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MULTIPLIERS AND INDUCED REPRESENTATIONS OF LOCALLY COMPACT GROUPS

by

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TO MY PARENTS

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INTRODUCTION

Groups first arose as groups of transformations, while now they are considered just as groups in the abstract. The theory of Lie groups was developed by Sophus Lie (1842-1899) in connection with the integration of systems of differential equations.

We can see how such a transformation group arises by considering the system of differential equations

(1) $\frac{dx_i}{dt} = a_i(x)$ $(i = 1, \ldots, n)$, where the x_i

are the Cartesian coordinates of a point x in real Euclidean n-space.

Assuming that the system of equations has a solution for all values of x, integrating the equations (1) we obtain

(2) $x_i' = f_i(x,t), (i = 1, \ldots, n)$ where f_i is some function depending on x and t, and $x'_i = x_i(t)$.

Taking n=3, we can interpret (2) as defining a transformation of the whole of the space; with each point x we can associate the point x' of the three-space. This can be expressed by putting $x'=xS_t$. (S_t) is thus a family of transformations of space, and it is easy to see that these transformations form a group:

$xS_tS_{t'} = xS_{t+t'}$, $xS_o = x$, $xS_tS_{-t} = x$

for any point x.

This group of transformations regarded as an abstract group, is isomorphic to the additive group of real numbers.

Group Representations:

The theory of group representations for finite non-commutative groups began in the middle 1890's with several important papers by Frobenius. His celebrated theorem, the so-called Frobenius Reciprocity Theorem [17] can be stated as follows:

Theorem: Let G be a compact group, and H a closed subgroup of G. Let U_o be an irreducible representation of H, and let V be an irreducible representation of G. If U is the representation of G, induced from the representation U_o of H, then U contains V as a discrete summand exactly as many times as the restriction of V to H contains U_o as a discrete summand. (See Sections 2.2, 2.3 and 4.2).

Frobenius's work was continued by others - notably Burnside and Schur. Until around 1919 group representations was exclusively concerned with representations of finite dimensional groups. Then however Schur pointed out that using integration on the group manifold, one could carry many results to compact Lie groups. This idea was taken up by Hermann Weyl in the 1920's and integrated with earlier work of E. Cartan on the structure and representation of Lie algebras. Weyl's results included a complete determination of the irreducible representations of all compact Lie groups having simple Lie algebras, and in collaboration with F. Peter he proved the so-called Peter-Weyl theorem allowing one to decompose the space of square summable functions on any compact Lie group into finite dimensional translation invariant subspaces indexed by the irreducible representations. If G is compact and commutative, then $\hat{G}$ is discrete and Peter-Weyl theorem reduces to Riesz-Fischer theorem ([15]; see also sections 1.3 and 2.1).

This can be seen as follows:

A representation of the commutative group G is irreducible if an only if it is one-dimensional (section 2.1). A one-dimensional representation V of G is of the form $V(x)=\chi(x)I$, where $x \in G$, $\chi \in \hat{G}$ and I is the identity in a one dimensional vector space (see section 1.3). Since G is also compact, it follows from Peter-Weyl theorem that the regular representation of G is a direct sum of irreducible representations (section 1.2). Thus $L^2(G,\mu)$ is a direct sum of one dimensional invariant subspaces, and V is a sub-representation of the regular representation acting in a one dimensional invariant subspace. Hence if Φ is an element in this subspace then $\chi(x)\Phi(y)=\phi(yx)$ for x,y in G. Thus $\phi(x)=c_\chi \chi(x)$ for some constant c_χ. Thus each one dimensional invariant subspace must be the set of all complex multiples of a fixed character χ of C. From Peter-Weyl theorem it now follows that each $f \in L^2(G,\mu)$ may be uniquely written in the form

$f=\Sigma c_\chi \chi$, where convergence is in the sense of the Hilbert space norm.

Assuming $\mu(G)=1$, we have $\langle f,\chi\rangle_{L^2(G,\mu)}=\int f(x)\overline{\chi(x)}d\mu(x)=c_\chi$.

The constants c_χ are, in fact, the Fourier coefficients of f in $L^2(G,\mu)$.

This is easily seen, if we let G to be the rotation group in the plane. Then for each integer n, we let $\chi_n(x)=e^{inx}$

Then it is easy to prove that χ_n is a character and there are no other characters. It now follows that every periodic function f on the line with period 2π can be written in the form $f(x) = \sum_n c_n e^{inx}$ where $c_n=1/2\pi\int_{-\pi}^{\pi} f(x)e^{-inx}dx$.

Seeing this link between group representation and Fourier analysis, it seems that a need to have an analogue of the character notion for non-commutative groups may have played a role in motivating Frobenius to create the theory of group representations.

Applications of Group Representations in Physics

The representation $t \longrightarrow U(t)$ of the additive group of the real line was the first example of a unitary representation of a non-compact group to be explicitly analyzed. This was worked out by Stone in a paper published in 1930. However before this took place group representations had entered into quantum mechanics through the fact that every symmetry of a physical system is reflected in an automorphism of the lattice of closed subspaces of the underlying Hilbert space and that this in turn is implemented either by a unitary or an anti-unitary operator. In the special case in which one is dealing with an atom with n electrons the compact group of rotations about the nucleus has natural unitary representations in the state space of the atom and the analysis of these turned out to be of great conceptual and computational importance in the understanding of the atom. This seems to have first been observed by Wigner in 1926 and 1927. Weyl became interested in this and published his Gruppentheorie und Quantenmechanik in 1928.

In his 1930 paper Stone showed that there always exists a unique self-adjoint operator H such that $U(t)=e^{-itH}$, and that every self-adjoint operator occurs for some unitary representation $t \longrightarrow U(t)$. Stone's theorem reduced the theory of the unitary representations of the additive group of the real line to the theory of self-adjoint operators in Hilbert space.

In the same paper Stone announced the fundamental theorem on the essential uniqueness of the systems of operators satisfying the fundamental Heisenberg commutation relations $p_k p_j = p_j p_k$, $q_k q_j = q_j q_k$ $j \neq k$, and $q_j p_j - p_j q_j = i1$, and von Neumann published a proof in 1931. Since p_j and q_j are unbounded, von Neumann replaced them by the so-called Weyl forms. (See section 2.3).

The Stone-von Neumann theorem seems to be rather artificial from purely a mathematical point of view. It has, however, several important generalizations from which the Stone-von Neumann theorem can be deduced as a corollary.

<u>The Generalizations of the Stone-von Neumann Theorem</u>:

<u>Definition</u>: Let $(X,\mathfrak{A})$ be a Borel space, and $\mathfrak{H}$ a Hilbert space. A <u>spectral measure</u> in X is a function E whose domain is $\mathfrak{A}$ and whose values are projections on $\mathfrak{H}$ such that $E(X)=1$, and $E(\bigcup_n M_n) = \sum_n E(M_n)$, whenever $\{M_n\}$ is a disjoint sequence of sets in $\mathfrak{A}$.

The <u>spectrum</u> of a spectral measure E is the complement in X of the union of all those open sets M for which $E(M)=0$.

A spectral measure is <u>compact</u>, if its spectrum is compact.

<u>Theorem 1</u>: Let $(X,\mathfrak{A})$ be a Borel space, and $\mathfrak{H}$ a Hilbert space. A projection-valued function E on $\mathfrak{A}$ is a spectral measure if and only if

Multiplicity. Chelsea Publishing Company, 1951.

In 1943 and 1944 Neumark, Ambrose and Godement independently discovered a generalization of Stone's theorem which can be stated as follows:

Let G be a separable locally compact commutative group and U a unitary representation of G in a separable Hilbert space $\mathcal{H}$.

Then every projection-valued measure P (see section 2.1) defined on the dual group $\hat{G}$ is a projection-valued measure canonically associated to a uniquely (up to unitary equivalence) defined unitary representation U of G.

Furthermore, for every vector f in $\mathcal{H}$, and $x \in G$, $\langle U(x)f,f\rangle = \int \chi(x)\,d\mu_f(x)$, where μ_f is the measure $E \longrightarrow \langle P_E(f),f\rangle$. If G is the additive group of the real line, then this reduces to Stone's theorem, since elements of $\hat{G}$ are, in this case, of the form $x \longrightarrow e^{itx}$ for $t \in \mathbb{R}$.

In 1948 Mackey discovered a generalization of Stone-von Neumann theorem to arbitrary separable locally compact groups.

For any separable locally compact commutative group G, he considered the equation

(A) - $U(x)V(\chi) = \chi(x)V(\chi)U(x)$, where $x \in G$, $\chi \in \hat{G}$, and U,V are unitary representations of G and $\hat{G}$, respectively (see sections 1.3 and 2.3). By Stone's generalization of the spectral measure theory, and Neumark-Ambrose-Godement theorem, he noticed that the representation V of $\hat{G}$ can be replaced by a projection-valued measure P on $\hat{\hat{G}}$ (Section 2.3). $\hat{\hat{G}}$ can be identified with G by Pontryagin-von Kampen Duality Theorem (Lemma 2.3.9).

CHAPTER I

A Brief Survey of the Theory of Unitary Group Representations

We give here definitions and some well-known theorems in group representation theory.

They are by no means complete and this survey will serve as an introduction to the following chapters.

The notation and terminology given here is consistent with those given by Mackey in [17].

Adequate references for this chapter are supplied in the bibliography.

1.1. G- Spaces:

1.1.1: Definition: Let X be a set. A collection $\mathfrak{A}$ of subsets of X is said to be a σ-algebra in X: if $\mathfrak{A}$ has the following properties:

1) $X \in \mathfrak{A}$;
2) $E \in \mathfrak{A}$ implies $cE \in \mathfrak{A}$, where cE is the complement of E relative to X;
3) If $E_i \in \mathfrak{A}$ for i=1,2..., then $\bigcup_{j=1}^{\infty} E_j \in \mathfrak{A}$

If $\mathfrak{A}$ is a σ-algebra in X, then the tuple $(X,\mathfrak{A})$ is called a measurable space, and members of $\mathfrak{A}$ are called measurable sets in X.

Let $(X,\mathfrak{A})$ be a measurable space, and Y a topological space. Then the map $f:X\to Y$ is said to be measurable if $f^{-1}(V)\in\mathfrak{A}$ for every open set V in Y.

Definition 1.1.2: Let $(X,\mathfrak{A})$ be a measurable space.

A measure μ on $(X,\mathfrak{A})$ is a set function $\mu:\mathfrak{A}\to[o,\infty]$, which is countably additive.

If $(X,\mathfrak{A})$ is a measurable space, and μ a measure defined on the measurable sets in $\mathfrak{A}$, then the triple $(X,\mathfrak{A},\mu)$ is called a measure space.

Definition 1.1.3: Let X be a topological space.

Let $\mathfrak{B}$ be the smallest σ-algebra in X containing all open sets in X. Members of $\mathfrak{B}$ are called Borel sets in X. The tuple $(X,\mathfrak{B})$ is called a Borel space.

Remarks: Open sets, closed sets, countable intersection of open sets, and countable union of closed sets are examples of Borel sets.

2) If $(X,\mathcal{B})$ is a Borel space, Y a topological space, $f:X \longrightarrow Y$ a continuous mapping, then $f^{-1}(V) \in \mathcal{B}$ for every open set V in Y. Hence every continuous mapping of X is Borel measurable.

Functions on X which are measurable relative to the Borel σ-algebra $\mathcal{B}$ of X are called Borel functions.

3) Let X be a locally compact Hausdorff space. A measure μ defined on the σ-algebra of all Borel sets in X is called a Borel measure.

4) Let X be a topological space. A base for X is a class $\underset{\sim}{B}$ of open sets such that for every x in X and every neighbourhood V of x, there exists a set B in $\underset{\sim}{B}$ such that $x \in B \subset V$. X is said to be separable, if there exists a countable base for its open sets.

Let X be a separable locally compact Hausdorff space, and μ a Borel measure defined on the σ-algebra of all Borel sets in X.

Then, 1) For any compact set $K \subset X$, $\mu(K) < \infty$;

2) For any Borel set E in X, $\mu(E) = \sup\{\mu(K) \mid K \subset E, K \text{ compact}\}$.

A Borel measure satisfying 2) is said to be regular.

<u>Definition 1.1.4</u>: Let G be a group which is also a Borel space. G is said to be a <u>Borel group</u>, if the map $(x,y)\to xy^{-1}$ of GxG into G is Borel.

<u>Definitions 1.1.5</u>: Let X be a Borel space, and G a separable Borel group. We say that X is a <u>G-space</u>, if for each $g\in G$, there exists a Borel automorphism of X, $\alpha_g: x \longrightarrow x\cdot g$ such that,

1) $\alpha_e = I$, the identity automorphism on X, where $e\in G$ is the identity element.

2) $\alpha_{g_1 g_2} = \alpha_{g_1}\alpha_{g_2}$ for $g_1, g_2 \in G$.

X is said to be a <u>Borel G-space</u> if the map $(x,g)\to x.g$ of X×G into X is Borel.

Let X be a separable locally compact Hausdorff space, and G a separable locally compact group. Suppose that X is a G-space and the map $(x,g)\to x.g$ of XxG into X is continuous. Then X is a Borel G-space and G is said to <u>act continuously</u> on X.

<u>Remark</u>: Let X be a separable locally compact Hausdorff space, and G a separable locally compact group. Suppose that G acts continuously on X.

1) μ is not identically zero;

2) $\mu(E.g)=\mu(E)$ for every Borel set E, and $g \in G$.

μ is said to be a <u>right invariant Haar measure</u> on G.

<u>Remark:</u> The Borel measure defined in 1.1.8 is regular.

<u>Theorem 1.1.9:</u> In every locally compact topological group there exists at least one right-invariant Haar measure.

<u>Theorem 1.1.10:</u> If μ and ν are right invariant Haar measures in a locally compact topological group G, then there exists a constant c, $0<c<\infty$, such that $\mu(E)=c\nu(E)$ for every Borel set E.

The last two theorems are proved in Halmos [8].

<u>1.2 Unitary Group Representations:</u>

Let $\mathcal{H}$ be a separable Hilbert space, and $\mathcal{U}(\mathcal{H})$ the set of all unitary operators on $\mathcal{H}$.

<u>Definition 1.2.1:</u> Let $\{U_n\}$ be a sequence of elements in $\mathcal{U}(\mathcal{H})$. Then $U_n \to V \in \mathcal{U}(\mathcal{H})$ <u>strongly</u>, if and only if $U_n\phi \to V\phi$ for

for all ϕ in $\mathcal{H}$.

The strong topology for $\mathcal{U}(\mathcal{H})$ is the smallest topology which makes all the maps $U \to U\phi$ ($\phi \in \mathcal{H}$) continuous.

Lemma 1.2.2: $\mathcal{U}(\mathcal{H})$ equipped with its strong topology is a separable, topological group.

Proof: Since $\mathcal{H}$ is separable, there exists a countable set $D \subset \mathcal{H}$ dense in $\mathcal{H}$. Then the strong topology for $\mathcal{U}(\mathcal{H})$ is the smallest topology which makes all the maps $U \to U\phi$ ($\phi \in D$) continuous. Since there are only countably many such maps it follows that $\mathcal{U}(\mathcal{H})$ is separable.

To prove that $\mathcal{U}(\mathcal{H})$ is a topological group, we have to show that the map $(U,V) \to UV^{-1}$ of $\mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H})$ into $\mathcal{U}(\mathcal{H})$ is continuous.

For any $\phi \in \mathcal{H}$, and elements U_o, V_o in $\mathcal{H}$, we have,

$$||UV^{-1}\phi - U_oV_o^{-1}\phi|| \leq ||V(V_o^{-1}\phi) - V_o(V_o^{-1}\phi)|| + ||U(V_o^{-1}\phi) - U_o(V_o^{-1}\phi)||$$

using the fact that UV^{-1} is a unitary operator.

This last inequality shows that $\mathcal{U}(\mathcal{H})$ is a topological group.

Definition 1.2.3: Let $\mathcal{H}$ be a separable Hilbert space and G a separable, locally compact topological group. By a unitary representation of G in $\mathcal{H}$ we mean a homomorphism $x \to U(x)$ of the group G into the group $\mathcal{U}(\mathcal{H})$ such that for any

$\phi \in \mathfrak{H}$ the function $x \longrightarrow U(x)\phi$ of G into $\mathfrak{H}$ is continuous.

Lemma 1.2.4: Let G be a separable, locally compact group and $\mathfrak{H}$ a separable Hilbert space.

Suppose that for each $x \in G, U(x)$ is a unitary operator on $\mathfrak{H}$, and for each $x, y \in G$ $U(xy)=U(x)U(y)$. Then U is a unitary representation of G in $\mathfrak{H}$, if $\langle U(x)\phi,\psi\rangle$ are measurable functions of x for all $\phi,\psi \in \mathfrak{H}$.

This lemma is proved in Varadarajan [29] (pages 34,55).

By theorem 1.1.9 there exists a right invariant Haar measure μ on G. We form the Hilbert space of all complex-valued Borel functions, such that

$\int_G |\phi(x)|^2 d\mu(x) < \infty$. Its scalar product is given by,
$\langle\phi,\psi\rangle = \int_G \phi(x)\overline{\psi(x)}d\mu(x)$.

For each $x,z \in G$ we define

$$U(x)\phi(z) = \phi(zx) \quad (\phi \in L^2(G,\mu)).$$

Then we have

1) for each $x,y \in G$, $U(xy)=U(x)U(y)$
2) for each $x \in G$, and $\phi,\psi \in L^2(G,\mu)$, we have

$$\langle U(x)\phi,U(x)\psi\rangle = \int_G U(x)\phi(y)\overline{U(x)\psi(y)}d\mu(y) =$$
$$= \int_G \phi(yx)\overline{\psi(yx)}d\mu(y) = \int_G \phi(y)\overline{\psi(y)}d\mu(yx^{-1}) =$$
$$= \int_G \phi(y)\overline{\psi(y)}d\mu(y) = \langle\phi,\psi\rangle,$$ since μ is right-invariant.

Hence $U(g)$ is a unitary operator on $L^2(G,\mu)$ for each $g \in G$.

3) For each $x \in G$, and $\phi, \psi \in L^2(G,\mu)$ we have $\langle U(x)\phi,\psi\rangle = \int_G U(x)\phi(y)\overline{\psi(y)}d\mu(y) = \int_G \phi(yx)\overline{\psi(y)}d\mu(y)$. The integrand is a Borel measurable function in both variables. It follows that, $\langle U(x)\phi,\psi\rangle$ is a Borel function on G for each ϕ,ψ in $L^2(G,\mu)$.

Hence by lemma 1.2.4 U is a unitary representation of G. We call this representation the right regular representation of G.

Definition 1.2.5: A representation U in $\mathfrak{H}$ is said to be equivalent to a representation U' in $\mathfrak{H}'$, if there exists a unitary isomorphism $W: \mathfrak{H} \longrightarrow \mathfrak{H}'$ of $\mathfrak{H}$ onto $\mathfrak{H}'$ such that $U'(g) = WU(g)W^{-1}$ for all $g \in G$.

Definition 1.2.6: Let U be a representation of G in $\mathfrak{H}$, and U' a representation of G in $\mathfrak{H}'$. Let V be a bounded linear transformation from $\mathfrak{H}$ to $\mathfrak{H}'$. V is said to be an intertwining operator for U and U' provided $VU(g) = U'(g)V$ for all g in G.

Remarks: If U=U', then the set of all intertwining operators for U and U' is the set of all bounded linear operators which commute with U(g) for all $g \in G$. It is an algebra which we call the commuting algebra $\mathfrak{A}'(U)$ of U.

In the following G will always be a separable locally compact group and $\mathfrak{H}$ a separable Hilbert space.

We say that G is a semi-direct product of N and K; we write $N \rtimes_{\phi} K$.

Conversely, let N and K be any two locally compact groups and $\alpha: K \to \mathrm{Aut}N$ be a homomorphism of K into the group of automophisms of N such that for $k \in K$, and $n \in N, \alpha(k)(n)$ is continuous on $N \times K$. The set $N \times K$ becomes a topological group if the group multiplication is defined by $(n_1, k_1)(n_2, k_2) = (n_1 \alpha(k_1)(n_2), k_1 k_2)$ for $(n_i, k_i) \in N \times K$, $i=1,2,...$ and $N \times K$ is equipped with the product topology.

<u>Definition 1.3.5</u>: Let N be a separable, locally compact, commutative group. Let $\hat{N}$ be the set of all continuous, complex-valued functions of modulus one on N satisfying,

$\chi(xy) = \chi(x)\chi(y)$ for all x,y in N.

$\hat{N}$ is a group called the <u>dual group</u> of N, and elements of $\hat{N}$ are said to be <u>characters</u> of N.

<u>Remark</u>: $\hat{N}$ is, in fact, a separable locally compact group. We shall assume this fact without proof.

2) $P(S_1 \cap S_2)=P(S_1)P(S_2)$ for any Borel sets S_1 and S_2 in $\mathbb{R}^3$.

This condition states that a system which is both in S_1 and S_2 is in $S_1 \cap S_2$. As an immediate consequence, it also follows that
$P(S_1)P(S_2)=P(S_2)P(S_1)$.

3) $P(S_1 \cup S_2)=P(S_1)+P(S_2)-P(S_1 \cap S_2)$ for any Borel sets S_1 and S_2 in $\mathbb{R}^3$.

Hence, if S_i, $i=1,2,\ldots$ are disjoint Borel sets in $\mathbb{R}^3$, then
$P(\bigcup_i S_i)=\sum_i P(S_i)$.

This states that the set of the states of the system for which it is localized in $S_1 \cup S_2$ is the closed subspace spanned by the states localized in S_1, and those localized in S_2.

4) $P(\mathbb{R}^3)=1$. Hence, the probability of finding the system somewhere in $\mathbb{R}^3$ is one.

For any $\underline{a}$ in $\mathbb{R}^3$, and a rotation R in three-space let $U(\underline{a},R)$ be the unitary operator whose application to a wave function ψ yields the wave function ψ rotated by R and translated by $\underline{a}$. Hence, $U(\underline{a},R)\psi(\underline{x})=\psi(\underline{x}^T R^{-1}+\underline{a})$ for $\underline{x}$ in $\mathbb{R}^3$. Without loss of physical generality, it can be assumed that the operators $U(\underline{a},R)$ form a representation of the Euclidean group up to a $\pm$ sign.

Hence,

$$U(\underline{a}_1,R_1)U(\underline{a}_2,R_2)=\omega((\underline{a}_1,R_1),(\underline{a}_2,R_2))U(\underline{a}_1+\underline{a}_2^T R_1^{-1},R_1R_2),$$

where $\omega=\pm 1$.

We state condition 5)

$P(SR^{-1}+\underline{a})=U(\underline{a},R)P(S)U(\underline{a},R)^{-1}$, where $SR^{-1}+\underline{a}$ is the set obtained from the Borel set S by carrying out the rotation R followed by the translation $\underline{a}$.

Condition 5) states that if Φ is a state in which the system is localized in S, then $U(\underline{a},R)\Phi$ is a state in which the system is localized in $SR^{-1}+\underline{a}$.

The above discussion motivates us to the following general definitions:

Definition 2.1.1: Let M be a Borel space, and let P be a function which carries each Borel subset E of M into a projection P_E in a Hilbert space $\mathcal{H}$ such that

(1) $P_{E_1}\cdot P_{E_2}=P_{E_1\cap E_2}$,

(2) if $E_i\cap E_j=\phi$ for $i\neq j$, then $P_{\cup_j E_j}=\sum_j P_{E_j}$

(3) $P_\phi=0$, and $P_M=1$.

P is called a projection-valued measure on M to $\mathcal{H}$.

Let N be a separable, locally compact, commutative group, and let U be any representation of N in a Hilbert space $\mathcal{H}$. We shall determine the projection-valued measure on $\hat{N}$, the dual group of N.

Lemma 2.1.2: Let G be a separable, locally compact, commutative group and U an irreducible unitary representation of G in a separable Hilbert space $\mathcal{H}$.

Then U is a one-dimensional representation of G in $\mathcal{H}$.

Proof: Let $g\in G$. Then U(g) commutes with all of the operators U(x) $(x\in G)$.
Hence $U(g)\in\mathfrak{A}(U)$.

But U is irreducible, and hence by corollary 1.2.12 U(g) is a multiple of the identity operator:
$U(g)=\chi(g)I (g\in G)$, for some $\chi\in\hat{G}$.

Thus every subspace of the representation space $\mathcal{H}$ is invariant under U so that $\mathcal{H}$ has to be one-dimensional.

Remark: If G is compact and commutative, it follows from Lemma 2.1.2 and Peter-Weyl theorem that $\hat{G}$ is discrete.

Definition 2.1.3: Let X be a separable, locally compact Hausdorff space, and let μ be a positive Borel measure

defined on the Borel subsets of X. For each $x \in X$, let $\mathcal{H}_x$ be a separable Hilbert space, whose dimension $\dim \mathcal{H}_x$ is a μ-measurable function of x.
The set $\int_{\oplus} \mathcal{H}_x d\mu(x)$ denotes the set of all functions defined on X such that

$f(x) \in \mathcal{H}_x$ for each $x \in X$.

Furthermore, its elements must satisfy $<f_1(x), f_2(x)>$ is a μ-measurable function of x for any two $f_1, f_2 \in \int_{\oplus} \mathcal{H}_x d\mu(x)$, and $\int_X <f(x), f(x)> d\mu(x) < \infty$ for any $f \in \int_{\oplus} \mathcal{H}_x d\mu(x)$.

The set $\int_{\oplus} \mathcal{H}_x d\mu(x)$ is a vector space under the usual operations of addition and scalar multiplication. It becomes a Hilbert space if we identify two functions differing on sets of μ-measure zero, and define the scalar product

$$(f_1, f_2) = \int_X <f_1(x), f_2(x)> d\mu(x).$$

The Hilbert space $\int_{\oplus X} \mathcal{H}_x d\mu(x)$ is called the <u>direct integral Hilbert space with measure μ</u>.

<u>Definition 2.1.4</u>: Let X be a separable, locally compact Hausdorff space, and let μ be a positive Borel measure defined on the Borel subsets of X. Let G be a separable, locally compact group. For each $x \in X$, suppose that $\mathcal{H}_x$ is a separable Hilbert space and U^x an irreducible, unitary representation of G in $\mathcal{H}_x$. We form the direct integral Hilbert space

$$\int_{\oplus X} \mathcal{H}_x d\mu(x).$$

Let $\int_{\oplus\Lambda} U^{\lambda}d\mu(\lambda)$ be a direct integral decomposition of U into irreducibles. For each Borel subset E of $\hat{N}$, let Λ_E be a subset of Λ consisting of all λ with $\chi_\lambda \in E$. (By theorem 2.1.5 each U^{λ} is of the form $x \longrightarrow \chi_\lambda(x)I$, $(x\in N$, and $\chi_\lambda \in \hat{N})$.

We consider the subrepresentation $\int_{\Lambda_E} U^{\lambda}d\mu(\lambda)$

This acts in some subspace M_E of $L^2(\Lambda)$. Clearly, $M_{\hat{N}\setminus E}=M_E^{\perp}$

Thus we have split U into two subrepresentations: the first is a direct integral of irreducible representations defined by characters in E, the second is a direct integral of irreducible representations defined by characters in $\hat{N}\setminus E$.

Let P_E denote the projection operator whose range is M_E.

Then $E \longrightarrow P_E$ assigns a projection operator to every Borel subset of $\hat{N}$, and it is easy to see that this assignment satisfies the following properties:

1) $P_{\hat{N}}=I$,

2) $P_{E\cap F}=P_E P_F$ for all E and F in $\hat{N}$,

3) $P_{E_1 \cup E_2 \cup \ldots}=P_{E_1}+P_{E_2}+\ldots$, for all $E_i \in \hat{N}$, $i=1,2,\ldots$, whenever $E_i \cap E_j=\emptyset$ for $i\neq j$, $i,j=1,2,\ldots$.

Hence P_E is a projection-valued measure associated with the representation U of N.

Let $\overset{o}{x} \in X$ be a fixed element. Consider the set of all g in G satisfying $\overset{o}{x}\cdot g=\overset{o}{x}$. This set is in fact the stability subgroup $G_{\overset{o}{x}}$ of G, which is a closed subgroup. We shall denote it by H ([17]; p. 131).

Clearly, $\overset{o}{x}\cdot g_1=\overset{o}{x}\cdot g_2$ if and only if $Hg_1=Hg_2$. Thus, the points of X are in one-to-one correspondence with the cosets in the right coset space $G/_H$.

Furthermore, the action of G on $G/_H$ is the canonical one: the group element g sends the right coset Hg_1 into the right coset Hg_1g.

Under the mapping $\overset{o}{x}\cdot g \leftrightarrow Hg$, the separable, locally compact topology of X corresponds to that of $G/_H$ induced from the separable locally compact topology of G.

Lemma 2.2.1: Let G be a separable, locally compact group, H a closed subgroup and $X=G/_H$ the quotient space.

Let μ be any quasi-invariant measure defined on the Borel sets in X. Then for any Borel set E in X, $\mu(E)=0$ if and only if $\pi^{-1}(E)$ has Haar measure zero, where $\pi: G \to X=G/_H$ denotes the natural map.

This lemma has been proved in [11].

<u>Corollary 2.2.2</u>: Let μ be any quasi-invariant measure in X. Then μ is unique (up to equivalence).

This corollary is a consequence of theorem 1.1.10.

Let μ be any quasi-invariant measure on $X=G/_H$. For each $g \in G$, we define a measure μ_g by $\mu_g(E)=\mu(E \cdot g)$, where $E \subseteq X$ is a Borel set.

Then for each g in G, μ_g and μ are absolutely continuous with respect to each other.

By Radon-Nikodym theorem for each $g \in G$, there exists a Borel function ρ_g on X such that

$$\mu_g(E)=\int_E \rho_g(x)d\mu(x) \text{ for any Borel set } E \subseteq X.$$

<u>Lemma 2.2.3</u>: The function ρ_g defined above has the following properties:

1) $\rho_g(x)$ is a Borel function on $G \times X$;
2) $\rho_{g_1 g_2}(x)=\rho_{g_1}(x)\rho_{g_2}(xg_1)$ for all g_1, g_2 in G, and $x \in X$.

A proof of this lemma is given by Mackey in [11].

2.2.4 Definition of Induced Representations:

Let G be a separable, locally compact group and H a closed subgroup. Suppose that μ is a quasi-invariant Borel measure on the quotient space $X=G/_H$.

Let U_o be a unitary representation of H in a separable Hilbert space $\mathcal{H}_o$.

Let K denote the set of all functions ϕ from G to $\mathcal{H}_o$ such that

1) $\langle\phi(g),\psi\rangle$ is a Borel function of g for each ψ in $\mathcal{H}_o$

2) For all h in H, and g in G, $\phi(hg)=U_o(h)\phi(g)$ holds everywhere except possibly on a set of μ-measure zero;

3) $\int_X\langle\phi(x),\phi(x)\rangle d\mu(x)<\infty$.

Then K becomes a separable Hilbert space if we identify functions differing on sets of μ-measure zero and define the scalar product,

$$(\phi,\psi)=\int_X\langle\phi(x),\psi(x)\rangle d\mu(x).$$

We shall denote this Hilbert space by $L^2_M(G,\mathcal{H}_o,d\mu)$

For each g in G, k in G, and $\phi\in L^2{}_M(G,\mathcal{H}_o d\mu)$ we define

$$U(g)\phi(k)=\phi(kg)\sqrt{\rho_g(k)}.$$

We shall verify that U is a unitary representation of G in $L^2_M(G,\mathcal{H}_0,d\mu)$.

U has the following properties:

1) For each g_1,g_2,k in G, $\phi\in L^2_M(G\ ,\mathcal{H}_0,d\mu)$

$$U(g_1)U(g_2)\phi(k) = U(g_2)\phi(kg_1)\sqrt{\rho_{g_1}(k)} =$$

$$=\phi(kg_1g_2)\sqrt{\rho_{g_1}(k)\rho_{g_2}(kg_1)} = \phi(kg_1g_2)\sqrt{\rho_{g_1g_2}(k)}$$

$=U(g_1g_2)\phi(k)$ by lemma 2.2.3.

2) For each ϕ,ψ in $L^2_M(G,\mathcal{H}_0,d\mu)$, $g\in G$ we have

$$(U(g)\phi,U(g)\psi) = \int_X <U(g)\phi(x),U(g)\psi(x)>d\mu(x)$$

$$=\int_X \rho_g(x)<\phi(xg),\ \psi(xg)>d\mu(x)=$$

$$=\int_X \rho_g(xg^{-1})<\phi(x),\psi(x)>d\mu(xg^{-1})=$$

$=\int_X <\phi(x),\psi(x)>d\mu(x)=(\phi,\psi)$ by the definition of ρ_g.

Hence $U(g)$ is a unitary operator on $L^2_M(G,\mathcal{H}_0,d\mu)$.

3) For each ϕ,ψ in $L^2_M(G,\mathcal{H}_0,d\mu)$ and $g\in G$

$$(U(g)\phi,\psi)=\int_X <U(g)\phi(x),\psi(x)>d\mu(x)=$$

$$=\int_X \sqrt{\rho_g(x)}<\phi(xg),\psi(x)>d\mu(x).$$

The integrand is a Borel measurable function in both variables. It follows that $(U(g)\phi,\psi)$ is a Borel function on G for each $\phi,\psi\in L^2_M(G,\mathcal{H}_0 d\mu)$.

By lemma 1.2.4., U is a unitary representation of G in $L^2_M\ (G,\mathcal{H}_0,d\mu)$.

$\| W\phi \|^2_{L^2(X,\mathcal{H}_0)} = \int_X \langle W\phi(x), W\phi(x)\rangle d\mu(x) =$

$= \int_X \langle \phi(b(x)), \phi(b(x))\rangle d\mu(x) = \| \phi \|^2_{L^2_M(G,\mathcal{H}_0)}$, since $\langle \phi(b(x)), \phi(b(x))\rangle$ is a Borel function on G which is constant on each right coset Hg.

For all $\psi \in L^2(X,\mathcal{H}_0,d\mu)$ and $x \in X$, $(WV\psi)(x) = V\psi(b(x)) = U_0(b(x)b^{-1}(\overline{b(x)}))\psi(x) = U_0(b(x)b^{-1}(x))\psi(x) = \psi(x)$.

Hence WV=I, the identity operator on the space $L^2(X,\mathcal{H}_0,d\mu)$.

Thus W maps the space $L^2_M(G,\mathcal{H}_0,d\mu)$ onto the space $L^2(X,\mathcal{H}_0,d\mu)$.

It follows that V and W are unitary and $W=V^{-1}$.

If $x \in X$ and $g \in G$, then $\overline{b(x)g} = \overline{b(xg)} = xg$, where $\bar{k}$ denotes Hk, the right coset of $k \in G$ with respect to H.

Let $\psi \in L^2(X,\mathcal{H}_0,d\mu)$. For each $x \in X$ and $g \in G$, we have

$$V^{-1}U(g)V\psi(x) = U(g)(V\psi)(b(x)) =$$
$$= V\psi(b(x)g) = U_0(b(x)gb^{-1}(\overline{b(x)g}))\psi(\overline{b(x)g}) =$$
$$= U_0(b(x)gb^{-1}(xg))\psi(xg).$$

Hence the representations U and U' of G are unitarily equivalent.

<u>Theorem 2.2.7.</u>: Let G be a separable, locally compact group, H a closed subgroup, and U_0 a unitary representation of H in a separable Hilbert space $\mathcal{H}_0$.

Let μ and ν be quasi-invariant measures on the space $X=G/H$.

We do the inducing construction to get unitary representations ${}^{\mu}U$ and ${}^{\nu}U$ in the Hilbert spaces $L^2_M(G,\mathcal{H}_0,d\mu)$ and $L^2_M(G,\mathcal{H}_0,d\nu)$, respectively.

Then ${}^{\mu}U$ and ${}^{\nu}U$ are unitarily equivalent.

Proof:

By Corollary 2.2.2. μ and ν are absolutely continuous with respect to each other. Let ρ be the Borel function which is the Radon-Nikodym derivative of μ with respect to ν, and let π be the natural projection of G onto X.

Then, clearly, for each $\phi \in L^2_M(G,\mathcal{H}_0,d\mu)$, $\sqrt{\rho\circ\pi}\,\phi$ is in $L^2_M(G,\mathcal{H}_0,d\nu)$.

Conversely, every f in $L^2_M(G,\mathcal{H}_0,d\nu)$ can be written in the form $\sqrt{\rho\circ\pi}\,\phi$ for some ϕ in $L^2_M(G,\mathcal{H}_0,d\mu)$.

Let V be the map $\phi \longrightarrow \sqrt{\rho\circ\pi}\,\phi$ of $L^2_M(G,\mathcal{H}_0,d\mu)$ into $L^2_M(G,\mathcal{H}_0,d\nu)$. Then clearly V is a unitary map.

The verification of the fact that $V\,{}^{\mu}U(g)V^{-1} = {}^{\nu}U(g)$ is also immediate.

2.3. The Imprimitivity Theorem and Some of Its Applications:

Definition 2.3.1: Let G be a separable locally compact group, and U a representation of G in a separable Hilbert space $\mathcal{H}$.

Let X be a separable locally compact Hausdorff space, and suppose that G acts on X such that the map $(x,g) \longrightarrow x\cdot g$ of $X \times G$ into X is a Borel function.

By a system of imprimitivity for U we mean a projection-valued measure P on X to $\mathcal{H}$, such that for all $g \in G$, $U(g)P_E U^{-1}(g) = P_{E\cdot g^{-1}}$ for all Borel sets E in X, where $E\cdot k$ is the image of E in X under the action of k.

Definition 2.3.2: Let G be a separable locally compact group, X a separable locally compact G-space, and U a unitary representation of G.

We say that the system of imprimitivity P for U is transitive, if there exists an orbit of X under the action of G whose complement has measure zero with respect to the projection valued measure P.

2.3.3: Let G be a separable locally compact group, H a closed subgroup and X=G/H the quotient space.

Let $\pi: G \longrightarrow X=G/H$ be the natural projection of G onto X.

Let U_0 be a unitary representation of G in a separable Hilbert space $\mathcal{H}_0$.

Suppose that U is the induced representation of G in the space $L^2_M(G,\mathcal{H}_0)$.

Let E be a Borel subset of X, and $E'=\pi^{-1}(E)$ the inverse image of E in G under the natural map $\pi:G \longrightarrow X$.

Let $\psi_{E'}$ be the characteristic function of E'. Then the map $\phi \longrightarrow \psi_{E'}\phi$ is a projection in the space $L^2_M(G,\mathcal{H}_0)$. We denote this projection by P_E.

Then $E \longrightarrow P_E$ is a projection-valued measure associated with the Borel space X.

Furthermore, we also have $U(g)P_E U^{-1}(g)=P_{E\cdot g^{-1}}$ for any Borel set E in X and $g \in G$.

Thus, P constitutes a system of imprimitivity <u>canonically associated</u> with the induced representation U of G.

<u>Remark</u>: The canonical system of imprimitivity associated with the induced representation U of G is transitive.

<u>Theorem 2.3.4 The Imprimitivity Theorem</u>

Let G be a separable locally compact group and H a closed subgroup.

Let V be a unitary representation of G in a separable Hilbert space $\mathcal{H}$, and P a transitive system of imprimitivity for V defined on the Borel space X=G/H.

Then there exists a unitary representation U_0 of H in a Hilbert space $\mathcal{H}_0$ and a unitary map $W:L^2_M(G,\mathcal{H}_0) \longrightarrow \mathcal{H}$ such that the following holds;

1) For all $g \in G$,

$WU(g)W^{-1}=V(g)$ where U is the induced representation of G in $L^2_M(G,\mathcal{H}_0)$;

2) For all Borel subsets E of X,

$WP'_E W^{-1}=P_E$,where P' is the projection valued measure canonically associated with the induced representation U of G.

A proof of the imprimitivity theorem will be given in the next section.

2.3.5 Remarks:

1) Let G be a separable locally compact group, X a separable locally compact Hausdorff space, and μ a quasi-invariant measure defined on the Borel subsets of X.

Suppose that μ is invariant, and G acts on X such that the map $(x,g) \longrightarrow x \cdot g$ of $X \times G$ into X is Borel.

We define a representation U of G in $L^2(X,d\mu)$ by putting

$$U(g)\phi(x)=\phi(x \cdot g) \text{ where } \phi \in L^2(X,d\mu)$$

$x \in X$ and $g \in G$.

We construct a projection-valued measure P in such a way that the P_E act on $L^2(X,d\mu)$. We define P_E to be the operator $\phi \longrightarrow \psi_E \phi$, where ψ_E is the characteristic function of the set E in X.

Then $E \longrightarrow P_E$ is a projection valued measure and for each $g \in G$, P_E and $U(g)$ satisfy

$U(g)P_E U^{-1}(g) = P_{E \cdot g^{-1}}$ for all $E \subseteq X$, where E is a Borel subset of X.

2) Let G and X be as in 1).

Let V be a unitary representation of G in a Hilbert space $\mathcal{H}$, and P a system of imprimitivity for V.

If E is a Borel subset of X such that $P_{(E \setminus E \cdot g) \cup (E \cdot g \setminus E)} = 0$ for all g in G, then the range of P_E is an invariant subspace of $\mathcal{H}$. If either E or $X \setminus E$ is not of P measure zero, then this projection gives rise to a direct sum decomposition of V.

3) Let U be a representation of G in a Hilbert space $\mathcal{H}$, and P a system of imprimitivity for U.

Then we say that the system (U,P) is <u>irreducible</u>, if there is no non-trivial subspace of $\mathcal{H}$ which is invariant under all U(g), and all P_E where $E \subseteq X$ is a Borel subset and $g \in G$.

Clearly, if U is an irreducible representation of G, then the system (U,P) is also irreducible.

However, if the system (U,P) is irreducible, then, in general, U is not an irreducible representation of G.

<u>Corollary 2.3.6</u>: Let G be a separable locally compact commutative group, and $\hat{G}$ its dual.

Let U and V be unitary representations of G and $\hat{G}$, respectively, in a separable Hilbert space $\mathcal{H}$.

Suppose further that

1) $U(x)V(\chi)=\chi(x)V(\chi)U(x)$ where $x\in G$, and $\chi\in\hat{G}$

2) $\mathcal{H}$ has no non-trivial subspaces invariant under the combined action of U and V.

Then U is unitarily equivalent to the right regular representation of G in $L^2(G)$ and V is unitarily equivalent to the representation V' in $L^2(G)$ given by,

$$V'(\chi)\phi(y)=\chi(y)\phi(y) \text{ where } y\in G,\ \chi\in\hat{G} \text{ and } \phi\in L^2(G).$$

Remark:

When $G=\hat{G}=\mathbb{R}$, the reals under addition, condition 1) in the corollary becomes $U(x)V(y)=e^{ixy}V(y)U(x)$ for $x,y\in\mathbb{R}$. Suppose that the one-parameter groups $\{U(x)\}$ and $\{V(y)\}$ are generated by hermitian operators p and q, respectively, so that $U(x)=e^{ixp}$ and $V(y)=e^{iyq}$. Then condition 1) of the corollary corresponds to the condition $qp-pq=i1$. But this is just the commutation condition imposed on the operator of position and momentum in a one-dimensional quantum-mechanical system.

Thus, corollary 2.3.6 implies Stone-von Neumann uniqueness theorem which states that these commutation relations have a unique irreducible solution.

Lemma 2.3.7: Let G be a separable locally compact group, $\hat{G}$ its dual group and $\mathcal{H}$ a separable Hilbert space.

It is easy to check the following:

<u>Sublemma</u>: τ maps $C_o(G)$ onto $C_o(M)$ as well as $\{\phi\in C_o(G)\,|\,\phi\geq 0\}$ onto $\{\psi\in C_o(M)\,|\,\psi\geq 0\}$ and it is continuous with respect to the topologies on $C_o(G)$ and $C_o(M)$ induced by an invariant metric ([19]; p.356).

Let U_0 be a unitary representation of H in a Hilbert space $\mathcal{H}_0$ and denote by $\mathfrak{F}$ the set of functions $f:G \longrightarrow \mathcal{H}_0$ satisfying

1) $f(hx)=\rho(h)^{\frac{1}{2}}\ U_0(h)f(x), h\in H$, and $x\in G$;

2) $\int_G <f(x),f(x)>_{\mathcal{H}_0} d\mu(x)<\infty$;

3) f is strongly measurable, that is, $\| f(x) \|$ is a measurable function of x for $x\in G$.

Condition 3) also implies that f is weakly measurable. Thus for each $\omega\in\mathcal{H}_0$, $<f(x),\omega>$ is a measurable function of x for $x\in G$.

<u>Remark</u>: We shall assume that $\mathcal{H}_0$ is a separable Hilbert space.

Ørsted indicated in [19] that this separability condition can be dropped.

<u>Lemma 2.4.2</u>: For $f\in\mathfrak{F}$ and $\phi\in C_o(G)$,

$$\mu_{f,f}:\tau\phi \longrightarrow \int_G <f(x),f(x)>_{\mathcal{H}_0} \phi(x)d\mu(x);$$

a (well-defined) Radon measure on M, that is a continuous linear functional on $C_o(M)$.

$\mathcal{F}$ defines a pre-Hilbert space with respect to the inner product given by the continuous linear functional in Lemma 2.4.2.

For $f \in \mathcal{F}$, let $||f||^2 = \mu_{f,f}(M)$, and let

$$\mathcal{H} = \{f \in \mathcal{F} \mid ||f|| < \infty\} \big/ \{f \in \mathcal{F} \mid ||f|| = 0\}$$

Then $\mathcal{H}$ is a Hilbert space, the completion of $\mathcal{F}$ with respect to $||\ ||$.

Then for $f \in \mathcal{H}$,

$U(x)f(y)=f(yx)$ $(x,y \in G)$ defines a unitary representation U of G in $\mathcal{H}$, the induced representation of U_0 from H to G.

We also let

$(P(\psi)f)(x)=\psi(\pi(x))f(x)$ for $f \in \mathcal{H}$, $\psi \in C_o(M)$ for each $x \in G$.

<u>Remark</u>: For any Hilbert space $\mathcal{K}$, $\mathcal{B}(\mathcal{K})$ the set of all bounded linear operators T on $\mathcal{K}$ is a Banach algebra normed by,

$$||T|| = \sup\{\, ||Tf|| \;\big|\; f \in \mathcal{K},\ ||f|| \leq 1\}.$$

If $T \in \mathcal{B}(\mathcal{K})$ and $f,g \in \mathcal{K}$, then there exists a unique $T^* \in \mathcal{B}(\mathcal{K})$ such that

$$\langle Tf,g \rangle = \langle f,T^*g \rangle.$$

It can easily be checked that the map $T \longrightarrow T^*$ is an involution on $\mathcal{B}(\mathcal{K})$, that is, that the following four properties hold:

1) $(T+S)^*=T^*+S^*$,
2) $(\alpha T)^*=\bar{\alpha}T^*$,
3) $(ST)^*=T^*S^*$,
4) $T^{**}=T$.

<u>Definition 2.4.3</u>: The system (U,P) is called an <u>induced system of imprimitivity</u>, and P is a homomorphism from $C_o(M)$ into $\mathcal{L}(\mathcal{H})$, all bounded linear operators on $\mathcal{H}$, satisfying,
$U(x)P(\psi)U(x)^{-1}=P(R(x)\psi)$, $x\in G$, where
$(R(x)\psi)(\pi(y))=\psi(\pi(yx))$ for $y\in G(\psi\in C_o(M))$.

We re-state the imprimitivity theorem:

<u>Theorem 2.4.4</u>: Let V be a unitary representation of G in a Hilbert space $\mathcal{H}'$, and $P':C_o(M) \longrightarrow \mathcal{B}(\mathcal{H}')$ a homomorphism with $P'(C_o(M))\mathcal{H}'$ dense in $\mathcal{H}'$, and

$$V(x)P'(\psi)V(x)^{-1}=P'(R(x)\psi), (x\in G \text{ and } \psi\in C_o(M)).$$

Let H be a closed subgroup of G.

Then there exists a unique (up to unitary equivalence) unitary representation U_0 of H in a Hilbert space $\mathcal{H}_0$, such that the induced system of imprimitivity (U,P) in $\mathcal{H}$ is unitarily equivalent to the pair (V,P'); that is, there exists a unitary operator $W:\mathcal{H}' \longrightarrow \mathcal{H}$ such that for each $x\in G$, and $\psi\in C_o(M)$,

$$W^{-1}U(x)W=V(x), \text{ and}$$
$$W^{-1}P(\psi)W=P'(\psi).$$

<u>Proof</u>: Let $\mathcal{D}$ denote the Gårding domain,

$\mathcal{D}$=span $\{V(\phi)x \mid x\in\mathcal{H}', \phi\in C_o(G)\}$,

where $V(\phi)=\int_G \phi(x)V(x^{-1})d\mu(x), (\phi\in C_o(G))$.

<u>Lemma</u>: For $x,y\in\mathcal{H}'$, the linear functional $\phi \longrightarrow \langle P'(\tau\phi)x,y\rangle$ $(\phi\in Co(G)$ is a Radon measure, denoted by $d\mu_{x,y}$.

So, for $x,y\in\mathcal{H}'$,

$$\langle P'(\tau\phi)x,y\rangle=\int_G \phi(g)d\mu_{x,y}(g) \quad (\phi\in Co(G)).$$

In particular, if $x,y\in\mathcal{D}$, $\langle P'(\tau\phi)x,y\rangle$ $(\phi\in Co(G))$ defines a Radon measure λ on $G\times G$.

We now return to the proof of theorem 2.4.4.

For $x,y\in\mathcal{D}$ and $g\in G$ we let $d\mu_{x,y}(g)=h_{x,y}(g)d\mu(g)$, and define $\beta(x,y)=h_{x,y}(e)$ where e is the identity in G.

Then β is a sesquilinear form on $\mathcal{D}\times\mathcal{D}$, and it can easily be checked that the following hold:

1) $\beta(x,x)\geq 0$, $x\in\mathcal{D}$;

2) $\beta(V(h)x,V(h)y)=\rho(h)\beta(x,y)$ for $x,y\in\mathcal{D}$, $h\in H$

3) $\langle P'(\tau\phi)x,y)=\int_G \phi(g)\beta(V(g)x,V(g)y)d\mu(g)$

for $x,y\in\mathcal{D}$, $\phi\in Co(G)$.

We now let $\mathcal{H}_0=\overline{(\mathcal{D}/\ker\beta)}$, the Hilbert space completion, and

$U_0(h)[x] = [\rho(h)^{-\frac{1}{2}} \; V(h)x]$, where $h\in H$, $x\in\mathcal{D}$, and $[x]$ is the equivalence class of x.

Then $\langle U_0(h)[x],[y]\rangle=\rho(h)^{-\frac{1}{2}} \; \beta(V(h)x,y)$ is a continuous function of h for each $x,y\in\mathcal{D}$, and hence U_0 is a unitary representation of H in $\mathcal{H}_0$.

The physical interpretation is that $<\underset{\sim}{\psi},\underset{\sim}{\phi}>$ is the transition probability, the probability of finding the system to be in the state $\underset{\sim}{\psi}$, when it is in the state $\underset{\sim}{\phi}$.

<u>Definition 3.1.1</u>: A bijective map $\underset{\sim}{T}:\underset{\sim}{\mathcal{H}} \longrightarrow \underset{\sim}{\mathcal{H}}$ is an <u>automorphism</u> of $\underset{\sim}{\mathcal{H}}$, if it preserves the transition probability, that is,

$$<\underset{\sim}{T}\underset{\sim}{\psi},\underset{\sim}{T}\underset{\sim}{\phi}>=<\underset{\sim}{\psi},\underset{\sim}{\phi}> \text{ for all } \underset{\sim}{\psi},\underset{\sim}{\phi} \text{ in } \underset{\sim}{\mathcal{H}}.$$

<u>Definition 3.1.2</u>: If $\underset{\sim}{T}:\underset{\sim}{\mathcal{H}} \longrightarrow \underset{\sim}{\mathcal{H}}$ is a ray map, we say that a linear or anti-linear map $T:\mathcal{H} \longrightarrow \mathcal{H}$ implements $\underset{\sim}{T}$, if $\underset{\sim}{T}\underset{\sim}{\psi}=\underset{\sim}{T\psi}$ for all $\psi\in\mathcal{H}$.

<u>Definition 3.1.3</u>: By an <u>anti-unitary operator</u> A on $\mathcal{H}$, we mean a map $A:\mathcal{H} \longrightarrow \mathcal{H}$ which satisfies:

1) $A(\psi+\phi)=A\psi+A\phi$ for all ϕ,ψ in $\mathcal{H}$;
2) $A(\lambda\psi)=\overline{\lambda}A\psi$, for all $\lambda\in\mathbb{C},\psi$ in $\mathcal{H}$;
3) $<A\psi,A\phi>=\overline{<\psi,\phi>}$ for all ψ,ϕ in $\mathcal{H}$.

Wigner proved the following theorem in [31];

<u>Theorem 3.1.4</u>: Let $\underset{\sim}{T}:\underset{\sim}{\mathcal{H}} \longrightarrow \underset{\sim}{\mathcal{H}}$ be an automorphism of $\underset{\sim}{\mathcal{H}}$.

Then there exists an operator T on $\mathcal{H}$, which is either unitary or anti-unitary such that T implements $\underset{\sim}{T}$.

Let G be a Lie group and, let $g \longrightarrow \underset{\sim}{T}_g$ be a representation of G in the group of automorphisms of $\underset{\sim}{\mathcal{H}}$. We suppose that $\underset{\sim}{T}_g$ is implemented by the operators U(g), which may be unitary or anti-unitary.

We shall assume that $U(g)$ are unitary for all $g \in G$.

For g_1, g_2 in G, since $U(g_1)U(g_2)$ and $U(g_1 g_2)$ implement the same automorphism $\underset{\sim}{T}_{g_1 g_2}$, it follows that there exist constants $\omega(g_1, g_2)$ of modulus unity such that,

$$U(g_1)U(g_2)=\omega(g_1,g_2)U(g_1 g_2) \text{ for all } g_1, g_2 \text{ in } G.$$

We say that U is a <u>projective (or multiplier) representation of G</u> in $\mathcal{H}$.

Since,

$U(g_1)[U(g_2)U(g_3)]=[U(g_1)U(g_2)]U(g_3)$ for all g_1, g_2, g_3 in G,

we have

(A) $\omega(g_1, g_2 g_3)\omega(g_2, g_3)=\omega(g_1, g_2)\omega(g_1 g_2, g_3)$ for all g_1, g_2, g_3 in G.

Furthermore, since $U(e)=I$ (e is the identity element in G) is the identity operator on $\mathcal{H}$, it follows that

(B) $\omega(g,e)=\omega(e,g)=1$ for all g in G.

<u>Definition 3.1.5</u>: Any function ω defined on $G \times G$ taking values in the multiplicative group of all complex numbers of modulus unity, and satisfying equations (A) and (B) is called a <u>multiplier</u> of G.

<u>Lemma 3.1.6</u>: Let U be a projective unitary representation of G with multiplier ω in a complex separable Hilbert space $\mathcal{H}$, implementing an automorphism $\underset{\sim}{T}_g$ of $\underset{\sim}{\mathcal{H}}$ for each g in G.

Furthermore, suppose that for g in G, U'(g) are unitary operators on $\mathcal{H}$ implementing the automorphism T_g of $\mathcal{H}$.

Then there exist complex numbers $\alpha(g)$ of modulus unity such that

$$U'(g)=\alpha(g)U(g) \text{ for all } g \text{ in } G.$$

Furthermore, for all g_1, g_2 in G, U' satisfies

$$U'(g_1)U'(g_2)=\omega'(g_1,g_2)U'(g_1g_2), \text{ where}$$

$$\omega'(g_1,g_2) = \frac{\alpha(g_1)\alpha(g_2)\omega(g_1,g_2)}{\alpha(g_1g_2)}$$

Multipliers ω and ω' of G related in this way are said to be <u>cohomologous</u>.

<u>3.2 Multipliers on Locally Compact Groups</u>

Throughout this section we shall assume that G is a separable locally compact topological group.

<u>Definition 3.2.1</u>: A function $\omega:G\times G \longrightarrow T$ is said to be a <u>Borel multiplier</u>, if ω is Borel measurable, and satisfies equations (A) and (B), where T denotes the multiplicative group of all complex numbers of modulus unity.

<u>Definition 3.2.2</u>: A Borel multiplier ω is said to be <u>trivial</u>, if there exists a Borel function $\alpha:G \longrightarrow T$ such that

$$\omega(g_1,g_2)=\alpha(g_1)\alpha(g_2)\alpha(g_1g_2)^{-1} \text{ for all } g_1,g_2 \text{ in } G.$$

If ω_1 and ω_2 are multipliers of G, then their product $\omega_1\omega_2$ is also a multiplier. If ω is a multiplier, then so is ω^{-1}. All multipliers of G thus constitute an abelian group A(G).

Two multipliers ω_1 and ω_2 are said to be equivalent (or cohomologous), if $\omega_1\omega_2^{-1}$ is trivial; we write $\omega_1 \sim \omega_2$ in symbols.

It is clear that "$\sim$" is indeed an equivalence relation. The set of all trivial multipliers constitute a subgroup $A_0(G)$ of A(g).

The factor group $A(G)/A_0(G)$ is the set of all equivalence classes of multipliers; it will be denoted by $H^2(G,\mathbb{C})$.

3.2.3: Let G be a locally compact group, and let ω be a multiplier of G. We define a new group G^ω to be the set of all pairs (λ,g), where $\lambda \in T$ and $g \in G$; its multiplication is given by,

(1) — $(\lambda_1,g_1)(\lambda_2,g_2)=\left(\lambda_1\lambda_2\omega(g_1,g_2),g_1g_2\right)$ for $(\lambda_i,g_i) \in G^\omega, i=1,2$.

Then G^ω is the semi-direct product of T and G with G acting on T depending on ω.

We shall assign a topology to G^ω, which makes G^ω a locally compact group, and it is such that the multiplication in G^ω is continuous in this topology:

By an analogous computation, we have

$$\eta''(g_1,g_2)=\int_G \eta'(g_1,s)\{f_2(g_2^{-1}s)-f_2(s)\}d\nu(s) \text{ for } g_1,g_2 \text{ in } N_2.$$

By inserting the last expression into (B) we obtain

$$\eta''(g_1,g_2)=\iint_{G\times G} \eta(k,s)\{f_1(kg_1^{-1})-f_1(k)\}\{f_2(g_2^{-1}s)-f_2(s)\}d\mu(k)d\nu(s)$$

for g_1,g_2 in N_2.

In this integral only f_1 and f_2 depend on g_1 and g_2. Thus the smoothness of η'' with respect to the coordinates of g_1 and g_2 follows from that of f_1 and f_2 and the analycity of group multiplication on N.

This completes the proof of the theorem.

3.3 Multipliers on Some Special Groups

Proposition 3.3.1: Let G be a locally compact group, and ω a multiplier for G. Then ω is said to be symmetric if $\omega(g_1,g_2)=\omega(g_2,g_1)$ for all g_1,g_2 in G.

Proposition 3.3.2: Let G be a separable, locally compact abelian group, and ω a symmetric multiplier for G. Then ω is locally trivial.

Proof: The symmetry of ω implies that G^ω is abelian. Let $(\lambda_0,e)\in G^\omega$ be a fixed point with $\lambda_0\neq 1$, and $|\lambda_0|=1$.

Then there exists a character χ on G^ω such that $\chi(\lambda_0,e)\neq 1$.

equivalent to the multiplier defined by the function $\exp iB(x,y)$.

Proof: It is trivial to verify that the function $\exp iB(x,y)$ is a multiplier for every real bilinear function on $V\times V$.

We state the following

Lemma: Let G be a separable, locally compact, connected, simply connected group. Let ω be a multiplier for G which is locally trivial. Then ω is globally trivial. A proof of this lemma is given in Parthasarathy [20].
We now return to the proof of proposition 3.3.3.

For any $x\in V$ we consider the subgroup of all points $tx, t\in\mathbb{R}$.

It is well-known that $\mathbb{R}$ has only trivial multipliers. Hence, it follows that there exists a function $\lambda_t(x)$ such that $\omega(tx,sx)=\lambda_{t+s}(x)\lambda_t(x)^{-1}\lambda_s(x)^{-1}$ for all $t,s\in\mathbb{R}$, $x\in V$, $x\neq 0$, and

$|\lambda_t(x)|=1$.

Thus, $\{(\lambda_t(x),tx)\,|\,t\in\mathbb{R}\}$ is a one-parameter subgroup of V^{ω}.

We consider the expression

(1) $\omega(y,tx)\omega(y+tx,-y)\omega(y,-y)^{-1}$, where $x,y\in V$, $t\in\mathbb{R}$.

By theorem 3.2.6, we can suppose that ω is a C^{∞}-function in in a neighbourhood N of the origin in V.

Assuming that y is in N, and differentiating the expression (1) with respect to t, and putting t=0, we get

$$\frac{d}{dt}\,\omega(y,tx)\Big|_{t=0}+\omega(y,-y)^{-1}\frac{d}{dt}\omega(y+tx,-y) \qquad \text{for all } x\in V.$$

We now put

$$(2)\quad iF(x,y)=\frac{d}{dt}\omega(y,tx)\Big|_{t=0}+\omega(y,-y)^{-1}\frac{d}{dt}\omega(y+tx,-y)\Big|_{t=0}$$

Integrating we get

(3) $\exp itF(x,y)=\omega(y,tx)\omega(y+tx,-y)\omega(y,-y)^{-1}$, and in particular this relation holds for any x,y in V and $t\in\mathbb{R}$.

Putting t=1, we get

(4) $\omega(y,x)\omega(y+x,-y)=\omega(y,-y)\exp iF(x,y)$.

We also have,

(5) $\omega(x,y)\omega(x+y,-y)=\omega(y,-y)$, and therefore

$\exp iF(x,y)=\omega(x,y)\omega(y,x)^{-1}$ for all x,y in V.

Thus, $F(x,y)=-F(y,x)$. From equation (2) it also follows that F is a linear function in x.

Hence, F is a skew-symmetric real bilinear functional on V×V.

Since V is simply connected it follows from the lemma, and proposition 3.3.2 that the symmetric multiplier $\omega(x,y)\omega(y,x)$ is trivial.

$\omega(x,y)^2=\omega(x,y)\omega(y,x)^{-1}\omega(y,x)\omega(x,y)$ is equivalent to $\omega(x,y)\omega(y,x)^{-1}=\exp -iF(x,y)$.

Hence, $\omega(x,y)\exp\frac{i}{2}F(x,y)$ is locally trivial and therefore by the lemma globally trivial.

We now put $B(x,y)=-\frac{1}{2}F(x,y)\ (x,y\in V)$.

This completes the proof of the proposition.

<u>Proposition 3.3.4</u>: Let G be a separable locally compact group which is a semi-direct product $N\underset{\alpha}{\times}K$, where N is a normal closed subgroup of G, and K is a closed subgroup of G.

Let ω be a multiplier for G.

Then there exists an equivalent multiplier of the form
(A) $\omega_1(n_1k_1,n_2k_2)=\sigma(n_1,\alpha(k_1)(n_2))\delta(k_1,k_2)\psi(n_2,k_1)$ for all $n_1,n_2\in N, k_1,k_2\in K$, where

σ is a multiplier for N, δ is a multiplier for K and ψ is a Borel function defined on $N\times K$ and taking values in T; furthermore, σ, δ and ψ satisfy the following conditions:

(1) $\sigma(\alpha(k)(n_1),\alpha(k)(n_2))=\sigma(n_1,n_2)\psi(n_1n_2,k)\psi(n_1,k)^{-1}\psi(n_2,k)^{-1}$ for all $k\in K$, $n_1,n_2\in N$, and

(2) $\psi(n,k_1k_2)=\psi(\alpha(k_2)(n),k_1)\psi(n,k_2)$ for all $n\in N$ $k_1,k_2\in K$.

Conversely, if σ, δ and ψ are functions satisfying the conditions described above then the function ω_1 defined by (A) is a multiplier for G.

Proof: The converse part of the proposition can be proved by direct verification.

To prove the first part, we note that

$$\omega(n_1k_1,n_2k_2)=\omega(n_1,\alpha(k_1)(n_2)) \frac{\omega(n_1\alpha(k_1)(n_2),k_1k_2)\,\omega(k_1,n_2k_2)}{\omega(n_1,k_1)\ \omega(\alpha(k_1)(n_2),k_1k_2)}, \text{ since}$$

ω is a multiplier for G.

For every $k\in K$, $\alpha(k)$ is an inner automorphism of N. Using this fact and putting $\omega(n,k)=a(nk)\,(n\in N, k\in K)$ we get

$$\omega(n_1k_1,n_2k_2)= \omega(n_1,\alpha(k_1)(n_2))\frac{a(n_1k_1n_2k_2)}{a(n_1k_1)\,a(n_2k_2)}\omega(k_1,k_2)\frac{\omega(k_1,n_2)}{\omega(\alpha(k_1)(n_2),k_1)}$$

which is equivalent to

$\omega(n_1, \alpha(k_1)(n_2))\ \omega(k_1,k_2)\psi(n_2,k_1)$, where

$\psi(n_2,k_1)=\omega(k_1,n_2)\ \omega(\alpha(k_1)(n_2),k_1)^{-1}$.

We shall denote by σ and δ the restrictions of ω to N and K respectively.

Thus ω is equivalent to a multiplier ω_1, where ω_1 is defined by,

$\omega_1(n_1k_1,n_2k_2)= \sigma(n_1,\alpha(k_1)(n_2)\ \delta(k_1,k_2)\psi(n_2,k_1)$.

Putting $k_1=n_2=e$; $k_1=k_2=e$; and $n_1=n_2=e$ successively in the above expression we get

$\psi(n_2,k_1)=\omega_1(k_1,n_2)$; $\sigma(n_1,n_2)=\omega_1(n_1,n_2)$; $\delta(k_1,k_2)=\omega_1(k_1,k_2)$.

<u>Lemma 3.3.7</u>: Let G be a Lie group and ω a multiplier for G, which is infinitely differentiable in a neighbourhood of the identity. Then G^{ω} itself admits a Lie structure.

<u>Proof</u>: It is easy to see that the product xy of any two elements x,y in G^{ω} can be expressed in some coordinate system as a C^{∞}-function of their arguments.

By the remark following theorem 2.6.2 in [3], G^{ω} itself admits a Lie structure.

<u>Proposition 3.3.8</u>: Let G be a Lie group, and ω a multiplier for G.

Then there exists a multiplier ω' which is equivalent to ω and analytic in a neighbourhood of the identity in G.

<u>Proof</u>: We assume that ω is infinitely differentiable in a neighbourhood of the identity in G. By lemma 3.3.7 G^{ω} admits a Lie structure.

The mapping $\beta:(\lambda,g)\longrightarrow g$ of G^{ω} onto G is an analytic homomorphism.

By the theory of semi-direct product extensions there exists an analytic homomorphism γ of an open set N containing e in G into G^{ω} such that $\beta\gamma(g)=g$ for all $g\in N$.

Then $\gamma(g)$ is of the form
$\gamma(g)=(\alpha(g),g)$ for all $g\in N$, where $|\alpha(g)|=1$. We have
$\gamma(g_1)\gamma(g_2)\gamma(g_1g_2)^{-1}= (\alpha(g_1)\alpha(g_2)\alpha(g_1g_2)^{-1}\omega(g_1,g_2),e)$
for all $g_1,g_2\in N$.

function on $V \times K$ satisfying conditions 1) and 2) of the proposition.

Furthermore, we can assume that σ, δ and ψ are analytic in a neighbourhood of the identity in the appropriate spaces.

By proposition 3.3.3, σ is equivalent to a multiplier of the form $\exp iB(v_1,v_2)$, where B is a real skew-symmetric bilinear function on $V \times V$.

We shall show that, in fact, any skew symmetric bilinear form invariant under K is identically zero.

We choose and fix any coordinate system in V. We let A be the matrix of the given symmetric bilinear form, and B' the matrix of any invariant skew symmetric bilinear form.

Let k^* denote the adjoint of k with respect to the Euclidean inner product, where k is an element of K.

The invariance conditions imply that $kAk^*=A$, and $kB'k^*=B'$ for all k in K.

Since the symmetric form is non-singular, A^{-1} exists, and we put $C=A^{-1}B'$.

Then

$$AC=B'=kB'k^*=kAk^*k^{*-1}Ck^*=Ak^{*-1}Ck^*, \text{ and hence}$$

$$C=k^{*-1}Ck^* \text{ for all } k \text{ in } K.$$

Thus k^* and C commute for all k in K. Since K is algebraically irreducible, C is a scalar times the identity operator, that is, $B'=tA$, for some $t\in\mathbb{R}$. Since A is symmetric, and B' skew symmetric this is impossible unless $t=0$.

Therefore, $\exp iB(v_1,v_2)\equiv 1$ for all v_1,v_2 in V.

This also implies that the function $\psi(v,k)$ on $V\times K$ satisfies the equation (A) $\psi(v_1v_2,k)=\psi(v_1,k)\psi(v_2,k)$ for all v_1,v_2 in V, and k in K.

The last relation shows that there exists a function $f:K\longrightarrow V$ such that

(B) $\psi(v,k)=\exp i\langle f(k),v\rangle$ for all $v\in V$, $k\in K$, $\langle,\rangle$ denotes the Euclidean scalar product.

From condition 2) of proposition 3.3.4, we also have,

(C) $\psi(v,k_1k_2)=\psi(k_2(v),k_1)\psi(v,k_2)$ for all v in V and k_1,k_2 in K.

The equation (B) together with (C) implies that

$\langle f(k_1k_2),v\rangle=\langle f(k_1),k_2(v)\rangle + \langle f(k_2),v\rangle$ for all $v\in V$, $k_1,k_2\in K$.

Denoting k^* the adjoint of k, we obtain $f(k_1k_2)=k_2^* f(k_1)+f(k_2)$ for all $k_1,k_2\in K$.

Then by Lemma 3.3.6 there exists a vector v' in V such that $f(k)=k^*v'-v'$ for all $k\in K$.

Thus from equation (B) we obtain

$\psi(v,k)=\exp i\langle k^{*}v'-v',v\rangle=\exp i\langle v',k(v)-v\rangle$ for all v in V and K.

Putting $\gamma(vk)=\exp -i\langle v',v\rangle$, we obtain

$\omega(v_1k_1,v_2k_2)=\delta(k_1,k_2)\psi(v_2,k_1)=$

$=\delta(k_1,k_2)\gamma(v_1k_1)\gamma(v_2k_2)\gamma(v_1k_1v_2k_2)^{-1}$ and therefore ω is a multiplier equivalent to the multiplier δ of K.

This completes the proof of the proposition.

Remark: We consider an example which is of great physical interest.

We let V to be the additive group of all quadruples of real numbers x_0,x_1,x_2,x_3, and K the connected component of the identity in the group of all linear transformations of V onto V which leave fixed the scalar product $\underset{\sim}{x}\cdot\underset{\sim}{y}=x_0y_0-x_1y_1-x_2y_2-x_3y_3$, where $\underline{x},\underline{y}$ is in V.

The resulting semi-direct product is then isomorphic to the so-called proper inhomogeneous Lorentz group, the connected component of the identity in the group of all relativistic automorphisms of space-time; it is usually denoted by $\mathcal{P}_+^{\uparrow}$.

Proposition 3.3.9 states that every multiplier of the semi-direct product $V\underset{\alpha}{\times}K$ is equivalent to a multiplier of K.

$\phi(hg)=\omega(h,g)^{-1}U_o(h)\phi(g)$ holds everywhere except possibly on a set of μ-measure zero.

3) $\int_X <\phi(x),\phi(x)>d\mu(x)<\infty$.

Then K becomes a separable Hilbert space, if we identify functions differing on sets of μ-measure zero, and define the scalar product

$$(\phi,\psi)=\int_X <\phi(x),\psi(x)>d\mu(x).$$

We shall denote this Hilbert space by $L^2_\omega(G,\mathcal{H}_o,d\mu)$.

For each g,k in G and $\phi\in L^2_\omega(G,\mathcal{H}_o,d\mu)$, we define

$$U(g)\phi(k)=\omega(k,g)\phi(kg)\sqrt{\rho_g(k)}.$$

We shall verify that U is a multiplier representation of G with multiplier ω in the space $L^2_\omega(G,\mathcal{H}_o,d\mu)$.

U has the following properties:

1) For each g_1,g_2,k in G, $\phi\in L^2_\omega(G,\mathcal{H}_o,d\mu)$

$U(g_1)U(g_2)\phi(k)=\omega(k,g_1)U(g_2)\phi(kg_1)\sqrt{\rho_{g_1}(k)}=$

$=\omega(k,g_1)\omega(kg_1,g_2)\phi(kg_1g_2)\sqrt{\rho_{g_1}(k)\rho_{g_2}(kg_1)}=$

$=\omega(k,g_1)\omega(kg_1,g_2)\phi(kg_1g_2)\sqrt{\rho_{g_1g_2}(k)}=\omega(g_1,g_2)U(g_1g_2)\phi(k)$

by lemma 2.2.3, and using the fact that ω satisfies the multiplier condition.

2) For each $\phi,\psi\in L^2_\omega(G,\mathcal{H}_o,d\mu)$, $g\in G$, we have

$(U(g)\phi, U(g)\psi) = \int_X \langle U(g)\phi(x), U(g)\psi(x)\rangle d\mu(x) =$

$= \int_X \rho_g(x) \langle \phi(xg), \psi(xg)\rangle d\mu(x) =$

$= \int_X \rho_g(xg^{-1}) \langle \phi(x), \psi(x)\rangle d\mu(xg^{-1}) =$

$= \int_X \langle \phi(x), \psi(x)\rangle d\mu(x)$, by the definition of ρ_g. Hence $U(g)$ are unitary operators on the space $L^2_\omega(G, \mathcal{H}_o, d\mu)$.

3) For each ϕ, ψ in $L^2_\omega(G, \mathcal{H}_o, d\mu)$ and g in G,
$(U(g)\phi, \psi) = \int_X \langle U(g)\phi(x), \psi(x)\rangle d\mu(x) =$

$= \int_X \sqrt{\rho_g(x)} \langle \phi(xg), \psi(x)\rangle d\mu(x)$. The integrand is a Borel measurable function in both variables. It follows that $(U(g)\phi, \psi)$ is a Borel function on G for each ϕ, ψ in $L^2_\omega(G, \mathcal{H}_o, d\mu)$.

By lemma 1.2.4 U is a unitary representation of G in $L^2_\omega(G, \mathcal{H}_o, d\mu)$.

<u>4.2.1 Induced Representations on the space $L^2(X, \mathcal{H}_0, d\mu)$:</u>

Let G be a separable locally compact group, and H a closed subgroup of G. Let U_o be a projective unitary representation of H with Borel multiplier ω in a separable Hilbert space $\mathcal{H}_o$. We shall assume that ω is a Borel multiplier for G. Let $X=G/H$ be the quotient space, and μ a quasi-invariant measure on X.

We shall assume that μ is invariant.

For each $x \in X$ let $\mathcal{H}_x$ be a separable Hilbert space. We form the direct integral Hilbert space

$$\int_{\oplus X} d\mu(x)\mathcal{H}_x.$$

We shall assume, in addition, that for each $x\in X$, $\mathcal{H}_x=\mathcal{H}_o$ μ-almost everywhere.

Then, $\int_{\oplus X} d\mu(x)\mathcal{H}_x=L^2(X,\mathcal{H}_o,d\mu)$.

Let $\mathring{x}\in X$ be a fixed element. Let $b:X\longrightarrow G$ be a Borel section such that $b(\mathring{x})=e\in G$, and $\mathring{x}b(x)=x$ for each $x\in X$.

We shall construct a projective unitary representation of G in $L^2(X,\mathcal{H}_o,d\mu)$, which is unitary equivalent to the induced representation U of G in $L^2_{\omega}(G,\mathcal{H}_o,d\mu)$.

Since μ is invariant, the induced representation U of G in $L^2_{\omega}(G,\mathcal{H}_o,d\mu)$ takes the form,

$$U(g)\phi(k)=\omega(k,g)\phi(kg) \text{ for } g,k\in G, \text{ and } \phi\in L^2_{\omega}(G,\mathcal{H}_o,d\mu).$$

We define operators V and W on the Hilbert spaces $L^2(X,\mathcal{H}_o,d\mu)$ and $L^2(G,\mathcal{H}_o d\mu)$, respectively by,

$V\psi(g)=\omega(g,b^{-1}(\overline{g}))U_o(gb^{-1}(\overline{g}))\psi(\overline{g})$, where $g\in G$ $\overline{g}=Hg$ is the right coset of g with respect to H and $\psi\in L^2(X,\mathcal{H}_o,d\mu)$.

$W\phi(\overline{g'})=\omega(b(\overline{g'}),b^{-1}(\overline{g}'))^{-1}\phi(b(\overline{g}'))$ for $g'\in G$ and $\phi\in L^2_{\omega}(G,\mathcal{H}_o,d\mu)$.

<u>Lemma</u>: V and W are unita y operators, and $W=V^{-1}$.

Proof: We show that V is well-defined. Let $\psi \in L^2(X, \mathcal{H}_o, d\mu)$.

For each g in G, and h in H, we have

$$V\psi(hg) = \omega(hg, b^{-1}(\overline{hg}) U_o(hgb^{-1}(\overline{hg}))\psi(\overline{hg}) =$$

$$= \omega(hg, b^{-1}(\overline{g}))U_o(hgb^{-1}(\overline{g}))\psi(\overline{g}) =$$

$$= \omega(hg, b^{-1}(\overline{g}))\omega(h, gb^{-1}(\overline{g}))^{-1}U_o(h)U_o(gb^{-1}(\overline{g}))\psi(\overline{g}) =$$

$$= \omega(hg, b^{-1}(\overline{g}))\omega(h, gb^{-1}(\overline{g}))^{-1}\omega(g, b^{-1}(\overline{g}))^{-1}U_o(h)\{\omega(g, b^{-1}(\overline{g}))U_o(gb^{-1}(\overline{g}))\}\psi(\overline{g}) =$$

$$= \omega(hg, b^{-1}(\overline{g}))\omega(h, gb^{-1}(\overline{g}))^{-1}\omega(g, b^{-1}(\overline{g}))^{-1}U_o(h)V\psi(g) =$$

$$= \omega(h, g)^{-1}U_o(h)V\psi(g).$$

Clearly, W is a one-to-one map. We show that W is an isometry.

$$\| W\phi \|^2_{L^2(X, \mathcal{H}_o)} = \int_X \langle W\phi(x), W\phi(x)\rangle d\mu(x) =$$

$$= \int_X \langle \phi(b(x)), \phi(b(x))\rangle d\mu(x) = \| \phi \|^2_{L^2_\omega(G, \mathcal{H}_o)}$$, since $\langle \phi(b(x)), \phi(b(x))\rangle$ is a Borel function on G constant on each right coset of b(x).

For all $\psi \in L^2(X, \mathcal{H}_o, d\mu)$ and $x \in X$

$$(WV\psi)(x) = \omega(b(x), b^{-1}(x))^{-1}V\psi(b(x)) =$$

$$= \omega(b(x), b^{-1}(x))^{-1}\omega(b(x), b^{-1}(\overline{b(x)}))\, U_o(b(x)b^{-1}(\overline{b(x)})\psi(\overline{b(x)})$$

$$= \omega(b(x), b^{-1}(x))^{-1}\omega(b(x), b^{-1}(x))U_o(b(x)b^{-1}(x))\psi(x) =$$

$= \psi(x)$, since $\overline{b(x)} = x$

Hence WV=I the identity operator on the space $L^2(X, \mathcal{H}_o, d\mu)$.

It follows that V and W are unitary and $W = V^{-1}$.

We define a multiplier representation U' of G in $L^2(X, \mathcal{H}_o d\mu)$ by,

$U'(g)\psi(x)=V^{-1}U(g)V\psi(x)=\omega(b(x),b^{-1}(x))^{-1}U(g)V\psi(b(x))=$
$=\omega(b(x),b^{-1}(x))^{-1}\omega(b(x),g)V\psi(b(x)g)=$
$=\omega(b(x),b^{-1}(x))^{-1}\omega(b(x),g)\omega(b(x)g,b^{-1}(xg))U_o(b(x)gb^{-1}(xg))\psi(xg)$
$=\lambda(g,x)U_o(b(x)gb^{-1}(xg))\psi(xg)$, since $\overline{b(x)g}=\overline{b(xg)}=xg$ where
$x\in X$, $g\in G$, $\psi\in L^2(X,\mathcal{H}_o,d\mu)$ and
$\lambda(g,x)=\omega(b(x),b^{-1}(x))^{-1}\omega(b(x),g)\omega(b(x)g,b^{-1}(xg))$.

Then by construction the representation U' of G in $L^2(X,\mathcal{H}_o,d\mu)$ is unitary equivalent to the representation U of G in $L^2(G,\mathcal{H}_o,d\mu)$ and therefore, the representation U' of G is a projective unitary representation of G with multiplier ω.

Definition 4.2.2: Let $\mathcal{A}$ and $\mathcal{B}$ be two involutive algebras.

A morphism (respectively isomorphism) of $\mathcal{A}$ into $\mathcal{B}$ is a map (respectively, a bijection) ϕ of $\mathcal{A}$ into $\mathcal{B}$ such that $\phi(x+y)=\phi(x)+\phi(y)$, $\phi(\lambda x)=\lambda\phi(x)$, $\phi(xy)=\phi(x)\phi(y)$, $\phi(x^*)=\phi(x)^*$ for any x,y in $\mathcal{A}$, λ in $\mathbb{C}$.

Remark: Let U be a unitary representation of a separable locally compact group G in a separable Hilbert space $\mathcal{H}$.

Then the commuting algebra $\mathfrak{A}'(U)$ of U is an algebra of bounded operators in a complex separable Hilbert space, which contains the identity operator, and is closed under the adjoint operation and in the weak operator topology.

Theorem 4.2.3: Let G be a separable locally compact group and H a closed subgroup.

Let V be a unitary representation of G in a separable Hilbert space $\mathcal{H}$, and P a transitive system of imprimitivity for V defined on the Borel space $X=G/H$.

Then by the imprimitivity theorem there exists a unique (up to unitary equivalence) unitary representation U_o of H in a Hilbert space $\mathcal{H}_o$ such that the induced representation U of G in $L^2_M(G,\mathcal{H}_o)$ is unitary equivalent to the representation V of G, and P is equivalent to the projection-valued measure canonically associated with the induced representation U of G.

Let $\mathfrak{A}'(U_o)$ be the commuting algebra of U_o.

Then $\mathfrak{A}'(U_o)$ is isomorphic to the algebra of operators in $\mathcal{H}$ which commute with both the range of V and the range of P.

Proof: Let S be the set of all operators in $L^2_M(G,\mathcal{H}_o)$ commuting both with the $U(g)$ $(g\in G)$ and with the range of the projection-valued measure associated with U.

For each A in $\mathfrak{A}'(U_o)$ let $(\tilde{A}\phi)(x)=A\phi(x)$ for ϕ in $L^2_M(G,\mathcal{H}_o)$, $x\in G$. Then it is easily seen that $\tilde{A}\in S$.

The map $A\longrightarrow\tilde{A}$ is clearly a $*$-morphism of $\mathfrak{A}'(U_o)$ into S. We have to show that this map is surjective, that is, that every operator in S is of the form $\tilde{A}$.

Let $B\in S$. Since the range of B commutes with the range of the projection-valued measure associated with U, we may decompose

Furthermore, the representation U_o of H is unique up to equivalence.

Theorem 4.2.3 also holds in the case V is a projective representation of G.

4.2.4 Induced Projective Representations in the Space $L^2(X,\mathcal{H}_o)$

Let G be a separable locally compact group, and H a closed subgroup of G. Let U_o be a projective representation of H with Borel multiplier ω in a separable Hilbert space $\mathcal{H}_o$. We shall assume that ω is a Borel multiplier for G. Let X=G/H be the quotient space, and μ a quasi-invariant measure on X.

We shall assume that μ is invariant.

For each $x\in X$, let $\mathcal{H}_x$ be a separable Hilbert space.

We form the direct integral Hilbert space

$$\int_{\oplus X} d\mu(x)\mathcal{H}_x.$$

We shall assume, in addition, that $\mathcal{H}_x=\mathcal{H}_o$ μ-almost everywhere.

Then $\int_{\oplus X} d\mu(x)\mathcal{H}_x=L^2(X,\mathcal{H}_o,d\mu).$

Let $\mathring{x}\in X$ be a fixed element. Let $b:X\longrightarrow G$ be a Borel section such that $b(\hat{x})= e\in G$, and $\mathring{x}b(x)=x$ for all $x\in X$.

Then the induced projective representation U' of G in $L^2(X,\mathcal{H}_o,d\mu)$ is given by,

(1) - $U'(g)\psi(x) = \lambda(g,x)U_o(b(x)gb^{-1}(xg))\psi(xg)$,

where

$x\in X$, $g\in G$, $\psi\in L^2(X,\mathcal{H}_o,d\mu)$, $\lambda(g,x)\in \mathfrak{U}'(U')$, and

$\lambda(g,x)=\omega(b(x),b^{-1}(x))^{-1}\omega(b(x),g)\omega(b(x)g,b^{-1}(xg))$.

In section 4.2.1, we have shown that the induced multiplier representation U' of G in $L^2(X,\mathcal{H}_o,d\mu)$ is unitary equivalent to the induced representation U of G in the Mackey space $L^2_\omega(G,\mathcal{H}_o,d\mu)$.

Thus, we have a multiplier representation unitary equivalent to the induced multiplier representation U of G in $L^2_\omega(G,\mathcal{H}_o,d\mu)$ for any choice of Borel section $b:X\longrightarrow G$ satisfying $b(\mathring{x})=e$ and $\mathring{x}b(x)=x$ for $x\in X$.

We note that each automorphism α_h of N has a dual α_h^* which is an automorphism of $\hat{N}$. Specifically, $[\chi]\alpha_h^*$ is the character $n \longrightarrow \chi(\alpha_h(n))$. Clearly, $\hat{N}$ becomes an H-space if we define $[\chi]\cdot h=[\chi]\alpha_h^*$.

Now, by theorem 2.1.7 U is determined by a projection-valued measure $E \longrightarrow P_E$ defined on the Borel subsets of the dual group $\hat{N}$ of N. It is readily verified U and V satisfy the identity (A), if and only if P and V satisfy $V(h)P_E V(h)^{-1}=P_{E\cdot h^{-1}}$ for all $h\in H$, and all Borel subsets E in $\hat{N}$.

Thus P is a system of imprimitivity for V.

In order to apply the imprimitivity theorem we must have a transitive system of imprimitivity and H does not usually act transitively on $\hat{N}$. On the other hand, H restricted to any orbit of H in $\hat{N}$ does act transitively and under appropriate circumstances we may concentrate on the restriction of P to an orbit. We define the orbit $\pi(\chi)$ of χ in $\hat{N}$ to be the set of all $[\chi]\cdot h$ with $h\in H$ and let $\tilde{\hat{N}}$ denote the space of all orbits. We define a subset F of $\tilde{\hat{N}}$ to be a Borel set if $\pi^{-1}(F)$ is a Borel subset of $\hat{N}$, and we say that $\tilde{\hat{N}}$ has a countably separated Borel structure if there exist countably many Borel sets which separate points. This condition holds, in particular, whenever there exists a Borel subset of $\hat{N}$ which meets each orbit just once. Whenever it does hold we say that G is a regular semi-direct product of N and H. The importance of this condition is that it implies $P_{\hat{N}\setminus O}=0$ for some unique orbit O whenever V is irreducible.

Thus every irreducible unitary representation of a regular semi-direct product is described by a pair U,V where P is a transitive system of imprimitivity for V based on an orbit of $\hat{N}$ under H.

We state the following

<u>Theorem</u>: Let G be a semi-direct product of N and H, where N is normal and commutative; N and H are separable and locally compact.

For each $\chi \in \hat{N}$, let H_χ denote the subgroup of all $h \in H$ for which $[\chi] \cdot h = \chi$.

Then H_χ is closed, and for each irreducible unitary representation U_o of H_χ, $n,h \longrightarrow \chi(n)U_o(h)$ is a unitary representation χU_o of the subgroup NH_χ.

We form the induced representation $U^{\chi U_o}$ of G. Let C be a set which meets each H orbit in $\hat{N}$ once and only once. Then

1) $U^{\chi U_o}$ is irreducible for all χ and U_o;

2) As χ varies over C and U_o varies over inequivalent irreducible representations of H_χ we get inequivalent irreducible representations of G and we get one equivalent to every $U^{\chi U_o}$ whether or not χ lies in C.

3) If G is a regular semi-direct product then every irreducible representation of G is equivalent to some $U^{\chi U_o}$.

4.3.2 Projective Representations and the Stone-von Neumann Theorem

Let G be a separable locally compact commutative group and $\hat{G}$ its dual.

Let U and V be unitary representations of G and $\hat{G}$ respectively and let U and V satisfy
$U(x)V(\chi)=\chi(x)V(\chi)U(x)$ for $x \in G$ and $\chi \in \hat{G}$.

Let $W(x,\chi)=U(x)V(\chi)$ for all x,χ in the product group $G\times\hat{G}$. Then
$W(x_1,\chi_1)W(x_2,\chi_2)=U(x_1)V(\chi_1)U(x_2)V(\chi_2)=$
$=U(x_1)U(x_2)V(\chi_1)V(\chi_2)\overline{\chi_1(x_2)}=$
$=U(x_1x_2)V(\chi_1\chi_2)\overline{\chi_1(x_2)}= W(x_1x_2,\chi_1\chi_2)\overline{\chi_1(x_2)}$

Thus W is a projective representation of $G\times\hat{G}$ whose multiplier ω is defined by the equation $\omega((x_1,\chi_1),(x_2,\chi_2))=\overline{\chi_1(x_2)}$.

Conversely, given any ω-representation W of $G\times\hat{G}$ we verify at once that $W(x,\chi)=U(x)V(\chi)$ where U and V are restrictions of W to $G\times e$ and $e\times\hat{G}$ respectively, and U,V satisfy the identity in question.

Thus the first generalization of Stone-von Neumann uniqueness theorem may be reinterpreted as stating that for the particular ω defined above the commutative group $G\times\hat{G}$ has to within equivalence just one irreducible ω-representation. It follows in particular that changing from one ω to another can have quite profound effects on the representation theory of a group.

The theory of representations of semi-direct products carries over to ω-representations without essential change whenever $\omega \equiv 1$ on the normal subgroup N . Applying it with $N=G\times e$ we arrive once more at the uniqueness theorem as well as the additional information that our unique irreducible ω-representation is equivalent to the ω-representation of $G\times\hat{G}$ induced by the identity representation of $G\times e$.

More generally, let H be a closed subgroup of G, and let $H^{\perp}$ be the group of all $\chi\in\hat{G}$ which reduce to 1 on H. Then $H\times H^{\perp}$ is a closed subgroup of $G\times\hat{G}$ on which $\omega\equiv 1$, and we may speak of the ω-representation of $G\times\hat{G}$ induced by the identity representation of $H\times H^{\perp}$.

It follows from theorem 4.3.1 and from the theory of projective representations that this ω-representation is also irreducible and hence equivalent to W.

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