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HAUSDORFF DIMENSION, PRO-p GROUPS, AND KAC-MOODY ALGEBRAS

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Abstract. Every finitely generated profinite group can be given the structure of a metric space, and as such it has a well defined Hausdorff dimension function. In this paper we study Hausdorff dimension of closed subgroups of finitely generated pro-p groups G. We prove that if G is p-adic analytic and $H \leq_c G$ is a closed subgroup, then the Hausdorff dimension of H is $\dim H / \dim G$ (where the dimensions are of H and G as Lie groups). Letting the spectrum $\mathrm{Spec}(G)$ of G denote the set of Hausdorff dimensions of closed subgroups of G, it follows that the spectrum of p-adic analytic groups is finite, and consists of rational numbers.

We then consider some non-p-adic analytic groups G, and study their spectrum. In particular we investigate the maximal Hausdorff dimension of non-open subgroups of G, and show that it is equal to $1 - \frac{1}{d+1}$ in the case of $G = SL_d(F_p[[t]])$ (where $p > 2$), and to $1/2$ if G is the so called Nottingham group (where $p > 5$). We also determine the spectrum of $SL_2(F_p[[t]])$ ($p > 2$) completely, showing that it is equal to $[0, 2/3] \cup \{1\}$.

Some of the proofs rely on the description of maximal graded subalgebras of Kac-Moody algebras, recently obtained by the authors in joint work with E. I. Zelmanov.

1. Introduction

The concept of Hausdorff dimension, which plays a key role in fractal geometry [F], was originally defined over the reals, but can be defined in exactly the same manner over any metric space. We give the exact definition in section 2 below. Recall that every subset S of a metric space has a well-defined Hausdorff dimension, which is denoted by $\dim_{\mathrm{H}}(S)$.

In this paper we focus on metric spaces arising from profinite groups. The study of Hausdorff dimension in profinite groups has recently been initiated by Abercrombie [A]. It is worthwhile mentioning that, while there is a canonical measure on profinite groups (namely, the Haar measure), there is no canonical metric. In fact, every filtration (namely, a descending chain of normal subgroups which form a base for the neighborhoods of the identity) G_n of G gives rise to an invariant metric on G, by setting $d(x, y) = \inf\{|G : G_n|^{-1} : xy^{-1} \in G_n\}$. It turns out that the Hausdorff dimension of a closed subgroup H of G with respect to this

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metric is given by

$$\dim_{\mathrm{H}}(H) = \liminf_{n\to\infty} \frac{\log |HG_n/G_n|}{\log |G/G_n|}. \tag{1}$$

See Theorem 2.4 below, whose proof relies heavily on a result of Abercrombie [A, Proposition 2.6]. Using (1) it is easy to see that the different metrics on G, while defining the same topology, can give rise to different Hausdorff dimension functions (see Example 2.5 below).

However, for finitely generated pro-p groups G, there is a rather natural way to define the metric, by using the filtration $G_n = G^{p^n} = \langle x^{p^n} : x \in G\rangle$. In the theorems below, the Hausdorff dimension is computed relative to this filtration. It can be shown, however, that our main results remain valid with respect to other natural filtrations, such as the lower p-series, or various congruence filtrations when available.

Given an infinite profinite group G (equipped with a metric induced by a prescribed filtration), we define the *dimension spectrum* (or simply the *spectrum*) of G by

$$\mathrm{Spec}(G) = \{\dim_{\mathrm{H}}(H) : H \text{ is a closed subgroup of } G\}.$$

Then the Hausdorff dimension of any open subgroup of G is 1, and finite subgroups of G have Hausdorff dimension 0. In particular we have

$$\{0, 1\} \subseteq \mathrm{Spec}(G) \subseteq [0, 1].$$

The main goal of this paper is to investigate the spectrum of pro-p groups lying in some important classes, such as p-adic analytic groups, more general analytic groups, and the so-called Nottingham group. We shall be particularly interested in groups G whose subgroup structure is far from clear. Thus, instead of finding all closed subgroups of G, we shall at least shed light on their possible 'sizes'. We shall also pose some problems which we consider natural, in the hope that they will stimulate further research in this area.

Our first result determines the Hausdorff dimension of subgroups of p-adic analytic groups (for background, see [DDMS]). Note that, if G is a p-adic analytic pro-p group, then any closed subgroup H of G is also p-adic analytic. The dimension of G as a p-adic analytic group is denoted by $\dim G$. The following result indicates that the concept of Hausdorff dimension generalizes (up to scaling) the concept of dimension for Lie groups.

Theorem 1.1. *Let G be a p-adic analytic pro-p group, and let $H \leq_c G$ be a closed subgroup. Then*

$$\dim_{\mathrm{H}}(H) = \frac{\dim H}{\dim G}.$$

Corollary 1.2. *Let G be as above. Then* $\mathrm{Spec}(G) \subseteq \{0, 1/d, 2/d, \dots, 1\}$*, where $d = \dim G$. In particular, G has a finite spectrum which consists of rational numbers.*

It is natural to ask whether the converse holds.

Problem 1. Let G be a finitely generated pro-p group such that $\mathrm{Spec}(G)$ is finite. Does it follow that G is p-adic analytic?

Even the case of a particularly small spectrum is not well understood.

Problem 2. Let G be a finitely generated pro-p group, and suppose $\mathrm{Spec}(G) = \{0, 1\}$. Does it follow that G is virtually pro-cyclic?

Using Theorem 1.1, it is easy to see that a positive answer to Problem 1 implies a positive answer to Problem 2.

Though we were not able to solve Problem 1, we can still obtain a characterization of p-adic analytic groups in terms of Hausdorff dimension. Indeed, combining Theorem 1.1 with the theory of p-adic analytic groups and with Zelmanov's theorem on the local finiteness of torsion pro-p groups [Z], we prove the following.

Theorem 1.3. *The following are equivalent for a finitely generated pro-p group G.*

(i) The Hausdorff dimension of a closed subgroup H of G is zero if and only if H is finite.

(ii) G is p-adic analytic.

Thus, a non-p-adic analytic pro-p group must contain an infinite subgroups whose Hausdorff dimension is 0.

Let us now turn to another family of pro-p groups, namely those with analytic structure over the power series ring $F_p[[t]]$. Background on such groups can be found in [S] and [LSh]. Focusing on the typical example of $SL_d(F_p[[t]])$ (or its first congruence subgroup), we prove the following:

Theorem 1.4. *Let $G = SL_d(F_p[[t]])$, where $d > 1$. Then*

(i) $\mathrm{Spec}(G)$ *contains intervals; in fact*

$$\mathrm{Spec}(G) \supseteq [0, \frac{d(d+1)-2}{2(d^2-1)}].$$

(ii) If $p > 2$ then 1 is an isolated point in $\mathrm{Spec}(G)$*; in fact*

$$\mathrm{Spec}(G) \cap (1 - \frac{1}{d+1}, 1) = \emptyset.$$

Part (ii) above is best possible in the sense that there exists a closed subgroup $H <_c G$ (namely, a suitable parabolic subgroup) such that $\dim_{\mathrm{H}}(H) = 1-(d+1)^{-1}$. As for part (i), it can be shown that, asymptotically, $\mathrm{Spec}(G)$ contains a larger interval of the form $[0, 1-O(d^{-1/2})]$, but this does not matter for our purpose here.

Theorem 1.4 enables us to determine $\mathrm{Spec}(G)$ completely in the case $d = 2$, $p > 2$.

Corollary 1.5. *Let $G = SL_2(F_p[[t]])$ $(p > 2)$. Then*

$$\mathrm{Spec}(G) = [0, 2/3] \cup \{1\}.$$

Note that $SL_d(F_p[[t]])$ is not a pro-p group, though it contains open pro-p subgroups, such as the first congruence subgroup which we denote by $SL_d^1(F_p[[t]])$. The results mentioned above are valid with respect to the congruence subgroups filtration on $SL_d(F_p[[t]])$, but they are also valid for $G = SL_d^1(F_p[[t]])$ with respect to our usual pro-p filtration G^{p^n}.

Our final result on Hausdorff dimensions concerns the so-called Nottingham group, namely the group of normalized automorphisms of the power series ring $F_p[[t]]$. This group, which we denote by $\mathrm{Aut}^1 F_p[[t]]$, has some remarkable properties and plays important role in the theory of pro-p groups; see, for instance, [J], [Yo], [Sh2], [C]. While we have not been able to determine its spectrum completely, we can prove the following.

Theorem 1.6. *Let* $G = \mathrm{Aut}^1(F_p[[t]])$*, where* $p \geq 5$*. Then*
(i) 1 *is an isolated point in* $\mathrm{Spec}(G)$*; in fact*

$$\mathrm{Spec}(G) \cap (1/2, 1) = \emptyset \text{ if } p > 5,$$

and $\mathrm{Spec}(G) \cap (3/5, 1) = \emptyset$ *if* $p = 5$.
(ii)

$$\{1/n : n \geq 1\} \subseteq \mathrm{Spec}(G) \subseteq [0, 3/p] \cup \{1/n : n \geq 1\}.$$

For $p > 5$ part (i) is best possible, in that there is a subgroup H of G whose Hausdorff dimension is $1/2$. Part (ii) suggests the following.

Problem 3. Does the spectrum of the Nottingham group coincide with the harmonic series $\{1/n : n \geq 1\}$?

In the context of Theorems 1.4 and 1.6, it is natural to associate with a finitely generated pro-p group G a parameter $l(G)$, defined by

$$l(G) = \sup\{\dim_{\mathrm{H}}(H) : H <_c G \text{ is not open}\}.$$

To determine $l(G)$ it suffices to compute $\dim_{\mathrm{H}}(H)$ for all closed non-open subgroups H of G which are maximal with respect to these properties. This is because every closed non-open subgroup of a finitely generated pro-p group is contained in such a subgroup.

Several fundamental problems in pro-p groups could be solved if one could determine the maximal non-open subgroups of groups such as $SL_d^1(F_p[[t]])$; however, this seems an incredibly difficult task, and Aschbacher type theorems (like those describing the maximal subgroups of finite groups of Lie type) seem out of reach here. Our approach to the determination of $l(G)$ is therefore more Lie-theoretic than group-theoretic. The main idea is to associate with G a graded Lie algebra $L(G)$ (using the given filtration G_n or some related series), and to note that non-open subgroups H of G give rise to graded subalgebras $L(H)$ of $L(G)$ which have infinite codimension, such that the Hausdorff dimension of H can be reconstructed from the Lie subalgebra $L(H)$.

For the groups under consideration $L(G)$ turns out to be the positive part of an affine (possibly twisted) Kac-Moody algebra associated with some finite simple Lie algebra $\mathcal{G}$ over F_p. More specifically, if $G = SL_d^1(F_p[[t]])$, then $L(G) \cong sl_d(F_p) \otimes tF_p[t]$, and if $G = \mathrm{Aut}^1(F_p[[t]])$ then $L(G)$ coincides with the positive part of a loop algebra associated with the first Witt algebra W_1 over F_p. We therefore need a description of the graded subalgebras of the positive part of affine Kac-Moody algebras which have infinite codimension and which are maximal with respect to these properties. Subalgebras of this kind – as well as maximal graded subalgebras of affine Kac-Moody algebras – are described in the recent work [BShZ]. This description plays a useful role in the proof of 1.4(ii) and 1.6. In particular, we show that $l(SL_d^1(F_p[[t]])) = 1-(d+1)^{-1}$ if $p > 2$, and that $l(\mathrm{Aut}^1(F_p[[t]])) = 1/2$ if $p > 5$. The proof of part (ii) of 1.6 is somewhat more subtle, and involves successive use of the results of [BShZ].

In order to prove part (ii) of Theorem 1.4, we also need information on maximal Lie subalgebras of $sl_d(F_p)$. In his classical works [D1] and [D2], Dynkin gave a complete description of maximal subalgebras of semisimple Lie algebras over an algebraically closed field F of characteristic 0. It would be interesting to obtain a characteristic p version of Dynkin's theorem. Here we prove a less ambitions result, which suffices for our purpose.

Theorem 1.7. *Let F be a field of characteristic $\neq 2$. Then the maximal dimension of a proper Lie subalgebra of $sl_d(F)$ is $d^2 - d$.*

We are not sure about the novelty of this result; for completeness, we give an elementary and self-contained proof of Theorem 1.7 in the appendix.

We note that, while $l(G) < 1$ for the groups discussed above, there exist finitely generated pro-p groups for which $l(G) = 1$. For example, if G is a free pro-p group of rank $d > 1$, and N is a closed normal subgroup such that G/N is infinite p-adic analytic (e.g. $N = G'$), then it is not difficult to check that $\dim_{\mathrm{H}}(N) = 1$, though N is not open.

We conclude the introduction with somewhat related problems.

Problem 4. What is the spectrum of the free pro-p group on d generators?

Problem 5. Is there a finitely generated pro-p group G such that $\mathrm{Spec}(G) = [0, 1]$?

This paper is organized as follows. Basic notions such as Hausdorff dimension and box dimension are discussed in Section 2, where formula (1) is established. Theorems 1.1-1.3 are proved in Section 3, which deals with p-adic analytic groups. In Section 4 we turn to $F_p[[t]]$-analytic groups, and prove part (i) of Theorem 1.4. The rest of our results except Theorem 1.7 are proved in Section 5, where the relevant descriptions of subalgebras of Kac-Moody algebras are presented. Theorem 1.7 is proved in the appendix.

2. Fractal dimensions

In this section we provide some background and basic definitions concerning Hausdorff dimension, and related fractal dimensions. See Chapters 2 and 3 of Falconer [F] for more details.

We start with some general notation. Unless otherwise stated, all profinite groups are assumed to be infinite. For profinite groups H, G we write $H \leq_c G$ if H is a closed subgroup of G. Group commutators are denoted by $(x, y) = x^{-1}y^{-1}xy$, to be distinguished from Lie products $[x, y]$. For subgroups $H, K \leq_c G$, (H, K) denotes the closed subgroup generated by all commutators (x, y) $(x \in H, y \in K)$. We denote by G^n the closed subgroup generated by all nth powers in G. A *filtration* of G is a descending chain of open normal subgroups $G = G_0 \geq G_1 \geq \cdots \geq G_n \geq \cdots$ which form a base for the neighborhoods of the identity. Note that we have $\bigcap_{n=0}^{\infty} G_n = \{1\}$ for such a series. A filtration G_n of G is said to be p-central if it satisfies $(G_n, G_m) \leq G_{n+m}$ and $G_n^p \leq G_{n+1}$ for all n, m. If the latter condition is replaced by $G_n^p \leq G_{pn}$ then G_n is said to be an N_p-series. We denote by $\mathbb{Z}_p$ the ring of p-adic integers, and by $F[[t]]$ the ring of formal power series over the field F. By a graded algebra we usually mean an $\mathbb{N}$-graded algebra, where $\mathbb{N}$ denotes the positive integers. The rank of a profinite group G is defined to be the minimal r (possibly infinity) such that every closed subgroup of G can be generated (topologically) by at most r elements. Additional notation will be introduced when required.

Let (X, d) be a metric space, let $Y \subseteq X$ and let α, ρ be positive numbers. Define

$$\mathcal{H}_\rho^\alpha(Y) = \inf \sum_i (\mathrm{diam}\ S_i)^\alpha,$$

where $\{S_i\}_{i=0}^{\infty}$ is a cover of Y by sets of diameter at most ρ, and the infimum is taken over all such covers. Note that $\mathcal{H}_\rho^\alpha(Y)$ is non-increasing with ρ, and so the

limit

$$\mathcal{H}^\alpha(Y) = \lim_{\rho\to\infty} \mathcal{H}^\alpha_\rho(Y)$$

exists. It is easy to verify that $\mathcal{H}^\alpha$ is an outer measure on X; it is usually referred to as the α-dimensional Hausdorff measure on X. The following can be readily verified.

Lemma 2.1. *If $\mathcal{H}^\alpha(Y) < \infty$ and $\alpha < \alpha'$, then $\mathcal{H}^{\alpha'}(Y) = 0$.*

We can now define the *Hausdorff dimension* of a set $Y \subseteq X$, as follows:

$$\dim_{\mathrm{H}}(Y) = \sup\{\alpha | \mathcal{H}^\alpha(Y) = \infty\} = \inf\{\alpha | \mathcal{H}^\alpha(Y) = 0\}.$$

Clearly, if $Y \subseteq Y'$ then $\dim_{\mathrm{H}}(Y) \leq \dim_{\mathrm{H}}(Y')$. It can also be shown that

$$\dim_{\mathrm{H}} \left(\bigcup_{n=0}^{\infty} Y_n \right) = \sup \dim_{\mathrm{H}}(Y_n)$$

for subsets $Y_n \subseteq X$.

Let us define additional fractal dimensions, which we shall call the *lower and upper box dimensions*. For $\rho > 0$ define $N_\rho(Y)$ to be the minimal number of sets of diameter at most ρ needed to cover Y. Now set

$$\underline{\dim}_{\mathrm{B}}(Y) = \liminf_{\rho\to 0} \frac{\log N_\rho(Y)}{-\log \rho}, \qquad \overline{\dim}_{\mathrm{B}}(Y) = \limsup_{\rho\to 0} \frac{\log N_\rho(Y)}{-\log \rho}.$$

The connection between the Hausdorff dimension and the box dimensions is given by

Lemma 2.2. *For every $Y \subseteq X$, $\dim_{\mathrm{H}}(Y) \leq \underline{\dim}_{\mathrm{B}}(Y)$.*

See [F, p. 43]. We point out that, in general, equality need not hold.

Now, let G be a profinite group, equipped with a filtration G_n. Define an invariant metric d on G by

$$d(x, y) = \inf\{\ |G : G_n|^{-1} \mid xy^{-1} \in G_n\}.$$

The balls in G, with respect to this metric, are the left (right) cosets of G_n, and the diameter of such a ball is $|G : G_n|^{-1}$. Moreover, every set of diameter $|G : G_n|^{-1}$ is contained in some coset of G_n.

Let $H \leq_c G$ be a closed subgroup. It follows from the above remarks that, if $\rho = |G : G_n|^{-1}$, then $N_\rho(H) = |HG_n : G_n| = |H : H \cap G_n|$. By the definition of the box dimensions we obtain

$$\underline{\dim}_{\mathrm{B}}(H) = \liminf_{n\to\infty} \frac{\log |HG_n/G_n|}{\log |G/G_n|}, \qquad \overline{\dim}_{\mathrm{B}}(H) = \limsup_{n\to\infty} \frac{\log |HG_n/G_n|}{\log |G/G_n|}.$$

In [A, Proposition 2.6] Abercrombie proved that the Hausdorff dimension of H is bounded below by $\liminf_{n\to\infty} \frac{\log |HG_n/G_n|}{\log |G/G_n|}$. We therefore have

Lemma 2.3. *Let H, G be as above. Then $\dim_{\mathrm{H}}(H) \geq \underline{\dim}_{\mathrm{B}}(H)$.*

Combining this result with Lemma 2.2, we can derive our main tool in this paper.

Theorem 2.4. *Let G be a profinite group with a filtration $\{G_n\}_{n=0}^\infty$ and let $H \leq_c G$ be a closed subgroup. Then*

$$\dim_{\mathrm{H}}(H) = \underline{\dim}_{\mathrm{B}}(H) = \liminf_{n\to\infty} \frac{\log |HG_n/G_n|}{\log |G/G_n|} = \liminf_{n\to\infty} \frac{\log |H : H \cap G_n|}{\log |G : G_n|},$$

where the Hausdorff dimension is computed with respect to the metric associated with the filtration $\{G_n\}$.

It is now clear that G and its open subgroups have Hausdorff dimension 1, and that finite subgroups of G have Hausdorff dimension zero (assuming G is infinite). As shown below, the Hausdorff dimension does depend on the filtration.

Example 2.5. Let $G = \mathbb{Z}_p \oplus \mathbb{Z}_p$ and $H = \{0\} \oplus \mathbb{Z}_p$. Then if the Hausdorff dimension is computed relative to the filtration $G_n = p^n\mathbb{Z}_p \oplus p^n\mathbb{Z}_p$ we obtain $\dim_{\mathrm{H}}(H) = \frac{1}{2}$, while if the Hausdorff dimension is computed relative to the filtration $G_n = p^{2n}\mathbb{Z}_p \oplus p^n\mathbb{Z}_p$ we obtain $\dim_{\mathrm{H}}(H) = \frac{1}{3}$.

Unless otherwise stated, if G is a pro-p group, it is considered as a metric space with respect to the filtration $G_n = G^{p^n}$.

3. p-adic analytic groups

This section is devoted to the proof of Theorems 1.1 and 1.3. The reader is referred to [DDMS] for background on p-adic analytic pro-p groups and powerful pro-p groups.

We need some ad-hoc notation. For a group G and a positive integer n let $G^{\{n\}}$ denote the set of all nth powers of elements of G (thus $G^{p^n} = \langle G^{\{p^n\}}\rangle$). We need the following somewhat technical result.

Lemma 3.1. *Let G be a finitely generated powerful pro-p group and let $H \leq_c G$ be a closed subgroup. Let $G_n = G^{p^n}$, $H_n = H^{p^n}$. Then*

(i) There exists a constant $c > 0$ such that $H \cap G_n = (H \cap G_c)^{\{p^{n-c}\}}$ for all $n \geq c$.

(ii) There exists a constant $c > 0$ such that $H \cap G_n \leq H_{n-c}$ for all $n \geq c$.

Proof. Let $U_n = G_n/G_{n+1}$ $(n \geq 1)$. Since G is powerful, the sections U_n are elementary abelian, and can be regarded as linear spaces over F_p. The map $x \mapsto x^p$ gives rise to well defined linear epimorphisms $\phi_n : U_n \to U_{n+1}$, which will play a useful role in the proof. Since $\dim U_n$ are bounded above by the rank, say r, of G, we see that $\dim U_n$ stabilizes and that ϕ_n is an isomorphism for all large n.

Define $V_n = (H \cap G_n)G_{n+1}/G_{n+1}$. Then $V_n \leq U_n$ and $\phi_n(V_n) \leq V_{n+1}$. Since ϕ_n are injective for large n, the series $\dim V_n$ is non-decreasing for large n. But $\dim V_n \leq r$ for all n. Hence the series $\{\dim V_n\}$ stabilizes. It follows that there is a constant c such that, if $n \geq c$, then ϕ_n induces an isomorphism from V_n to V_{n+1}. In particular, $\phi_n(V_n) = V_{n+1}$ for all $n \geq c$.

Claim. Let $n \geq c$. Then the map $x \mapsto x^p$ from $H \cap G_n$ to $H \cap G_{n+1}$ is surjective.

Let $h \in H \cap G_{n+1}$. Since $n \geq c$ there exist $h_n \in H \cap G_n$ and $g_{n+2} \in G_{n+2}$ such that $h = h_n^p g_{n+2}$. Note that $g_{n+2} \in H \cap G_{n+2}$.

We shall now show by induction on $k \geq 0$ that

$$h = (h_n h_{n+1} \cdots h_{n+k})^p g_{n+k+2}, \tag{2}$$

where $h_{n+m} \in H \cap G_{n+m}$ $(0 \leq m \leq k)$, and $g_{n+k+2} \in H \cap G_{n+k+2}$.

The case $k = 0$ has already been proved. Suppose (2) holds for k and let us prove it for $k+1$. Since $n + k + 2 > c$ there are elements $h_{n+k+1} \in H \cap G_{n+k+1}$ and $\widetilde{g} \in H \cap G_{n+k+3}$ such that $g_{n+k+2} = h_{n+k+1}^p \widetilde{g}$. Hence $h = (h_n h_{n+1} \cdots h_{n+k})^p h_{n+k+1}^p \widetilde{g}$.

Since G is powerful we have

$$\widetilde{h} = (h_{n+k+1}, h_n h_{n+1} \cdots h_{n+k}) \in (G_{n+k+1}, G) \leq G_{n+k+2}.$$

Furthermore, if $p = 2$ we even have $\widetilde{h} \in G_{n+k+3}$. It follows that, in G/G_{n+k+3}, the image of $\widetilde{h}$ commutes with the images of h_{n+k+1} and of $h_n h_{n+1} \cdots h_{n+k}$.

Applying the well known formula

$$(xy)^k = x^k y^k (y,x)^{k(k-1)/2},$$

which holds whenever (x,y) commutes with x and y, we see that

$$h = (h_n h_{n+1} \cdots h_{n+k} h_{n+k+1})^p \widetilde{h}^{p(p-1)/2} g' \widetilde{g},$$

where $g' \in H \cap G_{n+k+3}$. Since $\widetilde{h} \in G_{n+k+2}$ and $\widetilde{h} \in G_{n+k+3}$ if $p = 2$, we obtain that, in any case, $\widetilde{h}^{p(p-1)/2} \in G_{n+k+3}$. Hence we may write

$$h = (h_n h_{n+1} \cdots h_{n+k} h_{n+k+1})^p g_{n+k+3},$$

where $h_{n+m} \in H \cap G_{n+m}$ for $0 \le m \le k$ and $g_{n+k+3} \in H \cap G_{n+k+3}$.

This completes the proof of (2).

Clearly, the series $\{h_n h_{n+1} \cdots h_{n+k}\}_{k=0}^{\infty}$ is a Cauchy series, so it converges to some $h' \in H \cap G_n$. On the other hand it follows from (2) that

$$h = \lim_{k \to \infty} (h_n h_{n+1} \cdots h_{n+k})^p,$$

and since $x \mapsto x^p$ is continuous we have $h = (h')^p$.

Since h was an arbitrary element of $H \cap G_{n+1}$ we obtain $H \cap G_{n+1} = (H \cap G_n)^{\{p\}}$. By repeated use of this equality we obtain $H \cap G_n = (H \cap G_c)^{\{p^{n-c}\}}$ for all $n \ge c$. This proves part (i).

Part (ii) follows from part (i). Indeed

$$H \cap G_n = (H \cap G_c)^{\{p^{n-c}\}} \le H^{p^{n-c}} = H_{n-c}.$$

□

Applying part (ii) of the above result, we obtain the following 'Artin-Rees type' result for p-adic analytic groups.

Corollary 3.2. *Let G be a p-adic analytic pro-p group and let $H \le_c G$ be a closed subgroup. Let $G_n = G^{p^n}$ and $H_n = H^{p^n}$. Then there exists a constant c such that $H \cap G_n \le H_{n-c}$ for all $n \ge c$.*

Proof. Since G is p-adic analytic, there exists an integer $a > 0$ such that G_a is powerful. Applying 3.1(ii) for $H \cap G_a$ inside G_a, we see that for some constant b and for all $n \ge b$ we have

$$H \cap G_{a+n} = H \cap {G_a}^{p^n} \le (H \cap G_a)^{p^{n-b}} \le H_{n-b}.$$

The result follows with $c = a + b$. □

We can now prove our main results on the Hausdorff dimension in p-adic analytic pro-p groups. Recall that the Hausdorff dimension is computed relative to the filtration $G_n = G^{p^n}$.

Proof of Theorem 1.1. It is was shown by Lazard [La, p. 95] that

$$\dim G = \lim_{n \to \infty} \frac{\log_p |G : G^{p^n}|}{n}, \qquad \dim H = \lim_{n \to \infty} \frac{\log_p |H : H^{p^n}|}{n}.$$

By definition $H^{p^n} \le H \cap G_n$, and hence

$$\frac{\log_p |H : H \cap G_n|}{\log_p |G : G_n|} \le \frac{\log_p |H : H^{p^n}|/n}{\log_p |G : G_n|/n} \xrightarrow{n \to \infty} \frac{\dim H}{\dim G}.$$

By Theorem 2.4 we have

$$\dim_{\mathrm{H}}(H) = \liminf_{n\to\infty} \frac{\log_p |H : H \cap G_n|}{\log_p |G : G_n|} \leq \frac{\dim H}{\dim G}.$$

On the other hand, by Corollary 3.2 we have $H \cap G_{n+c} \leq H^{p^n} = H_n$ for all n, and hence

$$\dim_{\mathrm{H}}(H) = \liminf_{n\to\infty} \frac{\log_p |H : H \cap G_{n+c}|}{\log_p |G : G_{n+c}|} \geq \lim_{n\to\infty} \frac{\log_p |H : H_n|/n}{\log_p |G : G_{c+n}|/n} = \frac{\dim H}{\dim G}.$$

The result follows.

Proof of Theorem 1.3. Obviously, if H is a finite subgroup of the infinite pro-p group G, then $\dim_{\mathrm{H}}(H) = 0$. Suppose now that G is p-adic analytic, and that H is a closed subgroup satisfying $\dim_{\mathrm{H}}(H) = 0$. Applying Theorem 1.1 we obtain $\dim H = 0$ (as a p-adic Lie group), and this implies that H is finite. Therefore condition (ii) implies condition (i).

To prove the converse, suppose G is not p-adic analytic. Then G is a finitely generated infinite pro-p group, and so by Zelmanov's Theorem [Z], it cannot be periodic. So let $h \in G$ be an element of infinite order. Define $H = \overline{\langle h \rangle} \cong \mathbb{Z}_p$. We will show that $\dim_{\mathrm{H}}(H) = 0$, so that condition (i) fails to hold. Suppose, by contradiction, that $\dim_{\mathrm{H}}(H) > 0$. Then, by 2.4, there is $\epsilon > 0$ such that

$$\liminf_{n\to\infty} \frac{\log |H : H \cap G_n|}{\log |G : G_n|} > \epsilon.$$

Since $H_n = H^{p^n} \leq H \cap G_n$, we obtain

$$\liminf_{n\to\infty} \frac{\log |H : H_n|}{\log |G : G_n|} > \epsilon.$$

Note that $|H : H_n| = p^n$; hence, if n is sufficiently large, then

$$\frac{n}{\log_p |G : G_n|} > \epsilon.$$

This implies $|G : G_n| \leq p^{cn}$ for all n, where $c = 1/\epsilon$. It now follows from [La, p. 591] (see also [Sh1, Theorem B]) that G is p-adic analytic.

This contradiction completes the proof of Theorem 1.3.

Remark. Theorems 1.1 and 1.3 remain valid if the Hausdorff dimension function is replaced by the upper box dimension function.

4. $F_p[[t]]$-analytic groups

Let F_p be the field with p elements, and let $F_p[[t]]$ denote the local ring of formal power series over F_p. Recently there has been some interest in certain pro-p groups which have an analytic structure over $F_p[[t]]$ (see [LSh]). In this section we study Hausdorff dimension in such groups. Though our main interest is in finitely generated pro-p groups, it is necessary to consider certain groups which are not finitely generated, such as the additive and the multiplicative groups of $F_p[[t]]$. For such groups G the subgroups G^{p^n} are not open anymore, and so other filtrations have to be considered.

Define $G^+ = tF_p[[t]]$, considered as an additive group, and let $G_n^+ = t^n F_p[[t]]$ ($n \geq 1$). Then $\{G_n^+\}$ is a filtration of G^+.

Lemma 4.1. $\mathrm{Spec}(G^+) = [0,1]$ *with respect to the above filtration.*

Proof. Given $\alpha \in [0,1]$, choose a subset $S \subseteq \mathbb{N}$ of density α. Then the closed subgroup generated by t^n $(n \in S)$ is easily seen to have Hausdorff dimension α. □

A similar result holds for the direct sum of k copies of G^+.

Next, define $G^* = 1+tF_p[[t]]$, considered as a multiplicative group. Let $G_n^* = 1+t^nF_p[[t]]$ $(n \geq 1)$. Then $\{G_n^*\}$ is a filtration of G^*. Let $B = \mathbb{N}\backslash p\mathbb{N}$, the set of positive integers which are not divisible by p. The following result is straightforward.

Lemma 4.2. *The elements* $1 + t^n$ $(n \in B)$ *form a minimal generating set for* G^* *as a pro-p group.*

Using the above observation and density arguments, it is easy to obtain the following.

Lemma 4.3. $\mathrm{Spec}(G^*) = [0,1]$ *with respect to the above filtration.*

A similar result holds for a direct product of k copies of G^*.

Let us now turn to $G = SL_d(F_p[[t]])$ (where $p > 2$) and its congruence subgroups

$$G_n = SL_d^n(F_p[[t]]) = Ker(G \to SL_d(F_p[[t]]/t^nF_p[[t]])) \ \ (n \geq 1).$$

Note that G_1 is a pro-p group, $G/G_1 \cong SL_d(p)$, and for $n \geq 1$, G_n/G_{n+1} is elementary abelian of order p^{d^2-1}. In fact G_2 coincides with the Frattini subgroup of G_1, and so $d(G_1) = d^2 - 1$. In particular, G and G_1 are finitely generated profinite groups.

Since $(G_n, G_m) \leq G_{n+m}$, we can associate with G_1 a graded Lie algebra $L(G_1) = \bigoplus_{n\geq 1} G_n/G_{n+1}$, where the Lie product of homogeneous elements is induced by commutation in G. Moreover, since $G_n^p \leq G_{pn}$, the pth power map in G induces on $L(G)$ the structure of a restricted Lie algebra. It is known that

$$L(G) \cong \mathcal{G} \otimes_{F_p} tF_p[t], \tag{3}$$

as restricted Lie algebras, where $\mathcal{G} = sl_d(F_p)$. See [LSh] for this and for more details.

It is easy to see that $\mathcal{G} = \mathcal{G}^{[p]}$, namely, $\mathcal{G}$ is spanned by the pth powers of its elements. Using the isomorphism (3), it follows that pth powers of elements of G_n generate G_{pn} modulo G_{pn+1}. However, any subset of G_m which generates it modulo G_{m+1} generates G_m as a normal subgroup of G. The normality of G_n^p now implies that $G_n^p = G_{pn}$ for all n. In particular, $G_1^{p^n} = G_{p^n}$, so the filtration $\{G_1^{p^n}\}$ is a sub-filtration of the congruence filtration $\{G_n\}$ on G_1. This remark enables us to replace our canonical filtration $G_1^{p^n}$ by the congruence filtration G_n, which is easier to work with. The reader can now easily verify that all results below that deal with $\mathrm{Spec}(G)$ with respect to G_n are also valid for G_1 with respect to the filtration $G_1^{p^n}$.

We need some notation. Let I denote the identity $n \times n$ matrix. For $k < d$ define

$$T_k = \{I + (a_{ij})_{1\leq i,j\leq d} : a_{ij} \in tF_p[[t]] \text{ and } a_{ij} = 0 \text{ if } j - i < d - k\}.$$

Define

$$T = \{I + (a_{ij})_{1\leq i,j\leq d} \in G_1 : a_{ij} = 0 \text{ if } j < i\}.$$

Then

$$T_k = \begin{pmatrix} & & & & & \overbrace{\quad}^{k} & & \\ 1 & 0 & \cdots & \cdots & 0 & * & \cdots & * \\ 0 & \ddots & \ddots & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & & & & & \ddots & * \\ & & & & & & & 0 \\ & & & & & & & \vdots \\ & & & & & & \ddots & \vdots \\ \vdots & & & & & \ddots & \ddots & 0 \\ 0 & \cdots & & & & \cdots & 0 & 1 \end{pmatrix}$$

and

$$T = \begin{pmatrix} 1+* & * & \cdots & * \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & 1+* \end{pmatrix},$$

where the stars stand for arbitrary elements of the maximal ideal $tF_p[[t]]$. The congruence filtration G_n induces on the additive subgroups $I + tF_p[[t]]e_{ij} \cong G^+$ $(i < j)$ the filtration G_n^+ mentioned above. Similarly, G_n induces on the multiplicative subgroups $(1 + tF_p[[t]])e_{ii} \cong G^*$ the filtration G_n^* mentioned above.

Combining this observation with Theorem 2.4 and the equality $|G_1 : G_n| = p^{(n-1)(d^2-1)}$, we see that

$$\dim_{\mathrm{H}}(T_k) = \frac{(k+1)k}{2(d^2-1)} \quad \text{and} \quad \dim_{\mathrm{H}}(T) = \frac{d(d+1)-2}{2(d^2-1)}.$$

We can now show that $\mathrm{Spec}(G)$ contains intervals. Indeed we have

Proposition 4.4. *For every $0 \le \xi \le \frac{d(d+1)-2}{2(d^2-1)}$ there is a closed subgroup $H_\xi \le_c G$ (contained in T) such that $\dim_{\mathrm{H}}(H_\xi) = \xi$. Consequently, $\mathrm{Spec}(G)$ contains the closed interval $[0, \frac{d(d+1)-2}{2(d^2-1)}]$.*

Proof. Suppose first that $\xi \le \frac{d(d-1)}{2(d^2-1)}$, and choose k such that $\frac{k(k-1)}{2(d^2-1)} \le \xi \le \frac{(k+1)k}{2(d^2-1)}$. We will construct H_ξ so that $T_{k-1} \le_c H_\xi \le_c T_k$. Since $T_k/T_{k-1} \cong \underbrace{G^+ \oplus \cdots \oplus G^+}_{k}$, this can be done using Lemma 4.1 and the remark following it.

Suppose now that $\frac{d(d-1)}{2(d^2-1)} \le \xi \le \frac{d(d+1)-2}{2(d^2-1)}$. Since $T/T_{d-1} \cong \underbrace{G^* \oplus \cdots \oplus G^*}_{d-1}$, we can construct (using 4.3 and the remark following it) H_ξ such that $T_{d-1} \le_c H_\xi \le_c T$, whose Hausdorff dimension is ξ. □

This proves part (i) of Theorem 1.4.

Using similar constructions inside parabolic subgroups, it is possible to show that $\mathrm{Spec}(G_1)$ contains a closed interval of the form $[0, 1 - c \cdot d^{-1/2}]$, where $c > 0$ is some absolute constant.

5. Subalgebras of Kac-Moody algebras

Fix a filtration G_n of a pro-p group G, and suppose G_n is a central p-series. Then the associated Lie ring $L = L(G) = \bigoplus G_n/G_{n+1}$ is an infinite dimensional graded Lie algebra over F_p. Let H be a closed subgroup of G. Then H gives rise to the graded subalgebra $\bigoplus_{n\geq 1}(H \cap G_n)G_{n+1}/G_{n+1}$ of $L(G)$, which (by a slight abuse of notation) we denote by $L(H)$. If $|G : H| = \infty$ then $L(H)$ has infinite codimension in $L(G)$.

Theorem 2.4 enables us to reconstruct the Hausdorff dimension of H from the dimensions of the homogeneous components of $L(G)$ and $L(H)$. We need some notation. Let $L_n = L_n(G) = G_n/G_{n+1}$ and $L_n(H) = (H \cap G_n)G_{n+1}/G_{n+1}$. Define the (lower) *density* of a graded subalgebra $K = \bigoplus_{n\geq 1} K_n$ of $L(G)$ by

$$D(K) = \liminf_{m\to\infty} \frac{\sum_{n\leq m} \dim K_n}{\sum_{n\leq m} \dim L_n}.$$

Then we clearly have

Lemma 5.1. *With the above notation,*

$$\dim_{\mathrm{H}}(H) = D(L(H)),$$

where the Hausdorff dimension is computed in G with respect to the filtration G_n.

Denoting by $\mathrm{Spec}(L)$ the set of densities of graded subalgebras of L, we see that $\mathrm{Spec}(G) \subseteq \mathrm{Spec}(L(G))$. Equality need not hold, since in general not every graded subalgebra of $L(G)$ arises from some closed subgroup $H \leq G$. Similarly, it follows that $l(G)$ is bounded above by the maximal density of a graded subalgebra of infinite codimension in L, though equality need not hold. Anyhow, Lemma 5.1 indicates that the study of graded subalgebras of infinite codimension in $\mathbb{N}$-graded Lie algebras may be relevant for the computation of $\mathrm{Spec}(G)$ and of $l(G)$ in particular.

For the groups in question (where p does not divide d), the Lie algebras $L(G)$ take a rather simple form. They can be regarded as the positive part of (possibly twisted) Kac-Moody algebras, namely, of loop algebras associated with some finite-dimensional simple Lie algebras over F_p.

Let us first recall some definitions. Let $\mathcal{G}$ be a simple finite-dimensional Lie algebra over a field F. Let $F[t, t^{-1}]$ denote the ring of Laurent polynomials and set $L(\mathcal{G}) = \mathcal{G} \otimes F[t, t^{-1}]$. Let $m \geq 1$ and let α be a $\mathbb{Z}/k\mathbb{Z}$-grading of $\mathcal{G}$, $\mathcal{G} = \bigoplus_{i=0}^{k-1} \mathcal{G}_i$. Then the $\mathbb{Z}$-graded Lie subalgebra of $L(\mathcal{G})$ defined by

$$L(\mathcal{G}, k, \alpha) = \bigoplus_{n\in\mathbb{Z}} \mathcal{G}_{n \bmod k} \otimes t^n$$

is said to be a loop algebra associated with $\mathcal{G}$ (with respect to k and α). Its positive part is defined by

$$L^+(\mathcal{G}, k, \alpha) = \bigoplus_{n\geq 1} \mathcal{G}_{n \bmod k} \otimes t^n.$$

Note that $L(\mathcal{G})$ itself is a loop algebra on $\mathcal{G}$, and that $L^+(\mathcal{G}) = \mathcal{G} \otimes tF[t]$. Similar notation will be used for any (not necessarily simple) finite-dimensional Lie algebras $\mathcal{G}$.

It is known that, for $\mathcal{G}$ simple, any graded subalgebra of infinite codimension in $L^+(\mathcal{G}, k, \alpha)$ can be extended to a subalgebra which is maximal with respect to these properties (see [BShZ, §4]). Therefore, in order to compute the maximal density

of a graded subalgebra of infinite codimension in $L^+(\mathcal{G},k,\alpha)$ it suffices to compute the density of the maximal ones. In [BShZ, Theorem 4.2] those subalgebras are determined, under the assumption that $\mathcal{G}$ is central simple over F (i.e. its centroid coincides with F). In order to formulate the result, we associate with a graded subalgebra $\mathcal{H}=\bigoplus_{i=0}^{k-1}\mathcal{H}_i$ of $\mathcal{G}=\bigoplus_{i=0}^{k-1}\mathcal{G}_i$, a graded subalgebra of $L^+(\mathcal{G},k,\alpha)$, defined by

$$L^+(\mathcal{H},k,\alpha)=\bigoplus_{n\geq 1}\mathcal{H}_{n \bmod k}\otimes t^n\subseteq L^+(\mathcal{G},k,\alpha).$$

Theorem 5.2. *Let $\mathcal{G}$ be a central simple finite-dimensional Lie algebra over a field F. Let M be a graded subalgebra of infinite codimension in $L=L^+(\mathcal{G},k,\alpha)$ which is maximal with respect to these properties. Then one of the following holds:*

(i) $M=L^+(\mathcal{H},k,\alpha)$, where $\mathcal{H}=\bigoplus_{i=0}^{m-1}\mathcal{H}_i$ is a maximal graded subalgebra of $\mathcal{G}$.

(ii) $M=L^+(\mathcal{G},qk,\beta)$ for some prime q and a $\mathbb{Z}/qk\mathbb{Z}$-grading β of $\mathcal{G}$.

Note that the average dimension of a homogeneous component of a loop algebra of period k on $\mathcal{G}$ is $\frac{1}{k}\dim\mathcal{G}$. It follows that the density (in L) of the subalgebra of type (i) above is $\dim\mathcal{H}/\dim\mathcal{G}$, and that the density of the subalgebra of type (ii) above is $1/q$. We therefore have the following.

Corollary 5.3. *Let $L=L^+(\mathcal{G},k,\alpha)$ be as above, and let M be a graded subalgebra of L, maximal with respect to having infinite codimension. Then either $D(M)=1/q$ for some prime q, or $D(M)=\dim\mathcal{H}/\dim\mathcal{G}$ for some maximal graded subalgebra $\mathcal{H}$ of $\mathcal{G}$.*

In particular, we see that $D(M)$ is always rational.

Given the simple Lie algebra $\mathcal{G}$ and its cyclic grading α, define

$$l(\mathcal{G},\alpha)=\max\{\dim\mathcal{H}/\dim\mathcal{G}:\mathcal{H}\text{ is a proper graded subalgebra of }\mathcal{G}\}.$$

Note that, if $k=1$ – that is, if $L=L^+(\mathcal{G})$ – then $l(\mathcal{G},\alpha)=\max\{\dim\mathcal{H}/\dim\mathcal{G}\}$, where $\mathcal{H}$ ranges over all maximal subalgebras of $\mathcal{G}$; we denote this invariant by $l(\mathcal{G})$. Applying the preceding results, we obtain our main tool in the investigation of $l(G)$.

Corollary 5.4. *Let $L=L(\mathcal{G},k,\alpha)$ be as above. Then the density of a graded subalgebra M of L of infinite codimension is at most $\max\{l(\mathcal{G},\alpha),1/2\}$. Consequently, if G is a pro-p group satisfying $L(G)\cong L^+(\mathcal{G},k,\alpha)$, then*

$$l(G)\leq\max\{l(\mathcal{G},\alpha),1/2\}.$$

Now, let $G=SL_d^1(F_p[[t]])$ (where p is odd), and let G_n be the congruence filtration. As mentioned in the preceding section, the associated Lie algebra $L(G)$ is isomorphic to $L^+(\mathcal{G})$, where $\mathcal{G}=sl_d(F_p)$. Suppose first that p does not divide d. Then $sl_d(F_p)$ is central simple over F_p, and so the above results are applicable. We assume that $p>2$. Then Theorem 1.7 yields

$$l(sl_d(F_p))=\frac{d^2-d}{d^2-1}=1-\frac{1}{d+1}.$$

It follows from 5.4 that

$$l(G)\leq 1-\frac{1}{d+1}.$$

On the other hand, letting H be the stabilizer in G of a subspace of codimension 1 of the natural module, we have $\dim_{\mathrm{H}}(H) = 1 - \frac{1}{d+1}$. It follows that

$$l(SL_d^1(F_p[[t]])) = 1 - \frac{1}{d+1}.$$

This completes the proof of Theorem 1.4 in the case where p does not divide d. If p divides d then $L(G)$ is still isomorphic to $L^+(\mathcal{G})$, where $\mathcal{G} = sl_d(F_p)$, but $\mathcal{G}$ is no longer simple. However, the required result is easily obtained by factoring out the center of $\mathcal{G}$.

This completes the proof of Theorem 1.4.

It may be interesting to compute $l(G)$ for other classical groups G over power series rings, using essentially the same method.

Finally, let us turn to the proof of Theorem 1.6. Let $\mathcal{G} = W_1$ be the first Witt algebra over F_p. Then $\mathcal{G}$ is a simple Lie algebra with a basis $e_0, \dots, e_{p-1}$ satisfying

$$[e_i, e_j] = (j-i)e_{i+j \bmod p}.$$

Thus $\mathcal{G}$ admits $\mathbb{Z}/p\mathbb{Z}$-grading $\mathcal{G} = \bigoplus_{i=0}^{p-1} \mathcal{G}_i$, where $W_i = \langle e_i \rangle$. We denote this grading by α.

Now, let $G = \mathrm{Aut}^1(F_p[[t]])$ be the Nottingham group, and let G_n be its congruence filtration. Thus $G_1 = G$ and G_n is the collection of automorphisms $g \in G$ sending t to $t + \sum_{i>n} a_i t^i$ ($a_i \in F_p$). It is well known that G_n is an N_p-series and that the corresponding Lie algebra $L = L(G) = \bigoplus_{n \geq 1} G_n/G_{n+1}$ has the form $L^+(\mathcal{G}, \alpha, p)$.

It is easy to see that the maximal graded subalgebras of $\mathcal{G}$ (with respect to the grading α) have the form $F_p e_0 + F_p e_a + F_p e_{p-a}$ for some $1 \leq a \leq (p-1)/2$. Thus

$$l(W, \alpha) = \frac{3}{p}.$$

Suppose first that $p > 5$. Then it follows from Corollary 5.4 that $l(G) \leq 1/2$. However, for each integer m, G has a closed subgroup

$$G(m) = \{t \mapsto t + \sum_{n>1, n \equiv 1 \bmod m} a_n t^n : a_n \in F_p\},$$

whose associated subalgebra has the form $L(m) := \sum_{n \equiv 0 \bmod m} L_n$, where $L_n = G_n/G_{n+1}$. This implies $\dim_{\mathrm{H}}(G(m)) = D(L(m)) = 1/m$ by 5.1. In particular, $\dim_{\mathrm{H}}(H) = 1/2$ for some closed subgroup H of G. It follows that

$$l(G) = 1/2.$$

If $p = 5$ then Corollary 5.4 yields $l(G) \leq 3/5$, but it is not clear whether equality holds.

Part (i) of Theorem 1.6 is proved.

To prove part (ii), first note that, by the above discussion,

$$\mathrm{Spec}(G) \supseteq \{1/m : m \geq 1\}.$$

Now, let $L = \bigoplus_{n \geq 1} L_n = L^+(\mathcal{G}, p, \alpha)$ be as above, and let $M \subset L$ be a graded subalgebra of infinite codimension which is maximal with respect to these properties. Then, according to [BShZ, 4.4], one of the following holds.

(i) There exists $1 \leq a \leq (p-1)/2$ such that

$$M = \bigoplus_{n \equiv 0, a, -a \bmod p} L_n.$$

(ii) There exists a prime $q \neq p$ such that

$$M = \bigoplus_{n \equiv 0 \bmod q} L_n.$$

Moreover, the Lie algebras of type (ii) are isomorphic to the original Lie algebra L, by an isomorphism sending L_n to L_{qn}.

We now claim that, if M is a graded subalgebra of L, then either $D(M) = 1/m$ for some m, or $D(M) \leq 3/p$. Indeed, suppose $D(M) > 3/p$ and let us show that $D(M) = 1/m$ for some m. If $\dim L/M < \infty$ then $D(M) = 1$ and we are done. Otherwise M is contained in a maximal graded subalgebra of infinite codimension in L, say K_1. Clearly, K_1 cannot be of type (i), since this would imply $D(M) \leq D(K_1) = 3/p$. Hence K_1 is of type (ii). It follows that $D(K_1) = 1/q_1$ for some prime q_1 and that $K_1 \cong L$. If $\dim K_1/M < \infty$ then $D(M) = D(K_1) = 1/q_1$ and we are done. Otherwise there is a maximal graded subalgebra of infinite codimension K_2 of K_1 which contains M. Since $K_1 \cong L$ it follows that K_2 satisfies conditions (i) or (ii) above (with L replaced by K_1). However, K_2 cannot be of type (i) by density considerations. Hence K_2 is of type (ii), so its density in K_1 is $1/q_2$ for some prime q_2. This implies that the density of K_2 in L is $1/(q_1q_2)$. If M has finite codimension in K_2, then $D(M) = 1/(q_1q_2)$. Otherwise we can continue in this manner, based on the isomorphism $K_2 \cong L$. Note that the assumption $D(M) > 3/p$ implies that this process is finite. This completes the proof of the claim.

Having proved that $D(M) \in [0, 3/p] \cup \{1/m : m \geq 1\}$ for any graded subalgebra M of L, it follows from Lemma 5.1 that

$$\mathrm{Spec}(G) \subseteq [0, 3/p] \cup \{1/m : m \geq 1\}.$$

This concludes the proof of Theorem 1.6, provided the Hausdorff dimension is computed with respect to the congruence filtration G_n of the Nottingham group G. However, it is known that every open normal subgroup N of G lies between G_n and G_{n+2} for some n [Yo], and this implies that the theorem remains valid with respect to any other filtration.

6. Appendix

We give below a proof of Theorem 1.7. Let F be a field of characteristic $p \neq 2$. It is clear that $sl_d(F)$ has a (parabolic) subalgebra of dimension $d^2 - d$. It therefore suffices to show that, if L is a subalgebra of $sl_d(F)$ whose dimension exceeds $d^2 - d$, then $L = sl_d(F)$. For the rest of this discussion we fix p, d and L as above. We also let e_{ij} $(i, j = 1, \ldots, d)$ be the standard matrix units. Our Lie algebra notation is standard and follows [H]. Note that $sl_d(F)$ is a perfect Lie algebra, and so it follows that every maximal proper subalgebra of $sl_d(F)$ contains the center Z of $sl_d(F)$. We shall assume, for simplicity, that p does not divide d, in which case $Z = 0$. If p divides d then our arguments still apply after factoring out the center.

The first lemma follows from basic linear algebra.

Lemma 6.1. *Let $\varphi_1, \ldots, \varphi_f$ be linearly independent linear functionals on the linear space V. Let $v_1, \ldots, v_n$ be a basis for V. Then there are f basis elements $v_{i_1}, \ldots, v_{i_f}$ such that, for each $1 \leq m \leq f$, there is a linear combination $\psi_m = \sum_j c_{mj}\varphi_j$ satisfying $\psi_m(v_{i_k}) = \delta_{mk}$.*

We need some definitions. Let D denote the subalgebra of diagonal matrices in $sl_d(F)$. Then $\dim D = d-1$. Let $E = \sum_{i\neq j} Fe_{ij}$ and let P be the projection of $sl_d(F)$ onto E (whose kernel is D).

Lemma 6.2. *For each i there are $j,k \neq i$ with $e_{ij}, e_{ki} \in L$.*

Proof. Let $f = \dim(D \cap L)$. Then $f \leq d-1$, and since $\dim(sl_d(F)/L) < d-1$ we have $f \geq 1$. Choose a basis $h_1, \ldots, h_f$ for $L \cap D$. We can view each h_j as a functional on the roots space of $sl_d(F)$. Then $h_1, \ldots, h_f$ are linearly independent functionals.

Now, the subspace $P(L)$ has codimension $< f$ in E. This implies that, for any subset R of f matrix units $e_{ij} \in E$, some non-trivial linear combination $\sum_{e_{ij}\in R} c_{ij}e_{ij}$ lies in $P(L)$.

Fix i, $1 \leq i \leq d$, and let us show that, for some $j \neq i$ we have $e_{ij} \in L$. It is well known that the roots corresponding to the root elements $e_{i1}, \ldots, e_{ii-1}, e_{ii+1}, \ldots, e_{id}$ form a basis for the roots space of $sl_d(F)$. Apply Lemma 6.1 to the functionals $h_1, \ldots, h_f$ acting on the roots space, equipped with the above basis. Let $e_{ij_1}, \ldots, e_{ij_f}$ be as in the conclusion of 6.1.

Let $\sum_{k=1}^{f} c_{ij_k}e_{ij_k}$ be a non-trivial linear combination of these elements which lies in $P(L)$. Then there exists $b \in D$ such that $b + \sum_{k=1}^{f} c_{ij_k}e_{ij_k} \in L$. Now, suppose $c_{ij_m} \neq 0$. Consider the linear combination $h = \sum_{k=1}^{f} a_k h_k$ corresponding to φ_m in the conclusion of Lemma 6.1. Regarding h as an element of L, we have $c_{ij_m}e_{ij_m} = [h, b + \sum_{k=1}^{f} c_{ij_k}e_{ij_k}] \in L$. It follows that $e_{ij} \in L$, where $j = j_m$.

In a similar manner it follows that $e_{ki} \in L$ for some $k \neq i$. □

Lemma 6.3. *For each $i \in \{1, \ldots, d\}$ there is $k_i \neq i$ such that $e_{ik_i}, e_{k_i i}, e_{ii} - e_{k_i k_i} \in L$.*

Proof. Fix i and let $R = \{e_{ik} : i \neq k, e_{ik} \notin L\}$. If $R = \emptyset$ then the existence of k_i follows from Lemma 6.2. So suppose $R \neq \emptyset$. Then, since the root corresponding to e_{ij} is opposite to the root corresponding to e_{ji}, the roots corresponding to the set $S = R \cup \{e_{ki} : i \neq k, e_{ik} \in L\}$ form a basis for the roots space. As in the proof of Lemma 6.2, it follows that one of the elements of S lies in L. By definition, this element cannot lie in R. Hence we obtain k_i, as required. □

For the rest of the discussion we fix, for each i, an integer k_i as in Lemma 6.3, in such a way that $k_{k_i} = i$. Then the map $i \mapsto k_i$ is a fixed-point-free involution lying in the symmetric group S_d. Note that, by extension of scalars, we can (and will) assume that F is algebraically closed. Now, $D \cap L$ is a toral subalgebra of L. Hence we can write L as the direct sum of 1-dimensional subspaces L_α which are $\mathrm{ad}(D \cap L)$-invariant.

Lemma 6.4. *Let L_α be as above, and let $v \in L_\alpha$ be a non-zero vector. Then one of the following holds:*

(i) $v \in D \cap L$.

(ii) $v = c_{nm}e_{nm}$, where $0 \neq c_{nm} \in F$.

(iii) $v = c_{nm}e_{nm} + c_{k_m k_n}e_{k_m k_n}$, where $0 \neq c_{nm}, c_{k_m k_n}$.

Proof. Write $v = b + \sum_{i\neq j} c_{ij}e_{ij}$, where $b \in D$. If $c_{ij} = 0$ for all $i \neq j$, then $v = b \in D \cap L$. So suppose $c_{nm} \neq 0$ for some $n \neq m$.

Since $e_{nn} - e_{k_n k_n} \in D \cap L$, v is an eigenvector of $e_{nn} - e_{k_n k_n}$. Hence there is $l \in F$ such that

$$lv = [e_{nn} - e_{k_n k_n}, v] = [e_{nn} - e_{k_n k_n}, b] + \sum c_{ij}[e_{nn} - e_{k_n k_n}, e_{ij}].$$

This yields

$$lb + \sum lc_{ij}e_{ij} = \sum_j (c_{nj}e_{nj} - c_{k_n j}e_{k_n j}) + \sum_i (-c_{in}e_{in} + c_{ik_n}e_{ik_n})$$

$$= 2c_{nk_n}e_{nk_n} - 2c_{k_n n}e_{k_n n} + \sum_{j \neq n,k_n} c_{nj}e_{nj} + \sum_{i \neq n,k_n} c_{ik_n}e_{ik_n}$$

$$- \sum_{j \neq n,k_n} c_{k_n j}e_{k_n j} - \sum_{i \neq n,k_n} c_{in}e_{in}.$$

Suppose first that $m \neq k_n$. Then, by looking at the coefficients of e_{nm} in both sides of the above equality, we see that $l = 1$ and $b = 0$. Using the fact that $1 \neq -1$ in F, we conclude that

$$v = c_{k_n n}e_{k_n n} + \sum_{j \neq n,k_n} c_{nj}e_{nj} + \sum_{i \neq n,k_n} c_{ik_n}e_{ik_n}$$

(the first summand must vanish if $p \neq 3$).

We now use the fact that v is also an eigenvector of $e_{mm} - e_{k_m k_m}$, namely, $\mu v = [e_{mm} - e_{k_m k_m}, v]$ for some $\mu \in F$. Substituting the expression for v, and using the fact that n, m, k_n, k_m are all distinct, we obtain

$$\mu v = c_{mk_n}e_{mk_n} - c_{k_m k_n}e_{k_m k_n} - c_{nm}e_{nm} + c_{nk_m}e_{nk_m}.$$

As $c_{nm} \neq 0$, we must have $\mu = -1$, and

$$v = c_{nm}e_{nm} + c_{k_m k_n}e_{k_m k_n},$$

as required.

We are left with the case where $c_{nm} \neq 0$ for $n \neq m$ implies $m = k_n$. In this case we have

$$v = b + \sum_n c_{nk_n}e_{nk_n},$$

where $b \in D$. Expanding the equality $lv = [e_{nn} - e_{k_n k_n}, v]$ as above, we find that $l = 2, b = 0$, and $2v = 2c_{nk_n}e_{nk_n} - 2c_{k_n n}e_{k_n n}$. Since the characteristic of F is not equal to 2, it follows that $v = c_{nk_n}e_{nk_n}$, as in (ii). The result follows. □

An element $v = c_{ij}e_{ij} + c_{k_j k_i}e_{k_j k_i} \in L$ will be said to be *indecomposable* if v is in one of the subspaces L_α mentioned above, $c_{ij}, c_{k_j k_i} \neq 0$ and $e_{ij}, e_{k_j k_i} \notin L$. A matrix unit e_{ij} $(i \neq j)$ is said to be *missing* if $e_{ij} \notin L$ and e_{ij} does not occur in any indecomposable element.

Lemma 6.5. *There are no indecomposables elements in L, and the number of missing elements is less than $d - 1$.*

Proof. If e_{ij} occurs in an indecomposable element, then it must occur with $e_{k_j k_i}$. Since we fixed k_m for each m, e_{ij} can occur in at most one indecomposable element. For each indecomposable element choose one of the matrix units occurring in it. Let W be the linear space which is spanned by these chosen matrix units as well as all the missing elements. Then $\dim W$ is equal to the number of indecomposable elements plus the number of missing elements.

We claim that $W \cap L = 0$. Suppose otherwise, and let $v \in W \cap L$ be a non-zero vector. Since $v \in L \cap E$, it can be written as a linear combination of indecomposable elements and matrix units e_{ij} which lie in L. On the other hand, v is a linear combination of the elements spanning W. This yields a contradiction, and so the claim is proved.

Using the fact that $W \cap L = 0$ and that L has codimension less than $d-1$ in $sl_d(F)$, it follows that $\dim W < d-1$. In particular, the number of missing elements is less than $d-1$.

Suppose that $v = c_{ij}e_{ij} + c_{k_jk_i}e_{k_jk_i}$ is an indecomposable element. Then $i \neq k_j$, for otherwise $k_i = k_{k_j} = j$, which implies $v \in Fe_{ij}$, a contradiction to v being indecomposable. The same argument shows that $j \neq k_i$. Hence

$$[v, e_{ji}] = [c_{ij}e_{ij}, e_{ji}] = c_{ij}(e_{ii} - e_{jj}),$$

and

$$[v, e_{ji}, v] = c_{ij}[e_{ii} - e_{jj}, v] = c_{ij}[e_{ii} - e_{jj}, c_{ij}e_{ij}] = 2c_{ij}^2 e_{ij}.$$

Since $v \in L$ and $e_{ij} \notin L$, we conclude that $e_{ji} \notin L$.

Note that, if $m \neq j$, then e_{ij} and e_{im} cannot occur in the same indecomposable element (as this would imply $m = k_i = j$). Similarly, e_{ij} and e_{ni} cannot occur in the same indecomposable element (as this would imply $i = k_i$). Consider the $d-3$ pairs $\{e_{ik}, e_{ki}\}$ where $k \neq i, j, k_i$, and let S be the union of those pairs. By the above discussion any two elements of S cannot occur in the same indecomposable element. Let T be a set consisting of e_{ij}, e_{ji}, the elements of S which occur in some indecomposable element, as well as the missing elements. Let U be the linear space spanned by T. Then we easily obtain $U \cap L = 0$, and this yields $\dim U < d-1$. It follows that $|T| < d-1$, and so $|T \setminus \{e_{ij}, e_{ji}\}| < d-3$.

Since there are $d-3$ pairs of the form $\{e_{ik}, e_{ki}\}$ $(k \neq i, j, k_i)$, there exists k such that $e_{ik}, e_{ki} \notin T$. Thus e_{ik}, e_{ki} are not missing, and do not occur in indecomposable elements. It follows that $e_{ik}, e_{ki} \in L$. Therefore their Lie product $e_{ii} - e_{kk}$ also lies in L. We conclude that $[e_{ii} - e_{kk}, v] \in L$. However,

$$[e_{ii} - e_{kk}, v] = c_{ij}e_{ij} \text{ if } k \neq k_j,$$

and

$$[e_{ii} - e_{kk}, v] = c_{ij}e_{ij} - c_{k_jk_i}e_{k_jk_i} \text{ if } k = k_j.$$

Since $c_{ij}e_{ij} + c_{k_jk_i}e_{k_jk_i} \in L$, we conclude that in either case $e_{ij} \in L$, a contradiction.

The lemma is proved. □

Proof of Theorem 1.7. In view of 6.4 and 6.5, we can write L as a direct sum of $D \cap L$ and subspaces of the form Fe_{ij} for some i, j with $i \neq j$. We claim that $e_{ij} \in L$ for all $i \neq j$, namely, that there are no missing elements.

Suppose, by contradiction, that e_{ij} is missing. Then at most $d-3$ additional elements are missing. Consider the $d-2$ pairs $\{e_{ik}, e_{kj}\}$, where $k \neq i, j$. Then for some k we have $e_{ik}, e_{kj} \in L$. Therefore $e_{ij} = [e_{ik}, e_{kj}] \in L$, a contradiction.

Having proved that $e_{ij} \in L$ for all $i \neq j$ (namely, that $L \supseteq E$), it follows at once that $e_{ii} - e_{jj} \in L$ for all $i \neq j$, and so L contains D. Therefore $L = sl_d(F)$, and the theorem is proved.

References

[A] J.L. Abercrombie, Subgroups and subrings of profinite rings, *Math. Proc. Cambr. Phil. Soc.* **116** (1994), 209–222. MR **95h:**11078

[B] N. Bourbaki, *Lie groups and Lie algebras, Chapters 1-3*, Springer, Berlin, 1980.

[BShZ] Y. Barnea, A. Shalev and E.I. Zelmanov, Graded subalgebras of affine Kac-Moody algebras, to appear in *Israel J. Math.*

[C] R. Camina, *Subgroups of the Nottingham Group*, Ph.D. Thesis, QMW, London, 1996.

[D1] E.B. Dynkin, Semisimple subalgebras of semisimple Lie algebras, *Amer. Math. Soc. Transl.* (2) **6** (1957), 111–244. MR **13:**904c

[D2] E.B. Dynkin, Maximal subgroups of the classical groups, *Amer. Math. Soc. Transl.* (2) **6** (1957), 245–378. MR **14:**244d

[DDMS] J. Dixon, M.P.F. Du Sautoy, A. Mann and D. Segal, *Analytic Pro-p Groups*, London Math. Soc. Lecture Note Series **157**, Cambridge University Press, Cambridge, 1991. MR **94e:**20037

[F] K. Falconer, *Fractal Geometry: mathematical foundations and applications*, John Wiley & Sons, New York, 1990. MR **92j:**28008

[H] J.E. Humphreys, *Introduction to Lie Algebras and Represention Theory*, Springer, Berlin, 1972. MR **48:**2197

[J] D.L. Johnson, The group of formal power series under substitution, *J. Australian Math. Soc.* **45** (1988), 296–302 MR **89j:**13021

[La] M. Lazard, Groupes analytiques p-adiques, *Publ. Math. I.H.E.S.* **26** (1965), 389–603. MR **35:**188

[LM] A. Lubotzky and A. Mann, Powerful p-groups. I,II. *J. Algebra* **105** (1987), 484–515. MR **88f:**20045; MR **88f:**20046

[LSh] A. Lubotzky and A. Shalev, On some Λ-analytic pro-p groups, *Israel J. Math.* **85** (1994), 307–337. MR **95f:**20047

[S] J.-P. Serre, *Lie algebras and Lie groups* (new edition). Lecture Notes in Math. **1500**, Springer, Berlin, 1991. MR **93h:**17001

[Sh1] A. Shalev, Growth functions, p-adic analytic groups, and groups of finite coclass, *J. London Math. Soc.* **46** (1992), 111–122. MR **94a:**20047

[Sh2] A. Shalev, Some problems and results in the theory of pro-p groups, *Groups '93 - Galway/St Andrews*, eds: Campbell et al., London Math. Soc. Lecture Note Series **212**, Cambridge University Press, Cambrideg, 1995, pp. 528–542. MR **96k:**20040

[Yo] I.O. York, *The Group of Formal Power Series under Substitution*, Ph.D. Thesis, Nottingham, 1990.

[Z] E.I. Zelmanov, On periodic compact groups, *Israel J. Math.* **77** (1992), 83–95. MR **94e:**20055

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