TSP tour domination and Hamilton cycle decompositions of regular digraphs

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Abstract

In this paper, we solve a problem by Glover and Punnen (1997) from the context of domination analysis, where the performance of a heuristic algorithm is rated by the number of solutions that are not better than the solution found by the algorithm, rather than by the relative performance compared to the optimal value. In particular, we show that for the Asymmetric Traveling Salesman Problem (ATSP), there is a deterministic polynomial time algorithm that finds a tour that is at least as good as the median of all tour values. Our algorithm uses an unpublished theorem by Häggkvist on the Hamilton decomposition of regular digraphs.

Keywords: ATSP, domination analysis, Hamilton cycle decomposition, regular digraphs

1 Introduction, Terminology and Notation

It is well known that most combinatorial optimization problems are $\mathcal{NP}$-hard. Due to the lack of polynomial time algorithms to solve $\mathcal{NP}$-hard problems to optimality, the following two approaches to deal with such problems have been developed. The first one is the design of polynomial approximation algorithms that produce feasible solutions whose value is always within a constant factor of the optimum. Unfortunately, many important combinatorial optimization problems, including the Asymmetric Traveling Salesman Problem (ATSP), cannot have such approximation algorithms unless $\mathcal{P} = \mathcal{NP}$. The second approach is the use of various heuristics such as local search and

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genetic algorithms that usually provide good solutions for instances that arise in practice. The fact that heuristics do not have any theoretical guarantee implies not only that for some instances the value of heuristic solution is arbitrary far from the optimum, but also that we should have another theoretical way (in addition to computational experiments) to compare various heuristics using some objective measures.

With this state of affairs in mind, Glover and Punnen [9] suggested a new approach for evaluation of heuristics that compares heuristics according to their so-called domination ratio. We define this notion only for the ATSP since its extension to other problems is obvious. The *domination number*, $\mathrm{domn}(\mathcal{A}, n)$, of a heuristic $\mathcal{A}$ for the ATSP is the maximum integer $d = d(n)$ such that, for every instance $\mathcal{I}$ of the ATSP on n cities, $\mathcal{A}$ produces a tour T which is not worse than at least d tours in $\mathcal{I}$ including T itself. The ratio $\mathrm{domr}(\mathcal{A}, n) = \mathrm{domn}(\mathcal{A}, n)/(n-1)!$, i.e., the domination number divided by the total number of tours, is the *domination ratio* of $\mathcal{A}$. Clearly, the domination ratio is well defined for every heuristic and, for the same problem, a heuristic with higher domination ratio may be considered a better choice than a heuristic with lower domination ratio. (This kind of comparison is somewhat similar to the standard comparison of approximation algorithms, which continues to be the most popular choice of performance analysis.)

To explore the possible range of the domination ratio for the ATSP, Glover and Punnen [9] asked whether there exists a polynomial time (in n) algorithm $\mathcal{A}$ with domination ratio $\mathrm{domr}(\mathcal{A}, n) \geq 1/p$ for some p being a constant or even polynomial in n. They conjectured that, unless $\mathcal{P} = \mathcal{NP}$, the answer to this question is negative. (While the authors of [9] considered only the symmetric TSP, we will discuss here the more general ATSP.) In [13], we proved that the answer to the Glover-Punnen question is, in fact, positive (even for the ATSP). We showed the existence of a required algorithm for $p = n - 1$. The algorithm of [13] is of domination ratio $\Theta(\frac{1}{n-1})$, i.e., the Glover-Punnen question was still open for p being constant.

In this paper we show that, if there is a constant $r > 1$ such that, for every sufficiently large k, a k-regular digraph of order $n < rk$ can be decomposed into Hamilton cycles in time polynomial in n, then there exists an ATSP algorithm with the desired properties, for a constant p. This result is of interest due to the fact that Häggkvist [14, 15] demonstrated (not published yet) that the above Hamilton decomposition exists for every $1 < r \leq 2$, see also Alspach et al. [2]. His approach is constructive and implies a polynomial algorithm to find such a decomposition. Häggkvist's very deep graph-theoretical result and our Theorem 2.3 imply that, in polynomial time, one can always find a tour, which is not worse than 50% of all tours. Notice that the 50% threshold may seem to be easily achievable at first glance: just find the best in a large sample of randomly chosen tours. A random tour has approximately a 50% chance of being better than 50% of all tours. However, in this approach the probability that the best tour is longer than 50% of all tours is always positive (if we consider only polynomial size samples of random tours). The difficulty of the problem by Glover and Punnen is well illustrated by the problem [16] to find a tournament on n vertices with the number of hamiltonian cycles exceeding the average number of hamiltonian cycles in a tournament of order n. This problem formulated more than 30 years ago has not been solved yet.

Due to very complex and tedious arguments of Häggkvist [14, 15], which can only be applied to regular digraphs with an impractically large number of vertices (much more than one million), the algorithm that we suggest in this paper is apparently impractical. However, our result implies that a sufficiently fast tour construction heuristic with domination ratio $\Theta(1)$ could perhaps be obtained. Taking into consideration that widely used tour construction heuristics are proved to be of domination ratio $\Omega(\frac{1}{n-1})$ [19], the new heuristic would likely be of very high quality.

Polynomial algorithms with exponentially large domination number were suggested in a number of papers including [3, 6, 9, 10, 11, 12, 13, 17, 18, 19, 20]. The strongest results were obtained in [13] and [19]. In [13], we introduced an $\Theta(n^3)$ time heuristic for the ATSP, which, we proved to be of domination ratio $\Theta(\frac{1}{n-1})$. This algorithm has been evaluated by A. Zverovich (personal communication) using computational experiments. While the quality of the tours found by the algorithm was quite high especially for TSP instances taken from TSPlib (where the algorithm outperformed well-known construction heuristics), the time complexity of the algorithm was too high to apply it to instances of order $n \geq 500$. Thus the algorithm of [13] is also impractical.

However, there are several ATSP heuristics with exponentially large domination number that proved to be practical. Using one of the main results in [13], Punnen and Kabadi [19] managed to prove that several well-known and widely used ATSP construction heuristics, such as various vertex insertion algorithms and Karp's cycle patching algorithm, are of domination number at least $(n-2)!$. Glover et al. [8], in a series of computational experiments with several families of instances, showed that a combination of two algorithms with exponential domination number leads to a construction heuristic for the ATSP, which clearly outperforms well-known construction heuristics. Interestingly enough, Punnen and Kabadi (personal communication) reported that certain well-known TSP heuristics such as the greedy algorithm and Christofides' algorithm have domination number significantly lower than $(n-2)!$. These heuristics do not perform well for ATSP (see [8]). Thus, a large domination number may be a necessary requirement for a construction heuristic to be of high quality.

The *ATSP* is stated as follows. Given a weighted complete digraph $(\overleftrightarrow{K}_n, c)$, find a Hamilton cycle in $\overleftrightarrow{K}_n$ of minimum cost. Here the *cost function* c is a mapping from $A(\overleftrightarrow{K}_n)$, the set of arcs in $\overleftrightarrow{K}_n$, to the set of non-negative reals. The *cost* of an arc xy of $\overleftrightarrow{K}_n$ is $c(x,y)$. The *cost* $c(D)$ of a subdigraph D of $\overleftrightarrow{K}_n$ is the sum of the weights of arcs in D. In this paper, we will call a Hamilton cycle, i.e., a cycle containing all vertices in $\overleftrightarrow{K}_n$, a *tour*. A digraph D is *k-regular* if the in-degree and out-degree of every vertex in D equals k. A set $\{C_1, ..., C_k\}$ of k tours in a k-regular digraph D is a *tour decomposition* of D if $A(D) = \cup_{i=1}^{k} A(C_i)$. As $|A(D)| = kn$, $A(C_i) \cap A(C_j) = \emptyset$ for every pair of distinct i and j.

Further terminology and notation from graph theory and network flows used in this note can be found in [1, 4, 5].

2 Results

We start with a brief description of our algorithm applied to $(\overleftrightarrow{K}_n, c)$, where n is large enough.

1. Compute k such that every k-regular digraph of order n has a decomposition into Hamilton cycles (see Theorem 2.3 for details).

2. Find a minimum cost k-regular spanning subgraph M of $(\overleftrightarrow{K}_n, c)$.

3. Find the minimum cost tour Z in a Hamilton cycle decomposition of M. Return Z.

Now we shall study this algorithm.

Lemma 2.1 *(Tillson [21]) For every $n \geq 2$, $n \neq 4$, $n \neq 6$, there exists a decomposition of $\overleftrightarrow{K}_n$ into tours.*

Lemma 2.2 *For a positive integer $k < n$, a k-regular spanning subgraph of $(\overleftrightarrow{K}_n, c)$ of minimum cost can be found in polynomial time.*

Proof: Construct a network N from $(\overleftrightarrow{K}_n, c)$ by replacing every vertex x of $\overleftrightarrow{K}_n$ by the arc (x', x'') of cost zero, lower bound $\ell(x', x'') = k$ and upper bound (capacity) $u(x', x'') = k$, and every arc (x, y) of $\overleftrightarrow{K}_n$ by the arc (x'', y') of cost $c(x, y)$, lower bound zero and capacity one. In polynomial time, one can find a minimum cost circulation in N (due to the nature of the network N, we can use even non-strongly polynomial algorithms)[1]. This circulation corresponds to a minimum cost k-regular spanning subgraph H of $(\overleftrightarrow{K}_n, c)$. □

Theorem 2.3 *If there exists a constant $r > 1$ such that for every sufficiently large k a k-regular digraph of order $n < rk$ can be decomposed into tours and such a decomposition can be found in time polynomial in n, then there exists a polynomial ATSP approximation algorithm of domination number at least $(n - \lceil \frac{n+1}{r} \rceil)(n-2)!$.*

Proof: Let there exist constants $r > 1$ and k_0, such that each k-regular digraph ($k \geq k_0$) of order less than rk can be decomposed into tours and such a decomposition can be found in polynomial time. Let n_0 be the minimum integer such that $n_0 \geq 7$ and $n_0 \geq rk_0$.

We may assume that $n \geq n_0$ (otherwise, we can solve the ATSP to optimality by considering a restricted number of tours). Let $k = \lceil \frac{n+1}{r} \rceil$. By Lemma 2.2, we can, in polynomial time, find a minimum cost k-regular spanning subgraph M of $(\overleftrightarrow{K}_n, c)$. By our assumption we can find a tour decomposition of M in polynomial time. Choose the cheapest tour Z in this decomposition. Clearly, we can find Z in polynomial time. To complete our proof, it suffices to show that Z is not more expensive than at least $(n-k)(n-2)!$ distinct tours (including itself) in $\overleftrightarrow{K}_n$.

Let $D_1 = \{C_1, C_2, ..., C_{n-1}\}$ be a decomposition of $\overleftrightarrow{K}_n$ into tours such that $c(C_1) \geq c(C_2) \geq ... \geq c(C_{n-1})$. (Such a decomposition exists by Lemma 2.1.) Given a tour H in $\overleftrightarrow{K}_n$, clearly there is an automorphism of $\overleftrightarrow{K}_n$ that maps C_1 into H. Therefore, if we consider D_1 together with the decompositions $(D_1, ..., D_{(n-1)!})$ of $\overleftrightarrow{K}_n$ obtained from D_1 using all automorphisms of $\overleftrightarrow{K}_n$ which map the vertex 1 into itself, we will have every tour of $\overleftrightarrow{K}_n$ in one of the D_i's. Moreover, every tour is in exactly $n-1$ decompositions D_i's (by mapping a tour C_i into a tour C_j $(i, j \in \{1, 2, ..., n-1\})$ we fix the automorphism).

Choose the $(n-k)$ most expensive tours in each of the D_i and form a set $\mathcal{T}$ from all distinct tours obtained in this manner. Clearly, $|\mathcal{T}| \geq (n-k)(n-2)!$. It remains to show that $c(Z) \leq c(T)$ for every $T \in \mathcal{T}$. Without loss of generality, it suffices to prove that $c(Z) \leq c(C_{n-k})$. Since M is a minimum cost k-regular spanning subgraph of $\overleftrightarrow{K}_n$, $c(M) \leq c(\overleftrightarrow{K}_n - \cup_{i=1}^{n-k-1} A(C_i))$. Therefore,

$$c(Z) \leq \frac{c(M)}{k} \leq \frac{c(\overleftrightarrow{K}_n - \cup_{i=1}^{n-k-1} A(C_i))}{k} \leq c(C_{n-k}).$$

□

Corollary 2.4 *For the ATSP, there is a deterministic polynomial time algorithm of domination ratio at least* $1/2$.

Proof: Häggkvist's result shows that the assumption of Theorem 2.3 is valid for $r = 2$. If n is odd, we can directly use the algorithm described in Theorem 2.3. For $r = 2$, its domination ratio is at least

$$\frac{(n - \lceil (n+1)/2 \rceil)(n-2)!}{(n-1)!} \geq \frac{1}{2}.$$

In the rest of the proof, we use the operation of contraction of an arc $a = xy$ in $K = \overleftrightarrow{K}_n$. The result of this operation is the weighted complete digraph $K/a = \overleftrightarrow{K}_n/a$ with vertex set $V(K/a) = V(K) \cup \{v_a\} - \{x, y\}$, where $v_a \notin V(K)$, such that the cost $c_{K/a}(b)$ of an arc b in K/a is defined as follows: $c_{K/a}(uw) = c_K(uw), c_{K/a}(uv_a) = c_K(ux), c_{K/a}(v_a w) = c_K(yw)$, where $u, w \in V(K) - \{x, y\}$.

Suppose now that n is even and x is an arbitrary vertex in $\overleftrightarrow{K}_n$. For every arc a out of x, contract a and use the algorithm in Theorem 2.3 for $r = 2$ to find a tour T_a in $\overleftrightarrow{K}_n/a$. Let C_a be the tour in $\overleftrightarrow{K}_n$ obtained from T_a by replacing v_a with x, y, where $a = xy$. Let C be the best of all tours C_a. Observe that C is at least as good as $(n-2)!/2$ tours through any given arc out of x, implying that C is at least as good as $(n-1)(n-2)!/2 = (n-1)!/2$ tours. □

3 Remarks and Further Research

It is worth noting that Häggkvist's decomposition result cannot be improved, in a sense, since the digraph of order n with two connected components isomorphic to $\overleftrightarrow{K}_{n/2}$ is an $(n/2-1)$-regular digraph. This means that to improve the 50% threshold, another approach is needed. It would be very interesting to have a solution to the Glover-Punnen problem with p being constant, which does not rely heavily on previous results.

The algorithm suggested in this paper appears to be impractical due to the arguments of Häggkvist, which can only be applied to digraphs with impractically large number of vertices. However, the authors of [8] suggested a fast and high quality construction heuristic based on two algorithms of exponentially large domination number. This shows that investigation of heuristics of large domination number may well yield heuristics of practical interest and indicates the necessity of further research on domination analysis of ATSP heuristics.

It would be interesting to study the domination ratio of heuristics for other $\mathcal{NP}$-hard combinatorial optimization problems. We believe that the approaches obtained in this paper as well as in [13, 19] can be applied to many more such problems. For example, see certain results for the quadratic assignment problem (QAP) proved in [13]. These results show that heuristics with exponentially large domination numbers exist not only for the ATSP, but also for more complicated optimization problems such as the QAP. More generally, the idea of using polynomial algorithms to find the best among exponentially many feasible solutions of $\mathcal{NP}$-hard optimization problems[1] appears to be worth investigating in various settings: for local search heuristics, genetic algorithms, construction heuristics, exact algorithms, etc.

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