The number of pancyclic arcs in a k-strong tournament.

Anders Yeo
Department of Computer Science
Royal Holloway
University of London
Egham
Surrey, TW20 0EX
United Kingdom
anders@cs.rhul.ac.uk

Abstract

A *tournament* is a digraph, where there is precisely one arc between every pair of distinct vertices. An arc is *pancyclic* in a digraph D, if it belongs to a cycle of length l, for all $3 \leq l \leq |V(D)|$. Let $p(D)$ denote the number of pancyclic arcs in a digraph D and let $h(D)$ denote the maximum number of pancyclic arcs belonging to the same Hamilton cycle of D. Note that $p(D) \geq h(D)$. Moon showed that $h(T) \geq 3$ for all strong non-trivial tournaments, T, and Havet showed that $h(T) \geq 5$ for all 2-strong tournaments T. We will show that if T is a k-strong tournament, with $k \geq 2$, then $p(T) \geq \frac{1}{2}nk$ and $h(T) \geq \frac{k+5}{2}$. This solves a conjecture by Havet, stating that there exists a constant α_k, such that $p(T) \geq \alpha_k n$, for all k-strong tournaments, T, with $k \geq 2$. Furthermore the second results gives support for the conjecture $h(T) \geq 2k+1$, which was also stated by Havet. The previously best known bounds when $k \geq 2$ were $p(T) \geq 2k+3$ and $h(T) \geq 5$.

1 Introduction

A *tournament* is an orientation of the edges of a complete graph. An arc or vertex is *pancyclic* in a digraph D, if it belongs to a cycle of length l, for all $3 \leq l \leq |V(D)|$. As in the abstract let $p(D)$ denote the number of pancyclic arcs in a digraph D and let $h(D)$ denote the maximum number of pancyclic arcs belonging to the same Hamilton cycle of D. Note that $p(D) \geq h(D)$.

Let D be a digraph with vertex-set $V(D)$ and arc-set $A(D)$. A directed path, P, from x to y in D is denoted by (x,y)-*path*. The length of a path P is the number of arcs it contains. A path of length l is denoted by l-path. D is said to be *strong* if for all $x, y \in V(D)$ there is a (x,y)-path and a (y,x)-path in D. D is said to be k-*strong* if $D - X$ is strong for every set of vertices X in D with $|X| < k$. We define $p_k(n)$ and $h_k(n)$ as follows:

$$p_k(n) = min\{p(T) : T \text{ is a non-trivial } k\text{-strong tournament of order } n\}$$

$$h_k(n) = min\{h(T) : T \text{ is a non-trivial } k\text{-strong tournament of order } n\}$$

The only known bounds on $h_k(n)$, prior to this paper, were the following two results by Moon and Havet, respectively.

Theorem 1.1 *[5]* $h_1(n) = 3$.

Theorem 1.2 *[4]* $h_k(n) \geq 5$, *for all* $k \geq 2$.

In this paper we will prove that $h_k(n) \geq \frac{k+5}{2}$, which gives support for the following conjecture, which was stated in [4].

Conjecture 1.3 *[4]* $h_k(n) \geq 2k + 1$, *and for sufficiently large* n, $h_k(n) = 3k$.

As the following proposition was proved in [4], we can furthermore state that $h_k(n)$ grows linearly with respect to k.

Proposition 1.4 *[4]* $h_k(n) \leq 3k$.

In other words $\frac{k+5}{2} \leq h_k(n) \leq 3k$. In this paper we will also prove a bound on $p_k(n)$, which proves a conjecture from [4]. In [7] Yao, Guo and Zhang prove the following theorem.

Theorem 1.5 *[7] Every strong tournament contains a vertex* x *such that every arc out of* x *is pancyclic.*

This trivially implies that $p_k(n) \geq 2k + 2$ (as seen in [4]). In [4] this bound is improved to $p_k(n) \geq 2k+3$, when $k \geq 2$. In this paper we will prove that $p_k(n) \geq \frac{1}{2}kn$, which furthermore proves the following conjecture, stated in [4].

Conjecture 1.6 *[4] (Now proved) For all* $k \geq 2$, *there exists a constant* $\alpha_k > 0$, *such that* $p_k(n) \geq \alpha_k n$.

As Havet has proved Proposition 1.7 below, we note that $\frac{1}{2}kn \leq p_k(n) \leq 2kn$. It would be an interesting problem to narrow this gap further.

Proposition 1.7 *[4]* $p_k(n) \leq 2kn - 2k^2 - k$.

In the process of proving the bound $h_k(T) \geq \frac{k+5}{2}$ we will need Theorem 3.7 below, which states that every 3-strong tournament has two distinct vertices x and y, such that every arc out of x and every arc out of y are pancyclic. Note how this complements Theorem 1.5 above. However this result cannot be extended much further, due to the following result.

Theorem 1.8 *Let $k \geq 1$, be arbitrary. There exists an infinite class of k-strong tournaments, such that each tournament contains at most 3 vertices, with the property that all arcs out of them are pancyclic.*

A proof of Theorem 1.8 will be given after the definitions and terminology. However the following might hold.

Conjecture 1.9 *If T is a 2-strong tournament, then it has three distinct vertices, $\{x, y, z\}$, such that every arc out of x, y and z, is pancyclic.*

2 Definitions and Terminology

Let D be a digraph and let x and y be distinct vertices of D. If $xy \in A(D)$, then we say that x *dominates* y and that y is *dominated* by x. The set of all vertices which x dominates will be denoted by $N_D^+(x)$. Analogously the set of all vertices which dominate x will be denoted by $N_D^-(x)$. We will omit the subscript if D is known from the context. Furthermore $d^+(x) = |N^+(x)|$ is called the the *outdegree* of x, and $d^-(x) = |N^-(x)|$ is called the *indegree* of x. Analogously to an l-path, let an l-cycle denote a cycle of length l (i.e. a cycle with l arcs). If X and Y are sets of vertices in D such that there is no arc from a vertex in Y to a vertex in X, then we say that $X \Rightarrow Y$. A set X is called a *separating set* if $D - X$ is not strong. Furthermore X is called a *minimum separating set* if it is a separating set of minimum size (i.e if $D - X$ is not strong and D is $|X|$-strong). A *strong component* of a digraph D is a maximal set of vertices, which induce a strong component in D. If D is strong then $V(D)$ is the only strong component. If D is not strong, then we can partition the vertices in D into sets $X_1, X_2, \ldots, X_r$, such that $X_i \Rightarrow X_j$ if and only if $i < j$.

Let D be a digraph, let x and y be distinct vertices of D and let P be a (x, y)-path in D. We say that D' is obtained by contracting P into w, if the following holds. $V(D') = \{w\} \cup V(D) \backslash V(P)$, where w is a new vertex, not contained in D. Furthermore $N_{D'}^+(w) = N_D^+(y) \cap (V(D) \backslash V(P))$, $N_{D'}^-(w) = N_D^-(x) \cap (V(D) \backslash V(P))$ and an arc with both end-points in $V(D) \backslash V(P)$ belongs to D' if and only if it belongs to D. Note that if uwv is a path in D' then uPv is a path in D. Analogously if there exists a l-cycle in D', containing w then there exists an $(l + |A(P)|)$-cycle in D, containing P. We will often use the contraction operation on paths of length 1 (or 2).

We conclude this section with a well-known theorem by Camion (a non-trivial tournament is a tournament with at least 2 vertices).

Theorem 2.1 *[3] A non-trivial tournament has a Hamilton cycle, if and only if it is strong.*

3 Proofs

We can now prove Theorem 1.8.

Proof of Theorem 1.8: Let T_1, T_2 and T_3 be transitive tournaments of order at least k. Let T be the tournament obtained by adding arcs between the T_i's, such that

$T_1 \Rightarrow T_2 \Rightarrow T_3 \Rightarrow T_1$. Clearly T is k-strong. Note that the only vertex in T_i ($i = 1, 2, 3$) with the desired property, is the vertex with outdegree equal to zero in T_i, as any arc totally within T_i doesn't lie on a 3-cycle. $\square$

In the proofs of our main results, we will often use the following easy lemma.

Lemma 3.1 *Let D be a k-strong digraph, with $k \geq 1$, and let S be a separating set in D, such that $T = D - S$ is a tournament. Let $T_1, T_2, \ldots, T_r$ ($r \geq 2$) be the strong components of T, such that $T_1 \Rightarrow T_2 \Rightarrow \ldots \Rightarrow T_r$. Now the following holds:*

(i) At least k vertices in S dominate some vertices in T_1, and at least k vertices in S are dominated by some vertices in T_r.

(ii) For every $1 \leq l \leq |V(T)|$, $u \in T_1$ and $v \in T_r$, there exists a (u, v)-path of length l in T.

(iii) If $S = \{x\}$, then x is pancyclic in D.

Proof: Part (i) is well-known, and easy to prove, using the fact that D is k-strong. Part (ii) is also easy to prove using Theorem 2.1 on each component, T_i. Part (iii) follows immediately from parts (i) and (ii). $\square$

Lemma 3.2 *Let D be a strong digraph, containing a vertex x, such that $D - x$ is a tournament and $d_D^+(x) + d_D^-(x) \geq |V(D)|$. Then there is a l-cycle containing x in D, for all $2 \leq l \leq |V(D)|$.*

Proof: Let $T = D - x$ and let $n = |V(D)|$. Note that T is a tournament with $n - 1$ vertices. It is not difficult to see that $d_D^+(x) + d_D^-(x) \geq |V(D)|$, implies that x lies on a 2-cycle, so assume that $3 \leq l \leq n$. We will show that there is a l-cycle in D, containing x.

If T is not strong, then Lemma 3.1, part (iii), implies the desired result. So assume that T is strong. By Theorem 2.1 let $C = p_1 p_2 \ldots p_{n-1} p_1$ be a Hamilton cycle in T. We define $\chi^+(p_i)$ and $\chi^-(p_i)$ as follows.

$$\chi^+(p_i) = \begin{cases} 1 & : \text{ if } p_i \rightarrow x \\ 0 & : \text{ otherwise} \end{cases} \qquad \chi^-(p_i) = \begin{cases} 1 & : \text{ if } x \rightarrow p_i \\ 0 & : \text{ otherwise} \end{cases}$$

Note that $\sum_{i=1}^{n-1} (\chi^+(p_i) + \chi^-(p_{i-l+2})) = d_D^-(x) + d_D^+(x) \geq n$ (by definition, where all indices are modulo $n-1$). Since we only sum over $n-1$ numbers, we must have some i, where $(\chi^+(p_i) + \chi^-(p_{i-l+2})) = 2$. However $p_i x p_{i-l+2} p_{i-l+3} \ldots p_{i-1} p_i$ is now the desired l-cycle containing x. $\square$

Lemma 3.3 *Let T be a 2-strong tournament, containing an arc $e = xy$, such that $d^+(y) \geq d^+(x)$. Then e is pancyclic in T.*

Proof: Let $n = |V(T)|$ and let $e = xy$ be defined as in the lemma. Let D be the digraph obtained from T by contracting e, into a vertex, w_e. Note that $d_D^+(w_e) + d_D^-(w_e) = d_T^+(y) + d_T^-(x) = d_T^+(y) + (n - 1 - d_T^+(x)) \geq n - 1 = |V(D)|$, by definition.

If $\{x, y\}$ is a separating set in T, then we are done by Lemma 3.1, part (i) and (ii). So assume that $\{x, y\}$ is not a separating set in T, which implies that D is strong. As a r-cycle in D containing w_e corresponds to a $(r+1)$-cycle in T containing e, we are done by Lemma 3.2. □

Lemma 3.3 implies a result by Alspach (see [1]), which states that every arc in a regular tournament is pancyclic. This is the case as any regular tournament with at least 5 vertices is 2-strong, and the only regular tournament with less than 5 vertices is the 3-cycle.

Lemma 3.4 *Let T be a 2-strong tournament, containing a 3-cycle $xzyx$. Then at least 2 of the 3 arcs in $xzyx$ are pancyclic in T.*

Proof: If $d^+(x) = d^+(y) = d^+(z)$, then all three arcs are pancyclic, by Lemma 3.3, so assume that this is not the case. Now there is either a unique maximum element in $(d^+(x), d^+(y), d^+(z))$ or a unique minimum element. Without loss of generality assume that $d^+(y)$ is the unique maximum element, and note that this implies that $d^+(y) \geq d^+(x) + 1$.

Let $n = |V(T)|$ and let D be the digraph obtained from T by contracting xzy, into a vertex, w. As in Lemma 3.3 we get $d_D^+(w) + d_D^-(w) \geq (d_T^+(y) - 1) + (d_T^-(x) - 1) \geq d_T^+(y) - 1 + (n - 1 - d_T^+(x) - 1) \geq n - 2 = |V(D)|$. As in the proof of Lemma 3.2, this implies that w belongs to a 2-cycle, which furthermore implies that both xz and zy belongs to a 4-cycle. All the arcs xz, zy and yx belong to a 3-cycle $(xzyx)$.

If D is strong, then we are done by Lemma 3.2, so assume that D is not strong. This implies that $T - \{x, y, z\} = D - w$ is not strong (as w belongs to a 2-cycle in D), so let $T_1, T_2, \ldots, T_r$ $(r \geq 2)$ be the strong components of $T - \{x, y, z\}$, such that $T_1 \Rightarrow T_2 \Rightarrow \ldots \Rightarrow T_r$. As D is not strong either $T_1 \Rightarrow y$ or $x \Rightarrow T_r$ (or both).

Assume without loss of generality that $T_1 \Rightarrow y$ (the case $x \Rightarrow T_r$ can be handled analogously). As $T_1 \Rightarrow y$, we note that $\{x, z\}$ is a minimum separating set in T, which implies that both x and z dominate some vertices in T_1. Now consider the following cases.

(A): T_r has an arc into z. Let D_{zyx} be the digraph obtained from T by contracting zyx, into a vertex, w_{zyx}. Clearly D_{zyx} is strong, but $D_{zyx} - w_{zyx}$ is not strong, which by Lemma 3.1, part (iii), gives us l-cycles containing zy and yx, in T, for all $5 \leq l \leq n$. As we had already found the desired 3- and 4-cycles, containing zy, we note that zy is pancyclic.

(B): T_r has an arc into y. Let D_{yxz} be the digraph obtained from T by contracting yxz, into a vertex, w_{yxz}. Analogously to (A) we get that xz is pancyclic (D_{yxz} is strong, but $D_{yxz} - w_{yxz}$ is not strong).

Furthermore note that Lemma 3.1, part (iii), gives us l-cycles containing yx, in T, for all $5 \leq l \leq n$, as well.

(C): T_r has an arc into x. Let D_{xz} be the digraph obtained from T by contracting xz, into a vertex, w_{xz}. Clearly D_{xz} is strong, but $D_{xz} - w_{xz}$ is not strong (T_1 has no in-neighbours), which by Lemma 3.1, part (iii), gives us l-cycles containing xz, in T, for all $4 \leq l \leq n$.

As z is not a separating vertex in T, either x or y has an arc into it from T_r, which by (B) and (C) implies that xz is pancyclic. Assume that $z \Rightarrow T_r$, as otherwise we are done by (A). This implies that T_r has arcs into x and y. Note that by taking an out-neighbour, v_1, of x in T_1 and an in-neighbour, v_r, of y in T_r, we get the 4-cycle yxv_1v_ry, containing yx. As yx also lies on a 3-cycle, the last part of (B) implies that yx is pancyclic. □

Note that the following theorem proves that Conjecture 1.6 is true (with $a_k = k/2$).

Theorem 3.5 *$p_k(n) \geq \frac{1}{2}kn$, for all $k \geq 2$.*

Proof: Let T be a k-strong tournament, with $n = |V(T)|$ and let $x \in V(T)$ be arbitrary. We will show that there are at least k arc-disjoint 3-cycles containing x. As T is k-strong $d^+(x), d^-(x) \geq k$, and there exist k vertex-disjoint paths from $N^+(x)$ to $N^-(x)$. Assuming the paths are minimal, then they all have length 1, so they all give arc-disjoint 3-cycles, by adding the vertex x in the appropriate manner.

Lemma 3.4 now implies that every vertex in T is incident with at least k pancyclic arcs. This implies the Theorem (as any graph of order n, with degree at least k, has at least $\frac{1}{2}kn$ edges). □

Before we give a bound on $h_k(n)$ we need the following powerful theorem by Thomassen.

Theorem 3.6 *[6] If T is a 4-strong tournament, and x and y are distinct vertices in T, then there is a Hamilton path from x to y in T.*

Theorem 3.7 *If T is a 3-strong tournament, then there exists two distinct vertices x and y, such that all arcs out of x and all arcs out of y are pancyclic.*

Furthermore x and y can be chosen, such that $x \rightarrow y$ and $d^+(x) \leq d^+(y)$.

Proof: Let M contain all vertices in T, which have minimum out-degree. Lemma 3.3 implies that all arcs out of a vertex in M are pancyclic. If $|M| \geq 2$ we are done, so assume that $|M| = 1$ and $M = \{x\}$. Let $y \in N^+(x)$ have minimum possible $d^+(y)$ (of all vertices in $N^+(x)$), and let $w \in N^+(y)$ be arbitrary. We will show that the arc yw is pancyclic, which would prove the theorem.

If $d^+(w) \geq d^+(y)$, then we are done by Lemma 3.3, so assume that $d^+(w) < d^+(y)$. By the definition of y we see that w dominates x, and by the definition of x we get $d^+(x) < d^+(w)$.

Let D be the digraph obtained from T by contracting xyw, into a vertex, q. As in Lemma 3.3 we get $d^+_D(q) + d^-_D(q) \geq (d^+_T(w) - 1) + (d^-_T(x) - 1) \geq d^+_T(w) - 1 + (n - 1 - d^+_T(x) - 1) \geq n - 2 = |V(D)|$. D is strong, as T is 3-strong. Lemma 3.2 therefore implies that there exists l-cycles in D, containing q, for all $2 \leq l \leq |V(D)|$. Therefore there are l-cycles in T, containing yw, for all $4 \leq l \leq |V(T)|$. As $xywx$ is a 3-cycle in T, this implies that yw is pancyclic. □

Note that the following theorem gives support for Conjecture 1.3.

Theorem 3.8 *$h_k(n) \geq \frac{k+5}{2}$, for all $k \geq 1$.*

Proof: Note that by Theorem 1.1 and Theorem 1.2, $h_k(n) \geq \frac{k+5}{2}$ for all $k = 1, 2, 3, 4, 5$. We will now prove that $h_k(n) \geq \frac{k+5}{2}$, by induction, so let $k \geq 6$ and assume that the theorem is true for all values smaller than k. Let T be a k-strong tournament, with $n = |V(T)|$, and let x and y be defined as in Theorem 3.7. Let $T' = T - \{x, y\}$, and note that T' is $(k-2)$-strong. So by our induction hypothesis we will let H' be a Hamilton cycle in T', which contains at least $\frac{(k-2)+5}{2}$ pancyclic arcs in T'.

If the arc $e = uv$ is pancyclic in T', then it is pancyclic in T, because of the following. We will show that there is a cycle of length l in T containing e, for all $3 \leq l \leq n$. If $l \leq n-2$, then such a cycle exists as e is pancyclic in T', which is a subgraph of T. If $l = n$, then such a cycle exists by Theorem 3.6 (there is a (v, u)-Hamilton-path in T). If $l = n-1$, then delete any vertex from T, except u or v, and note that the result, D', is 4-strong, so, as before, there is a Hamilton cycle in D' containing e, which is the desired cycle in T.

So H' contains at least $\frac{k+3}{2}$ arcs, which are pancyclic in T. Note that $d^+(y)+d^-(x) = d^+(y) + n - 1 - d^+(x) \geq n-1$. Analogously to the proof of Lemma 3.2, we can find an arc, p_1p_2, on H', such that $p_1 \rightarrow x$ and $y \rightarrow p_2$ (as $|V(H')| = n-2$). By deleting p_1p_2 and adding the path p_1xyp_2, we obtain a Hamilton cycle, H in T. There is at least $\frac{k+3}{2} - 1$ pancyclic arcs on H', which still belong to H, as we have only deleted one arc from H'. However we have added two pancyclic arcs, namely the arc out of y and xy. Therefore H has at least $\frac{k+3}{2} - 1 + 2$ pancyclic arcs in T, which completes the proof. □

Note that Theorem 3.8 and Proposition 1.4 imply the following.

Corollary 3.9 *$\frac{k+5}{2} \leq h_k(n) \leq 3k$, for all $k \geq 1$. That is, $h_k(n) \in \Theta(k)$.*

4 Acknowledgement

We would like to thank Jørgen Bang-Jensen, Angela Koller, Frédéric Havet and the annonomous referees for helpful suggestions regarding this paper.

References

[1] B. Alspach. Cycles of each length in regular tournaments. *Canad. Math. Bull., 10:283-285, 1967.*

[2] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications*, Springer, London, 2000.

[3] P. Camion. Chemins et circuits hamiltoniens des graphes complets. *C. R. Acad. Sci. Paris, 249:2151-2152, 1959.*

[4] F. Havet. Pancyclic arcs and connectivity in tournaments. *Submitted to Journal of Graph Theory.*

[5] J. W. Moon. *Topics on Tournaments.* Holt, Rinehart and Winston, New York (1968).

[6] C. Thomassen. Hamiltonian-connected tournaments. *J. Combin. Theory Ser. B, 28(2):142-163, 1980.*

[7] T. Yao, Y. Guo and K. Zhang, Pancyclic out-arcs of a vertex in a tournament. *Discrete Appl. Math.,* **99** *(1-3) (2000), 245-249.*