^2 & u = 5 = v. \end{cases}$$

By (7.47), (&.48), $\phi'_{uu} < 0$ for $u < 5$, and $\phi'_{55} > 0$. Hence, on forming the corresponding integer form f', we obtain $f'_{uu} \leqslant 0$ for $u < 5$, and $f'_{55} \geqslant 0$. Thus the form $f'(T_1, \ldots T_5)$ is either indefinite or represents zero. We now apply Lemma 7.4 with $G_u = M_u^2 U^2$ for $u < 5$ and $g_5 = M_5^2 P^2$. There exist integers $T_1, \ldots T_5$, not all 0, such that $f'(T_1, \ldots T_5) = 0$, and $M_u^2 U^2 T_u^2 << P^{4\lambda} \alpha^4 U^8 P^2 M_1^2 \ldots M_5^2$ for $u \leqslant 4$, and $M_5^2 P^2 T_5^2 << P^{4\lambda} \alpha^4 U^8 P^2 M_1^2 \ldots M_5^2$. Hence, we obtain $M_u T_u << \alpha^2 M_1 \ldots M_5 U^3 P^{1+2\lambda}$ for $u \leqslant 4$, and $M_5 T_5 << \alpha^2 M_1 \ldots M_5 U^4 P^{2\lambda}$. Clearly, if we show that $\alpha^2 M_1 \ldots M_5 U^4 P^{2\lambda} = o(1)$, we will have shown that $T_1, \ldots T_5$ satisfy (7.62), (7.63). On substituting from (7.44), it clearly suffices to show that

(7.64) $$P^{20-5r+10\lambda+2n\lambda+A\varepsilon} U^{5r-n} = o(1).$$

We separate this into two cases

(i) Suppose $n \ngtr 5r$. Then $5r-n$ is positive and thus $U^{5r-n} \leqslant P^{5r-n}$. Therefore either (7.64) is true or

(7.65) $$\lambda > \frac{n-20}{2(n+5)} - A\varepsilon .$$

We may suppose (7.65) false for otherwise (7.13) is true and hence the lemma is proved. Thus (7.64) is proved.

(ii) Suppose $n > 5r$. Then $5r-n$ is negative, and therefore, on substituting from (7.46), it is enough to show

$$P^{-n+\frac{4n}{r}+\lambda(2n+\frac{2n}{r})+A\varepsilon} = o(1).$$

Clearly either this is true or

(7.66) $$\lambda > \frac{r-4}{2(r+1)} - A\varepsilon .$$

We may suppose (7.66) false for otherwise (7.13) is true and hence the lemma is proved. Thus (7.64) is satisfied. Thus, since $f'(T_1, \ldots T_5) = 0$,

$$2\alpha \leqslant \sum_{u=1}^{5} \sum_{v=1}^{5} |\psi'_{uv}|\,|T_u\,T_v| \ll U\,P^{-1} \sum_{u<5} \sum_{v=1}^{5} M_u\,M_v|T_u\,T_v| + M_5^2\,T_5^2$$

$$\ll \alpha^4\,M_1^2 \ldots M_5^2(U^7\,P^{1+4\lambda}+U^8\,P^{4\lambda}) \ll \alpha^4\,M_1^2 \ldots M_5^2\,U^7P^{1+4\lambda}$$

and thus $M_1 \ldots M_5 \gg \alpha^{-3/2}\,U^{-7/2}\,P^{-\frac{1}{2}-2\lambda} \gg P^{-\frac{1}{2}-2\lambda-A\varepsilon}\,U^{-7/2}$.

Clearly this holds independently of (7.52), and thus proves the lemma in this case.

This completes the proof of Lemma 7.20.

We have so far shown that either (7.13) holds, or that for each α in $\mathcal{J}$ - , for the value of U determined in Lemma 7.19, both (7.44) and (7.49) hold. Suppose that (7.13) is false, for otherwise we have proved the theorem. Then, by (7.44), (7.49)

$$P^{5(4-r+2\lambda)+A\varepsilon}\,U^{-5(n-r)} \gg P^{-2n\lambda-n/2-A\varepsilon}\,U^{-7/2}$$

i.e.

(7.67) $$U^{5r-3n/2} \gg P^{5r-20-2\lambda(5+n)-n/2-A\varepsilon}\,.$$

We now suppose $r \geqslant 7$ and split the argument into 2 cases.

(i) Suppose $10\,r \geqslant 3n$. Then the exponent of U in (7.67) is positive and since $U \leqslant P$, (7.67) implies

$$5r - 3n/2 \geqslant 5r - 20 - 2\lambda(5+n) - n/2 - A\varepsilon,$$

i.e. $$\lambda > \frac{n-20}{2(n+5)} - A\varepsilon\;,$$

which proves (7.13) and hence case (a) of the theorem.

(ii) Suppose $10r < 3n$. Then $5r - 3n/2$ is negative and thus, by (7.46), $U^{5r-3n/2} \ll P^{-3n/2 + 5r + 6n/r - 20 + \lambda(3n/r-10)+A\varepsilon}$. This with (7.67) implies

$$- 3n/2 + 5r + 6n/r - 20 + \lambda(3n/r - 10) \geqslant 5r - 20 - 2\lambda(n + 5) - n/2 - A\varepsilon,$$

i.e. $$\lambda > \frac{r - 6}{2r + 3} - A\varepsilon ,$$

which (since $k = r$) proves (7.13) and thus proves case (b) of the theorem.

In this section it only remains to prove case (f) of the theorem, i.e. the case $n = 21$, $k = 6$. In this case (7.46) becomes

$$U \geqslant P^{1/3-1/3\lambda-A\varepsilon}, \tag{7.68}$$

and (7.67) gives

$$U \ll P^{1/3 + \frac{104\lambda}{3} + A\varepsilon} . \tag{7.69}$$

On substituting from (7.68) into (7.44) we obtain

$$M_1(U) \ldots M_5(U) \ll P^{-5/3+5/3\lambda+A\varepsilon} , \tag{7.70}$$

while on substituting from (7.69) into (7.49) we obtain

$$M_1(U) \ldots M_5(U) \gg P^{-5/3- \frac{370\lambda}{3} - A\varepsilon} . \tag{7.71}$$

Furthermore, from (7.42), (7.68), (7.71), and on noting that since $s \leqslant n = 21$, $(M_1 \ldots M_s)^{-1} \ll (M_1 \ldots M_5)^{-21/5}$,

$$|S(P^{\lambda}\alpha)| \ll P^{19 + \frac{523}{2}\lambda + A\varepsilon} . \tag{7.72}$$

Now, for any α in $\mathcal{J}$ – , there is a non-zero integer point $\underline{x}^{(1)}$ in n dimensions which satisfies (7.50), (7.51) with $k = 1$. Since, by (7.70), $M_1(U) \ll P^{-1/3+\lambda/3+A\varepsilon}$, we obtain on substituting from (7.69),

$$(7.73) \qquad |x_j^{(1)}| \ll P^{35\lambda + A\varepsilon} \qquad (j = 1, \ldots n),$$

$$(7.74) \qquad ||2P^{\lambda}\alpha\, L_j(x_1^{(1)}, \ldots x_n^{(1)})|| \ll P^{-4/3+\lambda/3+A\varepsilon} \quad (j = 1, \ldots n).$$

Since $\underline{x}^{(1)}$ is a non-zero integer point and $L_1, \ldots L_n$ are fixed real linear forms of non-zero determinant, one at least of $|L_1(\underline{x}^{(1)})|, \ldots |L_n(\underline{x}^{(1)})|$ is $\gg 1$. It will suffice to consider the part of $\mathcal{J}-$ in which $|L_1(\underline{x}^{(1)})| \gg 1$. By (7.74) we have

$$(7.75) \qquad \left|\alpha - \frac{t}{2P^{\lambda} L_1(\underline{x}^{(1)})}\right| < \frac{P^{-4/3+\lambda/3+A\varepsilon}}{|2P^{\lambda} L_1(\underline{x}^{(1)})|}$$

where t is an integer. By (7.73) the number of integer points $\underline{x}^{(1)}$ that can arise from all α in $\mathcal{J}-$ is $\ll P^{735\lambda+A\varepsilon}$. For each $\underline{x}^{(1)}$ the number of possibilities for t in (7.75) is $\ll P^{\lambda+\varepsilon}|L_1(\underline{x}^{(1)})|$ since $\alpha < P^{\varepsilon}$. Hence, for each $\underline{x}^{(1)}$, the possible values of α are restricted to intervals of total length $\ll P^{\lambda+\varepsilon}\,|L_1\,(\underline{x}^{(1)})|\,\dfrac{P^{-4/3+\lambda/3+A\varepsilon}}{|2P^{\lambda} L_1(\underline{x}^{(1)})|}$

$$\ll P^{-4/3+\lambda/3+A\varepsilon}.$$

Thus the measure of $\mathcal{J}-$ is $\ll P^{-4/3+\frac{2206}{3}\lambda+A\varepsilon}$.

Hence, by Lemma 7.1 and (7.72)

$$\int_{\mathcal{J}-} |S(P^{\lambda}\alpha)\, K(\alpha)|\,d\alpha \ll \int_{\mathcal{J}-} |S(P^{\lambda}\alpha)|\,d\,\alpha$$

$$\ll P^{53/3 + \frac{5981}{6}\lambda + A\varepsilon}.$$

By (7.22) we must have

$$\frac{53}{3} + \frac{5981}{6}\lambda + A\varepsilon > 19 - \lambda - 2\varepsilon$$

i.e. $\lambda > \frac{8}{5987} - A\varepsilon$

which proves (7.13) and hence case (f) of the theorem.

As can quite easily be seen, the method of proof of case (f) can be generalised to a proof of the case $k = 6$, $n \geqslant 21$. The result obtained

$$f(n, k) = \frac{8\ (n - 20)}{15n^2 - 28n - 40} - \varepsilon$$

is asymptotically much worse than the result (7.7) since for f(n, k) defined by (7.7),

$$f\ (n, k) \to \tfrac{1}{2} \quad \text{as} \quad n \to \infty.$$

§3: Proof of parts (c) and (d) of Theorem 25: We start by making similar assumptions to those made at the beginning of §2. We suppose, until otherwise specified that $n \geqslant 21$ and $k \leqslant 5$ and without loss of generality we assume $k = r$. We further assume that (7.12) holds, under the same conditions as in §2, and we will show that (7.12) implies (7.13). Clearly this will prove the theorem in these cases.

We suppose $P = N^a$ where a is some positive real number, dependent at most upon n, which will be chosen later. We now choose 5 integer points $\underline{z}^{(1)}, \ldots \underline{z}^{(5)}$ in n dimensions. Suppose $\underline{z}^{(1)} = (1, 0, \ldots 0)$. We now use Lemma 7.7 to choose $\underline{z}^{(2)}, \underline{z}^{(3)}, \underline{z}^{(4)}, \underline{z}^{(5)}$, in that order. We suppose, in the choice of $\underline{z}^{(\rho)}$, $(\rho = 2, 3, 4, 5)$, that $m = \rho - 1$, $\gamma_{ij} = \sum_{k=1}^{n} \lambda_{jk} z_k^{(i)}$, $(i = 1, \ldots m)$, and $\theta = m$, $\theta_1 = \theta_2 = \ldots = \theta_m = n$. We then obtain integer points $\underline{z}^{(2)}, \underline{z}^{(3)}, \underline{z}^{(4)}, \underline{z}^{(5)}$. By the choice of $\underline{z}^{(1)}$, these 5 non-zero integer points satisfy

(7.76) $$|z_j^{(\rho)}| \leqslant N^{\rho-1} \qquad (j = 1, \ldots n, \rho = 1, \ldots 5),$$

and for $1 \leqslant \sigma < \rho \leqslant 5$

(7.77) $$\left|\sum_{j=1}^{n} \sum_{k=1}^{n} \lambda_{jk} z_k^{(\sigma)} z_j^{(\rho)}\right| = J_\sigma << N^{\rho-1-n} \sum_{j=1}^{n} \left|\sum_{k=1}^{n} \lambda_{jk} z_k^{(\sigma)}\right|$$

$$<< N^{\rho-1-n+\sigma-1} = N^{\rho+\sigma-2-n}.$$

Note that since $\theta, \theta_1, \ldots \theta_m$ depend upon n only, in the above expressions '$<<$' depends upon n only.

We now distinguish the two cases $r \leqslant 4$, which is part (c), and $r = 5$, which is part (d).

Case 1: $r \leqslant 4$: The linear substitution

(7.78) $$\underline{x} = t_1 \underline{z}^{(1)} + \ldots + t_5 \underline{z}^{(5)}$$

gives $Q(x_1, \ldots x_n) = \Phi(t_1, \ldots t_5) = \sum_{\rho=1}^{5} \sum_{\sigma=1}^{5} \phi_{\rho\sigma} t_\rho t_\sigma$

where $\phi_{\rho\sigma} = \sum_{j=1}^{n} \sum_{k=1}^{n} \lambda_{jk} z_k^{(\rho)} z_j^{(\sigma)}$, $(1 \leqslant \sigma, \rho \leqslant 5)$.

By (7.76), (7.77),

(7.79) $$|\phi_{\rho\rho}| << N^{2(\rho-1)} \qquad , \ 1 \leqslant \rho \leqslant 5 .$$

and for $1 \leqslant \sigma < \rho \leqslant 5$

(7.80) $$|\phi_{\rho\sigma}| << N^{\rho+\sigma-2-n} \qquad .$$

We suppose $a \geqslant 4$. Since $\phi_{\rho\rho}$ is a value of Q for integers $x_1, \ldots x_n$, not all 0, and satisfying $|\underline{x}| \leqslant P$, we have

(7.81) $$|\phi_{\rho\rho}| \geqslant P^{-\lambda} = N^{-a\lambda} \qquad , \ 1 \leqslant \rho \leqslant 5.$$

Now suppose

(7.82) $$\lambda < \frac{n-7}{a} - \varepsilon .$$

Then, by (7.80), (7,81), the form $\Phi(t_1, \ldots t_5)$ is almost diagonal. In particular its determinant is not zero and thus the rank of substitution (7.78) is 5. Hence

(7.83) $$|\Phi\ (t_1, \ldots t_5)| \geqslant N^{-a\lambda}$$

for all integers $t_1, \ldots t_5$, not all o, with

(7.84) $$|t_\rho| < \frac{1}{5} N^{a+1-\rho} \qquad (\rho = 1, \ldots, 5).$$

If the rank of Φ is r', then $r' \leqslant r \leqslant 4$ since Q represents Φ. Also $r' \geqslant 1$, since by the choice of $\underline{z}^{(1)}$, $\phi_{11} = \lambda_{11}$ and by Lemma 7.2, $\lambda_{11} > 0$. Hence Φ is indefinite and the same is true of the diagonal form $\phi_{11}t_1^2 + \ldots + \phi_{55}\, t_5^2$.

By (7.81) and Lemma 7.6 there exist integers $t_1, \ldots t_5$, not all o, satisfying

$$|2P^\lambda\, \phi_{11}t_1^2 + \ldots + 2\, P^\lambda\, \phi_{55}\, t_5^2| < 1$$

and $$|2P^\lambda\, \phi_{\rho\rho}\, t_\rho^2| << |P^{5\lambda}2^5\, \phi_{11} \ldots \phi_{55}|^{1+\varepsilon} .$$

Thus there exist integers $t_1, \ldots t_5$, not all o, satisfying

(7.85) $$|\phi_{11}t_1^2 + \ldots + \phi_{55}\, t_5^2| < \tfrac{1}{2} P^{-\lambda}$$

and

(7.86) $$|t_\rho| << N^{2a\lambda+10-(\rho-1)+A\varepsilon} .$$

Thus, by (7.80), (7.86), for $1 \leqslant \sigma < \rho \leqslant 5$,

(7.87) $$|\phi_{\rho\sigma}\, t_\rho\, t_\sigma| << N^{4a\lambda+20-n+A\varepsilon} .$$

Now suppose

(7.88) $$\lambda < \frac{a-10}{2a} - A\varepsilon .$$

Then these values of $t_1, \dots t_5$ satisfy (7.84) and therefore (7.83). Therefore, by (7.85), (7.87) and (7.12) we must have

$$4a\lambda + 20 - n + A\varepsilon > - a\lambda - \varepsilon ,$$

i.e.

(7.89) $$\lambda > \frac{n - 20}{5a} - A\varepsilon .$$

We suppose $a = \frac{1}{5}(2n + 10)$. Then $a > 10$. With this value of a, for

$$\lambda < \frac{n - 20}{2(n + 5)} - A\varepsilon ,$$

we have a contradiction of (7.12), since under this inequality (7.82) (by a wide margin) and (7.88) hold while (7.89) does not. Thus, for (7.12) to hold, (7.13) must be true, which proves part (c) of the theorem.

Case 2: $r = 5$: We use Lemma 7.7 to choose another integer point $\underline{z}^{(6)}$. We suppose $a = \dfrac{2n^2 + 9n - 2}{5n - 3}$. In Lemma 7.7 we let $m = 5, \gamma_{ij} = \sum_{k=1}^{n} \lambda_{jk} z_k^{(i)}$, $(1 \leqslant i \leqslant 5)$, and we choose $\theta_1 = \dots = \theta_4 = n$, $\theta = \dfrac{21 n + 8}{5n - 3}$; hence $\theta_5 = (\theta - 4) n > 0$. We then obtain a non-zero integer point $\underline{z}^{(6)}$ satisfying

(7.90) $$|z_j^{(6)}| \leqslant N^{\theta} \qquad (1 \leqslant j \leqslant n),$$

(7.91) $$\left|\sum_{j=1}^{n} \sum_{k=1}^{n} \lambda_{jk} z_k^{(\rho)} z_k^{(6)}\right| \ll \begin{cases} N^{\theta-n+\rho-1}, & 1 \leqslant \rho \leqslant 4, \\ N^{\theta+4-n(\theta-4)}, & \rho = 5. \end{cases}$$

The linear substitution

(7.92) $$\underline{x} = t_1 \underline{z}^{(1)} + \dots + t_6 \underline{z}^{(6)}$$

gives $Q(x_1, \dots x_n) = \Phi(t_1, \dots t_6) = \sum_{\rho=1}^{6} \sum_{\sigma=1}^{6} \phi_{\rho\sigma} t_\rho t_\sigma$,

where $\phi_{\rho\sigma} = \sum_{j=1}^{n} \sum_{k=1}^{n} \lambda_{jk} z_k^{(\rho)} z_j^{(\sigma)}$ $(1 \leqslant \rho \leqslant 6, 1 \leqslant \sigma \leqslant 6)$.

By (7.76), (7.77), (7.90), (7.91)

$$|\phi_{\rho\rho}| \ll \begin{cases} N^{2(\rho-1)} & 1 \leqslant \rho \leqslant 5, \\ N^{2\Theta} & \rho = 6, \end{cases} \tag{7.93}$$

and for $\sigma < \rho$,

$$|\phi_{\rho\sigma}| \ll \begin{cases} N^{\rho+\sigma-2-n} & 1 \leqslant \rho \leqslant 5, \\ N^{\Theta+\sigma-1-n} & \rho = 6,\ \sigma \leqslant 4, \\ N^{\Theta+4-n(\Theta-4)} & \rho = 6,\ \sigma = 5. \end{cases} \tag{7.94}$$

Clearly $a > \Theta$, and, thus, since $\phi_{\rho\rho}$ is a value of Q for integers $x_1, \ldots x_n$, not all o, and satisfying $|\underline{x}| \leqslant P$, we have

$$|\phi_{\rho\rho}| \geqslant P^{-\lambda} = N^{-a\lambda} . \tag{7.95}$$

We may suppose that

$$\lambda < \min \left\{\frac{n-(3+\Theta)}{a}, \frac{\Theta(n-1) - 4(n+1)}{a}\right\} - A\varepsilon = \frac{n^2 - 21n + 4}{2n^2 + 9n - 2} - A\varepsilon,$$

for otherwise (7.13) is true. Then, by (7.95) and (7.94), the form $\Phi(t_1, \ldots t_6)$ will be almost diagonal. In particular the determinant is not zero and consequently the rank of substitution (7.92) is 6. Hence

$$|\Phi\ (t_1, \ldots t_6)| \geqslant P^{-\lambda} = N^{-a\lambda} \tag{7.96}$$

for all integers $t_1, \ldots t_6$, not all o, satisfying

$$|t_\rho| < \begin{cases} \frac{1}{6} N^{a+1-\rho} & 1 \leqslant \rho \leqslant 5, \\ \frac{1}{6} N^{a-\Theta} & \rho = 6 \end{cases} . \tag{7.97}$$

As in Case 1, $\phi_{11} > o$, and we now distinguish two cases.

Case 2(a): $\phi_{\rho\rho} > 0$ for $1 \leqslant \rho \leqslant 5$. Since $r = 5$ and Q cannot represent a positive definite form in 6 variables, it follows that $\phi_{66} < 0$. We now consider the form $\Phi(t_1, t_2, \ldots t_4, 0, t_6)$. By Lemma 7.6 and (7.95) there exist integers t_1, t_2, t_3, t_4, t_6, not all o, satisfying

$$|2P^\lambda \phi_{11} t_1^2 + \ldots + 2P^\lambda \phi_{44} t_4^2 + 2P^\lambda \phi_{66} t_6^2| < 1$$

and $$|2P^\lambda \phi_{\rho\rho} t_\rho^2| \ll |2^5 P^{5\lambda} \phi_{11} \ldots \phi_{44} \phi_{66}|^{1+\varepsilon}, \ \rho = 1,2,3,4,6.$$

These integers then satisfy

(7.98) $$|\phi_{11} t_1^2 + \ldots + \phi_{44} t_4^2 + \phi_{66} t_6^2| < \tfrac{1}{2} P^{-\lambda}$$

and

(7.99) $$t_\rho \ll \begin{cases} N^{2a\lambda+6+\theta-(\rho-1)+A\varepsilon} & 1 \leqslant \rho \leqslant 4, \\ N^{2a\lambda+6+A\varepsilon} & \rho = 6 \end{cases} .$$

By (7.94), (7.99),

(7.100) $$|\phi_{\rho\sigma} t_\rho t_\sigma| \ll N^{4a\lambda+12+2\theta-n+A\varepsilon} \qquad (\sigma < \rho \leqslant 4, \ \rho = 6, \sigma = 4).$$

We may suppose that

$$\lambda < \frac{a - (6 + \theta)}{2a} - A\varepsilon = \frac{n^2 - 21n + 4}{2n^2 + 9n - 2} - A\varepsilon$$

for otherwise (7.13) is true. Then these values of $t_1, \ldots t_4, t_6$ satisfy (7.97) and since $t_5 = 0$, they also satisfy (7.96). Therefore, by (7.98), (7.100), and (7.12),

$$4a\lambda + 12 + 2\theta - n + A\varepsilon > - a\lambda - A\varepsilon$$

i.e. $$\lambda > \frac{n - (12 + 2\theta)}{5a} - A\varepsilon = \frac{n^2 - 21n + 4}{2n^2 + 9n - 2} - A\varepsilon ,$$

which proves (7.13) and hence the result in this case.

Case 2(b): $\phi_{\rho\rho} < o$ for some ρ, $2 \leqslant \rho \leqslant 5$. Since $\phi_{11} > o$, the form $\Phi(t_1, \dots t_5, 0)$ is clearly indefinite. As in the previous case, by Lemma 7.6 and (7.95) there exist integers $t_1, \dots t_5$, not all 0, satisfying

(7.101) $$|\phi_{11}t_1^2 + \dots + \phi_{55}\, t_5^2| < \tfrac{1}{2}\, P^{-\lambda}$$

and for $1 \leqslant \rho \leqslant 5$

(7.102) $$|t_\rho| < N^{2a\lambda+10-(\rho-1)+A\varepsilon} .$$

Thus, by (7.94), (7.102), for $1 \leqslant \sigma < \rho \leqslant 5$

(7.103) $$|\phi_{\rho\sigma}\, t_\rho\, t_\sigma| << N^{4a\lambda+20-n+A\varepsilon} .$$

We may suppose that

$$\lambda < \frac{a - 10}{2a} - A\varepsilon = \frac{2n^2 - 41n + 28}{2(2n^2 + 9n - 2)} - A\varepsilon$$

for otherwise (7.13) is true. Then the values of $t_1, \dots t_5$ satisfy (7.97) and since $t_6 = o$, they also satisfy (7.96). Therefore, by (7.101), (7.103, and (7.12),

$$4a\lambda + 20 - n + A\varepsilon > - a\lambda - A\varepsilon$$

i.e. $$\lambda > \frac{n - 20}{5a} - A\varepsilon = \frac{5n^2 - 103n + 60}{5(2n^2 + 9n - 2)} - A\varepsilon > f(n, k)$$

which proves (7.13) and hence completes the proof of part (d) of the theorem.

It is a remarkable coincidence that by using two rather dissimilar methods of proof one obtains, not only that the minimum value of n which can be treated is 21, in both cases, but that the values of $f(n, k)$ obtained in (a) and (c) are the same.

§4: Proof of part (e) of the theorem. The proof of this part of the theorem uses the same ideas as the proof of part (d), but it is written separately for simplicity. We make the same assumptions as made at the start of §2, i.e. we assume that (7.12) is true under the same conditions. As before, we will show that (7.12) implies (7.13) and hence the theorem in this case. Without loss of generality, we may assume that $k = r$ and thus $r = 6$. We note $n \geqslant 22$. We suppose $P = N^a$ where

$$a = \frac{2n^3 + 8n^2 - 5n - 2}{5n^2 - 6n - 3}.$$

As in §3 we choose 5 non-zero integer points $\underline{z}^{(1)}, \ldots \underline{z}^{(5)}$ which satisfy (7.76), (7.77), with $\underline{z}^{(1)} = (1, 0, \ldots 0)$. We now choose two more non-zero integer points $\underline{z}^{(6)}, \underline{z}^{(7)}$. First we choose $\underline{z}^{(6)}$ thus: in Lemma 7.7, let $m = 5$,

$$\gamma_{ij} = \sum_{k=1}^{n} \lambda_{jk} z_k^{(i)} \quad (i = 1, \ldots 5), \ \theta_1 = \theta_2 = \theta_3 = \theta_4 = n,$$

$$\theta_5 = \frac{n^2(n + 20)}{5n^2-6n-3};$$

thus $\theta = 4 + \frac{\theta_5}{n}$. We then obtain a non-zero integer point $\underline{z}^{(6)}$ satisfying

$$|z_j^{(6)}| \leqslant N^{4 + \frac{\theta_5}{n}} \quad (j = 1, \ldots n), \tag{7.104}$$

$$\left|\sum_{j=1}^{n} \sum_{k=1}^{n} \lambda_{jk} z_k^{(\sigma)} z_j^{(6)}\right| \ll \begin{cases} N^{3 + \frac{\theta_5}{n} - n + \sigma}, & 1 \leqslant \sigma \leqslant 4, \\ N^{8 + \frac{\theta_5}{n} - \theta_5}, & \sigma = 5. \end{cases} \tag{7.105}$$

We now choose $\underline{z}^{(7)}$ similarly, with $\theta_6 = \frac{n(n + 1)(n + 20)}{5n^2 - 6n - 3}$. We then obtain a non-zero integer point $\underline{z}^{(7)}$ satisfying

$$|z_j^{(7)}| \leqslant N^{4+ \frac{1}{n}(\theta_5 + \theta_6)} \quad (j = 1, \ldots n), \tag{7.106}$$

$$(7.107)\qquad \left|\sum_{k=1}^{n}\sum_{j=1}^{n}\lambda_{jk}\, z_k^{(\sigma)}\, z_j^{(7)}\right| \ll \begin{cases} N^{3-n+\sigma+\frac{\theta_5+\theta_6}{n}}, & 1 \leq \sigma \leq 4,\\ N^{8+\frac{2\theta_5}{n}+\theta_6\frac{(1-n)}{n}}, & \sigma = 6,\\ N^{8+\theta_5\frac{(1-n)}{n}+\frac{\theta_6}{n}}, & \sigma = 5.\end{cases}$$

The linear substitution

$$(7.108)\qquad \underline{x} = t_1\,\underline{z}^{(1)} + \dots + t_7\,\underline{z}^{(7)}$$

gives $Q(x_1, \dots x_n) = \Phi(t_1, \dots t_7) = \sum_{\rho=1}^{7}\sum_{\sigma=1}^{7}\phi_{\rho\sigma}\, t_\rho\, t_\sigma$,

where, for $1 \leq \rho, \sigma \leq 7$,

$$\phi_{\rho\sigma} = \sum_{j=1}^{n}\sum_{k=1}^{n}\lambda_{jk}\, z_k^{(\rho)}\, z_j^{(\sigma)} .$$

Therefore, by (7.76), (7,104), (7.106)

$$(7.109)\qquad |\phi_{\rho\rho}| \ll \begin{cases} N^{2(\rho-1)} & 1 \leq \rho \leq 5,\\ N^{8+\frac{2\theta_5}{n}} & \rho = 6,\\ N^{8+2/n(\theta_5+\theta_6)} & \rho = 7,\end{cases}$$

and for $1 \leq \sigma < \rho \leq 7$,

$$(7.110)\qquad |\phi_{\rho\sigma}| \ll \begin{cases} N^{\rho+\sigma-2-n} & \rho \leq 5,\\ N^{3+\frac{\theta_5}{n}-n+\sigma} & 1 \leq \sigma \leq 4,\ \rho = 6,\\ N^{8+\frac{\theta_5}{n}-\theta_5} & \sigma = 5,\ \rho = 6,\\ N^{3-n+\sigma+\frac{1}{n}(\theta_5+\theta_6)} & 1 \leq \sigma \leq 4,\ \rho = 7,\\ N^{8+\theta_5\frac{(1-n)}{n}+\frac{\theta_6}{n}} & \sigma = 5,\ \rho = 7,\\ N^{8+\frac{2\theta_5}{n}+\theta_6\frac{(1-n)}{n}} & \sigma = 6,\ \rho = 7.\end{cases}$$

Since, by the choice of a, θ_5, θ_6, $a > 4 + \frac{\theta_5+\theta_6}{n}$,

$0 < |\underline{z}^{(\rho)}| \leq P$, $(\rho = 1, \dots 7)$, and therefore, by the hypothesis

(7.12), for $1 \leqslant \rho \leqslant 7$,

(7.111) $$|\phi_{\rho\rho}| > N^{-a\lambda}.$$

We may suppose that

$$\lambda < \min\left\{ \frac{n - (7 + \frac{\theta_5 + \theta_6}{n})}{a}, \frac{\theta_5(1 - \frac{1}{n}) - (8 + \frac{\theta_6}{n})}{a}, \frac{\theta_6(1 - \frac{1}{n}) - (8 + \frac{2\theta_5}{n})}{a} \right\} - A\varepsilon$$

$$= \frac{n^3 - 22n^2 + 7n + 4}{2n^3 + 8n^2 - 5n - 2} - A\varepsilon ,$$

for otherwise (7.13) is true. Then by (7.110), (7.111) the form Φ is almost diagonal and in particular it is non-singular and consequently the rank of substitution (7.108) is 7. Therefore, by our initial hypothesis (7.12) and (7.76), (7.104), (7.106),

(7.112) $$|\Phi (t_1, \dots t_7)| \geqslant N^{-a\lambda} ,$$

for all integers $t_1, \dots t_7$, not all o, satisfying

(7.113) $$|t_\rho| < \begin{cases} \frac{1}{7} N^{a-\rho+1}, & 1 \leqslant \rho \leqslant 5, \\ \frac{1}{7} N^{a-(4 + \frac{\theta_5}{n})}, & \rho = 6, \\ \frac{1}{7} N^{a-(4+\frac{\theta_5}{n} + \frac{\theta_6}{n})}, & \rho = 7. \end{cases}$$

As in §3, we know that $\phi_{11} > o$. We now split the argument into 3 cases.

<u>Case A: $\phi_{\rho\rho} < o$ for some ρ, $2 \leqslant \rho \leqslant 5$.</u> Consider the form $\Phi\,(t_1, \ldots t_5, o, o)$. By Lemma 7.6 there exist integers $t_1, \ldots t_5$, (not all o) satisfying

(7.114) $$|\phi_{11}t_1^{\,2} + \ldots + \phi_{55}\, t_5^{\,2}| < \tfrac{1}{2} N^{-a\lambda} ,$$

with, for $1 \leqslant \rho \leqslant 5$,

(7.115) $$|t_\rho| << N^{10-\ (\rho-1)\ +\ 2a\lambda\ +\ A\varepsilon} .$$

Hence, by (7.110), (7.115), for $1 \leqslant \sigma < \rho \leqslant 5$,

(7.116) $$|\phi_{\rho\sigma}\ t_\rho\ t_\sigma| << N^{20-n+4a\lambda+A\varepsilon} .$$

Now, we may suppose that

$$\lambda < \frac{a-10}{2a} - A\varepsilon = \frac{2n^3 - 42n^2 + 55n + 28}{2(2n^3 + 8n^2 - 5n - 2)} - A\varepsilon ,$$

for otherwise (7.13) is true. Then, by (7.115), (7.113) is satisfied for these values of $t_1, \ldots t_5$ and since $t_6 = t_7 = o$, (7.112) is also satisfied. Therefore, by (7.114), (7.116),

$$20 - n + 4a\lambda + A\varepsilon > -a\lambda - A\varepsilon,$$

i.e. $$\lambda > \frac{n - 20}{5a} - A\varepsilon = \frac{5n^3 - 106n^2 + 117n + 60}{5(2n^3 + 8n^2 - 5n - 2)} - A\varepsilon$$

$$> f(n,\ 7)$$

which proves (7.13) and hence the result in this case.

We now suppose $\phi_{\rho\rho} > o$, $2 \leqslant \rho \leqslant 5$; since $r = 6$, this implies that either ϕ_{66} or ϕ_{77} is negative (or possibly both).

<u>Case B</u>: $\underline{\phi_{66} \leqslant o}$. Consider the form $\phi(t_1, \ldots t_4, o, t_6, o)$. As in Case A, by Lemma 7.6 there exist integers $t_1, \ldots t_4, t_6$ (not all o), satisfying

$$(7.117) \qquad |\phi_{11}t_1^2 + \ldots + \phi_{44}\, t_4^2 + \phi_{66}\, t_6^2| < \tfrac{1}{2} N^{-a\lambda}$$

with

$$(7.118) \qquad |t_\rho| \ll \begin{cases} N^{10 + \frac{\theta_5}{n} - (\rho-1) + 2a\lambda + A\varepsilon}, & 1 \leqslant \rho \leqslant 4, \\ N^{6 + 2a\lambda + A\varepsilon} & \rho = 6. \end{cases}$$

Then, by (7.110), (7.118), for $1 \leqslant \sigma \leqslant 4$, $\rho = 1, 2, 3, 4, 6$, $\sigma < \rho$, we obtain

$$(7.119) \qquad |\phi_{\rho\sigma}\, t_\rho\, t_\sigma| \ll N^{20 + \frac{2\theta_5}{n} - n + 4a\lambda + A\varepsilon} .$$

We may suppose that

$$\lambda < \frac{a - (10 + \frac{\theta_5}{n})}{2a} - A\varepsilon = \frac{2n^3 - 43n^2 + 35n + 28}{2(2n^3 + 8n^2 - 5n - 2)} - A\varepsilon$$

for otherwise (7.13) is true. Then, by (7.118), (7.113) is satisfied for these values of $t_1, \ldots t_4, t_6$, and, since $t_5 = t_7 = o$, (7.112) is also satisfied. Therefore, by (7.117), (7.119),

$$20 + \frac{2\theta_5}{n} - n + 4a\lambda + A\varepsilon > - a\lambda - A\varepsilon$$

$$\text{i.e. } \lambda > \frac{n - (20 + \frac{2\theta_5}{n})}{5a} - A\varepsilon = \frac{5n^3 - 108n^2 + 77n + 60}{5(2n^3 + 8n^2 - 5n - 2)} - A\varepsilon$$

$$> f(n, 7)$$

which proves (7.13) and hence the result in this case.

<u>Case C</u>: $\underline{\phi_{77} < o}$: Consider the form $\Phi(t_1, \ldots t_4, o, o, t_7)$. By Lemma 7.6 there exist integers $t_1, \ldots t_4, t_6$ (not all o), satisfying

(7.120) $$|\phi_{11}t_1^2 + \ldots + \phi_{44}t_4^2 + \phi_{77}t_7^2| < \tfrac{1}{2} N^{-a\lambda},$$

with

(7.121) $$|t_\rho| \ll \begin{cases} N^{10 + \frac{1}{n}(\theta_5 + \theta_6) - (\rho - 1) + 2a\lambda + A\varepsilon}, & 1 \leqslant \rho \leqslant 4, \\ N^{6 + 2a\lambda + A\varepsilon}, & \rho = 7. \end{cases}$$

By (7.110), (7.121), for $1 \leqslant \sigma \leqslant 4$, $\rho = 1, 2, 3, 4, 7$, $\sigma < \rho$,

(7.122) $$|\phi_{\rho\sigma} t_\rho t_\sigma| \ll N^{20 - n + 4a\lambda + \frac{2}{n}(\theta_5 + \theta_6) + A\varepsilon}.$$

We may suppose that

$$\lambda < \frac{a - \left(10 + \frac{\theta_5 + \theta_6}{n}\right)}{2} - A\varepsilon = \frac{n^3 - 22n^2 + 7n + 4}{2n^3 + 8n^2 - 5n - 2} - A\varepsilon$$

for otherwise (7.13) is true. Then, by (7.121), (7.113) is satisfied for these values of $t_1, \ldots t_4, t_7$, and, since $t_5 = t_6 = o$, (7.12) is also satisfied. Therefore, by (7.122), (7.120),

$$20 - n + 4a\lambda + \frac{2}{n}(\theta_5 + \theta_6) + A\varepsilon \geqslant -a\lambda - A\varepsilon,$$

i.e. $$\lambda > \frac{n - \left(20 + \frac{2}{n}(\theta_5 + \theta_6)\right)}{5a} - A\varepsilon = \frac{n^3 - 22n^2 + 7n + 4}{2n^3 + 8n^2 - 5n - 2} - A\varepsilon$$

$$= f(n, 7)$$

which proves (7.13) and hence completes the proof of part (e) of the theorem.

The proof of Theorem 25 is now complete. We note that the method of proof of parts (a), (b), (f) of the theorem follows closely the methods of Davenport [18] and Davenport and Ridout [21], and that the method of proof of parts (c), (d), (e) follow closely the methods of Birch and Davenport [1] and Ridout [31].

BIBLIOGRAPHY

1. BIRCH, B.J. & DAVENPORT H., Indefinite quadratic forms in many variables, Mathematika 5 (1958), 8-12.

2. BIRCH, B.J. & DAVENPORT, H., Quadratic equations in several variables, Proc. Camb. Phil. Soc. 54 (1958), 135-138.

3. BIRCH, B.J. & DAVENPORT, H., On a theorem of Davenport and Heilbronn, Acta Mathematica 100 (1958), 259-279.

4. CASSELS, J.W.S., An introduction to Diophantine approximation, Cambridge 1957.

5. COOK, R. J., On the fractional parts of a set of points, Mathematika 19 (1972), 63-68.

6. COOK, R.J., On the fractional parts of a set of points II, Pac. Journ. Maths. 45 (1973), 81-85.

7. COOK, R.J., On the fractional parts of a set of points III, Journ. Lond. Math. Soc. Ser. 2 9(1974), 490-494.

8. COOK, R.J., On the fractional parts of a set of points IV, (pre-print).

9. COOK, R.J., A note on diophantine approximation, (pre-print).

10. COOK, R.J., The fractional parts of a cubic form, (pre-print).

11. COOK, R.J., Simultaneous approximation to two cubic forms, (pre-print).

12. COOK, R.J., On the fractional parts of a polynomial, (pre-print).

13. DANICIC, I., Ph.D. thesis, London 1957.

14. DANICIC, I., An extension of a theorem of Heilbronn, Mathematika 5 (1958), 30-37.

15. DANICIC, I., On the fractional parts of θx^2 and ϕx^2, Journ. Lond. Math. Soc. 34 (1959), 353-357.

16. DANICIC, I., The distribution (mod 1) of pairs of quadratic forms with integer variables, Journ. Lond. Math. Soc. 42 (1967), 618-623.

17. DAVENPORT, H., On a theorem of Heilbronn, Quart. J. Math. Oxford (2) 18 (1967), 339-344.

18. DAVENPORT, H., Analytic methods for diophantine equations and diophantine inequalities, Ann Arbor, Michigan 1962.

19. DAVENPORT, H., Indefinite quadratic forms in many variables, Mathematika 3 (1956), 81-101.

20. DAVENPORT, H., Indefinite quadratic forms in many variables (II), Proc. Lond. Math. Soc. (3) 8 (1958), 109-126.

21. DAVENPORT, H. & RIDOUT.D., Indefinite quadratic forms, Proc. Lond. Math. Soc. (3) 9 (1959), 544-555.

22. HARDY, G.H. & LITTLEWOOD, J.E., Some problems of diophantine approximation, Acta. Math. 37 (1914), 155-191.

23. HEILBRONN, H., On the distribution of the sequence $n^2\theta$ (mod 1), Quart. J. Math. Oxford 19 (1948), 249-256.

24. HUA, L.K., An improvement of Vinogradov's mean-value theorem and several applications, Quart. J. Math. Oxford 20 (1949), 48-61.

25. HUA, L.K., Additive Theory of Prime Numbers, Amer. Math. Soc., Rhode Island, 1965.

26. LANDAU, E., Vorleslungen uber Zahlentheorie, Bd 1, Leipzig 1927.

27. LIU, M.C., On a theorem of Heilbronn concerning the fractional part of θn^2, Can. J. Math. 22 (1970), 784-788.

28. LIU, M.C., On the fractional parts of θn^k and ϕn^k, Quart. J. Math. Oxford (2) 21 (1970), 481-486.

29. LIU, M.C., Simultaneous approximation of two additive forms, Proc. Camb. Phil. Soc. 75 (1974), 77-82.

30. LIU, M.C., Simultaneous approximation of additive forms, Trans. Amer. Math. Soc. 206 (1975), 361-373.

31. RIDOUT, D., Indefinite quadratic forms, Mathematika 5 (1958), 122-124.

32. VINOGRADOV, I.M., Analytischer Beweis des Satzes über die Verteilung der Bruchteile eines ganzen Polynoms, Bull. Acad. Sci. U.S.S.R. (6) 21 (1927), 567-578.

33. VINOGRADOV, I.M., The method of trigonometric sums in the theory of numbers, Wiley - Interscience 1954.