Feedback and information gain in single atom cavity QED

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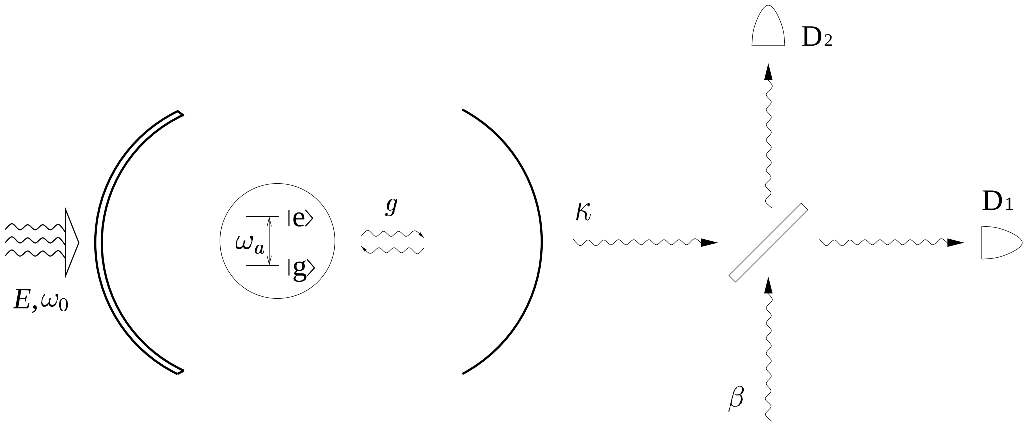

Figure 1: Dyne measurements in cavity QED. The cavity resonant frequency is ω_c and the cavity field decay rate is κ. The cavity is driven by an external laser field of frequency ω_0 and strength E. A two level atom with resonant frequency ω_a is trapped insight the cavity. The strength of the atom-cavity coupling is g. The cavity output field is analyzed by the detectors D_1 and D_2 after being added to the reference field β on the beam splitter.

1 System

The physical system under investigation consists of a single two-level atom located inside a high-finesse optical cavity, which is externally driven. This system can be characterized by five real parameters (see Fig 1). These are the strength of the atom-cavity coupling g, the cavity field decay rate κ, the driving strength E, and two detunings: the detuning of the atomic resonance ω_a from the driving laser frequency ω_0, $\Delta_a = \omega_a - \omega_0$, and the detuning of the cavity resonance ω_c from the driving laser frequency, $\Delta_c = \omega_c - \omega_0$. In the frame rotating at the driving laser frequency, the system evolution in the absence of measurements is described by the master equation

$$\dot{\rho} = -i[H, \rho] + \kappa(2a\rho a^\dagger - \{a^\dagger a, \rho\}) \,, \tag{1}$$

where we use square and curly brackets to denote commutator and anticommutator, respectively, ρ is the joint density operator for the atom and the intracavity field, and

$$H = \Delta_c a^\dagger a + \Delta_a \sigma^\dagger \sigma + iE(a^\dagger - a) + ig(a^\dagger \sigma - \sigma^\dagger a) \,. \tag{2}$$

The strength of the atom-cavity coupling, g, depends on the position of the atom inside the cavity, whereas the driving strength, E, can be easily varied during the experiment.

2 Problems of control

From the point of view of control theory [4], a physical system is a mapping that maps any *input* signal to some *output* signal. In addition to that the system is characterized by its state. A typical problem is to find a form of input signal that achieves the desired transformation of the system state given that the information about the state can only be obtained from the output signal.

In our case, for example, it is convenient to choose the strength of the driving laser E as the input signal. It can be easily modified during the course of experiment. The photocurrents registered by photodetectors D_1 and D_2 constitute the output signal. What we mean by the system state depends on the application. If, for example, we are interested in controlling atomic motion then atomic position inside the cavity should be chosen as the system state. Alternatively, one can consider the state of the joint quantum system consisting of the internal atomic degrees of freedom and the cavity mode.

2.1 Atomic motion

As the atom slowly moves inside the cavity, the strength g of the atom-cavity coupling changes depending on the position of the atom $\boldsymbol{r}= (x, y, z)$ through

$$g(\boldsymbol{r}) = g_0 \cos(2\pi x/\lambda) \exp(-(y^2 + z^2)/w^2) \,, \tag{3}$$

where the x-axis coincides with the optical axis of the cavity. At the time scale of the atomic motion the quantum system consisting of intracavity field and atomic internal degrees of freedom reaches the steady state, $\rho_{\rm ss}$, corresponding to the current value of g. The average force acting on the atom can therefore be calculated as

$$\boldsymbol{F} = -\boldsymbol{\nabla} \mathrm{tr}[ig(a^\dagger\sigma - \sigma^\dagger a)\rho_{\rm ss}] \,, \tag{4}$$

i.e. as a minus gradient of the potential energy of atom-cavity interaction. Because the potential along the x-axis is rather steep, the atomic motion is confined to the yz plane, and moreover, due to the cylindric symmetry of the setup, only the radial component of the force is relevant. Using (3) and (4) we have that the radial component of the force is given by

$$\begin{aligned} F_r &= -\frac{\partial}{\partial r}\mathrm{tr}[ig_0 e^{-r^2/w^2}(a^\dagger\sigma - \sigma^\dagger a)\rho_{\rm ss}] \\ &= i2g_0 r e^{-r^2/w^2}\langle a^\dagger\sigma - \sigma^\dagger a\rangle_{\rho_{\rm ss}}/w^2 \,, \end{aligned} \tag{5}$$

where $\langle a^\dagger\sigma - \sigma^\dagger a\rangle_{\rho_{\rm ss}} = \mathrm{tr}[(a^\dagger\sigma - \sigma^\dagger a)\rho_{\rm ss}]$. It is clear from this equation that the knowledge of an analytic expression of the steady state $\rho_{\rm ss}$ would give us an analytic expression for the force. The atomic motion could then be described in analytical terms simply by using Newton's Second Law. Thereby we reduce the problem of control of the atomic motion to a standard nonlinear problem of classical feedback control.

2.2 Optical information

Observability is an important concept of classical control theory which is used to describe the situation when the system state can be unambiguously determined from the output data [4]. From the point of view of information theory it appears natural that the concept of observability may be further developed by analyzing the amount of information in the output signal about the system state.

Considering our cavity QED setup we computed two, as it turns out, complementary quantities [9]. First, we calculated the optimal rate at which a homodyne measurement provides information about the quantum state of the system composed of the electromagnetic field and the internal state of the atom. Second, we calculated the optimal rate at which the measurement gives information about the coupling strength between the atom and the intra-cavity field. We found that the second quantity coincides with the so-called *optical information* [7, 3], which was introduced by Kimble to characterize the performance of the *atom-cavity microscope*. The main idea behind the atom-cavity microscope is to use the relation (3) to infer the atomic trajectory from the observed photocurrents. The photocurrents depend on the strength of the atom-cavity coupling g which, in its turn, depends on the location of the atom inside the cavity Eq. (3). The resulting dependence of the measured photocurrents on the atomic position gives all the necessary tools for infering the trajectory of the atom as it moves inside the cavity [6, 8]. Optical information [7, 3] measures the rate at which the measurement provides information about the system. In Refs. [7, 3], however, no formal definition of optical information is given, and only a heuristic derivation of its value is provided. We have established a precise mathematical framework in which optical information acquires the meaning of the information rate provided by the measurement about atom-cavity coupling. Further analysis reveals a tradeoff between the information gain about the quantum state of the system and the information gain about the atom-cavity coupling which is determined by the position of the atom inside the cavity. These results have been accepted for publication (see Ref. [9]). In section 4 of this report we present some further analysis on optical information which was not included in [9].

Like in the case of feedback control of atomic motion, our results on optical information make use of an approximate analytic expression for the steady state of equation Eq. (1).

3 Steady state

It is important to note that all our calculations can be easily performed for any parameter regime provided that an analytic expression for the corresponding steady state is available. In this section we present a systematic procedure for verifying whether a given density matrix represents a steady state of Eq. (1). Using this procedure we have found a better analytic approximation to the steady state than previously known. The problem of finding the exact analytical expression for the steady state for an arbitrary parameter regime, however, remains an open problem.

For any operator A the time derivative of the expectation $\langle A\rangle_\rho = \mathrm{tr}[A\rho]$ can be computed using the master equation (1)

$$\begin{aligned}\frac{d\langle A\rangle_\rho}{dt} &= \mathrm{tr}[A\dot{\rho}] = \mathrm{tr}\Big(-iA[H,\rho] + \kappa A(2a\rho a^\dagger - \{a^\dagger a, \rho\})\Big) \\ &= \big\langle -i[A,H] + \kappa(2a^\dagger A a - \{A, a^\dagger a\})\,\big\rangle_\rho\,,\end{aligned} \tag{6}$$

where we have used the cyclic property of the trace. A state ρ_{ss} is a steady state of Eq. (1) if and only if

$$\frac{d}{dt}\langle A\rangle_{\rho_{\mathrm{ss}}} = 0 \tag{7}$$

for any operator A. For any operator A there is a set of complex coefficients $\{\lambda_{mnk}\}$ such that

$$A = \sum_{m,n,k} \lambda_{mnk} (a^\dagger)^n a^m \sigma_k \tag{8}$$

where $m, n = 0, 1, 2, 3, \ldots$ and $\sigma_k = \mathbb{1}, \sigma_x, \sigma_y, \sigma_z$. Since, in general, the coefficients $\{\lambda_{mnk}\}$ are arbitrary, Eq. (7) is equivalent to

$$\frac{d}{dt}\langle (a^\dagger)^n a^m \sigma_k\rangle_{\rho_{\mathrm{ss}}} = 0\,, \quad m,n = 0,1,2,3,\ldots, \quad \sigma_k = \mathbb{1}, \sigma_x, \sigma_y, \sigma_z\,. \tag{9}$$

It turns out that the problem of finding an analytic expression for the steady state of Eq. (1) in the most general case is rather difficult. In what follows we focus our attention on an important case where the driving laser, the cavity, and the atom are all resonant, i.e. $\Delta_a = \Delta_c = 0$. In this case we know that in the strong driving regime ($E/g \gg 1$) the system approaches a steady state of the form [1]

$$\rho_{\mathrm{ss}}^\alpha = \frac{1}{2}(|\alpha;+\rangle\langle\alpha;+| + |\alpha^*;-\rangle\langle\alpha^*;-|)\,, \tag{10}$$

where $|\alpha;+\rangle$ and $|\alpha^*;-\rangle$ are two orthogonal quantum states

$$\begin{aligned}|\alpha;+\rangle &= \frac{1}{\sqrt{2}}|\alpha\rangle(|\mathrm{g}\rangle + i|\mathrm{e}\rangle)\,, \\ |\alpha^*;-\rangle &= \frac{1}{\sqrt{2}}|\alpha^*\rangle(|\mathrm{g}\rangle - i|\mathrm{e}\rangle)\,,\end{aligned} \tag{11}$$

where $|\alpha\rangle$ is the coherent field state with amplitude

$$\alpha = \frac{E}{\kappa}\left[1 - (\frac{g}{2E})^2 + i\frac{g}{2E}\sqrt{1 - (\frac{g}{2E})^2}\,\right]\,. \tag{12}$$

The main result of this section is a slight generalization of the above formula for the steady state. This is achieved by finding the parameters in an ansatz for the steady state that has a similar structure as $\rho_{\mathrm{ss}}^{\alpha}$ but is slightly more general. Let us define an auxiliary state

$$\tilde{\rho}_{\mathrm{ss}} = |\alpha\rangle\langle\alpha| \otimes |\theta,\phi\rangle\langle\theta,\phi| \,, \tag{13}$$

where $|\alpha\rangle$ is a coherent field state and

$$|\theta,\phi\rangle = \cos\theta\,|\mathrm{g}\rangle - e^{i\phi}\sin\theta\,|\mathrm{e}\rangle \tag{14}$$

is an arbitrary pure state of the two-level atom. Replacing H in the master equation with the all-on-resonance Hamiltonian

$$H_0 = iE(a^\dagger - a) + ig(a^\dagger\sigma - \sigma^\dagger a) \tag{15}$$

we have

$$\begin{aligned} E\frac{d\big\langle(a^\dagger)^n a^m \sigma_k\big\rangle_\rho}{dt} &= m\Big\langle (a^\dagger)^n a^{m-1}\sigma_k + \frac{g}{E}(a^\dagger)^n a^{m-1}\sigma_k\sigma - \frac{\kappa}{E}(a^\dagger)^n a^m\sigma_k\Big\rangle_\rho \\ &+ n\Big\langle (a^\dagger)^m a^{n-1}\sigma_k + \frac{g}{E}(a^\dagger)^m a^{n-1}\sigma_k\sigma - \frac{\kappa}{E}(a^\dagger)^m a^n\sigma_k\Big\rangle_\rho \\ &+ \frac{g}{E}\Big\langle (a^\dagger)^{n+1}a^m[\sigma_k,\sigma] + (a^\dagger)^n a^{m+1}[\sigma^\dagger,\sigma_k]\Big\rangle_\rho \,. \end{aligned} \tag{16}$$

Direct calculations show

<u>$\sigma_k := \mathbb{1}$</u>

$$\begin{aligned} E\frac{d}{dt}\langle(a^\dagger)^n a^m\rangle_{\tilde{\rho}_{\mathrm{ss}}} &= m\,(\alpha^*)^n\alpha^{m-1}\Big(1 - \frac{\kappa}{E}\alpha - \frac{g}{2E}e^{i\phi}\sin 2\theta\Big) \\ &+ n\,(\alpha^*)^{n-1}\alpha^m\Big(1 - \frac{\kappa}{E}\alpha^* - \frac{g}{2E}e^{-i\phi}\sin 2\theta\Big) \,. \end{aligned} \tag{17}$$

<u>$\sigma_k := \sigma_z = |\mathrm{g}\rangle\langle\mathrm{g}| - |\mathrm{e}\rangle\langle\mathrm{e}|$</u>

$$\begin{aligned} E\frac{d}{dt}\langle(a^\dagger)^n a^m\sigma_z\rangle_{\tilde{\rho}_{\mathrm{ss}}} &= m\,(\alpha^*)^n\alpha^{m-1}\Big((1 - \frac{\kappa}{E}\alpha)\cos 2\theta - \frac{g}{2E}e^{i\phi}\sin 2\theta\Big) \\ &+ n\,(\alpha^*)^{n-1}\alpha^m\Big((1 - \frac{\kappa}{E}\alpha^*)\cos 2\theta - \frac{g}{2E}e^{-i\phi}\sin 2\theta\Big) \\ &- \frac{g}{E}(\alpha^*)^n\alpha^m\big(\alpha^* e^{i\phi} + \alpha e^{-i\phi}\big) \,. \end{aligned} \tag{18}$$

<u>$\sigma_k := \sigma_x = \sigma^\dagger + \sigma$</u>

$$\begin{aligned} E\frac{d}{dt}\langle(a^\dagger)^n a^m\sigma_x\rangle_{\tilde{\rho}_{\mathrm{ss}}} &= m\,(\alpha^*)^n\alpha^{m-1}\Big(\frac{g}{E}\sin^2\theta - (1 - \frac{\kappa}{E}\alpha)\sin 2\theta\cos\phi\Big) \\ &+ n\,(\alpha^*)^{n-1}\alpha^m\Big(\frac{g}{E}\sin^2\theta - (1 - \frac{\kappa}{E}\alpha^*)\sin 2\theta\cos\phi\Big) \\ &- \frac{g}{E}(\alpha^*)^n\alpha^m\big(\alpha^* + \alpha\big)\cos 2\theta \,. \end{aligned} \tag{19}$$

$\underline{\sigma_k := \sigma_y = i(\sigma^\dagger - \sigma)}$

$$
\begin{aligned}
E\frac{d}{dt}\langle (a^\dagger)^n a^m \sigma_y\rangle_{\tilde\rho_{\rm ss}} = {} & m\,(\alpha^*)^n\alpha^{m-1}\Big(i\frac{g}{E}\sin^2\theta - (1-\frac{\kappa}{E}\alpha)\sin 2\theta\sin\phi\Big) \\
& + n\,(\alpha^*)^{n-1}\alpha^m\Big(-i\frac{g}{E}\sin^2\theta - (1-\frac{\kappa}{E}\alpha^*)\sin 2\theta\sin\phi\Big) \\
& + i\frac{g}{E}(\alpha^*)^n\alpha^m\big(\alpha-\alpha^*\big)\cos 2\theta\,.
\end{aligned}
\tag{20}
$$

A state of the form $\tilde\rho_{\rm ss}$ is a steady state of Eq. (1) if and only if

$$
\frac{d}{dt}\langle (a^\dagger)^n a^m \sigma_k\rangle_{\tilde\rho_{\rm ss}} = 0 \tag{21}
$$

for all $m, n = 0, 1, 2, 3, \ldots$ and for any $\sigma_k = \mathbb{1}, \sigma_x, \sigma_y, \sigma_z$. First we consider the cases when only field or only the atomic operators are used, i.e. the moments of the form $(a^\dagger)^n a^m$ and σ_k. From Eq. (17) we have

$$
\alpha = \frac{E}{\kappa}\Big(1 - \frac{g}{2E}e^{i\phi}\sin 2\theta\Big)\,. \tag{22}
$$

this is sufficient for all the moments of the form $\langle (a^\dagger)^n a^m\rangle_\rho$ to have zero time derivative at $\rho = \tilde\rho_{\rm ss}$. Analogous stationarity conditions for the atomic moments ($m = n = 0$) read

$$
\cos 2\theta = 0\,, \tag{23}
$$

from equations (19) and (20), and

$$
\alpha^* e^{i\phi} + \alpha e^{-i\phi} = 0\,, \tag{24}
$$

from Eq. (18). Together with Eq. (22) this implies

$$
\alpha = \frac{E}{\kappa}\, e^{i(\phi-\pi/2)}\sin\phi\,, \tag{25}
$$

where

$$
\cos\phi = \frac{g}{2E}\,. \tag{26}
$$

These equations give two possible values for $\tilde\rho_{\rm ss}$:

$$
\begin{aligned}
\tilde\rho^1_{\rm ss} &= |\alpha\rangle\langle\alpha| \otimes |\,\pi/4, \phi\rangle\langle\,\pi/4, \phi|\,, \\
\tilde\rho^2_{\rm ss} &= |\alpha^*\rangle\langle\alpha^*| \otimes |\,\pi/4, -\phi\rangle\langle\,\pi/4, -\phi|\,,
\end{aligned}
\tag{27}
$$

where α is given by Eq. (25) and where we fix

$$
\phi = \arccos(g/2E)\,. \tag{28}
$$

In summary, all the moments that are made up of either atomic *or* field operators have zero time derivative at any state of the form

$$\rho = p_1 \tilde{\rho}_{\mathrm{ss}}^1 + p_2 \tilde{\rho}_{\mathrm{ss}}^2 \, . \tag{29}$$

Choosing $p_1 = p_2 = 1/2$ we see that in the strong driving limit, $g/E \ll 1$, this state approaches the state (10) as required. Compared to Eq. (10) this gives us a better approximation of the steady state due to the exact satisfaction of an infinite subfamily of equations (21).

4 Information gain

In this section we present a systematic procedure for calculating time derivatives of an arbitrary fixed order of the information gain provided by the measurements about the quantum state of the "atom+intracavity mode" system. This procedure is essential for an in-depth analysis of the limitations as well as further generalizations of our results on optical information presented in Ref. [9].

The evolution of the state, ρ, of an open quantum system subject to a continuous measurement can often be described by a stochastic master equation of the form [2, 11, 10]

$$d\rho = \mathcal{L}(\rho)dt + \mathcal{M}(\rho)dW \, , \tag{30}$$

which is understood in the sense of the Itô stochastic differential calculus [5]. Any particular measurement record is represented by some realization of the stochastic process $W(t)$. The superoperator $\mathcal{L}$ is linear and defines the "unconditional" evolution, i.e., the evolution in the absence of measurements as given by Eq. (1). In the all-on-resonance case we have

$$\mathcal{L}(\rho) = \left[E(a^\dagger - a) + g(a^\dagger \sigma - \sigma^\dagger a), \rho\right] + \kappa(2a\rho a^\dagger - \{a^\dagger a, \rho\}) \, . \tag{31}$$

By contrast, the superoperator $\mathcal{M}$ is nonlinear and accounts for the effects of the measurement. In our case

$$\mathcal{M}(\rho) = \sqrt{2\kappa\eta} \left(e^{-i\phi} a\rho + e^{i\phi} \rho a^\dagger - \mathrm{tr}\left[\rho(e^{-i\phi} a + e^{i\phi} a^\dagger)\right] \rho \right) \, , \tag{32}$$

where η is the efficiency of the photodetection and $\phi = \arg \beta$ is the phase of the reference field.

By definition, the entropy of a system described by a density matrix ρ is

$$\begin{aligned} H(\rho) &\equiv -\mathrm{tr}(\rho \ln \rho) \\ &= \mathrm{tr}[\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \rho(\rho - \mathbb{1})^n] \end{aligned}$$

$$= \mathrm{tr}[\sum_{n=1}^{\infty}\frac{(-1)^n}{n}(\tilde{\rho}^{n+1}+\tilde{\rho}^n)]\,, \tag{33}$$

where $\tilde{\rho} = \rho - \mathbb{1}$. The average entropy change in the presence of continuous observations is given by

$$\langle H_1(\rho)\rangle = \lim_{\Delta t\to 0}\left\langle\frac{H(\rho+\delta)-H(\rho)}{\Delta t}\right\rangle \tag{34}$$

where the average $\langle\cdot\rangle$ is taken over all possible measurement outcomes observed over the time Δt, and

$$\delta = \mathcal{L}(\rho)\Delta t + \mathcal{M}(\rho)\Delta W\,. \tag{35}$$

Let ρ_0 be the state of the system at time t_0. Using the Fokker-Plank equation [5] that corresponds to the stochastic master equation (30) one can show that during the period of time $t - t_0 = \Delta t$ the average change of system entropy is given by the Taylor-like series

$$\langle\Delta H(\rho_0)\rangle = \langle H_1(\rho_0)\rangle\Delta t + \langle H_2(\rho_0)\rangle\frac{(\Delta t)^2}{2!} + \cdots + \langle H_n(\rho_0)\rangle\frac{(\Delta t)^n}{n!} + O(\Delta t^{n+1})\,, \tag{36}$$

where the coefficients obey the recursive relation

$$\Big\langle H_{n+1}(\rho)\Big\rangle = \lim_{\Delta t\to 0}\Big\langle\frac{\langle H_n(\rho+\delta)\rangle - \langle H_n(\rho)\rangle}{\Delta t}\Big\rangle\,. \tag{37}$$

In an earlier paper [9] we have shown that

$$\begin{aligned}\langle H_1(\rho)\rangle &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\mathrm{tr}\Big(\,[(n+1)\tilde{\rho}^n + n\tilde{\rho}^{n-1}]\mathcal{L}(\rho) \qquad (38)\\ &+ \sum_{s=0}^{n-1}(s+1)\tilde{\rho}^s\mathcal{M}(\rho)\tilde{\rho}^{n-1-s}\mathcal{M}(\rho) + \sum_{s=0}^{n-2}(s+1)\tilde{\rho}^s\mathcal{M}(\rho)\tilde{\rho}^{n-2-s}\mathcal{M}(\rho)\Big)\,.\end{aligned}$$

This formula is valid without any restrictions on the superoperators $\mathcal{L}$ and $\mathcal{M}$, and hence the above recursive formula provides a systematic procedure for calculating higher order corrections to $\langle\Delta H(\rho_0)\rangle$.

4.1 Higher order terms

As an illustration of the above recursive procedure let us compute $\langle H_2(\rho_{\mathrm{ss}})\rangle$. According to Eqs. (37) and (38)

$$\begin{aligned}\langle H_2(\rho)\rangle &= \lim_{\Delta t\to 0}\Big\langle\frac{\langle\dot{H}(\rho+\delta)\rangle - \langle\dot{H}(\rho)\rangle}{\Delta t}\Big\rangle\\ &= \lim_{\Delta t\to 0}\sum_{n=1}^{\infty}\frac{(-1)^n}{n\,\Delta t}\big(\lambda(\tilde{\rho},\delta,n) + \mu(\tilde{\rho},\delta,n) + \mu(\tilde{\rho},\delta,n-1)\big)\,, \qquad (39)\end{aligned}$$

where

$$\lambda(\tilde{\rho},\delta,n) \equiv \mathrm{tr}\Big\langle \big[(n+1)(\tilde{\rho}+\delta)^n + n(\tilde{\rho}+\delta)^{n-1}\big]\mathcal{L}(\rho+\delta) - \big[(n+1)\tilde{\rho}^n + n\tilde{\rho}^{n-1}\big]\mathcal{L}(\rho)\Big\rangle\,, \quad (40)$$

and

$$\mu(\tilde{\rho},\delta,n) \equiv \sum_{s=0}^{n-1}(s+1)\mathrm{tr}\Big\langle (\tilde{\rho}+\delta)^s\mathcal{M}(\rho+\delta)(\tilde{\rho}+\delta)^{n-1-s}\mathcal{M}(\rho+\delta) - \tilde{\rho}^s\mathcal{M}(\rho)\tilde{\rho}^{n-1-s}\mathcal{M}(\rho)\Big\rangle\,. \quad (41)$$

This expression for $\langle H_2(\rho)\rangle$ is valid for any state ρ and for any superoperators $\mathcal{L}$ and $\mathcal{M}$. Using the fact that $\mathcal{L}$ is linear and that $\mathcal{L}(\rho_{\rm ss})=0$ we find

$$\lambda(\tilde{\rho}_{\rm ss},\delta_{\rm ss},n) = \mathrm{tr}\Big\langle \big[(n+1)(\tilde{\rho}_{\rm ss}+\delta_{\rm ss})^n + n(\tilde{\rho}_{\rm ss}+\delta_{\rm ss})^{n-1}\big]\mathcal{L}(\delta_{\rm ss})\Big\rangle\,, \quad (42)$$

where

$$\delta_{\rm ss} = \mathcal{L}(\rho_{\rm ss})\Delta t + \mathcal{M}(\rho_{\rm ss})\Delta W = \mathcal{M}(\rho_{\rm ss})\Delta W. \quad (43)$$

In what follows we also assume that $[\rho_{\rm ss},\delta_{\rm ss}]=0$, which is true in the case of Eq. (10). Keeping terms to the second order in $\delta_{\rm ss}$ we thus obtain

$$\lambda(\tilde{\rho}_{\rm ss},\delta_{\rm ss},n) = \mathrm{tr}\Big\langle \big[(n+1)(\tilde{\rho}_{\rm ss}^n + n\tilde{\rho}_{\rm ss}^{n-1}\delta_{\rm ss}) + n(\tilde{\rho}_{\rm ss}^{n-1} + (n-1)\tilde{\rho}_{\rm ss}^{n-2}\delta_{\rm ss})\big]\mathcal{L}(\delta_{\rm ss}) + O(\delta_{\rm ss}^3)\Big\rangle\,. \quad (44)$$

Since $\langle\mathcal{L}(\delta_{\rm ss})\rangle = 0$,

$$\lambda(\tilde{\rho}_{\rm ss},\delta_{\rm ss},n) = \mathrm{tr}\Big\langle n\big[(n+1)\tilde{\rho}_{\rm ss}^{n-1} + (n-1)\tilde{\rho}_{\rm ss}^{n-2}\big]\delta_{\rm ss}\mathcal{L}(\delta_{\rm ss}) + O(\delta_{\rm ss}^3)\Big\rangle\,. \quad (45)$$

Because $\tilde{\rho}_{\rm ss}$ is proportional to the identity this equations has the form

$$\lambda(\tilde{\rho}_{\rm ss},\delta_{\rm ss},n) = \mathrm{tr}\Big\langle \delta_{\rm ss}\mathcal{L}(\delta_{\rm ss})\Big\rangle f(n) + O(\Delta t^2)\,, \quad (46)$$

where f is some function whose precise form we will not need in these notes. Using Eq. (31) it is easy to show that for any state of the form (10) we have $\mathrm{tr}\big\langle\delta_{\rm ss}\mathcal{L}(\delta_{\rm ss})\big\rangle = 0$ and so we can neglect $\lambda(\tilde{\rho},\delta,n)$ in Eq. (39). It now remains to calculate the μ-terms.

$$\begin{aligned}\mu(\tilde{\rho}_{\rm ss},&\delta_{\rm ss},n)\\ &= \sum_{s=0}^{n-1}(s+1)\mathrm{tr}\Big\langle\Big[\mathcal{M}(\rho_{\rm ss}+\delta_{\rm ss})\tilde{\rho}^s + s\delta_{\rm ss}\mathcal{M}(\rho_{\rm ss}+\delta_{\rm ss})\tilde{\rho}^{s-1} + \frac{s(s-1)}{2}\delta_{\rm ss}^2\mathcal{M}(\rho+\delta)\tilde{\rho}^{s-2}\Big]\\ &\qquad\times\Big[\tilde{\rho}_{\rm ss}^{n-1-s} + (n-s-1)\tilde{\rho}_{\rm ss}^{n-s-2}\delta_{\rm ss} + \frac{(n-s-1)(n-s-2)}{2}\tilde{\rho}_{\rm ss}^{n-s-3}\Big]\\ &\qquad\times\mathcal{M}(\rho_{\rm ss}+\delta_{\rm ss}) - \tilde{\rho}_{\rm ss}^{n-1}\big(\mathcal{M}(\rho_{\rm ss})\big)^2 + O(\delta_{\rm ss}^3)\Big\rangle\end{aligned} \quad (47)$$

Direct calculations show that

$$\mu(\tilde\rho_{\rm ss},\delta_{\rm ss},n) = \frac{n(n+1)}{2}\mathrm{tr}\Big(\tilde\rho_{\rm ss}^{n-1}\Big\langle\big(\mathcal{M}(\rho_{\rm ss}+\delta_{\rm ss})\big)^2-(\mathcal{M}(\rho_{\rm ss}))^2\Big\rangle \\ +(n-1)\tilde\rho_{\rm ss}^{n-2}\Big\langle\delta_{\rm ss}\big(\mathcal{M}(\rho_{\rm ss}+\delta_{\rm ss})\big)^2\Big\rangle\Big)+O(\Delta t^2) \quad (48)$$

The general expression for the second time derivative is obtained by substituting this result into Eq. (39) and calculating the sum over n, neglecting $\lambda(\tilde\rho_{\rm ss},\delta_{\rm ss},n)$ as explained above.

Now we will show that for any initial state of the form $\rho_{\rm ss}=\rho^\alpha_{\rm ss}$ we have $\langle H_2(\rho^\alpha_{\rm ss})\rangle=0$.

$$\mathcal{M}(\rho+\delta)=\mathcal{M}(\rho)+\mathcal{M}(\delta)+\mathcal{N}(\rho,\delta)\,, \quad (49)$$

where

$$\mathcal{N}(\rho,\delta)\equiv-\sqrt{2\kappa\eta}\left(\mathrm{tr}\big[\rho(e^{-i\phi}a+e^{i\phi}a^\dagger)\big]\,\delta+\mathrm{tr}\big[\delta(e^{-i\phi}a+e^{i\phi}a^\dagger)\big]\,\rho\right)\,, \quad (50)$$

Let $\alpha=|\alpha|e^{i\phi_0}$ and $\nu=\sin\phi\,\sqrt{2\kappa\eta}\,\mathrm{Im}(\alpha)$. In this notation we have by direct calculation

$$\begin{aligned}\mathrm{tr}\big[\rho^\alpha_{\rm ss}(e^{-i\phi}a+e^{i\phi}a^\dagger)\big] &= |\alpha|\Big(\cos(\phi_0+\phi)+\cos(\phi_0-\phi)\Big)\,,\\ \mathrm{tr}\big[\delta_{\rm ss}(e^{-i\phi}a+e^{i\phi}a^\dagger)\big] &= 2\nu|\alpha|\Big(\cos(\phi_0+\phi)-\cos(\phi_0-\phi)\Big)\,,\end{aligned} \quad (51)$$

and therefore in the basis (11) we have

$$\mathcal{N}(\rho^\alpha_{\rm ss},\delta_{\rm ss})=2\nu\sqrt{2\kappa\eta}|\alpha|\begin{pmatrix}\cos(\phi_0-\phi) & 0\\ 0 & -\cos(\phi_0+\phi)\end{pmatrix}\Delta W\,. \quad (52)$$

Similarly we find

$$\mathcal{M}(\delta_{\rm ss})=2\nu\sqrt{2\kappa\eta}\begin{pmatrix}\Lambda_1 & 0\\ 0 & \Lambda_2\end{pmatrix} \quad (53)$$

where

$$\begin{aligned}\Lambda_1 &= -|\alpha|\cos(\phi_0-\phi)\Delta W-2\nu\,\mathrm{Im}(\alpha)\sin\phi\;(\Delta W)^2\,,\\ \Lambda_2 &= |\alpha|\cos(\phi_0+\phi)\Delta W+2\nu\,\mathrm{Im}(\alpha)\sin\phi\;(\Delta W)^2\,.\end{aligned} \quad (54)$$

Puting everything together we find

$$\mathcal{M}(\rho^\alpha_{\rm ss}+\delta_{\rm ss})=\Big(\nu+4\nu^3(\Delta W)^2\Big)\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\,. \quad (55)$$

We therefore have

$$\langle\mathcal{M}^2(\rho^\alpha_{\rm ss}+\delta_{\rm ss})-\mathcal{M}^2(\rho^\alpha_{\rm ss})\rangle=\Big(8\nu^4\Delta t+O(\Delta t^2)\Big)\mathbb{1}\,. \quad (56)$$

Since for every odd k the average $\langle\Delta W^k\rangle=0$, we obtain $\langle\delta_{\rm ss}\mathcal{M}^2(\rho^\alpha_{\rm ss}+\delta_{\rm ss})\rangle=0$, and therefore

$$\mu(\tilde\rho^\alpha_{\rm ss},\delta_{\rm ss},n)=16\nu^4\,\frac{n(n+1)}{2^n}(-1)^{n-1}\Delta t+O(\Delta t^2)\,. \quad (57)$$

Substituting this result into Eq. (39) and neglecting $\lambda(\tilde\rho^\alpha_{\rm ss},\delta_{\rm ss},n)$ as explained above we obtain

$$\langle H_2(\rho^\alpha_{\rm ss})\rangle\propto\sum_{n=1}^{\infty}\frac{n^2-3n}{2^n}=0\,. \quad (58)$$

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