VIRTUALLY SOLUBLE GROUPS OF TYPE FP_∞

CONCHITA MARTINEZ-PÉREZ AND BRITA E.A. NUCINKIS

Abstract. We prove that a virtually soluble group G of type FP_∞ admits a finitely dominated model for $\underline{E}G$ of dimension the Hirsch length of G. This implies in particular that the Brown conjecture is satisfied for virtually torsion-free elementary amenable groups.

1. Introduction

A group is said to be of type FP_∞ if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ has a resolution with finitely generated projective $\mathbb{Z}G$-modules. Kropholler has shown that soluble groups of type FP_∞ are virtually torsion-free [17]. Torsion-free nilpotent groups of finite cohomological dimension equal to their Hirsch length, denoted by $\mathrm{h}G$, are finitely generated and in fact of type FP_∞, see [12, §8.8]. Gildenhuys and Strebel posed the question whether a similar result holds for soluble groups, which they partially answered [10]. The following result was then proved by Kropholler [15].

Theorem 1.1. [10, 15, 17] *Let G be a soluble group. Then the following are equivalent:*

(i) *G is of type* FP_∞
(ii) *G is a virtual duality group*
(iii) *G is virtually torsion-free and* $\mathrm{vcd}G = hG < \infty$
(iv) *G is virtually torsion-free and constructible.*

Here $\mathrm{vcd}G$ denotes the virtual cohomological dimension of G. A virtually torsion-free group has finite virtual cohomological dimension if there exists a finite-index subgroup of finite cohomological dimension. The class of constructible groups is the smallest class of groups containing the trivial group, which is closed under finite extensions and under HNN-extensions in which the base group and associated subgroups are constructible. Groups satisfying the conditions of the theorem are minimax, i.e. have a finite series of normal subgroups such that each factor has either min or max, see for example [2, 7.16] and are finitely presented [1]. Finite presentability of soluble groups of type FP_∞ implies in particular that these are of type VF: groups of type VF are those having a finite-index subgroup admitting a finite $K(G,1)$. With increased interest in classifying spaces for proper actions, new questions about soluble groups of type FP_∞ emerged. Let X be a G-CW-complex. X is called a classifying space for proper actions, or a model for $\underline{E}G$, if X^H is contractible if H is a finite subgroup of G and empty otherwise. It has been known for a while that virtually soluble groups of type FP_∞ admit a finite dimensional model for $\underline{E}G$ [18]. Results in [9] imply that we are able to bound the Bredon cohomological dimension $\underline{\mathrm{cd}}G$ of G in terms of the Hirsch length. In particular, for countable virtually soluble groups we have the following inequalities:

$$\mathrm{h}G \leq \underline{\mathrm{cd}}G \leq \mathrm{h}G + 1.$$

The Bredon-cohomological dimension is equal to the minimal dimension of a model for $\underline{E}G$ unless $\underline{\mathrm{cd}}G = 2$, see [21]. There is however, an example [5] of a virtually torsion-free group G for which $\underline{\mathrm{cd}}G = 2$, but which does not admit a 2-dimensional model for $\underline{E}G$. Hence, bar the possibility of such an Eilenberg-Ganea phenomenon, for a countable virtually soluble group there is always a model for $\underline{E}G$ of dimension equal to $\mathrm{h}G$ or equal to $\mathrm{h}G + 1$. As above, one wonders whether virtually soluble groups of type FP_∞ have a finite type model for $\underline{E}G$. Lück [22] gave an algebraic criterion for a group G to admit a finite type model for $\underline{E}G$:

Theorem 1.2. [22] *4.2 A group G admits a finite type model for $\underline{E}G$ if and only if the two following conditions are satisfied:*

(i) *G has finitely many conjugacy classes of finite subgroups.*
(ii) *For every finite subgroup F of G, the normalizer $N_G(F)$ is finitely presented and of type* FP_∞

For every group G of finite virtual cohomological dimension, Serre's construction yields a finite dimensional model for $\underline{E}G$ [7, VIII.11.2], but, except in trivial cases, the dimension is strictly greater that $\mathrm{vcd}G$. It was conjectured by K.S. Brown [6] that there is always a model for $\underline{E}G$ of dimension equal to $\mathrm{vcd}G$. Furthermore, even if G was of type VF, Serre's construction does not give a finite type model for $\underline{E}G$. It turns out [19], that there are examples of groups of type VF, which do not admit a finite type model for $\underline{E}G$. Some of these groups also provide a counterexample to Brown's conjecture. The construction used in [19] produces groups of type VF, which have either infinitely many conjugacy classes of finite subgroups or normalizers of finite subgroups, which are not of type FP_∞. If there is no restriction on the finite-index torsion-free subgroup, then the question regarding the number of conjugacy classes of finite subgroups has been completely answered. A result of Brown [7, IX.13.2] implies that any group of type VF has finitely many conjugacy classes of subgroups of prime power order. Leary [20] constructed, for any finite group Q not of prime power order, a group of type VF, which has infinitely many conjugacy classes of subgroups isomorphic to Q.

In this note we show that a virtually soluble group of type VF, i.e., a virtually soluble group of type FP_∞ indeed admits a finitely dominated model for $\underline{E}G$. In Section 2 we will show that such a group has only finitely many conjugacy classes of finite subgroups. Most of the remainder of this paper is then devoted to proving that centralizers of finite subgroups are also of type FP_∞. These centralizers are finitely presented [1] and have finite index in the corresponding normalizers hence the conditions of Theorem 1.2 are satisfied. Furthermore, since a virtually soluble group of type FP_∞ also admits a finite dimensional model for $\underline{E}G$, the claim that it admits a finitely dominated model for $\underline{E}G$ now follows from [22, 5.1,6.3].

To prove the result on the centralizers of finite subgroups we only have to consider finite group actions on groups which are nilpotent-by-abelian-by-finite, see part (a) of the proof of [25, 10.38]. We can therefore apply Bieri and Strebel's criterion for such a group to be of type FP_∞ [3, 5.2]. Let Q be a group acting on an abelian group A, we denote by $\Sigma^c_A(Q)$ the associated invariant, which will be defined in Section 3 below. Now let G be a nilpotent-by-abelian-by finite group, that is there is an extension $N \hookrightarrow G \twoheadrightarrow Q$ with N nilpotent and Q abelian-by-finite. To apply Theorem 5.2. of [3] it is sufficient to consider $\Sigma^c_A(Q)$, where $A = N/N'$.

As nice applications of our theorem we show that for virtually torsion-free elementary amenable groups Brown's conjecture is satisfied and that the property of admitting a finitely dominated $\underline{E}G$ is a quasi-isometry invariant within the class of virtually soluble groups.

Also note that our main result can be extended to elementary amenable groups of type FP_∞. A result of Hillman and Linnell [14] shows that elementary amenable groups of finite Hirsch length are locally finite-by -virtually soluble. In case these are of type FP_∞ they also have a bound on the orders of the finite subgroups [17] so we can reduce the above questions to questions on virtually soluble groups.

2. Conjugacy classes of finite subgroups

Lemma 2.1. *Let V be an abelian minimax group. Then any bounded section of V is finite.*

Proof: As any section of an abelian minimax group is again abelian minimax, we only have to see that a bounded abelian minimax group is finite, which follows immediately as any abelian minimax group is an extension of an abelian finitely generated group by a finite product of quasicyclic groups. □

Lemma 2.2. *Let F be a finite group acting on an abelian minimax group V. Then $H^1(F,V)$ is finite.*

Proof: As F is finite, $|F|$ annihilates $\mathrm{H}^1(F,V)$ so this is a bounded group. Moreover, we may choose a projective $\mathbb{Z}F$-resolution of $\mathbb{Z}$ with second term a finite sum $\oplus\mathbb{Z}F$. Therefore $H^1(F,V)$ is a section of the group

$$\mathrm{Hom}_{\mathbb{Z}F}(\oplus\mathbb{Z}F,V)=\oplus\mathrm{Hom}_{\mathbb{Z}F}(\mathbb{Z}F,V)=\oplus V.$$

Now it suffices to apply Lemma 2.1. □

Proposition 2.3. *Let F be a finite group acting on a soluble minimax group G. Then there is only a finite number of complements of G in the semidirect product $G\rtimes F$.*

Proof: Assume first that G is abelian. Then by Lemma 2.2 the group $\mathrm{H}^1(F,G)$ is finite and this implies the result, see [24, 11.13, 11.46].

Next we argue by induction on the derived length of the group. Let $\Gamma=G\rtimes F$ with G nonabelian and choose an F-invariant $K\trianglelefteq G$ with derived length of G/K and K both >0 and put

$$\Omega_1=\{\text{representatives of the conj. classes of complements of } G/K \text{ in } \Gamma/K\}.$$

By inductive hypothesis Ω_1 is finite. Put also

$$\Omega_2=\bigcup_{S\in\Omega_1}\{\text{representatives of the conj. classes of complements of } K \text{ in } S\}.$$

Note that for each $S\in\Omega_1$ either the previous set is empty or there is a complement of K in S and in this case S is isomorphic to the semidirect product of S/K and K so by the inductive hypothesis Ω_2 is also finite.

Now let M be a complement of G in Γ. Since

$$|MK/K|=|F|=|\Gamma/G|=|MK/G\cap MK|<\infty,$$

we deduce that KM/K is a complement of G/K in Γ/K and therefore for some $S\in\Omega_1$, $g\in\Gamma$, $KM^g=S$. Hence M^g is a complement of K in S implying $M^{gh}\in\Omega_1$

for some $h \in \Gamma$. Therefore Ω_2 contains a representative of each conjugacy class of complements of G in Γ. □

Theorem 2.4. *Let Γ be a virtually soluble group of type* FP_∞*. Then there is only a finite number of conjugacy classes of finite subgroups in* Γ*.*

Proof: There is a subgroup $G \triangleleft \Gamma$ which is soluble and of type FP_∞ and such that the index $|\Gamma : G|$ is finite. Therefore G is virtually-(torsion free soluble) so we may assume G itself is torsion free soluble. G is minimax.

Each finite subgroup $F \leq \Gamma$ is a complement of G in the group $G \rtimes F \leq \Gamma$. We may apply the previous result to the group $G \rtimes F \leq \Gamma$ and deduce that there is only a finite number of complements of G. The claim follows from the following: There are only finitely many subgroups S such that $G < S \leq \Gamma$. Proposition 2.3 implies that for each S there are only finitely many conjugacy classes of finite subgroups F of Γ such that $G \rtimes F = S$. Now let F be any finite subgroup of Γ, then $G \rtimes F$ is one of the subgroups counted before and F therefore falls into one of finitely many conjugacy classes of finite subgroups. □

Martin Hamilton has also shown this result for locally polycyclic-by-finite groups of finite virtual cohomological dimension [13, Lemma 1.2.].

3. Centralizers of finite subgroups

We will need the following definitions and notation, which can be found in [3]. A valuation of a finitely generated abelian group Q is a homomorphism

$$\nu : Q \to (\mathbb{R}, +).$$

The valuation sphere of Q is defined to be

$$S(Q) = \{[\nu] : 0 \neq \nu \text{ valuation of } Q\},$$

where $[\nu]$ denotes the equivalence class of ν with respect to the relationship "to be a positive scalar multiple of one another". This sphere is homeomorphic to the unit sphere in $\mathbb{R}^n$ with n the torsion-free rank of Q. Each $q \in Q$ defines an open hemisphere given by

$$H_q = \{[\nu] \in S(Q) : \nu(q) > 0\},$$

which is empty if q has finite order. For a subgroup $S \leq Q$ we denote

$$S(Q, S) = \{[\nu] \in S(Q) : \nu|_S = 0\}.$$

Now let A be a $\mathbb{Z}Q$-module, i.e. an abelian group with a Q-action. Put

$$\Sigma_A(Q) = \bigcup_{\lambda \in \mathrm{C}_{\mathbb{Z}Q}(A)} \{[\nu] : \nu(q) > 0 \text{ for any } q \in \mathrm{supp}\lambda\}$$

$$\Sigma_A^c(Q) = S(Q) \setminus \Sigma_A(Q).$$

A valuation is discrete if $\nu(Q) \subseteq \mathbb{Z}$ and for any subset $\Omega \subseteq \Sigma_A^c$ we denote by $\mathrm{dis}\Omega$ the set of equivalence classes in Ω containing a discrete valuation.

Let G be a nilpotent-by-abelian-by-finite group. It follows from [3, Theorem 2.3] that for all normal subgroups N and $H \triangleleft G$, such that G/H is finite, H/N is abelian and N nilpotent, $\Sigma^c_{N_{ab}}(H/N)$ is invariant up to homeomorphism of $S(H/N)$ (with $N_{ab} = N/N'$ the abelianization of N). Denote

$$\sigma(G) = \Sigma^c_{N_{ab}}(H/N).$$

Theorem 5.2 of [3] can be restated as follows:

Theorem 3.1. [3, 5.2] *Let G be a finitely generated soluble group. Then the following conditions are equivalent:*

(i) *G is constructible.*
(ii) *G is nilpotent-by-abelian-by-finite and $\sigma(G)$ is contained in an open hemisphere.*
(iii) *G is nilpotent-by-abelian-by-finite and* $\mathrm{dis}\sigma(G)$ *is contained in an open hemisphere.*

We shall use this characterization to prove that centralizers of finite subgroups of a virtually soluble group of type FP_∞ are also of type FP_∞.

Lemma 3.2. *Let $S \lhd Q$ be a normal subgroup which acts trivially on A. Then $\Sigma_A^c(Q)$ is contained in some open hemisphere if and only if $\Sigma_A^c(Q/S)$ is contained in an open hemisphere.*

Proof: For any $q \in S$, $q, q^{-1} \in \mathrm{C}_{\mathbb{Z}Q}(A)$. Hence the definition of $\Sigma_A^c(Q)$ implies

$$\Sigma_A^c(Q) \subseteq S(Q,S).$$

Let $\pi : Q \to Q/S$ be the projection and consider

$$\pi^* : S(Q/S) \to S(Q)$$
$$[\nu] \mapsto [\nu\pi]$$

as in [3, 1.1]. Now [3, 1.4] implies $\pi^*(\Sigma_A^c(Q/S)) \subseteq \Sigma_A^c(Q)$. Together with the fact that $\Sigma_A^c(Q) \subseteq S(Q,S)$ this yields the result. □

Lemma 3.3. *Assume $Q = Q_1 \times Q_2$ with both Q_1, Q_2 acting non trivially on A. Then for any $[v] \in \mathrm{dis}\Sigma_A^c(Q)$, $v = v_1 + v_2$ with $[v_i] \in \mathrm{dis}\Sigma_A^c(Q_i)$. Moreover, for any $[v_1] \in \mathrm{dis}\Sigma_A^c(Q_1)$ there is some $[v_2] \in \mathrm{dis}\Sigma_A^c(Q_2)$ with $[v_1 + v_2] \in \mathrm{dis}\Sigma_A^c(Q)$.*

Proof: Consider

$$\iota_i : Q_i \to Q,$$
$$\pi : Q \to Q_1$$

the inclusions and the first projection. They induce maps

$$\iota_i^* : S(Q) \to S(Q_i)$$
$$\nu \mapsto \nu\iota_i,$$
$$\pi^* : S(Q_1) \to S(Q)$$
$$\nu \mapsto \nu\pi$$

and by Proposition [3, 1.2]

$$\iota_i^*(\mathrm{dis}\Sigma_A^c(Q) \cap S(Q,Q_i)^c) = \mathrm{dis}\Sigma_A^c(Q_i),$$
$$\pi_i^*(\ \mathrm{dis}\Sigma_A^c(Q_1) \cap S(Q,Q_1)^c) = \mathrm{dis}\Sigma_A^c(Q).$$

□

Let F be a group acting on Q and on A so that this action is compatible with that of Q on A, i.e.

$$(a^q)^t = (a^t)^{q^t}$$

for any $a \in A$, $q \in Q$, $t \in F$. The group F acts also on $\mathrm{Hom}(Q, \mathbb{R})$ via

$$v^t(q) = v(q^{t^{-1}})$$

for $v : Q \to \mathbb{R}$, $t \in F$, $q \in Q$. Clearly this yields also an action of F on the valuation sphere $S(Q)$.

Lemma 3.4. *$\Sigma^c_A(Q)$ is F-invariant.*

Proof: Consider $\lambda \in \mathrm{C}_{\mathbb{Z}Q}(A)$. Then $\lambda a = a$ for any $a \in A$ and therefore for each $t \in F$ and $a \in A$, $a = (a^{t^{-1}})^t = (\lambda a^{t^{-1}})^t = \lambda^t a$. So we deduce that $\mathrm{C}_{\mathbb{Z}Q}(A)$ is F-invariant. Now let $v \in \Sigma^c_A(Q)$. Note that by definition

$$\Sigma^c_A(Q) = \bigcap_{\lambda \in \mathrm{C}_{\mathbb{Z}Q}(A)} \{[v] | v(q) \leq 0 \text{ for some } q \in \mathrm{supp}\lambda\}.$$

Let $[v] \in \Sigma^c_A(Q)$, $\lambda \in \mathrm{C}_{\mathbb{Z}Q}(A)$. Then $\lambda^{t^{-1}} \in \mathrm{C}_{\mathbb{Z}Q}(A)$ so for some $q^{t^{-1}} \in \mathrm{supp}\lambda^{t^{-1}}$, $v^t(q) = v(q^{t^{-1}}) \leq 0$. Thus $[v^t] \in \Sigma^c_A(Q)$. □

For the next results we fix a finite group F and let

$$e := \sum_{t \in F} t \in \mathbb{Z}F.$$

If F acts on an abelian group V (we use multiplicative notation) we put, for $v \in V$,

$$\mathrm{Ann}_V e := \{v \in Q : v^e = \prod_{t \in F} v^t = 1\}.$$

Lemma 3.5. *Assume F acts on an abelian group V and let $C = \mathrm{C}_V(F)$. Then*

$$V^{|F|} = C_1 T$$

for certain subgroups $T \leq Ann_V e$ and $C_1 \leq C$. Moreover if V is minimax, $V/V^{|F|}$, C/C_1 and $C \cap T$ are finite.

Proof: Put

$$T = V^{(|F|-e)} = \{v^{(|F|-e)} : v \in V\} = \{\prod_{t \in F} v(v^{-1})^t : v \in V\},$$

$$C_1 = V^e = \{v^e : v \in V\}.$$

The first assertion is a consequence of

$$|F| = e + (|F| - e)$$

and the fact that $V^e \leq C$. As

$$(|F| - e)e = 0$$

we also have $T \leq \mathrm{Ann}_V(e)$. For the rest use 2.1. □

Lemma 3.6. *Assume $Q = C \times T$ and F acts on Q so that the action is trivial on C and $T \leq \mathrm{Ann}_Q(e)$. Suppose $\mathrm{dis}\Sigma^c_A(Q)$ is contained in some open hemisphere. Then so is $\mathrm{dis}\Sigma^c_A(C)$.*

Proof: By 3.2 we may assume that the action of Q on A has trivial kernel. We have $\mathrm{Hom}(Q,\mathbb{R}) = \mathrm{Hom}(C,\mathbb{R}) \oplus \mathrm{Hom}(T,\mathbb{R})$ and both abelian groups are F-invariant. Moreover, F acts trivially on $\mathrm{Hom}(C,\mathbb{R})$ and $\mathrm{Hom}(T,\mathbb{R}) \leq \mathrm{Ann}_{\mathrm{Hom}(Q,\mathbb{R})}e$. Note also that the result is trivial if $\mathrm{dis}\Sigma^c_A(Q) = \emptyset$. Assume $\emptyset \neq \mathrm{dis}\Sigma^c_A(Q)$ is contained in some open hemisphere. This means that for some $q \in Q$, $v(q) > 0$ for any $v \in \mathrm{dis}\Sigma^c_A(Q)$. We may put $q = q_1q_2$ with $q_1 \in C$ and $q_2 \in T$. Now let $[v_1] \in \mathrm{dis}\Sigma^c_A(C)$. By 3.3 there is some $[v_2] \in \mathrm{dis}\Sigma^c_A(T)$ with $[v_1 + v_2] \in \mathrm{dis}\Sigma^c_A(Q)$. As $\mathrm{dis}\Sigma^c_A(Q)$ is F-invariant by 3.4, we get $[v_1 + v_2^t] \in \mathrm{dis}\Sigma^c_A(Q)$ for any $t \in F$. Therefore

$$v_1(q_1) + v_2^t(q_2) > 0.$$

Now, if $v_2^t(q_2) \leq 0$ for some $1 \neq t \in F$, then $v_1(q_1) > 0$. And in other case,

$$v_2(q_2) = -\sum_{1\neq t\in F} v_2^t(q_2) < 0$$

and again we get $v_1(q_1) > 0$. This means that also $\mathrm{dis}\Sigma^c_A(C)$ is contained in some open hemisphere. □

Lemma 3.7. *Let $N_1, N_2 \lhd G$ be subgroups such that G/N_1, G/N_2 are abelian-by-finite. Then the same holds for $N_1 \cap N_2$.*

Proof: Consider $H_1, H_2 \lhd G$ such that $N_i \leq H_i$ and G/H_i is finite, H_i/N_i is abelian for $i = 1, 2$. Then clearly $H = H_1 \cap H_2 \lhd G$ and G/H is finite. So we may assume G/N_i is abelian for $i = 1, 2$. Then $G' = [G,G] \leq N_1, N_2$ so $G' \leq N_1 \cap N_2$ thus $G/N_1 \cap N_2$ is also abelian. □

From now on we fix a soluble group G of type FP_∞ and consider a finite group F acting on G. Then by part (a) of the proof of [25, 10.38] there is some nilpotent $N \lhd G$ such that the group G/N is finitely generated abelian-by-finite. Moreover using the Lemma above and by considering $\bigcap_{t\in F} N^t$ we may assume that N is F-invariant. We also fix $A = N_{ab} = N/N'$.

Lemma 3.8. *A subgroup C of G with $N \leq C \leq G$ is finitely generated if and only if A is finitely generated as C/N-module.*

Proof. If C is finitely generated, then so is C/N'. As in the proof of [10, Corollary A2] we deduce that N/N' is a Noetherian $\mathbb{Z}C/N$-module so it is in fact finitely generated.

Conversely, assume that A is finitely generated as C/N-module and consider the i-th central factor $\gamma_iN/\gamma_{i+1}N$ of the central series of N. There is a C-epimorphism

$$A\otimes \overset{i}{\ldots} \otimes A \to \gamma_iN/\gamma_{i+1}N$$

which implies that also $\gamma_iN/\gamma_{i+1}N$ is finitely generated as C/N-module. As N is nilpotent we deduce that C is finitely generated. □

Proposition 3.9. *Let $C/N = \mathrm{C}_{G/N}(F)$. Then C is of type* FP_∞.

Proof: Let $Q = G/N$. There is a finite index subgroup H such that H/N is free abelian and we may assume that H is normal and F-invariant. As a group is of type FP_∞ if and only if a finite index subgroup is also, we may assume that Q is free abelian. So we have an action of F on a free abelian group. Then by 3.5 there is a finite index subgroup of G, say G_1 such that $G_1/N = C_1/N \times T/N$ with F acting trivially on C_1/N and $T/N \leq \mathrm{Ann}_Q(e)$. Moreover C_1 has finite index in C. So we may assume that $Q = C/N \times T/N$ and $T/N \leq \mathrm{Ann}_Q(e)$. The hypothesis that G is of type FP_∞ implies by [3, 5.2] that $\mathrm{dis}\Sigma_A(Q)$ is contained in some open hemisphere. Hence by 3.6 also

$$\mathrm{dis}\Sigma_A(C/N)$$

is contained in some open hemisphere. Moreover, the proof of 3.6 implies that there is some $q_1 \in C/N$ with $\upsilon_1(q_1) > 0$ for any $\upsilon_1 \in \mathrm{dis}\Sigma_A^c(C/N)$ thus also $\upsilon(q_1) > 0$ for any $\upsilon \in \mathrm{dis}\Sigma_A^c(Q)$. This means that

$$S(Q, C/N) \subseteq \Sigma_A(Q).$$

Therefore using [4, Theorem A] one deduces that A is finitely generated as a C-module. By Lemma 3.8, C is finitely generated. The result now follows by [3, 5.2]. □

Lemma 3.10. *Assume $V \lhd G$ is an abelian minimax F-invariant subgroup of G. Put $C/V = \mathrm{C}_{G/V}(F)$. Then*

$$|C : V\mathrm{C}_G(V)| < \infty.$$

Proof: Let $g \in C$. For any $t \in F$,

$$g^t = g\delta_g(t)$$

with $\delta_g(t) \in V$. Furthermore, for all $t_1 \in F$:

$$g\delta_g(tt_1) = g^{tt_1} = g^{t_1}\delta_g(t)^{t_1} = g\delta_g(t_1)\delta_g(t)^{t_1}.$$

Hence δ_g is a derivation

$$\delta_g : F \to V.$$

So we have a map

$$C \to \mathrm{Der}(F, V)$$
$$g \mapsto \delta_g,$$

which is not a group homomorphism. In fact for $g_1, g_2 \in C$, $t \in F$

$$g_1g_2\delta_{g_1g_2}(t) = (g_1g_2)^t = g_1^tg_2^t = g_1\delta_{g_1}(t)g_2\delta_{g_2}(t) = g_1g_2\delta_{g_1}(t)^{g_2}\delta_{g_1}(t).$$

Now, note that we may define an action of C on $\mathrm{Der}(F, V)$ via

$$\delta^g(t) = \delta(t)^g.$$

Thus the formula above can be written

$$\delta_{g_1g_2} = \delta_{g_1}^{g_2}\delta_{g_1}.$$

We claim that this action leaves the set $\mathrm{Inn}(F, V)$ of inner derivations setwise invariant. To see this, let $\delta \in \mathrm{Inn}(F, V)$ such that $\delta(t) = v^tv^{-1}$ for certain $v \in V$. Then for any $g \in C$ and any $t \in F$

$$\delta^g(t) = (v^t)^g(v^{-1})^g = (v^t)^{g^t}(v^g)^{-1} = (v^g)^t(v^g)^{-1}.$$

Therefore $\delta^g \in \mathrm{Inn}(F, V)$.

Finally note that for $g \in C$, $\delta_g \in \mathrm{Inn}(F, V)$ if and only if there is some $v \in V$ such that

$$g^t = gv^tv^{-1}.$$

This is equivalent to

$$(gv^{-1})^t = g^t(v^{-1})^t = gv^{-1},$$

i.e. equivalent to g belonging to $V\mathrm{C}_G(V)$.

This all implies that for $g_1, g_2 \in C$, $g_1 g_2^{-1} \in V\mathrm{C}_G(V)$ if and only if $\delta_{g_1 g_2^{-1}} \in \mathrm{Inn}(F, V)$. We have

$$\delta_{g_1} = \delta_{g_1 g_2^{-1} g_2} = \delta^{g_2}_{g_1 g_2^{-1}} \delta_{g_2}.$$

Thus $\delta_{g_1 g_2^{-1}} \in \mathrm{Inn}(F, V)$ if and only if $\delta_{g_1} = \delta\delta_{g_2}$ for some inner derivation δ. Therefore the map at the beginning of the proof induces a well defined injective map

$$C/V\mathrm{C}_G(V) \to \mathrm{H}^1(F, V)$$

where $C/V\mathrm{C}_G(V)$ denotes the set of cosets of $V\mathrm{C}_G(V)$ in C. The result now follows from the finiteness of $\mathrm{H}^1(F, V)$, see Lemma 2.2.

□

Proposition 3.11. *Let G, N, F be as above. The group $N\mathrm{C}_G(F)$ has finite index in C with $C/N = \mathrm{C}_{G/N}(F)$. Therefore $N\mathrm{C}_G(F)$ is of type* FP_∞*.*

Proof: We argue by induction on the nilpotency length of N. Let $V \lhd N$ be the last term in the lower central series of N. Then V is minimax and as this series is characteristic, V is F-invariant. Then by the inductive hypothesis $N\mathrm{C}_{G/V}(F)/N$ has finite index in $C/N = \mathrm{C}_{G/N}(F)$, i.e.

$$|C : N\mathrm{C}_{G/V}(F)| < \infty.$$

Now, by Lemma 3.10

$$|\mathrm{C}_{G/V}(F) : V\mathrm{C}_G(F)| < \infty$$

so

$$|N\mathrm{C}_{G/V}(F) : N\mathrm{C}_G(F)| < \infty$$

and the result follows. □

Lemma 3.12. *Let $S \leq G$ with $G = SN$. Then S is of type* FP_∞*.*

Proof: Note that $S \cap N$ is nilpotent and $Q = G/N \cong S/S \cap N$. Recall that $A = N/N'$ and put $B = S \cap N/(S \cap N)'$. If s is the nilpotency length of N and

$$1 = \gamma_s N \lhd \gamma_1 N \lhd \ldots \lhd \gamma_0 N = N$$

is the lower central series of N then B has a composition series with factors $B_1, \ldots, B_s$ such that each B_i is a subsection of $\otimes_i A$. Therefore if we consider the Bieri-Strebel invariants $\Sigma^c_A(Q)$, $\Sigma^c_B(Q)$, [3, Lemma 1.1 (c)] implies that

$$\Sigma^c_B(Q) = \Sigma^c_{B_1}(Q) \cup \ldots \cup \Sigma^c_{B_s}(Q)$$

and

$$\Sigma^c_{B_i}(Q) \subseteq \Sigma^c_{\otimes_i A}(Q).$$

We also have by [3, Theorem 1.3]

$$\mathrm{dis}\Sigma^c_{\otimes_i A}(Q) \subseteq \mathrm{dis}\Sigma^c_A(Q) + \overset{i}{\ldots} + \mathrm{dis}\Sigma^c_A(Q)$$

(convex sum as in [3, 1.1]). Now, if G is of type FP_∞,

$$\mathrm{dis}\Sigma^c_A(Q)$$

is contained in some open hemisphere and we deduce that also

$$\mathrm{dis}\Sigma^c_B(Q)$$

is. Thus S is of type FP_∞. □

Putting it all together yields:

Theorem 3.13. *The group* $C_G(F)$ *is of type* FP_∞.

Proof: It suffices to use Proposition 3.11 and Lemma 3.12. □

4. The main result

Theorem 4.1. *Let* G *be a virtually soluble group of type* FP_∞. *Then* G *admits a finited dominated model for* $\underline{E}G$.

Proof: As remarked in the introduction, constructible groups are finitely presented (see [1]) so also G and all its Weyl groups are. All that remains to prove is that for any finite subgroup F of G, the centralizer $C_G(F)$ and therefore the normalizer $N_G(F)$ is of type FP_∞. The result will then follow from Theorem [22, 5.1, 6.3]. Now let H be a finite-index torsion-free soluble subgroup of type FP. To show that $C_G(F)$ is of type FP_∞ it suffices to show that $C_H(F)$ is of type FP_∞. This is a consequence of Theorem 3.13. □

Corollary 4.2. *Let* G *be an elementary amenable group of type* FP_∞. *Then* G *admits a finited dominated model for* $\underline{E}G$.

Proof: Hillman and Linnell [14] have shown that elementary amenable groups are locally finite-by-virtually soluble. Furthermore, by a result of Kropholler [17], groups of type FP_∞ have a bound on the orders of the finite subgroups and hence G is finite-by-virtually soluble. By Theorem 4.1 this implies that the virtually soluble quotient admits a finitely dominated model for $\underline{E}G$. An application of [22, Theorem 3.2] yields the claim. □

We say a property $\mathcal{P}$ is geometric, if for any group G with property $\mathcal{P}$ and any group H quasi-isometric to G, it implies that H also has property $\mathcal{P}$.

Corollary 4.3. *In the class of virtually soluble groups, admitting a finitely dominated model for* $\underline{E}G$ *is a geometric property.*

Proof: Let H be quasi-isometric to G and let G admit a finitely dominated model for $\underline{E}G$. In particular, G is of type F_∞. By Gromov, [11, $1C_2$], F_∞ is a geometric property. Now apply Theorem 4.1. □

5. Brown's conjecture

As mentioned in the introduction, Theorem 4.1 also implies that Brown's conjecture holds for soluble groups. To apply the algebraic criteria of [9] we only need to show that an Eilenberg-Ganea type phenomenon as in [5] cannot happen for virtually (torsion-free soluble) groups.

Proposition 5.1. *Let* G *be a virtually (torsion-free soluble) group of* $\underline{\mathrm{cd}}G = 2$. *Then* G *admits a 2-dimensional model for* $\underline{E}G$.

Proof: First, let G be of type FP_∞. This implies that G is nilpotent-by-abelian-by finite, which means there is a short exact sequence

$$N \hookrightarrow G \twoheadrightarrow Q,$$

where N is nilpotent and Q is finitely generated abelian-by-finite. Suppose $\underline{\mathrm{cd}}N = 2$. Then, by [23, Remark 6.4] it follows that N is finitely generated and hence, G is polycyclic-by-finite and has a 2-dimensional model for $\underline{E}G$. Now suppose $\underline{\mathrm{cd}}N = 1$.

Since N is virtually torsion-free, it implies that N has an infinite cyclic subgroup of finite index and again, N is finitely generated. The condition $\underline{\mathrm{cd}}N = 0$ implies N finite and the claim follows, too.
This leaves the case when G is not of type FP_∞ . By Corollary 5.3 $\underline{\mathrm{cd}}G = 2$ yet $\mathrm{h}G = 1$. Hence G has a torsion-free subgroup H of finite index with homological dimension $\mathrm{hd}H = 1$. Hence H is isomorphic to a subgroup of $\mathbb{Q}$ and therefore is locally (infinite cyclic)-by-finite. Any (infinite cyclic)-by-finite group admits a 1-dimensional model for $\underline{E}G$. Since G is virtually torsion-free we can apply [8, Theorem 2.4] implying there is a 2-dimensional model for $\underline{E}G$ finishing the proof. □

Corollary 5.2. *Virtually torsion-free elementary amenable groups satisfy the Brown conjecture.*

Proof: Let us first assume that G is soluble-by-finite of finite virtual cohomological dimension. By the above, we only need to show that $\underline{\mathrm{cd}}G = vcdG$. For soluble groups of type FP_∞ this is a consequence of Theorem 4.1. Also note that virtually torsion-free soluble groups of finite Hirsch length are countable [2, Lemma 7.9]. If G is not of type FP_∞, then $\underline{\mathrm{hd}}G = hG = n-1$ where $n = \underline{\mathrm{cd}}G$. But let H denote a torsion-free subgroup of finite index. Then $hH = hG = hdH = n-1$ and $cdH = n$ as H is not of type FP_∞ either. Hence the claim follows for soluble-by-finite groups.

Let G now be virtually torsion-free elementary amenable. By [14] there is an extension

$$T \hookrightarrow G \twoheadrightarrow Q$$

with T locally finite and Q virtually soluble. Since G is virtually torsion-free, T is finite and Q is virtually torsion free of $\mathrm{vcd}Q = \mathrm{vcd}G$. Now [22, Theorem 3.1] implies that G and Q both have models for $\underline{E}G$ of the same dimension. □

Note finally that we can add one more equivalent algebraic condition to Kropholler's Theorem, see 1.1:

Corollary 5.3. *Let G be a virtually soluble group. Then the following are equivalent:*

(i) *G is of type* FP_∞.
(ii) *G is virtually torsion-free and* $\underline{\mathrm{cd}}G = hG < \infty$.

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Departamento de Matemáticas, Universidad de Zaragoza, Zaragoza 50009, Spain
E-mail address: conmar@unizar.es

School of Mathematics, University of Southampton, Southampton SO17 1BJ, UK,
E-mail address: B.E.A.Nucinkis@soton.ac.uk