ASPHERIC SURFACES OF REVOLUTION IN OPTICAL DESIGN

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CONTENTS

Aspheric Surfaces in Optical Design:- ABSTRACT

Part I. After a brief introduction to aspheric surfaces, the early work in the seventeenth century is described and the emergence of the optical path equality principle. The Abbe sine theorem in the nineteenth century is shown to lead to a number of aspheric aplanatic objectives in which this theorem is fulfilled. The Schmidt telescope, an anastigmatic system of high performance is considered briefly in conjunction with some variations of it.

Part II. The Schmidt camera is considered in more detail, including descriptions of a number of authors' methods for deriving the equation of the corrector plate, and the possibilities in balancing the aberrations over the whole field. A number of the Schmidt variations are described including field flattened and folded types.

Part III. The designs of two mirror and more complicated mirror plate systems are analysed by

first order and other methods, and the use of aspheric surfaces to provide field correctors for large paraboloid mirrors described.

Part IV. The general first order design of aspherics with the see-saw diagram and the application of differential methods of correction, are followed by methods for obtaining axial stigmatism and aplanatism. Lastly a number of ray tracing methods are examined, most of which involve the use of an electronic computor.

Pt. I Introduction and Early Development.

Spherical and Aspherical Surfaces.

Early Work of Descartes, Huygens and others in the seventeenth century.

The Sine theorem and the designs of Schwarzschild and those influenced by him on aplanatic two mirror systems.

The Schmidt Telescope.

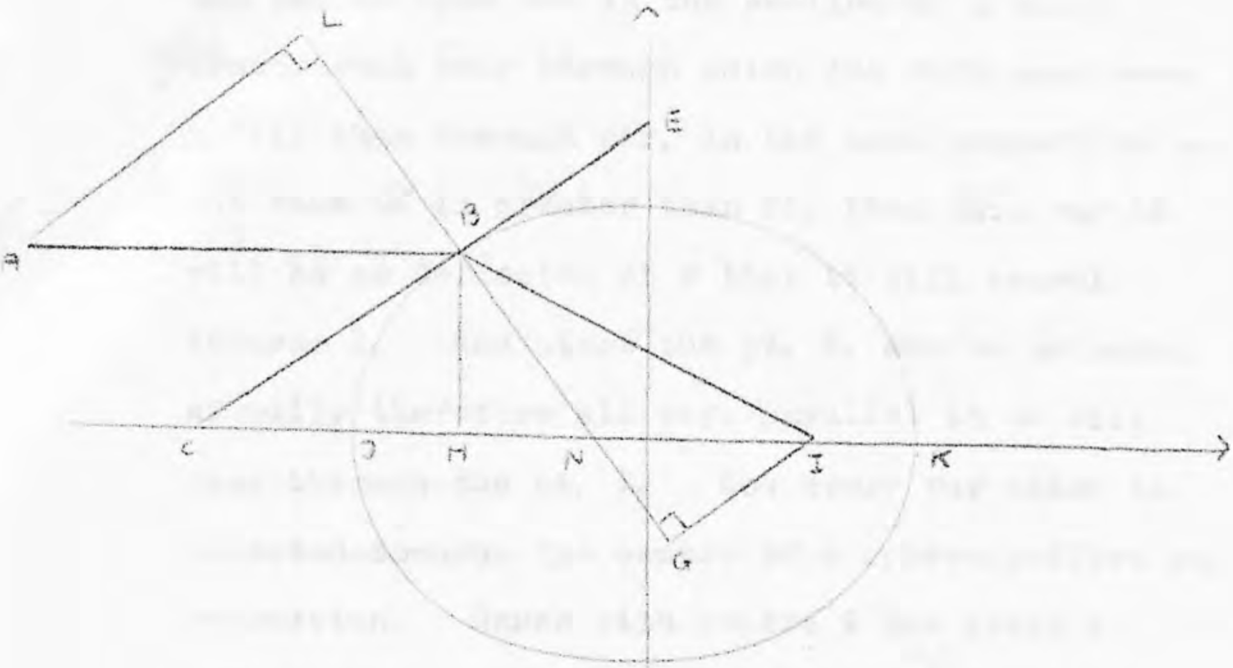
The Schmidt-Wright Short Telescope.

Spherical and Aspherical Surfaces.

Surfaces which may be used in Optical instruments, must be, in general, accurate in contour to the order of a wavelength, and the development of methods of producing spheres to this tolerance has resulted in the modern Optical industry's ability to make lenses of high quality in large numbers. Nevertheless, although spherical surfaces are the easiest to make, yet they will not, without correction, result in the formation of a good image, the axial monochromatic aberration they produce, in fact, being termed spherical.

Early Work of Descartes, Huygens and others in the seventeenth century.

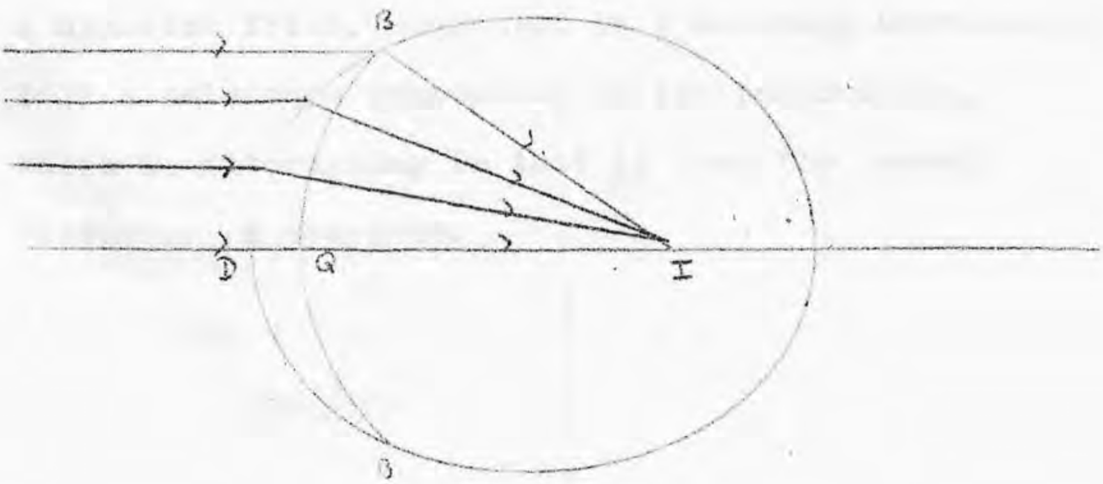
In the first half of the seventeenth century it was realised that to produce an image without Spherical aberration, more complicated curves would be required, and in 1637 Descartes, although labouring under the difficulty of having an insufficient knowledge of the nature of light, published in La Dioptique an account of the "shapes which transparent bodies should have to refract rays in every way serviceable for vision". He determined the curves, which furnish aberrationless image formation for a given pair of conjugate points, generally of the fourth degree, but degenerating in certain cases to conic sections, and in his determinations made no use of the principle of equal path lengths, essentially a wave theory conception, adopting geometrical methods only. The Construction is as follows:



If through the pt. B on the ellipse one draws the straight lines LBG and CBE, which cut each other at right angles and of which the one LG divides the angle HBI into two equal halves, the other CE will touch the ellipse at B.

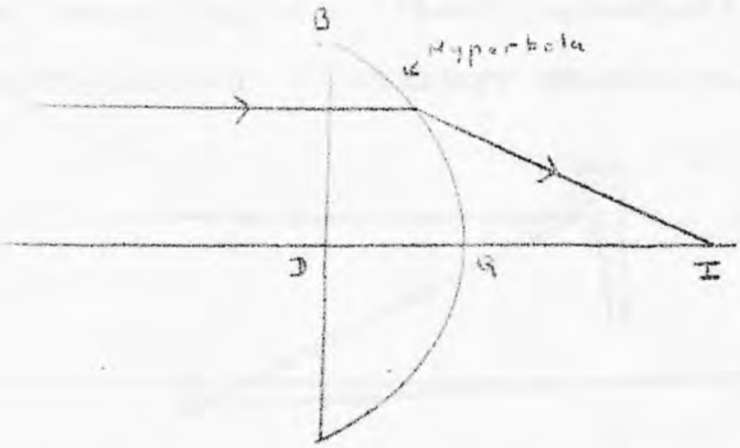
If through B is drawn a line BA parallel to the major axis DK and equal in length to BI, and if are drawn from the pts. A and I to LG the two perpendiculars AL and IG, these will stand in a fixed proportion to DK and HI, which is a constant ratio.

Hence if the line AB is a ray of light, and the ellipse DBK is the section of a solid transparent body through which the rays pass more easily than through air, in the same proportion as the line DK is greater than HI, then this ray AB will be so deflected at B that it will travel towards I. And since the pt. B. can be selected at will, therefore all rays parallel to DK will pass through the pt. I. Now every ray which is directed towards the centre of a sphere suffers no refraction. Hence with centre I one draws a circle of any required radius, then the line DB and QB turning about the axis DQ will describe the shape which a lens should have to focus in air, at the pt. I all the rays which were parallel before falling on the lens.



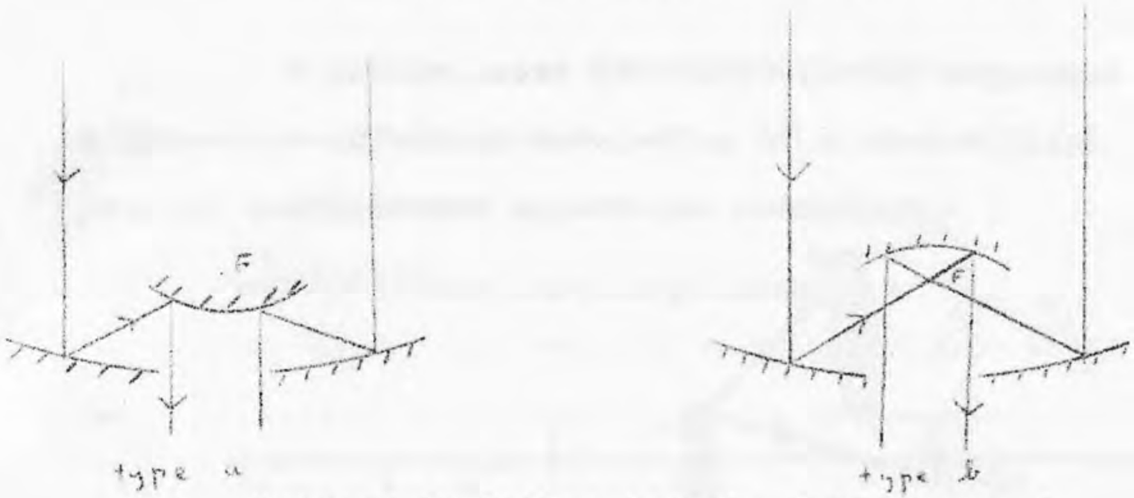
Just what Descartes meant by rays passing more easily through a solid body than air is obscure, he regarded light as an "instantaneously propagated statical pressure in a granulated continuum". He arrived however at the correct result.

By similar means Descartes gave the properties of the hyperbola in image forming lenses.

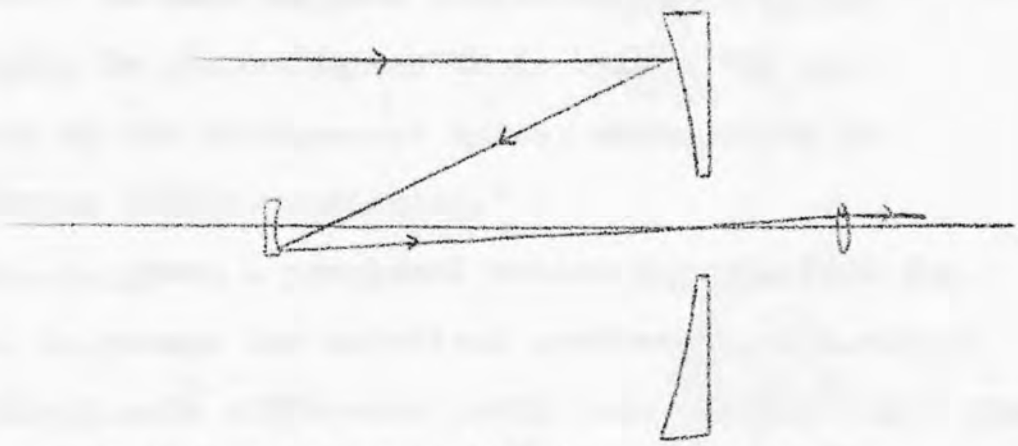


At about the same time Marin Mersenne, a Minorite friar, suggested in l'Harmonie Universelle 1636 a telescope consisting of two paraboloids, which is interesting in that it uses the second mirror as an eyepiece.

Mersenne telescopes of two confocal paraboloids

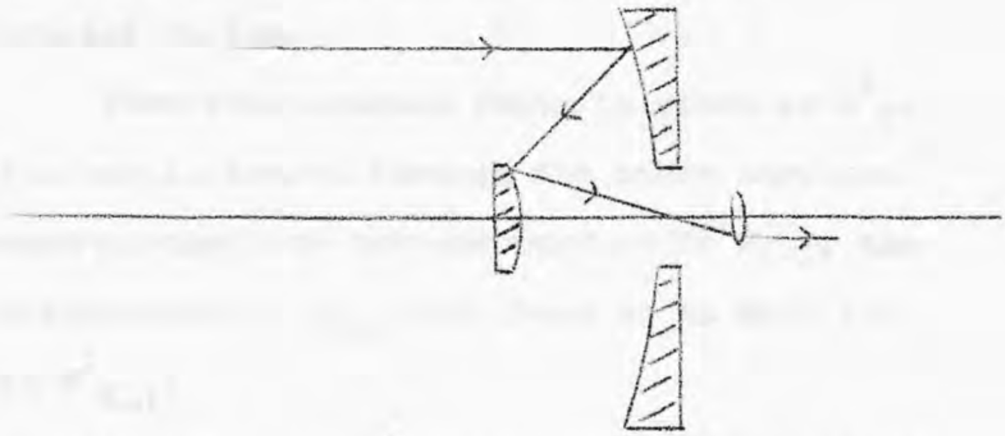


Gregory in Optica Promota 1663 proposed an objective consisting of a primary paraboloid centrally perforated and a secondary concave ellipsoid.

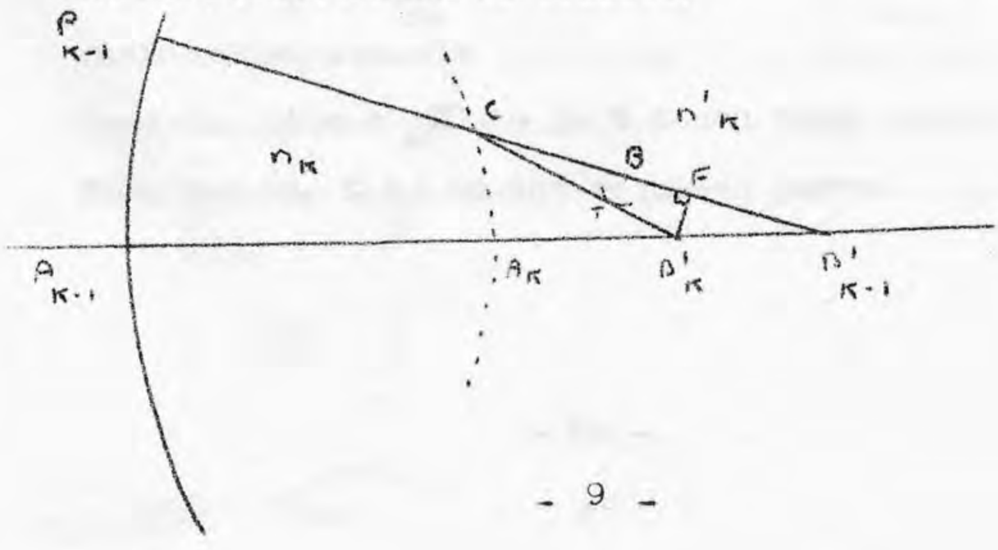


At about this time (1668) Newton, who had been studying Dispersion, concluding that it was not possible to correct chromatic aberration by combinations of glasses, turned his attention to the reflecting telescope and produced his type in which a paraboloid is combined with a small flat.

A little later Cassegrain (1672) suggested a telescope objective consisting of a paraboloidal primary and a convex hyperbolic secondary.



In 1690 Huygens[3] published his Treatise on Light, in which Chapter VI is headed "On the figures of the Transparent bodies which serve for refraction and for reflexion."
In this is given a graphical method for constructing curves to remove the spherical aberration, the method of optical path difference being used for the first time. The construction is as follows:-[4]



Suppose a system is to be corrected by means of its last surface.

Given the pole of this surface AK and its paraxial radius.

Then the paraxial focus is given by B^1_K. Now if a ray is traced through the other surfaces to arrive at the last but one surface at P_{K-1}, the ray is refracted at P_{K-1} and drawn on to meet the axis in B^1_{K-1}.

Computing Op the paraxial path and Om the marginal path up to P_{K-1}

Then $\frac{Op - Om}{n_K}$ equals the O.P.D. assuming that image is in n_K.

Construct the pt. B where $P_{K-1}\,B = \frac{Op - Om}{n_K}$

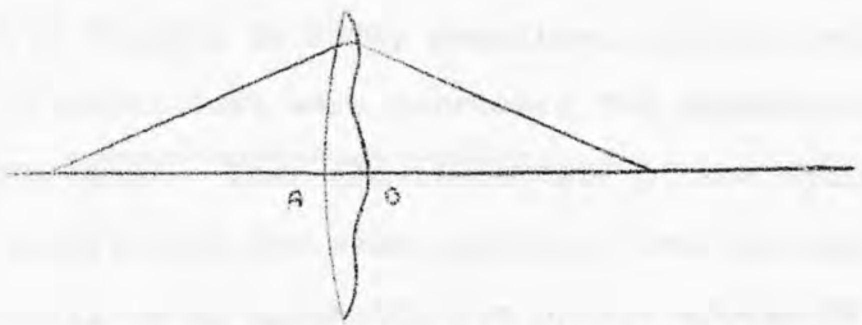
Drop a perpendicular on to $P_{K-1}\,B^1_{K-1}$ at F from B^1_K. Measure it.

Compute $B^1_K F \times \frac{n^1 K}{nK}$ and with centre B and this radius draw a circle

Draw the line $B^1_K TC$ to just touch this circle

Then the pt. C is on the required curve.

By constructing similar points for other rays a curve may be built up; using this method Huygens drew a lens which had one spherical surface and one aspherical surface.



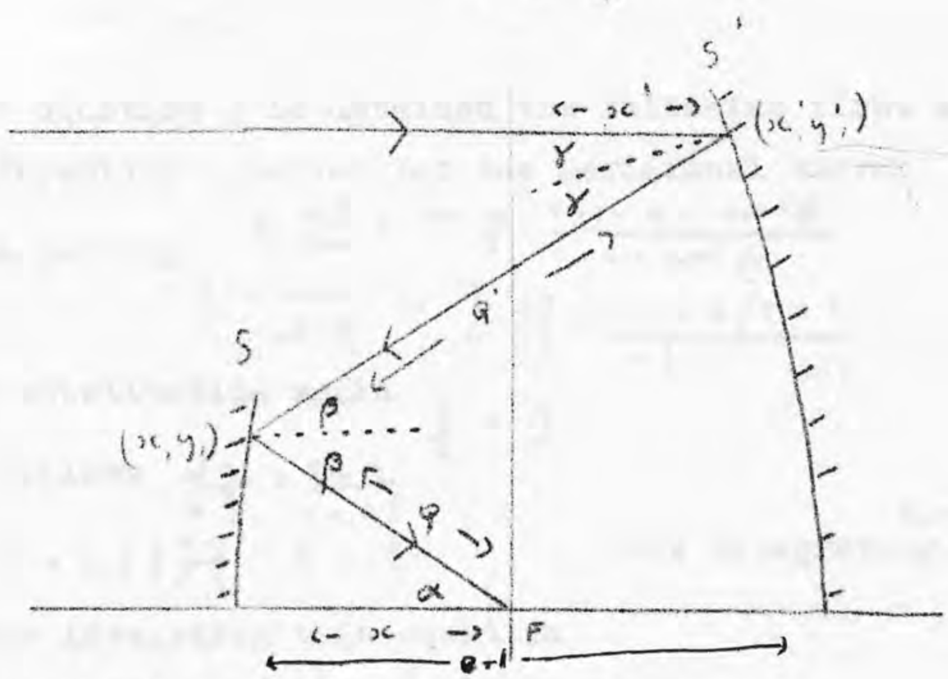
Such methods of construction, however, are not accurate enough for most optical work. Systems of this sort moreover are in general not aplanatic, for it is well known that the sine-condition is fulfilled for an infinitely distant object if the points of intersection of all incident rays parallel to the axis, with their corresponding emerging rays all lie in a circle of radius F^1 and centre F^1 the second focal point. Thus a lens of the Descartes type, for example, can never be aplanatic since the points of intersection of incident and refracted rays lie on the cartesian surface.

The Sine Theorem and the designs of Schwarzschild and those influenced by him, on aplanatic two mirror systems.

The discovery of the Abbe Helmholtz' sine theorem in 1873, permitted systems to be computed that were corrected for spherical aberration and coma. Thus the statements of the optical path equality and the sine condition are the starting points of an aplanatic two mirror system by Schwarzschild in 1905.[6] Schwarzschild starts by saying:

"We shall have to try to calculate a mirror system for an aperture of any size which is strictly free from any spherical aberration and fulfills at the same time the sine condition, as with the latter condition the disappearance of the coma is also secured. The focus of the required system must be according to Abbe's description, an aplanatic point, wherefore the entire system may be called aplanatic."

The two mirrors are represented in the figure.



Let the focal length be unity
Let the small mirror be S, and the large S'
For optical path equality $\quad Q + Q' + x' = 2(e+1)$ — 0
For the sine condition $\quad \dfrac{y'}{\sin\alpha} = \text{constant}$

Putting this constant equal to one

$$y' = \sin\alpha \qquad — 1$$

From the figure $\quad \dfrac{1}{Q}\dfrac{dQ}{d\alpha} = \tan\beta$ — 2

$$2\beta = \alpha + 2\gamma \qquad — 3$$

$$x' + Q\cos\alpha = Q'\cos 2\gamma \qquad — 4$$

$$y' = Q\sin\alpha + Q'\sin 2\gamma \qquad — 5$$

The equations above contain the information for the problem

Eliminating γ, x', y', using 1, 3, 4,

$$\sin\alpha = Q\sin\alpha + Q'\sin(2\beta - \alpha) \qquad — 6$$

$$Q + Q' + Q'\cos(2\beta - \alpha) - Q\cos\alpha = 2(e+1) \qquad — 7$$

Eliminating Q'

$$2(e+1) = Q(1-\cos\alpha) + (1-Q)\sin\alpha\cot\left(\beta - \frac{\alpha}{2}\right)$$

$$\tan\beta = \tan\frac{\alpha}{2}\,\frac{e + 1 - Q + \cos^2\frac{\alpha}{2}}{e + \cos^2\frac{\alpha}{2}}$$

From equation 2 is obtained the following first order differential equation for the meridional mirror

$$\frac{1}{q}\frac{dq}{d\alpha} = \tan\frac{\alpha}{2}\,\frac{e+1-q+\cos^2\frac{\alpha}{2}}{e+\cos^2\frac{\alpha}{2}}$$

Then putting

$$\xi = \frac{1}{\cos^2\frac{\alpha}{2}} \quad \text{so} \quad \frac{\xi}{Q}\frac{dQ}{d\xi} = \frac{(e+1-Q)\xi+1}{e\xi+1}$$

and substituting again

$$\frac{\xi}{Q} = \eta \qquad — 8$$

It follows $\frac{d\eta}{d\xi} = \frac{\xi-\eta}{1+e\xi}$

or $(1+e\xi)\frac{d\eta}{d\xi} + \eta = \xi$ (The integrating factor is $(1+e\xi)^{\frac{1}{e}-1}$)

Hence integratig this equation

$$\eta(1+e\xi)^{\frac{1}{e}} = \int d\xi\,\xi(1+e\xi)^{\frac{1}{e}-1}$$

$$= \int d\xi\,\frac{(1+e\xi)^{\frac{1}{e}} - (1+e\xi)^{\frac{1}{e}-1}}{e}$$

$$= \frac{(1+e\xi)^{\frac{1}{e}+1}}{e(e+1)} - \frac{(1+e)^{\frac{1}{e}}}{e} + C$$

$$\text{or } \eta = C(1+e\xi)^{-\frac{1}{e}} + \frac{\xi-1}{e+1}$$

Substituting back the original variables, the polar equation of the mirror S is obtained

$$\frac{1}{Q} = \frac{\sin^2\frac{\alpha}{2}}{e+1} + C\left(e+\cos^2\frac{\alpha}{2}\right)^{-\frac{1}{e}}\left(\cos^2\frac{\alpha}{2}\right)^{\frac{1+e}{e}} \qquad — 9$$

There now remains to express x' as a function of α and to obtain the form of the mirror S'.

From equation '0' $x' = 2(e+1) - Q - Q'$

From equations 6 and 7 it follows through elimination of β

and therewith

$$Q' = 1+e-Q\sin^2\frac{\alpha}{2} + \frac{\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}(1-Q)^2}{1+e-Q\sin^2\frac{\alpha}{2}}$$

$$x' = e+1-Q\cos^2\frac{\alpha}{2} - \frac{\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}(1-Q)^2}{1+e-Q\sin^2\frac{\alpha}{2}}$$

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$$\text{or} \quad x' = e+1 - \cos^2\frac{\alpha}{2} \cdot \frac{\varphi\left(1+e-2\sin^2\frac{\alpha}{2}\right)+\sin^2\frac{\alpha}{2}}{1+e-\varphi\sin^2\frac{\alpha}{2}}$$

Substituting for φ from equation 9

$$\begin{cases} x' = e+1 - \dfrac{\sin^2\alpha}{4(e+1)} - \dfrac{1}{e(e+1)^2}\left(e+\cos^2\dfrac{\alpha}{2}\right)^{2+\frac{1}{e}}\left(\cos^2\dfrac{\alpha}{2}\right)^{-\frac{1}{e}} \\ y' = \sin\alpha \end{cases} \quad \text{— 10}$$

These are the rectangular coordinates of the mirror S' as a function of α

To determine the constants e and c consider a paraxial ray where $\varphi_0 = \lambda$ = distance of the focal point from the mirror S

$\varphi_0 - x'_0 = \varphi'_0 = d =$ distance between the mirrors

Then it follows for $\alpha = 0$ in '9' & '10'

$$\frac{1}{\varphi} = c(1+e)^{-\frac{1}{e}} \qquad x' = e+1 - \frac{e+1}{c}$$

$$\text{so} \quad e+1 = d \qquad c = \frac{d^{\frac{d}{d-1}}}{\lambda} \quad \text{— 11}$$

Equation 9 is now developed as a power series

$$\frac{\varphi}{\lambda} = 1 + \sin^2\frac{\alpha}{2}\left(1+\frac{1-\lambda}{d}\right) + \sin^4\frac{\alpha}{2}\left[\left(1+\frac{1-\lambda}{d}\right)^2 - \frac{1}{2d}\right] + \text{etc} \quad \text{— 12}$$

From '10'

$$x' = d - \lambda - \frac{1-\lambda}{d}\sin^2\frac{\alpha}{2} + \frac{1}{d}\left(1-\frac{\lambda}{2}\right)\sin^4\frac{\alpha}{2} + \text{etc} \quad \text{— 13}$$

Now the rectangular coordinates for S and S' are

$$\begin{cases} x = \varphi\cos\alpha \\ y = \varphi\sin\alpha \end{cases} \qquad \begin{cases} x' = x' \\ y' = \sin\alpha \end{cases} \quad \text{— 14}$$

Using equation 9 a power series is developed for $\sin \frac{\varphi}{2}$ with respect to y and y'

Hence

$$\begin{cases} x = -\lambda - \left(\frac{1-\lambda}{d} - 1\right)\frac{y^2}{4\lambda} + \left\{\frac{1}{4d} - \frac{1-\lambda}{2d} + 2\left(\frac{1-\lambda}{2d}\right)^2\right\}\frac{y^4}{8\lambda^3} + \cdots \\ \\ x' = d - \lambda - \frac{1-\lambda}{4d}\, y'^2 + \frac{\lambda}{32\, d}\, y'^4 + \cdots \end{cases} \qquad \text{--- } 15$$

To obtain a practical application Schwarzschild puts

$$\lambda = .5 \qquad\qquad d = 1.25$$

For the mirror S

$$x = -Q\cos\alpha \quad , \quad y = Q\sin\alpha$$

$$\text{with} \quad \frac{1}{Q} = \frac{4}{5}\sin^2\frac{\alpha}{2} + \frac{2\left(\cos\frac{\alpha}{2}\right)^{10}}{\left(1 - \frac{4}{5}\sin^2\frac{\alpha}{2}\right)^4}$$

For the mirror S'

$$x' = \frac{5}{4} - \frac{\sin^2\alpha}{5} - \frac{\frac{1}{2}\left(1 - \frac{4}{5}\sin^2\frac{\alpha}{2}\right)^6}{\left(\cos\frac{\alpha}{2}\right)^8}$$

$$y = \sin\alpha$$

Using these equations Schwarzschild produces a pair of tables:-

Small mirror.

α^{0}	Relative opening	y in mm	x in mm	x_s sphere	x_e ellipsoid	$x - x_s$	$x - x_e$
5	1/6.6	43.694	.573	.573	.573	0	0
10	1/3.3	87.755	2.315	2.312	2.315	3	0
15	1/2.2	132.555	5.296	5.280	5.296	16	0
20	1/1.7	178.478	9.635	9.584	9.639	51	-4
25	1/1.4	225.922	15.508	15.383	15.525	125	-17
30	1/1.2	275.304	23.160	22.895	23.217	265	-57

large mirror.

α^{0}	Relative opening	y' in mm	x' in mm	x'_s sphere	x'_H hyperboloid	$x' - x'_s$	$x' - x'_H$
5	1/6.6	87.156	.759	.761	.759	-1	0
10	1/3.3	173.648	3.004	3.015	3.004	-11	0
15	1/2.2	258.819	6.641	6.702	6.643	-61	-2
20	1/1.7	342.020	11.518	11.714	11.532	-196	-14
25	1/1.4	422.618	17.431	17.893	17.479	-462	-48
30	1/1.2	500.000	24.128	25.063	24.264	-935	-136

These tables show that up to a Relative aperture of $^1/_3$ the mirrors may be ellipsoids, and hyperboloids & even beyond that up to a Relative aperture of 1/1.4., The deviation is restricted to a few hundredths of a millimetre. This is of importance from the manufacturing point of view. The field in this telescope is restricted by the astigmatism and from the practical point of view, as the photographic plate is situated midway between the two mirrors a sky baffle or extension is necessary to prevent exposure to direct light from the sky. Only two of Schwarzschild's designs seem to have been attempted, one of 24" aperture at the University of Indiana, and one of 18" at Brown University, Rhode Island. An alternative to Schwarzschild's system was proposed by A. Couder of the Paris Observatory in 1926 in which he computed the curve required to give zero astigmatism on a curved focal surface.

Linneman[7] of Gottingen (1905) gave the design for an aplanatic lens, which is founded in Schwarzschild's treatment, in which an aperture of F/2 is obtained.

A rather interesting contribution was made in 1908 by Arthur C. Lunn,[8] who investigated systems of Conic sections using mirrors. In his preamble he says "For convenience of testing it is desirable and in existing constructions apparently universal for each mirror separately to be free from axial aberration, through being a quadric surface of revolution having as conjugate optical foci, its own geometry foci. The investigation concerns the question, how far is it possible by suitable choice of the focal lengths of the component mirrors to diminish the errors due to departure from the sine-ratio", and after a proof by induction he states his result. "In any optical combination consisting of a centred system of reflecting surfaces of revolution, each of which is individually corrected for axial aberration, the relative zones of magnification at given points in the final annular aperture are identical with those of a single mirror giving the same paraxial magnification."

After Schwarzschild had drawn up general conditions for the aplanatism of reflecting systems, Siedentopf[7] explained a particular case and showed that for parallel incident rays aplanatic image formation occurs by reflection at a cardioid in conjunction with reflection at a spherical surface; these properties form the basis of the Zeiss cardioid condenser, in which the cardioid surface being a narrow one is sufficiently represented by a spherical ring. Aspheric surfaces at about this time were used to a limited extent where extreme accuracy was not required, special spectacles for cataract, eyepieces in which one surface was made Aspheric to reduce the spherical aberration, and of course, searchlights and condensers.

After the First World War Chretien,[9] a French optician, published the design for a new aplanatic telescope, produced at the request of, and in conjunction with Ritchy,[10] the famous American astronomer, in which the design is basically a Cassegrain type as opposed to the Schwarzschild Gregorian.

In the development, Chretien, after paying tribute to Schwarzschild's design, uses slightly different variables, but a very similar procedure, to arrive at a set of equations for the big mirror.

Chretien's equations for the mirror are

$$\begin{cases} x = e - \dfrac{t(1-t)}{e} - m e^{-(1-2e)} (e-t)^{\frac{1-2e}{1-e}} (1-t)^{\frac{1}{1-e}} \\ y = \sin u \end{cases}$$

where $t = \sin^2 \frac{u}{2}$

e = dist. between mirrors

m = dist. of plate behind mirror

Compared with Schwarzschild's equations.

$$\begin{cases} x' = e + 1 - \dfrac{\sin^2 \alpha}{4(e+1)} - \dfrac{1}{c(e+1)^2} \left(e + \cos^2 \frac{\alpha}{2}\right)^{2+\frac{1}{e}} \left(\cos^2 \frac{\alpha}{2}\right)^{-\frac{1}{e}} \\ y' = \sin \alpha \end{cases}$$

where $\alpha \equiv u$

e + 1 = dist. between mirrors

c = constant depending on distance between mirror and plate

Chretien then gets a power series for the equations in a similar manner and also a series for the small mirror.

The Ritchy-Chretien telescopes work at an aperture ratio of F/7 compared with F/3 or less for the Schwarzschild. They have, however, practical advantages over it. Several Chretien telescopes were made in America and one in France.

Bureau and Swings[11] have published a treatment similar to Chretien's but for two conjugate points, neither being at infinity.

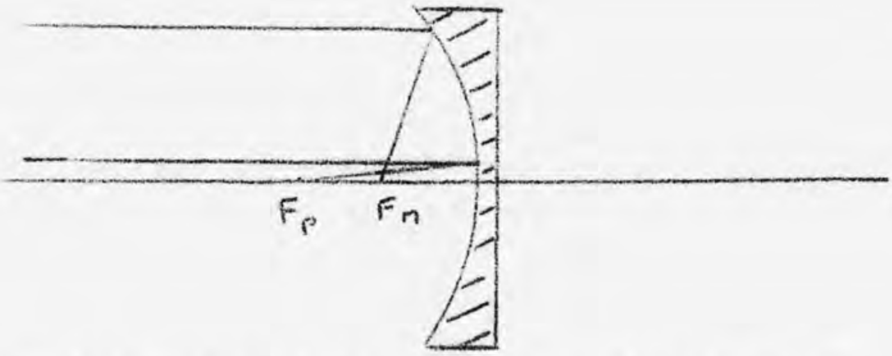
The Americans, both professional and amateur, have been foremost in the field of telescope making, and quite detailed descriptions have appeared in "Scientific American" of the Ritchy-Chretien telescopes.[10] A type of two mirror system with the secondary spherical and primary aspherical was proposed by Allan A. Kirkham.[12] This has the disadvantage of even more coma than the standard Cassegrain composed of conics. It is, however, easier to make.

<u>The Schmidt Telescope.</u>

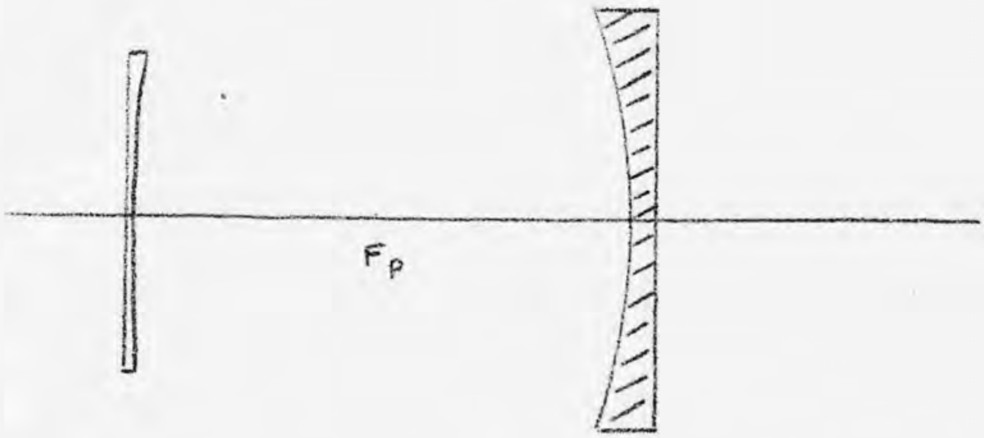
By far the most successful short-focus photographic telescope is the Schmidt, first described in 1932 in a paper by Bernhard Schmidt.[13] The system consists of a spherical mirror in front of which is placed a thin correction plate which corrects the spherical aberration of the mirror, and has nearly no power. Apparently the idea of using an aspheric plate to compensate spherical aberration in a mirror had been suggested before, although it seems doubtful if Schmidt knew this. Moreover Schmidt placed his plate at the centre of curvature of the mirror, thus eliminating at the same time coma, astigmatism and distortion; furthermore Schmidt was able to make his aspheric plate. There remains only curvature of field, and this may be overcome by using thin films or plates which can be sprung in a holder to follow the curve. In his paper on "A High Intensity Coma-free Mirror System", Schmidt did not give his method of construction, and he closes his paper "I have assumed in describing this telescope the technical ability to make the correction profile".

Stroemgren[14] in 1935 gave the third order aberrations and the best equation of the plate profile. As the Seidel terms for the Schmidt are zero, most of his discussion deals with the design of the correcting surface and its colour troubles.

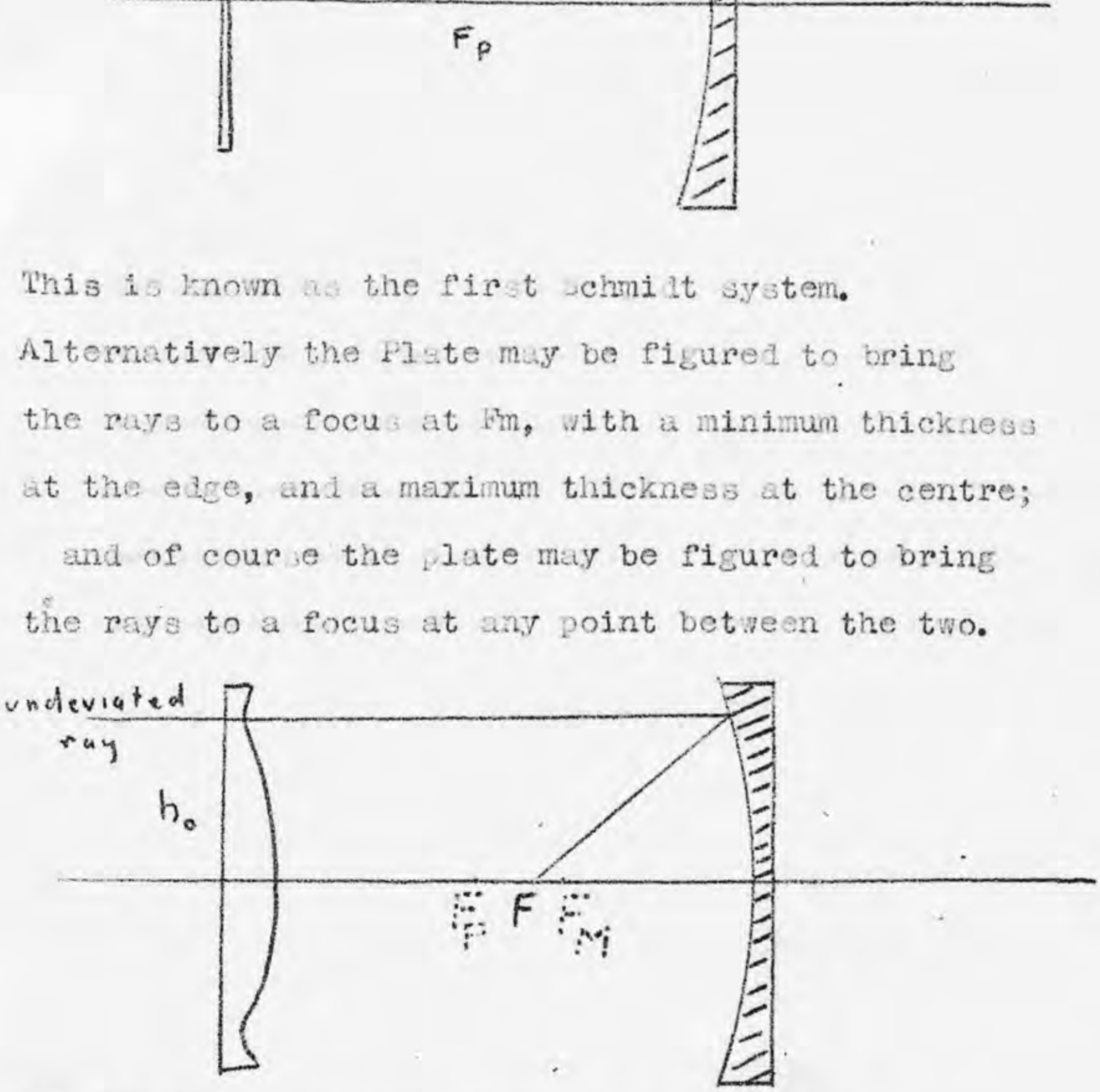
The spherical aberration of a mirror may be measured by the distance between the paraxial



focus and the marginal focus, the distance Fp Fm. The aspheric plate may be figured so as to correct the marginal rays and brings them to the point FP. In this case the contour takes the form of a plate of minimum thickness in the centre, increasing to a maximum at the edge.



This is known as the first Schmidt system.
Alternatively the Plate may be figured to bring the rays to a focus at Fm, with a minimum thickness at the edge, and a maximum thickness at the centre; and of course the plate may be figured to bring the rays to a focus at any point between the two.



This is known as the second Schmidt system.
Stroemgren chose the point F and the plate contour to give the minimum Chromatic difference of spherical aberration. The point F is where the disc of least

confusion lies and the height of the neutral zone h_o is at .86 of the plate diameter.

The Schmidt-Wright Short Telescope.

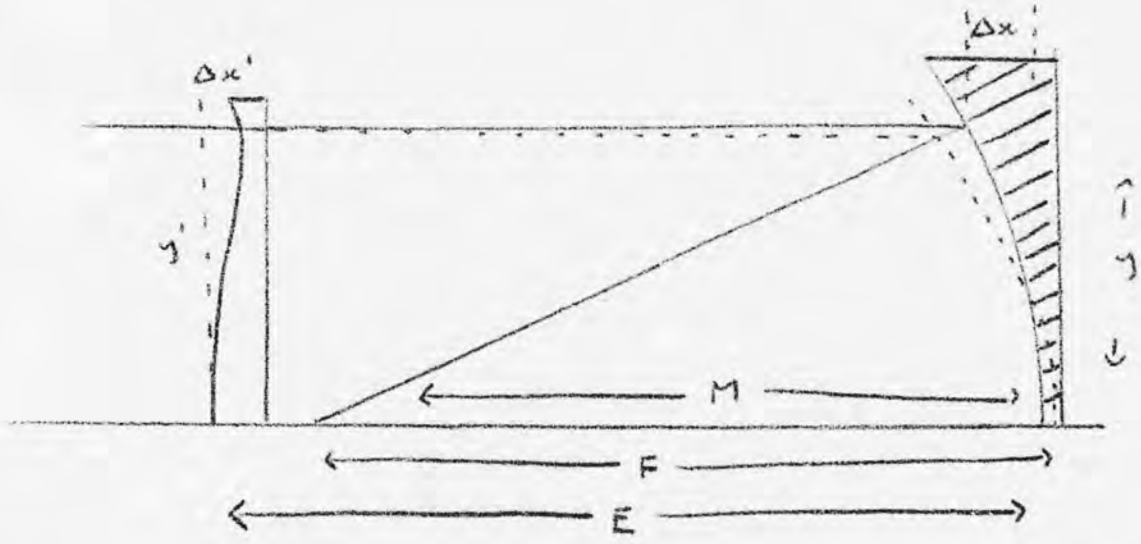
The disadvantages of the Schmidt are its long length in comparison with its focal length, and its curved field. Y Väisälä[15] of Turku University, Finland, proposed the use of a plano convex lens as a field flattener, and the same writer and F.B. Wright[16] of California, apparently independently invented a modified Schmidt with a length equal to the focal length, possessing, however, astigmatism and greater chromatic aberration, both the mirror and the plate being aspherical.

"The theory of the Schmidt Telescope has been based on a consideration of the properties of a spherical mirror. If the theory is developed along more general lines with no assumptions as to the shape of the mirror and the position of the correcting lens, a whole family of telescopes may be obtained."

One of this family has the property that the focal plane is in the surface of best definition

The correcting lens is mounted at a distance from the mirror nearly equal to the focal length of the system.

The equations are derived from the Schwarzschild equations by multiplying the equation for the first optical component by $\frac{-2}{n-1}$.



$$\Delta x' = -\frac{F-M}{2(n-1)E}\cdot\frac{y'^2}{F} + \frac{M}{16(n-1)E}\,\frac{y'^4}{F^3} + \frac{(4E+F)M}{192(n-1)E^2}\cdot\frac{y'^6}{F^5} +$$

$$\Delta x = \frac{E-(F-M)}{4E}\cdot\frac{y^2}{M} + \frac{M-(F-M)}{32E}\,\frac{y^4}{M^3} + \frac{(E+M)M}{384E^2}\cdot\frac{y^6}{M^5} +$$

The Schmidt case is E=F+M and in this case equation (2) reduces to a sphere as E becomes smaller (2) approaches an oblate spheroid.

Wright then writes down the third order astigmatism and shows that for his telescope the tangential and sagital lines are nearly equally spaced about the Gauss plane. He estimates that for a 3° field star images will be 9" in diameter. The length of the camera is thus reduced at the cost of a further asphericity and a considerable reduction in the field, in comparison with the Schmidt, Dimitroff and Baker[17] list three Wright type cameras in use in America, all working at a relative aperture of F/4.

Pt. II Development of the Schmidt.

The Design of the Classical Schmidt.

Aberration balancing over the whole field.

Field Flattening.

Solid and thick mirror types.

Composite or meniscus Schmidt.

The Design of the Classical Schmidt.

The Schmidt Camera, as indicated in part I, was described and produced as long ago as 1930 by B. Schmidt.[13] Consisting essentially of a spherical mirror at whose centre of curvature is placed an aspherical plate which corrects the spherical aberration, the system combines simplicity with very high performance. However, in common with many optical designs, the simple first principles tend to become obscured in the complicity of the algebra required in their realisation. Linfoot in a number of papers[15] and in a chapter in his "Recent Advances in Optics"[19] has developed a comprehensive treatment of the classical Schmidt, using a method originally due to Caratheodory.[20] The latter obtained the leading aberrations of the instrument using the fact that the mirror and field surface are parts of concentric spheres whose centre lies on the pole of the corrector surface. This method of spherical symmetry is acknowledged in turn by Caratheodory to have been suggested to him just before the Second World War when he was in America. Caratheodory, however, in his treatment considers only the first type of Schmidt system, except for a short section on the chromatic aberration, where the second Schmidt type is introduced and a mention when the oblique ray aberrations of the two types are compared.

Using rectangular axes t, y

The Equation of a circle touching the axis y, centre t=0 may be written $(t-2F)^2 + y^2 = (2F)^2$

$$\text{or } 4Ft = y^2 + t^2$$

Expanding by the binomial $t = \frac{y^2}{4F} + \frac{y^4}{64F^3} + \quad$ — (1)

The paraboloid whose vertex touches the sphere formed by revolving the circle about its axis and whose focal length is also F, is given by

$$t = \frac{y^2}{4F}$$

and it will be seen that the equations differ by the y^4 and higher terms.

Neglecting the higher terms, the difference between the two curves horizontally is given by $\frac{y^4}{64F^3}$. The paraboloid mirror has the property that plane wavefronts incident along its axis are reflected as spherical. Consider now a paraboloid with focal length F^1, whose equation is

$$t = \frac{y^2}{4F'} \qquad \text{—} (2)$$

Subtracting (1) from (2) $\xi = y^2\left(\frac{1}{4F'} - \frac{1}{4F}\right) - \frac{y^4}{64F^3}$

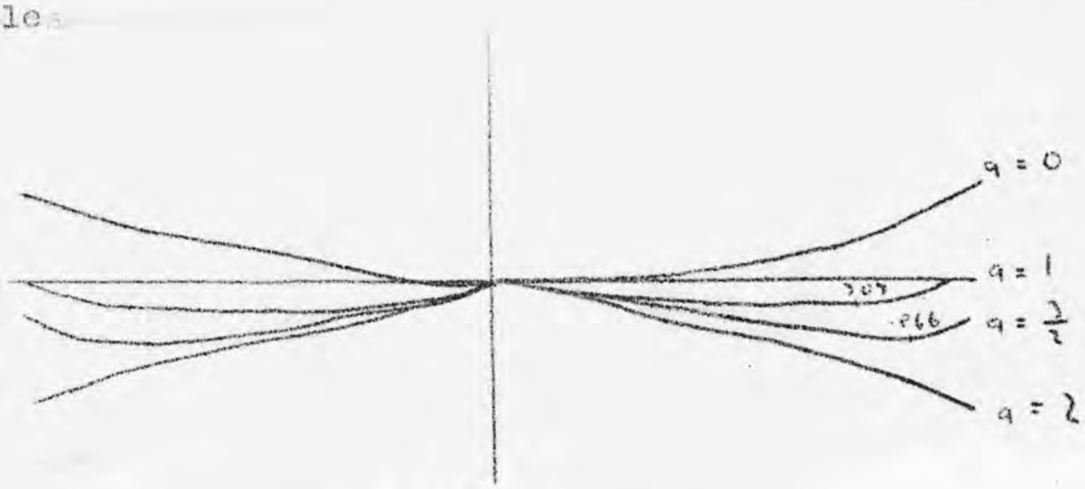
This difference may be produced by a glass plate figured to the profile

$$\eta = \frac{2\xi}{(n-1)} - \frac{y^2}{(n-1)}\left(\frac{1}{2F'} - \frac{1}{2F}\right) + \frac{y^4}{32(n-1)F^3} \qquad \text{—} (3)$$

putting $\quad a = \frac{32F^3}{y_0^2}\left(\frac{1}{2F} - \frac{1}{2F'}\right)$

$$\eta = \frac{y^4 - a y^2 y_0^2}{32F^3(n-1)} \qquad \text{—} (4)$$

The strength of an aspheric plate is defined as (n-1) times the coefficient of y^4 and here equals $\frac{1}{32F^3}$. The value of 'a' may be fixed to give various plate profiles.



As indicated earlier, the profile of the plate may be chosen to minimise the colour error. For a crown-glass the dispersion may be considered as 1/60th the deviation produced by the prism effect of the plate.

Thus from (4) $(n-1)\dfrac{d\eta}{dy} = \dfrac{y^3 - \frac{1}{2} a y_0^2 y}{8F^3}$

And for the minimum colour error the value of $y = y_0$ should be equal to the value when $y^3 - \frac{1}{2} a y_0^2 y$ is greatest. This gives $a = \frac{3}{2}$.

Angular colour split maximum $= \frac{1}{60} \times \frac{1}{8F^3} \times \frac{1}{4} y_0^3 = \frac{1}{15{,}360 F^3}$ (F the Aperture Ratio)

This corresponds to a neutral zone at height $.866 = \frac{1}{2}\sqrt{3}$.

Although the third order aberrations of the Schmidt are equal to zero, it is necessary to consider the higher order aberrations. Several writers derive these, Martin[4], Bouwers[21], Linfoot[19], Caratheodory[20], in varying degrees of generallity. Linfoot is most general but probably Bouwers the neatest. Because of the spherical symmetry of the system, in order to determine the image errors at an angular distance from the centre of the field, it is sufficient to calculate the effect on the axial image by tilting the plate. Following Bouwers

suppose the corrector is of the form $T = F(y)$

Then a meridian ray incident at an angle ϕ at point y will be deviated (from the formula for an inclined prism)

$$(n-1)F'(y)\left(1 + \frac{n+1}{2n}\phi^2 + \text{etc}\right) \quad \left(\begin{array}{l}\phi \text{ small and} \\ F'(y) \text{ small}\end{array}\right)$$

but this ray is distant $y \cos\phi$ from the central ray. Therefore the correction required is

$$(n-1)F'(y\cos\phi)$$

$$= (n-1)F'\left(y - y\frac{\phi^2}{2} + \text{etc}\right)$$

Expanding by Taylor's Theorem, this becomes

$$(n-1)\left[F'(y) - y\frac{\phi^2}{2}F''(y) + \text{etc}\right]$$

The difference between the deviation and the deviation required

$$(n-1)\frac{\phi^2}{2}\left[\frac{n+1}{n}F'(y) + yF''(y)\right] \quad \text{neglecting high powers.} \quad \text{— (5)}$$

This equation may be applied to any form of correction plate, and one may see from it that the separation is of a radial character and of a symmetrical nature and small. Linfoot[14] shows that these residual errors represent lateral spherical aberration and in addition a species of higher order astigmatism. Provided the focal ratio is less than F/3, it is enough to know the coefficients of the squared and fourth power terms in the expansion of the plate. However, for higher apertures it is necessary to consider the sixth powers as well.

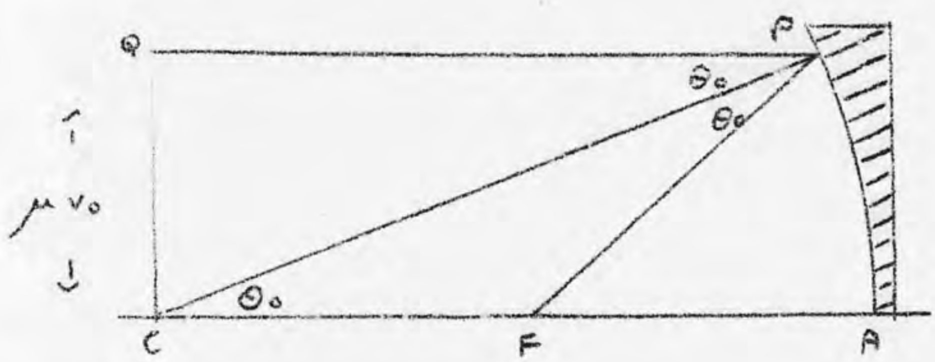
Take axes in the plane of the plate x and y, and z along the optical axis.

In addition put $\mu = H$, $x = \mu u$, $y = \mu v$, $h = \mu r$

Let the equation of the plate be of the form

$$\varsigma = a_1 \mu^2 v^2 + a_2 \mu^4 v^4 + a_3 \mu^6 v^6 +$$

The coefficients may be obtained by considering a geometrical figure, with neutral zone at height μv_0



From the geometry of the figure and the Gaussian optics, it may be shown

$$a_1 = - \frac{(1 - \cos \theta_0)}{n - 1}$$

and then by using the equal path length principle, one obtains, after some reduction,

$$\begin{cases} q_1 = \dfrac{-1}{n-1}\left(\frac{1}{2}\mu^2 v_0^2 + \frac{1}{8}\mu^4 v_0^4 + \qquad\right) \\ q_2 = \dfrac{1}{n-1}\left(\frac{1}{4} - \frac{1}{2}\mu^2 v_0^2 + \qquad\right) \\ q_3 = \dfrac{1}{n-1}\left(\frac{3}{8} + \qquad\right) \end{cases} \qquad \text{--- (6)}$$

The equation thus is written

$$(n-1)s = \tfrac{1}{4}\mu^4(v^4 - 2v^2v_0^2) + \mu^6\left(\tfrac{3}{8}v^6 - \tfrac{1}{2}v^4v_0^2 - \tfrac{1}{8}v^2v_0^4\right) + O(\mu^8) \qquad \text{--- (7)}$$

or $(n-1)s = \tfrac{1}{4}\mu^4(v^2 - v_0^2)^2(1 + \tfrac{5}{2}v_0^2\mu^2) + \tfrac{3}{8}\mu^6(v^2 - v_0^2)^3 + \text{const} + O(\mu^8)$ — (8)

Now let $v_0 = \tfrac{1}{2}\sqrt{3}\,(1 + b\mu^2 + \text{etc} \quad)$ — (9)

As before the slope at the edge of the plate should equal the steepest slope on the bulge.

By (8)

$$(n-1)\frac{ds}{dv} = \mu^4 v(v^2 - v_0^2)(1 + \tfrac{5}{2}v_0^2\mu^2) + \tfrac{9}{4}\mu^6 v(v^2 - v_0^2)^2 + \text{etc} \qquad \text{--- (10)}$$

$$(n-1)\frac{d^2s}{dv^2} = \mu^4(2v^2 - v_0^2)(1 + \tfrac{5}{2}v_0^2\mu^2) + \tfrac{9}{4}\mu^6(v^2 - v_0^2)(5v^2 - v_0^2) \qquad \text{--- (11)}$$

Substituting from (9) into (11) and neglecting powers of μ higher than the sixth, the stationary slope is found when $b = \frac{3}{16}$.

This $v_0 = \tfrac{1}{2}\sqrt{3}\,(1 + \tfrac{3}{16}\mu^2)$

and substituting back in (6)

$$\begin{cases} q_1 = -\dfrac{1}{n-1}\left(\frac{3}{8}\mu^2 + \frac{27}{128}\mu^4 \cdots \right) \\ q_2 = \dfrac{1}{n-1}\left(\frac{1}{4} - \frac{3}{8}\mu^2 \qquad\right) \\ q_3 = \dfrac{1}{n-1}\left(\frac{3}{8} \qquad\right) \end{cases} \qquad \text{--- (12)}$$

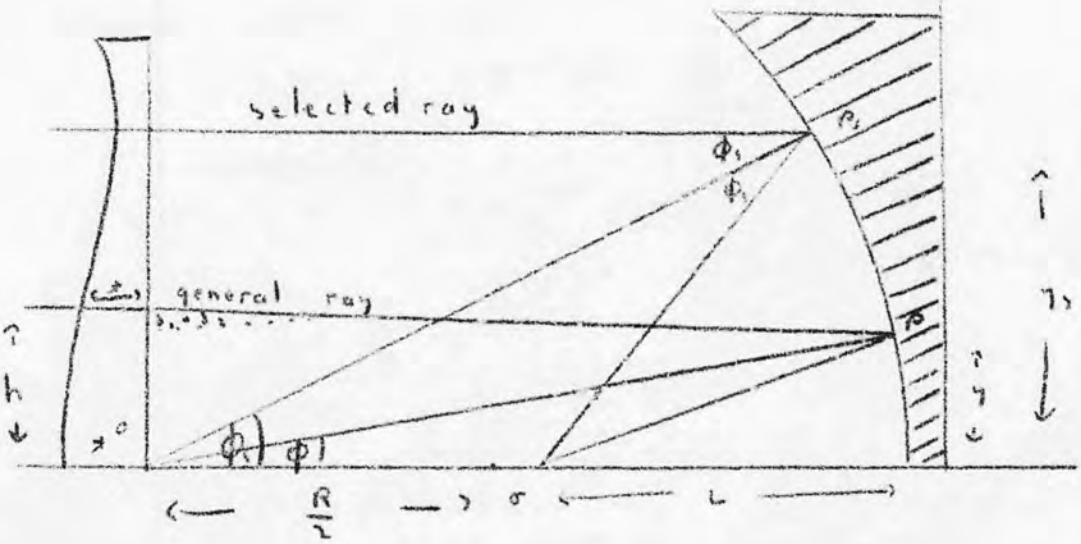
also

$$f = \tfrac{1}{2}\sec\Theta_0 = \tfrac{1}{2} + \tfrac{3}{16}\mu^2 + \tfrac{45}{256}\mu^4 - \qquad \text{--- (13)}$$

These formulae were first given by Baker[22] in1940 for use at focal ratios shorter than F/1. F.A. Lucy[23] in "Exact and Approximate Computation of Schmidt Cameras" uses the path difference method applied directly to the camera.

Suppose equation of the plate is of the form

$x = \alpha h^2 + \beta h^4 + \text{etc}$ — (14)



Let the paraxial quantities be indicated by the suffix o

Then the focal length $F = \dfrac{\Theta_0 - \Phi_0}{\Theta_0} = 1 - \dfrac{\Phi_0}{\Theta_0}$

Let σ be the spherical aberration of the selected ray

$L_0 = \frac{1}{2} + \sigma$ and $\dfrac{\Phi_0}{L_0} = \Theta_0$ as R = 1

Therefore $F = \frac{1}{2} + \sigma$

The optical length of a paraxial ray is.

$\Pi_0 = n t_0 + 2 - F$ — (15)

From geometry $F = \dfrac{L_1}{\cos 2\Phi_1}$ — (16)

$$= (2\cos\phi_s)^{-1} \qquad \text{— (17)}$$

Using Pythagoras $FP = (1 - 2F\cos\phi - F^2)^{\frac{1}{2}}$ — (18)

General ray path $\pi = x + \frac{nt}{\cos\delta_1} + \frac{\cos\phi}{\cos(\delta_1 + \delta_2)} + (1 - 2F\cos\phi + F^2)^{\frac{1}{2}}$ — (19)

Equating (14) and (19)

$$x\left(\frac{n}{\cos\delta_1} - 1\right) = \frac{\cos\phi}{\cos(\delta_1 + \delta_2)} + (1 - 2F\cos\phi + F^2)^{\frac{1}{2}} + F - 2 - nt_0\left(1 - \frac{1}{\cos\delta_1}\right) \qquad \text{— (20)}$$

Since it is possible to compute δ_1 and δ_2 as given below (20) could be used as it stands.

Various less exact but simpler equations are obtainable. Differentiating (14)

$$dx = 2\alpha h\,dh - 4\beta h^3\,dh + \text{etc} \qquad \text{— (21)}$$

at the incidence point of a selected ray

$$\frac{dx_s}{dh_s} = 0 \qquad \text{whence } \alpha = 2\beta h_s^2 \text{ or } \beta = \frac{\alpha}{2h_s^2}$$

$$x_s \simeq \alpha h_s^2 - \frac{\alpha h_s^2}{2} \simeq \frac{\alpha h_s^2}{2}$$

$$x \simeq \frac{2x_s h^2}{h_s^2} - \frac{x_s h^4}{h_s^4} \qquad \text{— (22)}$$

From (20) x_s may be determined exactly since for the selected ray

$$x_s = \frac{\cos\phi_s + \sec\phi_s - 2}{n - 1} \qquad \text{— (23)}$$

By expansion $x_s = \frac{\phi_s^4}{4(n-1)} + \frac{\phi^6}{12(n-1)} + \text{etc}$ — (24)

For most purposes the term in ϕ_s^4 would be enough. Another equation of intermediate precision suitable for any telescope likely to be constructed is

$$(n-1)x \simeq (1 - 2f\cos\phi + f^2)^{\frac{1}{2}}\left[1 + \frac{1}{1 - \frac{f\cos 2\phi}{\cos\phi}}\right] + f - 2 \qquad \text{— (25)}$$

obtained from (20) by substituting for $\delta_1 + \delta_2$ and cancelling terms affecting only the eighth or further significant figures. Certain details of the derivation of (25) are of interest, as δ_1 occurs in (20) early as a division of n, and equating it to unity corresponds to correcting the ray for a slightly different wavelength.

From Figure I $\quad \delta_1 + \delta_2 = \Theta - 2\phi$

$$\cos(\delta_1 + \delta_2) = (\cos\phi - f\cos 2\phi) / (1 - 2f\cos\phi + f^2)^{\frac{1}{2}} .$$

Substitution of this in (20) with $\cos\ \delta_1 = 1$ gives (25)

It has been shown that for the least possible departure from flatness at any point of the plate $h = \frac{1}{2}\sqrt{3}$ for a plate correcting up to fourth order terms.

Another choice for h would be to obtain the smallest average deviation, thus concentrating the chromacity blur near its centre and getting the best possible resolving power.

This gives $h, \simeq .79$

Stroemgren gives for his formula

$$x = 5.86 \times 10^{-3}h^2 - 6.25 \times 10^{-2}h^4 \quad \text{— (26)}$$

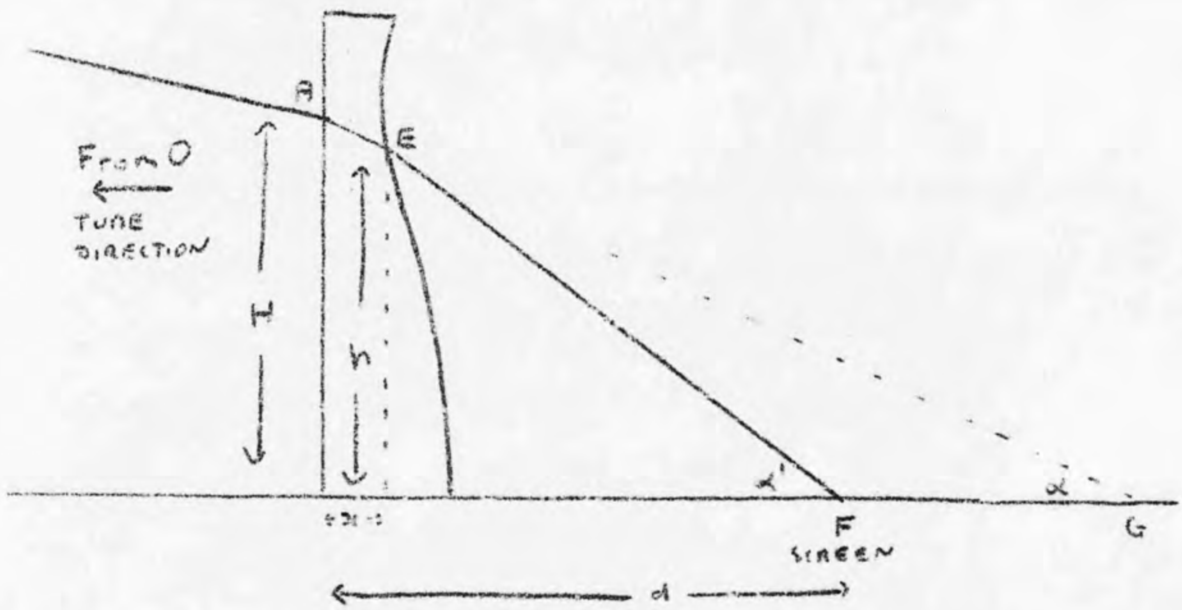
Using equations (24) and (22)

$$x = 5.903 \times 10^{-3}h^2 - 6.302 \times 10^{-2}h^4 \quad \text{— (27)}$$

Equation	x at h edge	x at h_s	x at h inflexion
(20)	123.9	138.95	76.8
(25)	123.9	138.95	76.8
(26)	122.1	137.6	76.3
(27)	122.7	138.4	76.6

This table of course shows Lucy's equations are slightly superior to Stroemgren's. Because of their high luminosity, Schmidt systems have been used in television projection systems, their curvature of field making them especially suitable.[24] H.S. Friedman has given a method of computing the Schmidt Plate required for near projection using a ray tracing method, although essentially, as with the previous methods, the optical paths are used.

Rays are computed at first from the tube to the screen, thus making the aspheric surface the last. The spherical aberration is removed at the centre of the field by equalising the optical path lengths. Chromatic aberration is minimised by a proper selection of the constants of the system. For a 6 X magnification the height of the neutral zone is about 85% of the edge.



The optical path from O to F is computed and the optical path of any other ray from O as well. The difference K between the two is found. Then if K is the difference

For no optical path difference

$$K = nAE + EF \qquad -(28)$$

$$nAE = \frac{nx}{\cos\alpha} \qquad -(29)$$

$$EF^2 = (H - x\tan\alpha)^2 + (d - x)^2 \qquad -(30)$$

$$x^2 + 2\left[\frac{\cos^2\alpha}{n^2-1}\right]\left[H\tan\alpha + d - \frac{nK}{\cos\alpha}\right]x + \left[\frac{\cos^2\alpha}{n^2-1}\right]\left[K^2 - H^2 - d^2\right] = 0 \quad \text{— (31)}$$

Since α, H & K are determined by the ray trace, and d and n are constants of the system. The equation is a quadratic in x.

Then $h = H - x\tan\alpha$ — (32)

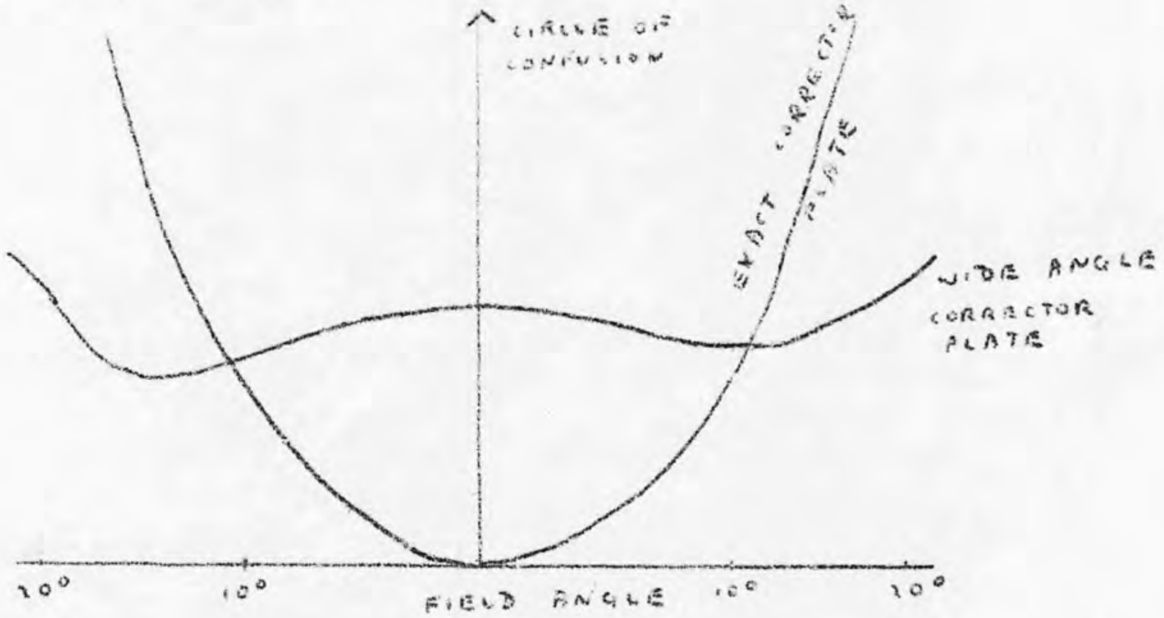
Thus the coordinates of the correction plate may be found.

For oblique rays the trace is started at the screen, and rays directed from a point on the screen to a point on the curve which is already known. As the normal at the point is known, a ray is easily traced.

Maloff and Epstein[25] have written on the use of the Schmidt in projective television. They consider the system gives a gain of 6 or 7 to 1 on an ordinary F/2 lens, wth a quality of image comparable with an ordinary projection lens.

<u>Aberration balancing over the whole field.</u>

Corrector plates designed to give the best error-free axial image in this way, although giving a performance rated as high by ordinary standards, possess the disadvantage that they do not balance the aberrations over the whole field. The ~~images~~ aberrations are smallest in the centre of the field and increase in size towards the edge. A number of papers describe methods for balancing the aberrations over a wider field. E.M. Wormser[26] applies such a method to a F/.7 Schmidt for 40° using Lucy's approximate method.



Linfoot and Wolf[27] discuss the problem of designing the aspheric surface so as to obtain optimum performance. The equation of the plate as has already been seen may be written as

$$(n-1)s = \frac{\mu^4}{4}(v^2 - v_0^2)^2\left(1 + \frac{5}{2} r_0^2 \mu^2\right) + \frac{7}{8}\mu^6 (v^2 - v_0^2)^3 + \text{constant} + O(\mu^8) \quad (33)$$

Neglecting terms of μ^6

$$(n-1)s = \frac{\mu^4}{4}(v^4 - av^2) + O(\mu^6) \qquad — (34)$$

or if 2λ = numerical aperture $\triangleq 2\mu$, $a = 2\,v_0^2$

$$(n-1)s = \frac{\lambda^4}{4}(v^4 - av^2) + O(\lambda)^6 \qquad — (35)$$

If the off axis aberrations are δX and δY measured in seconds of arc.

$$\delta X + i\,\delta Y = \frac{\kappa}{4}\mu^3\phi^2\left[\frac{N+1}{2N}\frac{\partial}{\partial u} + \frac{1}{2}u\frac{\partial^2}{\partial u^2} + \frac{i}{2N}\frac{\partial}{\partial v} + \frac{1}{2}iu\frac{\partial^2}{\partial u\,\partial v}\right](v^4 - av^2)$$

$$+ O(\kappa\mu^7) \qquad — (36)$$

For light of a different wavelength (refractive index n) the rays of the axial 11" pencils are no longer brought to a sharp focus. If $N - n = O(\mu^2)$ the aberrations are now given by

$$\delta X + i\,\delta Y = \frac{\kappa\mu^3\phi^2}{4}\,\frac{n-1}{N-1}\left[\frac{n+1}{2n}\frac{\partial}{\partial u} + \frac{1}{2}u\frac{\partial^2}{\partial u^2} + \frac{i}{2n}\frac{\partial}{\partial v} + \frac{1}{2}iu\frac{\partial^2}{\partial u\,\partial v}\right](v^4 - av^2)$$

$$+ \frac{\kappa\mu^3}{4}\,\frac{n-N}{N-1}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)(v^4 - av^2) + O(\kappa\mu^7) \qquad — (37)$$

Suppose n_0 be some selected value, and suppose that $n - n_0$ and $N - n_0$ are $O(\mu^6)$

Then in place of δX and δY, use the quantities defined by the equation $\delta X^* = \delta Y^*$ such that

$$\delta X^* + i\,\delta Y^* = \frac{\kappa\mu^3\phi^2}{4}\left[\frac{n_0+1}{2n_0}\frac{\partial}{\partial u} + \frac{1}{2}u\frac{\partial^2}{\partial u^2} + \frac{i}{2n_0}\frac{\partial}{\partial v} + \frac{i}{2}\,u\frac{\partial^2}{\partial u\,\partial v}\right](v^4 - av^2)$$

$$+ \frac{1}{4}\kappa\mu^3\,\frac{n-N}{n_0-1}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)(v^4 - av^2) \qquad — (38)$$

the error is only $O(\kappa\mu^7)$ in each case.

From 38 as n_0 can only be varied by $O(\mu^2)$ it follows to within an accuracy $O(\kappa\mu^4)$ that δx and δy depend only on $n - N$, a, μ, and ϕ, and are independent of n_0.

The two fifth order aberrations of lateral spherical aberration and higher astigmatism, may be balanced out by figuring the plate for undercorrection of the spherical aberration on axis for the first, and increasing the central bulge for the second, i.e. by increasing a. However, too big a departure from the determined value of a will give excessive chromatism and spoil the performance of the system. From 38 it appears in wide angle systems where $\frac{n-N}{n_0-1} \ll \phi_0^2$ that chromatism is not so important, but in large astromonical Schmidts working at F/2.5 or larger, over a 5° or 6° field, considerations of chromatism play the larger part in determining the profile.

The idea of optimising the performance over a given field can be interpreted in various ways. One method is to take the greatest image diameter for all object points in the given field and for all wavelengths in the given range, and make the diameter as small as possible. This method has certain drawbacks, and Linfoot and Wolf define the effective radius as the square root of

$$\frac{1}{\pi H^2}\iint_{x^2+y^2 \leq H^2} [\delta X^2 + \delta Y^2]\, dx\, dy \qquad - (39)$$

(its dynamic analogy would be radius of gyration)

and the effective monochromatic image-radius over the field Φ_0, as the square root of

$$E = \frac{1}{\pi H^2 \phi_0^2}\int_0^{\phi_0} \phi\, d\phi \iint [\delta X^2 + \delta Y^2]\, dx\, dy$$

$$= \frac{1}{\pi \phi_0^2}\int_0^{\phi_0} \phi\, d\phi \iint_{u^2+v^2 \leq 1} [(\delta X)^2 + (\delta Y)^2]\, du\, dv$$

(mean square average over the whole field)

The value of $E = E(\phi_0, a, N - n)$ is different for various wavelengths.

After lengthy analysis the value of E* where E* is defined by $E = E^*[1 + O(\mu^2)]$ is arrived at.

$$E^* = \frac{1}{5}\kappa^2\mu^6\phi_0^4\left\{(a^2 - \frac{8}{3}a + 2)\left[\frac{1}{6}(1 + 2\alpha + 2\alpha^2) + \frac{1}{2}p(1 + 2\alpha) + p^2\right] + 2\left(\frac{1}{2} - \frac{1}{3}a\right)\left[\frac{1}{6}(3 + 4\alpha) + p\right] + \frac{1}{6}\right\}$$

where $\alpha = \frac{1}{2n_0}$ and $p = \frac{n - N}{n_0 - 1} \times \frac{1}{\phi_0^2}$

Thus the effective image radius in "n light" is

$$E^{\frac{1}{2}} = (E^*)^{\frac{1}{2}}\{1 + O(\mu^2)\}$$

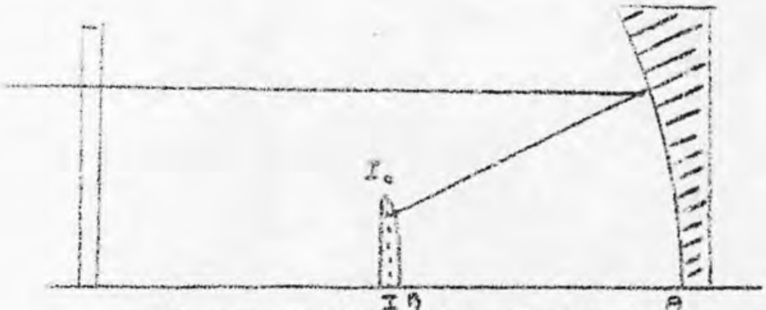
If we define e* such that

$$E^{\frac{1}{2}} = \frac{1}{2}\left(\frac{e^*}{2}\right)^{\frac{1}{2}}\kappa\mu^3\phi_0^2$$

A graph is plotted of $(e_2^*)^{\frac{1}{2}}$ as a function of a and n . From this graph the minimum value of $(e_2^*)^{\frac{1}{2}}$ may be measured off and thus a value of 'a' obtained to give a profile equation.
Two examples are quoted by Linfoot and Wolf, which show that at apertures near F/3 the optimum plate may be obtained by slightly decreasing the strength of the ordinary colour minimised plate. Thus the example given of an F/3.5 plate puts 'a' at about 1.3; while in wide angle systems at F/1 it is better to use a plate with the neutral zone at the edge, the example given a = 2. At the same time the graph shows the diminishing importance of the chromatism at the larger field and F numbers.

Field Flattening.

As was seen in part I, it was suggested by Väisälä[15] in 1935 and F.E. Ross in 1940 that the curvature of field of the Schmidt could be removed by a field flattener lens. This should be a plano convex lens with the curved surface facing the mirror and with a radius of about one sixth of it. Linfoot[14] has considered the aberrations of the field flattened Schmidt.



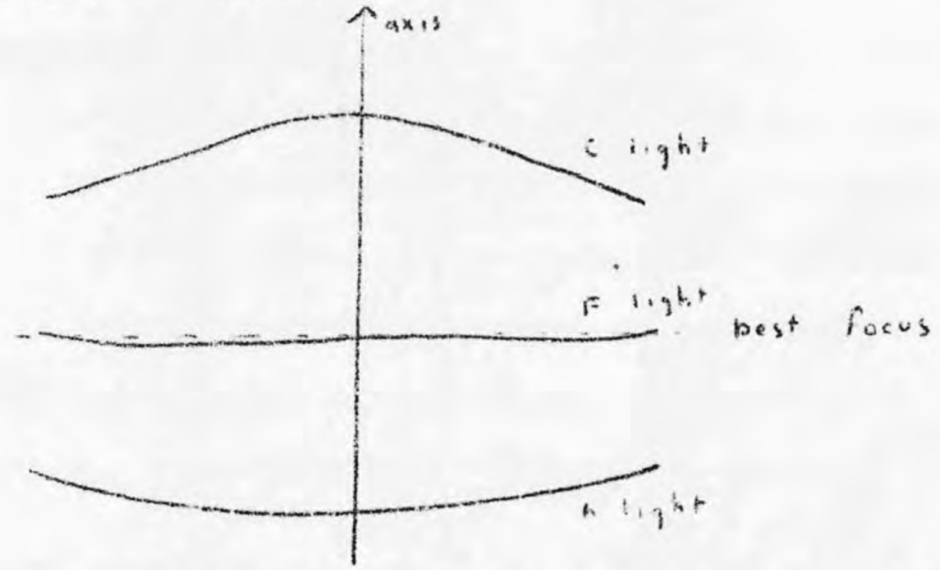
Disregarding the aberrations for the time being, of the converging pencils, suppose all the rays are converging on points in the curve I Io', Linfoot finds the image spreads due to the field flattener alone, which are of the same order as those of the camera, and adds the two. By consideration of the geometry of the figure, the fifth order aberration function of the system may be obtained, from which the only asymmetrical contribution is represented by the term

$$8 R \mu^4 c^3 U(\xi \cdot U^2) \frac{n_1+1}{n_1^2(n_1-1)} \mu r^2 \qquad \text{— (40)}$$

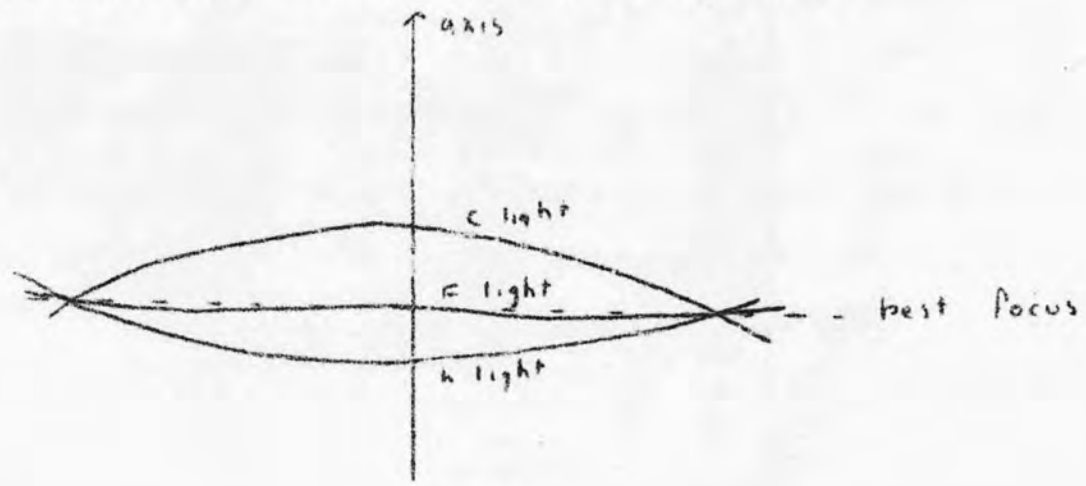
This coma being proportional to $U(\xi \cdot U^2)$ vanishes at the centre of the field and is very small in practice, and may be reduced to less than one third its value by moving the plate along the axis towards the mirror a distance

$$\frac{11}{3} \frac{n_1+1}{n_1^2(n_1-1)} R c^2 \mu^2 \qquad \text{— (41)}$$

A further improvement can be made by adjusting the focal setting according to the figure, which gives the chromatic aberration. The best position for F light is dotted.



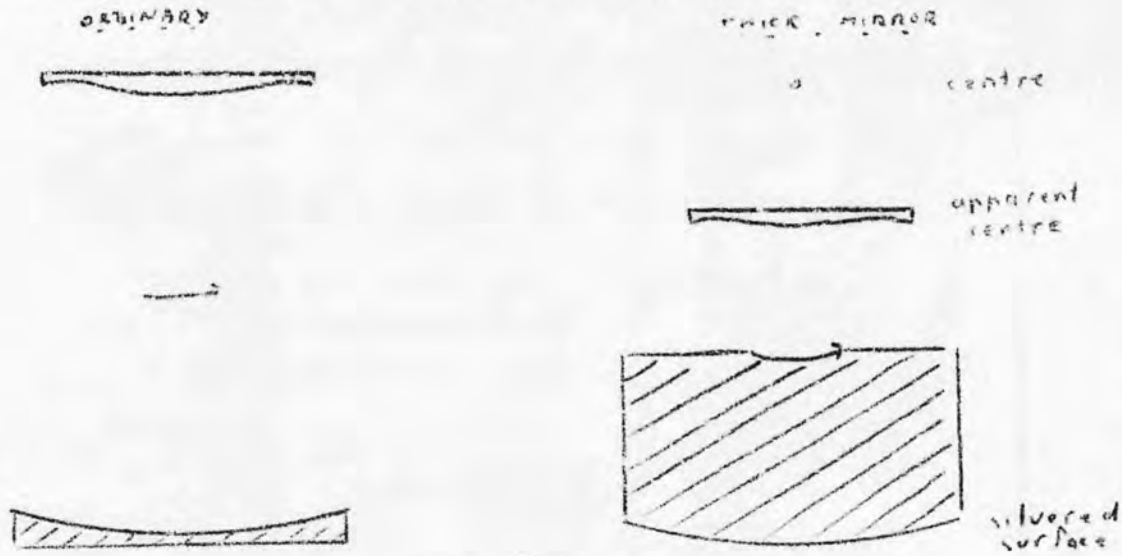
A better colour correction may be obtained by designing the system from the first as a field flattened Schmidt, instead of taking a plain Schmidt, adding a flattener, and moving the plate to balance the coma. Thus, if 'a' is increased from $^3/2$ to 1.705 the colour lines move closer together.



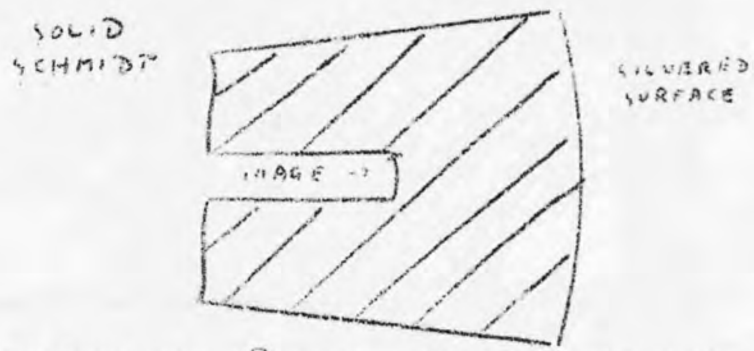
This improved system gives aberrations not much larger in size than the classical Schmidt.

Solid and Thick Mirror Types.

The design of extremely fast Schmidt-type cameras for astronomical spectroscopy has produced a number of variations. The thick mirror Schmidt has been described by Hendrix[48] and compared with the ordinary type. The type is shown in the following figure and compared with the ordinary Schmidt.

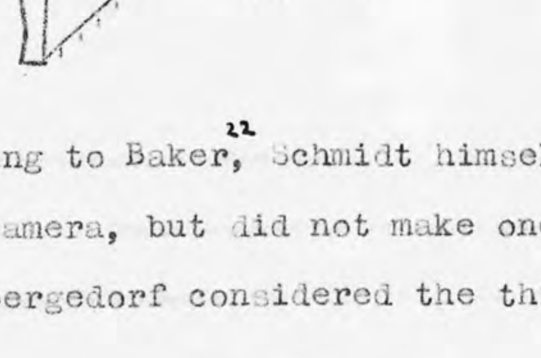


If a solid mirror $^{R}/2$ in thickness, silvered on the back surface, is used in place of the ordinary Schmidt mirror, the speed is increased by a factor of $2\frac{1}{2}$-3 times, depending on n. Thus the speed of an F/.66 camera may be obtained with the field and correction plate of an F/1. Better still, the solid type Schmidt[29] can be made of a solid piece of glass or quartz with worked end faces.



This type becomes n^2 times faster, and thus a solid Schmidt of diamond could have an aperture ratio of F/.2.

A subvariation is the folded Schmidt.



According to Baker,[22] Schmidt himself contemplated a solid glass camera, but did not make one, while his colleagues at Bergedorf considered the thick mirror type afforded the best answer to the problem of obtaining access to the focal surface. Unfortunately the thick mirror type is optically inferior to the solid, and cannot be used at the extreme speeds available to the latter. F/.3 is practicable.

For the solid Schmidt the formulae as previously developed may be used, replacing $\frac{1}{n-1}$ by $\frac{n}{n-1}$ The form of the plate is given by

$$\text{~~Thus~~} \quad x = ay^2 + by^4 + cy^6 +$$

$$a = \frac{3}{8}\frac{n}{n-1} h^2 \left[1 + \frac{a}{16} h^2 +\right.$$

$$b = -\frac{1}{4}\frac{n}{n-1} \left[1 - \frac{3}{2} h^2 +\right.$$

$$c = -\frac{3}{8}\frac{n}{n-1} \left[1\right.$$

The equivalent focal length is given by

$$F^1 = \frac{1}{2n}\left[1 + \frac{3}{8}h^2 + \frac{45}{128}h^4 + \text{etc}\right]$$

As the oblique rays are refracted at the figured surface the field is compressed towards the axis according to the expression $\sin\theta = n \sin\theta'$ and the focal length is reduced by $\frac{1}{n}$ reducing exposure time by n^2. The excavation is multiplied by n^4 so that it is necessary to use the exact form of the profile equation. By a similar treatment Baker obtains

$$x(n\sec\delta - 1) = n\left[\cos\phi\sec\delta + (\cos\phi - f)\sec\theta + f - 2\right] \qquad \text{— (42)}$$

$$x(n\sec\delta - 1) = n\left\{(1 - 2f\cos\phi + f^2)^{\frac{1}{2}}\left[1 + \frac{1}{1 - \frac{f\cos 2\phi}{\cos\phi}}\right] + f - 2\right. \qquad \text{— (43)}$$

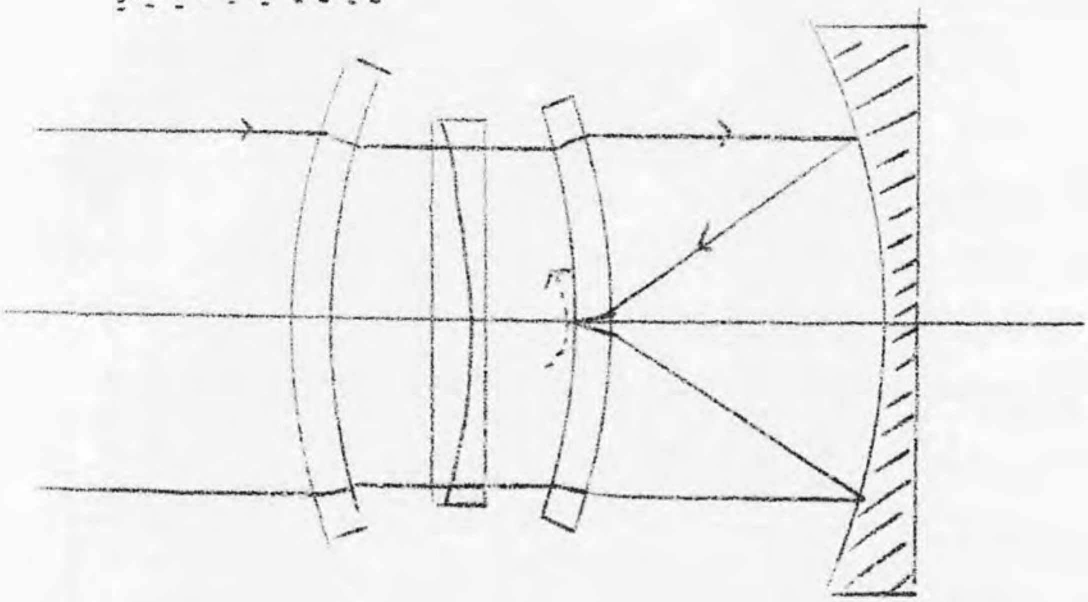
$$h_1 = \sin\phi - (\cos\phi - n)\tan\delta \qquad \text{— (48)}$$

The occurrence of the factor n on the right side of (42) shows that for a given ϕ, the solid Schmidt requires deeper figuring than the classical, but this is no real disadvantage as the classical form must be made more compact by the same factor to obtain a given actual speed. Further, because of the angular compression of the field in the solid form, equivalent focal ratios are possible, which would have not been attainable otherwise.

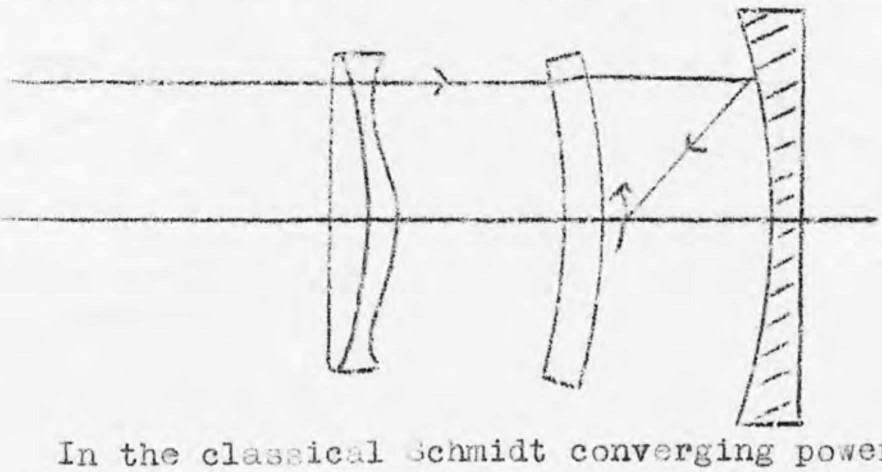
<u>Composite or Meniscus Schmidts</u>.

By substituting for the aspheric plate of the Schmidt a combination of aspheric plate and meniscus lens concentric with the mirror surface, a system may be obtained whose performance is much superior to either the Maksutov or Schmidt camera. Bouwers,[21] and Hawkins and Linfoot[30] have published information on these systems, while in America the so-called "Super-Schmidt Cameras" have been constructed to the designs of Baker,[2] having an effective aperture of 12½" working at F/.82 and covering 55° field.

SUPER SCHMIDT



These cameras are used for Meteor research, at the two Harvard meteor-observing stations, which being 18 miles apart, permit meteor height determinations by simultaneous photographs.
Hawkins and Linfoot consider a system slightly simpler employing one lens meniscus.



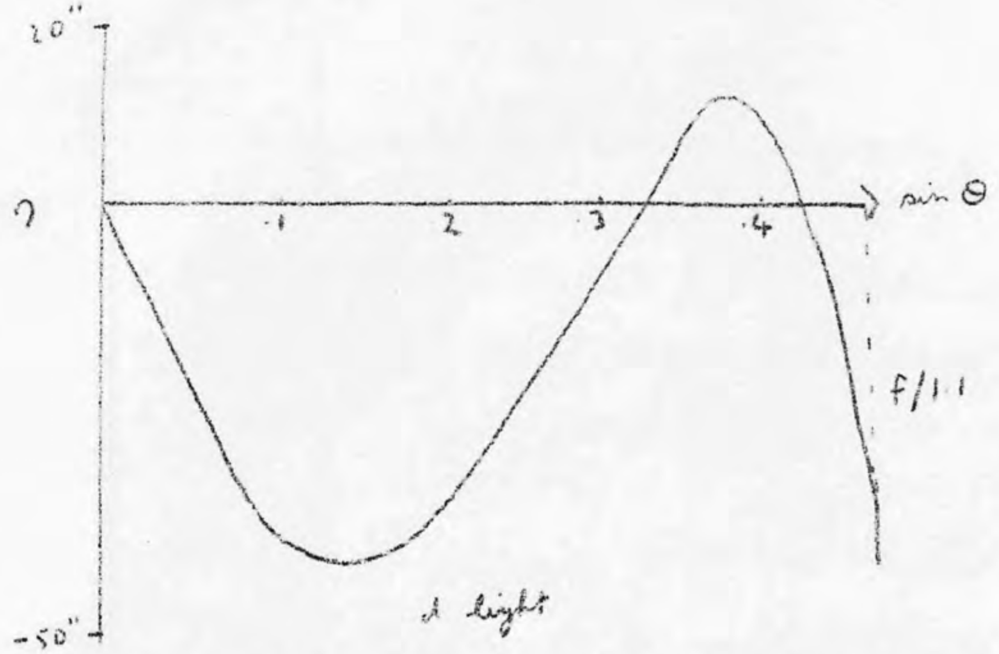
In the classical Schmidt converging power is provided by the spherical concave mirror, which it is the function of the Schmidt plate to correct. This it does at the cost of introducing certain higher order errors, lateral spherical aberration and higher astigmatism, and a certain amount of chromatism, the good performance being due to the intrinsic smallness.

Using a meniscus the Seidel aberrations of the mirror may be corrected, and using a stop at C the system suffers then from higher order spherical aberration, and colour, but is uniform over the whole field. The residual spherical aberration may now be removed with a plate of the form

$$s = cr^6$$

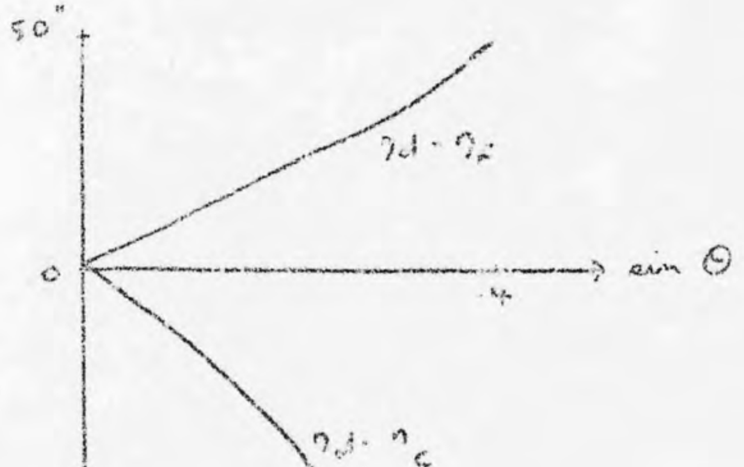
and the colour corrected without introducing too much other aberration by making the plate in the form of a cementable doublet, from two glasses of the same refractive index, and different dispersions.

In the design, a concentric meniscus was found by trial so that the angular deviations of the d light were as small as possible. Then if the focus be at a point, say F, rays were traced outwards from F through the system at angle θ, and their inclination η to the axis found.



The colour may be illustrated by plotting $\eta_d - \eta_F$

$\eta_d - \eta_c$ against $\sin\Theta$



The profile of the plate was determined by numerical integration of the table using the equation

$$(n'_d - 1)\frac{ds}{dr} = \frac{\pi}{648{,}000}\,\eta_d r \qquad r = 13 \cdot 9995 \sin\Theta$$

By choosing hard crown and telescope flint glasses the colour error may be reduced to 8" of arc on 18° field. The off axis aberration may be analysed by the same method as before used, by considering the plate to be tilted and calculating the effect on a ray passing through a thin, nearly parallel prism.

As before the expressions

$$(n-1)\sin^{1}\Phi\left[\frac{n+1}{2n}\frac{\partial T}{\partial x}+\frac{x}{2}\frac{\partial^2 T}{\partial x^2}\right], \quad (n-1)\sin^{1}\Phi\left[\frac{1}{2n}\frac{\partial T}{\partial y}+\frac{x}{2}\frac{\partial^2 T}{\partial x\partial y}\right]$$

represent the components of angular aberration in the ray through the point x y.

The linear deviation components ξ, η, are given by multiplying by f and since the plate is a solid of revolution $T = T(x^2 + y^2) = Tr^2$

$$\xi + i\eta = (n-1)f\sin^{1}\Phi\left\{2xT'(r^2)+(x+iy)\left(\frac{1}{n}T(r^2)+2x^2T''(r^2)\right)\right\}$$

introducing $\rho(t^2) = \frac{648000}{\pi f}(n-1)\left[T(r^2) - T(0)\right]$

whose t derivations $\eta(t) = 2t\rho'(t^2)$ measured in seconds of arc the radial deviation imposed on the plate by a ray meeting it normally at a zone of radius $r = ft$

and also δX and δY $\delta X = \frac{648000}{\pi f}\xi$, $\delta Y = \frac{648000}{\pi f}\eta$

so that δX and δY are the apparent angular aberrations in seconds of arc in the incoming ray.

Then

$$\delta X + i\,\delta Y = \sin^2\phi\left\{\frac{3}{2}u\,\eta(t) + (u+iv)\left(\frac{\alpha}{2t}\eta'(t) + \frac{\alpha^2u^2}{t^3}\left(t\eta'(t) - \eta(t)\right)\right)\right\}$$

where $\alpha = \frac{R}{F}$ R = radius of the aperture stop

$u = \frac{x}{R}$, $v = \frac{y}{R}$ $t^2 = \alpha^2(u^2+v^2)$

It may be shown that the function

$\eta^*(t) = -457t + 63504t^3 - 133404t^5 - 43520t^7$ agrees with the function η of the graph to within .8" of arc.

Thus $P = \left\{(\delta x)^2 + (\delta y)^2\right\}^{\frac{1}{2}} / \sin^2\phi$ may be calculated for each point u, v, of the aperture stop.

By this means it may be seen that the error spreads of the meniscus Schmidt over an 18^o field are everywhere less than 8" and that 97% of the light falls in a circle of less than 35" of arc. Thus the total error spreads are less than $^1/16$th of those of a classical Schmidt of the same field and aperture ratio.

Part III

Two Mirror Systems and More Complicated Mirror Plate Systems.

Two mirror telescopes and their variations.

Schmidt.Cassegraine.

Reflecting microscopes.

Paraboloid field correctors.

Two Mirror Systems and more complicated Mirror Plate Systems.

Two Mirror Telescopes and their variations.

As has been described in part I the standard forms of the Cassegrainian and Gregorian telescopes are confocal conic sections and subject to the error or Coma, the amount of which, as shown by A.C. Lunn,[9] is the same for different confocal conic sections of the same focal length. Various modifications proposed for practical reasons, such as making either primary or secondary mirror a sphere, and figuring the other for zero spherical aberration, only tend to make the coma worse. Robert E. Jones[31] has derived formulae, which show the effect of small modifications on two mirror systems.

In the case of a single parabolic reflector the departure from the sine condition is given by

$$\left(\frac{F_y}{F_0} - 1\right) = \frac{y^2}{4F_0^2} \qquad \text{— 1}$$

Let A be the ratio of the focal length of the combination to the focal length of the primary mirror. To consider the effects of various modifications on the Cassegrain the equation of the primary is written

$$x_1 = \frac{y_1^2}{2R_1} + \frac{P\,y_1^4}{(2R_1)^3} \qquad \text{— 2}$$

and for the secondary

$$x_2 = \frac{y_2^2}{2R_2} + \left[S - \frac{4A}{(A-1)^2}\right]\frac{y_2^4}{(2R_2)^3} \qquad \text{— 3}$$

the quantities $\Delta x_1 = P\left[\frac{y_1^4}{(2R_1)^3}\right]$ — 4

$$\Delta x_2 = S\left[\frac{y_2^4}{(2R_2)^3}\right] \qquad \text{— 5}$$

represent deviations of the primary and secondary mirrors from their original parabolic and hyperbolic forms. Thus putting P = 1 makes the primary spherical. The spherical aberration will remain corrected if

$$\Delta x_1 = \Delta x_2 \qquad \text{— 6}$$

If D_2 is the diameter of the secondary and D_1 that of the primary

$$\frac{D_2}{D_1} = \frac{y_2}{y_1} \qquad \text{— 7}$$

From equations 4, 5, 6 & 7 $\quad P = S\left(\frac{A_1}{A_2}\right)^3\left(\frac{D_2}{D_1}\right)^4$ — 8

The offence against the sine conditions may be written

$$\left(\frac{F_y}{F_0} - 1\right) = \frac{C y^2}{4F_0^2} \qquad — 9$$

where $C = 1 + \frac{(A-1)^3}{2A}\left(1 - \frac{D_2}{D_1}\right)S$ — 10

or in terms of P $\quad C = 1 + \frac{A^2}{2}\left(\frac{D_1}{D_2} - 1\right)P$ — 11

For Schwarzschild's condition of aplanation C = 0

$$\text{and} \quad S = \frac{2A}{(A-1)^3\left(1 - \frac{D_2}{D_1}\right)}, \quad P = \frac{2}{A^2\left(\frac{D_1}{D_2} - 1\right)} \qquad — 12 \; — 13$$

When these values are put in 2 and 3 they produce a slight overcorrection of the primary and an increase in the eccentricity of the hyperbolic secondary as seen in part I.

For the construction of Dale and Kirkham employing a spherical secondary

$$S = 1 + \frac{4A}{(A-1)^2} \qquad — 14$$

and $\frac{F_y}{F_0} - 1 = \left[1 + \frac{(A-1)(A+1)^2}{2A}\left(1 - \frac{D_2}{D_1}\right)\right]\frac{y^2}{4F_0^2}$ — 15

if $\frac{D_2}{D_1} = \frac{1}{4}$ and A = 4 $\qquad$ C = 8. Thus the coma is increased eight times over the standard Cassegrain by this modification.

If the primary mirror is spherical $\quad$ P = 1

$$\text{and} \quad S = \left(\frac{A}{A-1}\right)^3\frac{D_1}{D_2} \qquad — 16$$

$$\left(\frac{F_y}{F_0} - 1\right) = \left[1 + \frac{A^2}{2}\left(\frac{D_1}{D_2} - 1\right)\right]\frac{y^2}{4F_0^2} \qquad — 17$$

for $\frac{D_1}{D_2} = 4$ and A = 4 $\qquad$ C = 24. Thus the coma is three times as great as in the previous example.

For modification of the Gregorian

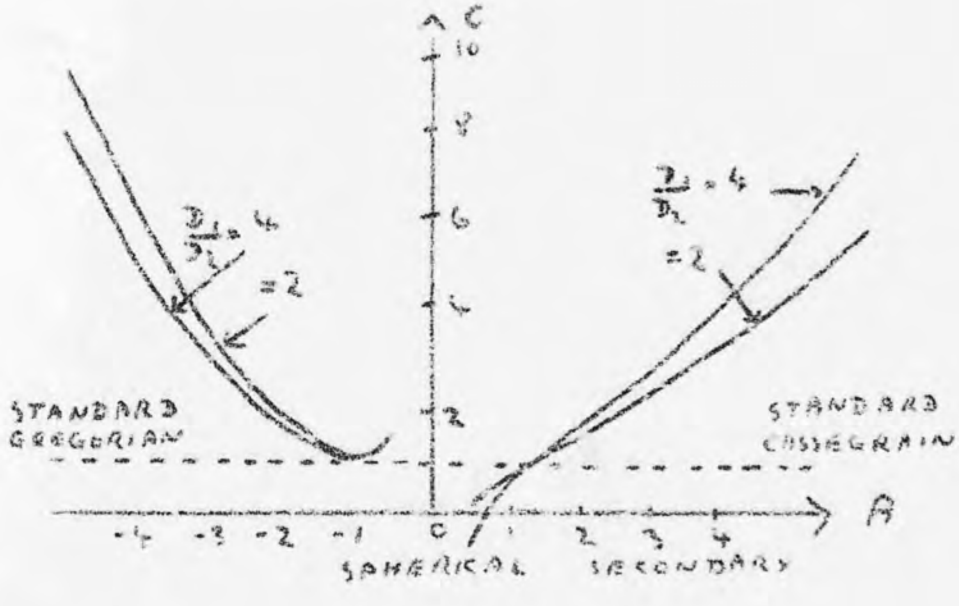
$$x_2 = -\frac{y_2^2}{2R_2} \left[5 + \frac{4A}{(A+1)^2}\right]\frac{y_2^4}{(2R_2)^3} \qquad -\underline{18}$$

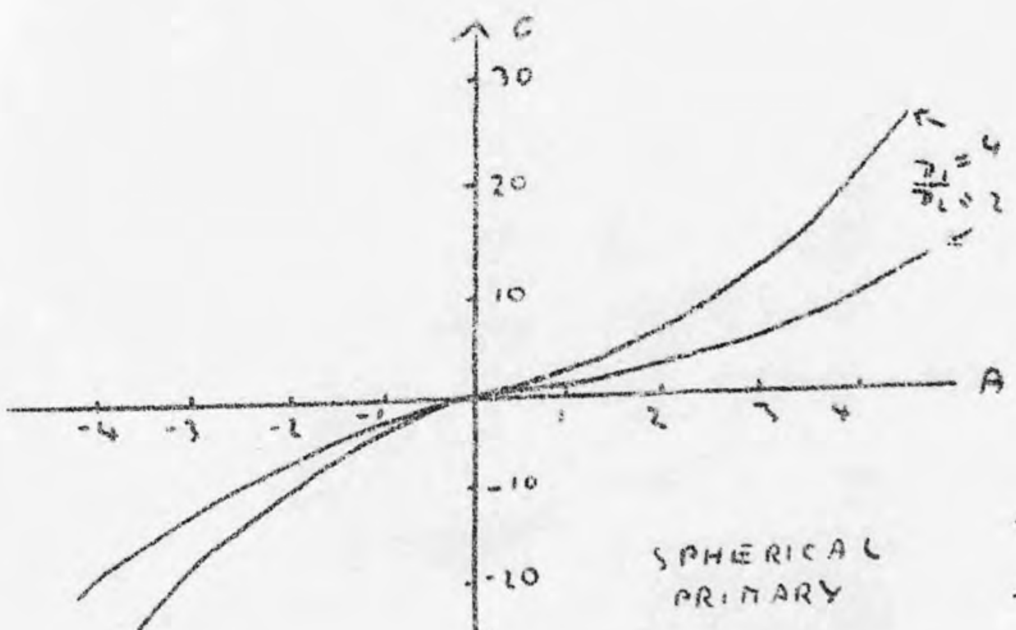
while the factor of increase of zonal magnification

$$C = 1 - \frac{A^2}{2}\left(1 + \frac{D_1}{D_2}\right)\rho \qquad -\underline{19}$$

The coma of two mirror systems is shown by the

following graphs:

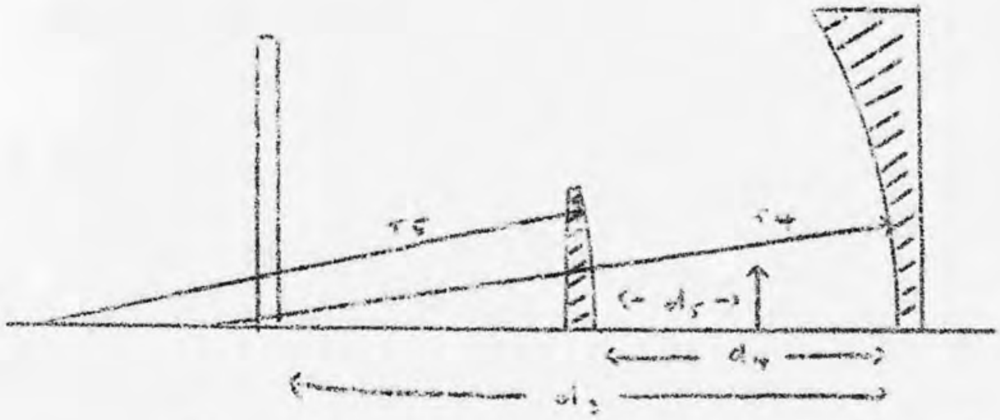




From these graphs it can be seen that the attempts to make the type easier to construct only introduce further amounts of coma. For wide field photographic telescopes it is of course necessary to considerfurther aberrations. The Schwarzschild telescope, although aplanatic, retains the defect of astigmatism, which is distributed about a nearly plane focal surface and, furthermore, which is small enough to obtain a moderately large useful field. Wright, using the Schwarzschild equations, has generalised the Schmidt camera into a n entire family, and has shown that the two surfaces of astigmatism cannot coalesce in a flat surface by means of a correcting pla te and an aspherical mirror.

<u>Schmidt Cassegrains</u>

J.G. Baker[32] in his important paper on "Flat fielded cameras equivalent in performance to the Schmidt camera" first suggested the idea of the Schmidt-Cassegrain type of camera, that is to say, a system of a correcting plate and two mirrors. "One can see that such a system properly designed will be free not only of spherical aberration and coma of all orders, but will be anastigmatic on a flat field to the third order."



Even after all the conditions for exact aplanatism and after the third order equations for anastigmatism and flat field are satisfied, there remain two free parameters, being the distance of the correcting plate from the primary mirror d_3 , and the distance of the photo plate from the secondary mirror d_5. All the constants of the system can be expressed as rational functions of the two parameters. In the following equations the equivalent focal length = 1.

In order to compute the constants of a given system, one must first decide upon the value of d_5

Then $\quad d_4 = (1 - d_5)^2$

and if $\quad \beta = 0$

$$r_4 + r_5 = -2(1 - d_5)$$

also $\quad A = \alpha_1 + \beta_1 S_5 + \gamma_1 S_4$

$$B = \alpha_2 + \beta_2 S_5$$

$$C = \alpha_3 + \beta_3 S_5$$

$$D = \alpha_4 + \beta_4 S_5$$

where $\quad \alpha_1 = -\dfrac{1 + 2d_5 - d_5^2}{8(1 - d_5)}, \quad \beta_1 = 8 d_5^4, \quad \gamma_1 = -8$

$$\alpha_2 = \frac{6 + 6d_5 - 12 d_5^2 + 3 d_5^3}{8(1 - d_5)}, \quad \beta_2 = 24 d_5^2 (1 - d_5)^2$$

$$\alpha_3 = -\frac{3}{2} - \frac{3}{2} d_5 + \frac{15}{8} d_5^2 - \frac{3}{8} d_5^3, \quad \beta_3 = 24 d_5^2 (1 - d_5)^4$$

$$\alpha_4 = \frac{10 + d_5 - 13 d_5^2 + 7 d_5^3 - d_5^4}{8}, \quad \beta_4 = 8 d_5 (1 - d_5)$$

The quantities of A, B, C, D contain the two unknowns S_4 and S_5 which have the following significance:- In the coordinate system of the figure, let the equation of the ith surface be

$$x_i = a_i y_i^2 + b_i y_i^4 + \text{etc}$$

$$a_1 = \frac{1}{2 r_1}$$

and if the surface is spherical $\quad b_1 = a_1^3$

The quantity S_1 represents the departure from a sphere such that for an aspheric surface the expansion runs $\quad x_1 = a_1 y_1^2 + (a_1^3 - S_1) y_1^4 + \text{etc}$

thus $b_1 = q_1^3 - S_2$

In order to evaluate S_4 and S_5 the following equations must be satisfied.

$$C + B\alpha_3 = 0$$

$$B + 3A\alpha_3 = 0$$

Thus for each value of α_3 one obtains particular values for S_4 and S_5.

If distortion is also tobe zero (a condition usually unimportant in celestial photography)

then $$D + A\alpha_3^3 = \frac{1}{2}$$

$$b_3 = -\frac{A}{4(n-1)}$$

Considerations of tube length and manufacturing difficulties identify four important single parameter families of nearly equivalent performance.

<u>Case A.</u> Correcting plate close to the secondary mirror. The tube length reaches a low value but the mirrors depart far from spheres and curvature of correcting plate is also quite large.

<u>Case B.</u> Criterion that the secondary mirror shall be spherical, the departure of the primary mirror is also small.

<u>Case C.</u> Primary is a sphere.

Case D. Distortion to be made zero.

The correcting plate of each of the new type of cameras is of higher curvature than the Schmidt of the same focal length and aperture, consequently the dispersion causes a more serious colour error, depending upon the value of b_3.

As in the Schmidt it is desirable to introduce a central bulge into the correcting plate so that the maximum slope of any part of the plate is minimised. In order to illustrate the performance of the systems the author has traced a number of rays accurately through camera B. The correcting plate was taken without a central bulge for ease of computation. A differential correction made the system extremely aplanatic.

C.R. Burch[33] in 1942 used the Optical See-Saw diagram (about which more will be said in part IV) to design a Baker camera. Briefly, the See-saw diagram represents the Seidel properties of systems of spherical and aspherical surfaces of revolution, by placing at the centre of curvature a figured plate which corrects, to the fourth power, the spherical aberrations, and by its position the off axis aberration of coma and astigmatism as in the Schmidt camera, and then images into any convenient representative space all the plates which the system possesses. Thus the system is replaced by an anastigmatic system and a collection of figured plates, some of these being negatives of missing anastigmatising plates, and other figurings on surfaces which may be asphérised. The plate diagram is situated in an image space for which the object is at infinity. Burch calls this "star space". The strength of the plates has already been defined as the retardation = K (zonal radius)^4 and is altered by imaging, inversely as the fourth power of the magnification. If the plates be regarded as masses proportional to their strength poised on a see-saw in star space whose point is the pupil of the system, then

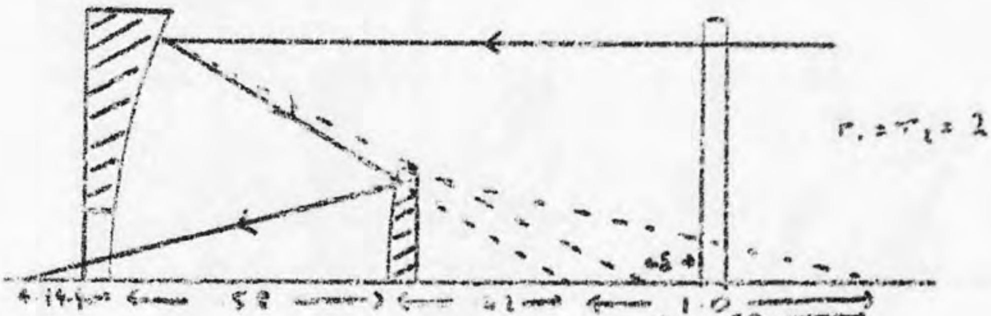
(1) Spherical aberration is proportional to the total weight on the pivot.

(2) Coma is proportional to the unbalanced moment about the pivot point.

(3) Astigmatism is proportional to moment of inertia.

(4) That part of the distortion which is not associable with the representation of spheres on tangent planes is proportional to the third moment of mass about the pivot point.

Taking a Baker camera for example, putting $P = 0$ $f = 1$ the figure is constructed as shown



To construct the see-saw diagram five plates are involved.

(1) Anastigmatising plate lacked by concave mirror

(2) Anastigmatising plate lacked by convex mirror.

(3) Figuring on surface of concave mirror.

(4) Figuring on surface of convex mirror.

(5) Figured plate.

Plate (1) is in star space already, Plate (2) is at the centre of the convex, i.e. 2.58 in front of the concave mirror. Thus it images into star space 1.63291 in front of the concave with magnification $\frac{1.63291}{2.58}$. Plate (3) is also in star space already. Plate (4) is .58 in front of the concave it images 1.38095 behind it. Plate (5) is in star space already.

The strength of the anastigmatising plate lacked by a concave mirror of radius of curvature ρ , object distance u , at zonal radius r

$$r = \left(\frac{u}{u-\rho}\right)^2 \frac{r^4}{4\rho^3}$$

Taking as unit of strength $\frac{r^4}{4\rho^3}$ so that the factor $\frac{r^4}{4\rho^3}$ may be dropped.

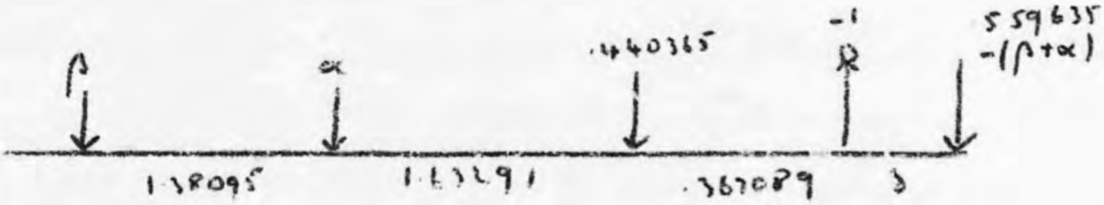
Plate (1) has strength -1 retardation units.

Plate (2) has strength $\left(\frac{.42}{1.58}\right)^2$ since the convex is used at u = .42 and images into star space with

strength $\left(\frac{.42}{1.58}\right)^2 \times \left(\frac{3.58}{1.6329}\right)^4$ = .440365 (retardations)

Let plate (3) have strength α retardations and plate (4) β in star space.

The plate diagram may be then shown



The strength of plate (5) $.559635 - (\beta + \alpha)$ has of course been chosen so that $\sum m = 0$, for zero spherical aberration.

Now setting $\sum mx = 0$ for aplanatism and $\sum mx^2 = 0$ for anastigmatism

$$3.38095\,\beta + 2\alpha + .161653 - \{.559635 - (\beta + \alpha)\}\,\delta = 0 \quad -20$$

$$3.380951^2\beta + 4\alpha + .059341 + \{.559635 - (\beta + \alpha)\}\,\delta^2 = 0 \quad -21$$

Eliminating δ

$$9.08721\,\beta + 2.82571\,\alpha + .059341 - 1.186677\,\alpha\beta = 0$$

It is possible to control the contribution to distortion $\sum mx^3$ and still satisfy this equation by a suitable choice of α and β. This gives the Baker type D.

Suppose $\beta = 0$

Then $\alpha = -.02100$ $\delta = .20607$ and plate 5 has strength $= .538635$ retardations.

Suppose $\alpha = 0$

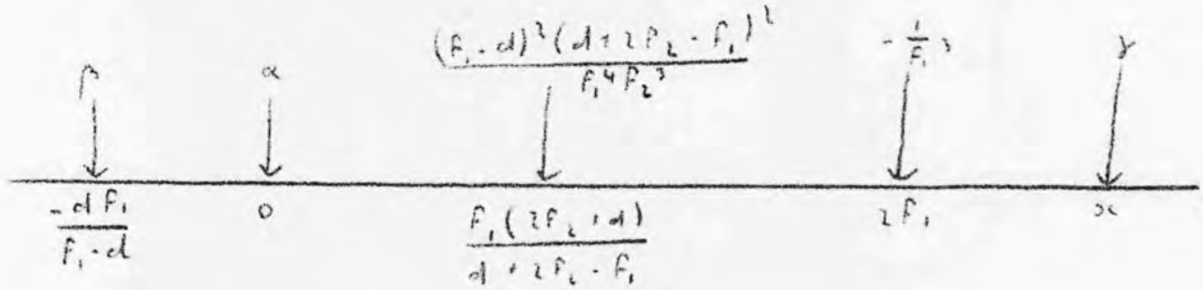
$\beta = -.00653$ $\delta = .24653$ and plate 5 has strength $= .553105$ retardations.

These examples give, of course, Baker B and C types.

Burch then considers mono-centric types, which have not zero petzval curvature but have both mirrors spherical and are anastigmats, using the same method.

[34,35,19.] Linfoot has extended and generalised the method of plate-diagram analysis and applied it to the Schmidt-Cassegrain systems.

Suppose the two mirrors M_1 and M_2 have radii ρ_1 and ρ_2, with poles d apart, focii f_1 and f_2. Then the plate diagram may be represented.



For zero spherical aberration

$$\alpha+\beta+\gamma+\frac{(f_1-d)^2(d+2f_2-f_1)^2}{f_1^4 f_2^3}-\frac{1}{f_1^3}=0$$

For zero coma

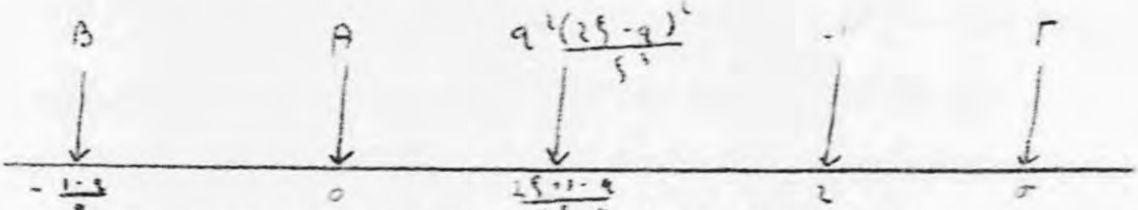
$$-\frac{d f_1}{f_1-d}\beta+x\gamma+\frac{f_1(2f_2+d)}{d+2f_2-f_1}\cdot\frac{(f_1-d)^2(d+2f_2-f_1)^2}{f_1^4 f_2^3}-\frac{2}{f_1^2}=0$$

For zero astigmatism

$$\left(\frac{d f_1}{f_1-d}\right)^2\beta+x^2\gamma+\frac{f_1^2(2f_2+d)^2}{(d+2f_2-f_1)^2}\cdot\frac{(f_1-d)^2(d+2f_2-f_1)^2}{f_1^4 f_2^3}-\frac{4}{f_1}=0$$

Then putting $A = \alpha f_1^3$, $B = \beta f_1^3$, $\Gamma = \gamma f_1^3$, $\sigma = \frac{s}{f_1}$, $q = 1 - \frac{d}{f_1}$, $\xi = \frac{f_2}{f_1}$

The plate diagram becomes



The conditions for the first three Seidel errors become

$$\begin{cases} A + B + \Gamma + \dfrac{q^2(2\xi - q)^2}{\xi^3} = 1 \\ -\dfrac{1-q}{q} B + \sigma \Gamma + \dfrac{q^2(2\xi - q)(2\xi + 1 - q)}{\xi^3} = 2 \\ \left(\dfrac{1-q}{q}\right)^2 B + \sigma^2 \Gamma + \dfrac{q^2(2\xi + 1 - q)^2}{\xi^3} = 4 \end{cases}$$

These three equations specify all the Schmidt-Cassegrain anastigmats in terms of six parameters.

A B Γ are the figuring depths on primary, secondary and plate expressed in terms of parabolic correction of the primary as a unit.

σ is the distance of the plate in front of the primary expressed in terms of f_1 as a unit.

ξ is the ratio $\frac{f_2}{f_1}$.

q is the obstruction ratio for the axis pencil, alternatively 1 - q is the separation between the mirrors expressed in terms of f_1 as a unit.

On setting A = B = 0 the case of the anastigmat with two spheres and one plate may be obtained. The equations become

$$\begin{cases} \Gamma = 1 - \dfrac{q^2(2\xi - q)^2}{\xi^3} \\[2ex] \sigma\Gamma = 2 - \dfrac{q^2(2\xi - q)(2\xi + 1 - q)}{\xi^3} \\[2ex] \sigma^2\Gamma = 4 - \dfrac{q^2(2\xi + 1 - q)^2}{\xi^3} \end{cases}$$

on eliminating σ and Γ

$$q^2(2\xi - 1 - q)^2/\xi^3 = 0$$

hence $q = 2\xi$ or $q = 0$ that is to say $d = 2f_1 - 2f_2$ or $d = f_1$.

Case 1 is when the spheres are concentric and in this case the petzval curvature cannot be zero and hence the systems are not flat fielded.

Case 2 is when the second sphere is at the paraxial focus of the first. This in fact corresponds to the flat fielded Schmidt, in which the convex secondary mirror is replaced by a field flattening lens.

If the first two equations only are retained, and the mirrors made spherical, A = 0 and B = 0 and

$$\Gamma = 1 - \frac{q^2(2\xi - q)^2}{\xi^3}$$

$$\sigma\Gamma = 2 - \frac{q^2(2\xi + 1 - q)(2\xi - q)}{\xi^3}$$

The petzval curvature is given by

$$\frac{1}{f_p} = -\frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{f_1}\,\frac{1-\xi}{\xi}$$

In Schmidt-Cassegrain aplanats of high definition, the off axis astigmatism should be so small as to cause no observable effect on their image size for most practical cases. This means the petzval curvature should be small, otherwise the astigmatism needed to flatten the ~~field~~ will spoil the definition.

Flat fielded anastigmats described by Baker are characterised by having $\frac{1}{f_p} = 0$.

Thus $\frac{1}{f_1} = -\frac{1}{f_2}$ and $\xi = 1$

<u>Type A</u>. Short tube. This has the corrector plate at the focus of the primary. Hence $\sigma = 1$

$$\text{and } A = -1 - 2q$$

$$B = \frac{q^2}{1-q}\left[2 - q^2(3-q)\right]$$

$$\Gamma = 2(1+q) - 2q^2(3-q)$$

The distortion is about $^{1}/5\%$ pincushion at 3° off axis ~~pincushion.~~

<u>Type B</u>. Spherical secondary. i.e. B = 0

$$A = -\frac{q^3(1-q)^2}{R}$$

$$\Gamma = \frac{Q^2}{R} \qquad \sigma = \frac{R}{Q}$$

Distortion $^{1}/20\%$ barrel at 3° ~~barrel.~~

Type C. Spherical primary. i.e. $A = 0$

$$B = -\frac{q^4(1-q)}{1+3q}$$

$$\Gamma = \rho + \frac{q^4(1-q)}{1+3q}$$

Type D. Distortion free. The condition for zero distortion from the plate diagram is given by

$$\left(\frac{1-q}{q}\right)^3 \rho - \sigma^3 \Gamma + \frac{3-q}{1-q} B = 0$$

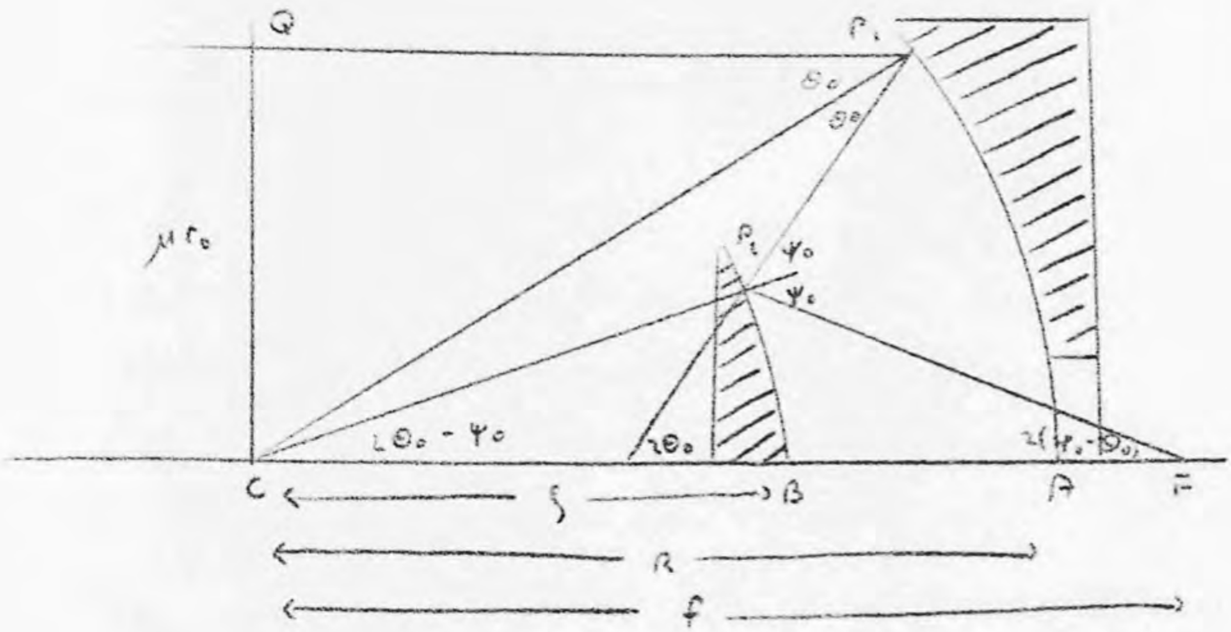
$$A = \rho - B - \Gamma$$

$$B = \frac{q^2}{1-q}\,\frac{\rho - \sigma Q}{q\sigma + 1 - q}$$

$$\Gamma = \frac{1}{\sigma}\left(\rho + \frac{1-q}{q} B\right)$$

By similar methods Linfoot extends the argument to two plate systems to reduce the chromatic error, restricting his examples to two sphere, two plate systems.

P.A. Wayman[36] has obtained the higher aberrations in the case of the monocentric Schmidt-Cassegrain cameras using the spherical ~~symetry~~ symmetry method evolved by Caratheodory and developed by Linfoot. This method cannot be applied to the general Schmidt-Cassegrain system but only to the monocentric type.



Let f_1 and f_2 be the foci of the two mirrors and if the radius of the primary be R = 1

$$f_1 = \tfrac{1}{2}, \quad f_2 = \frac{\xi}{2} \qquad \xi = \text{ratio of } \frac{f_1}{f_2}$$

Putting $\mu = \frac{H}{R}$ where H is the radius of the aperture stop and suppose μ is small.

Setting coordinate axes with $x^2 + y^2 = H^2 r^2$

The plate profile is of the form

$$S(r) = \frac{T(r) - T(0)}{R} = a_1 \mu^2 r^2 + a_2 \mu^4 r^4 + a_3 \mu^6 r^6 + O(\mu^8) \qquad \text{— 22}$$

The paraxial focal length of the plate is given by

$$f_p = -\frac{1}{2(n_0 - 1)} \cdot \frac{1}{a_1} \qquad \text{— 23}$$

In the figure the neutral ray is given by QP_1 with

$r = r_0$.

Therefore $$\frac{1}{f_r} = 2\,\frac{\xi - 1}{\xi} + \frac{1}{\xi}\,\frac{\sin 2(\psi_0 - \theta_0)}{\sin \psi_0} \qquad \text{— 24}$$

Since ψ_0 and θ_0 are functions of r_0

$$\frac{1}{f_r} = P(r_0) \qquad \text{— } \underline{25}$$

Thus equations 23 and 25 relate a_1 and r_0.

Then making use of the fact

(a) that the plate slope vanishes at $r = r_0$

(b) that under perfect axial stigmatism the optical path difference is zero

Wayman, after considerable analysis, obtains the plate coefficients in the following form

$$\left.\begin{aligned}
(n_0-1)a_1 &= -\left(\frac{\xi^3-4\xi^2+4\xi-1}{2\xi^3}\right)\mu^2 r_0^2 - \frac{\xi^5-8\xi^3+8\xi^2-1}{8\xi^5}\mu^4 r_0^4 + O(\mu^6) \\
(n_0-1)a_2 &= \frac{\xi^3-4\xi^2+4\xi-1}{4\xi^3} - \left(\frac{\xi^5-6\xi^4+13\xi^3-13\xi^2+6\xi-1}{2\xi^5}\right)\mu^2 r_0^2 + O(\mu^4) \\
(n_0-1)a_3 &= \frac{3\xi^5-16\xi^4+32\xi^3-32\xi^2+16\xi-3}{8\xi^5} + O(\mu^2)
\end{aligned}\right\} \underline{26}$$

Combining 26 with 22

$$\delta(r) = \frac{\xi^3-4\xi^2+4\xi-1}{4(n_0-1)\xi^3}\mu^4(r^4 - a r^2) + O(\mu^6) \qquad \text{— } \underline{27}$$

To minimise the colour spread over the plate,

taking $a = \frac{3}{2}$ $\quad \frac{\partial^2\delta}{\partial r^2} = 0$ at $r = \frac{1}{2}$

so $\quad \left|\frac{\partial\delta}{\partial r}\right|_{r=1} = \left|\frac{\partial\delta}{\partial r}\right|_{r=\frac{1}{2}}$ is the condition required as before

and thus $r_0 = \frac{\sqrt{3}}{2}$ $= .866$ as in the colour minimised Schmidt camera.

Using the argument of Hawkins and Linfoot[30] the leading monochromatic terms in the components of angular aberration are given by

$$\delta X + i\,\delta Y = (n-1)\sin^2\Phi\left(\frac{n^2-1}{2n}\frac{\partial T}{\partial x} + \frac{x}{2}\frac{\partial^2 T}{\partial x^2} + \frac{i\,\partial T}{2n\,\partial y} + \frac{ix}{2}\frac{\partial^2 T}{\partial x\,\partial y}\right) \quad \text{— } \underline{28}$$

where, as before, T(xy) is the thickness of the corrector plate, ϕ the angle off axis, and the marginal zone of the corrector is given by

$r^2 = u^2 + v^2 = 1$

so that $\frac{\partial T}{\partial x} = \frac{1}{\mu}\frac{\partial S}{\partial v}$ etc.

From 27 and 28 the angular aberrations in seconds of arc are given by

$$\delta X + i\,\delta Y = \frac{\xi^3-4\xi^2+4\xi-1}{4\xi^3}K\mu^3\phi^2\left[\frac{n_0+1}{2n_0}\frac{\partial}{\partial u} + \frac{u}{2}\frac{\partial^2}{\partial u^2} + \frac{i}{2n_0}\frac{\partial}{\partial v} + \frac{iu}{2}\frac{\partial^2}{\partial u\,\partial v}\right] \times (r^4 - a r^2) + O(K\mu^7) \quad \text{— } \underline{29}$$

This equation represents the leading monochromatic aberrations of the monocentric Schmidt-Cassegrain cameras which are seen to be, to a first approximation, dependent only on the second order fourth power terms of the plate profile expansion. The choice of the parameter ξ is restricted to a small range, for while it is desirable that it should be low to reduce obstruction, the focal surface must lie behind the primary mirror to obtain the fully accessible image surface.

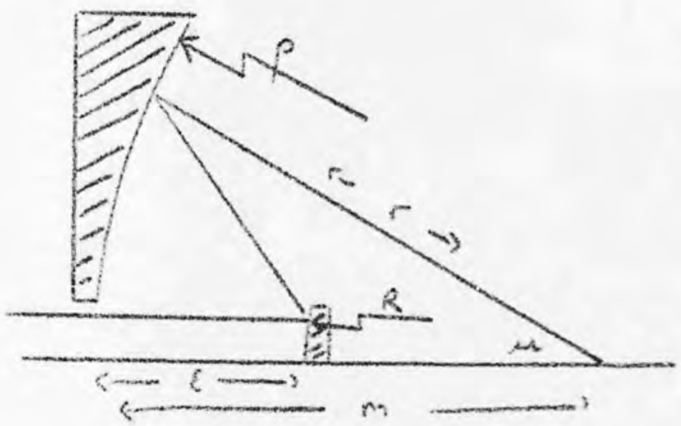
Allowing for the thickness of the mirror $\frac{f}{f} = .69$ seems to be the lowest permissable value. Because of the curvature of the image surface the system cannot conveniently be used over a greater field diameter than about 5^0 at an aperture ratio of f/3.5

Linfoot[57] has given details for the fine correction of a Baker B type camera, that is one with a spherical secondary mirror. Having by the means previously specified fixed the dimensions and figurings to annul the Seidel errors of the system, the neutral zone of the plate is fixed and a ray traced through it on to the mirrors to determine the focal point F. F thus being determined, a meridional fan of rays is traced out from F at numèrical apertures of .01, .02, .03 up to .25 for an f/2 system, up to the plate. Let $h_1(\theta)$ denote the height at which a ray cuts the plate having left the axis inclined at θ to it. Then $\frac{h_1(\theta)}{\sin\theta}$ is found to be nearly constant for the different values of θ (since the Seidel condition is already satisfied). The asphericity on the big mirror is adjusted to make $\frac{h_1(\theta)}{\sin\theta}$ constant as θ varies, and the neutral ray is traced again and the procedure repeated till no change is found. The plate profile is then adjusted to bring all the rays out parallel to the axis.

Further improvement to the design might be made after exploring the field with a microscope, the residual field curvature could be balanced out by changing the radius of curvature of the secondary, the coma reduced by a slight shift in the position of the plate along the axis.

<u>Reflecting Microscopes.</u>

In part I, the early attempts to use aspheric surfaces in telescopes were described, and at the same time it was realised that if the light was sent backwards through a telescope objective, it became a microscope objective. The various attempts to make practical instruments, however, failed because of the difficulty of making the aspheric surfaces. Schwarzschild's equations for the two mirror aplanat may be used to form the basis of the design for a two mirror reflecting microscope objective and instruments based on them have been designed and constructed by C.R. Burch.[58, 639] The figure represents the Schwarzschild type of objective.



The paraxial radii of the large and small mirrors are ρ and R respectively, separated by distance ε.

The image distance from the large mirror is m.

Taking an objective with infinite tube length.

The Schwarzschild equation for the large mirror is

$$\frac{1}{r} = \frac{t}{\varepsilon} + \frac{1}{m}\,\frac{\left(1-\frac{t}{\varepsilon}\right)^{\frac{1}{1-\varepsilon}}}{(1-t)^{\frac{\varepsilon}{1-\varepsilon}}} \qquad - 30$$

where $t = \sin^2\frac{u}{2}$ — 31

The incidence point of this ray on the small mirror may be expressed

$$\left.\begin{aligned} x &= \varepsilon - (1-t)\left[\frac{r(\varepsilon-2t)+t}{\varepsilon-rt}\right] \\ y &= 2t^{\frac{1}{2}}(1-t)^{\frac{1}{2}} = \sin u \end{aligned}\right\} \qquad - 32$$

and the incidence angle i^1 of this ray on the large mirror is given by

$$\tan i' = \frac{\varepsilon - r + 1}{\varepsilon - r}\left(\frac{t}{1-t}\right)^{\frac{1}{2}} \qquad - 33$$

The primary astigmatism is given by

$$\frac{2-\varepsilon}{2m}\,\theta^2 \qquad - 34$$

and the fraction of N.A. shadowed = $\frac{1}{m-\varepsilon}$ or $m\cdot\varepsilon$ — 35

Equations 30 and 31 may be written as a series:

$$x = m + \left[\frac{1-m}{\varepsilon} - 1\right]\frac{y^2}{4m} - \left\{\frac{(1-m)^2}{\varepsilon} - \frac{1-m}{\varepsilon} + \frac{1}{2\varepsilon}\right\}\frac{y^4}{16m^3}$$
$$+ \left[2\left(\frac{1-m}{\varepsilon}\right)^3 - 2\left(\frac{1-m}{\varepsilon}\right)^2 + \frac{3}{\varepsilon}\left(\frac{1-m}{\varepsilon}\right) - \frac{1+\varepsilon}{6\varepsilon^2}\right]\frac{y^6}{64m^5} \qquad \text{— 36}$$

disregarding terms above y^6 the non-elliptic part of this is

$$\frac{-m}{128\varepsilon^2}\left[\frac{1+\varepsilon}{3} + \frac{\varepsilon}{1-m-\varepsilon}\right]\left(\frac{y}{m}\right)^6 \qquad \text{— 37}$$

while the remainder is an ellipse of paraxial radius

$$\rho = \frac{2m\varepsilon}{m+\varepsilon-1} \qquad \text{— 38}$$

the eccentricity being given by

$$e'^2 = \left(\frac{1-m+\varepsilon}{1-m-\varepsilon}\right)^2 + \frac{2\varepsilon^2}{1-m-\varepsilon} = e_0'^2 - \frac{1}{4\varepsilon}\left(\frac{\rho}{m}\right)^3 \qquad \text{— 39}$$

e_0 is eccentricity of the ellipse with foci object and paraxial image point.

for the small mirror $x = A + By^2 + Cy^4 + Dy^6 + Ey^8$

$$\left.\begin{array}{l} \text{where } A = m - \varepsilon, \; B = \dfrac{1-m}{4\varepsilon}, \; C = -\dfrac{1}{8}\dfrac{m}{4\varepsilon} \\ D = -\dfrac{1}{96}\dfrac{1+4\varepsilon}{\varepsilon}\dfrac{m}{4\varepsilon}, \; E = -\dfrac{1}{1536}\dfrac{2+11\varepsilon+30\varepsilon^2}{\varepsilon^2}\dfrac{m}{4\varepsilon} \end{array}\right\} \qquad \text{— 40}$$

This is approximately an ellipse of paraxial radius

$$R = \frac{2\varepsilon}{m-1} \qquad \text{— 41}$$

and of eccentricity e given by $e^2 = 1 + \dfrac{2\varepsilon^2 m}{(1-m)^3}$ — 42

The non-elliptic part of the sixth power coefficient is

$$D - \frac{2C^2}{B} \qquad \text{— 43}$$

If both mirrors are made spherical e^1 and e re zero. This gives ε = 2 and the system is anastigmatic. As previously seen the system is monocentric with $m = 2 + \sqrt{5}$, $\rho = 1 + \sqrt{5}$, $R = \sqrt{5} - 1$ The obstruction ratio, however, is too high $1/\sqrt{5}$. Retaining anastigmatism and setting $\epsilon = 2$, except in the previous case both mirrors must be asphericed. If the system be aplanatic only, putting e = zero gives $\epsilon^2 = \frac{(m-1)^2}{2m}$ then m may be selected to give as small an obstruction ratio $\frac{1}{m-\epsilon}$ as required. Alternatively e' = zero, but this leads to a higher astigmatic coefficient for a given obstruction ratio. The small mirror may be spherical in objectives of small obstruction ratio for NA up to .65 and in the special case of the sphere cardioid, pair up to NA 1, used in aplanatic dark ground condensers usually in approximation as two spheres. Burch has made two such objectives, one of NA .58 and a second of NA .65. The visual performance is comparable with refractors of similar N.A.

4. Paraboloid Field Correctors.

From the twin mirror and mirror plate systems which have been considered, it may be seen that there are nearly no aplanatic systems which

. . . .

require a parabolic primary mirror. This means that in astronomy the primary mirror cannot be used alone, in such systems, as a viewing device. From the other point of view, parabolic mirrors, and this includes most of the very large ones, cannot be used over large field angles.

To overcome this defect field correctors have been designed by Ross consisting of spherical lenses for the 100" and 200" telescopes. More recently C.G. Wynne[40] has investigated this subject including the use of aspheric corrector plates. He shows, by considering the Seidel sums of a system of a parabolic mirror and doublet lens, that the spherical aberration can only be corrected simultaneously with coma and astigmatism by a system of thin lenses in contact in the converging beam if their combined power is such as to give, in combination with the primary mirror, an afocal system.

For a pair of aspheric plates in the converging beam, the total uncorrected spherical aberration (coma and astigmatism having been corrected) depends on the distance from the mirror of the point midway between the plates, and becomes smaller as this distance approaches the focal length of the mirror. The spherical aberration coefficients of the individual surfaces are of opposite signs and increase in magnitude as the separation between the plates is reduced.

The Seidel aberrations for a system of surfaces and the conditions for the removal are

Spherical aberration	$S_1 = \Sigma A = 0$
Coma	$S_2 = \Sigma AB = 0$
Astigmatism	$S_3 = \Sigma AB^2 = 0$
Field curvature	$S_4 = \Sigma P = 0$
Distortion	$S_5 = \Sigma B(AB^2 + P) = 0$

The quantity B contains a factor E which is changed when the Stop position is moved.

Suppose the Stop is moved to a new position giving a change $E \rightarrow E - \delta E$.

Then denoting the new values of the aberration coefficients by S_1^{1} S_2^{1} etc.

$$S_1^{1} = S_1$$
$$S_2^{1} = S_2 + S_1 \delta E$$
$$S_3^{1} = S_3 + 2S_2 \delta E + S_1(\delta E)^2$$
$$S_4^{1} = S_4$$

$$S_5' = S_5 + (S_4 + 3S_3)\delta E + 3S_2(\delta E)^2 + S_1(\delta E)^3$$

For a parabolic mirror with radius of curvature at the vertex r_m and semi-aperture h, the spherical aberration for an infinite object distance is zero, so that the coma coefficient is independent of stop position. Its value $= -\frac{2h^2}{r_m^2}$. The astigmatism coefficient varies with stop position and is zero for a stop at distance $\frac{1}{2}r_m$ in front of the mirror.

If A_1 A_2 be the Seidel spherical aberration of the two plates at distances d_1, $d_1 + d_2$ from the mirror then the conditions for the removal of coma and astigmatism from the whole system are

$$A_1E_1 + A_2E_2 = \frac{2h^2}{r_m}$$
$$A_1E_1^2 + A_2E_2^2 = 0$$

Thus the spherical aberration of the system is given by

$$A_1 + A_2 = \frac{2h^2}{r_m^2} \cdot \frac{E_1 + E_2}{E_1 E_2}$$

which can only be zero if E_1 and E_2 have opposite signs. Putting in the values for E_1 and E_2

$$A_1 + A_2 = \frac{8h^4(r_m - 2d_1 - d_2)}{r_m^4}$$
$$= 4S(1 - D_1 - \tfrac{1}{2}D_2)$$

where S is the spherical aberration of a spherical mirror radius r_m and D the ratio of $\frac{2d}{r_m}$.

A similar treatment of the case of two Schmidt plates shows that the spherical aberration, coma and astigmatism may be corrected by the use of two plates so placed that the sum of their distances from the mirror is twice the focal length.

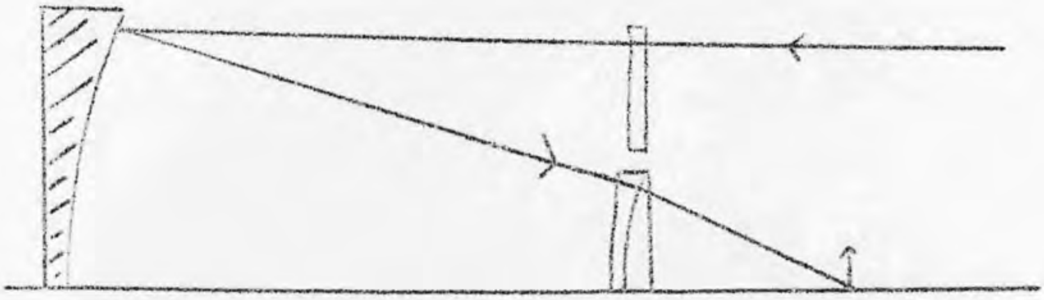
With one plate in the converging beam and one in the parallel beam, the latter must have a distance from the mirror of more than twice the focal length, this distance increasing as the second plate is moved nearer the focal plane.

The length of all such systems therefore is such as to introduce considerable vignetting unless the plates are very large.

It is necessary therefore to have three or more correcting elements to give an image free of spherical aberration, coma, astigmatism, curvature and chromatic errors.

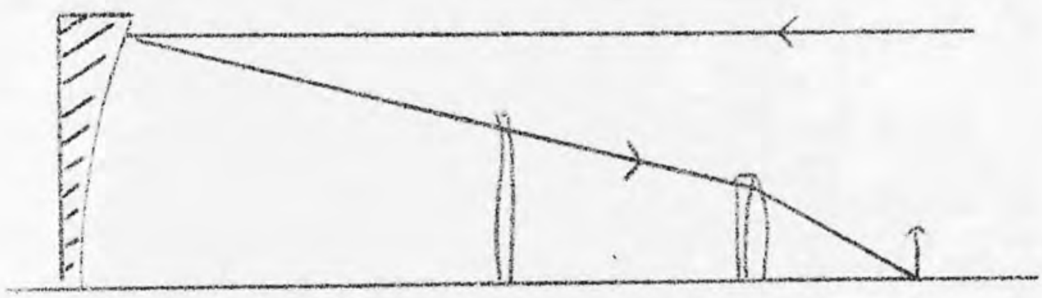
Wynne goes on to discuss by similar methods a number of three-element systems.

1. The first suggested by J.G. Baker, consisting of a doublet lens and a figured plate.



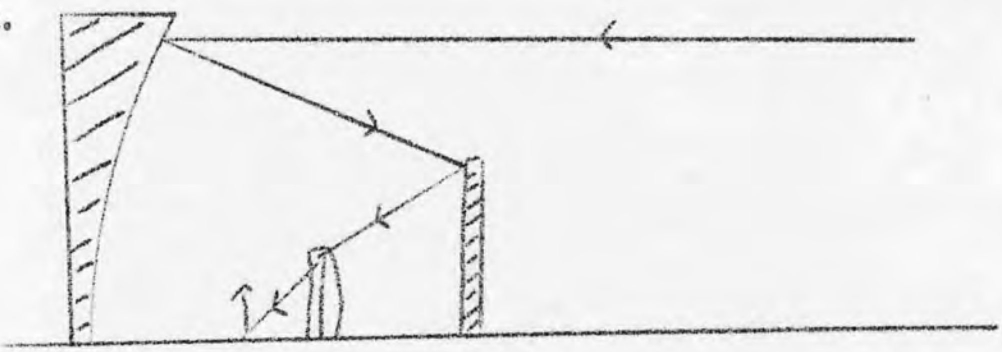
The disadvantages are the difficulties in making such a large plate, and also the vignetting produced.

2. A correction plate between the mirror and lens.



This type of design gives best results when the lens and plate are far apart. Some advantage in compactness would result from replacing the plate by a figured mirror.

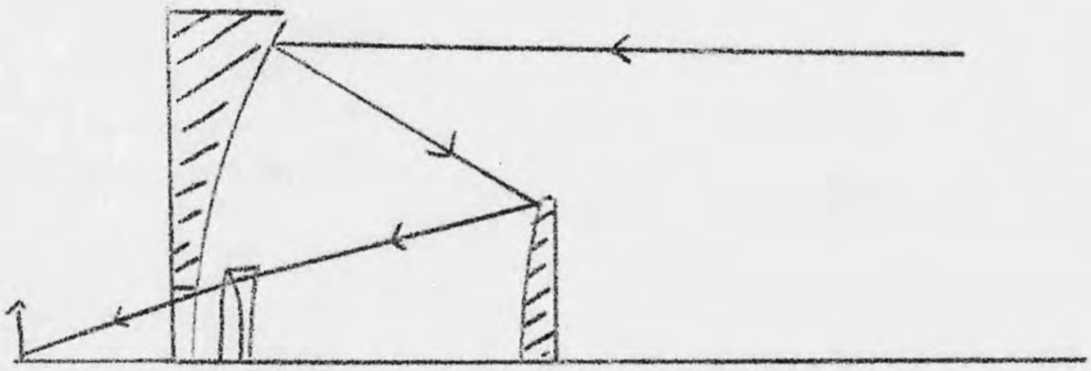
2a.



The field curvature may be eliminated by a convex secondary mirror with an afocal lens pair.

3.

3.



Four examples have been computed. One each of 1. and 2. and two of 3. in the fourth example the secondary mirror is nearly spherical, and small changes would give an exactly spherical arc, which is not any easier to make but less sensitive to centering errors.

Part IV. General Aspheric Design and Ray Tracing.

First order design.

Differential correction.

Methods for obtaining axial stigmatism.

Aplanatism.

Ray Tracing.

Conclusion.

First order design.

In part III the first order design of the Baker Schmidt Cassegrain cameras, was described by the use of the method of plate diagram analysis. The method is, of course, a means of describing the Seidel aberrations in terms which (at least according to C.R. Burch[41]) may be more easily visualised, although the method of analysis by H.H. Hopkins[42] as applied first to spherical surfaces and extended to aspheric surfaces may be considered to have equal clarity without the need of the analogue. However, Hopkins is primarily concerned with aberrations of spherical surfaces, and specifically does not include lens design, whereas Burch is mainly interested in the use of the Seidel aberrations in aspheric design.

Burch has considered the conditons for anastigmatism in a four plate system. Suppose a system of four plates is represented in the see-saw diagram by m_1, m_2, m_3 m_4, separated by distances a, b and c. Then the necessary and sufficient

condition for anastigmatism is that the total weight, and the first and second moments all be zero. This is satisfied if the strengths are in the ratio

$$bc(b + c), \quad -c(a + b)(a + b + c), \quad a(b + c)(a + b + c),$$

$$-ab(a + b)$$

A system of two surfaces is in general a four plate system, and this result may be applied to it. Consider a sphere of radius r, refractive index N^1, immersed in index N. An image is produced distance v from the pole in N space, by an object distance u from the pole in N^1 space.

Then the strength of the plate required at the centre of curvature in N space is given by

Retardation at height h

$$= \frac{h^4}{8r^3}(N'-N)\left(\frac{N}{N'}\right)^2\left(\frac{u}{u-r}\right)^2\left(\frac{Nu-(N+N')r}{Nu}\right)$$

$$= \frac{h^4}{8r^3}(N'-N)\left(\frac{N}{N'}\right)^2\left(\frac{v}{v-r}\right)^2\left(\frac{Nv-(N+N')r}{Nv}\right)$$

In a two surface system, let the first surface be a figured sphere of radius ρ, refractive index N^1 immersed in N, and let the second surface distance d inside the first be an asphere of radius $-R$, index N^1 immersed in N. The position of the object is specified by means of the intermediate image as this is easily referred to either surface, and let the intermediate image be L in N^1 space inside the ρ asphere.

The strength of the ρ plate

$$= \frac{(N'-N)}{8\rho^3}h'^4\left(\frac{N'}{N}\right)^2\left(\frac{L}{L-\rho}\right)^2\left(\frac{L-\left(1+\frac{N}{N'}\right)\rho}{L}\right)$$

of the R plate

$$= -\frac{(N'-N)}{8R^3}h'^4\left(\frac{N'}{N}\right)^2\left(\frac{L-d}{R+d-L}\right)^2\left(\frac{L-d-\left(1+\frac{N}{N'}\right)R}{L-d}\right)$$

To create a star space in which to construct the see-saw diagram, a perfect anastigmat of unit focal length is placed such that its posterior focus is at the intermediate image.

The surfaces and para-centres are then imaged into star space giving

$$a = \frac{R+d-\rho}{(\rho-L)(R+d-L)} \qquad a+b = \frac{\rho}{L(\rho-L)}$$

$$b = \frac{R+d}{L(R+d-L)} \qquad b+c = \frac{R}{(L-d)(R+d-L)}$$

$$c = \frac{d}{L(L-d)} \qquad a+b+c = \frac{\rho-d}{(L-d)(\rho-L)}$$

The process of imaging the plates into star space increases the strengths of the -4th power of the magnification, that is, by the +4th power of the distance of each plate from the posterior focus of the unit focal length anastigmat.

Thus the strengths in star space of the plates are given by

$$\rho \text{ plate } \quad m_1 = \frac{H^4(N'-N)}{8\rho^3}\left(\frac{N'}{N}\right)^2 L^2 (L-\rho)^2 \left\{\frac{L-\left(1+\frac{N}{N'}\right)\rho}{L}\right\} \quad — (1)$$

$$-R \text{ plate } \quad m_2 = -\frac{H^4(N'-N)}{8R^3}\left(\frac{N'}{N}\right)^2 (L-d)^2 (L-d-R)^2 \left\{\frac{L-d-\left(1+\frac{N}{N'}\right)R}{L-d}\right\} \quad — (2)$$

H being the general height in star space.

Since m_3 and m_4, the values of the figuring on the surfaces, may be any value, the possibility of anastigmatism by figuring reduces to whether $\frac{m_1}{m_2}$ has the value demanded.

From (1) and (2)

$$\frac{m_1}{m_2} = -\frac{R^3}{\rho^3}\left(\frac{L}{L-d}\right)^2 \frac{(L-\rho)^2}{(R+d-L)^2}\left\{\frac{1-\left(1+\frac{N}{N_1}\right)\frac{\rho}{L}}{1-\left(1+\frac{N}{N^1}\right)\frac{R}{(L-d)}}\right\} \quad — (3)$$

while the ratio $\frac{-b(b+c)}{(a+b)(a+b+c)}$ is $-\frac{R}{\rho}\left(\frac{R+d}{\rho-d}\right)\frac{(L-\rho)^2}{(R+d-L)^2}$ — (4)

so that $\left(\frac{L-d}{L}\right)^2 = \frac{R^2}{\rho^2}\left(\frac{\rho-d}{R+d}\right)\left[\frac{1-\left(1+\frac{N}{N_1}\right)\frac{\rho}{L}}{1-\left(1+\frac{N}{N_1}\right)\frac{R}{L-d}}\right]$ — (5)

Then the two mirror system may be deduced as a special case, putting $N = -N^1$. (5) becomes

$$\frac{L-d}{L} = \frac{R}{\rho}\sqrt{\frac{\rho-d}{R+d}} \quad — (6)$$

and further putting the tube length infinite i.e. $L - d = \frac{R}{2}$ it may be seen that focal length must be half the distance between the mirrors or infinite as was first seen by Schwarzschild. Burch goes on to discuss the design of anastigmatic singlet lenses by the use of these formulae.

Rewriting (5) in the form

$$\frac{1}{\rho}\left\{\frac{L}{\rho} - \left(\frac{1+N}{N^1}\right)\right\}(\rho-d) = \frac{L-d}{R}\left\{\left(\frac{L-d}{R}\right) - \left(\frac{1+N}{N^1}\right)\right\} R+d \quad — (7)$$

it may be seen that this is a quadratic in L and thus there are two object distances for which a given singlet may be figured anastigmatic. For a flat fieldedanastigmat $R = \rho$ and for objects at infinity

$L = \rho \frac{N'}{(N'-N)} = \frac{\rho n}{n-1}$ on putting $\frac{N'}{N} = n$ Then (7) gives

$$L\left(L - \left(\frac{n+1}{n}\right)\left(\frac{n-1}{n}\right)L\right)(\rho - d) = (L-d)\left(L - d - \left(\frac{n+1}{n}\right)\left(\frac{n-1}{n}\right)L\right)\rho + d \quad — (8)$$

$$\left\{\frac{(L-d)}{L}\right\}\left(\frac{L-n^2 d}{L}\right) = \frac{(n-1)L - nd}{(n-1)L + nd} \quad — (9)$$

or $$\left(\frac{n\rho - (n-1)d}{n\rho}\right)\left(\frac{n\rho - (n-1)n^2 d}{n\rho}\right) = \frac{\rho - d}{\rho + d} \quad — (10)$$

Putting $x = d/\rho$

$$\left\{1 - \left(\frac{n-1}{n}\right)x\right\}\left\{1 - (n-1)nx\right\} = \frac{1-x}{1+x} \quad — (11)$$

or $$(n-1)^2 x^2 - (n-1)\left(\frac{n+1}{n}\right)x + 2 - (n-1)\left(n + \frac{1}{n}\right) = 0 \quad — (12)$$

Solving for x

$$x = \frac{n+1}{2n(n-1)} \pm \frac{1}{2(n-1)}\left[\left(\frac{n-1}{n}\right)^2 - 4\left\{1 - n(n-1)\right\}\right]^{\frac{1}{2}} \quad — (13)$$

Two real values of x exist if $\frac{n-1}{2n} > 1 - n(n-1)$

i.e. for x greater than 1.602. For the critical value $x = \frac{n+1}{2n(n-1)} \simeq 1.35$

The product of the two roots, $2 - (n - 1)(n + \frac{1}{n})$, is positive for the usual range of n so that both roots will be positive. As d must be positive no flat field anastigmat exists with a concave front face for the usual range of n. For a convex front face putting $n(n - 1) = 1$ so that $x = 1$ or n the lens assumes a special form, with $x = 1$ the intermediate image is in the Amici position with respect to the second surface, Hence, only that surface need be figured and $n = 1.618$. When x = n the second surface is concentric with the final image. Burch finally considers the possibility of achromatising for magnification by suitably placing the diaphragm.

Differential Correction.

The plate diagram therefore gives a method of determining the aspheric profiles which a system must have to obtain a required standard of definition over a finite field to the approximation of the third order. As a rule, of course, higher aberrations must be taken into account, and there is at present no similar general method for dealing with them. The differential methods of lens correction of M'Aulay and Cruickshank were used with great success during the Second World War for adjustment of rough lens designs and D. S. Volosov[43] has published a method of differential correction of spherical surfaces into aspherical ones, and by this means improves several aberrations without actually having to trace rays through aspheric surfaces. He states the objects of his discourse as follows:

1. To find from the analysis of the structure of axial as well as oblique pencils, which spherical surface of the system is to be shaped into a non-spherical one.

2. To find from the analysis mentioned above the shape of the non-spherical surface without having recourse to any supplementary ray tracing through non-spherical surfaces.

3. To evaluate, though approximately, the influence of the introduced non-spherical surface upon the remaining aberrations of the system.

The equation of an aspheric surface may be given as

$$f(x) = y^2 = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \qquad \text{— (14)}$$

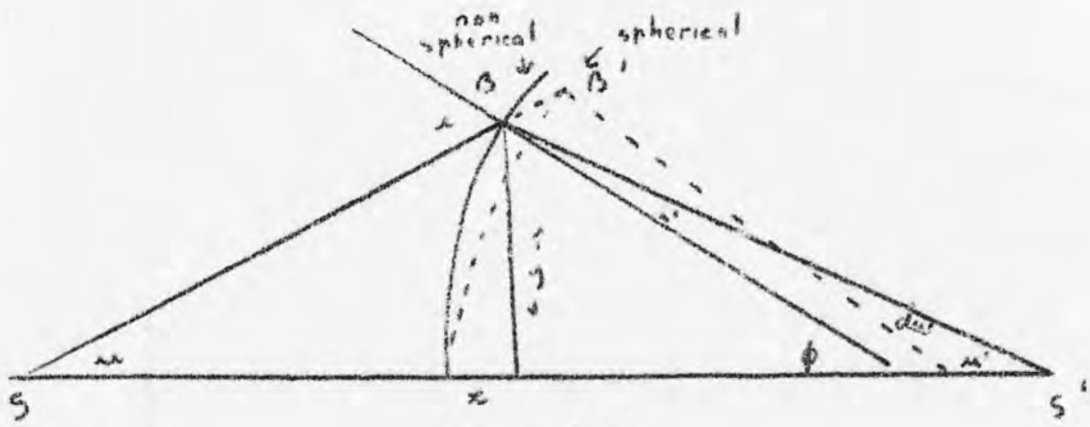
$$\left.\begin{array}{l} a_1 = 2r \\ a_2 = -(1+b) \end{array}\right\} \qquad \text{— (15)}$$

where b is the deformation coefficient of Schwarzschild.

(for conic sections $b = -e^2$)

The coefficients a_1 and a_2 are parameters for the correction of aberrations in the Seidel region.

Consider the figure which represents a ray passing through an aspheric surface.



$$y = (s - x)\tan u \qquad — (16)$$

Let this ray pass through the system and reach some place in the object space the ordinate of which is l^1. By taking a system of spheric or aspheric surfaces of the second order corrected for third order aberrations, the two coefficients a_1 and a_2 may be found. Then variations of the abscissa $l^1(dl^1)$ in terms of the variations of the coefficients a_3 a_4 of the non-spherical surface are determined $\frac{dl'}{da_K}$ for several rays oblique and axial, and from these results the inverse problem, assuming any desired changes in l^1 may be solved. Suppose the variations in the image space be denoted by

$$D_m = \frac{\partial l'}{\partial a_m}, \qquad D_n = \frac{\partial l'}{\partial a_n}, \text{ and so on.}$$

Then

$$dl' = D_1 da_1 + D_2 da_2 + D_3 da_3 \text{ \&c}$$

In the proposed method the most laborious part is the determination of two initial partials, say, D_k and D_{k-1}. Once these are determined the remainder may be found from

$$D_{n+1} = 2x_0 D_n - x_0^2 D_{n-1} \qquad (17)$$

Volosov proposed five methods for determining the two initial partials:-

1. A method in which the non-aspherical surface is the last one.

2. Application of the laws of optics of colinear relationships to the meridinal pencils.

3. A simplified method where the deformations are small.

4. A method of immediate determination of the partials D_{k-1} and D_k based upon the application of tables describing how changes of the parameters of a system influence the aberrations; the tables being composed for systems of spherical surfaces.

Volosov considers the last method to be the best. Method 2 is really the method of Cruickshank and M'Aulay, and in addition the advent of electronic computing has removed the difficulties of aspheric ray tracing, at least for those who have access to a machine.

Methods for obtaining axial stigmatism.

In some systems involving aspheric surfaces it is sufficient to ensure that the image is axially stigmatic, the geometry of the system ensuring the automatic removal or near removal of the off axis aberrations. The Schmidt system is the obvious example and methods for computing the corrector plate, generally depending on the optical path difference method have been described in part II. Some other more general methods have been described. M. Herzberger and H.O. Hoadley[44] desribe a method for the calculation of the aspheric correcting surface for an optical system in which the surface is adjacent to object of image and an extension for cases where the aspheric surface is in the interior, the rays being refracted to match a given non-spherical wave surface.

The first part of the computation is the tracing of a series of rays from the object point up to the final correcting surface of the form

$$z = \alpha\eta^2 + \beta\eta^4 + \gamma\eta^6 \text{ etc} \qquad \text{— (18)}$$

The power of the surface and therefore the coefficient α

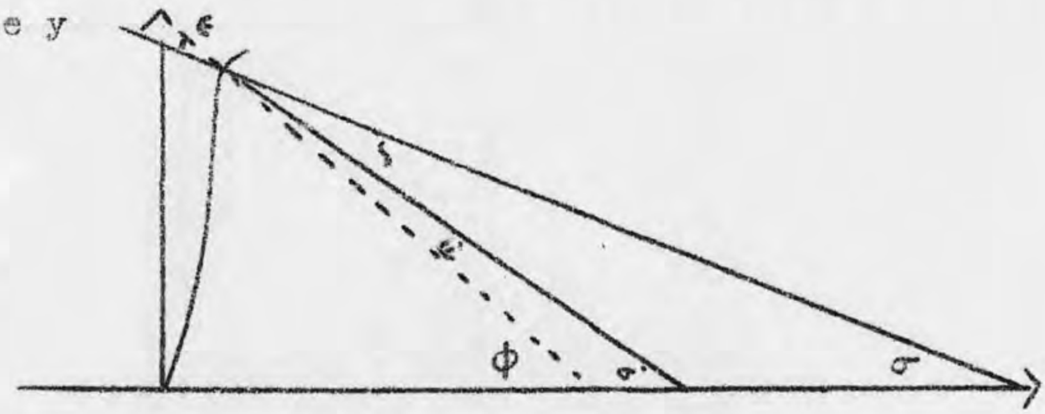
is determined by the necessity of bringing the paraxial ray to the focal point.

From the well known formula $\frac{n'}{s'_o} - \frac{n}{s_o} = \frac{n'-n}{r_o}$ — (19)

ro may be determined and hence α since $\alpha = \frac{1}{2r_o}$

The first approximation to the correcting surface is now assumed, and the intersection points of the rays with the surface are calculated. A suitable surface would usually be a sphere radius r_o or for a Schmidt camera a plane. Let the coordinate of the intersection point be z_1 and y_1 and σ' the desired angle that the refracted ray makes with the axis.

The figure represents a non-spheric refracting surface y



Then $\tan \sigma' = \frac{y_1}{s'-z_1}$ — (20)

also $\delta = \sigma' - \sigma = \epsilon - \epsilon'$ $\quad n' \sin \epsilon' = n \sin \epsilon$

$$\sin t = \sin \delta \frac{n'}{\sqrt{n^2 + n'^2 - 2nn' \cos \delta}}$$ — (21)

The required slope angle ϕ is given by

$$\phi = \sigma + \epsilon = \sigma' + \epsilon' \qquad -(21)$$

The actual angle ϕ_1, will in general not agree with the value and may be determined from equation (18)

Since $\tan \phi_1 = \dfrac{dz_1}{dy_1} = 2y_1(\alpha + 2\beta y_1^2 + 3\gamma y_1^4 \text{ etc})$ $\qquad -(23)$

A surface which will have the required ϕ S at the specified value y_1 can be calculated..

Suppose several rays have been traced, and are denoted by a, b, c, etc. The coefficients of such a surface can be found from the following set of simultaneous linear equations derived from equation (23).

$$\left.\begin{aligned} 2\beta_1 + 3\gamma_1 y_{a_1}^2 + 4\delta_1 y_{a_1}^4 + \text{etc} &= R_{a_1} \\ 2\beta_1 + 3\gamma_1 y_{b_1}^2 + 4\delta_1 y_{b_1}^4 + \text{etc} &= R_{b_1} \\ 2\beta_1 + 3\gamma_1 y_{c_1}^2 + 4\delta_1 y_{c_1}^4 + \text{etc} &= R_{c_1} \end{aligned}\right\} \qquad -(24)$$

Thus the number of coefficients in the surface which can be determined is equal to the number of rays used. Rays are then traced through the surface thus determined and new intersection points y_2, Z_2, determined. The whole procedure being repeated until the refracted rays pass through the desired point with the required accuracy.

In principle the calculation of an interior correcting surface is similar, but a modification is needed as the refracted rays will in most cases not be required to meet in a point and the desired σ' cannot be found from equation (20). The wave surface required is described by giving the slope of the emerging rays. The normals to the wave surface, as a function of the intersection heights of the rays in the tangent plane of the correcting surface. To find this a number of rays are traced backwards through the system and by means of a set of equations similar to (24) the slopes of the rays are approximated by a function of the form

$$\tan \sigma' = Ah' + Bh'^3 + Ch'^5 + \text{etc} \qquad \text{— (25)}$$

and it is now required to find a value of σ' and h' such that they fulfil the equations

$$\tan \sigma' = Ah' + Bh'^3 + Ch'^5 + \text{etc} \qquad \text{— (26)a}$$

and

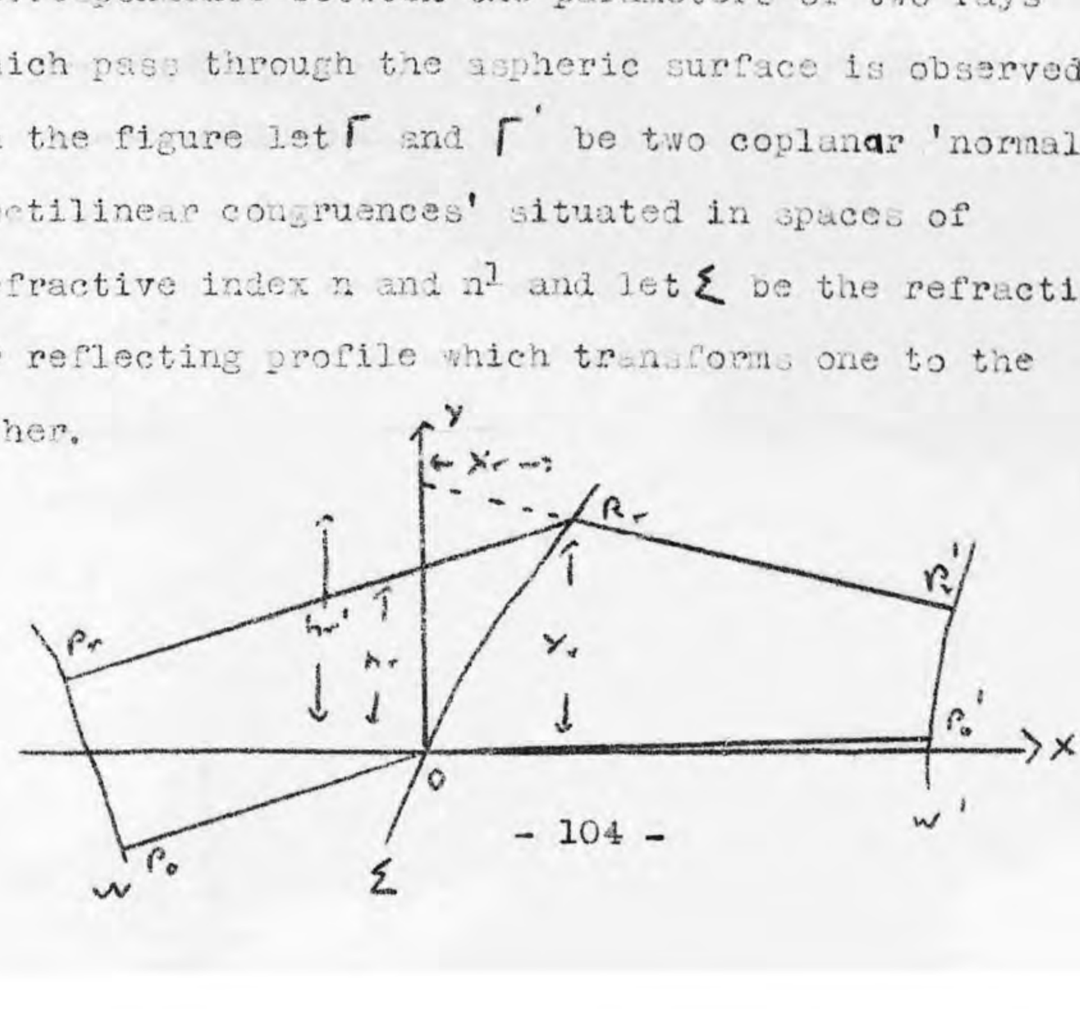
$$\delta = h' - y - z \tan \sigma' = 0 \qquad \text{— (26)b}$$

The procedure then continues in a similar manner as for the simpler case and equation (20) may be considered a simple case of equations (26) in which the refracted rays all pass through a point.

The paper also gives a ray tracing procedure which will be discussed later. W. S. Preddy and E. Wolf,[45] and later E. Wolf[46] alone, have written papers on the axial correction of systems by means of aspheric surfaces. The paper by Wolf is an extension of the earlier one and treats the subject more generally.

Wolf first states a theorem A "If two rays h_1 and h_2 intersect a wave front W of Γ in P_1 and P_2 and OY in Q_1 and Q then the optical path difference $P_2Q_2 - P_1Q_1$ is given by $n \int_{h_1}^{h_2} \sin w dh$" This result is proved by Preddy and Wolf.

Next, with the use of this theorem the correspondence between the parameters of two rays which pass through the aspheric surface is observed. In the figure let Γ and Γ' be two coplanar 'normal rectilinear congruences' situated in spaces of refractive index n and n^1 and let Σ be the refracting or reflecting profile which transforms one to the other.



Let h_r and h_r^1 be any two corresponding rays in this transformation, and let W and W^1 denote the wave fronts of Γ and Γ'. Let Pr and Qr be the points of intersection of the ray hr with W and OY respectively and similarly for the dashed letters. Let $x_r y_r$ denote the coordinates of the point Rr. Then after consideration of this figure and the optical paths involved, use of theorem A results in the following equation which is stated then as Theorem B.

In the transformation Γ to Γ', the parameters of corresponding rays when referred to any set of rectangular axes with their origin placed at a point of the transforming profile satisfy the following relation;

$$\sin(\omega_r - \omega_r')\left\{n\int_0^{h_r} \sin\omega\, dh - n'\int_0^{h_r'} \sin\omega'\, dh'\right\}$$

$$+ (n\cos\omega_r' - n'\cos\omega_r)(h_r' - h_r) = 0$$

Suppose the equation of the first congruence is given by

$$w = w(hr)$$

In practice it is often more convenient to use a polynomial approximation of the form

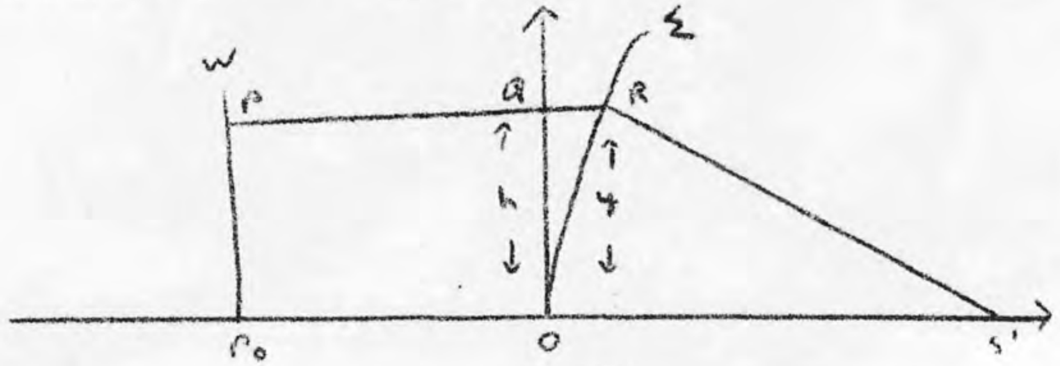
$$\sin W = a_1 h + {}_3 h^3 +$$

which may be obtained by tracing a number of rays into the space which precedes Σ , and computing for each ray the quantities w and h and fitting a polynomial.

In the space which follows Σ the rays are traced backwards from the image point and a similar equation may be obtained.

The function of the surface Σ is to transform Γ into Γ' and Theorem B permits the calculation of the profile. The proof of Theorem B then follows somewhat similar lines to that proposed by Herzberger and Hoadly, being an iterative method.

In the special case where the corrector is the first or last surface, the profile calculation is considerably simplified. In this case the surface Σ is the last before S' the image. The figure represents this.



Since the optical path from the wave front W to S^1 is constant

$$PQ + p x \sec\omega + n'[(s'-x)^2 + (h + x\tan\omega)^2]^{\frac{1}{2}} = [P_0 O] + n's'$$

From Theorem A

$$n\int_0^h \sin\omega\, dh + n x \sec\omega + n'[(s'-x)^2 + (h + x\tan\omega)^2]^{\frac{1}{2}} - n's' = 0$$

or $Ax^2 + 2Bx\cos\omega + C\cos^2\omega = 0$

where $A = n^2 - n'^2$

$$B = n'\int_0^h \sin\omega\, dh - nn's' + n'^2(s'\cos\omega - h\sin\omega)$$

$$C = \left(n\int_0^h \sin\omega\, dh\right)\left(n\int_0^h \sin\omega\, dh - 2n's'\right) - n'^2 h^2$$

Solving and using $y = h + x\tan\omega$

$$x \text{ and } y = -\left(\frac{\cos\omega}{A}\right)\left[B \pm (B^2 - AC)^{\frac{1}{2}}\right] + \text{ and } h$$

The integral which occurs in B and C may be evaluated by tracing rays and applying Theorem A.

A. Warmisham[47] describes a dioptric Schmidt in which converging power is provided by three doublets and the spherical aberration corrected by an aspheric plate, the image lying on a curved field as against the flat one normally considered as essential in lenses of this type. However, the aperture is raised to f/.7 and an interesting method of calculating the plate profile is used. The method is as follows.

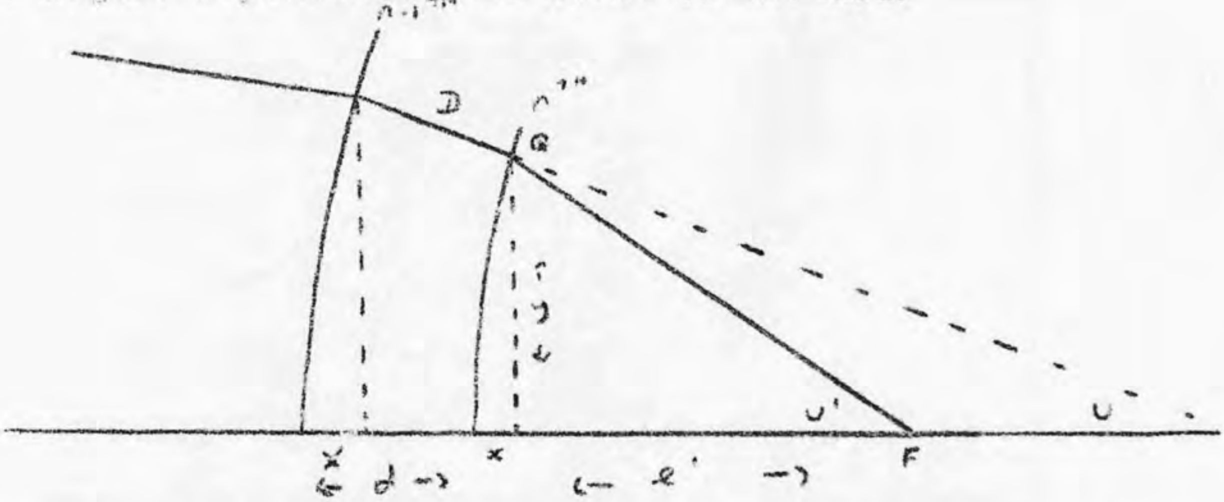
Suppose the equation of the plate is of the form

$$X = a_2y^2 + a_4y^4 + a_6y^6 + a_8y^8$$

In the figure the last aspherical surface is the n^{th} surface and the inner or $n{-}1^{th}$ the first plate surface which is spherical.

A paraxial ray is traced through the system, and in addition a zonal ray can be brought through to the $n{-}1^{th}$ surace.

The figure represents this state of affairs.



Q(x,y) is the unknown point where the ray meets the aspheric surface and D the unknown path length.

U^1 is also unknown.

From the figure $x = X - d + D\cos U$ and $y = Y - D \sin U$

Therefore $QF^2 = (Y - D \sin U)^2 + (Z - D \cos U)^2$

where $Z = l^1 - X + d$

Equating the zonal and paraxial optical paths

$$p = P + ND + \left[(Y - D \sin U)^2 + (Z - D \cos U)^2\right]^{\frac{1}{2}}$$

O

On solving for D, x and y may be found and from these $\tan U^{1} = \frac{y}{\ell' - x}$

and then $\tan I = \frac{\sin(U' - U)}{\cos(U' - U) - N}$ also $\frac{dy}{dx} = \tan(I + U)$

If the value of U^{1} gives a satisfactory sine condition, the coefficients may be calculated.

a_2 is known from the paraxial equations and a_4 from the first order aberration.

Hence $a_2 y^2 + a_4 y^4 = x_1$ say and differentiating

$$2a_2 y + 4a_4 y^3 = V_1 \text{ say.}$$

It follows that $a_6 y^6$ and $a_8 y^8 = x - x_1 = q$

$$\text{and } 6a_6 y^5 + 8a_8 y^7 = \frac{dx}{dy} - V_1 = V \text{ say}$$

Thus a_6 and a_8 may be calculated

$$a_6 = \frac{8q - Vy}{2y^6} \qquad a_8 = \frac{Vy - 6q}{2y^8}$$

and the equation of the plate profile is completed. If the sine condition is not good enough the plate is bent slightly to correct for this. A zonal ray of .83 full aperture gives satisfactory results for apertures not exceeding f/1.0.

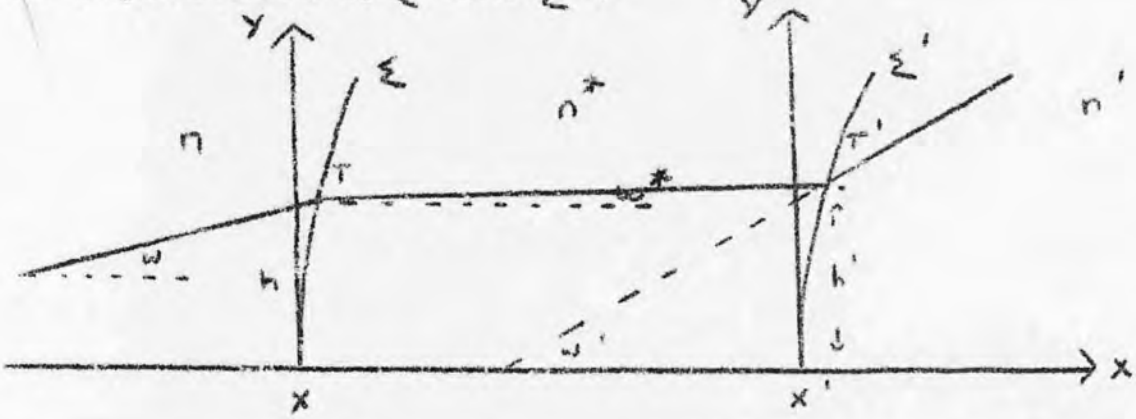
<u>Aplanatism.</u>

In systems which do not depend on geometrical considerations to secure small off axis aberrations, and this includes the majority, axial stigmatism is not enough, and as was seen in part 1, Schwarzschild and Chretien were able to produce aplanatic two mirror systems and Linnemann an aplanatic lens, by introducing the additional sine condition, although the sine condition, of course, does not obtain the release from coma of all orders.

L.C. Martin[48] has worked out the data for two aplanat singlet lenses using Chretien's theory of the aplanatic telescope, the statement of optical path equality and the sine condition forming the basis of the argument. Two numerical examples are given, one an aplanat, the other an anastigmat with zero petzval sum, which follows Burch's example given earlier. It is interesting to see that although the lens is spherically corrected for a very large aperture, owing to the high order coma and astigmatism the correction is only reasonably good over a semi-field of $7\frac{1}{2}^{\circ}$ and at an aperture of f/5. These results are determined by ray tracing.

G.D. Wassermann and E. Wolf[49] have given methods whereby the aplanatism of a system may be ensured by the asphERization of two of the surfaces.

In the figure suppose the two aspherised surfaces are represented by Σ and Σ'.



The other quantities are similar to those used by Wolf in the paper previously discussed.

Suppose a normal rectilinear congruence Γ is transformed by the two surfaces into Γ', Γ can be specified by the relations,

$$W = W(t) \qquad h = h(t) \qquad -(27)$$

If the object point is at a finite distance t is chosen as

$$= \sin H \qquad -(28)$$

where H is the angle made by the ray in the object space.

If the object is at infinity it is put equal to H, the distance of the ray from the axis.

The relations (27) can rarely be obtained and Wolf and Wassermann favour the forming of a table based on a ray trace as against a polynomial approximation.

By tracing rays backwards from the object Γ' can be specified by similar means.

$$W^1 = W^1(\lambda') \qquad H^1 = h^1(\lambda') \qquad -(29)$$

The sine relation is given by

$$\frac{\lambda}{\lambda'} = \text{constant} \qquad -(29)b$$

If T and T^1 in the figure are the points (x, y) and (x^1, y^1), Snell's law gives

$$n\left(\cos W \frac{dx}{dt} + \sin W \frac{dy}{dt}\right) = n^*\left(\cos W^* \frac{dx}{dt} + \sin W^* \frac{dy}{dy}\right) \qquad -(30)$$

where $\sin W^* = \frac{Ry}{R}$ $\qquad \cos W^* = \frac{Rx}{R}$ $\qquad -(31)$

where $Rx = x^1 - x + d$, $Ry = y^1 - y$ $\qquad R^2 = Rx^2 + Ry^2$ $\qquad -(32)$

$$y = h + x \tan w \qquad -(33)$$

$$y^1 = h^1 + x^1 \tan w^1 \qquad -(34)$$

Substituting for $\cos W^*$ and $\sin W^*$ and $\frac{dy}{dt}$ from (33) (31) and (30)

$$\frac{dx}{dt} = -\left[\frac{(n^* Rx - nR\cos W)}{(n^* Ry - nR\sin W)} + \tan W\right]^{-1}\left[\frac{dh}{dt} + x\frac{d}{dt}\tan W\right] \qquad -(35)$$

Similarly $\frac{dx^1}{dt} = -\left[\frac{n^* Rx - n' R\cos W'}{n^* Ry - n' R\sin W'} + \tan W'\right]^{-1}\left[\frac{dh'}{dt} + x'\frac{d}{dt}\tan W'\right] \qquad -(36)$

These equations permit ξ and ξ' to be computed for by means of (32) (33) (34) y and y^1 could be eliminated from (35) and (36) to obtain two first order differential equations which could be integrated numerically. However, since y and y^1 are required it is preferable to avoid elimination and solve for unknown quantities step by step.

Wassermann and Wolf then proceed to the design of a reflecting microscope which may be summarized as follows:-

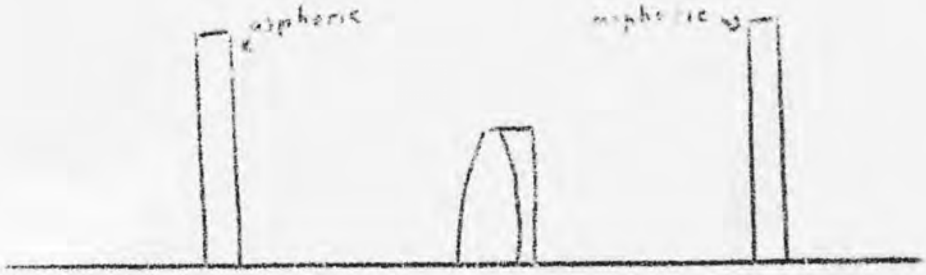
First. Rays are traced from the object point for a set of selected values of the parameter t. For each value of t the quantities w and h are determined. The values of $\frac{d}{dt}$ tan w and $\frac{dh}{dt}$ calculated by numerical differentiation.

Second. For each value of t the corresponding value of t' is calculated and the values w^1 and h^1 determined by tracing rays backwards. Hence, in a similar manner $\frac{d}{dt}$ tan w^1 and $\frac{dh'}{dt}$ are found.

Third. Equations (35), (36) are integrated numerically step by step, i.e. for each value of x and x^1 the corresponding value of y and y^1 is found from (33) and (34). In this way the two profiles are obtained in tabulated form.

In the table given, the results are compared with Schwarzschild's formulae, the maximum error being 2×10^{-6} cms in x and 1×10^{6} cms for y.

Joseph Meiron has designed a double aspheric curved field anastigmat, the object of the design the extension of the field of view of a doublet telescope objective.



The preliminary design is carried out by the third order methods of Burch and H. H. Hopkins, and Meiron says that in the design of a doublet, any spherical aberration and coma can be fully corrected. An aspheric plate, the object of which is to correct the astigmatism, must be placed away from the aperture stop, but with the stop in this position distortion will be introduced, which may be overcome by placing a second aspheric plate on the other side of the doublet. This, of course, assumes the stop to be on the doublet. In the dioptric Schmidt, as previously discussed, the stop is on the aspheric plate and at the same time the lens is giving a residual of spherical aberration. The lens is compared with a petzval lens where a slightly superior astigmatic performance is claimed, other factors, spherical aberration, coma, field curvature, being similar. The lens was tested by ray tracing on an electronic machine and hence some attempt at control of higher order aberrations could be made, the final lens, however, appeared to show few advantages over the petzval lens with which it was compared.

Ray Tracing.

The difficulty involved in tracing rays through aspheric surfaces has often been considered to be one of the greatest stumbling blocks to aspheric design and only the advent of electronic computing has made it possible to trace the large numbers of rays which seem to be required. A number of writers have given methods for aspheric tracing with the electronic machines in mind. Most of the methods are algebraic, usually iterative in solution.

T. Smith[51] in 1945 gave a method for tracing rays through an axially symmetrical optical system with aspheric refracting or reflecting surfaces. The equation of a non-spherical surface is specified by $s = f(z)$

where x, y, z are the coordinates of the surface whose pole is the origin, z the optic axis and s the subnormal.

ξ, η, ζ , are modified direction cosines, the direction cosines times the refractive index.

The ray tracing equations are given as

$$\begin{aligned} \xi_r &= \xi_{r-1} - x_r A_r \\ \eta_r &= \eta_{r-1} - y_r A_r \end{aligned} \qquad \text{Refraction}$$

$$\begin{aligned} x_{r+1} &= x_r - \xi_r D_r \\ y_{r+1} &= y_r - \eta_r D_r \end{aligned} \qquad \text{Transfer}$$

D is the distance between surfaces divided by the refractive index and

A is the generalised power at the point of incidence.

They are given by.

$$-D_r = \frac{t_r - z_r + z_{r-1}}{\zeta_r} \qquad \text{where } t_r \text{ is the axial distance}$$

$$A_r = \frac{\zeta_r - \zeta_{r-1}}{s_r}$$

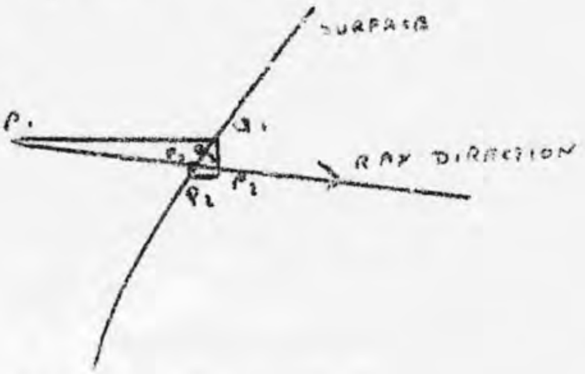
The ray trace consists in using approximate values z_r and ζ_r and putting these in the equations, and on completing the trace, improved values of z_r and ζ_r are computed from equations

$$z' = \frac{1}{2s}\left\{ x^2 + y^2 + 2\int_0^z z\,ds \right.$$

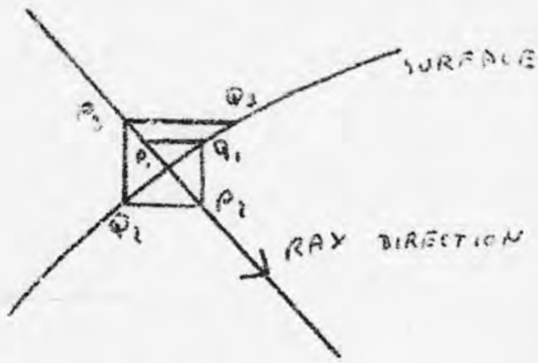
$$\zeta' = \frac{1}{2}\left\{ \zeta + \frac{w^2 - \xi^2 - \eta^2}{2s} \right\}$$

These improved values are put in the equations and the procedure repeated until the values of the coordinates and direction cosines remain unchanged to the desired number of places. If the first approximate values of z_r and ζ_r are the paraxial ones, the first round gives the paraxial properties, and the following rounds approximate to the first, second and so on orders of aberrations.

W. Weinstein,[52] in tracing rays through a reflecting microscope noticed a case in which the iteration did not converge but diverged. This led him to an investigation of the conditions for convergence of such systems and he found that the complete iteration converges when the rays are only moderately inclined to the axis and the refracting surfaces not too steep, but divergence can occur for systems of large numerical aperture. This is illustrated in the two following figures, the first showing convergence, the second divergence. P_1, Q_1, P_2, Q_2, etc. are the points of the iteration.

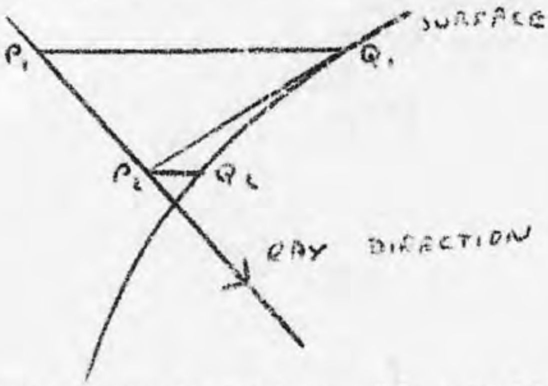


(a) Convergence



(b) Divergence.

Weinstein gives an improved iteration which always converges, represented in the figure.



(c) Convergent under all conditions.

The improved values of Z and ζ somewhat more complicated are given by

$$z^1 = \frac{2(\xi x + \eta y) - \zeta(x^2 + y^2 + z\zeta_c^2 \ldots)}{\xi x + \eta y + \zeta s}$$

$$\zeta' = \zeta \frac{\{s(N^2 - \xi^2 - \eta^2) - 3(\xi x + \eta y)}{\frac{1}{2}s(\zeta^2 + N^2 - \xi^2 - \eta^2) - 3(\xi x + \eta y)}$$

Weinstein claims that this iteration better represents the aberrations of increasing order than does that given by Smith.

Marx[53] has produced a further rearrangement in which the formulae are made linear, by using approximation in which the tertiary aberrations are neglected.

Herzberger and Hoadly[44] include a ray tracing scheme, as has been mentioned, for tracing through surfaces expressed in the form of a power series.

$$z_1 = \alpha y_1^2 + \beta y_1^4 + \gamma y_1^6 \text{ etc}$$

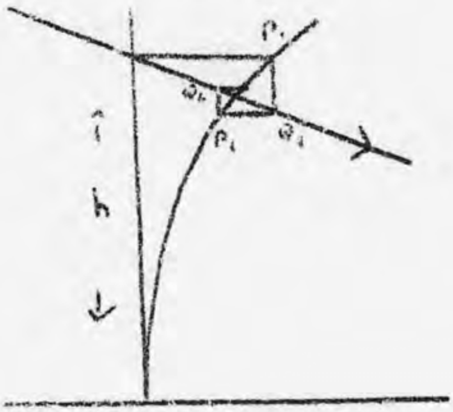
a form which is very often used in defining an aspheric surface.

The starting data are the angle σ of the incident ray, and the heighth of its intersection point in the plane tangent to the surface at the pole.

If the coordinates of the point of incidence are y, z, it may be seen that

$$y = h - z \tan \sigma$$

Now if a value of y is assumed and substituted in the first equation, a value of z may be found, which in turn is put in the second equation to find a new value of y and so on. The process is represented in the figure.



It may be seen that this system suffers from the same disadvantages for steep rays as the T. Smith method.

The computation may be shortened if the value of the rectidual Δ be calculated where

$$\Delta = y - h + z \tan \sigma$$

Δ must of course be zero if y and z are actually the point of incidence, and if Δ be found for two different values of y as explained before, the third approximation of y should be calculated by linear interpolation to make $\triangle = 0$. The iteration then converges more rapidly. When the point of incidence has been located the direction of the surface normal is found by differentiation

$$\tan\phi = \frac{dz}{dy} = y(2\alpha + 4\beta y^2 + 6\gamma y^4 \text{ etc })$$

and then the angles of incidence and refraction and the direction of the refracted ray are found. The method, of course, has not the generality of T. Smith's method, being confined to the meridional plane, but uses the surface formulae in a convenient form.

T.Y. Baker[54] in 1944 gave a method for tracing shew-rays through second degree surfaces in which spherical trigonometric formulae are used for logarithmic work. He divides the trace into four parts.

(1) determination of coordinates of the point where the ray cuts the surface.

(2) determination of the direction cosines of the normal to the surface at this point.

(3) the refraction equation.

(

(4) determination of the direction cosines of the refracted ray.

The equation of the ray is given by

$$\frac{x-\alpha}{\cos\theta} = \frac{y-\beta}{\sin\theta\sin\phi} = \frac{z-\gamma}{\sin\theta\cos\phi}$$

and the equation of the surface by

$$y^2 + z^2 = 2rx - px^2$$

where (α, β, γ) is a point on the ray, and $\cos\theta$, $\sin\theta \sin\phi$, $\sin\theta \cos\phi$ the direction cosines. From these equations a quadratic may be found to determine x.

For small values of x Baker uses an iterative method for solving. Spherical trigonometry is used to evaluate the direction cosines and a special table of logarithms is appended to assist in the evaluation of the equations involved. Baker considers that the method is at least three times as long as for an ordinary ray, and this would seem to be an understatement.

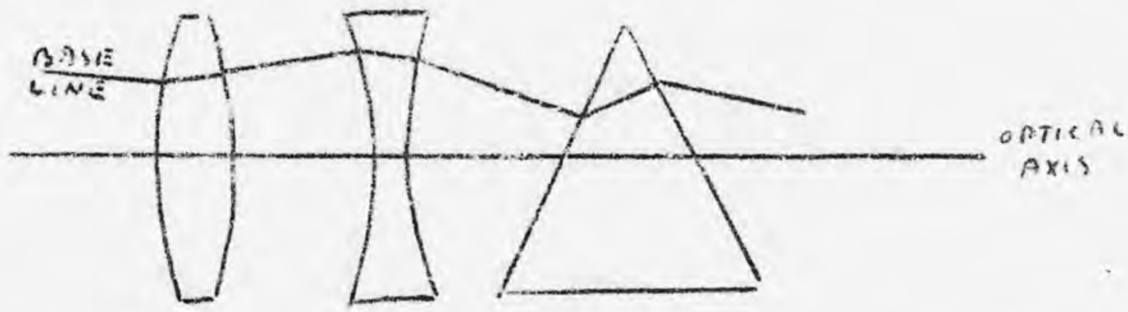
Most of the modern systems for tracing rays through aspheric surfaces are intended for use with electronic machines with their advantages and disadvantages. The advantages, of course, are the extremely high rate of calculation and the ability to make simple "either or" decisions; the disadvantages, the difficulty of using systems involving tables (this includes most of the older types of ray tracing). A machine would normally work out the value from first principles, and the fact that the machines have a technology of their own, ~~which~~ must be fully understood by the designer before attempting a computation.

For these reasons the systems evolved tend to be extremely general so that one programme may be used for all traces and, as a rule, involve direction cosines. Solution of equations is often by iterative means.

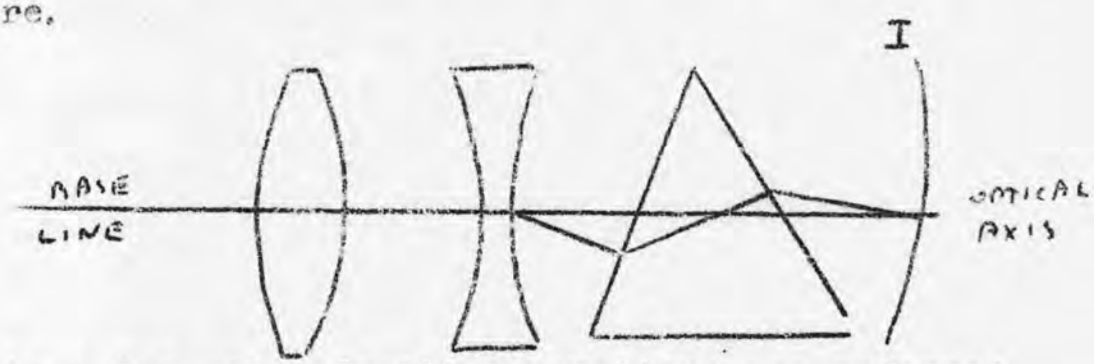
D. P. Feder[55] has given a method for calculating aspheric surfaces; it involves first finding the point of incidence on the surface using an iterative method, and second, finding the direction of the refracted ray. Vectors are used to define the problem but the equations used are set out in algebraic form.

Allen and Snyder[56] have extended

equations for use with prisms, mirrors, centred aspherics and even to uncentred spherical surfaces; of considerable interest to optical designers. Their system uses base lines surveyed through the optical system, each successive segment being formed by dropping a perpendicular to the next interface, as in the figure.



The base line may be conveniently the optical axis, and if it departs from this, the correspondence may be reintroduced by bringing into the system one or more fictitious interfaces such as I in the figure.



The transfer and refraction equations are referred to the base line, the computing scheme is algebraic, and, as in Feder's method, the determination of the point of contact iterative and possibly divergent for steep angles. The equations contain an expression for the optical path length and this gives a means of getting axial stigmatism in an aspheric system, which the authors use to correct a reflecting coma.

G. Black[57] has used the algebraic method for tracing skew rays on the Manchester machine and the basic formulae are given by him.

If an incident ray has direction cosines L, M, N, the refracted ray L^1, M^1, N^1, and the normal at the point of incidence direction cosines A, B, C, the law of refraction can be expressed.

$$\begin{vmatrix} L & M & N \\ L^1 & M^1 & N^1 \\ A & B & C \end{vmatrix} = 0$$

The cosine of the angle of incidence is given by

$$\cos I = LA + MB + NC$$

and $\Delta(n \sin I) = 0 \quad L^2 + M^2 + N^2 = L^{1^2} + M^{1^2} + N^{1^2} = 1$,

hence the direction cosines of the refracted ray are determinable. The intersection point of the refracted ray comes from the equation

$$\frac{X - x}{L'} = \frac{Y - y}{M'} = \frac{Z - z}{N'} = D$$ (D the distance along the ray between X, Y, Z, and x, y, z)

and the equation of the form of the surface. The form of the solution of these equations varies with the authors, the iterative form being favoured.

For central quadrics of the form

$$\alpha x^2 + \beta y^2 + \gamma z^2 = 1$$

the coordinates of he intersection point are $X + L^1 D_1 Y + M^1 D_1 Z + N^1 D_1$ and the parameter D may be found by substitution.

The equation of the normal at the intersection point may be written

$$\frac{X - x}{\alpha x} = \frac{Y - y}{\beta y} = \frac{Z - z}{\gamma z}$$

For paraboloids of the form

$$\alpha x^2 + \beta y^2 + 2\gamma^2 = 0$$

the equation of the normal is

$$\frac{X - x}{\alpha x} = \frac{Y - y}{\beta y} = \frac{Z - z}{\gamma}$$

The equation of the refracted ray may thus be determined.

Peltier[58] has given a similar treatment for ray tracing through quadrics of revolution, stating in his preamble, that although trigonometric methods are more suitable for meridinal rays traced through spherical surfaces, skew rays through aspheric surfaces are better traced algebraically.

The use of machines as analogues was suggested by C.W. Harris[59] who has developed a series of matrices to describe the third order aberrations. The matrices could be represented by electrical networks so that the setting up and analysing of a design could be done in a few minutes. Unfortunately, the matrices are extremely cumbersome, and an analogue computor could be more readily adapted from the third order equations themselves. A number of graphical ray trace constructions have been developed, and P.E. Lewis[60] uses a geometric construction for aspheric surfaces. Such systems can, only be a guide to the type of surface required and as most aspheric systems can readily be understood the need for a graphical idea is seldom required.

Conclusions.

Although a discussion of the methods of aspheric design does not include the practice of the manufacture of the aspheric surfaces required, the fact that such manufacture is extremely difficult and costly, must necessarily influence the design itself. Aspheric surfaces will only be used where spheric ones cannot and the number of these surfaces kept to the minimum. In a design employing spherical surfaces only, most designers will probably proceed by laying down a paraxial scheme, and then determining the Seidel coefficients for the system. By examination of the Seidel coefficients using some system such as H.H. Hopkins, an experienced designer can probably arrive at a required result fairly rapidly, the sizes of the zones being estimated by the size of the individual coefficients, and the final residuals. If a similar method is applied to the design of aspheric systems using the primary aberration method of C.R. Burch, the final design will not necessarily be better than a corresponding design of spherical surfaces, although it may have fewer surfaces.

These considerations indicate why the use of aspheric surfaces has mainly been confined to astronomical systems, where economic considerations are not of primary importance, and where mirrors have long been established because of their achromatic properties. In a reflecting system the number of components must be a minimum because of obscuration and in addition the aberrational zones are low because of the high refractive index.

In fact no general methods are available which enable the designer to determine the shapes of the surfaces needed to give a required performance over a finite field, the most successful systems being those in which the field aberrations are controlled by virtue of the inherent symmetry, e.g. Schmidts and mono-centric Schmidts. These systems lend themselves to analysis due to the fact that the mirrors and field surfaces are parts of concentric spheres at whose centre the aspheric plate lies. This method, originally due to Caratheodory, and developed by Linfoot, gives the higher order aberrations, but is not applicable to others. The information is therefore limited to the aperture ratio and field angle which is permissable. In any case, information of this sort is better obtained by ray tracing, especially as iterative procedures give as steps the aberration orders, or at least close approximations to them.

The differential method of fine correction is one which has been applied by Baker to the cameras known by his name, and also described by Volosov as a method which is most suitable for figuring spherical surfaces. In these cases where the difference is as implied, small, the method is extremely good, but for large differences it might easily be possible to bend a lens the wrong way to try to find a large change. Volosov's paper is not clear on some points regarding the determination of the polynomial constants which determine the new form of the surface, although the general method is easy to follow.

When axial stigmatism only is required, as in the Schmidt systems, variations of the optical path difference method, qualified by the need for minimum chromatic aberration, are sufficient to obtain the required result. Where the aspheric surface is first or last, the method is straightforward in theory at least, but for interior surfaces, some such method as Herzberger and Hoadly's is required, which involves the determination of the waveform before and after the surface in question, and thus for a complex surface involving a large number of constants in the polynomial which determines it, a large number of rays must be traced. The method of Wassermann and Wolf for obtaining aplanatic systems involving two aspheric surfaces also requires a number of rays to be traced.

The use of aspheric surfaces for the control of zones in lenses such as the Cook triplet has been considered, and it was at first thought that the figuring of one of the negative surfaces, which are both very close to the iris, would have this effect without materially affecting the other aberrations, but in fact a large amount of higher order coma appeared. The best solution was found in adjusting the primary coma and figuring the last surface. In any case in aspheric designs it is always expedient to trace skew rays as designs which have seemed excellent with regard to the usual meridional rays have been poor in practice.

The large amount of aspheric ray tracing required by some of these methods can only be undertaken by an electronic calculating machine, and methods involving their use have been developed both in this country and America. Although they are very fast, they require a large amount of money to instal and operate, and a large amount of effort on the part of the would-be designer to utilise this speed. Methods which involve iteration would seem to offer advantages as the aberrational orders are obtained as part of the trace.

The promise that optical plastics gave of being moulded into aspheric forms and so provide cheap wide aperture photo lenses, did not materialise for physical reasons, but with the advent of electronic techniques for measuring lengths of optical magnitude, it is possible that aspheric surfaces might be produced fairly cheaply in glass. At the moment, however, it would seem that interest in aspheric design, which was high shortly after the Second World War, has diminished, is diminishing, and will continue to diminish.

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