"ON RELATIONS BETWEEN TWO OR MORE GENERALISED HYPERGEOMETRIC SERIES OF THE ORDINARY AND BASIC TYPES."

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(A thesis presented for the degree of Master of Arts at the University of London.)

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March 1949

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§ (1.1). Scheme of the thesis.

After a short historical introduction, (§ (1.2)-(1.3)) and the proof, § (1.4), of Theorem (1), the fundamental theorem, on which the rest of the thesis is based, a few of the simplest summation theorems for hypergeometric series are deduced from first principles (§ (2.1)-(2.3)). In § (3.1) the form of the general transformation between two generalised hypergeometric series, is deduced from theorem (1), and the convergence of such series is discussed. Then, in § (3.2)-(9.7), an attempt is made to deduce in a logical order some of the main transformations and summation theorems for hypergeometric series, from theorem (1), and from the simple sums of § (2.1)-(2.3). This is followed by the deduction of some new transformations between such series by using theorem (1), and supposing every alternate term of one of the series involved, to be zero. The idea for this method was suggested by Professor W.N.Bailey. It seems originally to have been used by Professor L.J.Rogers, who applied it to some results in trigonometric series. There are two ways of using this idea. In § (10.1)-(14.2), the first way is used, and this generally gives relations between three hypergeometric series. In (15.1)-(19.9), the second method is used, and this involves a slight extension of the usual definition of the symbol $(a)_n$. this usually leads to transformations between two hypergeometric series, and several of these transformations, given here, are believed to be new.

In § (20.1), the notation for basic hypergeometric series is defined, and some relations between products of the type $(a)_{x,n}$ are given. Then, in § (20.2), a group of summation theorems for basic series is proved, and, in § (20.3), the form of the general transformation between such series is deduced from the fundamental theorem. In the following paragraphs, § (20.4)-(24.3), some of the known transformations between these series are deduced in logical order, together with those transformations which arise by supposing every alternate term of one of the series to be zero. Most of these basic transformations have already been given by Professor W.N.Bailey, in his paper "Some identities in combinatory analysis" (Proceedings of the London Mathematical Society, (2). Vol. 49 (1947).).

In § (25.1), four particular relations between basic series are deduced. These were given in Professor Bailey's paper "Identities of the Rogers-Ramanujan type" which was read before the London Mathematical Society, on 15th June, 1944. In § (25.1)-(26.8), two of these particular relations are considered in detail, and from them are deduced some identities of the Rogers-Ramanujan type. In the following paragraphs, two new relations are given, and more results similar to the Rogers-Ramanujan identities are deduced. The thesis concludes with a list of those results which are believed to be new. There are also four appendices for reference purposes:-

(1) Relations between products of the type $(a)_n$.

(2) Relations between products of the type $(a)_{x,n}$.

(3) Summation theorems for ordinary hypergeometric series.

(4) Summation theorems for basic series.

§ (1.2) Historical introduction, ordinary hypergeometric series.

In his work "Arithmetica Infinitorum", (1655), the English preacher, John Wallis, (1616-1703), first used the term "hypergeometric" (Greek, ὑπερ, above or beyond.) to denote the series of which the n-th term is $a(a+b)(a+2b)\ldots\ldots(a+(n-1)b)$.

L. Euler, (1707-1783), the famous Swiss mathematician, later used the term "hypergeometric" for such a series, and, among others, he gave the relation

$$ {}_2F_1[a,b;c;x] = (1-x)^{c-a-b} {}_2F_1[c-a,c-b;c;x], $$

between two hypergeometric series as we know them today.

A.T.Vandermonde (1735-1796)(French) also worked on such series.

The Göttingen school of mathematicians under Professor C.F.Hindenburg (1741-1808) were working on various complicated extensions of the Binomial and Multinomial theorems, when C.F.Gauss (1777-1855, German), was a young student. On Jan. 30th 1812, Gauss gave his famous paper, "Disquisitiones generales circa seriem infinitam" before the Royal Scientific Society of Göttingen, and he defined the series

$$ 1 + \frac{a.b.x}{1.c} + \frac{a(a+1)b(b+1).x^2}{1.2.c(c+1)} + \ldots $$

(the modern hypergeometric series), introducing the notation $F(a,b;c;x)$ for it. He also proved his summation theorem for the series ${}_2F_1(a,b;c;1)$, and he deduced many relationships between two or more of these series. In a note, added to his paper on 10th Feb. 1812, he discussed the convergence of the hypergeometric series, this being the first time that such a rigorous discussion had been given by any mathematician.

The next important work was done by E.E.Kummer (1810-1893, German), in his paper of 1836, "Ueber die hypergeometrische Reihe", (Journal fur Maths. Vol. 15). The term "hypergeometric" was applied by him to series of the Gaussian type for the first time, and he proved his summation theorem for the series ${}_2F_1[a,b;1+a-b;-1]$. Kummer also showed that the differential equation satisfied by the function ${}_2F_1[a,b;c;x]$ has, in all, twenty-four solutions of this type, and he studied as well, the confluent hypergeometric functions.

G.F.B.Riemann (1826-1866, German) studied the hypergeometric differential equation, and he constructed his P-function which is a generalisation of the Gaussian $F[a,b;c;x]$.

Louis Saalschutz, (German) in his paper, "Uber einen Spezialfall der hypergeometrischen Reihe dritter Ordnung", (Zeitschrift fur Maths. und Phys. Vol. 36, 1891) gave his summation theorem for the series ${}_3F_2[a,b,c;d,e;1]$ under certain restrictions. Many other mathematicians, notably A.C.Dixon, (English), and J. Thomae (German), studied relations among these generalised hypergeometric series during the later part of the 19th century.

In 1907, John Dougall, in his paper "On Vandermonde's theorem and some more general expansions" (Proceedings Edinburgh Maths. Soc. Vol. 25) made a notable advance on the existing knowledge of these series by summing the series

$$ {}_7F_6\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c, & d, & e, & f; & 1 \\ & \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f; & \end{matrix}\right] $$

under the conditions that it terminates, and that $1+2a = b+c+d+e+f$. His method was fundamentally a very simple one, based on the symmetry in 'a' and 's' possessed by such a function as

$$ \frac{(c-1)(c-2)\ldots\ldots(c-a-s)}{(c-1)\ldots\ldots(c-s)(c-1)\ldots\ldots(c-a)} $$

where 'a' and 's' are positive integers.

Since 1907, some of the leading mathematicians who have worked on the hypergeometric series are E.W.Barnes with his representation of such series by contour integrals, M.J.M.Hill, F.J.W.Whipple, E.T.Whittaker with his work on confluent hypergeometric series, G.N.Watson and W.N.Bailey.

§ (1.3) Historical Introduction. Basic Hypergeometric Series.

A different view of the study of such series was taken by E. Heine (German), in his book "Theorie der Kugelfunctionen" (Vol. 1, 1878). In this work, he defined the basic hypergeometric series as

$$ 1 + \frac{(1-q^a)(1-q^b)}{(1-q)(1-q^c)}.x + \frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})}{(1-q)(1-q^2)(1-q^c)(1-q^{c+1})}.x^2 + \ldots\ldots $$

where $|q|<1$.

Obviously if $q \to 1$, this series reduces to Gauss's series. In his book Heine also proved a number of elementary relations between such series.

Later some interesting work was done on these basic series by L.J.Rogers. In his papers "Memoirs on the expansion of certain infinite products"(1894) and "On two theorems of combinatory analysis" (1917), in the Proceedings of the London Mathematical Society, he gave the famous"Rogers-Ramanujan" identities:

$$1 + \frac{q}{(1-q)} + \frac{q^4}{(1-q)(1-q^2)} + \frac{q^9}{(1-q)(1-q^2)(1-q^3)} + \dots$$

$$\dots + \frac{q^{n^2}}{(1-q)\dots\dots(1-q^n)} + \dots$$

$$= \frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})(1-q^{14})\dots\dots(1-q^{5n+1})(1-q^{5n+4})\dots}$$

and

$$1 + \frac{q^2}{1-q} + \frac{q^6}{(1-q)(1-q^2)} + \frac{q^{12}}{(1-q)(1-q^2)(1-q^3)} + \dots$$

$$\dots + \frac{q^{n(n+1)}}{(1-q)\dots\dots(1-q^n)} + \dots$$

$$= \frac{1}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)(1-q^{12})(1-q^{13})\dots\dots(1-q^{5n+2})(1-q^{5n+3})\dots}$$

together with many similar relations.

The brilliant young Indian mathematician S.Ramanujan

(1887–1920) rediscovered these identities in 1917, and G.N.Watson in his paper "A new proof of the Rogers-Ramanujan identities" (Journal Lond. Maths. Soc. 4(1929)) deduced them as special limiting forms of a relation proved by him in the same paper, between two basic hypergeometric series.

The study of the work of Rogers on these identities led indirectly to the discovery of the following theorem which has many applications to infinite integrals as well as in the study of series. This theorem is fundamental in the development of both basic and ordinary hypergeometric series.

<u>Theorem (1).</u> If $\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{r+n}$,

and $\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n}$

then $\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n$,

for
$$\sum_{r=0}^{\infty} \beta_r \delta_r = \sum_{r=0}^{\infty} \sum_{n=0}^{r} \alpha_n \delta_r u_{r-n} v_{r+n}$$
$$= \sum_{n=0}^{\infty} \sum_{r=n}^{\infty} \delta_r \alpha_n u_{r-n} v_{r+n}$$
$$= \sum_{n=0}^{\infty} \alpha_n \gamma_n ,$$

provided that the series involved are convergent, or finite, and that the change in the order of summation is justifiable. This theorem was given by W.N.Bailey in his paper "Identities of the Rogers-Ramanujan type" (1944).

§ (2.1) Notation.

Let $(a)_0 \equiv 1$, and let

$$(a)_n \equiv a(a+1)(a+2)(a+3)\ldots\ldots(a+n-1),$$

where 'a' is a complex number and 'n' is a positive integer. Also let the generalised hypergeometric series

$$\sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n (a_3)_n \ldots\ldots (a_k)_n}{(b_1)_n (b_2)_n (b_3)_n \ldots\ldots (b_{k-1})_n n!} . x^n$$

be denoted by

$${}_kF_{k-1}\left[\begin{matrix} a_1, a_2, a_3, \ldots a_k ; x \\ b_1, b_2, \ldots b_{k-1} ; \end{matrix}\right]$$

where $a_1, a_2, \ldots\ldots a_k, b_1, b_2, \ldots\ldots b_{k-1}$, and x are all complex numbers. This series is convergent

if $|x| < 1$,

if $x = 1$, provided that $Rl(b_1+b_2+\ldots\ldots+b_{k-1}-a_1-a_2-\ldots-a_k) > 0$,

or, if $x = -1$, provided that

$$Rl(1+b_1+b_2+\ldots\ldots+b_{k-1}-a_1-a_2-\ldots\ldots-a_k) > 0.$$

It is assumed that none of the products

$(b_1)_n, (b_2)_n, (b_3)_n, \ldots\ldots (b_{k-1})_n$ vanish.

The familiar Binomial theorem then assumes the form

$$(1-x)^{-a} = {}_1F_0[\, a ; \; ; x \,]$$

The first complete proof of this result was given in 1826 by N.H.Able (1802-1829, Norwegian).

§ (2.2). Some preliminary results.

It can easily be shown that the series $_2F_1(a,b;c;x)$ is a solution of the hypergeometric differential equation

$$x(1-x)\frac{d^2y}{dx^2} + (c-(a+b+1)x)\frac{dy}{dx} - aby = 0, \qquad \ldots(1)$$

provided that $|x|<1$.

A second solution is $x^{1-c}\,{}_2F_1[1+a-c,1+b-c;2-c;x]$. By changing the dependent variable y, in equation (1) it can be shown that $(1-x)^{c-a-b}\,{}_2F_1[c-a,c-b;c;x]$ is also a solution. But, since equation (1) is of order two, it can have only two linearly independent solutions. Hence there exist constants A and B such that

$$(1-x)^{c-a-b}\,{}_2F_1[c-a,c-b;c;x]$$
$$= A\,{}_2F_1[a,b;c;x] + x^{1-c}\,{}_2F_1[1+a-c,1+b-c;2-c;x]$$

Now the function on the left of this equation can be expanded in integral powers of x, and c is not necessarily an integer. Therefore B = 0, and, putting x = 0, we get A = 1. Therefore

$$(1-x)^{a+b-c}\,{}_2F_1[a,b;c;x] = {}_2F_1[c-a,c-b;c;x] \qquad \ldots(2)$$

(Euler).

Equating the coefficients of x^n in (2), we obtain

$$\sum_{r=0}^{n}\frac{(a)_r(b)_r(c-a-b)_{n-r}}{r!\,(c)_n(n-r)!} = \frac{(c-a)_n(c-b)_n}{n!\,(c)_n}$$

Hence we have

<u>Theorem (2)</u> $${}_3F_2[a,b,-n;c,1+a+b-c-n;1] = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n} \qquad \ldots(3)$$

This is the result known as Saalschutz's theorem.

Since $\frac{\Gamma(a+n)}{\Gamma(a)} \equiv (a)_n$, this theorem can be written

in the form $${}_3F_2[a,b,c;d,e;1] = \frac{\Gamma(d)\Gamma(1+a-e)\Gamma(1+b-e)\Gamma(1+c-e)}{\Gamma(1-e)\Gamma(d-a)\Gamma(d-b)\Gamma(d-c)}$$

provided a,b, or c is a negative integer and

$$d+e = a+b+c+1.$$

Now let $n \to \infty$ in equation (3) above.

Then $\frac{(-n)_r}{(1+a+b-c-n)_r} \to 1$ as $n \to \infty$.

Hence, using Tannery's theorem, we get

<u>Theorem (3)</u> $${}_2F_1[a,b;c;1] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad \ldots(4)$$

which is Gauss's theorem.

When 'b' is a negative integer $-n$, equation (4) becomes

$${}_2F_1[a,-n;c;1] = \frac{(c-a)_n}{(c)_n} \qquad \ldots(5)$$

In this form, the summation is known as Vandermonde's theorem.

<u>§ (3.1)</u> <u>A general transformation between two generalised hypergeometric series.</u>

To investigate hypergeometric series, by the application of theorem (1), let us suppose that

$$u_r = \frac{(e_1)_r(e_2)_r\ldots}{(E_1)_r(E_2)_r\ldots}$$

$$v_r = \frac{(f_1)_r(f_2)_r\ldots}{(F_1)_r(F_2)_r\ldots}$$

and $$\delta_r = \frac{(d_1)_r(d_2)_r\ldots}{(D_1)_r(D_2)_r\ldots}.t^r$$

In these three expressions, there can be any number of the parameters $e_1, e_2, \ldots f_1, f_2, \ldots d_1, d_2, \ldots E_1, E_2, \ldots F_1, F_2, \ldots D_1, D_2, \ldots$. All these parameters, together with 't' must be independent of r. With these values of u_r, v_r, and δ_r,

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n}$$

$$= \sum_{s=0}^{\infty} \delta_{s+n} u_s v_{s+2n}, \qquad \text{(putting } s = r+n.\text{)}$$

Hence,

$$\gamma_n = \sum_{s=0}^{\infty} \frac{(d_1)_{s+n}(d_2)_{s+n}\ldots(e_1)_s(e_2)_s\ldots(f_1)_{s+2n}(f_2)_{s+2n}\ldots}{(D_1)_{s+n}(D_2)_{s+n}\ldots(E_1)_s(E_2)_s\ldots(F_1)_{s+2n}(F_2)_{s+2n}\ldots}.t^{s+n}$$

$$\therefore \gamma_n = \frac{(d_1)_n(d_2)_n\ldots(f_1)_{2n}(f_2)_{2n}\ldots}{(D_1)_n(D_2)_n\ldots(F_1)_{2n}(F_2)_{2n}\ldots}.t^n$$

$$\times \sum_{s=0}^{\infty} \frac{(d_1+n)_s(d_2+n)_s\ldots(e_1)_s(e_2)_s\ldots(f_1+2n)_s(f_2+2n)_s\ldots}{(D_1+n)_s(D_2+n)_s\ldots(E_1)_s(E_2)_s\ldots(F_1+2n)_s(F_2+2n)_s\ldots}.t^s$$

For this series to be a hypergeometric series, one of the denominator parameters must be unity. Originally the parameters were all independent of r. Hence now they must all be independent of both n and s.

$\therefore$ $D_m + n \neq 1$, and $F_m + 2n \neq 1$, for any value of m, and for all values of n.

Hence there must be at least one E parameter and this must be taken equal to unity.

Suppose then that $E_1 = 1$.

It is also necessary that the number of numerator parameters shall be one more than the number of denominator parameters (excluding $E_1 = 1$).

Then,

$$\gamma_n = \frac{(d_1)_n (d_2)_n \ldots (f_1)_{2n} (f_2)_{2n} \ldots}{(D_1)_n (D_2)_n \ldots (F_1)_{2n} (F_2)_{2n} \ldots} \cdot t^n$$
$$\times \; {}_kF_{k-1}\left[\begin{matrix} e_1, e_2, \ldots d_1+n, d_2+n, \ldots f_1+2n, f_2+2n, \ldots ; \; t \\ E_2, \ldots D_1+n, D_2+n, \ldots F_1+2n, F_2+2n, \ldots ; \end{matrix}\right] \quad \ldots(1)$$

Let this series occurring in the expression for γ_n be denoted by F_γ.

F_γ is convergent (case (1)) if $|t| < 1$,

(case (2)) if $t = 1$, and $Rl(E_2 + \ldots + D_1 + D_2 + \ldots + F_1 + F_2 + \ldots - e_1 - e_2 - \ldots - d_1 - d_2 - \ldots - f_1 - f_2 - \ldots + f(n)) > 0$,

where $f(n) = n.$(the number of D parameters + twice the number of F parameters - the number of d parameters - twice the number of f parameters.)

For this to be true for all values of n, $f(n) \geq 0$,

i.e. f(n) must have one of the values o,n,2n,3n,...

Or, F_γ is convergent, (case (3)) if t = -1, and

$Rl(1+E_2+..+D_1+D_2+..+F_1+F_2+..-e_1-e_2-..-d_1-d_2-..-f_1-f_2-..+f(n))>0$

where f(n) is defined as above.

For this to be true for all n, $f(n) \geq -1$,

i.e., as before, f(n) must have one of the values o,n,2n,...

With the same values of u_r and v_r

$$\beta_n = \sum_{r=o}^{n} \alpha_r \frac{(e_1)_{n-r}(e_2)_{n-r}..(f_1)_{n+r}(f_2)_{n+r}..}{(n-r)!(E_2)_{n-r}..(F_1)_{n+r}(F_2)_{n+r}..}$$

$$\therefore \beta_n = \frac{(e_1)_n(e_2)_n..(f_1)_n(f_2)_n..}{n!(E_2)_n..(F_1)_n(F_2)_n..}$$

$$\times \sum_{r=o}^{n} \alpha_r \frac{(-n)_r(1-n-E_2)_r..(f_1+n)_r(f_2+n)_r..(-1)^{pr}}{(1-e_1-n)_r(1-e_2-n)_r..(F_1+n)_r(F_2+n)_r..} \quad ..(2)$$

where p is the difference between the number of e parameters and 1 + the number of E parameters.

Here the value of α_r remains to be chosen, but must be independent of n. Let the series occurring in the expression for β_n be denoted by F_β.

Obviously F_β always terminates.

For F_β to be a hypergeometric series of the type under consideration α_n must have the form

$$\frac{(a_1)_n(a_2)_n..}{(A_1)_n(A_2)_n}.t'^n$$

where there can be any number of the parameters $a_1, a_2, \ldots$ $A_1, A_2, \ldots$, provided that the total number of numerator parameters is one more than the total number of denominator parameters in F_β, and that $t', a_1, a_2, \ldots A_1, A_2, \ldots$ are independent of both n and r.

The general transformation given by theorem (1) is then

$$\sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \ldots t'^n (d_1)_n (d_2)_n \ldots (f_1)_{2n} (f_2)_{2n} \ldots t^n}{n! (A_2)_n \ldots (D_1)_n (D_2)_n \ldots (F_1)_{2n} (F_2)_{2n} \ldots}$$

$$\times\ {}_kF_{k-1}\left[\begin{matrix} e_1, e_2, \ldots d_1+n, d_2+n, \ldots f_1+2n, f_2+2n, \ldots; \ t \\ E_2, \ldots D_1+n, D_2+n, \ldots F_1+2n, F_2+2n, \ldots; \end{matrix}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(e_1)_n (e_2)_n \ldots (f_1)_n (f_2)_n \ldots (d_1)_n (d_2)_n \ldots}{n!\ (E_2)_n \ldots (F_1)_n (F_2)_n \ldots (D_1)_n (D_2)_n \ldots} \cdot t^n$$

$$\times \sum_{r=0}^{n} \frac{(-n)_r (1-n-E_2)_r \ldots (f_1+n)_r (f_2+n)_r \ldots (-1)^p (a_1)_r (a_2)_r \ldots t'^r}{(1-e_1-n)_r (1-e_2-n)_r \ldots (F_1+n)_r\ (F_2+n)_r \ldots\ r!\ (A_2)_r \ldots} \qquad \ldots(3)$$

where p has the same definition as above.

Thus, putting r for n and n for r on the left of this equation,

$$\sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a_1)_r \ldots t'^r (d_1)_{n+r} \ldots (f_1)_{2r+n} (f_2)_{2r+n} \ldots (e_1)_n (e_2)_n \ldots t^{n+r}}{r!\ (A_2)_r \ldots (D_1)_{n+r} \ldots (F_1)_{2r+n} \ldots (E_2)_n \ldots\ n!}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(d_1)_n \ldots (e_1)_{n-r} \ldots (f_1)_{n+r} \ldots t^n (a_1)_r \ldots t'^r}{(D_1)_n \ldots (n-r)!\ (E_2)_{n-r} \ldots (F_1)_{n+r} \ldots\ r! (A_2)_r \ldots} \qquad \ldots(4)$$

provided the change in the order of summation can be justified.

This process is equivalent to the summation of a double series by diagonals instead of by rows. If one of the series in equation (4) is convergent, when $t = t' = 1$, the process is justifiable, since all the terms on either side of equation (4) are then ultimately of the same sign. The process can also be justified if t or $t' = -1$, when the convergence is absolute, and also if $|t|<1$, $|t'|<1$, by the use of a comparison test.

§ (3.2) Application of the Binomial Theorem.

Suppose that F_γ is summable by the binomial theorem. There are three possibilities for F_γ. These are:-

(1) ${}_1F_0[f+2n; ; x]$

(2) ${}_1F_0[d+n; ; x]$

and (3) ${}_1F_0[e; ; x]$

The corresponding series for F_β are:-

(1) $\sum_{r=o}^{n}(-n)_r(f+n)_r(-1)^r\alpha_r$

(2) $\sum_{r=o}^{n}(-n)_r(-1)^r\alpha_r$

and (3) $\sum_{r=o}^{n}\frac{(-n)_r}{(1-e-n)_r}\alpha_r$

all of which are summable for some value of α_r.

§ (3.3) Case one.

If $\delta_n = x^n$, $\gamma_n = \frac{(f)_{2n}x^n}{(1-x)^{f+2n}}$

If $\alpha_n = \dfrac{(-1)^n}{n!(1+f-a)_n}$, summing F_β by Vandermonde's theorem,

$$\beta_n = \frac{(f)_n}{n!}{}_2F_1(f+n,-n;1+f-a;1)$$

$$\therefore\ \beta_n = \frac{(f)_n(a)_n(-1)^n}{n!\,(1+f-a)_n}$$

Hence, $\dfrac{1}{(1-x)^f}\,{}_2F_1\left[\begin{matrix}\tfrac{1}{2}f,\tfrac{1}{2}(f+1);\ \dfrac{-4x}{(1-x)^2}\\ 1+f-a\end{matrix}\right] = {}_2F_1\left[\begin{matrix}f,a;\ -x\\ 1+f-a;\end{matrix}\right]$ ...(1)

This result was given by Gauss (Disquisitiones, (100)).

If $\alpha_n = \dfrac{(-1)^n(a)_n}{n!(A)_n(1+f+a-A)_n}$, summing F_β by Saalschutz's theorem,

$$\beta_n = \frac{(f)_n(A-a)_n(1+f-A)_n}{n!\ (A)_n(1+f+a-A)_n}$$

Hence

$\dfrac{1}{(1-x)^f}\,{}_3F_2\left[\begin{matrix}a,\tfrac{1}{2}f,\tfrac{1}{2}(f+1);\ \dfrac{-4x}{(1-x)^2}\\ A,1+f+a-A\end{matrix}\right] = {}_3F_2\left[\begin{matrix}f,A-a,1+f-A;\ x\\ A,\ 1+f+a-A;\end{matrix}\right]$ ..(2)

This formula is proved by W.N.Bailey in his paper of Nov. 10th 1927 "Products of generalised hypergeometric series". (Proc.Lond.Maths.Soc.Vol.28 p.250 eq. 4.12)

Putting $a \equiv \tfrac{1}{2}(f+1)-b$ and $A \equiv \tfrac{1}{2}(f+1)$ this formula becomes

$\dfrac{1}{(1-x)^f}\,{}_2F_1\left[\begin{matrix}\tfrac{1}{2}f,\tfrac{1}{2}(f+1)-b;\ \dfrac{-4x}{(1-x)^2}\\ 1+f-b\end{matrix}\right] = {}_2F_1\left[\begin{matrix}f,\ b;\ x\\ 1+f-b;\end{matrix}\right]$ ...(3)

which is also given by Gauss in his Disquisitiones.

These formulae hold for $|x|<3-2\sqrt{2}$, and in general, by analytic continuation in that loop of the curve $|4x| = |1-x|^2$ which surrounds the origin.

Now let $x \to -1$, in equation (3). Then, writing a for f, we get $2^{-a} {}_2F_1\left[\begin{matrix}\frac{1}{2}a, \frac{1}{2}(a+1)-b; & 1 \\ 1+a-b & ;\end{matrix}\right] = {}_2F_1\left[\begin{matrix}a, b; & -1 \\ 1+a-b; & \end{matrix}\right]$ ...(4)

Summing the series on the left by Gauss's theorem, we have

<u>Theorem (4)</u>, <u>Kummer's theorem</u>,

$$ {}_2F_1\left[\begin{matrix}a, b; & -1 \\ 1+a-b; & \end{matrix}\right] = \frac{\Gamma(1+a-b)\Gamma(1+\frac{1}{2}a)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)} \qquad \ldots(5)$$

<u>§ (3.4)</u> <u>Case two</u>.

If $\delta_n = (d)_n x^n$, $\gamma_n = \dfrac{(d)_n x^n}{(1-x)^{d+n}}$

If $\alpha_n = \dfrac{(a)_n (-1)^n}{n!\,(A)_n}$, summing F_β by Vandermonde's theorem

$$\beta_n = \frac{(A-a)_n}{n!(A)_n}$$

Hence $\dfrac{1}{(1-x)^d}\, {}_2F_1\left[\begin{matrix}a, d; & \frac{-x}{1-x} \\ A; & \end{matrix}\right] = {}_2F_1\left[\begin{matrix}A-a, d; & x \\ A; & \end{matrix}\right]$ ...(1)

provided $|x|<1$, and $Rl(x)<\frac{1}{2}$. (Gauss, Disquisitiones Formula 22)

Now let $x \to -1$ and this relation will become

$$ {}_2F_1\left[\begin{matrix}a, d; & \frac{1}{2} \\ A; & \end{matrix}\right] = 2^d\, {}_2F_1\left[\begin{matrix}d, A-a; & -1 \\ A; & \end{matrix}\right] \qquad \ldots(2)$$

If $A = \frac{1}{2}(a+d+1)$ the series on the right of equation (2) above can be summed by Kummer's theorem, giving

<u>Theorem (5)</u>

$$ {}_2F_1\left[\begin{matrix}a, d; & \frac{1}{2} \\ \frac{1}{2}(a+d+1); & \end{matrix}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(1+a+d))}{\Gamma(\frac{1}{2}(1+a))\Gamma(\frac{1}{2}(1+d))} \qquad \ldots(3)$$

This is Gauss's second summation theorem.

Again, if $a+d = 1$, the series on the right of equation (2) can be summed by Kummer's theorem, giving

Theorem (6)

$$ {}_2F_1\left[\begin{matrix} a, 1-a; & \tfrac{1}{2} \\ A\ ; & \end{matrix}\right] = \frac{\Gamma(\tfrac{1}{2}A)\Gamma(\tfrac{1}{2}(A+1))}{\Gamma(\tfrac{1}{2}(A+a))\Gamma(\tfrac{1}{2}(1+A-a))} \quad \ldots(4) $$

This formula is given in W.N.Bailey's Cambridge Tract "Generalised hypergeometric series" (§ 2.4 (3))

§ (3.5) Case three

$$ \text{If } \delta_n = x^n \text{ then } \gamma_n = \frac{x^n}{(1-x)^e} $$

If $\alpha_n = \dfrac{(a_1)_n(a_2)_n}{n!(a_1+a_2+e)_n}$ summing F_β by Saalschutz's theorem, we get the same relation as § 2.2 (2) above, from which Saalschutz's theorem was initially deduced.

§ (4.1) Applications of Gauss's theorem.

Suppose now that F_γ can be summed either by Gauss's theorem, as an infinite ${}_2F_1(1)$, or by Vandermonde's theorem as a finite ${}_2F_1(1)$ series. There are eighteen possibilities for F_γ. Of the corresponding F_β series only eleven are summable. The summable cases are:-

(1) $F_\gamma = {}_2F_1\left[\begin{matrix} f_1+2n, f_2+2n; 1 \\ F+2n\ ; \end{matrix}\right] \quad F_\beta = \sum_{r=0}^{n} \dfrac{(-n)_r(f_1+n)_r(f_2+n)_r(-1)^r}{(F+n)_r} \alpha_r$

(2) $F_\gamma = {}_2F_1\left[\begin{matrix} f+2n, d+n; & 1 \\ F+2n; & \end{matrix}\right] \quad F_\beta = \sum_{r=0}^{n} \dfrac{(-n)_r(f+n)_r(-1)^r}{(F+n)_r} \alpha_r$

(3) $F_\gamma = {}_2F_1\left[\begin{matrix} d_1+n, d_2+n; & 1 \\ F+2n & ; \end{matrix}\right]$ $F_\beta = \sum_{r=0}^{n} \frac{(-n)_r(-1)^r}{(F+n)_r} \alpha_r$

(4) $F_\gamma = {}_2F_1\left[\begin{matrix} f+2n, d+n; & 1 \\ D+n; & \end{matrix}\right]$ $F_\beta = \sum_{r=0}^{n} (-n)_r(f+n)_r(-1)^r \alpha_r$

(5) $F_\gamma = {}_2F_1\left[\begin{matrix} f+2n, e; & 1 \\ F+2n; & \end{matrix}\right]$ $F_\beta = \sum_{r=0}^{n} \frac{(-n)_r(f+n)_r}{(1-e-n)_r(F+n)_r} \alpha_r$

(6) $F_\gamma = {}_2F_1\left[\begin{matrix} f+2n, e; & 1 \\ D+n; & \end{matrix}\right]$ $F_\beta = \sum_{r=0}^{n} \frac{(-n)_r(f+n)_r}{(1-e-n)_r} \alpha_r$

(7) $F_\gamma = {}_2F_1\left[\begin{matrix} d_1+n, d_2+n; 1 \\ D+n \quad ; \end{matrix}\right]$ $F_\beta = \sum_{r=0}^{n} (-n)_r(-1)^r \alpha_r$

(8) $F_\gamma = {}_2F_1\left[\begin{matrix} f+2n, e; & 1 \\ E; & \end{matrix}\right]$ $F_\beta = \sum_{r=0}^{n} \frac{(-n)_r(1-E-n)_r(f+n)_r(-1)^r}{(1-e-n)_r} \alpha_r$

(9) $F_\gamma = {}_2F_1\left[\begin{matrix} d_1+n, d_2+n; 1 \\ E; \end{matrix}\right]$ $F_\beta = \sum_{r=0}^{n} (-n)_r (1-E-n)_r \alpha_r$

(10) $F_\gamma = {}_2F_1\left[\begin{matrix} d+n, e; & 1 \\ D+n; & \end{matrix}\right]$ $F_\beta = \sum_{r=0}^{n} \frac{(-n)_r}{(1-e-n)_r} \alpha_r$

(11) $F_\gamma = {}_2F_1\left[\begin{matrix} d+n, e; & 1 \\ E; & \end{matrix}\right]$ $F_\beta = \sum_{r=0}^{n} \frac{(-n)_r(1-E-n)_r(-1)^r}{(1-e-n)_r} \alpha_r$

Of these, only cases (3), (5), and (10) satisfy the condition "Rl(c-a-b) > 0" for all values of n, and they give convergent series for F_γ. In the remaining cases, one of the numerator parameters of F_γ must be a negative integer $-N$, and F_γ can be summed only by Vandermonde's theorem.

§ (4.2) Cases one and two

In case one $\delta_n = 1$. Summing F_γ by Vandermonde's theorem, $\gamma_n = \dfrac{(F-f)_N (f)_{2n} (-N)_{2n}}{(F)_N \ (1+f-F-N)_{2n}}$

Let $\alpha_n = \dfrac{(-1)^n}{n!(1+f-F-N)_n}$. Then, summing F_β by Saalschutz's theorem, $\beta_n = \dfrac{(f)_n(-N)_n(F-f)_n(F+N)_n(-1)^n}{n! \ (F)_{2n}(1+f-F-N)_n}$

Hence $${}_4F_3\left[\begin{matrix} f, F-f, \ F+N, \ -N; \ -\tfrac{1}{4} \\ \tfrac{1}{2}F, \tfrac{1}{2}(F+1), 1+f-F-N; \end{matrix}\right]$$
$$= \frac{(F-f)_N}{(F)_N}\,{}_4F_3\left[\begin{matrix} \tfrac{1}{2}f, \tfrac{1}{2}(f+1), \ -\tfrac{1}{2}N, \tfrac{1}{2}(-N+1); -4 \\ \tfrac{1}{2}(1+f-F-N), 1+\tfrac{1}{2}(f-F-N), 1+f-F-N; \end{matrix}\right] \quad \ldots(1)$$

In case two, summing both F_γ and F_β by Vandermonde's theorem, $\delta_n = (d)_n$, and hence,

$$\gamma_n = \frac{\Gamma(F)\Gamma(F-f-d)}{\Gamma(F-f)\Gamma(F-d)} \frac{(d)_n(f)_{2n}(-1)^n}{(F-d)_n(1+f+d-F)_n}$$

If $$\alpha_n = \frac{(-1)^n}{n!}$$

$$\beta_n = \frac{(f)_n(F-f)_n}{n! \ (F)_{2n}}$$

Hence $${}_3F_2\left[\begin{matrix} d, \ f, \ F-f; \ \tfrac{1}{4} \\ \tfrac{1}{2}F, \tfrac{1}{2}(F+1); \end{matrix}\right]$$
$$= \frac{\Gamma(F)\Gamma(F-f-d)}{\Gamma(F-f)\Gamma(F-d)}\,{}_3F_2\left[\begin{matrix} d, \ \tfrac{1}{2}f, \ \tfrac{1}{2}(f+1); \ 4 \\ F-d, 1+f+d-F; \end{matrix}\right] \quad \ldots(2)$$

provided either f or d is a negative integer.

§ (4.3) Case three

Summing F_γ by Gauss's theorem,

if $\delta_n = (d_1)_n(d_2)_n$

$$\gamma_n = \frac{\Gamma(F)\Gamma(F-d_1-d_2)}{\Gamma(F-d_1)\Gamma(F-d_2)} \frac{(d_1)_n(d_2)_n}{(F-d_1)_n(F-d_2)_n}$$

provided $Rl(F-d_1-d_2) > 0$.

Here $\beta_n = \dfrac{1}{n!\,(F)_n} \sum_{r=0}^{n} \dfrac{(-n)_r(-1)^r}{(F+n)_r} \alpha_r$

First sum F_β by Vandermonde's theorem.

Then, if $\alpha_n = \dfrac{(-1)^n(a)_n}{n!}$

$$\beta_n = \frac{(F-a)_{2n}}{n!(F-a)_n(F)_{2n}}$$

Hence,

$$ {}_4F_3\left[\begin{matrix}\tfrac{1}{2}(F-a), & d_1, & d_2, & \tfrac{1}{2}(F-a+1); & 1 \\ & F-a, & \tfrac{1}{2}F, & \tfrac{1}{2}(F+1) ; & \end{matrix}\right] = \frac{\Gamma(F)\Gamma(F-d_1-d_2)}{\Gamma(F-d_1)\Gamma(F-d_2)} \, {}_3F_2\left[\begin{matrix}a, & d_1, & d_2; & -1 \\ & F-d_1, & F-d_2; & \end{matrix}\right] \quad \ldots(1)$$

provided $Rl(F-d_1-d_2) > 0$, and $Rl(1-a) > 0$.

This gives a nearly-poised series ${}_3F_2(-1)$, (i.e. a series in which all the pairs of parameters are well-poised except one pair,) in terms of a ${}_4F_3(1)$ series, (Bailey, Tract, § 4.6 (3)).

Next sum F_β by Kummer's theorem.

Then, if $\alpha_n = \dfrac{(F-1)_n}{n!}$

$$\beta_n = \frac{1}{n!\,(\tfrac{1}{2}(F+1))_n}$$

Hence,

$$ {}_2F_1\left[\begin{matrix}d_1, & d_2; & 1 \\ \tfrac{1}{2}(F+1); & & \end{matrix}\right] = \frac{\Gamma(F)\Gamma(F-d_1-d_2)}{\Gamma(F-d_1)\Gamma(F-d_2)} \, {}_3F_2\left[\begin{matrix}F-1, & d_1, & d_2; & 1 \\ & F-d_1, & F-d_2; & \end{matrix}\right] \quad \ldots(2)$$

provided $Rl(F-d_1-d_2)>0$ and $Rl(1-d_1-d_2)>0$.

Summing this ${}_2F_1(1)$ by Gauss's theorem, and putting $1+f \equiv F$, $a \equiv d_1$, $b \equiv d_2$, we get

Theorem (7)

$$
{}_3F_2\left[\begin{matrix} f, & a, & b; 1 \\ & 1+f-a, & 1+f-b; \end{matrix}\right]
$$
$$
= \frac{\Gamma(1+\tfrac{1}{2}f)\Gamma(1+\tfrac{1}{2}f-a-b)\Gamma(1+f-a)\Gamma(1+f-b)}{\Gamma(1+\tfrac{1}{2}f-a)\Gamma(1+\tfrac{1}{2}f-b)\Gamma(1+f)\Gamma(1+f-a-b)} \quad \ldots(3)
$$

where $Rl(1+\tfrac{1}{2}f-a-b)>0$ (as in equation (2) above).

This is Dixon's theorem, first given by A.C.Dixon in his paper "Summation of a certain series" (Proc.Lond. Maths.Soc. Vol.(2) series (2) (1903)). It gives the value of a ${}_3F_2(1)$ series in which the sum of each pair of numerator and denominator parameters is equal to the first numerator parameter + 1. Such a series is called a well-poised series.

Next, summing F_β by Dixon's theorem,

$$
\text{if } \alpha_n = \frac{(F-1)_n (a)_n (-1)^n}{n!\ (F-a)_n}
$$
$$
\beta_n = \frac{(\tfrac{1}{2}(1+F)-a)_n}{n!(\tfrac{1}{2}(1+F))_n (F-a)_n}
$$

Hence,

$$
{}_3F_2\left[\begin{matrix} \tfrac{1}{2}(1+F)-a, & d_1, & d_2; 1 \\ & \tfrac{1}{2}(1+F), & F-a; \end{matrix}\right]
$$
$$
= \frac{\Gamma(F)\Gamma(F-d_1-d_2)}{\Gamma(F-d_1)\Gamma(F-d_2)}\ {}_4F_3\left[\begin{matrix} F-1, & d_1, & d_2, & a; -1 \\ & F-d_1, & F-d_2, & F-a; \end{matrix}\right] \quad ..(4)
$$

provided $Rl(F-d_1-d_2)>0$ and $Rl(1-a)>0$.

Let $d_1 = \frac{1}{2}(1+F)$, $d_2 = b$, and $1+f = F$, then the ${}_3F_2(1)$ series becomes ${}_2F_1\left[\begin{matrix}1+\frac{1}{2}f-a, & b; & 1\\ & 1+f-a; & \end{matrix}\right]$. Summing this series by Gauss's theorem, we get,

Theorem (8)

$$ {}_4F_3\left[\begin{matrix}f, & 1+\frac{1}{2}f, & a, & b; & -1\\ & \frac{1}{2}f, & 1+f-a, & 1+f-b; & \end{matrix}\right] = \frac{\Gamma(1+f-a)\Gamma(1+f-b)}{\Gamma(1+f)\Gamma(1+f-a-b)} \qquad \ldots(5) $$

where $Rl(1+\frac{1}{2}f-a-b)>0$ (as in equation (4) above).

This gives the sum of a well-poised ${}_4F_3(-1)$ series with the special form of the second numerator parameter.

Now sum F_β as a well-poised ${}_4F_3(-1)$, letting

$$ \alpha_n = \frac{(F-1)_n(\frac{1}{2}(F+1))_n(a)_n}{n!(\frac{1}{2}(F-1))_n(F-a)_n} $$

Then, $\beta_n = \dfrac{1}{n!\,(F-a)_n}$

Hence, ${}_2F_1\left[\begin{matrix}d_1, & d_2; & 1\\ & F-a; & \end{matrix}\right] =$

$$ =\frac{\Gamma(F)\Gamma(F-d_1-d_2)}{\Gamma(F-d_1)\Gamma(F-d_2)}\,{}_5F_4\left[\begin{matrix}F-1, & \frac{1}{2}(F+1), & a, & d_1, & d_2; & 1\\ & \frac{1}{2}(F-1), & F-a, & F-d_1, & F-d_2; & \end{matrix}\right] \qquad \ldots(6) $$

provided $Rl(F-d_1-d_2)>0$ and $Rl(a)<0$.

In this relation, let $1+f = F$, $b = d_1$, $c = d_2$, and sum the ${}_2F_1(1)$ series by Gauss's theorem. Then we get

Theorem (9)

$$ {}_5F_4\left[\begin{matrix}f, & 1+\frac{1}{2}f, & a, & b, & c; & 1\\ & \frac{1}{2}f, & 1+f-a, & 1+f-b, & 1+f-c; & \end{matrix}\right] $$

$$ =\frac{\Gamma(1+f-a)\Gamma(1+f-b)\Gamma(1+f-c)\Gamma(1+f-a-b-c)}{\Gamma(1+f)\Gamma(1+f-a-b)\Gamma(1+f-a-c)\Gamma(1+f-b-c)} \qquad \ldots(7) $$

where $Rl(1+f-a-b-c)>0$ (as in equation (6) above).

This result gives the sum of a well-poised ${}_5F_4(1)$ with the special form of the second numerator parameter.

Now sum F_β as a well-poised ${}_5F_4(1)$, letting

$$\alpha_n = \frac{(F-1)_n(\frac{1}{2}(F+1))_n(a_1)_n(a_2)_n(-1)^n}{n!\,(\frac{1}{2}(F-1))_n(F-a_1)_n(F-a_2)_n}$$

Then $\beta_n = \dfrac{(F-a_1-a_2)_n}{n!(F-a_1)_n(F-a_2)_n}$

Hence,

$$
{}_3F_2\left[\begin{matrix}F-a_1-a_2, & d_1, & d_2; & 1\\ & F-a_1, & F-a_2; & \end{matrix}\right]
$$

$$
= \frac{\Gamma(F)\Gamma(F-d_1-d_2)}{\Gamma(F-d_1)\Gamma(F-d_2)}\,{}_6F_5\left[\begin{matrix}F-1, & \frac{1}{2}(F+1), & a_1, & a_2, & d_1, & d_2; & -1\\ & \frac{1}{2}(F-1), & F-a_1, & F-a_2, & F-d_1, & F-d_2; & \end{matrix}\right] \qquad \ldots(8).
$$

provided $Rl(F-d_1-d_2)>0$ and $Rl(1+F-2a_1-2a_2)>0$.

If $1 = d_1+a_2 = d_2+a_1$, the ${}_3F_2(1)$ series can be summed by Dixon's theorem. Let $1+f \equiv F$, $a \equiv a_1$, $b \equiv a_2$, $c \equiv d_1$, and $d \equiv d_2$. Then equation (8) gives

<u>Theorem (10)</u>

$$
{}_6F_5\left[\begin{matrix}f, & 1+\frac{1}{2}f, & a, & b, & c, & d; & -1\\ & \frac{1}{2}f, & 1+f-a, & 1+f-b, & 1+f-c, & 1+f-d; & \end{matrix}\right]
$$

$$
= \frac{\Gamma(1+f-a)\Gamma(1+f-b)\Gamma(1+f-c)\Gamma(1+f-d)\Gamma(1+\frac{1}{2}(1+f-a-b))}{\Gamma(f)\Gamma(1+f)\Gamma(1+f-c-d)\Gamma(f+b+d)\Gamma(1+\frac{1}{2}(f-a-c))} \qquad \ldots(9)
$$

$$
\times \frac{\Gamma(\frac{1}{2}(1+f-c-d))}{\Gamma(1+\frac{1}{2}(f-b-d))}
$$

where, for convergence, $Rl(f)>0$.

This gives the sum of a well-poised ${}_6F_5(-1)$ with the special second numerator parameter, under the two conditions $1 = a+d = b+c$.

These results were given by Whipple, in his paper "On well-poised series" (Proc.Lond.Maths.Soc. (2) Vol. 24. (1924)), equations (3.4),(4.5),(6.3).

<u>§ (4.41)</u> <u>Case four.</u>

Summing F_γ by Vandermonde's theorem,

if $\delta_n = \dfrac{(d)_n}{(D)_n}$

$$\gamma_n = \frac{\Gamma(D)\Gamma(D-f-d)}{\Gamma(D-f)\Gamma(D-d)} \frac{(d)_n(1+f-D)_n(f)_{2n}(-1)^n}{(1+f+d-D)_n}$$

If $\alpha_n = \dfrac{(-1)^n}{n!\,(a)_n}$, summing F_β by Vandermonde's theorem,

$$\beta_n = \frac{(f)_n(1+f-a)_n(-1)^n}{n!\,(a)_n}$$

Hence,

$${}_3F_2\left[\begin{matrix}f, 1+f-a, d; & -1 \\ a, D; & \end{matrix}\right]$$

$$= \frac{\Gamma(D)\Gamma(D-f-d)}{\Gamma(D-f)\Gamma(D-d)}\ {}_4F_3\left[\begin{matrix}d, 1+f-D, & \tfrac{1}{2}f, & \tfrac{1}{2}(f+1); & 1 \\ a, \tfrac{1}{2}(1+f+d-D), & 1+\tfrac{1}{2}(f+d-D); & & \end{matrix}\right] \quad ..(1)$$

provided d or f is a negative integer $-N$.

If $a = 1+f$, the ${}_3F_2(-1) = 1$, and we have

<u>Theorem (11)</u>

$${}_4F_3\left[\begin{matrix}d, 1+f-D, & \tfrac{1}{2}f, & \tfrac{1}{2}(f+1); & 1 \\ 1+f, \tfrac{1}{2}(1+f+d-D), & 1+\tfrac{1}{2}(f+d-D); & & \end{matrix}\right]$$

$$= \frac{\Gamma(D-f)\Gamma(D-d)}{\Gamma(D)\Gamma(D-f-d)} \quad ...(2)$$

where d or f is a negative integer.

This gives the sum of a finite special Saalschutzian ${}_4F_3(1)$ series.

Next, let $\alpha_n = \dfrac{(a)_n(-1)^n}{n!\,(A)_n(1+f+a-A)_n}$

Then, summing F_β by Saalschutz's theorem,

$$\beta_n = \frac{(f)_n(A-a)_n(1+f-A)_n}{n!\,(A)_n(1+f+a-A)_n}$$

Hence,

$$_4F_3\left[\begin{matrix}f, 1+f-A, & A-a, d; \; 1\\ & A, 1+f+a-A, D;\end{matrix}\right]$$

$$= \frac{\Gamma(D)\Gamma(D-f-d)}{\Gamma(D-f)\Gamma(D-d)}\ {}_5F_4\left[\begin{matrix}a, d, & 1+f-D, & \tfrac{1}{2}f, & \tfrac{1}{2}(1+f); \; 1\\ A, & 1+f+a-A, & \tfrac{1}{2}(1+f+d-D), & 1+\tfrac{1}{2}(f+d-D);\end{matrix}\right] \qquad \ldots(3)$$

This is a transformation between a nearly-poised $_4F_3(1)$ series and a Saalschutzian $_5F_4(1)$ series. It was given by Whipple in his paper "Some transformations of generalised hypergeometric series" (Proc.Lond.Maths.Soc. (2) Vol.26.1927, equ.(3.5)). Either f or d must be a negative integer. If d = −N, a negative integer, the other parameters in the $_5F_4(1)$ series can be chosen in four ways, so that this series becomes a summable Saalschutzian $_3F_2(1)$. Carrying out these summations, we have, putting a ≡ f, and b ≡ D,

<u>Theorem (12)</u>

$$_3F_2\left[\begin{matrix}a, 1+\tfrac{1}{2}a, -N; \; 1\\ \tfrac{1}{2}a, \; b;\end{matrix}\right] = \frac{(a+2-b)_N(b-a-1)_N}{(b)_N\,(1+a-b)_N} \qquad \ldots(4)$$

Similarly,

<u>Theorem (13)</u>

$$_3F_2\left[\begin{matrix}a, & b, & -N; \; 1\\ & 1+a-b, & 1+2b-N;\end{matrix}\right] = \frac{(a-2b)_N(1+\tfrac{1}{2}a-b)_N(-b)_N}{(1+a-b)_N(\tfrac{1}{2}a-b)_N(-2b)_N} \qquad \ldots(5)$$

Theorem (14)

$$ {}_4F_3\left[\begin{matrix} a, 1+\tfrac{1}{2}a, & b, & -N; & 1 \\ \tfrac{1}{2}a, & 1+a-b, & 1+2b-N; & \end{matrix}\right] = \frac{(a-2b)_N(-b)_N}{(1+a-b)_N(-2b)_N} \qquad \ldots(6) $$

Theorem (15)

$$ {}_4F_3\left[\begin{matrix} a, 1+\tfrac{1}{2}a, & b, & -N; & 1 \\ \tfrac{1}{2}a, & 1+a-b, & 2+2b-N; & \end{matrix}\right] = \frac{(a-2b-1)_N(\tfrac{1}{2}(1+a)-b)_N(-b-1)_N}{(1+a-b)_N(\tfrac{1}{2}(a-1)-b)_N(-2b-1)_N} \qquad \ldots(7) $$

When $\beta_n = \dfrac{1}{n!}\displaystyle\sum_{r=0}^{n}(-n)_r(-1)^r\alpha_r$,

if $\alpha_n = \dfrac{(f)_n(1+\tfrac{1}{2}f)_n(-1)^n}{n!(\tfrac{1}{2}f)_n(A)_n}$ F_β can be summed by theorem (12),

giving $\beta_n = \dfrac{(f+2-A)_n(A-f-1)_n}{n!\,(A)_n(1+f-A)_n}$

Hence, from § (3.4) where $\delta_n = (d)_n x^n$,

and $\gamma_n = \dfrac{(d)_n x^n}{(1-x)^{d+n}}$, we have

$$ \frac{1}{(1-x)^d}\,{}_3F_2\left[\begin{matrix} f, 1+\tfrac{1}{2}f, d; & \dfrac{-x}{(1-x)} \\ \tfrac{1}{2}f, A; & \end{matrix}\right] = {}_3F_2\left[\begin{matrix} f+2-A, A-f-1, d; & x \\ 1+f-A, A; & \end{matrix}\right] \qquad \ldots(8) $$

Summing F_β by theorem (12) in § (4.3) case (3),

if $\alpha_n = \dfrac{(f)_n(1+\tfrac{1}{2}f)_n(-1)^n}{n!\,(\tfrac{1}{2}f)_n}$

$$ \beta_n = \frac{(F-f-1)_{2n}}{n!\,(F)_{2n}(F-f)_n} $$

Hence, $\dfrac{\Gamma(F)\Gamma(F-d_1-d_2)}{\Gamma(F-d_1)\Gamma(F-d_2)}\,{}_4F_3\left[\begin{matrix} f, 1+\tfrac{1}{2}f, & d_1, & d_2; & -1 \\ \tfrac{1}{2}f, & F-d_1, & F-d_2; & \end{matrix}\right]$

$$ = {}_4F_3\left[\begin{matrix} \tfrac{1}{2}(F-f-1), \tfrac{1}{2}(F-f), & d_1, & d_2; & 1 \\ \tfrac{1}{2}(F+1), & \tfrac{1}{2}F, & F-f; & \end{matrix}\right] \qquad \ldots(9) $$

provided $Rl(F-d_1-d_2)>0$ and $Rl(f)<0$.

Summing F_β by theorem (13), if

$$\alpha_n = \frac{(a)_n(-\frac{1}{2}e)_n}{n!\,(1+a+\frac{1}{2}e)_n}$$

then $$\beta_n = \frac{(a+e)_n(1+\frac{1}{2}a+\frac{1}{2}e)_n(\frac{1}{2}e)_n}{(1+a+\frac{1}{2}e)_n(\frac{1}{2}a+\frac{1}{2}e)_n\, n!}$$

Hence, from § (3.5),

$$\frac{1}{(1-x)^e}\,{}_2F_1\left[\begin{matrix}a, & -\frac{1}{2}e; & x\\ & 1+a+\frac{1}{2}e; & \end{matrix}\right] = {}_3F_2\left[\begin{matrix}a+e, & 1+\frac{1}{2}a+\frac{1}{2}e, & \frac{1}{2}e; & x\\ & 1+a+\frac{1}{2}e, & \frac{1}{2}a+\frac{1}{2}e; & \end{matrix}\right] \quad \ldots(10)$$

This transforms a particular well-poised ${}_3F_2(x)$ into a well-poised ${}_2F_1(x)$. It is a combination of two results due to W.N.Bailey (Products of Generalised Hypergeometric Series, Proc.Lond.Maths.Soc. (2) Vol. 28, 1927, equs. 4.07,4.08.)

Summing the same series for F_β by theorem (14) gives the same result.

Summing the same series for F_β by theorem (15),

if $$\alpha_n = \frac{(f)_n(1+\frac{1}{2}f)_n(-\frac{1}{2}(1+e))_n}{n!\,(\frac{1}{2}f)_n(\frac{1}{2}(3+e)+f)_n}$$

$$\beta_n = \frac{(e+f)_n(1+\frac{1}{2}(e+f))_n(\frac{1}{2}(e-1))_n}{n!\,(\frac{1}{2}(3+e)+f)_n(\frac{1}{2}(f+e))_n}$$

Hence, from § (3.5),

$$\frac{1}{(1-x)^e}\,{}_3F_2\left[\begin{matrix}f, & 1+\frac{1}{2}f, & -\frac{1}{2}(e+1); & x\\ & \frac{1}{2}f, & \frac{1}{2}(3+e)+f; & \end{matrix}\right]$$

$$= {}_3F_2\left[\begin{matrix}f+e, & 1+\frac{1}{2}(f+e), & \frac{1}{2}(e-1); & x\\ & \frac{1}{2}(f+e), & \frac{1}{2}(3+e)+f; & \end{matrix}\right] \quad \ldots(11)$$

This is a transformation between two well-poised ${}_3F_2(x)$'s, both with the special form of the second numerator parameter.

§ (4.42) Case Five

Summing F_γ by Gauss's theorem, if $Rl(F-f-e)>0$

$$\delta_n = 1, \text{ and } \gamma_n = \frac{\Gamma(F)\Gamma(F-f-e)\,(f)_{2n}}{\Gamma(F-f)\Gamma(F-e)(F-e)_{2n}}$$

If $\alpha_n = \frac{(a)_n}{n!}$, summing F_β by Saalschutz's theorem,

$$\beta_n = \frac{(f)_n(\tfrac{1}{2}(f+e))_n(\tfrac{1}{2}(1+f+e))_n(F-f)_n}{(f-e)_n(\tfrac{1}{2}F)_n(\tfrac{1}{2}(F+1))_n\, n!}$$

Hence, $$\frac{\Gamma(F)\Gamma(F-f-e)}{\Gamma(F-f)\Gamma(F-e)}\,{}_3F_2\left[\begin{matrix}F-f-e, & \tfrac{1}{2}f, & \tfrac{1}{2}(f+1); & 1\\ & \tfrac{1}{2}(F-e), & \tfrac{1}{2}(1+F-e); & \end{matrix}\right]$$

$$= {}_4F_3\left[\begin{matrix}f, \tfrac{1}{2}(f+e), \tfrac{1}{2}(1+f+e), F-f; & 1\\ f+e, \quad \tfrac{1}{2}(F+1), \quad \tfrac{1}{2}F; & \end{matrix}\right] \qquad \ldots(1)$$

This is a transformation between a ${}_3F_2(1)$ and a ${}_4F_3(1)$, in both of which the sum of the numerator parameters equals the sum of the denominator parameters. Under this condition, both the series must terminate.

Next, summing F_β by theorem (13), if $\alpha_n = \frac{(f-e)_n}{n!}$ and $F = 1+2f$,

$$\beta_n = \frac{(f)_n(\tfrac{1}{2}(e+f))_n(\tfrac{1}{2}(1+e+f))_n}{n!\,(\tfrac{1}{2}+f)_n(1+f+e)_n}$$

Hence, $$\frac{\Gamma(1+2f)\,\Gamma(1+f-e)}{\Gamma(1+f)\,\Gamma(1+2f-e)}\,{}_3F_2\left[\begin{matrix}\tfrac{1}{2}f, & \tfrac{1}{2}(f+1), & f-e; & 1\\ & \tfrac{1}{2}(1-e)+f, & 1-\tfrac{1}{2}e+f; & \end{matrix}\right]$$

$$= {}_3F_2\left[\begin{matrix}f, \tfrac{1}{2}(e+f), \tfrac{1}{2}(1+e+f); & 1\\ \tfrac{1}{2}+f, \quad 1+e+f; & \end{matrix}\right] \qquad \ldots(2)$$

provided $Rl(F-f-e)>0$.

This is a transformation between two Saalschutzian ${}_3F_2(1)$'s.

A similar result is obtained by summing F_β by theorem (14).

Summing F_β by theorem (15),

$$\frac{\Gamma(2+2f)\Gamma(2+f-e)}{\Gamma(2+f)\Gamma(2+2f-e)}\,{}_4F_3\left[\begin{matrix}f-e,\ 1+\tfrac{1}{2}(f-e),\ \tfrac{1}{2}f,\ \tfrac{1}{2}(f+1);\ 1\\ \tfrac{1}{2}(f-e),\ 1+f-\tfrac{1}{2}e,\ \tfrac{1}{2}(3-e)+f;\end{matrix}\right]$$

$$= {}_4F_3\left[\begin{matrix}f,\ f+2,\ \tfrac{1}{2}(1+f+e),\ 1+\tfrac{1}{2}(f-e);\ 1\\ 1+f,\ 3/2+f,\ 2+e+f;\end{matrix}\right] \qquad \ldots(3)$$

where $Rl(F-f-e) > 0$.

This is a transformation between two Saalschutzian ${}_4F_3(1)$'s.

<u>§ (4.51)</u> <u>Case six</u>

Summing both F_γ and F_β by Vandermonde's theorem,

$$ {}_3F_2\left[\begin{matrix}f,\ \tfrac{1}{2}(f-N),\ \tfrac{1}{2}(1+f-N);\ 4\\ f-N,\ D;\end{matrix}\right]$$

$$= \frac{(D)_N}{(D-f)_N}\,{}_3F_2\left[\begin{matrix}1+f-D,\ \tfrac{1}{2}f,\ \tfrac{1}{2}(1+f);\ 4\\ D+N,\ 1+f-N-D;\end{matrix}\right]$$

<u>§ (4.52)</u> <u>Case seven</u>

Summing both F_γ and F_β by Vandermonde's theorem, as in § (3.4) above,

$$ {}_3F_2\left[\begin{matrix}A-a,\ d,\ -N;\ 1\\ A,\ D;\end{matrix}\right] = \frac{(D-d)_N}{(D)_N}\,{}_3F_2\left[\begin{matrix}a,\ d,\ -N;\ 1\\ A,\ 1+d-D-N;\end{matrix}\right] \qquad \ldots(1)$$

This is a relation between two general ${}_3F_2(1)$'s. Relations of this kind were investigated by J. Thomae (Journal fur Maths. Vol. 87 (1879)) and by Whipple ("A group of generalised hypergeometric series: relations between 120 allied series of the type F(a,b,c;d,e; 1),

Proc.Lond.Maths.Soc. (2) Vol. 23. (1925)).
Here Whipple introduced a contracted notation for this type of series.

Summing F_β by theorem (12),

$$ {}_4F_3\left[\begin{matrix} a, 1+\frac{1}{2}a, d, -N; 1 \\ \frac{1}{2}a, A, 1+d-D-N; \end{matrix}\right] = \frac{(D)_N}{(D-d)_N}\, {}_4F_3\left[\begin{matrix} A-a-1, 2+a-A, d, -N; 1 \\ 1+a-A, A, D; \end{matrix}\right] \quad \ldots(2) $$

§ (4.61) Case eight.

Summing F_γ by Vandermonde's theorem, where $f = -N$, and $\delta_n = 1$, then

$$ \gamma_n = \frac{(E-e)_N(-N)_{2n}(1-N-E)_{2n}}{(E)_N\,(1-N+e-E)_{2n}} $$

If $\alpha_n = \dfrac{(-1)^n}{n!\,(1-N+e-E)_n}$, summing F_β by Saalschutz's theorem

$$ \beta_n = \frac{(-N)_n(e-N)_{2n}(E-e)_n(-1)^n}{n!\,(E)_n(e-N)_n(1-N+e-E)_n} $$

Hence,

$$ {}_4F_3\left[\begin{matrix} -N, E-e, \frac{1}{2}(e-N), \frac{1}{2}(1-N+e); -4 \\ E, e-N, 1-N+e-E; \end{matrix}\right] $$

$$ = \frac{(E-e)_N}{(E)_N}\, {}_4F_3\left[\begin{matrix} -\frac{1}{2}N, \frac{1}{2}(1-N), \frac{1}{2}(1-N-E), 1-\frac{1}{2}(N+E); -4 \\ \frac{1}{2}(1-N+e-E), 1+\frac{1}{2}(e-N-E), 1-N+e-E; \end{matrix}\right] $$

§ (4.62) Case nine

Summing both F_γ and F_β by Vandermonde's theorem, if $\delta_n = (d)_n(-N)_n$

$$ \gamma_n = \frac{(E-d)_N(d)_n(-N)_n(1+d-E)_n(1-E-N)_n}{(E)_N(\frac{1}{2}(1+d-E-N))_n(1+\frac{1}{2}(d-E-N))_n}\cdot\frac{1}{4^n} $$

Let $\alpha_n = \frac{1}{n!(A)_n}$. Then $\beta_n = \frac{(A+E-1)_{2n}}{n!\ (E)_n(A)_n(A+E-1)_n}$

Hence,

$$ {}_4F_3\left[\begin{matrix} d, \tfrac{1}{2}(A+E-1), \tfrac{1}{2}(A+E), -N; \ 4 \\ A+E-1, \quad E, \ A; \end{matrix}\right] $$

$$ = \frac{(E-d)_N}{(E)_N} {}_4F_3\left[\begin{matrix} 1+d-E, 1-E-N, d, -N; \ \tfrac{1}{4} \\ 1+\tfrac{1}{2}(d-E-N), A, \tfrac{1}{2}(1+d-E-N); \end{matrix}\right] \qquad \ldots(1) $$

Next let $\alpha_n = \frac{(-1)^n}{n!(2-E)_n}$

Then $F_\beta = {}_2F_1\left[\begin{matrix} 1-E-n, -N; \ -1 \\ 2-E; \end{matrix}\right]$

Summing F_β by Kummer's theorem, if n is even,

$$ \beta_{2n} = \frac{(-1)^n}{n!(\tfrac{1}{2})_n(\tfrac{1}{2}(E+1))_n(\tfrac{1}{2}(3-E))_n\, 4^n} $$

and if n is odd,

$$ \beta_{2n+1} = \frac{2(1-E)\ (-1)^n}{(2-E)E\ \ n!(3/2)_n(1+\tfrac{1}{2}E)_n(2-\tfrac{1}{2}E)_n 4^n} $$

Hence, $\frac{(E-d)_N}{(E)_N} {}_4F_3\left[\begin{matrix} d, \quad 1+d-E, \quad 1-E-N, \ -N; \ -\tfrac{1}{4} \\ \tfrac{1}{2}(1+d-E-N), \ 1+\tfrac{1}{2}(d-E-N), 2-E; \end{matrix}\right]$

$$ = {}_4F_3\left[\begin{matrix} \tfrac{1}{2}d, -\tfrac{1}{2}N, \tfrac{1}{2}(d+1), \tfrac{1}{2}(1-N); \ -4 \\ \tfrac{1}{2}, \tfrac{1}{2}(E+1), \tfrac{1}{2}(3-E); \end{matrix}\right] - \frac{2(1-E)}{(2-E)E} $$

$$ \times\ {}_4F_3\left[\begin{matrix} \tfrac{1}{2}(d+1), 1+\tfrac{1}{2}d, \tfrac{1}{2}(1-N), 1-\tfrac{1}{2}N; \ -4 \\ 3/2, 1+\tfrac{1}{2}E\ , \ 2-\tfrac{1}{2}E; \end{matrix}\right] \qquad \ldots(2) $$

§ (4.7) Case ten

Summing F_γ by Gauss's theorem,

if $\delta_n = \frac{(d)_n}{(D)_n}$

then $\gamma_n = \frac{\Gamma(D)\Gamma(D-d-e)(d)_n}{\Gamma(D-d)\Gamma(D-e)(D-d)_n}$

where $Rl(D-d-e) > 0$.

Then, if $\alpha_n = \dfrac{(a_1)_n (a_2)_n}{n!\,(a_1+a_2+e)_n}$, summing F_β by Saalschutz's theorem,

$$\beta_n = \frac{(a_1+e)_n (a_2+e)_n}{n!\,(a_1+a_2+e)_n},$$ as in § (3.3) above.

Hence,

$$ {}_3F_2\left[\begin{matrix} a_1+e, & a_2+e, d; & 1 \\ & a_1+a_2+e, D; & \end{matrix}\right] = \frac{\Gamma(D)\Gamma(D-d-e)}{\Gamma(D-e)\Gamma(D-d)}\,{}_3F_2\left[\begin{matrix} a_1, & a_2, & d; & 1 \\ a_1+a_2+e, & D-e; & & \end{matrix}\right] \qquad \ldots(1)$$

provided $Rl(D-d) > 0$ and $Rl(e) < 0$.

If $D = 1+d$, $a_1 = a$, $a_2 = b$, and either a or b is a negative integer, the second series can be summed by Saalschutz's theorem, to give:-

<u>Theorem (16)</u>

$$ {}_3F_2\left[\begin{matrix} a+e, b+e, & d; & 1 \\ a+b+e, 1+d; & & \end{matrix}\right]$$

$$= \frac{\Gamma(1+d)\Gamma(1-a-e)\Gamma(1-b-e)\Gamma(1+d-a-b-e)}{\Gamma(1+d-a-e)\Gamma(1+d-b-e)\Gamma(1-e-a-b)} \qquad \ldots(2)$$

provided either a or b is a negative integer, d is not a negative integer, and $Rl(1-e) > 0$.

<u>Note.</u> The equation corresponding to eq.(4) § (3.1) is here

$$\sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_r (b)_r (d)_{n+r} (e)_n}{r!\,(a+b+e)_r (1+d)_{n+r}\, n!}$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{(d)_n (e)_{n-r} (a)_r (b)_r}{(1+d)_n (n-r)!\, r!\,(a+b+e)_r}$$

and this change in the order of summation can be justified under the given conditions.

Summing F_β by theorem (13),

$$\frac{\Gamma(D)\Gamma(D-d-e)}{\Gamma(D-e)\Gamma(D-d)}\,{}_3F_2\left[\begin{matrix} a, & -\tfrac{1}{2}e, & d; & 1 \\ & 1+a+\tfrac{1}{2}e, & D-e; & \end{matrix}\right] = {}_4F_3\left[\begin{matrix} a+e, 1+\tfrac{1}{2}(a+e), \tfrac{1}{2}e, d; & 1 \\ \tfrac{1}{2}(a+e), 1+a+\tfrac{1}{2}e, D; & \end{matrix}\right] \qquad \ldots(3)$$

provided $Rl(D-d+1)>0$, and $Rl(D-d-e)>0$.

This transforms a nearly-poised ${}_4F_3(1)$ series into a nearly-poised ${}_3F_2(1)$ series.

Summing F_β by theorem (14),

$$\frac{\Gamma(D)\Gamma(D-d-e)}{\Gamma(D-d)\Gamma(D-e)}{}_4F_3\left[\begin{matrix}f,1+\tfrac{1}{2}f, & -\tfrac{1}{2}e, & d; & 1\\ \tfrac{1}{2}f, & 1+f+\tfrac{1}{2}e, & D-e; & \end{matrix}\right]$$

$$= {}_3F_2\left[\begin{matrix}f+e, & \tfrac{1}{2}e, d; & 1\\ & 1+f+\tfrac{1}{2}e, D; & \end{matrix}\right] \qquad \ldots(4)$$

provided $Rl(D-d)>0$ and $Rl(1-e)>0$.

Summing F_β by theorem (15),

$$\frac{\Gamma(D)\Gamma(D-d-e)}{\Gamma(D-e)\Gamma(D-d)}{}_4F_3\left[\begin{matrix}f,1+\tfrac{1}{2}f, & -\tfrac{1}{2}-\tfrac{1}{2}e, & d; & 1\\ \tfrac{1}{2}f, & \tfrac{1}{2}(3+e)+f, & D-e; & \end{matrix}\right]$$

$$= {}_4F_3\left[\begin{matrix}f+e,1+\tfrac{1}{2}(f+e), & \tfrac{1}{2}(e-1), & d; & 1\\ \tfrac{1}{2}(f+e), & \tfrac{1}{2}(3+e)+f, & D; & \end{matrix}\right] \qquad \ldots(5)$$

under the same conditions.

§ (4.8) Case eleven

Summing F_γ and F_β by Vandermonde's theorem,

if $\delta_n = (d)_n$, then $\gamma_n = \dfrac{\Gamma(E)\Gamma(E-d-e)}{\Gamma(E-d)\Gamma(E-e)}\dfrac{(d)_n(1+d-E)_n}{(1+d+e-E)_n}$

Let $\alpha_n = \dfrac{(-1)^n}{n!}$. Then $\beta_n = \dfrac{(E-e)_n(-1)^n}{n!\,(E)_n}$

Hence, $${}_2F_1\left[\begin{matrix}E-e,d; & -1\\ E; & \end{matrix}\right] = \frac{\Gamma(E)\Gamma(E-d-e)}{\Gamma(E-e)\Gamma(E-d)}{}_2F_1\left[\begin{matrix}d, & 1+d-E; & -1\\ & 1+d+e-E; & \end{matrix}\right]$$

provided d is a negative integer. $\qquad \ldots(1)$

Summing F_β by Dixon's theorem,

if $\alpha_n = \dfrac{(1+e-E)_n(-1)^n}{n!\,(2-E)_n}$ and n is even,

$$\beta_{2n} = \frac{(\tfrac{1}{2}(1+E)-e)_n(\tfrac{1}{2}(1-E)+e)_n}{n!\,(\tfrac{1}{2})_n(\tfrac{1}{2}(E+1))_n(\tfrac{1}{2}(3-E))_n 4^n}$$

If n is odd,

$$\beta_{2n+1} = \frac{(E-1)(E-2e)}{E(2-E)} \frac{(1+\frac{1}{2}E-e)_n(1-\frac{1}{2}E+e)_n}{n!(3/2)_n(2-\frac{1}{2}E)_n(1+\frac{1}{2}E)_n 4^n}$$

Hence,

$$\frac{(E-e)_N}{(E)_N}\ {}_3F_2\left[\begin{matrix}-N,1-N-E,1+e-E; & -1\\ 2-E,1-N+e-E; & \end{matrix}\right]$$

$$= {}_4F_3\left[\begin{matrix}\frac{1}{2}(1+E)-e,\frac{1}{2}(1-E)+e,-\frac{1}{2}N,\frac{1}{2}(1-N); & 1\\ \frac{1}{2},\frac{1}{2}(1+E),\frac{1}{2}(3-E); & \end{matrix}\right]$$

$$+\frac{(E-1)(E-2e)}{E\ (2-E)}\ {}_4F_3\left[\begin{matrix}1+\frac{1}{2}E-e,1-\frac{1}{2}E+e,\frac{1}{2}(1-N),1-\frac{1}{2}N; & 1\\ 3/2,\ 2-\frac{1}{2}E,1+\frac{1}{2}E; & \end{matrix}\right] \qquad \ldots(2)$$

§ (5.1) Applications of Saalschutz's theorem

Let us now suppose F_γ can be summed by Saalschutz's theorem. There are ten possibilities for F_γ, of which only four lead to summable series for F_β. These are:-

(1) $F_\gamma = {}_3F_2\left[\begin{matrix}d_1+n,d_2+n,d_3+n; & 1\\ F+2n,\ D+n; & \end{matrix}\right]$

$$F_\beta = \sum_{r=0}^{n}\frac{(-n)_r(-1)^r}{(F+n)_r}\alpha_r$$

(2) $F_\gamma = {}_3F_2\left[\begin{matrix}f+2n,d+n,\ e; & 1\\ D+n,F+2n; & \end{matrix}\right]$

$$F_\beta = \sum_{r=0}^{n}\frac{(-n)_r(f+n)_r}{(1-e-n)_r(F+n)_r}\alpha_r$$

(3) $F_\gamma = {}_3F_2\left[\begin{matrix}d_1+n,d_2+n,\ e; & 1\\ D_1+n,D_2+n; & \end{matrix}\right]$

$$F_\beta = \sum_{r=0}^{n}\frac{(-n)_r}{(1-e-n)_r}\alpha_r$$

(4) $F_\gamma = {}_3F_2\left[\begin{matrix} d+n, & e_1, e_2; & 1 \\ & D+n, E; & \end{matrix}\right]$

$$F_\beta = \sum_{r=0}^{n} \frac{(-n)_r (1-E-n)_r}{(1-e_1-n)_r (1-e_2-n)_r} \alpha_r$$

In all these series for F_γ, the parameters are subject to the usual restrictions, that one of the numerator parameters is a negative integer, and that the sum of the numerator parameters exceeds that of the denominator parameters by one.

§ (5.2) Case one

$$\text{If } \delta_n = \frac{(d_1)_n (d_2)_n (-N)_n}{(D)_n}$$

$$\gamma_n = \frac{(D-d_1)_N (D-d_2)_N (d_1)_n (d_2)_n (-N)_n (-1)^n}{(D)_N (D-d_1-d_2)_N (1+d_1-D-N)_n (1+d_2-D-N)_n (1+d_1+d_2-D)_n}$$

Let $\alpha_n = \dfrac{(a_1)_n (a_2)_n (1+\frac{1}{2}(F-1))_n (F-1)_n (-1)^n}{n!\ (F-a_1)_n (F-a_2)_n (\frac{1}{2}(F-1))_n}$

Then, summing F_β as a well-poised ${}_5F_4(1)$ series, where

$F \equiv 1+d_1+d_2-N-D$,

$\beta_n = \dfrac{(F-a_1-a_2)_n}{n!(F-a_1)_n (F-a_2)_n}$ and we have

$${}_4F_3\left[\begin{matrix} F-a_1-a_2, & d_1, & d_2, -N; & 1 \\ & F-a_1, F-a_2, & D; & \end{matrix}\right]$$

$$= \frac{(D-d_1)_N (D-d_2)_N}{(D)_N (D-d_1-d_2)_N} {}_7F_6\left[\begin{matrix} F-1, \frac{1}{2}(1+F), & a_1, & a_2, & d_1, & d_2, & -N; & 1 \\ \frac{1}{2}(F-1), & F-a_1, & F-a_2, & F-d_1, & F-d_2, & F+N; & \end{matrix}\right] \quad \ldots(1)$$

This transforms a Saalschutzian ${}_4F_3(1)$ series into a well-poised ${}_7F_6(1)$ series, with the special form of the second numerator parameter. It was given by Whipple in his paper "On well-poised series" (Proc.Lond.Maths.Soc. (2) Vol 24 (1926)).

Put $a \equiv F-1$, $b \equiv a_1$, $c \equiv a_2$, $d \equiv d_1$, and $e \equiv d_2$. Then if $1+2a = b+c+d+e-N$, the ${}_4F_3(1)$ series reduces to a ${}_3F_2(1)$ series which can be summed by Saalschutz's theorem. This gives:--

Theorem (17) (Dougall's theorem)

$$
{}_7F_6\left[\begin{matrix} a, 1+\tfrac{1}{2}a, & b, & c, & d, & -N; & 1 \\ & \tfrac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a+N; & \end{matrix}\right]
$$

$$
= \frac{(1+a)_N(1+a-b-c)_N(1+a-b-d)_N(1+a-c-d)_N}{(1+a-b)_N(1+a-c)_N(1+a-d)_N(1+a-b-c-d)_N} \qquad \ldots(2)
$$

where $1+2a = b+c+d+e-N$.

(J. Dougall. "On Vandermonde's theorem" Proc. Edinburgh Maths.Soc. Vol.25 (1907)).

§ (5.3) Case two

Summing F_γ and F_β by Saalschutz's theorem,

$$
\delta_n = \frac{(d)_n}{(1+f+d+e-F)_n}
$$

and

$$
\gamma_n = \frac{\Gamma(1+f+d+e-F)\Gamma(1+f-F)\Gamma(1+d-F)\Gamma(1+e-F)}{\Gamma(1-F)\Gamma(1+d+e-F)\Gamma(1+f+e-F)\Gamma(1+f+d-F)} \times \frac{(d)_n(F-d-e)_n(\tfrac{1}{2}f)_n(\tfrac{1}{2}(f+1))_n}{(F-d)_n(1+f+d-F)_n(\tfrac{1}{2}(F-e))_n(\tfrac{1}{2}(1+F-e))_n}
$$

provided f or d is a negative integer.

Let

$$
\alpha_n = \frac{(F-f-e)_n}{n!}
$$

then

$$
\beta_n = \frac{(f)_n(\tfrac{1}{2}(f+e))_n(\tfrac{1}{2}(1+f+e))_n(F-f)_n}{(f+e)_n(\tfrac{1}{2}F)_n(\tfrac{1}{2}(F+1))_n n!}
$$

~~summing F_β as a well-poised ${}_5F_4(1)$ series, where $F = 1+f-e$.~~

Hence,

$$\frac{\Gamma(1+f+d+e-F)\Gamma(1+f-F)\Gamma(1+d-F)\Gamma(1+e-F)}{\Gamma(1-F)\Gamma(1+d+e-F)\Gamma(1+e+f-F)\Gamma(1+d+f-F)}$$

$$\times\ {}_5F_4\left[\begin{matrix}F-f-e, & d, & F-d-e, & \tfrac{1}{2}f, & \tfrac{1}{2}(1+f); & 1\\ & F-d, & 1+f+d-F, & \tfrac{1}{2}(1+F-e), & \tfrac{1}{2}(F-e); & \end{matrix}\right]$$

$$= {}_5F_4\left[\begin{matrix}f, & \tfrac{1}{2}(f+e), & \tfrac{1}{2}(1+f+e), & F-f, & d; & 1\\ & \tfrac{1}{2}(1+F), & \tfrac{1}{2}F, & f+e, & 1+f+d+e-F; & \end{matrix}\right]$$

i.e. putting $F = 1+f-e$,

$$\frac{\Gamma(d+2e)\Gamma(e)\Gamma(d+e-f)\Gamma(2e-f)}{\Gamma(e-f)\Gamma(d+2e-f)\Gamma(2e)\Gamma(d+e)}\,{}_5F_4\left[\begin{matrix}1-2e, & d, & 1+f-d-2e, & \tfrac{1}{2}f, & \tfrac{1}{2}(1+f); & 1\\ & 1+f-d-e, & d+e, & 1+\tfrac{1}{2}f-e, & \tfrac{1}{2}(1+f)-e; & \end{matrix}\right]$$

$$= {}_5F_4\left[\begin{matrix}f, & \tfrac{1}{2}(1+f+e), & \tfrac{1}{2}(f+e), & 1-e, & d; & 1\\ & \tfrac{1}{2}(1+f-e), & 1+\tfrac{1}{2}(f-e), & f+e, & d+2e; & \end{matrix}\right] \qquad \ldots(1)$$

This transforms a nearly-poised terminating ${}_5F_4(1)$ series into another nearly-poised terminating ${}_5F_4(1)$ series. (Bailey, "Some identities involving generalised hypergeometric series", § (6.4), Proc.Lond.MathsSoc. (2) 1929.)

Next, summing F_β by Dougall's theorem,

if $$\alpha_n = \frac{(f-e)_n(1+\tfrac{1}{2}(f-e))_n(a_1)_n(a_2)_n(a_3)_n}{(\tfrac{1}{2}(f-e))_n(1+f-e-a_1)_n(1+f-e-a_2)_n(1+f-e-a_3)_n\,n!}$$

$$\beta_n = \frac{(f)_n(e+a_1)_n(e+a_2)_n(e+a_3)_n}{n!\,(1+f-e-a_1)_n(1+f-e-a_2)_n(1+f-e-a_3)_n}$$

provided $1+f-e = a_1+a_2+a_3+e$.

Hence,

$${}_5F_4\left[\begin{matrix}f, & e+a_1, & e+a_2, & e+a_3, & d; & 1\\ & 1+f-e-a_1, & 1+f-e-a_2, & 1+f-e-a_3, & d+2e; & \end{matrix}\right]$$

$$= \frac{\Gamma(d+2e)\Gamma(e)\Gamma(d+e-f)\Gamma(2e-f)}{\Gamma(e-f)\Gamma(d+2e-f)\Gamma(2e)\Gamma(d+e)}$$

$$\times\ {}_9F_8\left[\begin{matrix}f-e, & 1+\tfrac{1}{2}(f-e), & a_1, & a_2, & a_3, & d, & 1+f-d-2e, & \tfrac{1}{2}f, & \tfrac{1}{2}(1+f); & 1\\ & \tfrac{1}{2}(f-e), & 1+f-e-a_1, & 1+f-e-a_2, & 1+f-e-a_3, & 1+f-e-d, & d+e, & 1+\tfrac{1}{2}f-e, & \tfrac{1}{2}(1+f)-e; & \end{matrix}\right] \qquad \ldots(2)$$

provided f or d is a negative integer. This is a transformation between a nearly-poised $_5F_4(1)$ series and a well-poised $_9F_8(1)$ series both of which terminate. (Bailey. Loc. cit. § 8.1)

Taking $a_1 = -e$, this $_5F_4(1) = 1$, and the $_9F_8(1)$ series reduces to a $_7F_6(1)$ series, since $a_1+a_2+a_3+e = 1+f-e$. Hence we get,

<u>Theorem (18)</u>

$$ {}_7F_6\left[\begin{matrix} f-e, 1+\tfrac{1}{2}(f-e), -e, & d, 1+f-2e-d, & \tfrac{1}{2}f, & \tfrac{1}{2}(1+f); \; 1 \\ \tfrac{1}{2}(f-e), 1+f, & 1+f-d-e, d+e & , 1+\tfrac{1}{2}f-e, & \tfrac{1}{2}(1+f)-e \; ; \end{matrix}\right] $$

$$ = \frac{\Gamma(e-f)\Gamma(d+2e-f)\Gamma(2e)\Gamma(d+e)}{\Gamma(d+2e)\Gamma(e)\Gamma(d+e-f)\Gamma(2e-f)} \quad (\text{where } d = -n) \qquad \ldots(3) $$

This sum of a special $_7F_6(1)$ series is not a particular case of Dougall's theorem unless $\frac{1}{2}(f+1) = e$. (Bailey Loc. cit. equ.(7.42))

Applying the transformation of a $_7F_6(1)$ into a $_4F_3(1)$ series, given in § (5.2) equ. (5), we deduce

<u>Theorem (19)</u>

$$ {}_4F_3\left[\begin{matrix} \tfrac{1}{2}(f-e), \tfrac{1}{2}(1+f-e), & f+n, & -n; \; 1 \\ \tfrac{1}{2}f, \tfrac{1}{2}(1+f), & 1+f-e; & \end{matrix}\right] $$

$$ = \frac{(e)_n}{(f)_n} \qquad \ldots(4) $$

(Bailey. Loc. cit. § (8) equ. C.)

Summing F_β by theorem (13),

$$\frac{\Gamma(d+e-f)\Gamma(-f)\Gamma(d-2f)\Gamma(e-2f)}{\Gamma(-2f)\Gamma(d+e-2f)\Gamma(e-f)\Gamma(d-f)}\,{}_5F_4\left[\begin{matrix} f-e, & d, & 1+2f-d-e, & \tfrac{1}{2}f, & \tfrac{1}{2}(1+f); & 1 \\ & d-f, & 1+2f-d, & \tfrac{1}{2}(1-e)+f, & 1-\tfrac{1}{2}e+f; & \end{matrix}\right]$$

$$= {}_4F_3\left[\begin{matrix} f, & \tfrac{1}{2}(1+f+e), & \tfrac{1}{2}(e+f), & d; & 1 \\ & \tfrac{1}{2}+f, & 1+f+e, & d+e-f; & \end{matrix}\right] \qquad \ldots(5)$$

where f or d is a negative integer.

Similarly, summing F_β by theorem (14),

$$\frac{\Gamma(d+e-f)\Gamma(-f)\Gamma(d-2f)\Gamma(e-2f)}{\Gamma(-2f)\Gamma(d+e-2f)\Gamma(e-f)\Gamma(d-f)}$$

$$\times\ {}_6F_5\left[\begin{matrix} f-e, & 1+\tfrac{1}{2}(f-e), & d, & 1+2f-d-e, & \tfrac{1}{2}f, & \tfrac{1}{2}(f+1); & 1 \\ & \tfrac{1}{2}(f-e), & 1+2f-d, & d-f, & \tfrac{1}{2}(1-e)+f, & 1-\tfrac{1}{2}e+f; & \end{matrix}\right]$$

$$= {}_4F_3\left[\begin{matrix} f, & \tfrac{1}{2}(1+e+f), & 1+\tfrac{1}{2}(e+f), & d; & 1 \\ & \tfrac{1}{2}+f, & 1+e+f, & d+e-f; & \end{matrix}\right] \qquad \ldots(6)$$

provided f or d is a negative integer.

Summing F_β by theorem (15),

$$\frac{\Gamma(d+e-f-1)\Gamma(-f-1)\Gamma(d-2f-1)\Gamma(e-2f-1)}{\Gamma(-2f-1)\Gamma(d+e-2f-1)\Gamma(e-f-1)\Gamma(d-f-1)} \times$$

$${}_6F_5\left[\begin{matrix} f-e, & 1+\tfrac{1}{2}(f-e), & d, & 2+2f-d-e, & \tfrac{1}{2}f, & \tfrac{1}{2}(1+f); & 1 \\ & \tfrac{1}{2}(f-e), & 2+2f-d, & d-f-1, & 1+f-\tfrac{1}{2}e, & 3/2+f-\tfrac{1}{2}e; & \end{matrix}\right]$$

$$= {}_5F_4\left[\begin{matrix} f, & f+2, & \tfrac{1}{2}(1+e+f), & 1+\tfrac{1}{2}(e+f), & d; & 1 \\ & f+1, & 3/2+f, & 2+e+f, & d+e-f-1; & \end{matrix}\right] \qquad \ldots(7)$$

where f or d is a negative integer.

§ (5.4) Case three

Summing F_γ and F_β by Saalschutz's theorem,

if $\delta_n = \dfrac{(d)_n(-N)_n}{(D)_n(1+e+d-N-D)_n}$

then $\gamma_n = \dfrac{(D-d)_N(D-e)_N(d)_n(-N)_n}{(D)_N(D-d-e)_N(D-e)_n(1+d-N-D)_n}$

Taking $\alpha_n = \frac{(a)_n (b)_n}{n!\,(a+b+e)_n}$

then $\beta_n = \frac{(a+e)_n (b+e)_n}{(a+b+e)_n\, n!}$

Hence,

$${}_4F_3\left[\begin{matrix} a+e, b+e, d, & -N; \; 1 \\ a+b+e, D, 1+e+d-N-D; & \end{matrix}\right]$$

$$= \frac{(D-d)_N (D-e)_N}{(D)_N (D-d-e)_N}\, {}_4F_3\left[\begin{matrix} a, \quad b, \quad d, & -N; \; 1 \\ a+b+e, D-e, 1+d-N-D; & \end{matrix}\right] \qquad \ldots(1)$$

This is a transformation between two terminating Saalschutzian ${}_4F_3(1)$'s.

Summing F_β by theorem (13),

$$\frac{(D-e)_N (D-d)_N}{(D)_N (D-d-e)_N}\, {}_4F_3\left[\begin{matrix} a, \quad -\tfrac{1}{2}e, \quad d, & -N; \; 1 \\ 1+a+\tfrac{1}{2}e, D-e, 1+d-N-D; & \end{matrix}\right]$$

$$= {}_5F_4\left[\begin{matrix} a+e, 1+\tfrac{1}{2}(a+e), \quad \tfrac{1}{2}e, d, & -N; \; 1 \\ \tfrac{1}{2}(a+e), 1+a+\tfrac{1}{2}e, D, 1+e+d-N-D; & \end{matrix}\right] \qquad \ldots(2)$$

Summing F_β by theorem (14),

$$\frac{(D-e)_N (D-d)_N}{(D)_N (D-d-e)_N}\, {}_5F_4\left[\begin{matrix} f, 1+\tfrac{1}{2}f, \quad -\tfrac{1}{2}e, \quad d, & -N; \; 1 \\ \tfrac{1}{2}f, 1+f+\tfrac{1}{2}e, D-e, 1+d-N-D; & \end{matrix}\right]$$

$$= {}_4F_3\left[\begin{matrix} f+e, \quad \tfrac{1}{2}e, d, & -N; \; 1 \\ 1+f+\tfrac{1}{2}e, D, 1+e+d-N-D; & \end{matrix}\right] \qquad \ldots(3)$$

Summing F_β by theorem (15),

$$\frac{(D-e)_N (D-d)_N}{(D)_N (D-d-e)_N}\, {}_5F_4\left[\begin{matrix} f, 1+\tfrac{1}{2}f, \quad -\tfrac{1}{2}(1+e), \quad d, & -N; \; 1 \\ \tfrac{1}{2}f, \tfrac{1}{2}(3+e)+f, D-e, 1+d-N-D; & \end{matrix}\right]$$

$$= {}_5F_4\left[\begin{matrix} f+e, 1+\tfrac{1}{2}(f+e), \quad \tfrac{1}{2}(e-1), d, & -N; \; 1 \\ \tfrac{1}{2}(f+e), \tfrac{1}{2}(3+e)+f, D, 1+e+d-N-D; & \end{matrix}\right] \qquad \ldots(4)$$

§ (5.5) Case four

Summing F_γ and F_β by Saalschutz's theorem, if $\delta_n = \dfrac{(-N)_n}{(D)_n}$

then $\gamma_n = \dfrac{(1-E)_N(D-e)_N(1-E-N)_n(-N)_n}{(D)_N(1+e-E)_N(D-e)_n(1+e-E-N)_n}$

Let $\alpha_n = \dfrac{(1-D-N)_n}{n!}$

Then, $\beta_n = \dfrac{(E-e)_n(1+e-D-N)_n}{n!\ (E)_n}$.

Hence,

$$\frac{(1-E)_N(D-e)_N}{(D)_N(1+e-E)_N}\,{}_3F_2\left[\begin{matrix}1-E-N, & 1-D-N, & -N; & 1\\ & 1+e-E-N, & D-e; & \end{matrix}\right]$$

$$= {}_3F_2\left[\begin{matrix}E-e, & 1+e-D-N, & -N; & 1\\ & E, & D; & \end{matrix}\right]$$

§ (6.1) Applications of Dougall's theorem

Suppose F_γ can be summed by Dougall's theorem as a ${}_7F_6(1)$ which is well-poised in a d parameter, an e parameter, or an f parameter. Now, F_γ cannot be well-poised in a d parameter or terms involving $\frac{1}{2}n$ would arise, in the ${}_7F_6(1)$ series. Also F_γ cannot be well-poised in an e parameter , for, if it were, the corresponding series for F_β would not be able to be summed for any values of α_r independent of n, by any of the known summation theorems. Thus, F_γ must be well-poised in an f parameter, and the parameters are subject to a restriction of the type

$$(d_1+n)+\ldots\ldots+e_1+\ldots\ldots+(f_2+2n)+\ldots = 1+2f_1+4n.$$

There are three possible cases in which F_γ can be summed as a ${}_7F_6(1)$ which is well-poised in an f parameter. Of these, one results in a series for F_β which is **not** summable. The two remaining cases are:-

(1) $$F_\gamma = {}_7F_6\left[\begin{matrix} f_1+2n, & 1+\tfrac{1}{2}f_1+n, & f_2+2n, & e_1, & e_2, & d+n, & -N+n;\ 1 \\ & \tfrac{1}{2}f_1+n, & 1+f_1-f_2, & 1+f_1-e_1+2n, & 1+f_1-e_2+2n, & 1+f_1-d+n, & 1+f_1+N+n; \end{matrix}\right]$$

$$F_\beta = \sum_{r=0}^{n} \frac{(-n)_r(f_1+n)_r\,(f_2+n)_r(f_2-f_1-n)_r}{(1-e_1-n)_r(1-e_2-n)_r(1+f_1-e_1+n)_r(1+f_1-e_2+n)_r}\alpha_r$$

where $1+2f_1 = e_1+e_2+d-N+f_2$.

(2) $$F_\gamma = {}_7F_6\left[\begin{matrix} f+2n, & 1+\tfrac{1}{2}f+n, & d_1+n, & d_2+n, & d_3+n, & e, & n-N;\ 1 \\ & \tfrac{1}{2}f+n, & 1+f-d_1+n, & 1+f-d_2+n, & 1+f-d_3+n, & 1+f-e+2n, & 1+f+N+n; \end{matrix}\right]$$

$$F_\beta = \sum_{r=0}^{n} \frac{(-n)_r\,(f+n)_r}{(1-e-n)_r\,(1+f-e+n)_r}\alpha_r$$

where $1+2f = e+d_1+d_2+d_3-N$.

§ (6.2) The first case

Summing both F_γ and F_β by Dougall's theorem,

if $$\delta_n = \frac{(d)_n(-N)_n(1+\tfrac{1}{2}f_1)_n}{(\tfrac{1}{2}f_1)_n(1+f_1-d)_n(1+f_1+N)_n}$$

then $$\gamma_n = \frac{(1+f_1)_N(1+f_1-f_2-d)_N(1+f_1-f_2-e_1)_N(1+f_1-d-e_1)_N}{(1+f_1-f_2)_N(1+f_1-d)_N(1+f_1-e_1)_N(1+f_1-f_2-d-e_1)_N}$$

$$\times\frac{(1+f_1-e_1-e_2-d)_n(1+f_1-e_1-e_2+N)_n(f_2)_{2n}(d)_n(-N)_n}{(1+f_1-e_1-e_2)_{2n}(1+f_1-d-e_2)_n(1+f_1+N-e_2)_n(1+f_1+N-e_1)_n(1+f_1-d-e_1)_n}$$

where $e_1+e_2+d-N+f_2 = 1+2f_1$.

Let $f_2-e_2 = f_1-e_1$, and

$$\alpha_n = \frac{(f_1-e_1)_n(1+\frac{1}{2}(f_1-e_1))_n(a)_n}{n!\ (\frac{1}{2}(f_1-e_1))_n(1+f_1-e_1-a)_n}$$

where $1+2f_1-2e_1 = 2f_2+a$. Then $a = 1-2e_2$, and

$$\beta_n = \frac{(f_1)_n(f_2)_n(1+e_1-2e_2)_n(1-e_2)_n(\frac{1}{2}(e_1+f_2))_n(\frac{1}{2}(1+e_1+f_2))_n}{n!(1+f_1-f_2)_n(f_2+e_2)_n(f_2+e_1)_n(1+\frac{1}{2}(f_1-e_2))_n(\frac{1}{2}(1+f_1-e_2))_n}$$

provided e_1, e_2, and hence f_2 are not negative integers.

Hence,

$$\frac{(1+f_1)_N(1+f_1-e_1-d)_N(1+f_1-f_2-e_1)_N(1+f_1-f_2-d)_N}{(1+f_1-f_2)_N(1+f_1-d)_N(1+f_1-e_1)_N(1+f_1-f_2-d-e_1)_N}$$

$$\times\ {}_9F_8\left[\begin{matrix}f_1-e_1, 1+\frac{1}{2}(f_1-e_1), 1-2e_2, f_2-N-f_1, f_2+d-f_1,\\ \frac{1}{2}(f_1-e_1), f_2+e_2, 1+f_1+N-e_2, 1+f_1-d-e_2,\end{matrix}\right.$$

$$\left.\begin{matrix}d, \frac{1}{2}f_2, \frac{1}{2}(f_2+1), -N; 1\\ 1+f_1-e_1-d, 1+\frac{1}{2}(f_1-e_1-e_2), \frac{1}{2}(1+f_1-e_1-e_2), 1+f_1-e_1+N;\end{matrix}\right]$$

$$= {}_9F_8\left[\begin{matrix}f_1, 1+\frac{1}{2}f_1, f_2, 1+e_1-2e_2, 1-e_2, \frac{1}{2}(e_1+f_2),\\ \frac{1}{2}f_1, 1+f_1-f_2, f_2+e_2, f_2+e_1, 1+\frac{1}{2}(f_1-e_2),\end{matrix}\right.$$

$$\left.\begin{matrix}\frac{1}{2}(1+e_1+f_2), d, -N; 1\\ \frac{1}{2}(1+f_1-e_2), 1+f_1-d, 1+f_1+N;\end{matrix}\right] \quad \ldots(1)$$

where $f_1-e_1 = f_2-e_2$ and $1+2f_1 = e_1+e_2+f_2+d-N$.

(W.N.Bailey, "Some identities involving hypergeometric series" Proc.Lond.Maths.Soc. (2) Vol. 29 (1928) equ. 7.41.))

§ (6.3) Second case

Summing both F_γ and F_β by Dougall's theorem,

if $$\delta_n = \frac{(1+\frac{1}{2}f)_n(d_1)_n(d_2)_n(d_3)_n(-N)_n}{(\frac{1}{2}f)_n(1+f-d_1)_n(1+f-d_2)_n(1+f-d_3)_n(1+f+N)_n}$$

then $$\gamma_n = K_N\frac{(d_1)_n(d_2)_n(d_3)_n(-N)_n}{(1+f-e-d_1)_n(1+f-e-d_2)_n(1+f-e-d_3)_n(1+f-e+N)_n}$$

where $K_N = \dfrac{(1+f)_N(1+f-e-d_1)_N(1+f-e-d_2)_N(1+f-e-d_3)_N}{(1+f-e)_N(1+f-d_1)_N(1+f-d_2)_N(1+f-d_1-d_2-e)_N}$

provided $e+d_1+d_2+d_3-N = 1+2f.$

If $\alpha_n = \dfrac{(f-e)_n(1+\frac{1}{2}(f-e))_n(a_1)_n(a_2)_n(a_3)_n}{n!(\frac{1}{2}(f-e))_n(1+f-e-a_1)_n(1+f-e-a_2)_n(1+f-e-a_3)_n}$

then $\beta_n = \dfrac{(f)_n(e+a_1)_n(e+a_2)_n(e+a_3)_n}{n!(1+f-e-a_1)_n(1+f-e-a_2)_n(1+f-e-a_3)_n}$

Hence,

$$K_N \cdot {}_9F_8\left[\begin{matrix} f-e, & 1+\frac{1}{2}(f-e), & a_1, & a_2, & a_3, \\ & \frac{1}{2}(f-e), & 1+f-e-a_1, & 1+f-e-a_2, & 1+f-e-a_3, \end{matrix}\right.$$

$$\left.\begin{matrix} d_1, & d_2, & d_3, & -N; & 1 \\ 1+f-e-d_1, & 1+f-e-d_2, & 1+f-e-d_3, & 1+f-e+N; & \end{matrix}\right]$$

$$= {}_9F_8\left[\begin{matrix} f, & 1+\frac{1}{2}f, & e+a_1, & e+a_2, & e+a_3, & d_1, \\ & \frac{1}{2}f, & 1+f-e-a_1, & 1+f-e-a_2, & 1+f-e-a_3, & 1+f-d_1; \end{matrix}\right.$$

$$\left.\begin{matrix} d_2, & d_3, & -N; & 1 \\ 1+f-d_2, & 1+f-d_3, & 1+f+N; & \end{matrix}\right]$$

§ (6.4) Summation of F_γ as a well-poised ${}_5F_4(1)$ series.

The only case in which summation of both F_γ and F_β is possible, and which is not included as a special case of those given in § (6.1), is

$$F_\gamma = {}_5F_4\left[\begin{matrix} f+2n, & 1+\frac{1}{2}f+n, & d_1+n, & d_2+n, & -N+n; & 1 \\ & \frac{1}{2}f+n, & 1+f-d_1+n, & 1+f-d_2+n, & 1+f+N+n; & \end{matrix}\right]$$

Here $F_\beta = \sum_{r=o}^{n} (-n)_r(f+n)_r(-1)^r\alpha_r$

Let $\alpha_n = \dfrac{(-1)^n}{n!(A)_n}$. Then, summing F_β by Vandermonde's theorem,

$$\beta_n = \frac{(f)_n(1+f-A)_n(-1)^n}{n!\,(A)_n}$$

Hence,

$$\frac{(1+f)_N(1+f-d_1-d_2)_N}{(1+f-d_1)_N(1+f-d_2)_N}\, {}_3F_2\left[\begin{matrix}d_1, d_2, -N;\ 1\\ A, d_1+d_2-N-f;\end{matrix}\right]$$

$$= {}_6F_5\left[\begin{matrix}f, 1+\frac{1}{2}f, d_1, d_2, -N, 1+f-A;\ -1\\ \frac{1}{2}f, 1+f-d_1, 1+f-d_2, 1+f+N, A;\end{matrix}\right] \qquad \ldots(1)$$

If $-N = 1+f-2d_2$ and $A = d_1+2d_2-f$, or if $A = 1+d_1-d_2$ and $d_2 = 1+f+2N$, the ${}_3F_2(1)$ series is well-poised and can be summed by Dixon's theorem, giving

Theorem (20)

$${}_6F_5\left[\begin{matrix}f, 1+\frac{1}{2}f, d, \frac{1}{2}(1+f+N), -d-N, -N;\ -1\\ \frac{1}{2}f, 1+f-d, \frac{1}{2}(1+f-N), 1+f+d+N, 1+f+N;\end{matrix}\right]$$

$$= \frac{(1+f)_N(1+f-c-d)_N(1+d)_N(1+\frac{1}{2}d-c)_N}{(1+f-c)_N(1+f-d)_N(1+d-c)_N(1+\frac{1}{2}d)_N} \qquad \ldots(2)$$

where $c \equiv 1+f+2N$.

Next, let $\alpha_n = \dfrac{(-1)^n(a)_n}{n!(A)_n(1+f+a-A)_n}$

Then, summing F_β by Saalschutz's theorem,

$$\beta_n = \frac{(f)_n(A-a)_n(1+f-A)_n}{n!\,(A)_n(1+f+a-A)_n}$$

Hence,

$$\frac{(1+f)_N(1+f-d_1-d_2)_N}{(1+f-d_1)_N(1+f-d_2)_N}\, {}_4F_3\left[\begin{matrix}d_1, d_2, a, -N;\ 1\\ 1+f+a-A, A, d_1+d_2-f-N;\end{matrix}\right]$$

$$= {}_7F_6\left[\begin{matrix}f, 1+\frac{1}{2}f, A-a, 1+f-A, d_1, d_2, -N;\ 1\\ \frac{1}{2}f, 1+f+a-A, A, 1+f-d_1, 1+f-d_2, 1+f+N;\end{matrix}\right] \qquad ..(3)$$

(Whipple, "On well-poised series, etc." Proc.Lond.Maths. Soc. (2) Vol. 24 (1926) equ. 7.7)

There are no cases in which F_γ can be summed as a well-poised ${}_4F_3(-1)$ series, which are not already included as special cases of the ${}_5F_4(1)$ and ${}_7F_6(1)$ sums.

§ (6.51) Summation of F_γ by Dixon's theorem

The only cases which are not included as special forms of the previous transformations, are:-

(1) $F_\gamma = {}_3F_2\left[\begin{matrix} d_1+n, d_2+n, & -N+n; \ 1 \\ 1+d_1-d_2, 1+d_1+N; & \end{matrix}\right]$

$$F_\beta = \sum_{r=0}^{n} (-n)_r \ (d_2-d_1-n)_r(-N-d_1-n)_r(-1)^r\alpha_r$$

and

(2) $F_\gamma = {}_3F_2\left[\begin{matrix} d+n, & e, \ -N+n; \ 1 \\ & 1+d-e+n, 1+d+N; \end{matrix}\right]$

$$F_\beta = \sum_{r=0}^{n} \frac{(-n)_r(-N-d-n)_r(-1)^r}{(1-e-n)_r}\alpha_r$$

§ (6.52) Case one

Summing both F_γ and F_β by Dixon's theorem, if $\delta_n = (d_1)_n(d_2)_n(-N)_n$, and n is even,

$$\gamma_{2n} = \frac{(d_1)_{2n}(d_2)_{2n}(-N)_{2n}(d_2-d_1-N)_{2n}(d_2-\tfrac{1}{2}d_1)_n(-\tfrac{1}{2}d_1-N)_n}{(1+d_1)_{2n}(d_2-\tfrac{1}{2}d_1-N)_{3n}(1+d_1-d_2)_N(1+\tfrac{1}{2}d_1)_N}$$

$$\times (1+\tfrac{1}{2}d_1)_n(-1)^n(1+d_1)_N(1+\tfrac{1}{2}d_1-d_2)_N$$

and if n is odd,

$$\gamma_{2n+1} = \frac{d_1d_2(-N)(d_1-d_2)}{2(1-d_1)(\tfrac{1}{2}(d_1-1)-d_2)} \ \frac{(1+d_1)_N(\tfrac{1}{2}(d_1-1)-d_2)_N}{(\tfrac{1}{2}(1+d_1))_N(d_1-d_2)_N}$$

$$\times \frac{(-1)^n(1-d_1+d_2-N)_{2n}(\tfrac{1}{2}(1-d_1)+d_2)_n(\tfrac{1}{2}(1-d_1)-N)_n}{(2+d_1)_{2n}(\tfrac{1}{2}(3-d_1)+d_2-N)_{3n}}$$

$$\times (\tfrac{1}{2}(3+d_1))_n(1+d_1)_{2n}(1+d_2)_{2n}(1-N)_{2n}$$

Let $\alpha_n = \dfrac{(-1)^n}{n!(1+d_2+N)_n(1+d_2-d_1)_n}$.

Then, if n is even, say $n = 2m$,

$$F_\beta = \frac{(\tfrac{1}{2}(1+d_1-d_2))_m(1/3.(1+\tfrac{1}{2}(d_1+d_2)+N))_m(1/3.(2+\tfrac{1}{2}(d_1+d_2)+N))_m}{(\tfrac{1}{2}(1+d_2+N))_m(1+\tfrac{1}{2}(d_2+N))_m(1+\tfrac{1}{2}(d_1+d_2)+N)_m(1+\tfrac{1}{2}(d_2-d_1))_m}$$
$$\times (1+1/3.(\tfrac{1}{2}(d_1+d_2)+N))_m(-1)^m 3^{3m}$$

and if n is odd, say $n = 2m+1$

$$F_\beta = \frac{(d_2-d_1)(3+d_1+d_2+2N)(-1)^m 3^{3m}(1+\tfrac{1}{2}(d_1-d_2))_m}{(1+d_2+N)(1+d_2-d_1)(1+\tfrac{1}{2}(d_2+N))_m(\tfrac{1}{2}(3+d_1+d_2)+N)_m}$$
$$\times \frac{(1/6.(5+d_1+d_2)+1/3.N)_m(1/6.(7+d_1+d_2)+1/3.N)_m}{(\tfrac{1}{2}(3+d_2+N))_m(\tfrac{1}{2}(3+d_2-d_1))_m}$$
$$\times (1/6.(9+d_1+d_2)+1/3.N)_m$$

Hence,
$$\frac{(1+d_1)_N(1+\tfrac{1}{2}d_1-d_2)_N}{(1+d_1-d_2)_N(1+\tfrac{1}{2}d_1)_N}\; {}_9F_8\left[\begin{matrix}\tfrac{1}{2}d_1, \tfrac{1}{2}d_2, \tfrac{1}{2}(d_2+1), \tfrac{1}{2}(d_2-d_1-N), \tfrac{1}{2}(d_2-d_1-N+1), d_2-\tfrac{1}{2}d_1, -\tfrac{1}{2}d_1-N, \tfrac{1}{2}(1-N), -\tfrac{1}{2}N; \\ \tfrac{1}{2}, \tfrac{1}{2}(1+d_2+N), 1/6.(2d_2-d_1-2N), 1/6.(2+2d_2-d_1-2N), 1/6.(4+2d_2-d_1-2N), 1+\tfrac{1}{2}(d_2-d_1), \tfrac{1}{2}(1+d_2-d_1), 1+\tfrac{1}{2}(d_2+N);\end{matrix}\ \frac{-1}{27}\right]$$

$$+\frac{d_1d_2N(1+d_1)_N(\tfrac{1}{2}(d_1-1)-d_2)_N}{2(1+d_2+N)(1+d_2-d_1)(\tfrac{1}{2}(d_1-1)-d_2)(\tfrac{1}{2}(1+d_1))_N(1+d_1-d_2)_{N-1}}$$

$$\times\, {}_9F_8\left[\begin{matrix}1+\tfrac{1}{2}d_2, \tfrac{1}{2}(1+d_2), \tfrac{1}{2}(d_1+1), \tfrac{1}{2}(1-d_1)+d_2, \tfrac{1}{2}(1-d_1)-N, \tfrac{1}{2}(1-d_1+d_2-N), 1+\tfrac{1}{2}(d_2-d_1-N), 1-\tfrac{1}{2}N, \tfrac{1}{2}(1-N); \\ 3/2, \tfrac{1}{2}(3+d_2-d_1), 1+\tfrac{1}{2}(d_2-d_1), 1/6.(3-d_1+2d_2-2N), 1/6.(5+2d_2-d_1-2N), 1/6.(7+2d_2-d_1-2N), \tfrac{1}{2}(3+d_2+N), 1+\tfrac{1}{2}(d_2+N);\end{matrix}\ \frac{-1}{27}\right]$$

$$= {}_9F_8\left[\begin{matrix}\tfrac{1}{2}d_1, \tfrac{1}{2}(1+d_1), \tfrac{1}{2}d_2, \tfrac{1}{2}(d_2+1), 1/6.(2+d_1+d_2+2N), 1/6.(4+d_1+d_2+2N), 1+1/6(d_1+d_2+2N), \tfrac{1}{2}(1-N), -\tfrac{1}{2}N; \\ \tfrac{1}{2}, 1+\tfrac{1}{2}(d_1-d_2), 1+\tfrac{1}{2}(d_2-d_1), \tfrac{1}{2}(1+d_2+N), 1+\tfrac{1}{2}(d_2+N), 1+\tfrac{1}{2}(d_1+d_2)+N, \tfrac{1}{2}(1+d_1+N), 1+\tfrac{1}{2}(d_1+N);\end{matrix}\ -27\right]$$

$$+\frac{d_1d_2N(d_2-d_1)(3+d_1+d_2+2N)}{(1+d_1-d_2)(1+d_1+N)(1+d_2+N)(1+d_2-d_1)}\; {}_9F_8\left[\begin{matrix}1+\tfrac{1}{2}d_1, \tfrac{1}{2}(1+d_1), \tfrac{1}{2}(1+d_2), 1+\tfrac{1}{2}d_2, 1/6.(5+d_1+d_2+2N), 1/6.(7+d_1+d_2+2N), 1/6(9+d_1+d_2+2N), \tfrac{1}{2}(1-N), 1-\tfrac{1}{2}N; \\ 3/2, \tfrac{1}{2}(3+d_1-d_2), \tfrac{1}{2}(3+d_2-d_1), 1+\tfrac{1}{2}(d_2+N), \tfrac{1}{2}(3+d_2+N), \tfrac{1}{2}(3+d_1+d_2+2N), \tfrac{1}{2}(3+d_1+N), 1+\tfrac{1}{2}(d_1+N);\end{matrix}\ -27\right]$$

This is a relation between four special terminating $_9F_8$ series.

§ (6.53) Case two

Summing F_γ by Dixon's theorem, and F_β by Vandermonde's theorem,

if $\delta_n = \dfrac{(d)_n(-N)_n}{(1+d-e)_n}$, and n is even, then

$$\gamma_{2n} = \frac{(1+d)_N(1+\tfrac{1}{2}d-e)_N(\tfrac{1}{2}d)_n(-N)_{2n}(-N-\tfrac{1}{2}d)_n}{(1+d-e)_N(1+\tfrac{1}{2}d)_N(1+\tfrac{1}{2}d-e)_n(e-N-\tfrac{1}{2}d)_n}$$

and, if n is odd,

$$\gamma_{2n+1} = -\frac{(\tfrac{1}{2}(1+d))_n(\tfrac{1}{2}(1-d)-N)_n(1-N)_{2n}\,dN(1+d)_N(\tfrac{1}{2}(1+d)-e)_N}{(\tfrac{1}{2}(d+3)-e)_n(e+\tfrac{1}{2}(1-d)-N)_n(1+d-2e)(1+d-e)_N(\tfrac{1}{2}(1+d))_N}$$

Let $\alpha_n = \dfrac{(-1)^n}{n!}$.

Then, $\beta_n = \dfrac{(-1)^n(1+d-e+N)_n}{n!\,(1+d+N)_n}$.

Hence, $${}_3F_2\left[\begin{matrix}1+d-e+N, & d, & -N; & -1\\ & 1+d-e, & 1+d+N & ;\end{matrix}\right]$$

$$= \frac{(1+d)_N(1+\tfrac{1}{2}d-e)_N}{(1+d-e)_N(1+\tfrac{1}{2}d)_N}\,{}_4F_3\left[\begin{matrix}\tfrac{1}{2}d, -N-\tfrac{1}{2}d, & \tfrac{1}{2}(1-N), & -\tfrac{1}{2}N; & 1\\ \tfrac{1}{2}, & 1+\tfrac{1}{2}d-e, & -N-\tfrac{1}{2}d+e; & \end{matrix}\right]$$

$$+\frac{dN(1+d)_N(\tfrac{1}{2}(1+d)-e)_N}{(1+d-2e)(1+d-e)_N(\tfrac{1}{2}(1+d))_N}$$

$$\times\ {}_4F_3\left[\begin{matrix}\tfrac{1}{2}(1+d), \tfrac{1}{2}(1-d)-N, & \tfrac{1}{2}(1-N), & 1-\tfrac{1}{2}N; & 1\\ 3/2, & \tfrac{1}{2}(3+d)-e, & e+\tfrac{1}{2}(1-d)-N\ ; & \end{matrix}\right]$$

§ (6.61) Applications of Kummer's theorem to F_γ

There are three cases in which both F_γ and F_β are summable, which are not included as special cases of the previous work. These are:-

(1) $F_\gamma = {}_2F_1\left[\begin{matrix} d+n, & -N+n; & -1 \\ & 1+d+N; & \end{matrix}\right]$ $F_\beta = \sum_{r=0}^{n} (-n)_r (-d-N-n)_r \alpha_r$

(2) $F_\gamma = {}_2F_1\left[\begin{matrix} d+n, & e; & -1 \\ & 1+d-e+n; & \end{matrix}\right]$ $F_\beta = \sum_{r=0}^{n} \frac{(-n)_r}{(1-e-n)_r} \alpha_r$

(3) $F_\gamma = {}_2F_1\left[\begin{matrix} e_1, & e_2; & -1 \\ & 1+e_1-e_2; & \end{matrix}\right]$ $F_\beta = \sum_{r=0}^{n} \frac{(-n)_r (e_2-e_1-n)_r}{(1-e_1-n)_r (1-e_2-n)_r} \alpha_r$

§ (6.62) The first case

$$\text{If } \delta_n = (d)_n(-N)_n(-1)^n$$

then $\gamma_{2n} = \frac{(1+d)_N}{(1+\frac{1}{2}d)_N} (\tfrac{1}{2}d)_n(-N-\tfrac{1}{2}d)_n(-N)_{2n}(-1)^n$

and $\gamma_{2n+1} = \frac{(1+d)_N}{(\frac{1}{2}(1+d))_N} . \tfrac{1}{2}dN(1-N)_{2n}(\tfrac{1}{2}(1-d)-N)_n(\tfrac{1}{2}(1+d))_n(-1)^n$

Let $\alpha_n = \frac{1}{n!\,(A)_n}$ and sum F_β by Vandermonde's theorem.

Then, $\beta_n = \frac{(A+d+N)_{2n}}{n!(1+d+N)_n(A)_n(A+d+N)_n}$

Hence,

$$\frac{(1+d)_N}{(1+\frac{1}{2}d)_N} {}_4F_3\left[\begin{matrix} \frac{1}{2}d, -N-\frac{1}{2}d, \frac{1}{2}(1-N), -\frac{1}{2}N; & -\frac{1}{4} \\ \frac{1}{2}, \frac{1}{2}(1+A), \frac{1}{2}A; & \end{matrix}\right]$$

$$+ \frac{(1+d)_N dN}{(\frac{1}{2}(1+d))_N 2A} {}_4F_3\left[\begin{matrix} \frac{1}{2}(1+d), \frac{1}{2}(1-d)-N, 1-\frac{1}{2}N, \frac{1}{2}(1-N); & -\frac{1}{4} \\ 3/2, 1+\frac{1}{2}A, \frac{1}{2}(1+A); & \end{matrix}\right]$$

$$= {}_4F_3\left[\begin{matrix} d, \frac{1}{2}(A+d+N), \frac{1}{2}(A+d+N+1), -N; & -4 \\ A+d+N, \quad 1+d+N, \quad A; & \end{matrix}\right] \qquad ...(1)$$

Next, let $\alpha_n = \frac{(-1)^n}{n!(2-E)_n}$. Then, summing F_β by Kummer's theorem, as in § 4.62, if $E \equiv 1+d+N$,

$$\beta_{2n} = \frac{(-1)^n}{n!(\tfrac{1}{2})_n(\tfrac{1}{2}(1+E))_n(\tfrac{1}{2}(3-E))_n 4^n}$$

and $$\beta_{2n+1} = \frac{(-1)^n 2(1-E)}{(2-E)E\; n!(3/2)_n(1+\tfrac{1}{2}E)_n(2-\tfrac{1}{2}E)_n 4^n}$$

Hence,

$$\frac{(1+d)_N}{(1+\tfrac{1}{2}d)_N}\, {}_4F_3\left[\begin{matrix}\tfrac{1}{2}d, -\tfrac{1}{2}d-N, \tfrac{1}{2}(1-N), -\tfrac{1}{2}N; & -\tfrac{1}{4}\\ \tfrac{1}{2}, \tfrac{1}{2}(3-E), 1-\tfrac{1}{2}E; & \end{matrix}\right]$$

$$- \frac{(1+d)_N\, d\, N}{(\tfrac{1}{2}(1+d))_N 2(2-E)}\, {}_4F_3\left[\begin{matrix}\tfrac{1}{2}(d+1), \tfrac{1}{2}(1-d)-N, 1-\tfrac{1}{2}N, \tfrac{1}{2}(1-N); & -\tfrac{1}{4}\\ 3/2, 2-\tfrac{1}{2}E, \tfrac{1}{2}(3-E); & \end{matrix}\right]$$

$$= {}_4F_3\left[\begin{matrix}\tfrac{1}{2}d, \tfrac{1}{2}(1+d), \tfrac{1}{2}(1-N), -\tfrac{1}{2}N; & -4\\ \tfrac{1}{2}, \tfrac{1}{2}(1+E), \tfrac{1}{2}(3-E); & \end{matrix}\right]$$

$$+ \frac{2(1-E)dN}{(2-E)E}\, {}_4F_3\left[\begin{matrix}1+\tfrac{1}{2}d, \tfrac{1}{2}(1+d), \tfrac{1}{2}(1-N), 1-\tfrac{1}{2}N; & -4\\ 3/2, 1+\tfrac{1}{2}E, 2-\tfrac{1}{2}E; & \end{matrix}\right] \qquad \ldots(2)$$

where $1+d+N \equiv E$.

§ (6.63) Second case

Summing F_γ by Kummer's theorem, and F_β by Saalschutz's theorem, if $\delta_n = \dfrac{(-1)^n (d)_n}{(1+d-e)_n}$

then $$\gamma_{2n} = \frac{\Gamma(1+d-e)\Gamma(1+\tfrac{1}{2}d)}{\Gamma(1+d)\Gamma(1+\tfrac{1}{2}d-e)}\frac{(\tfrac{1}{2}d)_n}{(1+\tfrac{1}{2}d-e)_n}$$

and $$\gamma_{2n+1} = \frac{-d\,\Gamma(1+d-e)\Gamma(3/2+\tfrac{1}{2}d)}{\Gamma(1+d)\;(1+d)\Gamma(3/2+\tfrac{1}{2}d-e)}\frac{(\tfrac{1}{2}(1+d))_n}{(\tfrac{1}{2}(3+d)-e)_n}$$

Let $\alpha_n = \dfrac{(a_1)_n (a_2)_n}{n!\,(e+a_1+a_2)_n}$. Then,

$$\beta_n = \frac{(a_1+e)_n (a_2+e)_n}{n!\,(a_1+a_2+e)_n}$$

Hence, $$\frac{\Gamma(1+d-e)\Gamma(1+\tfrac{1}{2}d)}{\Gamma(1+d)\Gamma(1+\tfrac{1}{2}d-e)}\, {}_5F_4\left[\begin{matrix}\tfrac{1}{2}a_1, \tfrac{1}{2}(1+a_1), \tfrac{1}{2}a_2, \tfrac{1}{2}(1+a_2), \tfrac{1}{2}d; & 1\\ \tfrac{1}{2}, \tfrac{1}{2}(a_1+a_2+e), \tfrac{1}{2}(1+a_1+a_2+e), 1+\tfrac{1}{2}d-e; & \end{matrix}\right]$$

$$- \frac{\Gamma(1+d-e)\Gamma(3/2+\tfrac{1}{2}d)\, a_1 a_2\, d}{\Gamma(1+d)\Gamma(3/2+\tfrac{1}{2}d-e)(a_1+a_2+e)(1+d)}$$

$$\times\, {}_5F_4\left[\begin{matrix}\tfrac{1}{2}(1+a_1), 1+\tfrac{1}{2}a_1, \tfrac{1}{2}(1+a_2), 1+\tfrac{1}{2}a_2, \tfrac{1}{2}(1+d); & 1\\ 3/2, \tfrac{1}{2}(a_1+a_2+e+1), 1+\tfrac{1}{2}(a_1+a_2+e), \tfrac{1}{2}(3+d)-e; & \end{matrix}\right]$$

$$= {}_3F_2\left[\begin{matrix} a_1+e, & a_2+e, & d; & -1 \\ & a_1+a_2+e, & 1+d-e; & \end{matrix}\right] \qquad \ldots(1)$$

Summing F_β by theorem (13),

$$\frac{\Gamma(1+d-e)\Gamma(1+\frac{1}{2}d)}{\Gamma(1+d)\Gamma(1+\frac{1}{2}d-e)}\,{}_5F_4\left[\begin{matrix} \frac{1}{2}a, \frac{1}{2}(a+1), & -\frac{1}{4}e, & \frac{1}{4}(2-e), & \frac{1}{2}d; & 1 \\ \frac{1}{2}, & \frac{1}{4}(2+2a+e), & 1+\frac{1}{4}(2a+e), & 1+\frac{1}{2}d-e; & \end{matrix}\right]$$

$$+\frac{\Gamma(1+d-e)\Gamma(\frac{1}{2}(3+d))}{\Gamma(1+d)\Gamma(\frac{1}{2}(3+d)-e)}\,\frac{a\,d\,e}{(1+d)\,(2+2a+e)}$$

$$\times\,{}_5F_4\left[\begin{matrix} \frac{1}{2}(a+1), 1+\frac{1}{2}a, & -\frac{1}{4}(e+2), & 1-\frac{1}{4}e, & \frac{1}{2}(d+1); & 1 \\ \frac{3}{2}, & 1+\frac{1}{4}(2a+e), & \frac{1}{4}(6+2a+e), & \frac{1}{2}(3+d)-e; & \end{matrix}\right]$$

$$= {}_4F_3\left[\begin{matrix} a+e, 1+\frac{1}{2}(a+e), & \frac{1}{2}e, & d; & -1 \\ \frac{1}{2}(a+e), & 1+a+\frac{1}{2}e, & 1+d-e; & \end{matrix}\right] \qquad \ldots(2)$$

Summing F_β by theorem (14),

$$\frac{\Gamma(1+d-e)\Gamma(1+\frac{1}{2}d)}{\Gamma(1+d)\Gamma(1+\frac{1}{2}d-e)}\,{}_6F_5\left[\begin{matrix} \frac{1}{2}f, 1+\frac{1}{4}f, \frac{1}{2}(1+f), & -\frac{1}{4}e, & \frac{1}{4}(2-e), & \frac{1}{2}d; & 1 \\ \frac{1}{4}f, \frac{1}{2}, & \frac{1}{4}(2+2f+e), & 1+\frac{1}{4}(2f+e), & 1+\frac{1}{2}d-e; & \end{matrix}\right] + \frac{\Gamma(1+d-e)\Gamma(\frac{1}{2}(3+d))}{\Gamma(1+d)\Gamma(\frac{1}{2}(3+d)-e)}\,\frac{d\,e\,(2+f)}{(1+d)(2+2f+e)}$$

$$\times\,{}_6F_5\left[\begin{matrix} 1+\frac{1}{2}f, \frac{1}{4}(6+f), \frac{1}{2}(1+f), & \frac{1}{4}(2-e), & 1-\frac{1}{4}e, & \frac{1}{2}(d+1); & 1 \\ \frac{1}{4}(2+f), \frac{3}{2}, & 1+\frac{1}{4}(2f+e), & \frac{1}{4}(6+2f+e), & \frac{1}{2}(3+d)-e; & \end{matrix}\right]$$

$$= {}_3F_2\left[\begin{matrix} f+e, & \frac{1}{2}e, & d; & -1 \\ & 1+f+\frac{1}{2}e, & 1+d-e; & \end{matrix}\right] \qquad \ldots(3)$$

Summing F_β by theorem (15),

$$\frac{\Gamma(1+d-e)\Gamma(1+\frac{1}{2}d)}{\Gamma(1+d)\Gamma(1+\frac{1}{2}d-e)}\,{}_6F_5\left[\begin{matrix} \frac{1}{2}f, 1+\frac{1}{4}f, \frac{1}{2}(1+f), & -\frac{1}{4}(1+e), & \frac{1}{4}(1-e), & \frac{1}{2}d; & 1 \\ \frac{1}{4}f, \frac{1}{2}, & \frac{1}{4}(3+2f+e), & \frac{1}{4}(5+2f+e), & 1+\frac{1}{2}d-e; & \end{matrix}\right] + \frac{\Gamma(1+d-e)\Gamma(\frac{1}{2}(3+d))\,(2+f)\,(1+e)}{\Gamma(1+d)\Gamma(\frac{1}{2}(3+d)-e)(3+e+2f)}$$

$$\times\,{}_6F_5\left[\begin{matrix} 1+\frac{1}{2}f, \frac{1}{4}(6+f), \frac{1}{2}(1+f), & \frac{1}{4}(1-e), & \frac{1}{4}(3-e), & \frac{1}{2}(d+1); & 1 \\ \frac{1}{4}(2+f), \frac{3}{2}, & \frac{1}{4}(2f+e+5), & \frac{1}{4}(2f+e+7), & \frac{1}{2}(3+d)-e; & \end{matrix}\right]$$

$$= {}_4F_3\left[\begin{matrix} f+e, 1+\frac{1}{2}(f+e), & \frac{1}{2}(e-1), & d; & -1 \\ \frac{1}{2}(f+e), & \frac{1}{2}(3+e)+f, & 1+d-e; & \end{matrix}\right] \qquad \ldots(4)$$

§ (6.64). Third case

Summing F_γ by Kummer's theorem and F_β by Saalschutz's theorem, leads to a relation which implies Kummer's theorem again.

§ (7.1) Summation of F_γ by theorem (12)

There are eighteen possible series for F_γ, of which only seven lead to summable series for F_β. These are :-

(1) $$F_\gamma = {}_3F_2\left[\begin{matrix} e, 1+\tfrac{1}{2}e, -N+n; & 1 \\ \tfrac{1}{2}e, D+n\ ; & \end{matrix}\right]$$

$$F_\beta = \sum_{r=0}^{n} \frac{(-n)_r(1-\tfrac{1}{2}e-n)_r}{(1-e-n)_r(-\tfrac{1}{2}e-n)_r}\alpha_r$$

(2) $$F_\gamma = {}_3F_2\left[\begin{matrix} f+2n, 1+\tfrac{1}{2}f+n, & -N; & 1 \\ \tfrac{1}{2}f+n, & E; & \end{matrix}\right]$$

$$F_\beta = \sum_{r=0}^{n} \frac{(-n)_r(f+n)_r(1-E-n)_r(-1)^r}{(1-N-n)_r}\alpha_r$$

(3) $$F_\gamma = {}_3F_2\left[\begin{matrix} f+2n, 1+\tfrac{1}{2}f+n, -N; & 1 \\ \tfrac{1}{2}f+n, D+n; & \end{matrix}\right]$$

$$F_\beta = \sum_{r=0}^{n} \frac{(-n)_r(f+n)_r}{(1-N-n)_r}\alpha_r$$

(4) $$F_\gamma = {}_3F_2\left[\begin{matrix} f+2n, 1+\tfrac{1}{2}f+n, & -N; & 1 \\ \tfrac{1}{2}f+n, F+2n; & \end{matrix}\right]$$

$$F_\beta = \sum_{r=0}^{n} \frac{(-n)_r(f+n)_r}{(1-N-n)_r(F+n)_r}\alpha_r$$

(5) $$F_\gamma = {}_3F_2\left[\begin{matrix} f+2n, 1+\tfrac{1}{2}f+n, -N+n; & 1 \\ \tfrac{1}{2}f+n, D+n; & \end{matrix}\right]$$

$$F_\beta = \sum_{r=o}^{n} (-n)_r (f+n)_r (-1)^r \alpha_r$$

(6). $F_\gamma = {}_3F_2\left[\begin{matrix} f+2n, 1+\frac{1}{2}f+n, -N+n; \; 1 \\ \frac{1}{2}f+n, F+2n; \end{matrix}\right]$

$$F_\beta = \sum_{r=o}^{n} \frac{(-n)_r (f+n)_r}{(F+n)_r} \alpha_r$$

(7). $F_\gamma = {}_3F_2\left[\begin{matrix} f+2n, 1+\frac{1}{2}f+n, -N+2n; \; 1 \\ \frac{1}{2}f+n, \; F+2n; \end{matrix}\right]$

$$F_\beta = \sum_{r=o}^{n} \frac{(-n)_r (f+n)_r \; (-1)^r (-N+n)_r}{(F+n)_r} \alpha_r$$

§ (7.21) Case one

If $\delta_n = \frac{(-N)_n}{(D)_n}$ then,

$$\gamma_n = \frac{(e+2-D)_N (D-e-1)_N}{(D)_N (1+e-D)_N} \frac{(-N)_n}{(D-e)_n}$$

Summing F_β by Saalschutz's theorem, leads to a trivial result , a restatement of Vandermonde's theorem.

§ (7.22) Second case

If $\delta_n = \frac{(1+\frac{1}{2}f)_n}{(\frac{1}{2}f)_n}$

then $\gamma_n = \frac{(f-E+2)_N (E-f-1)_N (1+\frac{1}{2}f)_n (\frac{1}{2}(1+f))_n (f-E+2+N)_{2n}}{(E)_N (f-E+1)_N (\frac{1}{2}(f-E+1+N))_n (\frac{1}{2}(f-E+N)+1)_n}$

$$\times \frac{(1+f-E)_{2n}}{(f-E+2-N)_{2n}}$$

Let $\alpha_n = \frac{(-1)^n}{n! \; (1+f-E-N)_n}$

Then, summing F_β by Saalschutz's theorem,

$$\beta_n = \frac{(f)_n (E+N)_n (f-N)_{2n} (-1)^n}{n!\,(E)_n (1+f-E-N)_n (f-N)_n}$$

Hence,

$$\frac{(f+2-E)_N (E-f-1)_N}{(E)_N (1+f-E)_N}\, {}_5F_4\left[\begin{matrix} 1+\tfrac{1}{2}f, \tfrac{1}{2}(1+f), 1+\tfrac{1}{2}(f-E+N+1), & \tfrac{1}{2}(f-E+1), \\ \tfrac{1}{2}(f-E+N+1), \tfrac{1}{2}(f-E-N+1)+1, 1+\tfrac{1}{2}(f-E-N), \end{matrix}\right.$$

$$\left.\begin{matrix} 1+\tfrac{1}{2}(f-E); & -4 \\ 1+f-E-N; & \end{matrix}\right] = {}_5F_4\left[\begin{matrix} f, 1+\tfrac{1}{2}f, E+N, \tfrac{1}{2}(f-N), \tfrac{1}{2}(f-N+1); & -4 \\ \tfrac{1}{2}f, \quad E, \quad -f-N, \quad 1+f-E-N; & \end{matrix}\right]$$

Here both the series must terminate.

§ (7.23) Third case

If $\delta_n = \dfrac{(1+\frac{1}{2}f)_n}{(\frac{1}{2}f)_n (D)_n}$

then $\gamma_n = \dfrac{(f-D+2)_N (D-f-1)_N (1+f)_{2n} (f-D+2+N)_n (f-D+1)_n}{(D)_N (1+f-D)_N\ (2+f-D-N)_n (1+f-D+N)_n\ (D+N)_n}$

If $\alpha_n = 1$, summing F_β by Vandermonde's theorem,

then $\beta_n = \dfrac{(f)_n (f-N)_{2n}}{n!\,(f-N)_n}$.

Hence,

$$\frac{(f-D+2)_N (D-f-1)_N}{(D)_N\ (1+f-D)_N}\ {}_4F_3\left[\begin{matrix} \tfrac{1}{2}(1+f), & 1+\tfrac{1}{2}f, f-D+2+N, f-D+1; & 4 \\ & f-D+2-N, f-D+1+N, \quad D+N; & \end{matrix}\right]$$

$$= {}_4F_3\left[\begin{matrix} f, 1+\tfrac{1}{2}f, \tfrac{1}{2}(f-N), \tfrac{1}{2}(1+f-N); & 4 \\ \tfrac{1}{2}f, \quad f-N, \quad D; & \end{matrix}\right]$$

where both the series must, of course, terminate.

§ (7.24) Case four

If $\delta_n = \dfrac{(1+\frac{1}{2}f)_n}{(\frac{1}{2}f)_n}$, then,

$$\gamma_n = \frac{(f-F+2)_N(F-f-1)_N(1+\tfrac{1}{2}f)_n(f)_{2n}}{(F)_N(f+1-F)_N(\tfrac{1}{2}f)_n(F+N)_{2n}}$$

If $\alpha_n = \dfrac{(F+N-f)_n}{n!}$, summing F_β by Saalschutz's theorem, then $\beta_n = \dfrac{(f)_n(\tfrac{1}{2}(f-N))_n(\tfrac{1}{2}(f-N+1))_n(F-f)_n}{n!\,(f-N)_n(\tfrac{1}{2}F)_n(\tfrac{1}{2}(F+1))_n}$

Hence,

$$\frac{(f+2-F)_N(F-f-1)_N}{(F)_N(1+f-F)_N}\; {}_3F_2\left[\begin{matrix}\tfrac{1}{2}(1+f), F+N-f, & 1+\tfrac{1}{2}f; \; 1\\ \tfrac{1}{2}(F+N), \tfrac{1}{2}(F+N+1); & \end{matrix}\right]$$

$$= {}_5F_4\left[\begin{matrix}f, 1+\tfrac{1}{2}f, \tfrac{1}{2}(f-N), \tfrac{1}{2}(f-N+1), F-f; \; 1\\ \tfrac{1}{2}f, \tfrac{1}{2}(F+1), \qquad \tfrac{1}{2}F, f-N;\end{matrix}\right]$$

§ (7.25) Case five

If $\delta_n = \dfrac{(1+\tfrac{1}{2}f)_n(-N)_n}{(\tfrac{1}{2}f)_n(D)_n}$

then, $\gamma_n = \dfrac{(2+f-D)_N(D-f-1)_N}{(D)_N(1+f-D)_N}\,\dfrac{(1+f)_{2n}(-N)_n(1+f-D)_n(-1)^n}{(2+f-D-N)_{2n}}$

Summing F_β by Saalschutz's theorem, if

$$\alpha_n = \frac{(a)_n(-1)^n}{n!(A)_n(1+f+a-A)_n} \quad \text{then,}$$

$$\beta_n = \frac{(A-a)_n(1+f-A)_n(f)_n}{n!(A)_n(1+f+a-A)_n}$$

Hence,

$$\frac{(2+f-D)_N(D-f-1)_N}{(D)_N(1+f-D)_N}\,{}_5F_4\left[\begin{matrix}a, \tfrac{1}{2}(1+f), & 1+\tfrac{1}{2}f, & 1+f-D, -N; \; 1\\ 1+f+a-A, 1-\tfrac{1}{2}(D+N), & \tfrac{1}{2}(3+f-D-N), & A;\end{matrix}\right]$$

$$= {}_5F_4\left[\begin{matrix}f, 1+\tfrac{1}{2}f, & A-a, 1+f-A, -N; \; 1\\ \tfrac{1}{2}f, 1+f+a-A, & A, D;\end{matrix}\right]$$

This is a transformation between a nearly-poised ${}_5F_4(1)$ and a Saalschutzian ${}_5F_4(1)$ series.

§ (7.26) Case six

If $\delta_n = \frac{(1+\frac{1}{2}f)_n(-N)_n}{(\frac{1}{2}f)_n}$

then $\gamma_n = \frac{(f+2-F)_N(F-f-1)_N(f)_{2n}(1+\frac{1}{2}f)_n(-N)_n(F-f-N)_n(-1)^n}{(1+f-F)_N(F)_N\, n!\,(\frac{1}{2}f)_n(F-f-1-N)_n\,(2+f-F-N)_n(F+N)_n}$

If $\alpha_n = \frac{(-1)^n}{n!}$ then, summing F_β by Vandermonde's theorem,

$$\beta_n = \frac{(f)_n(F-f)_n}{n!\,(F)_{2n}}$$

Hence,

$$\frac{(2+f-F)_N(F-f-1)_N}{(1+f-F)_N(F)_N}\,{}_4F_3\left[\begin{matrix}\frac{1}{2}(1+f), 1+\frac{1}{2}f, & F-f-N, & -N; & 4\\ & F-f-1-N, 2+f-F-N, F+N\ ; & \end{matrix}\right]$$

$$= {}_4F_3\left[\begin{matrix}f, 1+\frac{1}{2}f, F-f, & -N; & \frac{1}{4}\\ \frac{1}{2}f, \frac{1}{2}F, \frac{1}{2}(1+F); & \end{matrix}\right]$$

§ (7.27) Case seven

If $\delta_n = \frac{(1+\frac{1}{2}f)_n}{(\frac{1}{2}f)_n}$ then

$$\gamma_n = \frac{(2+f-F)_N(F-f-1)_N\ (f)_{2n}(-N)_{2n}(F-f-N)_{2n}}{(F)_N(1+f-F)_N(F-f-1-N)_{2n}(2+f-F-N)_{2n}}$$

Let $\alpha_n = \frac{(-1)^n}{n!(1+f-N-F)_n}$. Then, summing F_β by Saalschutz's theorem, $\beta_n = \frac{(f)_n(F-f)_n(-N)_n(F+N)_n(-1)^n}{n!\,(F)_{2n}(1+f-N-F)_n}$

Hence,

$$\frac{(2+f-F)_N(F-f-1)_N}{(F)_N(1+f-F)_N}\,{}_5F_4\left[\begin{matrix}1+\frac{1}{2}f, \frac{1}{2}(1+f), \frac{1}{2}(1-f+F-N), & & -\frac{1}{2}N,\\ 1+f-N-F, \frac{1}{2}(F-f-1-N), 1+\frac{1}{2}(f-F-N), & & \end{matrix}\right.$$

$$\left.\begin{matrix}\frac{1}{2}(1-N); & -4\\ \frac{1}{2}(3+f-F-N); & \end{matrix}\right] = {}_5F_4\left[\begin{matrix}f, 1+\frac{1}{2}f, F-f, & F+N, & -N; & -\frac{1}{4}\\ \frac{1}{2}f, \frac{1}{2}F, \frac{1}{2}(1+F), 1+f-F-N; & & \end{matrix}\right]$$

§ (7.3) Summation of F_γ by theorem (13).

There are eleven possible cases for F_γ, of which only six give series for F_β which are summable. These are:–

(1) $$F_\gamma = {}_3F_2\left[\begin{matrix} e_1, & e_2, & -N+n; & 1 \\ & 1+e_1-e_2, & 1+2e_2-N+n; & \end{matrix}\right]$$

$$F_\beta = \sum_{r=0}^{n} \frac{(-n)_r (e_2-e_1-n)_r}{(1-e_1-n)_r (1-e_2-n)_r} \alpha_r$$

(2) $$F_\gamma = {}_3F_2\left[\begin{matrix} d+n, & e, & -N; & 1 \\ & 1+d-e+n, & 1+2e-N; & \end{matrix}\right]$$

$$F_\beta = \sum_{r=0}^{n} \frac{(-n)_r (N-2e-n)_r}{(1-e-n)_r (1+N-n)_r} \alpha_r$$

(3) $$F_\gamma = {}_3F_2\left[\begin{matrix} d+n, & e, & -N+n; & 1 \\ & 1+d-e+n, & 1+2e-N+n; & \end{matrix}\right]$$

$$F_\beta = \sum_{r=0}^{n} \frac{(-n)_r}{(1-e-n)_r} \alpha_r$$

(4) $$F_\gamma = {}_3F_2\left[\begin{matrix} d+n, & e, & -N+2n; & 1 \\ & 1+d-e+n, & 1+2e-N+2n; & \end{matrix}\right]$$

$$F_\beta = \sum_{r=0}^{n} \frac{(-n)_r (-N+n)_r}{(1-e-n)_r (1+2e-N+n)_r} \alpha_r$$

(5) $$F_\gamma = {}_3F_2\left[\begin{matrix} f+2n, & e, & -N+n; & 1 \\ & 1+f+2n-e, & 1+2e-N+n; & \end{matrix}\right]$$

$$F_\beta = \sum_{r=0}^{n} \frac{(-n)_r (f+n)_r}{(1-e-n)_r (1+f-e+n)_r} \alpha_r$$

(6) $$F_\gamma = {}_3F_2\left[\begin{matrix} f+2n, & d+n, & -N; & 1 \\ & 1+f-d+n, & 1+2d+2n-N; & \end{matrix}\right]$$

$$F_\beta = \sum_{r=0}^{n} \frac{(-n)_r (f+n)_r}{(1+N-n)_r (1+2d-N+n)_r} \alpha_r$$

§ (7.41). Case one

If $\delta_n = \dfrac{(-N)_n}{(1+2e_2-N)_n}$ then

$$\gamma_n = \frac{(e_1-2e_2)_N(1+\tfrac{1}{2}e_1-e_2)_N(-e_2)_N(-N)_n(e_2-e_1-N)_n(e_2-\tfrac{1}{2}e_1+1-N)_n}{(1+e_1-e_2)_N(\tfrac{1}{2}e_1-e_2)_N(-2e_2)_N(1+2e_2-e_1-N)_n(e_2-\tfrac{1}{2}e_1-N)_n(1+e_2-N)_n}$$

Summing F_β by Saalschutz's theorem,

if $\alpha_n = \dfrac{(1-2e_2)_n}{n!}$ then

$$\beta_n = \frac{(1-e_2)_n(1+e_1-2e_2)_n}{n!\,(1+e_1-e_2)_n}$$

Hence,

$$\frac{(e_1-2e_2)_N(1+\tfrac{1}{2}e_1-e_2)_N(-e_2)_N}{(1+e_1-e_2)_N(\tfrac{1}{2}e_1-e_2)_N(-2e_2)_N}\,{}_4F_3\left[\begin{matrix}e_2-e_1-N,\ 1-\tfrac{1}{2}e_1-e_2-N,\ 1-2e_2,\ -N;\ 1\\ e_2-\tfrac{1}{2}e_1-N,\ 1+2e_2-e_1-N,\ 1+e_2-N;\end{matrix}\right]$$

$$= {}_3F_2\left[\begin{matrix}1-e_2,\ 1+e_1-2e_2,\ -N;\ 1\\ 1+e_1-e_2,\ 1+2e_2-N;\end{matrix}\right]$$

§ (7.42). Case two

If $\delta_n = \dfrac{(d)_n}{(1+d-e)_n}$ then

$$\gamma_{2n} = \frac{(-e)_N(d-2e)_N(1+\tfrac{1}{2}d-e)_N(d-2e-N)_{2n}(1+\tfrac{1}{2}d-e+N)_n(\tfrac{1}{2}d-e)_n}{(-2e)_N(1+d-e)_N(\tfrac{1}{2}d-e)_N(1+\tfrac{1}{2}d-e)_n(d-2e)_{2n}(1+d-e+N)_{2n}} \times \frac{(d)_{2n}}{(\tfrac{1}{2}d-e+N)_n}$$

and

$$\gamma_{2n+1} = \frac{d(-e)_N(d-2e+1)_N(\tfrac{1}{2}(3+d)-e)_N(d-2e+1+N)_{2n}(d+1)_{2n}}{(1+d-e)(2+d-e)_N(\tfrac{1}{2}(d+1)-e)_N(-2e)_N(d-2e+1)_{2n}} \times \frac{(\tfrac{1}{2}(3+d)-e+N)_n(\tfrac{1}{2}(d+1)-e)_n}{(\tfrac{1}{2}(3+d)-e)_n(2+d-e+N)_{2n}(\tfrac{1}{2}(d+1)-e+N)_n}$$

Let $\alpha_n = \dfrac{(1+e)_n}{n!}$. Then, summing F_β by Saalschutz's theorem,

$$\beta_n = \frac{(1+e-N)_n(1+2e)_n}{n!\,(1+2e-N)_n}$$

Hence,

$$\frac{(-e)_N(d-2e)_N(1+\tfrac{1}{2}d-e)_N}{(-2e)_N(1+d-e)_N(\tfrac{1}{2}d-e)_N}\ {}_7F_6\left[\begin{matrix}\tfrac{1}{2}(d-N)-e,\tfrac{1}{2}(d-N+1)-e,1+\tfrac{1}{2}d-e-N, & \tfrac{1}{2}d, & \tfrac{1}{2}(1+d), & \tfrac{1}{2}(1+e), & 1+\tfrac{1}{2}e;\ 1\\ \tfrac{1}{2},\tfrac{1}{2}(d+1)-e, & 1+\tfrac{1}{2}d-e, & \tfrac{1}{2}(1+d-e+N), & 1+\tfrac{1}{2}(d-e+N), & \tfrac{1}{2}d-e+N;\end{matrix}\right]$$

$$+\frac{d(1+e)}{(1+d-e)}\frac{(-e)_N(d-2e+1)_N(\tfrac{1}{2}(3+d)-e)_N}{(2+d-e)_N(\tfrac{1}{2}(d+1)-e)_N(-2e)_N}\ {}_7F_6\left[\begin{matrix}\tfrac{1}{2}(d+1+N)-e, & \tfrac{1}{2}(d+N)+1-e,\tfrac{1}{2}(3+d)-e+N, & \tfrac{1}{2}(d+1), & 1+\tfrac{1}{2}d, & 1+\tfrac{1}{2}e, & \tfrac{1}{2}(3+e);\ 1\\ & 1+\tfrac{1}{2}d-e, & \tfrac{1}{2}(3+d)-e, & 1+\tfrac{1}{2}(d-e+N), & \tfrac{1}{2}(3+d-e+N), & \tfrac{1}{2}(d+1)-e+N, & 3/2;\end{matrix}\right]$$

$$= {}_3F_2\left[\begin{matrix}1+e-N, & 1+2e, & d;\ 1\\ & 1+2e-N, & 1+d-e;\end{matrix}\right]$$

<u>§ (7.43)</u> <u>Case three</u>

If $\delta_n = \dfrac{(d)_n(-N)_n}{(1+d-e)_n(1+2e-N)_n}$ then

$$\gamma_{2n} = \frac{(d-2e)_N(1+\tfrac{1}{2}d-e)_N(-e)_N(1-\tfrac{1}{2}d+e-N)_n(\tfrac{1}{2}d-e)_n(d)_{2n}(-N)_{2n}}{(1+d-e)_N(\tfrac{1}{2}d-e)_N(-2e)_N(e-\tfrac{1}{2}d-N)_n(d-2e)_{2n}(1+\tfrac{1}{2}d-e)_n}$$

$$\times\frac{1}{(1+e-N)_{2n}} \quad \text{and}$$

$$\gamma_{2n+1} = \frac{(d-2e+1)_{N-1}(\tfrac{1}{2}(3+d)-e)_{N-1}(-e)_{N-1}\ d\ (-N)}{(2+d-e)_{N-1}(\tfrac{1}{2}(d+1)-e)_{N-1}(-2e)_{N-1}\ (1+d-e)(1+2e-N)}$$

$$\times\frac{(d+1)_{2n}(1-N)_{2n}(\tfrac{1}{2}(d+1)-e)_n(e+\tfrac{1}{2}(1-d)+1-N)_n}{(d-2e+1)_{2n}(\tfrac{1}{2}(3+d)-e)_n(2+e-N)_{2n}(e-\tfrac{1}{2}(d+1)-N+1)_n}$$

Let $\alpha_n = \dfrac{(a_1)_n(a_2)_n}{n!(a_1+a_2+e)_n}$, then, summing F_β by Saalschutz's theorem $\beta_n = \dfrac{(e+a_1)_n(e+a_2)_n}{n!\,(a_1+a_2+e)_n}$

Hence,

$$\frac{(d-2e)_N(1+\tfrac{1}{2}d-e)_N(-e)_N}{(1+d-e)_N(\tfrac{1}{2}d-e)_N(-2e)_N}\ {}_9F_8\left[\begin{matrix}\tfrac{1}{2}a_1, & \tfrac{1}{2}(a_1+1), & \tfrac{1}{2}a_2, & \tfrac{1}{2}(a_2+1), & 1+e-\tfrac{1}{2}d-N, & \tfrac{1}{2}d,\tfrac{1}{2}(d+1), & -\tfrac{1}{2}N, & \tfrac{1}{2}(1-N);\ 1\\ & \tfrac{1}{2}(a_1+a_2+e), & \tfrac{1}{2}, & \tfrac{1}{2}(a_1+a_2+e+1), & \tfrac{1}{2}d-e+1, & e-\tfrac{1}{2}d-N,1+\tfrac{1}{2}d-e, & \tfrac{1}{2}(1+e-N), & 1+\tfrac{1}{2}(e-N);\end{matrix}\right]$$

$$+\frac{a_1a_2dN(d-2e+1)_{N-1}(\tfrac{1}{2}(3+d)-e)_{N-1}(-e)_{N-1}}{(a_1+a_2+e)(\tfrac{1}{2}(d+1)-e)_{N-1}(1+d-e)_N(-2e)_N}\ {}_9F_8\left[\begin{matrix}\tfrac{1}{2}(a_1+1), & 1+\tfrac{1}{2}a_1, & \tfrac{1}{2}(a_2+1), & 1+\tfrac{1}{2}a_2, & e+\tfrac{1}{2}(3-d)-N, & \tfrac{1}{2}(d+1), & 1+\tfrac{1}{2}d, & \tfrac{1}{2}(1-N), & 1-\tfrac{1}{2}N; & 1\\ & 3/2, & \tfrac{1}{2}(a_1+a_2+e+1), & 1+\tfrac{1}{2}(a_1+a_2+e), & \tfrac{1}{2}d-e+3/2, & e-\tfrac{1}{2}(d-1)-N, & 1+\tfrac{1}{2}(e-N), & \tfrac{1}{2}(3+e-N), & \tfrac{1}{2}(d+3)-e; & \end{matrix}\right]$$

$$= {}_4F_3\left[\begin{matrix}e+a_1, & e+a_2, & d, & -N; & 1\\ & a_1+a_2+e, & 1+d-e, & 1+2e-N; & \end{matrix}\right]$$

<u>§ (7.44)</u> <u>Case four</u>

If $\delta_n = \dfrac{(d)_n}{(1+d-e)_n}$ then

$$\gamma_{2n} = \frac{(d-2e)_N(1+\tfrac{1}{2}d-e)_N(-e)_N(e-d-N)_{2n}(1+\tfrac{1}{2}d-e+N)_n(\tfrac{1}{2}d-e)_n}{(1+d-e)_N(\tfrac{1}{2}d-e)_N(-2e)_N(1+2e-d-N)_{2n}(d-2e)_{2n}(1+\tfrac{1}{2}d-e)_n}$$

$$\times\frac{(1-\tfrac{1}{2}d+e-N)_{3n}(d)_{2n}(-N)_{4n}}{(e-\tfrac{1}{2}d-N)_{3n}(\tfrac{1}{2}d-e+N)_n(1+e-N)_{4n}} \quad \text{and}$$

$$\gamma_{2n+1} = \frac{(d-2e+1)_{N-1}(\tfrac{1}{2}(3+d)-e)_{N-1}(-e)_{N-1}(e-d-N)_{2n}dN(1-N)}{(1+d-e)_N(\tfrac{1}{2}(1+d)-e)_{N-1}(-2e)_N(2+2e-N)(2e-d-N+1)_{2n}}$$

$$\times\frac{(\tfrac{1}{2}(1+d)-e+N)_n(\tfrac{1}{2}(3-d)+e-N)_{3n}(\tfrac{1}{2}(d+1)-e)_n(d+1)_{2n}}{(1+d-2e)_{2n}(\tfrac{1}{2}(3+d)-e)_n(e-\tfrac{1}{2}d+\tfrac{1}{2}-N)_{3n}(\tfrac{1}{2}(d-1)-e+N)_n}$$

$$\times\frac{(2-N)_{4n}}{(3+e-N)_{4n}}$$

If $\alpha_n = \dfrac{(1+e)_n}{n!}$, summing F_β by Saalschutz's theorem, then

$$\beta_n = \frac{(-N)_n(\tfrac{1}{2}(e-N))_n(\tfrac{1}{2}(1-N+e))_n(1+2e)_n}{(e-N)_n(\tfrac{1}{2}(1-N)+e)_n(1-\tfrac{1}{2}N+e)_n\, n!}$$

Hence,

$$\frac{(d-2e)_N(1+\tfrac{1}{2}d-e)_N(-e)_N}{(1+d-e)_N(\tfrac{1}{2}d-e)_N(-2e)_N}\ {}_{12}F_{11}\left[\begin{matrix}\tfrac{1}{2}(1+e), & 1+\tfrac{1}{2}e, & \tfrac{1}{2}(e-d-N), & 1+\tfrac{1}{2}d-e+N, & \tfrac{1}{2}(1+e-d-N), & \tfrac{1}{2}d, & \tfrac{1}{2}(1+d), & 1+1/6.(2e-2N-d), & -\tfrac{1}{4}N, & \tfrac{1}{4}(1-N), & \tfrac{1}{4}(2-N), & \tfrac{1}{4}(3-N); & 1\\ & \tfrac{1}{2}, & \tfrac{1}{2}(1+2e-d-N), & 1+e-\tfrac{1}{2}(d+N), & \tfrac{1}{2}d-e+N, & \tfrac{1}{2}(1+d)-e, & 1+\tfrac{1}{2}d-e, & 1/6.(2e-d-2N), & \tfrac{1}{4}(1+e-N), & \tfrac{1}{4}(2+e-N), & \tfrac{1}{4}(3+e-N), & 1+\tfrac{1}{4}(e-N); & \end{matrix}\right]$$

$$+\frac{dN(1-N)(d-2e+1)_{N-1}(\tfrac{1}{2}(3+d)-e)_{N-1}(-1-e)_N}{(2+2e-N)(1+d-e)_N(\tfrac{1}{2}(1+d)-e)_{N-1}(-2e)_N}$$

$$\times\ {}_{12}F_{11}\left[\begin{matrix}1+\tfrac{1}{2}e,\tfrac{1}{2}(3+e), & \tfrac{1}{2}(e-d-N), & \tfrac{1}{2}(1+e-d-N),\tfrac{1}{2}(1+d)-e+N,\\ & 3/2,\tfrac{1}{2}(2e-d-N+1), & 1+e-\tfrac{1}{2}(d+N),1+\tfrac{1}{2}d-e,\end{matrix}\right.$$

$$\begin{matrix}1/6.(7-d+2e-2N), & \tfrac{1}{2}(d+1), & 1+\tfrac{1}{2}d, & \tfrac{1}{4}(2-N), & \tfrac{1}{4}(3-N),\\ 1/6.(2e-d+1-2N), & \tfrac{1}{2}(3+d)-e, & \tfrac{1}{2}(d-1)-e+N, & \tfrac{1}{4}(3+e-N), & 1+\tfrac{1}{4}(e-N),\end{matrix}$$

$$\left.\begin{matrix}1-\tfrac{1}{4}N, & \tfrac{1}{4}(5-N); & 1\\ \tfrac{1}{4}(5+e-N), & \tfrac{1}{4}(6+e-N); & \end{matrix}\right]$$

$$= {}_5F_4\left[\begin{matrix}d,\tfrac{1}{2}(e-N),\tfrac{1}{2}(1-N+e), & 1+2e, & -N; & 1\\ 1+d-e,\tfrac{1}{2}(1-N)+e,1-\tfrac{1}{2}N+e, & -N+e; & & \end{matrix}\right].$$

§ (7.45) Case five

If $\delta_n = \frac{(-N)_n}{(1+2e-N)_n}$, then

$$\gamma_n = \frac{(f-2e)_N(1+\tfrac{1}{2}f-e)_N(-e)_N(f)_{2n}(f-2e+N)_n(-N)_n}{(1+f-e)_N(\tfrac{1}{2}f-e)_N(-2e)_N(1+f-e+N)_n(1+f-2e)_{2n}(1+e-N)_n}$$

Let $\alpha_n = \frac{(1-2e)_n}{n!}$. Then, summing F_β by Saalschutz's theorem,

$$\beta_n = \frac{(f)_n(\tfrac{1}{2}(f+e))_n(1-e)_n(\tfrac{1}{2}(1+f+e))_n}{(f+e)_n(\tfrac{1}{2}(1+f-e))_n(1+\tfrac{1}{2}(f-e))_n n!}$$

Hence,

$$\frac{(f-2e)_N(1+\tfrac{1}{2}f-e)_N(-e)_N}{(1+f-e)_N(\tfrac{1}{2}f-e)_N(-2e)_N}\,{}_5F_4\left[\begin{matrix}1-2e,f-2e+N, & \tfrac{1}{2}f, & \tfrac{1}{2}(1+f), & -N; & 1\\ & 1+e-N,1+\tfrac{1}{2}f-e, & \tfrac{1}{2}(1+f)-e, & 1+f-e+N; & \end{matrix}\right]$$

$$= {}_5F_4\left[\begin{matrix}f,1-e, & \tfrac{1}{2}(f+e),\tfrac{1}{2}(1+f+e), & -N; & 1\\ f+e,1+\tfrac{1}{2}(f-e), & \tfrac{1}{2}(1+f-e),1+2e-N; & & \end{matrix}\right] \qquad \ldots(1)$$

This is a transformation between two nearly-poised ${}_5F_4(1)$ series.

Summing F_β as a well-poised ${}_7F_6(1)$ series, if

$$\alpha_n = \frac{(f-e)_n(1+\tfrac{1}{2}(f-e))_n(a_1)_n(a_2)_n(a_3)_n}{n!\,(\tfrac{1}{2}(f-e))_n(1+f-e-a_1)_n(1+f-e-a_2)_n(1+f-e-a_3)_n}$$

where $a_1+a_2+a_3+e = 1+f-e$. Then,

$$\beta_n = \frac{(f)_n(e+a_1)_n(e+a_2)_n(e+a_3)_n}{n!(1+f-e-a_1)_n(1+f-e-a_2)_n(1+f-e-a_3)_n}$$

Hence,

$$\frac{(f-2e)_N(1+\tfrac{1}{2}f-e)_N(-e)_N}{(1+f-e)_N(\tfrac{1}{2}f-e)_N(-2e)_N}\,{}_9F_8\left[\begin{matrix}f-e, 1+\tfrac{1}{2}(f-e), & a_1, \\ \tfrac{1}{2}(f-e), & 1+f-e-a_1,\end{matrix}\right.$$

$$\left.\begin{matrix}a_2, & a_3, & \tfrac{1}{2}f, & \tfrac{1}{2}(1+f), & f-2e+N, & -N; & 1 \\ 1+f-e-a_2, & 1+f-e-a_3, & 1+\tfrac{1}{2}f-e, & \tfrac{1}{2}(1+f)-e, & 1+e-N, & 1+f-e+N; & \end{matrix}\right]$$

$$= {}_5F_4\left[\begin{matrix}f, & e+a_1, & e+a_2, & e+a_3, & -N; & 1 \\ & 1+f-e-a_1, & 1+f-e-a_2, & 1+f-e-a_3, & 1+2e-N; & \end{matrix}\right] \qquad \ldots(2)$$

This is a transformation between a well-poised ${}_9F_8(1)$ series and a nearly-poised ${}_5F_4(1)$ series.

§ (7.46) Case six

If $\delta_n = \dfrac{(d)_n}{(1+f-d)_n}$ then

$$\gamma_n = \frac{(f-2d)_N(1+\tfrac{1}{2}f-d)_N(-d)_N(d)_n(\tfrac{1}{2}f)_n(\tfrac{1}{2}(1+f))_n}{(1+f-d)_N(\tfrac{1}{2}f-d)_N(-2d)_N(1+d-N)_n(\tfrac{1}{2}+d)_n(1+f-d+N)_n}$$

Let $\alpha_n = \dfrac{(1+2d-f)_n}{n!}$ and sum F_β by Saalschutz's theorem,

Then, $$\beta_n = \frac{(f)_n(\tfrac{1}{2}(f-N))_n(\tfrac{1}{2}(1+f-N))_n(1+2d-f-N)_n}{n!(f-N)_n(\tfrac{1}{2}(1-N)+d)_n(1+d-\tfrac{1}{2}N)_n}$$

Hence,

$$\frac{(f-2d)_N(1+\tfrac{1}{2}f-d)_N(-d)_N}{(1+f-d)_N(\tfrac{1}{2}f-d)_N(-2d)_N}\,{}_4F_3\left[\begin{matrix}1+2d-f, & d, & \tfrac{1}{2}f, & \tfrac{1}{2}(1+f); & 1 \\ & 1+f-d+N, & d+\tfrac{1}{2}, & 1+d-N\;; & \end{matrix}\right] =$$

$${}_5F_4\left[\begin{matrix}f, & \tfrac{1}{2}(f-N), & \tfrac{1}{2}(1+f-N), & 1+2d-f-N, & d; & 1 \\ & f-N, & \tfrac{1}{2}(1-N)+d, & 1-\tfrac{1}{2}N+d, & 1+f-d; & \end{matrix}\right]$$

§ (7.5) Summation of F_γ by theorems (14) and (15)

There are seven possible cases for F_γ. Of these, only two give summable series for F_β. These are:-

(1) $F_\gamma = {}_4F_3\left[\begin{matrix} f+2n, 1+\frac{1}{2}f+n, & e, & -N+n; & 1 \\ \frac{1}{2}f+n, & 1+f+2n-e, & 1+2e-N+n; & \end{matrix}\right]$

$$F_\beta = \sum_{r=0}^{n} \frac{(-n)_r (f+n)_r}{(1-e-n)_r (1+f-e+n)_r} \alpha_r$$

(2) $F_\gamma = {}_4F_3\left[\begin{matrix} f+2n, 1+\frac{1}{2}f+n, & d+n, & -N; & 1 \\ \frac{1}{2}f+n, & 1+f-d+n, & 1+2d+2n-N; & \end{matrix}\right]$

$$F_\beta = \sum_{r=0}^{n} \frac{(-n)_r (f+n)_r}{(1+N-n)_r (1+2d-N+n)_r} \alpha_r$$

§ (7.61) First case

If $\delta_n = \dfrac{(-N)_n (1+\frac{1}{2}f)_n}{(1+2e-N)_n (\frac{1}{2}f)_n}$, summing F_γ by theorem (14), then

$$\gamma_n = \frac{(f-2e)_N (-e)_N (1+\frac{1}{2}f)_n (-N)_n (\frac{1}{2}(f+1))_n (f-2e+N)_n}{(1+f-e)_N (-2e)_N (\frac{1}{2}f-e)_n (\frac{1}{2}(1+f)-e)_n (1+f-e+N)_n (1+e-N)_n}$$

Let $\alpha_n = \dfrac{(1-2e)_n}{n!}$, and sum F_β by Saalschutz's theorem,

Then, $\beta_n = \dfrac{(f)_n (\frac{1}{2}(f+e))_n (\frac{1}{2}(1+f+e))_n (1-e)_n}{(f+e)_n (\frac{1}{2}(1+f-e))_n (1+\frac{1}{2}(f-e))_n \, n!}$

Hence,

$$\frac{(f-2e)_N (-e)_N}{(1+f-e)_N (-2e)_N} {}_5F_4\left[\begin{matrix} 1-2e, \frac{1}{2}(f+1), & f-2e+N, & -N, & 1+\frac{1}{2}f; & 1 \\ 1+e-N, & 1+f-e+N, & \frac{1}{2}f-e, & \frac{1}{2}(1+f)-e; & \end{matrix}\right]$$

$$= {}_6F_5\left[\begin{matrix} f, \frac{1}{2}(f+e), \frac{1}{2}(1+f+e), & 1-e, 1+\frac{1}{2}f, & -N; & 1 \\ f+e, \frac{1}{2}(1+f-e), 1+\frac{1}{2}(f-e), & 1+2e-N, & \frac{1}{2}f; & \end{matrix}\right] \quad \ldots(1)$$

This transforms a nearly-poised ${}_5F_4(1)$ into a nearly-poised ${}_6F_5(1)$ series.

Summing F_β as a well-poised ${}_7F_6(1)$ series, if $a_1+a_2+a_3+e = 1+f-e$, then

$$\frac{(f-2e)_N(-e)_N}{(1+f-e)_N(-2e)_N}\,{}_9F_8\left[\begin{matrix} f-e, & 1+\frac{1}{2}(f-e), & a_1, & a_2, & a_3, & 1+\frac{1}{2}f, & \frac{1}{2}(1+f), & f-2e+N, & -N; \; 1 \\ & \frac{1}{2}(f-e), & 1+f-e-a_1, & 1+f-e-a_2, & 1+f-e-a_3, & \frac{1}{2}f-e, & \frac{1}{2}(1+f)-e, & 1+e-N, & 1+f-e+N \; ; \end{matrix}\right]$$

$$= {}_6F_5\left[\begin{matrix} f, & 1+\frac{1}{2}f, & e+a_1, & e+a_2, & e+a_3, & -N; \; 1 \\ & \frac{1}{2}f, & 1+f-e-a_1, & 1+f-e-a_2, & 1+f-e-a_3, & 1+2e-N; \end{matrix}\right] \quad \ldots(2)$$

Similar results are given by summing by theorem (15).

§ (7.62) Second case

Summing F_γ by theorem (14), if

$$\delta_n = \frac{(d)_n(1+\frac{1}{2}f)_n}{(1+f-d)_n(\frac{1}{2}f)_n}, \text{ then,}$$

$$\gamma_n = \frac{(f-2d)_N(-d)_N(d)_n(1+f)_{2n}(1+d)_n}{(1+f-d)_N(-2d)_N(1+d-N)_n(1+2d)_{2n}(1+f-d+N)_n}.$$

Let $\alpha_n = \frac{(1+2d-f)_n}{n!}$, and sum F_β by Saalschutz's theorem,

then, $\beta_n = \frac{(f)_n(\frac{1}{2}(f-N))_n(\frac{1}{2}(1+f-N))_n(1+2d-N-f)_n}{(f-N)_n(\frac{1}{2}(1+2d-N))_n(1+d-\frac{1}{2}N)_n\; n!}$

Hence,

$$\frac{(f-2d)_N(-d)_N}{(1+f-d)_N(-2d)_N}\,{}_4F_3\left[\begin{matrix} 1+2d-f, & \frac{1}{2}(1+f), & d, & 1+\frac{1}{2}f; \; 1 \\ & 1+d-N, & 1+f-d+N, & \frac{1}{2}+d; \end{matrix}\right]$$

$$= {}_6F_5\left[\begin{matrix} f, & 1+\frac{1}{2}f, & d, & \frac{1}{2}(f-N), & \frac{1}{2}(1+f-N), & 1+2d-f-N; \; 1 \\ & \frac{1}{2}f, & 1+f-d, & f-N, & \frac{1}{2}(1+2d-N), & 1+d-\frac{1}{2}N \; ; \end{matrix}\right] \quad \ldots(1)$$

Summing F_γ by theorem (15) and F_β by Saalschutz's theorem,

if $\delta_n = \frac{(1+\frac{1}{2}f)_n(d)_n}{(\frac{1}{2}f)_n(1+f-d)_n}$ then

$$\gamma_n = \frac{(f-2d-1)_N(\frac{1}{2}f-d-\frac{1}{2})_N(-d-1)_N(\frac{1}{2}(3-f)+d-N)_n(2+d)_n(d)_n}{(1+f-d)_N(\frac{1}{2}(f-1)-d)_N(-2d-1)_N(2+2d-f-N)_n(\frac{1}{2}(1-f)+d-N)_n}$$

$$\times \frac{(2+2d-N)_{3n}(1+f)_{2n}(-1)^n}{(2+d-N)_{2n}(2+2d)_{2n}(2+2d-N)_{2n}}$$

Hence,

$$\frac{(f-2d-1)_N\left(\tfrac{1}{2}(1+f)-d\right)_N(-d-1)_N}{(1+f-d)_N\left(\tfrac{1}{2}(f-1)-d\right)_N(-2d-1)_N}\;{}_9F_8\left[\begin{matrix}2+2d-f,\ \tfrac{1}{2}(3-f)+d-N,\ 2+d,\ 1/3.(2+2d-N),\ 1+1/3.(2d-N),\ 1/3.(4+2d-N),\ 1+\tfrac{1}{2}f,\ \tfrac{1}{2}(1+f),\ d;\ \frac{-27}{16}\\ \tfrac{1}{2}(3+d-N),\ 1+\tfrac{1}{2}(d-N),\ \tfrac{1}{2}(1-f)+d-N,\ 2+2d-f-N,\ 1+d,\ 3/2+d,\ 1+d-\tfrac{1}{2}N,\ \tfrac{1}{2}(3-N)+d;\end{matrix}\right]$$

$$= {}_6F_5\left[\begin{matrix}f,\ \tfrac{1}{2}(f-N),\ \tfrac{1}{2}(1+f-N),\ 2+2d-f-N,\ d,\ 1+\tfrac{1}{2}f;\ 1\\ f-N,\ \tfrac{1}{2}(2+2d-N),\ \tfrac{1}{2}(3+2d-N),\ 1+f-d,\ \tfrac{1}{2}f;\end{matrix}\right] \qquad \ldots(2)$$

§ (8.1) Summation of F_γ by theorem (18)

There are five possible cases for F_γ, of which only one leads to a summable series for F_β. In this case

$$F_\gamma = {}_7F_6\left[\begin{matrix}f+2n,\ 1+\tfrac{1}{2}f+n,\ f-d,\ \tfrac{1}{2}d+n,\ \tfrac{1}{2}(d+1)+n,\ 1+2f-d+N+n,\ -N+n;\ 1\\ \tfrac{1}{2}f+n,\ 1+d+2n,\ 1+f+n-\tfrac{1}{2}d,\ \tfrac{1}{2}(1-d)+f+n,\ d-f-N+n,\ 1+f+N+n;\end{matrix}\right],$$

and

$$F_\beta = \sum_{r=0}^{n}\frac{(-n)_r(f+n)_r}{(1-f+d-n)_r(1+d+n)_r}\alpha_r$$

If

$$\delta_n = \frac{(1+\tfrac{1}{2}f)_n(\tfrac{1}{2}d)_n(\tfrac{1}{2}(d+1))_n(1+2f-d+N)_n(-N)_n}{(\tfrac{1}{2}f)_n(1+f-\tfrac{1}{2}d)_n(\tfrac{1}{2}(1-d)+f)_n(d-f-N)_n(1+f+N)_n}$$

then

$$\gamma_n = \frac{(1+f)_N(1+2f-2d)_N(\tfrac{1}{2}d)_n(-N)_n}{(1+f-d)_N(1+2f-d)_N(1+\tfrac{1}{2}d)_n(2d-2f-N)_n}$$

Let $\alpha_n = \dfrac{(1+2d-2f)_n}{n!}$. Then, summing F_β by Saalschutz's theorem,

$$\beta_n = \frac{(f)_n\,(2f-d)_{2n}\,(1+d-f)_n}{(1+d)_{2n}\,(2f-d)_n\,n!}$$

Hence,

$$\frac{(1+f)_N(1+2f-2d)_N}{(1+f-d)_N(1+2f-d)_N}\;{}_3F_2\left[\begin{matrix}1+2d-2f,\ \tfrac{1}{2}d,\ -N;\ 1\\ 1+\tfrac{1}{2}d,\ 2d-2f-N;\end{matrix}\right]$$

$$= {}_7F_6\left[\begin{matrix}f,\ 1+\tfrac{1}{2}f,\ 1+d-f,\ f-\tfrac{1}{2}d,\ \tfrac{1}{2}d,\ 1+2f-d+N,\ -N;\ 1\\ \tfrac{1}{2}f,\ 2f-d,\ 1+\tfrac{1}{2}d,\ 1+f-\tfrac{1}{2}d,\ d-f-N,\ 1+f+N;\end{matrix}\right]$$

§ (8.2) Summation of F_γ by theorem (19).

There are eight possible ways of summing F_γ by theorem (19). Of these, only four lead to summable series for F_β. These are:-

(1) $$F_\gamma = {}_4F_3\left[\begin{matrix}\tfrac{1}{2}f+n, \tfrac{1}{2}(1+f)+n, & d+N+2n, & -N; 1 \\ \tfrac{1}{2}(1+d)+n, & \tfrac{1}{2}d+n, & 1+f+2n; \end{matrix}\right]$$

$$F_\beta = \sum_{r=0}^{n} \frac{(-n)_r (d+N+n)_r}{(1+N-n)_r (1+f+n)_r} \alpha_r$$

(2) $$F_\gamma = {}_4F_3\left[\begin{matrix}\tfrac{1}{2}e, \tfrac{1}{2}(1+e), & f+N+n, -N+n; 1 \\ \tfrac{1}{2}f+n, \tfrac{1}{2}(1+f)+n, & 1+e; \end{matrix}\right]$$

$$F_\beta = \sum_{r=0}^{n} \frac{(-n)_r (-e-n)_r}{(1-\tfrac{1}{2}e-n)_r (\tfrac{1}{2}(1-e)-n)_r} \alpha_r$$

(3) $$F_\gamma = {}_4F_3\left[\begin{matrix}\tfrac{1}{2}f+n, \tfrac{1}{2}(1+f)+n, d+N+n, -N+n; 1 \\ \tfrac{1}{2}d+n, \tfrac{1}{2}(1+d)+n, 1+f+2n; \end{matrix}\right]$$

$$F_\beta = \sum_{r=0}^{n} \frac{(-n)_r (-1)^r}{(1+f+n)_r} \alpha_r$$

(4) $$F_\gamma = {}_4F_3\left[\begin{matrix}\tfrac{1}{2}f+n, \tfrac{1}{2}(1+f)+n, d+N, -N+2n; 1 \\ \tfrac{1}{2}d+n, \tfrac{1}{2}(1+d)+n, 1+f+2n; \end{matrix}\right]$$

$$F_\beta = \sum_{r=0}^{n} \frac{(-n)_r (-N+n)_r}{(1-d-N-n)_r (1+f+n)_r} \alpha_r$$

§ (8.3) First case

If $\delta_n = \dfrac{(\tfrac{1}{2}f)_n (\tfrac{1}{2}(1+f))_n}{(\tfrac{1}{2}d)_n (\tfrac{1}{2}(1+d))_n}$, then

$$\gamma_n = \frac{(d-f)_N (\tfrac{1}{2}f)_n}{(d)_N (1+\tfrac{1}{2}f)_n}.$$

Let $\alpha_n = \dfrac{(1+f-d)_n}{n!}$, and sum F_β by Saalschutz's theorem.

Then, $\beta_n = \frac{(d+N)_n(1+f-d-N)_n(d)_{2n}}{(1+f)_{2n}(d)_n\ n!}$

Hence, $\frac{(d-f)_N}{(d)_N}\,{}_2F_1\left[\begin{matrix}1+f-d, & \frac{1}{2}f; & 1\\ & 1+\frac{1}{2}f; & \end{matrix}\right]$

$= {}_3F_2\left[\begin{matrix}\frac{1}{2}f, & d+N, 1+f-d-N; & 1\\ 1+\frac{1}{2}f, & d; & \end{matrix}\right]$ ...(1)

Summing the ${}_2F_1(1)$ series by Gauss's theorem, we get a special ${}_3F_2(1)$ sum, namely

Theorem (21)

$${}_3F_2\left[\begin{matrix}\frac{1}{2}f, & d+N, 1+f-d-N; & 1\\ 1+\frac{1}{2}f, & d; & \end{matrix}\right] = \frac{\Gamma(1+\frac{1}{2}f)\Gamma(d-f+N)\Gamma(d)}{\Gamma(d-\frac{1}{2}f)\Gamma(d+N)}$$ ...(2)

provided $Rl(d-f)>0$, and the change in the order of summation can be justified.

§ (8.4) Second case

If $\delta_n = \frac{(f+N)_n(-N)_n}{(\frac{1}{2}f)_n(\frac{1}{2}(1+f))_n}$ then

$\gamma_n = \frac{(f-e)_N(-N)_n(f-e+N)_n}{(f)_N(\frac{1}{2}(f-e))_n(\frac{1}{2}(1+f-e))_n}$

Let $\alpha_n = \frac{(\frac{1}{2})_n}{n!}$, and sum F_β by Saalschutz's theorem, then,

$\beta_n = \frac{(1+\frac{1}{2}e)_n(\frac{1}{2}(1+e))_n}{n!\ (1+e)_n}$

Hence, $\frac{(f-e)_N}{(f)_N}\,{}_3F_2\left[\begin{matrix}\frac{1}{2}, & f-e+N, & -N; & 1\\ & \frac{1}{2}(f-e), & \frac{1}{2}(1+f-e); & \end{matrix}\right]$

$= {}_4F_3\left[\begin{matrix}1+\frac{1}{2}e, \frac{1}{2}(1+e), f+N, & -N; & 1\\ 1+e, \frac{1}{2}f, \frac{1}{2}(1+f); & \end{matrix}\right]$

§ (8.5) Third case

If $\delta_n = \dfrac{(\frac{1}{2}f)_n(\frac{1}{2}(1+f))_n(d+N)_n(-N)_n}{(\frac{1}{2}d)_n(\frac{1}{2}(1+d))_n}$

then $\gamma_n = \dfrac{(d-f)_N(\frac{1}{2}f)_n(-N)_n(-1)^n}{(d)_N(1+f-d-N)_n(1+\frac{1}{2}f)_n}$

Let $\alpha_n = \dfrac{(a)_n(-1)^n}{n!}$. Then, summing F_β by Vandermonde's theorem, $\beta_n = \dfrac{(1+f-a)_{2n}}{(1+f-a)_n(1+f)_{2n}\, n!}$.

Hence, $\dfrac{(d-f)_N}{(d)_N}{}_3F_2\left[\begin{matrix}a, & \frac{1}{2}f, & -N; \; 1\\ & 1+\frac{1}{2}f, & 1+f-d-N;\end{matrix}\right]$

$$= {}_5F_4\left[\begin{matrix}d+N, & \frac{1}{2}f, & \frac{1}{2}(1+f-a), & 1+\frac{1}{2}(f-a), & -N; \; 1\\ & 1+\frac{1}{2}f, & \frac{1}{2}(1+d), & \frac{1}{2}d, & 1+f-a;\end{matrix}\right] \qquad \ldots(1)$$

Now, let $\alpha_n = \dfrac{(f)_n(1+\frac{1}{2}f)_n(a)_n(b)_n(-1)^n}{n!\,(1+f-a)_n(1+f-b)_n(\frac{1}{2}f)_n}$, and sum F_β as a well-poised ${}_5F_4(1)$ series. Then,

$\beta_n = \dfrac{(1+f-a-b)_n}{n!(1+f-a)_n(1+f-b)_n}$. Hence,

$\dfrac{(d-f)_N}{(d)_N}{}_4F_3\left[\begin{matrix}f, & a, & b, & -N; \; 1\\ & 1+f-a, & 1+f-b, & 1+f-d-N;\end{matrix}\right]$

$$= {}_5F_4\left[\begin{matrix}\frac{1}{2}f, & \frac{1}{2}(1+f), & 1+f-a-b, & d+N, & -N; \; 1\\ & \frac{1}{2}d, & \frac{1}{2}(1+d), & 1+f-a, & 1+f-b;\end{matrix}\right] \qquad \ldots(2)$$

This is a transformation between a nearly-poised terminating ${}_4F_3(1)$ series and a Saalschutzian ${}_5F_4(1)$ series.

§ (8.6) Fourth case

If $\delta_n = \dfrac{(\frac{1}{2}f)_n(\frac{1}{2}(1+f))_n}{(\frac{1}{2}d)_n(\frac{1}{2}(1+d))_n}$, then

$\gamma_n = \dfrac{(d-f)_N(\frac{1}{2}f)_n(-N)_{2n}}{(d)_N(1+f-d-N)_{2n}(1+\frac{1}{2}f)_n}$.

Let $\alpha_n = \dfrac{(1+f-d)_n}{n!}$ and sum F_β by Saalschutz's theorem.

then $\beta_n = \frac{(d)_{2n}(1+f+N)_n(-N)_n}{n!(1+f)_{2n}(d)_n}$

Hence, $\frac{(d-f)_N}{(d)_N}\,{}_4F_3\left[\begin{matrix}1+f-d, & -\frac{1}{2}N, & \frac{1}{2}f, \frac{1}{2}(1-N); & 1\\ \frac{1}{2}(1+f-d-N), & 1+\frac{1}{2}(f-d-N), & 1+\frac{1}{2}f & ; \end{matrix}\right]$

$= {}_3F_2\left[\begin{matrix}\frac{1}{2}f, 1+f+N, -N; & 1\\ 1+\frac{1}{2}f, \; d; & \end{matrix}\right]$

§ (9.1). Summation of F_γ by theorem (5).

There are four possible series for F_γ which can be summed by theorem (5), Gauss's second summation theorem, and all of these series lead to series for F_β which are summable. The four cases are :-

(1) $F_\gamma = {}_2F_1\left[\begin{matrix}e_1, & e_2; & \frac{1}{2}\\ \frac{1}{2}(1+e_1+e_2); & \end{matrix}\right]$

$$F_\beta = \sum_{r=0}^{n} \frac{(-n)_r(1-\frac{1}{2}(e_1+e_2+1)-n)_r}{(1-e_1-n)_r(1-e_2-n)_r}\alpha_r$$

(2) $F_\gamma = {}_2F_1\left[\begin{matrix}e, & f+2n; & \frac{1}{2}\\ \frac{1}{2}(1+e+f)+n; & \end{matrix}\right]$

$$F_\beta = \sum_{r=0}^{n} \frac{(-n)_r(f+n)_r}{(1-e-n)_r}\alpha_r$$

(3) $F_\gamma = {}_2F_1\left[\begin{matrix}d_1+n, & d_2+n; & \frac{1}{2}\\ \frac{1}{2}(d_1+d_2+1)+n; & \end{matrix}\right]$

$$F_\beta = \sum_{r=0}^{n} (-n)_r(-1)^r\alpha_r$$

(4) $F_\gamma = {}_2F_1\left[\begin{matrix}f_1+2n, & f_2+2n; & \frac{1}{2}\\ \frac{1}{2}(f_1+f_2+1)+2n & ; \end{matrix}\right]$

$$F_\beta = \sum_{r=0}^{n} \frac{(-n)_r(f_1+n)_r(f_2+n)_r(-1)^r}{(\frac{1}{2}(f_1+f_2+1)+n)_r}\alpha'_r$$

§ (9.2) First case

If $\delta_n = (\tfrac{1}{2})^n$, then

$$\gamma_n = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(1+e_1+e_2))}{\Gamma(\frac{1}{2}(1+e_1))\Gamma(\frac{1}{2}(1+e_2))}(\tfrac{1}{2})^n.$$

Let $\alpha_n = \dfrac{(\frac{1}{2}(1-e_1-e_2))_n}{n!}$, and sum F_β by Saalschutz's theorem.

Then $\beta_n = \dfrac{(\frac{1}{2}(1+e_2-e_1))_n(\frac{1}{2}(1+e_1-e_2))_n}{n!\,(\frac{1}{2}(1+e_1+e_2))_n}$

Hence, $\dfrac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(1+e_1+e_2))}{\Gamma(\frac{1}{2}(1+e_1))\Gamma(\frac{1}{2}(1+e_2))}\,{}_1F_0\left[\tfrac{1}{2}(1-e_1-e_2);\ ;\ \tfrac{1}{2}\right]$

$$= {}_2F_1\left[\begin{matrix}\frac{1}{2}(1+e_2-e_1),\frac{1}{2}(1+e_1-e_2);\ \frac{1}{2}\\ \frac{1}{2}(1+e_1+e_2);\end{matrix}\right]$$

§ (9.3) Second case

If $\delta_n = \dfrac{1}{(\frac{1}{2}(1+e+f))_n 2^n}$, then

$$\gamma_n = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(1+e+f))\ (\frac{1}{2}f)_n 2^n}{\Gamma(\frac{1}{2}(1+e))\Gamma(\frac{1}{2}(1+f))}$$

If $\alpha_n = \dfrac{1}{n!}$, summing F_β by Vandermonde's theorem,

then $\beta_n = \dfrac{(f)_n(\frac{1}{2}(e+f))_n(\frac{1}{2}(1+e+f))_n 4^n}{n!\,(e+f)_n}$

Hence, $\dfrac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(1+e+f))}{\Gamma(\frac{1}{2}(1+e))\Gamma(\frac{1}{2}(1+f))}\ {}_1F_0\left[\tfrac{1}{2}f;\ ;\ 2\right]$

$$= {}_2F_1\left[\begin{matrix}f,\frac{1}{2}(e+f);\ 2\\ e+f;\end{matrix}\right]$$

provided f is an even negative integer.

§ (9.4) Third case

If $\delta_n = \dfrac{(d_1)_n(d_2)_n}{(\frac{1}{2}(1+d_1+d_2))_n 2^n}$, then

$$\gamma_{2n} = \frac{\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}(1+d_1+d_2))\,(\tfrac{1}{2}d_1)_n(\tfrac{1}{2}d_2)_n 2^{2n}}{\Gamma(\tfrac{1}{2}(1+d_1))\Gamma(\tfrac{1}{2}(1+d_2))}$$

and $$\gamma_{2n+1} = \frac{\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}(1+d_1+d_2))\,(\tfrac{1}{2}(1+d_1))_n(\tfrac{1}{2}(1+d_2))_n 2^{2n+1}}{\Gamma(\tfrac{1}{2}d_1)\Gamma(\tfrac{1}{2}d_2)}$$

Let $\alpha_n = \dfrac{(a)_n(-1)^n}{n!\,(A)_n}$. Then, summing F_β by Vandermonde's theorem, $\beta_n = \dfrac{(A-a)_n}{n!\,(A)_n}$.

Hence, $$\frac{\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}(1+d_1+d_2))}{\Gamma(\tfrac{1}{2}(1+d_1))\Gamma(\tfrac{1}{2}(1+d_2))}\,{}_4F_3\left[\begin{matrix}\tfrac{1}{2}a, \tfrac{1}{2}(1+a), \tfrac{1}{2}d_1, \tfrac{1}{2}d_2; & -1\\ \tfrac{1}{2}(1+A), \tfrac{1}{2}, \tfrac{1}{2}A; & \end{matrix}\right]$$

$$-\frac{2a\,\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}(1+d_1+d_2))}{A\,\Gamma(\tfrac{1}{2}d_1)\Gamma(\tfrac{1}{2}d_2)}\,{}_4F_3\left[\begin{matrix}\tfrac{1}{2}(1+a), 1+\tfrac{1}{2}a, \tfrac{1}{2}(1+d_1), \tfrac{1}{2}(1+d_2); & -1\\ \tfrac{1}{2}(A+1), 1+\tfrac{1}{2}A, \tfrac{3}{2}; & \end{matrix}\right]$$

$$= {}_3F_2\left[\begin{matrix}A-a, d_1, d_2; & \tfrac{1}{2}\\ \tfrac{1}{2}(1+d_1+d_2), A; & \end{matrix}\right]$$

§ (9.5) Fourth case

If $\delta_n = (\tfrac{1}{2})^n$, then

$$\gamma_n = \frac{\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}(1+f_1+f_2))(\tfrac{1}{2}f_1)_n(\tfrac{1}{2}f_2)_n 8^n}{\Gamma(\tfrac{1}{2}(1+f_1))\Gamma(\tfrac{1}{2}(1+f_2))}$$

Let $\alpha_n = \dfrac{(-1)^n}{n!\,(\tfrac{1}{2}(1+f_1+f_2))_n}$. Summing F_β by Saalschutz's theorem, then $$\beta_n = \frac{(\tfrac{1}{2}(1+f_2-f_1))_n(\tfrac{1}{2}(1+f_1-f_2))_n(f_1)_n(f_2)_n(-\tfrac{1}{4})^n}{n!\,(\tfrac{1}{4}(1+f_1+f_2))_n(\tfrac{1}{4}(3+f_1+f_2))_n(\tfrac{1}{2}(1+f_1+f_2))_n}$$

Hence, $$\frac{\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}(1+f_1+f_2))}{\Gamma(\tfrac{1}{2}(1+f_1))\Gamma(\tfrac{1}{2}(1+f_2))}\,{}_2F_1\left[\begin{matrix}\tfrac{1}{2}f_1, \tfrac{1}{2}f_2; & -8\\ \tfrac{1}{2}(1+f_1+f_2); & \end{matrix}\right]$$

$$= {}_4F_3\left[\begin{matrix}\tfrac{1}{2}(1+f_2-f_1), \tfrac{1}{2}(1+f_1-f_2), f_1, f_2; & -\tfrac{1}{8}\\ \tfrac{1}{4}(1+f_1+f_2), \tfrac{1}{4}(3+f_1+f_2), \tfrac{1}{2}(1+f_1+f_2); & \end{matrix}\right]$$

provided both series terminate.

§ (9.6) Summation of F_γ by theorem six

There are three ways of summing F_γ by theorem

six, of which only one leads to a summable series for F_β.
In this case $F_\gamma = {}_2F_1\left[\begin{matrix} e, 1-e; & \frac{1}{2} \\ E; & \end{matrix}\right]$ and

$$F_\beta = \sum_{r=0}^{n} \frac{(-n)_r (1-E-n)_r}{(1-e-n)_r (e-n)_r} \alpha_r .$$

If $\delta_n = (\frac{1}{2})^n$, then

$$\gamma_n = \frac{\Gamma(\frac{1}{2}E)\Gamma(\frac{1}{2}(1+E))\ (\frac{1}{2})^n}{\Gamma(\frac{1}{2}(E+e))\Gamma(\frac{1}{2}(1+E-e))}$$

Let $\alpha_n = \dfrac{(E-1)_n}{n!}$. Summing F_β by Saalschutz's theorem,

then $\beta_n = \dfrac{(E+e-1)_n (E-e)_n}{n!\ (E)_n}$

Hence, $$\frac{\Gamma(\frac{1}{2}E)\Gamma(\frac{1}{2}(1+E))}{\Gamma(\frac{1}{2}(E+e))\Gamma(\frac{1}{2}(1+E-e))}\ {}_1F_0\left[E-1;\ ;\ \tfrac{1}{2}\right]$$

$$= {}_2F_1\left[\begin{matrix} E+e-1, E-e; & \frac{1}{2} \\ E; & \end{matrix}\right]$$

Summing the ${}_1F_0(\frac{1}{2})$ series by the binomial theorem gives theorem five again.

The results of this section (§(3.2)–§(9.6)) are all deduced from the fundamental theorem (theorem one), and from the simple theorems of § (2.2), showing some of the remarkable possibilities of theorem one. The argument, in every case, involves an interchange of the order of summation of a double series. Hence, when this double series is infinite, as for example in § (4.7) the result cannot be regarded as proved, until the convergence of the double series involved has been investigated.

§ (10.1) A method of obtaining futher results

Suppose now that we replace some of the numbers α_r by zero. Thus, if $\alpha_{2r+1} = 0$, but $\alpha_{2r} \neq 0$,

then $$\beta_n = \sum_{r=0}^{n} \alpha_{2r} \frac{(e_1)_{n-2r} \cdots \quad (f_1)_{n+2r} \cdots}{(n-2r)!(E_2)_{n-2r} \cdots \quad (F_1)_{n+2r} \cdots}$$

$$= \frac{(e_1)_n \cdots \quad (f_1)_n \cdots}{n! \ (E_2)_n \cdots \ (F_1)_n \cdots} \sum_{r=0}^{[n/2]} \alpha_{2r} \frac{(-n)_{2r}(1-n-E_2)_{2r} \cdots}{(1-e_1-n)_{2r} \cdots} \times \frac{(f_1+n)_{2r} \cdots}{(F_1+n)_{2r} \cdots}$$

where $[\frac{1}{2}n]$ is the greatest integer $\leq \frac{1}{2}n$.

But $(a)_{2r} \equiv (\frac{1}{2}a)_r(\frac{1}{2}(1+a))_r 4^r$

Hence, $$\beta_n = \frac{(e_1)_n \cdots \quad (f_1)_n \cdots}{n! \ (E_2)_n \cdots \ (F_1)_n \cdots}$$

$$\times \sum_{r=0}^{[n/2]} \frac{(-\frac{1}{2}n)_r \ (\frac{1}{2}(1-n))_r(\frac{1}{2}(1-n-E_2))_r(1-\frac{1}{2}(n+E_2))_r \cdots}{(\frac{1}{2}(1-e_1-n))_r(1-\frac{1}{2}(e_1+n))_r \cdots}$$

$$\times 2^q \alpha_{2r} \frac{(\frac{1}{2}(f_1+n))_r \ (\frac{1}{2}(1+f_1+n))_r \cdots}{(\frac{1}{2}(F_1+n))_r(\frac{1}{2}(1+F_1+n))_r \cdots} \qquad \ldots(1)$$

where q is the difference between the number of numerator parameters and the number of denominator parameters in F_β. The value of α_{2r} remains to be chosen but it must be independent of n.

Similarly, if $\alpha_{kr} \neq 0$, but $\alpha_r = \alpha_{2r} = \ldots = \alpha_{(k-1)r} = 0$, then a similar series for F_β is obtained by using the formula $(a)_{kr} = (a/k)_r((a+1)/k)_r \cdots \cdots ((a+k-1)/k)_r(k)^{kr}$ ...(2)

This formula is analogous to the multiplication theorem for the Gamma function,

$$\Gamma(\tfrac{1}{2})\Gamma(nx) = \Gamma(nx/n)\Gamma((nx+1)/n)\ldots\ldots\Gamma((nx+n-1)/n)\,2^{nx-1} \quad \ldots(3)$$

(Gauss, Disquisitiones.)

§ (10.2) Application of the Binomial theorem when $\alpha_{2r+1} = 0$

Let us consider the three series for F_β, which result from the summation of F_γ by the Binomial theorem. In the first case, when $\alpha_{2r+1} = 0$ and $\alpha_{2r} \neq 0$, the resulting series for F_β is not summable by any of the known summation theorems.

In the second case, $\beta_n = \dfrac{1}{n!}\sum_{r=0}^{[\frac{1}{2}n]} (-\tfrac{1}{2}n)_r(\tfrac{1}{2}(1-n))_r 4^r \alpha_{2r}$.

If $\alpha_{2n} = \dfrac{1}{n!\,(A)_n\,4^n}$, summing F_β by Vandermonde's theorem, then $\beta_{2n} = \dfrac{(A-\frac{1}{2})_{2n}}{2n!\,(A)_n(A-\frac{1}{2})_n}$ and

$$\beta_{2n+1} = \frac{(A+\frac{1}{2})_{2n}}{(2)_{2n}(A)_n(A+\frac{1}{2})_n}$$

If $\delta_n = (d)_n x^n$, then $\gamma_n = \dfrac{(d)_n x^n}{(1-x)^{d+n}}$, as in § (3.4)

Hence, $\dfrac{1}{(1-x)^d}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}d, \frac{1}{2}(1+d); & \frac{x^2}{(1-x)^2}\\ A\ ; & \end{matrix}\right]$

$$= {}_4F_3\left[\begin{matrix}\frac{1}{4}(2A-1), \frac{1}{4}(2A+1), \frac{1}{2}d, \frac{1}{2}(1+d); & 4x^2\\ A, \frac{1}{2}, A-\frac{1}{2}; & \end{matrix}\right]$$

$$+ \; xd\;{}_4F_3\left[\begin{matrix}\frac{1}{4}(2A+1), \frac{1}{4}(2A+3), \frac{1}{2}(1+d), 1+\frac{1}{2}d; & 4x^2\\ A, 3/2, A+\frac{1}{2}; & \end{matrix}\right] \quad \ldots(1)$$

In the third case, if $\alpha_n = 0$ when n is odd, then

$$\beta_n = \frac{(e)_n}{n!}\sum_{r=0}^{[\frac{1}{2}n]}\frac{(-\frac{1}{2}n)_r(\frac{1}{2}(1-n))_r}{(\frac{1}{2}(1-e-n))_r(1-\frac{1}{2}(e+n))_r}\alpha_{2r}$$

If $\alpha_{2n} = \frac{(-e)_n}{n!}$, summing F_β by Saalschutz's theorem, then

$$\beta_{2n} = \frac{(-\frac{1}{2}e)_n(\frac{1}{2}(1-e))_n}{n!\,(\frac{1}{2})_n} \quad \text{and} \quad \beta_{2n+1} = \frac{e\,(1-\frac{1}{2}e)_n(\frac{1}{2}(1-e))_n}{n!\,(3/2)_n}$$

Hence, if $\delta_n = x^n$ and $\gamma_n = \frac{x^n}{(1-x)^e}$, as in § (3.5),

then $$(1+x)^e = {}_2F_1\left[\begin{matrix}\frac{1}{2}(1-e),-\frac{1}{2}e;\\ \frac{1}{2};\end{matrix}\;x^2\right] + ex\,{}_2F_1\left[\begin{matrix}1-\frac{1}{2}e,\frac{1}{2}(1-e);\\ 3/2;\end{matrix}\;x^2\right] \qquad \ldots(2)$$

which is obviously true.

§ (11.1) Applications of Gauss's theorem: Case three, when $\alpha_{2r+1} = 0$.

Consider now the series for F_β which result from the summation of F_γ by Gauss's theorem, or by Vandermonde's theorem.

If $$\beta_n = \frac{1}{n!\,(F)_n}\sum_{r=0}^{[\frac{1}{2}n]}\frac{(-\frac{1}{2}n)_r(\frac{1}{2}(1-n))_r}{(\frac{1}{2}(F+n))_r(\frac{1}{2}(1+F+n))_r}\alpha_{2r}$$

when $\alpha_{2n} = \frac{(\frac{1}{2}(F-1))_n(\frac{1}{4}(3+F))_n(a)_n}{n!(\frac{1}{4}(F-1))_n(\frac{1}{2}(F+1)-a)_n}$ then F_β can be summed as a well-poised ${}_5F_4(1)$ series. Hence,

$$\beta_{2n} = \frac{(\frac{1}{4}(F-2a))_n(\frac{1}{4}(F+2-2a))_n}{2n!(\frac{1}{2}F-a)_n(\frac{1}{2}(1+F)-a)_n(\frac{1}{2}F)_{2n}} \quad \text{and}$$

$$\beta_{2n+1} = \frac{(\frac{1}{4}(F+2-2a))_n(1+\frac{1}{4}(F-2a))_n}{F\,(2)_{2n}(1+\frac{1}{2}F-a)_n(1+\frac{1}{2}F)_n(\frac{1}{2}(1+F)-a)_n}.$$

Thus, if $\delta_n = (d_1)_n(d_2)_n$ and

$$\gamma_n = \frac{\Gamma(F)\Gamma(F-d_1-d_2)\,(d_1)_n(d_2)_n}{\Gamma(F-d_1)\Gamma(F-d_2)\,(F-d_1)_n(F-d_2)_n} \quad \text{as in § (4.3),}$$

we get $$\frac{\Gamma(F)\Gamma(F-d_1-d_2)}{\Gamma(F-d_1)\Gamma(F-d_2)} \times {}_7F_6\left[\begin{matrix}\tfrac{1}{2}(1-F), \tfrac{1}{4}(F+3), & a, & \tfrac{1}{2}d_1, \tfrac{1}{2}(1+d_1), & \tfrac{1}{2}d_2, \tfrac{1}{2}(1+d_2); & 1 \\ \tfrac{1}{4}(F-1), & \tfrac{1}{2}(F+1)-a, & \tfrac{1}{2}(F-d_1+1), \tfrac{1}{2}(F-d_1), & \tfrac{1}{2}(F-d_2+1), \tfrac{1}{2}(F-d_2); & \end{matrix}\right]$$

$$= {}_6F_5\left[\begin{matrix}\tfrac{1}{4}(F-2a), \tfrac{1}{4}(F+2-2a), \tfrac{1}{2}d_1, \tfrac{1}{2}(1+d_1), \tfrac{1}{2}d_2, \tfrac{1}{2}(1+d_2); & 1 \\ \tfrac{1}{2}(F+1)-a, \tfrac{1}{2}F-a, \tfrac{1}{4}(F+2), \tfrac{1}{4}F, \tfrac{1}{2}; & \end{matrix}\right]$$

$$+ \frac{d_1 d_2}{F}\, {}_6F_5\left[\begin{matrix}\tfrac{1}{4}(F+2-2a), 1+\tfrac{1}{4}(F-2a), \tfrac{1}{2}(d_1+1), 1+\tfrac{1}{2}d_1, \tfrac{1}{2}(1+d_2), 1+\tfrac{1}{2}d_2; & 1 \\ \tfrac{1}{2}(F+1)-a, 1+\tfrac{1}{2}F-a, \tfrac{1}{4}(F+2), 1+\tfrac{1}{4}F, 3/2; & \end{matrix}\right]$$

<u>§ (11.2)</u> <u>Case three, when $\alpha_{3r+1} = \alpha_{3r+2} = 0$</u>

Now suppose that, in the previous case, $\alpha_r = 0$, when r is not a multiple of three.

Then $$F_\beta = \sum_{r=0}^{[n/3]} \frac{(-n)_{3r}(-1)^{3r}}{(F+n)_{3r}}\alpha_{3r}$$

Let $$\alpha_{3n} = \frac{(1/3.(F-1))_n(1+1/6.(F-1))_n(-1)^n}{n!\,(1/6.(F-1))_n}$$

Then, summing F_β as a well-poised ${}_5F_4(1)$ series,

$$\beta_{3n} = \frac{(1/9.(F-1))_n(1/9.(F+2))_n(1/9.(F+5))_n}{n!\,(1/3)_n(2/3)_n(1/3.(1+F))_{2n}(1/3.F)_{2n}(1/3.(F-1))_{2n}3^{3n}}$$

$$\beta_{3n+1} = \frac{(1/9.(2+F))_n(1/9.(5+F))_n(1/9.(8+F))_n}{F\,(2/3)_n n!(4/3)_n(1+1/3.F)_{2n}(1/3.(F+1))_{2n}(1/3.(2+F))_{2n}3^{3n}}$$

$$\beta_{3n+2} = \frac{(1/9.(5+F))_n(1/9.(8+F))_n(1/9.(11+F))_n}{2F(F+1)(4/3)_n(5/3)_n n!(1+1/3.F)_{2n}(1/3.(4+F))_{2n}}$$

$$\times \frac{1}{(1/3.(5+F))_{2n}\,3^{3n}}$$

Hence,

$$\frac{\Gamma(F)\Gamma(F-d_1-d_2)}{\Gamma(F-d_1)\Gamma(F-d_2)}\;{}_8F_7\left[\begin{matrix} 1/3.(F-1),\ 1+1/6.(F-1),\ 1/3.d_1,\ 1/3.(d_1+1),\ 1/3.(d_1+2),\ 1/3.d_2,\ 1/3.(d_2+1),\ 1/3.(d_2+2);\ -1 \\ 1/6.(F-1),\ 1/3.(F-d_1+2),\ 1/3.(F-d_1+1),\ 1/3.(F-d_1),\ 1/3.(F-d_2+2),\ 1/3.(F-d_2+1),\ 1/3.(F-d_2); \end{matrix}\right]$$

$$= {}_9F_8\left[\begin{matrix} 1/3.d_1,\ 1/3.(1+d_1),\ 1/3.(2+d_1),\ 1/3.d_2,\ 1/3.(1+d_2),\ 1/3.(2+d_2),\ 1/9.(F-1),\ 1/9.(F+2),\ 1/9.(F+5);\ \frac{27}{64} \\ 1/3,\ 2/3,\ 1/6(F-1),\ 1/6.F,\ 1/6.(F+1),\ 1/6.(F+2),\ 1/6.(F+3),\ 1/6.(F+4)\ ; \end{matrix}\right]$$

$$+\frac{d_1d_2}{F}\,{}_9F_8\left[\begin{matrix} 1/3.(d_1+1),\ 1/3.(d_1+2),\ 1+1/3.d_1,\ 1/3.(1+d_2),\ 1/3.(2+d_2),\ 1+1/3.d_2,\ 1/9.(F+2),\ 1/9.(F+5),\ 1/9.(F+8);\ \frac{27}{64} \\ 2/3,\ 4/3,\ 1/6.(1+F),\ 1/6.(F+2),\ 1/6.(F+3),\ 1/6.(F+4),\ 1/6.(F+5),\ 1+1/6.F\ ; \end{matrix}\right]$$

$$+\frac{d_1(d_1+1)d_2(d_2+1)}{2F(F+1)}\,{}_9F_8\left[\begin{matrix} 1/3.(d_1+2),\ 1+1/3.d_1,\ 1/3.(d_1+4),\ 1/3.(d_2+2),\ 1+1/3.d_2,\ 1/3.(d_2+4),\ 1/9.(F+5),\ 1/9.(F+8),\ 1/9.(11+F);\ \frac{27}{64} \\ 4/3,\ 5/3,\ 1/6.(F+3),\ 1/6.(F+4),\ 1/6.(F+5),\ 1+1/6.F,\ 1/6.(F+7),\ 1/6.(8+F);\ \end{matrix}\right]$$

§ (11.3) Case five; when $\alpha_{2r+1} = 0$.

In this case we have

$$F_\beta = \sum_{r=0}^{[\frac{1}{2}n]} \frac{(-n)_{2r}(f+n)_{2r}}{(1-e-n)_{2r}(1+f-e+n)_{2r}}\,\alpha_{2r}, \quad \text{if } F = 1+f-e, \text{ and}$$

$\alpha_r = 0$ when r is odd.

If $\alpha_{2n} = \dfrac{(\frac{1}{2}(f-e))_n(1+\frac{1}{4}(f-e))_n(-e)_n}{n!\,(\frac{1}{4}(f-e))_n(1+\frac{1}{2}(f+e))_n}$, this series for F_β can be summed by Dougall's theorem, as a well-poised ${}_7F_6(1)$ series in $1+\frac{1}{2}(f-e)$.

Then, $$\beta_{2n} = \frac{(\frac{1}{2}f)_n(\frac{1}{2}(1+f))_n(-\frac{1}{2}e)_n(\frac{1}{2}(1-e))_n(\frac{1}{4}(1+f+e))_n}{n!\,(\frac{1}{2})_n(1+\frac{1}{2}(f+e))_n(\frac{1}{2}(1+f+e))_n(\frac{1}{4}(3+f-e))_n} \times \frac{(\frac{1}{4}(3+f+e))_n}{(\frac{1}{4}(1+f-e))_n}.$$

and $$\beta_{2n+1} = \frac{ef}{(1+f-e)\,n!\,(3/2)_n} \frac{(1+\frac{1}{2}f)_n(\frac{1}{2}(1+f))_n(\frac{1}{2}(1-e))_n(1-\frac{1}{2}e)_n}{(\frac{1}{2}(3+f+e))_n(1+\frac{1}{2}(f+e))_n} \times \frac{(\frac{1}{4}(3+f+e))_n(\frac{1}{4}(5+f+e))_n}{(\frac{1}{4}(5+f-e))_n(\frac{1}{4}(3+f-e))_n}$$

If $\delta_n = 1$, then $$\gamma_n = \frac{\Gamma(1+f-e)\Gamma(1-2e)\,(f)_{2n}}{\Gamma(1-e)\Gamma(1+f-2e)(1+f-2e)_{2n}},$$

(as in § 4.42).

Hence, $$\frac{\Gamma(1+f-e)\Gamma(1-2e)}{\Gamma(1-e)\Gamma(1+f-2e)}\ {}_7F_6\left[\begin{matrix}\frac{1}{2}(f-e), 1+\frac{1}{4}(f-e), & -e, \frac{1}{4}(2+f), \\ \frac{1}{4}(f-e), 1+\frac{1}{2}(f+e), \frac{1}{4}(2+f-2e),\end{matrix}\right.$$

$$\left.\begin{matrix}\frac{1}{4}f, \quad \frac{1}{4}(1+f), \quad \frac{1}{4}(3+f); \ 1 \\ 1+\frac{1}{4}(f-2e), \frac{1}{4}(3+f-2e), \frac{1}{4}(1+f-2e);\end{matrix}\right]$$

$$= {}_6F_5\left[\begin{matrix}\frac{1}{2}f, \frac{1}{2}(1+f), & -\frac{1}{2}e, \ \frac{1}{2}(1-e), \frac{1}{4}(1+f+e), \frac{1}{4}(3+f+e); \ 1 \\ & \frac{1}{2}, 1+\frac{1}{2}(f+e), \frac{1}{2}(1+f+e), \frac{1}{4}(3+f-e), \frac{1}{4}(1+f-e);\end{matrix}\right]$$

$$+ \frac{ef}{1+f-e}\ {}_6F_5\left[\begin{matrix}1+\frac{1}{2}f, \frac{1}{2}(1+f), & \frac{1}{2}(1-e), & 1-\frac{1}{2}e, \frac{1}{4}(3+f+e), \\ & 3/2, \frac{1}{2}(3+f+e), 1+\frac{1}{2}(f+e), \frac{1}{4}(5+f-e),\end{matrix}\right.$$

$$\left.\begin{matrix}\frac{1}{4}(5+f+e); \ 1 \\ \frac{1}{4}(3+f-e);\end{matrix}\right]$$

<u>§ (11.4)</u> <u>Case seven; when $\alpha_{2r+1} = 0$.</u>

In this case $$F_\beta = \sum_{r=0}^{[n/2]} (-\tfrac{1}{2}n)_r(\tfrac{1}{2}(1-n))_r 4^r \alpha_{2r}$$

Let $\alpha_{2n} = \dfrac{1}{n!(A)_n 4^n}$, and sum F_β by Vandermonde's theorem,

then $$\beta_{2n} = \frac{(\frac{1}{4}(2A-1))_n(\frac{1}{4}(2A+1))_n}{n!(\frac{1}{2})_n(A-\frac{1}{2})_n(A)_n}$$

and $\beta_{2n+1} = \dfrac{(\frac{1}{4}(2A+1))_n(\frac{1}{4}(2A+3))_n}{n!(3/2)_n(A)_n(A+\frac{1}{2})_n}$ (as in § 10.2)

Also, if $\delta_n = \dfrac{(d)_n(-N)_n}{(D)_n}$ then $\gamma_n = \dfrac{(D-d)_N(d)_n(-N)_n}{(D)_N\ (1+d-D-N)_n}$

Hence, $\dfrac{(D-d)_N}{(D)_N}{}_4F_3\left[\begin{matrix}\frac{1}{2}d, \frac{1}{2}(1+d), & -\frac{1}{2}N, & \frac{1}{2}(1-N); \ 1\\ A, \frac{1}{2}(1+d-D-N), & 1+\frac{1}{2}(d-N-D); & \end{matrix}\right]$

$= {}_6F_5\left[\begin{matrix}\frac{1}{4}(2A-1), \frac{1}{4}(2A+1), \frac{1}{2}d, \frac{1}{2}(1+d), -\frac{1}{2}N, \frac{1}{2}(1-N); \ 4\\ \frac{1}{2}, A-\frac{1}{2}, A, \frac{1}{2}D, \frac{1}{2}(1+D); \end{matrix}\right]$

$- \dfrac{dN}{D}{}_6F_5\left[\begin{matrix}\frac{1}{4}(2A+1), \frac{1}{4}(2A+3), \frac{1}{2}(d+1), 1+\frac{1}{2}d, \frac{1}{2}(1-N), 1-\frac{1}{2}N; \ 4\\ A, 3/2, A+\frac{1}{2}, \frac{1}{2}(1+D), 1+\frac{1}{2}D; \end{matrix}\right]$

<u>§ (11.5)</u> <u>Case ten; when $\alpha_{2r+1} = 0$</u>

Here $F_\beta = \displaystyle\sum_{r=0}^{[n/2]} \frac{(-\frac{1}{2}n)_r(\frac{1}{2}(1-n))_r}{(\frac{1}{2}(1-e-n))_r(1-\frac{1}{2}(e+n))_r}\alpha_{2r}$

Thus, taking $\alpha_{2n} = \dfrac{(-e)_n}{n!}$, and summing F_β by Saalschutz's theorem, (as in § 10.3), we have

$\beta_{2n} = \dfrac{(\frac{1}{2}(1-e))_n(-\frac{1}{2}e)_n}{n!\ (\frac{1}{2})_n}$ and $\beta_{2n+1} = \dfrac{e(1-\frac{1}{2}e)_n(\frac{1}{2}(1-e))_n}{n!\ (3/2)_n}$

Also, if $\delta_n = \dfrac{(d)_n}{(D)_n}$, then $\gamma_n = \dfrac{\Gamma(D-d-e)\Gamma(D)\ (d)_n}{\Gamma(D-d)\Gamma(D-e)\ (D-e)_n}$

(as in § 4.71).

Hence, $\dfrac{\Gamma(D)\Gamma(D-d-e)}{\Gamma(D-d)\Gamma(D-e)}{}_3F_2\left[\begin{matrix}-e, & \frac{1}{2}d, & \frac{1}{2}(1+d); \ 1\\ & \frac{1}{2}(D-e), & \frac{1}{2}(1+D-e); \end{matrix}\right]$

$= {}_4F_3\left[\begin{matrix}\frac{1}{2}(1-e), -\frac{1}{2}e, \frac{1}{2}d, \frac{1}{2}(1+d); \ 1\\ \frac{1}{2}, \frac{1}{2}D, \frac{1}{2}(1+D); \end{matrix}\right]$

$+ \dfrac{de}{D}{}_4F_3\left[\begin{matrix}1-\frac{1}{2}e, \frac{1}{2}(1-e), \frac{1}{2}(1+d), 1+\frac{1}{2}d; \ 1\\ 3/2, \frac{1}{2}(1+D), 1+\frac{1}{2}D; \end{matrix}\right]$

There are no other series for F_β in the group of possible summations of F_γ by Gauss's theorem, which can be summed when $\alpha_{2r+1} = 0$, and $\alpha_{2r} \neq 0$.

§ (12.1) Applications of Saalschutz's theorem;

Case one, when $\alpha_{2r+1} = 0$

Now let us consider the cases in which $F_{\beta\gamma}$ can be summed by Saalschutz's theor em, when $\alpha_{2r+1} = 0$. In the first case, F_β can be summed as a well-poised ${}_5F_4(1)$ series, if

$$\alpha_{2n} = \frac{(\frac{1}{2}(F-1))_n(\frac{1}{4}(3+F))_n(a)_n}{n!(\frac{1}{4}(F-1))_n(\frac{1}{2}(F+1)-a)_n}, \text{ (as in § 11.1).}$$

Then $$\beta_{2n} = \frac{(\frac{1}{4}(F-2a))_n(\frac{1}{4}(F+2-2a))_n}{2n!\,(\frac{1}{2}F-a)_n(\frac{1}{2}(1+F)-a)_n(\frac{1}{2}F)_n}$$

and $$\beta_{2n+1} = \frac{(\frac{1}{4}(F+2-2a))_n(1+\frac{1}{4}(F-2a))_n}{F\,(2)_{2n}(1+\frac{1}{2}F-a)_n(1+\frac{1}{2}F)_n(\frac{1}{2}(1+F)-a)_n}$$

Also, if $\delta_n = \dfrac{(d_1)_n(d_2)_n(-N)_n}{(1+d_1+d_2-N-F)_n}$, then

$$\gamma_n = \frac{(F-d_1)_N(F-d_2)_N(d_1)_n(d_2)_n(-N)_n(-1)^n}{(F)_N(F-d_1-d_2)_N(F-d_1)_n(F-d_2)_n(F+N)_n}, \text{ (as in § 5.2)}$$

Hence,

$$\frac{(F-d_1)_N(F-d_2)_N}{(F)_N(F-d_1-d_2)_N}\,{}_9F_8\left[\begin{matrix}\frac{1}{2}(F-1),\frac{1}{4}(3+F), a, \frac{1}{2}d_1,\frac{1}{2}(1+d_1), \frac{1}{2}d_2,\frac{1}{2}(1+d_2), -\frac{1}{2}N,\frac{1}{2}(1-N);\ 1\\ \frac{1}{4}(F-1),\frac{1}{2}(1+F)-a, \frac{1}{2}(F-d_1+1),\frac{1}{2}(F-d_1),\frac{1}{2}(F-d_2+1),\frac{1}{2}(F-d_2),\frac{1}{2}(F+N+1),\frac{1}{2}(F+N);\end{matrix}\right]$$

$$= {}_8F_7\left[\begin{matrix}\frac{1}{4}(F-2a),\frac{1}{4}(F+2-2a),\frac{1}{2}d_1,\frac{1}{2}(1+d_1),\frac{1}{2}d_2,\frac{1}{2}(1+d_2), -\frac{1}{2}N, \frac{1}{2}(1-N);\ 1\\ \frac{1}{2},\frac{1}{2}F-a,\frac{1}{2}(1+F)-a, \frac{1}{4}F, \frac{1}{4}(F+2), \frac{1}{2}(1+d_1+d_2-F-N), 1+\frac{1}{2}(d_1+d_2-F-N);\end{matrix}\right] - \frac{d_1d_2N}{F(1+d_1+d_2-F-N)}$$

$$\times\ {}_8F_7\left[\begin{matrix}\frac{1}{4}(F+2-2a), 1+\frac{1}{4}(F-2a),\frac{1}{2}(1+d_1), 1+\frac{1}{2}d_1, \frac{1}{2}(1+d_2), 1+\frac{1}{2}d_2, \frac{1}{2}(1-N), 1-\frac{1}{2}N;\ 1\\ \frac{3}{2},\frac{1}{2}F-a+1,\frac{1}{4}(F+2), 1+\frac{1}{4}F, \frac{1}{2}(F+1)-a, 1+\frac{1}{2}(d_1+d_2-F-N),\frac{1}{2}(3+d_2+d_1-F-N);\end{matrix}\right]$$

§ (12.2) Case one; when $\alpha_{2r+1} = \alpha_{2r+2} = 0$.

Next suppose that $\alpha_r = 0$ if r is not a multiple of three. Then, summing the F_β series as three well-poised $_5F_4(1)$ series, we have (from § 11.1)

$$\frac{(F-d_1)_N(F-d_2)_N}{(F)_N(F-d_1-d_2)_N}\,{}_{11}F_{10}\left[\begin{matrix}1/3.(F-1), & 1+1/6.(F-1), & 1/3.d_1, \\ & 1/6.(F-1), & 1/3.(F-d_1+2),\end{matrix}\right.$$

$$\begin{matrix}1/3.(d_1+1), & 1/3.(d_1+2), & 1/3.d_2, & 1/3.(d_2+1), \\ 1/3.(1+F-d_1), & 1/3.(F-d_1), & 1/3.(F-d_2+2), & 1/3.(F-d_2+1),\end{matrix}$$

$$\left.\begin{matrix}1/3.(d_2+2), & -1/3.N, & 1/3.(1-N), & 1/3.(2-N); & 1 \\ 1/3.(F-d_2), & 1/3.(2+F+N), & 1/3.(1+F+N), & 1/3.(F+N) & ;\end{matrix}\right]$$

$$= {}_{12}F_{11}\left[\begin{matrix}1/9.(F-1), & 1/9.(F+2), & 1/9.(F+5), & 1/3.d_1, & 1/3.(1+d_1), \\ & 1/3, & 2/3, & 1/6.(1+F), & 1/6.(4+F),\end{matrix}\right.$$

$$\begin{matrix}1/3.(2+d_1), & 1/3.d_2, & 1/3.(1+d_2), & 1/3.(d_2+2), & -1/3.N, \\ 1/6.F, & 1/6.(3+F), & 1/6.(F-1), & 1/6.(F+2), & 1/3.(1+d_1+d_2-F-N),\end{matrix}$$

$$\left.\begin{matrix}1/3.(1-N), & 1/3.(2-N); & \frac{27}{64} \\ 1/3.(2+d_1+d_2-F-N), & 1+1/3.(d_1+d_2-F-N) & ;\end{matrix}\right]$$

$$-\frac{d_1d_2N}{(1+d_1+d_2-F-N)F}\,{}_{12}F_{11}\left[\begin{matrix}1/9.(2+F), & 1/9.(5+F), & 1/9.(8+F), \\ & 2/3, & 4/3,\end{matrix}\right.$$

$$\begin{matrix}1/3.(1+d_1), & 1/3.(2+d_1), & 1+1/3.d_1, & 1/3.(1+d_2), & 1/3.(2+d_2), \\ 1/6.(3+F), & 1+1/6.F, & 1/6.(1+F), & 1/6.(4+F), & 1/6.(2+F),\end{matrix}$$

$$\begin{matrix}1+1/3.d_2, & 1/3.(1-N), & 1/3.(2-N), & 1-1/3.N \\ 1/6.(5+F), & 1/3.(2+d_1+d_2-F-N), & 1+1/3.(d_1+d_2-F-N), & 1/3.(4+d_1+d_2-F-N)\end{matrix}$$

$$\left.\begin{matrix}; & \frac{27}{64} \\ ; & \end{matrix}\right]$$

$$+\frac{d_1(d_1+1)d_2(d_2+1)N(N-1)}{2(1+d_1+d_2-F-N)(2+d_1+d_2-F-N)F(F+1)}\,{}_{12}F_{11}\left[\begin{matrix}1/9.(5+F), \\ \end{matrix}\right.$$

$$\begin{matrix}1/9.(8+F), & 1/9.(11+F), & 1/3.(2+d_1), & 1+1/3.d_1, & 1/3.(4+d_1), \\ 4/3, & 5/3, & 1/6.(3+F), & 1+1/6.F, & 1/6.(5+F),\end{matrix}$$

$$\begin{matrix}1/3.(2+d_2), & 1+1/3.d_2, & 1/3.(4+d_2), & 1/3.(2-N), \\ 1/6.(8+F), & 1/6.(4+F), & 1/6.(7+F), & 1+1/3.(d_1+d_2-F-N),\end{matrix}$$

$$\left.\begin{matrix}1-1/3.N, & 1/3.(4-N); & \frac{27}{64} \\ 1/3.(4+d_1+d_2-F-N), & 1/3.(5+d_1+d_2-F-N); & \end{matrix}\right]$$

<u>§ (12.3)</u> <u>Case two; when $\alpha_{2r+1} = 0$.</u>

In this case, if $F = 1+f-e$, then, taking

$$\alpha_{2n+1} = 0, \quad F_\beta = \sum_{r=0}^{[n/2]} \frac{(-n)_{2r}(f+n)_{2r}}{(1-e-n)_{2r}(1+f-e+n)_{2r}} \alpha_{2r}$$

Let $\alpha_{2n} = \dfrac{(\frac{1}{2}(f-e))_n(1+\frac{1}{4}(f-e))_n(-e)_n}{n!\ (\frac{1}{4}(f-e))_n(1+\frac{1}{2}(f+e))_n}$ and sum the series for F_β by Dougall's theorem, (as in § 11.3).

Then,
$$\beta_{2n} = \frac{(\frac{1}{2}f)_n(\frac{1}{2}(f+1))_n(-\frac{1}{2}e)_n(\frac{1}{2}(1-e))_n(\frac{1}{4}(f+e+1))_n(\frac{1}{4}(3+f+e))_n}{n!(\frac{1}{2})_n(1+\frac{1}{2}(f+e))_n(\frac{1}{2}(1+f+e))_n(\frac{1}{4}(3+f-e))_n(\frac{1}{4}(1+f-e))_n}$$

and
$$\beta_{2n+1} = \frac{e\ f\ (1+\frac{1}{2}f)_n(\frac{1}{2}(1+f))_n(1-\frac{1}{2}e)_n(\frac{1}{2}(1-e))_n}{(1+f-e)n!(3/2)_n(\frac{1}{2}(3+f+e))_n(1+\frac{1}{2}(f+e))_n} \times \frac{(\frac{1}{4}(5+f+e))_n(\frac{1}{4}(3+f+e))_n}{(\frac{1}{4}(5+f-e))_n(\frac{1}{4}(3+f-e))_n}$$

Also, if $\delta_n = \dfrac{(-N)_n}{(2e-N)_n}$, then

$$\gamma_n = \frac{(1-e)_N(1+f-2e)_N(-N)_n(1+f+N-2e)_n(\frac{1}{2}f)_n(\frac{1}{2}(1+f))_n}{(1+f-e)_N(1-2e)_N(1+f+N-e)_n(e-N)_n(1+\frac{1}{2}f-e)_n(\frac{1}{2}(1+f)-e)_n}$$

Hence,
$$\frac{(1+f-2e)_N(1-e)_N}{(1+f-e)_N(1-2e)_N}\,{}_{11}F_{10}\left[\begin{matrix}\frac{1}{2}(f-e), 1+\frac{1}{4}(f-e), & -e, \\ \frac{1}{4}(f-e), & 1+\frac{1}{2}(f+e),\end{matrix}\right.$$

$$\begin{matrix}\frac{1}{4}f, & \frac{1}{4}(f+1), & \frac{1}{4}(f+2), & \frac{1}{4}(f+3), & 1+\frac{1}{2}(f+N)-e, & \frac{1}{2}(1+f+N)-e, \\ 1+\frac{1}{4}(f-2e), & \frac{1}{4}(3+f-2e), & \frac{1}{4}(2+f-2e), & \frac{1}{4}(1+f-2e), & \frac{1}{2}(e-N), & \frac{1}{2}(1-N+e),\end{matrix}$$

$$\left.\begin{matrix}\frac{1}{2}(1-N), & -\frac{1}{2}N; & 1 \\ \frac{1}{2}(1+f+N-e), & 1+\frac{1}{2}(f+N-e); & \end{matrix}\right]$$

$$= {}_8F_7\left[\begin{matrix}\frac{1}{2}f, \frac{1}{2}(1+f), & -\frac{1}{2}e, & \frac{1}{2}(1-e), & \frac{1}{4}(1+f+e), & \frac{1}{4}(3+f+e), & -\frac{1}{2}N, \\ & \frac{1}{2}, 1+\frac{1}{2}(f+e), & \frac{1}{2}(1+f+e), & \frac{1}{4}(3+f-e), & \frac{1}{4}(1+f-e), & e-\frac{1}{2}N,\end{matrix}\right.$$

$$\left.\begin{matrix}\frac{1}{2}(1-N); & 1 \\ e+\frac{1}{2}(1-N); & \end{matrix}\right] - \frac{N\ e\ f}{(2e-N)(1+f-e)}\,{}_8F_7\left[\begin{matrix}1+\frac{1}{2}f, \frac{1}{2}(1+f), & \frac{1}{2}(1-e), \\ 3/2, & \frac{1}{2}(3+f+e),\end{matrix}\right.$$

$$\left.\begin{matrix}1-\frac{1}{2}e, & \frac{1}{4}(3+f+e), & \frac{1}{4}(5+f+e), & \frac{1}{2}(1-N), & 1-\frac{1}{2}N; & 1 \\ 1+\frac{1}{2}(f+e), & \frac{1}{4}(5+f-e), & \frac{1}{4}(3+f-e), & e+\frac{1}{2}(1-N), & e+1-\frac{1}{2}N; & \end{matrix}\right]$$

<u>§ (12.4)</u> <u>Case three; when $\alpha_{2r+1} = 0$.</u>

In this case, $F_\beta = \displaystyle\sum_{r=0}^{[n/2]} \frac{(-n)_{2r}}{(1-e-n)_{2r}}\alpha_{2r}$

Thus, taking $\alpha_{2n} = \frac{(-e)_n}{n!}$, and summing F_β by Saalschutz's theorem, (as in § 11.5), $\beta_{2n} = \frac{(\frac{1}{2}(1-e))_n(-\frac{1}{2}e)_n}{n!\,(\frac{1}{2})_n}$

and $\beta_{2n+1} = \frac{e\,(1-\frac{1}{2}e)_n(\frac{1}{2}(1-e))_n}{n!\,(3/2)_n}$

Also, if $\delta_n = \frac{(d)_n(-N)_n}{(D)_n(1+e+d-N-D)_n}$, then

$$\gamma_n = \frac{(D-d)_N(D-e)_N\,(d)_n(-N)_n}{(D)_N(D-d-e)_N(D-e)_n(1+d-N-D)_n}.$$

Hence,

$$\frac{(D-d)_N(D-e)_N}{(D)_N(D-d-e)_N}\,{}_5F_4\left[\begin{matrix}-e, & \frac{1}{2}d, & \frac{1}{2}(1+d), & \frac{1}{2}(1-N), & -\frac{1}{2}N;\ 1\\ & \frac{1}{2}(D-e), & \frac{1}{2}(1+D-e), & \frac{1}{2}(1+d-N-D), & 1+\frac{1}{2}(d-N-D);\end{matrix}\right] = {}_6F_5\left[\begin{matrix}\frac{1}{2}(1-e), -\frac{1}{2}e, & \frac{1}{2}d, & \frac{1}{2}(1+d), & \frac{1}{2}(1-N), & -\frac{1}{2}N;\ 1\\ & \frac{1}{2}, & \frac{1}{2}D, & \frac{1}{2}(1+D), & \frac{1}{2}(1+e+d-N-D), & 1+\frac{1}{2}(e+d-N-D);\end{matrix}\right]$$

$$-\frac{d\,N\,e}{D(1+e+d-N-D)}\,{}_6F_5\left[\begin{matrix}1-\frac{1}{2}e, \frac{1}{2}(1-e), & \frac{1}{2}(1+d), & 1+\frac{1}{2}d, & \frac{1}{2}(1-N), & -\frac{1}{2}N;\ 1\\ & 3/2, & \frac{1}{2}(1+D), & 1+\frac{1}{2}D, & \frac{1}{2}(3+e+d-N-D), & 1+\frac{1}{2}(e+d-N-D);\end{matrix}\right]$$

In the last case no summation of the series F_β is possible when $\alpha_{2r+1} = 0$.

§ (13.1) The application of Dougall's theorem; when $\alpha_{2r+1} = 0$.

The first of the two series for F_β which result from summing F_γ by Dougall's theorem, cannot be summed for any values of α_{2r}, when $\alpha_{2r+1} = 0$. The second series for F_β, however, can be summed by Dougall's theorem, as a ${}_7F_6(1)$ series, well-poised in $1+\frac{1}{2}(f-e)$, (as in § 11.3), by taking

$$\alpha_{2n} = \frac{(\frac{1}{2}(f-e))_n(1+\frac{1}{4}(f-e))_n(-e)_n}{n!\,(\frac{1}{4}(f-e))_n(1+\frac{1}{2}(f+e))_n}.$$

Then,

$$\beta_{2n} = \frac{(\tfrac12 e)_n(\tfrac12(1-e))_n(\tfrac12 f)_n(\tfrac12(1+f))_n}{n!\,(\tfrac12)_n(\tfrac12(1+f+e))_n(1+\tfrac12(f+e))_n} \times \frac{(\tfrac14(1+f+e))_n(\tfrac14(3+f+e))_n}{(\tfrac14(3+f-e))_n(\tfrac14(1+f-e))_n}$$

and

$$\beta_{2n+1} = \frac{e\ f}{(1+f-e)n!}\frac{(\tfrac12(1+f))_n(1+\tfrac12 f)_n(\tfrac12(1-e))_n(1-\tfrac12 e)_n}{(3/2)_n(1+\tfrac12(f+e))_n(\tfrac12(3+f+e))_n} \times \frac{(\tfrac14(3+f+e))_n(\tfrac14(5+f+e))_n}{(\tfrac14(5+f-e))_n(\tfrac14(3+f-e))_n}$$

Also, if

$$\delta_n = \frac{(1+\tfrac12 f)_n(d_1)_n(d_2)_n(d_3)_n(-N)_n}{(\tfrac12 f)_n(1+f-d_1)_n(1+f-d_2)_n(1+f-d_3)_n(1+f+N)_n}$$

then

$$\gamma_n = \frac{(1+f)_N(1+f-d_1-e)_N(1+f-d_2-e)_N(1+f-d_1-d_2)_N}{(1+f-d_1)_N(1+f-d_2)_N(1+f-e)_N(1+f-d_1-d_2-e)_N} \times \frac{(d_1)_n(d_2)_n(d_3)_n(-N)_n}{(1+f-e-d_1)_n(1+f-e-d_2)_n(1+f-e-d_3)_n(1+f-e+N)_n}$$

(as in § (6.3)). Hence,

$$\frac{(1+f)_N(1+f-d_1-e)_N(1+f-d_2-e)_N(1+f-d_1-d_2)_N}{(1+f-d_1)_N(1+f-d_2)_n(1+f-e)_N(1+f-d_1-d_2-e)_N}$$

$$\times\ {}_{11}F_{10}\left[\begin{matrix}\tfrac12(f-e), 1+\tfrac14(f-e), & -e, & \tfrac12 d_1, & \tfrac12(1+d_1), \\ \tfrac14(f-e), & 1+\tfrac12(f-e-d_1), & 1+\tfrac12(f+e), & \tfrac12(1+f-e-d_1),\end{matrix}\right.$$

$$\begin{matrix}\tfrac12 d_2, & \tfrac12(1+d_2), & \tfrac12 d_3, & \tfrac12(1+d_3), \\ 1+\tfrac12(f-e-d_2), & \tfrac12(1+f-e-d_2), & 1+\tfrac12(f-e-d_3), & \tfrac12(1+f-e-d_3),\end{matrix}$$

$$\left.\begin{matrix}-\tfrac12 N, & \tfrac12(1-N); & 1 \\ 1+\tfrac12(f-e+N), & \tfrac12(1+f-e+N); & \end{matrix}\right]$$

$$= {}_{15}F_{14}\left[\begin{matrix}\tfrac12 f, 1+\tfrac14 f, \tfrac12(1+f), & -\tfrac12 e, & \tfrac12(1-e), & \tfrac14(1+f+e), & \tfrac14(3+f+e), \\ \tfrac14 f, & \tfrac12, 1+\tfrac12(f+e), & \tfrac12(1+f+e), & \tfrac14(3+f-e), & \tfrac14(1+f-e),\end{matrix}\right.$$

$$\begin{matrix}\tfrac12 d_1, & \tfrac12(1+d_1), & \tfrac12 d_2, & \tfrac12(1+d_2), & \tfrac12 d_3, \\ 1+\tfrac12(f-d_1), & \tfrac12(1+f-d_1), & 1+\tfrac12(f-d_2), & \tfrac12(1+f-d_2), & 1+\tfrac12(f-d_3),\end{matrix}$$

$$\left.\begin{matrix}\tfrac12(1+d_3), & -\tfrac12 N, & \tfrac12(1-N); & 1 \\ \tfrac12(f+1-d_3), & 1+\tfrac12(f+N), & \tfrac12(1+f+N); & \end{matrix}\right]$$

$$-\frac{d_1 d_2 d_3 e\ f\ (1+\tfrac12 f)\ N}{(1+f-e)\tfrac12 f(1+f-d_1)(1+f-d_2)(1+f-d_3)(1+f+N)} \times$$

$$ {}_{15}F_{14}\left[\begin{matrix}1+\tfrac{1}{2}f, & \tfrac{1}{4}(6+f), & \tfrac{1}{2}(1+f), & \tfrac{1}{4}(3+f+e), & \tfrac{1}{4}(5+f+e), & \tfrac{1}{2}(1-e), \\ & \tfrac{1}{4}(2+f), & 3/2, & \tfrac{1}{4}(5+f-e), & \tfrac{1}{4}(3+f-e), & \tfrac{1}{2}(3+f+e),\end{matrix}\right. $$

$$ \begin{matrix}1-\tfrac{1}{2}e, & \tfrac{1}{2}(1+d_1), & 1+\tfrac{1}{2}d_1, & \tfrac{1}{2}(1+d_2), & 1+\tfrac{1}{2}d_2, \\ 1+\tfrac{1}{2}(f+e), & \tfrac{1}{2}(3+f-d_1), & 1+\tfrac{1}{2}(f-d_1), & \tfrac{1}{2}(3+f-d_2), & 1+\tfrac{1}{2}(f-d_2),\end{matrix} $$

$$ \left.\begin{matrix}\tfrac{1}{2}(1+d_3), & 1+\tfrac{1}{2}d_3, & \tfrac{1}{2}(1-N), & 1-\tfrac{1}{2}N; & 1 \\ \tfrac{1}{2}(3+f-d_3), & 1+\tfrac{1}{2}(f-d_3), & \tfrac{1}{2}(3+f+N), & 1+\tfrac{1}{2}(f+N); & \end{matrix}\right] $$

where $e+d_1+d_2+d_3-N = 1+2f$.

The three series involved in this relation are all well-poised. The similar relations which can be obtained by summing F_β as a well-poised ${}_5F_4(1)$ series, or as a well-poised ${}_4F_3(-1)$ series, or by Dixon's theorem, or by Kummer's theorem, are all included as special cases of this result.

§ (13.2) Application of theorem thirteen; case three, when $\alpha_{2r+1} = 0$.

If F_γ is summable by theorem thirteen, as a nearly-poised ${}_3F_2(1)$ series, and $\alpha_r = 0$, when r is odd, (as in § 7.43) then, when $\alpha_{2n} = \frac{(-e)_n}{n!}$, F_β is summable by Saalschutz's theorem, giving,

$$ \beta_{2n} = \frac{(\tfrac{1}{2}(1-e))_n(-\tfrac{1}{2}e)_n}{n!\,(\tfrac{1}{2})_n}, \text{ and } \beta_{2n+1} = \frac{e(1-\tfrac{1}{2}e)_n(\tfrac{1}{2}(1-e))_n}{n!\,(3/2)_n} $$

Also, if $\delta_n = \frac{(d)_n(-N)_n}{(1+d-e)_n(1+2e-N)_n}$, then

$$ \gamma_{2n} = \frac{(d-2e)_N(1+\tfrac{1}{2}d-e)_N(-e)_N(1+e-\tfrac{1}{2}d-N)_n(\tfrac{1}{2}d-e)_n(d)_{2n}(-N)_{2n}}{(1+d-e)_N(\tfrac{1}{2}d-e)_N(-2e)_N(d-2e)_{2n}(e-\tfrac{1}{2}d-N)_n(1+\tfrac{1}{2}d-e)_n(1+e-N)_{2n}} $$

Hence,

$$\frac{(d-2e)_N(1+\tfrac{1}{2}d-e)_N(-e)_N}{(1+d-e)_N(\tfrac{1}{2}d-e)_N(-2e)_N}$$

$$\times\ {}_6F_5\left[\begin{matrix}-e,1+e-\tfrac{1}{2}d-N, & \tfrac{1}{2}d,\tfrac{1}{2}(1+d), & \tfrac{1}{2}(1-N), & -\tfrac{1}{2}N;\ 1\\ \tfrac{1}{2}(1+d)-e,e-\tfrac{1}{2}d-N, & 1+\tfrac{1}{2}d-e, & \tfrac{1}{2}(1+e-N), & 1+\tfrac{1}{2}(e-N);\end{matrix}\right]$$

$$= {}_6F_5\left[\begin{matrix}\tfrac{1}{2}(1-e),-\tfrac{1}{2}e, & \tfrac{1}{2}d, & \tfrac{1}{2}(1+d), & \tfrac{1}{2}(1-N), & -\tfrac{1}{2}N;\ 1\\ & \tfrac{1}{2},\tfrac{1}{2}(1+d-e), & 1+\tfrac{1}{2}(d-e), & \tfrac{1}{2}(1-N)+e, & 1-\tfrac{1}{2}N+e;\end{matrix}\right]$$

$$- \frac{d\ e\ N}{(1+d-e)(1+2e-N)}\,{}_6F_5\left[\begin{matrix}1-\tfrac{1}{2}e,\tfrac{1}{2}(1-e),\tfrac{1}{2}(1+d),1+\tfrac{1}{2}d,\tfrac{1}{2}(1-N), & 1-\tfrac{1}{2}N;\ 1\\ 3/2,1+\tfrac{1}{2}(d-e),\tfrac{1}{2}(3+d-e),1+e-\tfrac{1}{2}N, & \tfrac{1}{2}(3-N)+e;\end{matrix}\right]$$

<u>§ (13.3)</u> <u>Case five; when $\alpha_{2r+1}=0$.</u>

Summing F_γ by theorem thirteen, (as in § 7.45), and summing F_β by Dougall's theorem,

if $\alpha_{2n} = \dfrac{(\tfrac{1}{2}(f-e))_n(1+\tfrac{1}{4}(f-e))_n(-e)_n}{(\tfrac{1}{4}(f-e))_n(1+\tfrac{1}{2}(f+e))_n\ n!}$

then $\beta_{2n} = \dfrac{(\tfrac{1}{2}f)_n(\tfrac{1}{2}(1+f))_n(-\tfrac{1}{2}e)_n(\tfrac{1}{2}(1-e))_n(\tfrac{1}{4}(1+f+e))_n}{n!\ (\tfrac{1}{2})_n(1+\tfrac{1}{2}(f+e))_n(\tfrac{1}{2}(1+f+e))_n(\tfrac{1}{4}(3+f-e))_n} \times \dfrac{(\tfrac{1}{4}(3+f+e))_n}{(\tfrac{1}{4}(1+f-e))_n}$

and $\beta_{2n+1} = \dfrac{e\ f\ (1+\tfrac{1}{2}f)_n(\tfrac{1}{2}(1+f))_n(\tfrac{1}{2}(1-e))_n(1-\tfrac{1}{2}e)_n}{(1+f-e)\ n!\ (3/2)_n(\tfrac{1}{2}(3+f+e))_n(1+\tfrac{1}{2}(f+e))_n} \times \dfrac{(\tfrac{1}{4}(3+f+e))_n(\tfrac{1}{4}(5+f+e))_n}{(\tfrac{1}{4}(5+f-e))_n(\tfrac{1}{4}(3+f-e))_n}$

Also, if $\delta_n = \dfrac{(-N)_n}{(1+2e-N)_n}$, then

$$\gamma_n = \frac{(f-2e)_N(1+\tfrac{1}{2}f-e)_N(-e)_N(-N)_n(f)_{2n}(f-2e+N)_n}{(1+f-e)_N(\tfrac{1}{2}f-e)_N(-2e)_N(1+f-e+N)_n(1+f-2e)_{2n}(1+e-N)_n}$$

Hence, $\dfrac{(f-2e)_N(1+\tfrac{1}{2}f-e)_N(-e)_N}{(1+f-e)_N(\tfrac{1}{2}f-e)_N(-2e)_N}\,{}_{11}F_{10}\left[\begin{matrix}\tfrac{1}{2}(f-e),1+\tfrac{1}{4}(f-e), & -e,\\ \tfrac{1}{4}(f-e), & 1+\tfrac{1}{2}(f+e),\end{matrix}\right.$

$$\begin{matrix}\tfrac{1}{4}f, & \tfrac{1}{4}(1+f), & \tfrac{1}{4}(2+f), & \tfrac{1}{4}(3+f), & \tfrac{1}{2}(f-2e+N),\\ 1+\tfrac{1}{4}(f-2e), & \tfrac{1}{4}(3+f-2e), & \tfrac{1}{4}(2+f-2e), & \tfrac{1}{4}(1+f-2e), & 1+\tfrac{1}{2}(e-N),\end{matrix}$$

$$\left.\begin{matrix}\tfrac{1}{2}(1+f-2e+N), & -\tfrac{1}{2}N, & \tfrac{1}{2}(1-N);\ 1\\ \tfrac{1}{2}(1+e-N), & 1+\tfrac{1}{2}(f-e+N), & \tfrac{1}{2}(1+f-e+N)\ ;\end{matrix}\right] =$$

$${}_8F_7\left[\begin{matrix}\tfrac12 f, \tfrac12(1+f), -\tfrac12 e, \tfrac12(1-e), \tfrac14(1+f+e), \tfrac14(3+f+e), -\tfrac12 N, \\ \tfrac12, 1+\tfrac12(f+e), \tfrac12(1+f+e), \tfrac14(3+f-e), \tfrac14(1+f-e), \tfrac12(1+2e-N), \end{matrix}\right.$$

$$\left.\begin{matrix}\tfrac12(1-N); 1 \\ 1+e-\tfrac12 N; \end{matrix}\right] - \frac{e\ f\ N}{(1+f-e)(1+2e-N)}\,{}_8F_7\left[\begin{matrix}1+\tfrac12 f, \tfrac12(1+f), \tfrac12(1-e), \\ 3/2, \tfrac12(3+f+e), \end{matrix}\right.$$

$$\left.\begin{matrix}1-\tfrac12 e, \tfrac14(3+f+e), \tfrac14(5+f+e), \tfrac12(1-N), 1-\tfrac12 N; 1 \\ 1+\tfrac12(f+e), \tfrac14(5+f-e), \tfrac14(3+f-e), 1+e-\tfrac12 N, \tfrac12(3+2e-N); \end{matrix}\right].$$

§ (13.4) Application of theorem fourteen; case one, when $\alpha_{2r+1} = 0$.

Similarly, summing F_γ by theorem fourteen, as a nearly-poised ${}_4F_3(1)$ series, (as in § 7.61) and summing F_β by Dougall's theorem, when $\alpha_{2n+1} = 0$, we have,

$$\frac{(f-2e)_N(-e)_N}{(1+f-e)_N(-2e)_N}\,{}_{11}F_{10}\left[\begin{matrix}\tfrac12(f-e), 1+\tfrac14(f-e), -e, \tfrac12(1+f), \\ \tfrac14(f-e), 1+\tfrac12(f+e), \tfrac14(3+f-2e), \end{matrix}\right.$$

$$\begin{matrix}\tfrac14(2+f), \tfrac14(3+f), 1+\tfrac14 f, \tfrac12(f-2e+N), \tfrac12(1+f-2e+N), \\ \tfrac14(2+f-2e), \tfrac14(1+f-2e), \tfrac14(f-2e), 1+\tfrac12(e-N), \tfrac12(1+e-N), \end{matrix}$$

$$\left.\begin{matrix}-\tfrac12 N, \tfrac12(1-N); 1 \\ 1+\tfrac12(f-e+N), \tfrac12(1+f-e+N); \end{matrix}\right] = {}_9F_8\left[\begin{matrix}\tfrac12 f, 1+\tfrac14 f, \tfrac12(1+f), -\tfrac12 e, \\ \tfrac14 f, \tfrac12, 1+\tfrac12(f+e), \end{matrix}\right.$$

$$\left.\begin{matrix}\tfrac12(1-e), \tfrac14(1+f+e), \tfrac14(3+f+e), -\tfrac12 N, \tfrac12(1-N); 1 \\ \tfrac12(1+f+e), \tfrac14(3+f-e), \tfrac14(1+f-e), 1+e-\tfrac12 N, \tfrac12(1+2e-N); \end{matrix}\right]$$

$$- \frac{e\ N\ (2+f)}{(1+f-e)(1+2e-N)}\,{}_9F_8\left[\begin{matrix}1+\tfrac12 f, \tfrac14(6+f), \tfrac12(1+f), \tfrac12(1-e), \\ \tfrac14(2+f), 3/2, \tfrac12(3+f+e), \end{matrix}\right.$$

$$\left.\begin{matrix}1-\tfrac12 e, \tfrac14(3+f+e), \tfrac14(5+f+e), \tfrac12(1-N), 1-\tfrac12 N; 1 \\ 1+\tfrac12(f+e), \tfrac14(5+f-e), \tfrac14(3+f-e), \tfrac12(3-N)+e, 1+e-\tfrac12 N; \end{matrix}\right].$$

A similar result is given by summing F_γ by theorem fifteen as a nearly-poised ${}_4F_3(1)$ series, (as in § 7.61) and summing F_β by Dougall's theorem, when $\alpha_{2r+1} = 0$.

§ (14.1) Application of theorem nineteen; case three, when $\alpha_{2r+1} = 0$.

Now suppose that F_γ is summable by theorem nineteen, (as in § 8.5) and $\alpha_{2r+1} = 0$. Then, if

$$\delta_n = \frac{(\tfrac12 f_1)_n(\tfrac12(1+f_1))_n(f_2+N)_n(-N)_n}{(\tfrac12 f_2)_n(\tfrac12(1+f_2))_n}$$

$$\gamma_{2n} = \frac{(f_2-f_1)_N(\tfrac12 f_1)_{2n}(-N)_{2n}}{(f_2)_N(1+f_1-f_2-N)_{2n}(1+\tfrac12 f_1)_{2n}} .$$

Let $\alpha_{2n} = \dfrac{(\tfrac12 f_1)_n(1+\tfrac14 f_1)_n(a)_n}{n!(\tfrac14 f_1)_n(1+f_1-a)_n}$. Then, summing F_β as a well-poised ${}_5F_4(1)$ series,

$$\beta_{2n} = \frac{(\tfrac14(1+f_1-2a))_n(\tfrac14(3+f_1-2a))_n}{2n!\,(\tfrac12(1+f_1)-a)_n(1+\tfrac12 f_1-a)_n(\tfrac12(1+f_1))_n}$$

and $$\beta_{2n+1} = \frac{(\tfrac14(3+f_1-2a))_n(\tfrac14(5+f_1-2a))_n}{(2)_{2n}\,(\tfrac12(3+f_1)-a)_n(\tfrac12(3+f_1))_n(1+\tfrac12 f_1-a)_n}$$

Hence, $$\frac{(f_2-f_1)_N}{(f_2)_N}\,{}_4F_3\left[\begin{matrix}\tfrac12 f_1, & a, & \tfrac12(1-N), & -\tfrac12 N; \; 1\\ & 1+\tfrac12 f_1-a, & \tfrac12(1+f_1-f_2-N), & 1+\tfrac12(f_1-f_2-N) \; ;\end{matrix}\right]$$

$$= {}_8F_7\left[\begin{matrix}\tfrac14(1+f_1-2a), \tfrac14(3+f_1-2a), \tfrac14 f_1, \tfrac14(2+f_1), \tfrac12(f_2+N), \tfrac12(1+f_2+N), -\tfrac12 N, \tfrac12(1-N); \; 1\\ \tfrac12(1+f_1)-a, \tfrac12, 1+\tfrac12 f_1-a, \tfrac14 f_2, \tfrac14(1+f_2), \tfrac14(2+f_2), \tfrac14(3+f_2);\end{matrix}\right] - \frac{(f_2+N)f_1 N}{f_2(1+f_2)}\,{}_8F_7\left[\begin{matrix}\tfrac14(3+f_1-2a), \tfrac14(5+f_1-2a), \tfrac14(2+f_1), 1+\tfrac14 f_1, \tfrac12(f_2+N+1), 1+\tfrac12(f_2+N), \tfrac12(1-N), 1-\tfrac12 N; \; 1\\ 3/2, \tfrac12(3+f_1)-a, 1+\tfrac12 f_1-a, \tfrac14(f_2+2), \tfrac14(3+f_2), 1+\tfrac14 f_2, \tfrac14(5+f_2);\end{matrix}\right]$$

<u>§ (14.2)</u> <u>Case three; when $\alpha_{3r+1} = \alpha_{3r+2} = 0$.</u>

Next let $\alpha_r = 0$, if r is not a multiple of three. Then, if $\alpha_{3n} = \dfrac{(-1)^n(1/3.f_1)_n(1+1/6.f_1)_n}{n!\,(1/6.f_1)_n}$

summing the series for F_β as three well-poised ${}_5F_4(1)$ series, we have, $$\frac{(f_2-f_1)_N}{(f_2)_N}\,{}_4F_3\left[\begin{matrix}1/3.f_1, & -1/3.N, & 1/3.(1-N), & 1/3.(2-N); \; 1\\ & 1/3.(1+f_1-f_2-N), & 1/3.(2+f_1-f_2-N), & 1/3.(3+f_1-f_2-N);\end{matrix}\right] =$$

$$ {}_9F_8\left[\begin{matrix} 1/9.f_1,\ 1/9.(3+f_1),\ 1/9.(6+f_1),\ 1/3.(f_2+N),\ 1/3.(1+f_2+N), \\ 1/3,\quad 2/3,\quad 1/6.f_2,\ 1/6.(1+f_2), \end{matrix}\right. $$

$$ \left.\begin{matrix} 1/3.(2+f_2+N),\ -1/3.N,\ 1/3.(1-N),\ 1/3.(2-N);\ \frac{27}{64} \\ 1/6.(2+f_2),\ 1/6.(3+f_2),\ 1/6.(4+f_2),\ 1/6.(5+f_2); \end{matrix}\right] $$

$$ -\frac{f_1(f_2+N)N}{f_2(1+f_2)}\,{}_9F_8\left[\begin{matrix} 1/9.(3+f_1),\ 1/9.(6+f_1),\ 1+1/9.f_1, \\ 2/3,\quad 4/3, \end{matrix}\right. $$

$$ \begin{matrix} 1/3.(f_2+N+1),\ 1/3.(f_2+N+2),\ 1+1/3.(f_2+N),\ 1/3.(1-N), \\ 1/6.(1+f_2),\ 1/6.(2+f_2),\ 1/6.(3+f_2),\ 1/6.(4+f_2), \end{matrix} $$

$$ \left.\begin{matrix} 1/3.(2-N),\ 1-1/3.N\ ;\ \frac{27}{64} \\ 1/6.(5+f_2),\ 1+1/6.f_2; \end{matrix}\right] + \frac{f_1(f_2+N)N(N-1)(f_2+N+1)(f_1+3)}{2\ f_2(1+f_2)(2+f_2)(3+f_2)} $$

$$ \times\,{}_9F_8\left[\begin{matrix} 1/9.(5+f_1),\ 1/9.(8+f_1),\ 1/9.(11+f_1),\ 1/3.(2+N+f_2), \\ 4/3,\quad 5/3,\quad 1/6.(2+f_2), \end{matrix}\right. $$

$$ \left.\begin{matrix} 1+1/3.(f_2+N),\ 1/3.(f_2+N+4),\ 1/3.(2-N),\ 1-1/3.N,\ 1/3.(4-N);\ \frac{27}{64} \\ 1/6.(f_2+3),\ 1/6(f_2+4),\ 1/6.(f_2+5),\ 1+1/6.f_2,\ 1/6.(f_2+7); \end{matrix}\right] $$

§ (14.3) Application of theorem five; case three, when $\alpha_{2r+1} = 0$.

Next suppose $\alpha_{2n+1} = 0$, and F_γ is summable by theorem, five, (as in § 9.4). If $\delta_n = \dfrac{(d_1)_n(d_2)_n}{(\frac{1}{2}(1+d_1+d_2))_n 2^n}$

then $\gamma_{2n} = \dfrac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(1+d_1+d_2))(\frac{1}{2}d_1)_n(\frac{1}{2}d_2)_n 2^{2n}}{\Gamma(\frac{1}{2}(1+d_1))\Gamma(\frac{1}{2}(1+d_2))}$

Let $\alpha_{2n} = \dfrac{1}{n!\ (A)_n 4^n}$, and sum F_β by Vandermonde's

theorem, then $\beta_{2n} = \dfrac{(\frac{1}{4}(2A-1))_n(\frac{1}{4}(2A+1))_n 2^{2n}}{n!\ (\frac{1}{2})_n(A-\frac{1}{2})_n(A)_n}$

and $\beta_{2n+1} = \dfrac{(\frac{1}{4}(2A+1))_n(\frac{1}{4}(2A+3))_n 2^{2n}}{n!\ (3/2)_n(A)_n(A+\frac{1}{2})_n}$

Hence, $\dfrac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(1+d_1+d_2))}{\Gamma(\frac{1}{2}(1+d_1))\Gamma(\frac{1}{2}(1+d_2))}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}d_1, \frac{1}{2}d_2;\ 1 \\ A\ ;\end{matrix}\right]$

$$ = {}_6F_5\left[\begin{matrix} \frac{1}{4}(2A-1),\ \frac{1}{4}(2A+1),\ \frac{1}{2}d_1,\ \frac{1}{2}(1+d_1),\ \frac{1}{2}d_2, \\ \frac{1}{2},\ A-\frac{1}{2},\ \frac{1}{4}(1+d_1+d_2),\ \frac{1}{4}(3+d_1+d_2), \end{matrix}\right. $$

$$ \left.\begin{matrix} \frac{1}{2}(1+d_2);\ 1 \\ A\ ; \end{matrix}\right] + \frac{d_1d_2}{1+d_1+d_2}\times $$

$${}_6F_5\left[\begin{matrix}\frac{1}{4}(2A+1), \frac{1}{4}(2A+3), \frac{1}{2}(1+d_1), 1+\frac{1}{2}d_1, \frac{1}{2}(1+d_2), 1+\frac{1}{2}d_2; 1 \\ \frac{3}{2}, A, A+\frac{1}{2}, \frac{1}{4}(3+d_1+d_2), \frac{1}{4}(5+d_1+d_2);\end{matrix}\right]$$

§ (15.1) A new definition of $(a)_n$.

In those results obtained by supposing that $\alpha_r = 0$, when r is odd, (§ 10.2 to § 14.3) it is possible to replace the two series for β_{2n} and β_{2n+1} by a single series for β_n. To do this, it is necessary to make use of a slightly extended definition of the symbol $(a)_n$. When $(a)_n$ is defined as $a(a+1)\ldots(a+n-1)$, it follows that $\frac{\Gamma(a+n)}{\Gamma(a)} = (a)_n$. Hence, the meaning of $(a)_n$ can be extended for values of n other than positive int-eger-s, by letting $(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)}$ for all real and complex values of a and b provided that a is not a negative int-eger.

All the relations between these products hold with the extended definition. In particular, $(a)_{-n} = \frac{\Gamma(a-n)}{\Gamma(a)} = \frac{(-1)^n}{(1-a)_n}$, and $(a)_{n/2} = \frac{\Gamma(a+\frac{1}{2}n)}{\Gamma(a)}$.

Hence, $(a)_{n/2}(a+\frac{1}{2})_{n/2} = 2^{-n}(a)_n$.

§ (15.2) Use of the new definition of $(a)_n$; when F_γ is summed by the Binomial theorem.

Let us now reconsider the previous results. Thus, summing F_γ by the Binomial theorem, (as in § 10.2 case two) and F_β by Vandermonde's theorem, if $\delta_n = (d)_n x^n$, then $\gamma_n = \dfrac{(d)_n x^n}{(1-x)^{d+n}}$, and if $\alpha_{2n} = \dfrac{1}{n!(A)_n 4^n}$, but $\alpha_{2n+1} = 0$, then

$$\beta_n = \frac{1}{n!}\,{}_2F_1(-\tfrac{1}{2}n, \tfrac{1}{2}(1-n); A; 1)$$

$$= \frac{(A-\frac{1}{2}+\frac{1}{2}n)_{n/2}}{n!\,(A)_{n/2}}$$

$$= \frac{(A-\frac{1}{2})_n}{n!(A-\frac{1}{2})_{n/2}(A)_{n/2}}$$

$$\therefore \quad \beta_n = \frac{(A-\frac{1}{2})_n 2^n}{n!(2A-1)_n}.$$

Hence,

$$\frac{1}{(1-x)^d}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}d, \frac{1}{2}(1+d); & \dfrac{x^2}{(1-x)^2} \\ A; & \end{matrix}\right] = {}_2F_1\left[\begin{matrix}A-\frac{1}{2}, d; & 2x \\ 2A-1; & \end{matrix}\right]$$

This result was originally due to Kummer, (Journal für Maths. Vol.15 (1836) 78 (52)).

Similarly, if $\delta_n = x^n$, then $\gamma_n = \dfrac{x^n}{(1-x)^e}$,

and if $\alpha_{2n} = \dfrac{(-e)_n}{n!}$ but $\alpha_{2n+1} = 0$, then

$$\beta_n = \frac{(e)_n}{n!}\,{}_3F_2\left[\begin{matrix}-e, & -\frac{1}{2}n, & \frac{1}{2}(1-n); & 1 \\ & \frac{1}{2}(1-e-n), & 1-\frac{1}{2}(e+n); & \end{matrix}\right]$$

$$= \frac{(e)_n(\frac{1}{2}(1+e-n))_{n/2}(-\frac{1}{2}e)_{n/2}}{n!(\frac{1}{2}(1-e-n))_{n/2}(\frac{1}{2}(1+e-n)-(1-n)\frac{1}{2})_{n/2}}$$

summing F_β by Saalschutz's theorem,.

$$\therefore \quad \beta_n = \frac{(e)_n(-\frac{1}{2}e)_{n/2}(\frac{1}{2}(1-e))_{n/2}(-1)^n}{n!(\frac{1}{2}e)_{n/2}(\frac{1}{2}(1+e))_{n/2}}$$

since

$$\frac{(\frac{1}{2}(1+e)-\frac{1}{2}n)_{n/2}}{(\frac{1}{2}(1-e)-\frac{1}{2}n)_{n/2}} = \frac{\Gamma(\frac{1}{2}(1+e))\Gamma(\frac{1}{2}(1-e)-\frac{1}{2}n)}{\Gamma(\frac{1}{2}(1+e)-\frac{1}{2}n)\Gamma(\frac{1}{2}(1-e))}$$

$$= \frac{\Gamma(\frac{1}{2}(1+e))\Gamma(\frac{1}{2}(1-e)+\frac{1}{2}n)(\frac{1}{2}(1+e)+\frac{1}{2}n-1)\ldots}{\Gamma(\frac{1}{2}(1+e)+\frac{1}{2}n)\Gamma(\frac{1}{2}(1-e))(\frac{1}{2}(1-e)+\frac{1}{2}n-1)\ldots}$$

$$\times \frac{\ldots(\frac{1}{2}(1+e)+\frac{1}{2}n-n)}{\ldots(\frac{1}{2}(1-e)+\frac{1}{2}n-n)}$$

$$= \frac{(\frac{1}{2}(1-e))_{n/2}(-1)^n}{(\frac{1}{2}(1+e))_{n/2}}$$

and $\alpha_{3n} = \dfrac{(-1)^n (1/3.(F-1))_n (1+1/6.(F-1))_n}{n!(1/6.(F-1))_n}$, (as in § 11.2)

then, summing F_β as a well-poised ${}_5F_4(1)$ series,

$$\beta_n = \frac{1}{n!(F)_n}\,{}_5F_4\left[\begin{matrix}1/3.(F-1), 1+1/6.(F-1), & -1/3.n, & 1/3.(1-n), 1/3.(2-n); \; 1\\ 1/6.(F-1), & 1/3.(F+n+2), & 1/3.(F+n+1), 1/3.(F+n);\end{matrix}\right]$$

$$= \frac{(1/3.(F-1))_n (1/3.F)_{n/3} (1/3.(F+1))_{n/3} (1/3.(F+2))_{n/3}}{n!(F)_n (1/3.F)_{2n/3} (1/3.(1+F))_{2n/3} (1/3.(F-1))_{2n/3}}$$

$$= \frac{(1/3.(F-1))_n 3^{2n}}{n!\; 3^n\; (F-1)_{2n}}$$

$$\therefore\; \beta_n = \frac{(1/3.(F-1))_n}{n!(\tfrac{1}{2}(F-1))_n (\tfrac{1}{2}F)_n}\left(\frac{3}{4}\right)^n, \text{ and we have}$$

$$\frac{\Gamma(F)\Gamma(F-d_1-d_2)}{\Gamma(F-d_1)\Gamma(F-d_2)}\,{}_8F_7\left[\begin{matrix}1/3.(F-1), 1+1/6.(F-1), & 1/3.d_1, & 1/3.(1+d_1), & 1/3.(2+d_1), & 1/3.d_2, & 1/3.(1+d_2), & 1/3.(2+d_2); \; -1\\ 1/6.(F-1), & 1/3.(F-d_1+2), & 1/3.(F-d_1+1), & 1/3.(F-d_1), & 1/3.(F-d_2+2), & 1/3.(F-d_2+1), & 1/3.(F-d_2);\end{matrix}\right] = {}_3F_2\left[\begin{matrix}1/3.(F-1), & d_1, d_2; \; \frac{3}{4}\\ \tfrac{1}{2}F, \tfrac{1}{2}(F-1); & \end{matrix}\right] \quad \ldots(2)$$

provided that $Rl(F-d_1-d_2) > 0$, and that $Rl(1-d_1-d_2) > 0$.

This is a transformation between a well-poised ${}_8F_7(-1)$ series and a ${}_3F_2(\frac{3}{4})$ series.

§ (16.2) Gauss's theorem; case five.

Summing F_γ by Gauss's theorem, if $\delta_n = 1$,

then $\gamma_n = \dfrac{\Gamma(1+f-e)\Gamma(1-2e)\;(f)_{2n}}{\Gamma(1+f-2e)\Gamma(1-e)\;(1+f-2e)_{2n}}$ (as in § 11.3)

Summing F_β by Dougall's theorem, if

$\alpha_{2n} = \dfrac{(\frac{1}{2}(f-e))_n (1+\frac{1}{4}(f-e))_n (-e)_n}{n!\;(\frac{1}{4}(f-e))_n (1+\frac{1}{2}(f+e))_n}$ and $\alpha_{2n+1} = 0$, then

$$\beta_n = \frac{(f)_n(-e)_n(\frac{1}{2}(1+f+e))_n(-1)^n}{n!\ (1+f+e)_n(\frac{1}{2}(1+f-e))_n}.$$

Hence, $\frac{\Gamma(1+f-e)\Gamma(1-2e)}{\Gamma(1-e)\Gamma(1+f-2e)}\ {}_7F_6\left[\begin{matrix}\frac{1}{2}(f-e), 1+\frac{1}{4}(f-e), & -e, & \frac{1}{4}f, & \frac{1}{4}(2+f), & \frac{1}{4}(3+f), & \frac{1}{4}(1+f);\ 1\\ \frac{1}{4}(f-e), & 1+\frac{1}{2}(f+e), & 1+\frac{1}{4}(f-2e), & \frac{1}{4}(2+f-2e), & \frac{1}{4}(1+f-2e), & \frac{1}{4}(3+f-2e);\end{matrix}\right]$

$= {}_3F_2\left[\begin{matrix}f, & -e, \frac{1}{2}(1+f+e);\ -1\\ & 1+f+e, \frac{1}{2}(1+f-e);\end{matrix}\right]$ provided that $Rl(1-2e)>0$.

This transforms a well-poised ${}_7F_6(1)$ into a well-poised ${}_3F_2(-1)$ series.

§ (16.3) Gauss's theorem; case seven.

Summing F_γ by Vandermonde's theorem, if $\delta_n = \frac{(d)_n(-N)_n}{(D)_n}$ then $\gamma_n = \frac{(D-d)_N(d)_n(-N)_n}{(D)_N(1+d-D-N)_n}$, (as in § 11.4). If $\alpha_{2n} = \frac{1}{n!(A)_n 4^n}$, and $\alpha_{2n+1} = 0$, then

$$\beta_n = \frac{(A-\frac{1}{2})_n 2^n}{n!\ (2A-1)_n}.$$

Hence, $\frac{(D-d)_N}{(D)_N}\ {}_4F_3\left[\begin{matrix}\frac{1}{2}d, \frac{1}{2}(1+d), & -\frac{1}{2}N, & \frac{1}{2}(1-N);\ 1\\ A, & \frac{1}{2}(1+d-D-N), & 1+\frac{1}{2}(d-D-N);\end{matrix}\right]$

$= {}_3F_2\left[\begin{matrix}A-\frac{1}{2}, & d, -N;\ 2\\ 2A-1, & D;\end{matrix}\right].$

§ (16.4) Gauss's theorem; case ten.

Summing F_γ by Gauss's theorem, if $\delta_n = \frac{(d)_n}{(D)_n}$ then $\gamma_n = \frac{\Gamma(D)\Gamma(D-d-e)}{\Gamma(D-e)\Gamma(D-d)}\frac{(d)_n}{(D-e)_n}$. If $\alpha_{2n} = \frac{(-e)_n}{n!}$, and $\alpha_{2n+1} = 0$, (as in § 11.5), then, summing F_β by Saalschutz's theorem, $\beta_n = \frac{(-1)^n(-e)_n}{n!}$.

Hence, $\dfrac{\Gamma(D)\Gamma(D-d-e)}{\Gamma(D-d)\Gamma(D-e)}\,{}_3F_2\left[\begin{matrix}-e, & \tfrac{1}{2}d, & \tfrac{1}{2}(1+d); & 1\\ & \tfrac{1}{2}(D-e), & \tfrac{1}{2}(1+D-e); & \end{matrix}\right]$

$$= {}_2F_1\left[\begin{matrix}-e, d; & -1\\ D; & \end{matrix}\right]$$

provided that $Rl(D-d)>0$, and that $Rl(e)<0$.

If $D = 1-d-e$, summing the ${}_2F_1(-1)$ series by Kummer's theorem, we get a special case of Dixon's theorem.

§ (16.5) Summation of F_γ by Saalschutz's theorem; case one.

Summing F_γ by Saalschutz's theorem, if

$$\delta_n = \frac{(d_1)_n(d_2)_n(-N)_n}{(1+d_1+d_2-N-F)_n}, \text{ then}$$

$$\gamma_n = \frac{(F-d_1)_N(F-d_2)_N(d_1)_n(d_2)_n(-N)_n(-1)^n}{(F)_N(F-d_1-d_2)_N(F-d_1)_n(F-d_2)_n(F+N)_n}$$, (as in § 12.1). Summing F_β as a well-poised ${}_5F_4(1)$ series, if $\alpha_{2n+1} = 0$, and $\alpha_{2n} = \dfrac{(\tfrac{1}{2}(F-1))_n(\tfrac{1}{4}(3+F))_n(a)_n}{n!(\tfrac{1}{4}(F-1))_n(\tfrac{1}{2}(F+1)-a)_n}$

then $\beta_n = \dfrac{(\tfrac{1}{2}F-a)_n}{n!(F-2a)_n(\tfrac{1}{2}F)_n}$.

Hence, $\dfrac{(F-d_1)_N(F-d_2)_N}{(F)_N(F-d_1-d_2)_N}\,{}_9F_8\left[\begin{matrix}\tfrac{1}{2}(F-1), \tfrac{1}{4}(3+F), & a, \\ & \tfrac{1}{4}(F-1), \tfrac{1}{2}(1+F)-a,\end{matrix}\right.$

$\left.\begin{matrix}\tfrac{1}{2}d_1, \tfrac{1}{2}(1+d_1), & \tfrac{1}{2}d_2, \tfrac{1}{2}(1+d_2), & -\tfrac{1}{2}N, \tfrac{1}{2}(1-N); & 1\\ \tfrac{1}{2}(F-d_1+1), \tfrac{1}{2}(F-d_1), & \tfrac{1}{2}(F-d_2+1), \tfrac{1}{2}(F-d_2), & \tfrac{1}{2}(F+N+1), \tfrac{1}{2}(F+N); & \end{matrix}\right]$

$= {}_4F_3\left[\begin{matrix}\tfrac{1}{2}F-a, d_1, d_2, & -N; & 1\\ \tfrac{1}{2}F, F-2a, & 1+d_1+d_2-F-N; & \end{matrix}\right]$

This is a transformation between a well-poised ${}_9F_8(1)$ series, and a ${}_4F_3(1)$ series.

If $\alpha_n = 0$, when n is not a multiple of three, and

$$\alpha_{3n} = \frac{(-1)^n(1/3.(F-1))_n(1+1/6.(F-1))_n}{n!\,(1/6.(F-1))_n}$$

similarly, we have

$$\beta_n = \frac{(1/3.(F-1))_n(\frac{3}{4})^n}{n!(\frac{1}{2}(F+1))_n(\frac{1}{2}F)_n}.$$ Hence,

$${}_{11}F_{10}\left[\begin{matrix}1/3.(F-1),1+1/6.(F-1), & 1/3.d_1, 1/3.(1+d_1), \\ & 1/6.(F-1), 1+1/3.(F-d_1), 1/3.(2+F-d_1),\end{matrix}\right.$$

$$\begin{matrix}1/3.(2+d_1), & 1/3.d_2, & 1/3.(1+d_2), & 1/3.(2+d_2), \\ 1/3.(1+F-d_1), & 1+1/3.(F-d_2), & 1/3.(2+F-d_2), & 1/3.(1+F-d_2),\end{matrix}$$

$$\left.\begin{matrix}-1/3.N, & 1/3.(1-N), & 1/3.(2-N); & 1 \\ 1+1/3.(F+N), & 1/3.(2+F+N), & 1/3.(1+F+N); & \end{matrix}\right]$$

$$= \frac{(F)_N(F-d_1-d_2)_N}{(F-d_1)_N(F-d_2)_N}\;{}_4F_3\left[\begin{matrix}1/3.(F-1), & d_1, d_2, & -N; & \frac{3}{4} \\ & \frac{1}{2}(F+1), \frac{1}{2}F, & 1+d_1+d_2-N-F; & \end{matrix}\right]$$

This is a transformation between a well-poised ${}_{11}F_{10}(1)$ series and a ${}_4F_3(\frac{3}{4})$ series.

§ (16.6) Summation by Saalschutz's theorem; case two.

Summing F_γ by Saalschutz's theorem, if

$$\delta_n = \frac{(-N)_n}{(2e-N)_n}, \text{ then } \gamma_n = \frac{(1-e)_N(1+f-2e)_N(-N)_n(\frac{1}{2}f)_n(\frac{1}{2}(1+f))_n}{(1-2e)_N(1+f-e)_N\,(1+f+N-e)_n(e-N)_n} \times \frac{(1+f+N-2e)_N}{(1+\frac{1}{2}f-e)_n(\frac{1}{2}(1+f)-e)_n}.$$

Summing F_β by Dougall's theorem, if $\alpha_{2n+1} = 0$, and

$$\alpha_{2n} = \frac{(\frac{1}{2}(f-e))_n(1+\frac{1}{4}(f-e))_n(-e)_n}{n!(\frac{1}{4}(f-e))_n(1+\frac{1}{2}(f+e))_n}$$ then,

$$\beta_n = \frac{(f)_n(-e)_n(\frac{1}{2}(1+f+e))_n}{n!(1+f+e)_n(\frac{1}{2}(1+f-e))_n}.$$

Hence, $$\frac{(1-e)_N(1+f-2e)_N}{(1-2e)_N(1+f-e)_N}\;{}_{11}F_{10}\left[\begin{matrix}\frac{1}{2}(f-e), 1+\frac{1}{4}(f-e), & -e, \\ \frac{1}{4}(f-e), & 1+\frac{1}{2}(f+e),\end{matrix}\right.$$

$$\begin{matrix}\frac{1}{2}(1+f+N)-e, 1+\frac{1}{2}(f+N)-e, & \frac{1}{4}f, & \frac{1}{4}(f+1), & \frac{1}{4}(f+2), \\ \frac{1}{2}(1+e-N), \quad \frac{1}{2}(e-N), & 1+\frac{1}{4}(f-2e), & \frac{1}{4}(3+f-2e), & \frac{1}{4}(2+f-2e),\end{matrix}$$

$$\left.\begin{matrix}\frac{1}{4}(f+3), & \frac{1}{2}(1-N), & -\frac{1}{2}N; & 1 \\ \frac{1}{4}(1+f-2e), & \frac{1}{2}(1+f+N-e), & 1+\frac{1}{2}(f+N-e); & \end{matrix}\right]$$

$$= {}_4F_3\left[\begin{matrix} f, & -e, & \frac{1}{2}(1+f+e), & -N; & -1 \\ & 1+f+e, & \frac{1}{2}(1+f-e), & 2e-N; & \end{matrix}\right]$$

This transforms a nearly-poised ${}_4F_3(-1)$ series into a well-poised ${}_{11}F_{10}(1)$ series.

§ (16.7) Summation by Saalschutz's theorem; case three.

Summing both F_γ and F_β by Saalschutz's theorem, (as in § 12.4) if $\delta_n = \dfrac{(d)_n(-N)_n}{(D)_n(1+e+d-N-D)_n}$, then

$$\gamma_n = \frac{(D-e)_N(D-d)_N(d)_n(-N)_n}{(D)_N(D-d-e)_N(D-e)_n(1+d-N-D)_n},$$ and if $\alpha_{2n+1} = 0$,

and $\alpha_{2n} = \dfrac{(-e)_n}{n!}$, then $\beta_n = \dfrac{(-e)_n(-1)^n}{n!}$.

Hence,
$$\frac{(D-e)_N(D-d)_N}{(D)_N(D-d-e)_N}\, {}_5F_4\left[\begin{matrix} -e, & \frac{1}{2}d, & \frac{1}{2}(1+d), & -\frac{1}{2}N, & \frac{1}{2}(1-N); & 1 \\ & \frac{1}{2}(D-e), & \frac{1}{2}(1+D-e), & \frac{1}{2}(1+d-N-D), & 1+\frac{1}{2}(d-N-D); & \end{matrix}\right] = {}_3F_2\left[\begin{matrix} -e, & d, & -N; & -1 \\ & D, & 1+e+d-N-D; & \end{matrix}\right]$$

§ (17.1) Summation by Dougall's theorem; case two.

Summing F_γ and F_β by Dougall's theorem, (as in § 13.1), if $\delta_n = \dfrac{(1+\frac{1}{2}f)_n(d_1)_n(d_2)_n(d_3)_n(-N)_n}{(\frac{1}{2}f)_n(1+f-d_1)_n(1+f-d_2)_n(1+f-d_3)_n(1+f+N)_n}$

then
$$\gamma_n = \frac{(1+f)_N(1+f-d_1-e)_N(1+f-d_2-e)_N(1+f-d_1-d_2)_N}{(1+f-d_1)_N(1+f-d_2)_N(1+f-e)_N(1+f-d_1-d_2-e)_N}$$

$$\times \frac{(d_1)_n(d_2)_n(d_3)_n(-N)_n}{(1+f-e-d_1)_n(1+f-e-d_2)_n(1+f-e-d_3)_n(1+f-e+N)_n}$$

Also, if $\alpha_{2n} = \dfrac{(\frac{1}{2}(f-e))_n(1+\frac{1}{4}(f-e))_n(-e)_n}{n!(\frac{1}{4}(f+e))_n(1+\frac{1}{2}(f+e))_n}$, and $\alpha_{2n+1} = 0$,

then $\beta_n = \dfrac{(f)_n(\frac{1}{2}(1+f+e))_n(-e)_n(-1)^n}{n!(\frac{1}{2}(1+f-e))_n(1+f+e)_n}$.

Hence,

$$\frac{(d-2e)_N(1+\tfrac{1}{2}d-e)_N(-e)_N}{(1+d-e)_N(\tfrac{1}{2}d-e)_N(-2e)_N}\;{}_6F_5\left[\begin{matrix}-e,\,1+e-\tfrac{1}{2}d-N,\,\tfrac{1}{2}d,\,\tfrac{1}{2}(1+d),\,\tfrac{1}{2}(1-N),\,-\tfrac{1}{2}N;\;1\\ \tfrac{1}{2}(1+d)-e,\,e-\tfrac{1}{2}d-N,\,1+\tfrac{1}{2}d-e,\,\tfrac{1}{2}(1+e-N),\,1+\tfrac{1}{2}(e-N);\end{matrix}\right] = {}_3F_2\left[\begin{matrix}-e,\,d,\,-N;\;-1\\ 1+d-e,\,1+2e-N;\end{matrix}\right]$$

§ (17.3) Summation by theorem thirteen; case five.

Summing F_β by Dougall's theorem, and F_γ by theorem thirteen (as in § 13.3), if $\delta_n = \dfrac{(-N)_n}{(1+2e-N)_n}$

then $\gamma_n = \dfrac{(f-2e)_N(1+\frac{1}{2}f-e)_N(-e)_N(f)_{2n}(f-2e+N)_n(-N)_n}{(1+f-e)_N(\frac{1}{2}f-e)_N(-2e)_N(1+f-e+N)_n(1+f-2e)_n(1+e-N)_n}$

and if $\alpha_{2n} = \dfrac{(\frac{1}{2}(f-e))_n(1+\frac{1}{4}(f-e))_n(-e)_n}{n!\,(\frac{1}{4}(f-e))_n(1+\frac{1}{2}(f+e))_n}$, but $\alpha_{2n+1} = 0$,

then $\beta_n = \dfrac{(f)_n(-e)_n(\frac{1}{2}(f+e+1))_n(-1)^n}{n!\,(1+f+e)_n(\frac{1}{2}(1+f-e))_n}$

Hence,

$$\frac{(f-2e)_N(1+\tfrac{1}{2}f-e)_N(-e)_N}{(1+f-e)_N(\tfrac{1}{2}f-e)_N(-2e)_N}\;{}_{11}F_{10}\left[\begin{matrix}\tfrac{1}{2}(f-e),\,1+\tfrac{1}{4}(f-e),\,-e,\,\tfrac{1}{4}f,\,\tfrac{1}{4}(1+f),\,\tfrac{1}{4}(2+f),\,\tfrac{1}{4}(3+f),\\ \tfrac{1}{4}(f-e),\,1+\tfrac{1}{2}(f+e),\,1+\tfrac{1}{4}(f-2e),\,\tfrac{1}{4}(3+f-2e),\,\tfrac{1}{4}(2+f-2e),\,\tfrac{1}{4}(1+f-2e),\end{matrix}\right.$$

$$\left.\begin{matrix}\tfrac{1}{2}(f-2e+N),\,\tfrac{1}{2}(1+f-2e+N),\,-\tfrac{1}{2}N,\,\tfrac{1}{2}(1-N);\;1\\ 1+\tfrac{1}{2}(e-N),\,\tfrac{1}{2}(1+e-N),\,1+\tfrac{1}{2}(f-e+N),\,\tfrac{1}{2}(1+f-e+N);\end{matrix}\right]$$

$$= {}_4F_3\left[\begin{matrix}f,\,-e,\,\tfrac{1}{2}(1+f+e),\,-N;\;-1\\ 1+f+e,\,\tfrac{1}{2}(1+f-e),\,1+2e-N;\end{matrix}\right]$$

This transforms a well-poised ${}_{11}F_{10}(1)$ series into a nearly-poised ${}_4F_3(-1)$ series.

§ (17.4) Summation by theorem fourteen; case one.

Summing F_γ by theorem fourteen, and F_β by Dougall's theorem, (as in § 13.4), if $\delta_n = \dfrac{(1+\frac{1}{2}f)_n(-N)_n}{(\frac{1}{2}f)_n(1+2e-N)_n}$

then $\gamma_n = \dfrac{(f-2e)_N(-e)_N(\frac{1}{2}(f+1))_n(1+\frac{1}{2}f)_n(f-2e+N)_n(-N)_n}{(1+f-e)_N(-2e)_N(\frac{1}{2}f-e)_n(\frac{1}{2}(1+f)-e)_n(1+f-e+N)_n(1+e-N)_n}$

and if $\alpha_{2n} = \dfrac{(\frac{1}{2}(f-e))_n(1+\frac{1}{4}(f-e))_n(-e)_n}{n!\,(\frac{1}{4}(f-e))_n(1+\frac{1}{2}(f+e))_n}$ but

$\alpha_{2n+1} = 0$, then $\beta_n = \dfrac{(f)_n(\frac{1}{2}(1+f+e))_n(-e)_n(-1)^n}{n!\,(\frac{1}{2}(1+f-e))_n(1+f+e)_n}$

Hence,

$$\frac{(f-2e)_N(-e)_N}{(1+f-e)_N(-2e)_N}\;{}_{11}F_{10}\left[\begin{matrix}\frac{1}{2}(f-e), 1+\frac{1}{4}(f-e), & -e, & \frac{1}{4}(1+f), & \frac{1}{4}(2+f), & \frac{1}{4}(3+f), & 1+\frac{1}{4}f, \frac{1}{2}(f-2e+N), & \frac{1}{2}(1+f-2e+N), & -\frac{1}{2}N, & \frac{1}{2}(1-N); 1\\ \frac{1}{4}(f-e), & 1+\frac{1}{2}(f+e), & \frac{1}{4}(3+f-2e), & \frac{1}{4}(2+f-2e), & \frac{1}{4}(1+f-2e), & \frac{1}{4}(f-2e), 1+\frac{1}{2}(e-N), & \frac{1}{2}(1+e-N), & 1+\frac{1}{2}(f-e+N), & \frac{1}{2}(1+f-e+N);\end{matrix}\right]$$

$$= {}_5F_4\left[\begin{matrix}f, 1+\frac{1}{2}f, & -e, \frac{1}{2}(1+f+e), & -N; -1\\ \frac{1}{2}f, 1+f+e, & \frac{1}{2}(1+f-e), & 1+2e-N;\end{matrix}\right].$$

This transforms a well-poised ${}_{11}F_{10}(1)$ series into a nearly-poised ${}_5F_4(-1)$ series.

A similar result is given by summing F_β by Dougall's theorem, and F_γ by theorem fifteen.

§ (17.5) Summation by theorem nineteen; case three.

Summing F_γ by theorem nineteen and F_β as a well-poised ${}_5F_4(1)$ series, (as in § 14.1), if

$\delta_n = \dfrac{(\frac{1}{2}f_1)_n(\frac{1}{2}(1+f_1))_n(f_2+N)_n(-N)_n}{(\frac{1}{2}f_2)_n(\frac{1}{2}(1+f_2))_n}$ then

$\gamma_n = \dfrac{(f_2-f_1)_N(\frac{1}{2}f_1)_n(-N)_n(-1)^n}{(f_2)_N(1+f_1-f_2-N)_n(1+\frac{1}{2}f_1)_n}$

and if $\alpha_{2n} = \dfrac{(\frac{1}{2}f_1)_n(1+\frac{1}{4}f_1)_n(a)_n}{n!(\frac{1}{4}f_1)_n(1+\frac{1}{2}f_1-a)_n}$ but $\alpha_{2n+1} = 0$,

then $\beta_n = \dfrac{(\frac{1}{2}(1+f_1)-a)_n}{n!(1+f_1-2a)_n(\frac{1}{2}(1+f_1))_n}$.

Hence,

$$\frac{(f_2-f_1)_N}{(f_2)_N}\,{}_4F_3\left[\begin{matrix}a, & -\frac{1}{2}N, & \frac{1}{2}(1-N), & \frac{1}{2}f_1; 1\\ & 1+\frac{1}{2}f_1-a, & \frac{1}{2}(1+f_1-f_2-N), & 1+\frac{1}{2}(f_1-f_2-N);\end{matrix}\right]$$

$$= {}_4F_3\left[\begin{matrix}\frac{1}{2}(1+f_1)-a, & \frac{1}{2}f_1, f_2+N, & -N; 1\\ & 1+f_1-2a, \frac{1}{2}f_2, \frac{1}{2}(1+f_2); & \end{matrix}\right]$$

<u>Theorem (21)</u> (Watson's theorem)

$$ {}_3F_2\left[\begin{matrix} d_1, & d_2, & a;\; 1 \\ & \tfrac{1}{2}(1+d_1+d_2), & 2a; \end{matrix}\right] $$

$$ = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(1+d_1+d_2))\Gamma(a+\frac{1}{2})\Gamma(a+\frac{1}{2}(1-d_1-d_2))}{\Gamma(\frac{1}{2}(1+d_1))\Gamma(\frac{1}{2}(1+d_2))\Gamma(a+\frac{1}{2}(1-d_1))\Gamma(a+\frac{1}{2}(1-d_2))} $$

This theorem was first given by G.N.Watson, in his paper "A note on generalised hypergeometric series" (Proc. Lond. Maths. Soo. (2) Vol. 23(1925). p. xiii-xv. Records 8th Nov. 1923), for terminating series, and the general case, for infinite series, was given by F.J.W.Whipple, in his paper "A group of generalised hypergeometric series" (Proc. Lond. Maths. Soc. (2) Vol. 23 (1925)).

<u>§ (18.2)</u> <u>Applications of Watson's theorem to the summation of F_γ.</u>

There are, in all, seven possible ways of summing F_γ by Watson's theorem, of which only three give summable series for F_β . These are:-

(1) $$ F_\gamma = {}_3F_2\left[\begin{matrix} e, f+2n, & d+n;\; 1 \\ \tfrac{1}{2}(e+f+1)+n, & 2d+2n; \end{matrix}\right] $$

$$ F_\beta = \sum_{r=0}^{n} \frac{(-n)_r (f+n)_r}{(1-e-n)_r (2d+n)_r} \alpha_r $$

(2) $$ F_\gamma = {}_3F_2\left[\begin{matrix} d_1+n, d_2+n, & e;\; 1 \\ \tfrac{1}{2}(1+d_1+d_2)+n, & 2e; \end{matrix}\right] $$

$$ F_\beta = \sum_{r=0}^{n} \frac{(-n)_r (1-2e-n)_r (-1)^r}{(1-e-n)_r} \alpha_r $$

(3) $F_\gamma = {}_3F_2\left[\begin{matrix} d_1+n, & d_2+n, & d_3+n; 1 \\ & \frac{1}{2}(1+d_1+d_2)+n, & 2d_3+2n; \end{matrix}\right]$

$$F_\beta = \sum_{r=0}^{n} \frac{(-n)_r(-1)^r}{(2d_3+n)_r}\alpha_r$$

§ (18.3) Case one.

If $\delta_n = \dfrac{(d)_n}{(\frac{1}{2}(1+f+e))_n}$, then

$$\gamma_n = \frac{\Gamma(\frac{1}{2})\Gamma(d+\frac{1}{2})\Gamma(\frac{1}{2}(1+f+e))\Gamma(\frac{1}{2}(1-f-e)+d)\ (\frac{1}{2}f)_n}{\Gamma(\frac{1}{2}(1+e))\Gamma(\frac{1}{2}(1+f))\Gamma(\frac{1}{2}(1-e)+d)\Gamma(\frac{1}{2}(1-f)+d)\ (\frac{1}{2}(1-e)+d)_n}$$

Summing F_β by Saalschutz's theorem merely gives Watson's theorem again, in another form.

Summing F_β by Dougall's theorem, if $d = \frac{1}{2}(1+f-e)$ and

$$\alpha_n = \frac{(f-e)_n(1+\frac{1}{2}(f-e))_n(a_1)_n(a_2)_n(a_3)_n}{n!\ (\frac{1}{2}(f-e))_n(1+f-e-a_1)_n(1+f-e-a_2)_n(1+f-e-a_3)_n}$$

then $\beta_n = \dfrac{(f)_n(e+a_1)_n(e+a_2)_n(e+a_3)_n}{n!(1+f-e-a_1)_n(1+f-e-a_2)_n(1+f-e-a_3)_n}$.

Hence $\dfrac{\Gamma(\frac{1}{2})\Gamma(1+\frac{1}{2}(f-e))\Gamma(\frac{1}{2}(1+f+e))\Gamma(1-e)}{\Gamma(\frac{1}{2}(1+e))\Gamma(\frac{1}{2}(1+f))\Gamma(1+\frac{1}{2}f-e)\Gamma(1-\frac{1}{2}e)} \times$

$${}_6F_5\left[\begin{matrix} f-e, 1+\frac{1}{2}(f-e), & a_1, & a_2, & a_3, & \frac{1}{2}f; 1 \\ \frac{1}{2}(f-e), & 1+f-e-a_1, & 1+f-e-a_2, & 1+f-e-a_3, & 1+\frac{1}{2}f-e; \end{matrix}\right]$$

$$= {}_5F_4\left[\begin{matrix} f, \frac{1}{2}(1+f-e), & e+a_1, & e+a_2, & e+a_3; 1 \\ \frac{1}{2}(1+f+e), & 1+f-e-a_1, & 1+f-e-a_2, & 1+f-e-a_3; \end{matrix}\right] \quad \ldots(1)$$

provided $1+f-2e = a_1+a_2+a_3$.

Let $a_3 = 1+\frac{1}{2}f-e$, then the ${}_6F_5(1)$ series becomes a ${}_4F_3(1)$ series and the ${}_5F_4(1)$ series is summable. This gives, writing a for f-e, b for a_1, and c for $\frac{1}{2}f$,

Theorem (22)

$${}_4F_3\left[\begin{matrix} a, 1+\frac{1}{2}a, & b, & c; 1 \\ \frac{1}{2}a, & 1+a-b, & 1+a-c; \end{matrix}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(\frac{1}{2}(1+a)-b-c)}{2^a\Gamma(1+\frac{1}{2}a)\Gamma(\frac{1}{2}(1+a)-b)\Gamma(\frac{1}{2}(1+a)-c)}$$

$$\times \frac{1}{\Gamma(1+a-b-c)} \quad \ldots(2)$$

If we let $c \to \infty$ in this result, or, alternatively, if we sum F_β as a well-poised ${}_4F_3(-1)$ series, we get

Theorem (23)

$$ {}_3F_2\left[\begin{matrix} a, 1+\tfrac{1}{2}a, & b; & -1 \\ \tfrac{1}{2}a, 1+a-b; & & \end{matrix}\right] = \frac{\Gamma(1+a-b)\Gamma(\tfrac{1}{2}(1+a))}{\Gamma(\tfrac{1}{2}(1+a)-b)\Gamma(1+a)} \qquad \ldots(3) $$

(Whipple, Proc. Lond. Maths. Soc. (2) Vol. 24 1924, equations 9.6 and 9.5 respectively.)

Next suppose that $\alpha_{2n} = \dfrac{(\tfrac{1}{2}(f-e))_n(1+\tfrac{1}{4}(f-e))_n(-e)_n}{n!\,(\tfrac{1}{4}(f-e))_n(1+\tfrac{1}{2}(f+e))_n}$

but that $\alpha_{2n+1} = 0$, then, summing F_β by Dougall's theorem,

$$ \beta_n = \frac{(\tfrac{1}{2}(1+f+e))_n(f)_n(-e)_n(-1)^n}{n!\,(\tfrac{1}{2}(1+f-e))_n(1+f+e)_n} $$

and we get the simple result

$$ \frac{\Gamma(\tfrac{1}{2})\Gamma(1+\tfrac{1}{2}(f-e))\Gamma(\tfrac{1}{2}(1+f+e))\Gamma(1-e)}{\Gamma(\tfrac{1}{2}(1+e))\Gamma(\tfrac{1}{2}(1+f))\Gamma(1+\tfrac{1}{2}f-e)\Gamma(1-\tfrac{1}{2}e)} \times $$

$$ {}_5F_4\left[\begin{matrix} \tfrac{1}{2}(f-e), 1+\tfrac{1}{4}(f-e), & -e, & \tfrac{1}{4}f, & \tfrac{1}{4}(2+f); & 1 \\ \tfrac{1}{4}(f-e), & 1+\tfrac{1}{2}(f+e), & \tfrac{1}{4}(2+f-2e), & \tfrac{1}{4}(f-2e)+1; & \end{matrix}\right] $$

$$ = {}_2F_1\left[\begin{matrix} f, & -e; & -1 \\ & 1+f+e; & \end{matrix}\right] \qquad \ldots(4) $$

§ (18.4) The second case

If $\delta_n = \dfrac{(d_1)_n(d_2)_n}{(\tfrac{1}{2}(1+d_1+d_2))_n}$ then

$$ \gamma_{2n} = \frac{\Gamma(\tfrac{1}{2})\Gamma(e+\tfrac{1}{2})\Gamma(\tfrac{1}{2}(1+d_1+d_2))\Gamma(\tfrac{1}{2}(1-d_1-d_2)+e)\,(\tfrac{1}{2}d_1)_n}{\Gamma(\tfrac{1}{2}(1+d_1))\Gamma(\tfrac{1}{2}(1+d_2))\Gamma(\tfrac{1}{2}(1-d_1)+e)\Gamma(\tfrac{1}{2}(1-d_2)+e)} \times \frac{(\tfrac{1}{2}d_2)_n(\tfrac{1}{4}(1+d_1+d_2-2e))_n(\tfrac{1}{4}(3+d_1+d_2-2e))_n\,2^{6n}}{(\tfrac{1}{2}(1+d_1)-e)_n(\tfrac{1}{2}(1+d_2)-e)_n} $$

and

$$ \gamma_{2n+1} = \frac{\Gamma(\tfrac{1}{2})\Gamma(e+\tfrac{1}{2})\Gamma(\tfrac{1}{2}(1+d_1+d_2))\Gamma(\tfrac{1}{2}(1-d_1-d_2)+e)}{\Gamma(1+\tfrac{1}{2}d_1)\Gamma(1+\tfrac{1}{2}d_2)\Gamma(\tfrac{1}{2}(1-d_1)+e)\Gamma(\tfrac{1}{2}(1-d_2)+e)} $$

$$ \times \frac{d_1d_2(\tfrac{1}{2}(1+d_1))_n(\tfrac{1}{2}(1+d_2))_n(\tfrac{1}{4}(3+d_1+d_2)-\tfrac{1}{2}e)_n(\tfrac{1}{4}(5+d_1+d_2)-\tfrac{1}{2}e)_n\,2^{6n}}{(1+\tfrac{1}{2}d_1-e)_n(1+\tfrac{1}{2}d_2-e)_n} $$

Let $\alpha_m = \dfrac{(-1)^n}{n!}$, and sum F_β by Vandermonde's theorem, then $\beta_n = \dfrac{(e)_n(-1)^n}{n!\,(2e)_n}$.

Hence, $\dfrac{\Gamma(\frac{1}{2})\Gamma(e+\frac{1}{2})\Gamma(\frac{1}{2}(1+d_1+d_2))\Gamma(\frac{1}{2}(1-d_1-d_2)+e)}{\Gamma(\frac{1}{2}(1+d_1))\Gamma(\frac{1}{2}(1+d_2))\Gamma(\frac{1}{2}(1-d_1)+e)\Gamma(\frac{1}{2}(1-d_2)+e)}$

$$\times\ {}_4F_3\left[\begin{matrix}\frac{1}{2}d_1, & \frac{1}{2}d_2, & \frac{1}{4}(1+d_1+d_2-2e), & \frac{1}{4}(3+d_1+d_2-2e); & 16\\ \frac{1}{2}(1+d_1)-e, & \frac{1}{2}(1+d_2)-e, & & 1/2 & ;\end{matrix}\right]$$

$$+\ \frac{d_1d_2\,\Gamma(\frac{1}{2})\Gamma(e+\frac{1}{2})\Gamma(\frac{1}{2}(1+d_1+d_2))\Gamma(e-\frac{1}{2}(1+d_1+d_2))}{\Gamma(1+\frac{1}{2}d_1)\Gamma(1+\frac{1}{2}d_2)\Gamma(e-\frac{1}{2}d_1)\Gamma(e-\frac{1}{2}d_2)}$$

$$\times\ {}_4F_3\left[\begin{matrix}\frac{1}{2}(1+d_1), & \frac{1}{2}(1+d_2), & \frac{1}{4}(3+d_1+d_2-2e), & \frac{1}{4}(5+d_1+d_2-2e); & 16\\ & 3/2, & 1+\frac{1}{2}d_1-e, & 1+\frac{1}{2}d_2-e & ;\end{matrix}\right]$$

$$= {}_3F_2\left[\begin{matrix}d_1, & d_2, & e; & -1\\ \frac{1}{2}(1+d_1+d_2), & 2e; & &\end{matrix}\right]$$

provided that all the series involved are finite.

§ (18.5) The third case

If $\delta_n = \dfrac{(d_1)_n(d_2)_n(d_3)_n}{(\frac{1}{2}(1+d_1+d_2))_n}$ then

$$\gamma_{2m} = \frac{\Gamma(\frac{1}{2})\Gamma(d_3+\frac{1}{2})\Gamma(\frac{1}{2}(1+d_1+d_2))\Gamma(\frac{1}{2}(1-d_1-d_2)+d_3)\,(\frac{1}{2}d_1)_n}{\Gamma(\frac{1}{2}(1+d_1))\Gamma(\frac{1}{2}(1+d_2))\Gamma(\frac{1}{2}(1-d_1)+d_3)\Gamma(\frac{1}{2}(1-d_2)+d_3)}$$
$$\times\ \frac{(\frac{1}{2}d_2)_n}{(\frac{1}{2}(1-d_1)+d_3)_n(\frac{1}{2}(1-d_2)+d_3)_n}$$

and
$$\gamma_{2n+1} = \frac{\Gamma(\frac{1}{2})\Gamma(d_3+\frac{1}{2})\Gamma(\frac{1}{2}(1+d_1+d_2))\Gamma(\frac{1}{2}(1-d_1-d_2)+d_3)}{\Gamma(1+\frac{1}{2}d_1)\Gamma(1+\frac{1}{2}d_2)\Gamma(1-\frac{1}{2}d_1+d_3)\Gamma(1-\frac{1}{2}d_2+d_3)}$$
$$\times\ \frac{d_1d_2(\frac{1}{2}(1+d_1))_n(\frac{1}{2}(1+d_2))_n}{(1-\frac{1}{2}d_1+d_3)_n(1-\frac{1}{2}d_2+d_3)_m}$$

Summing F_β as a well-poised ${}_5F_4(1)$ series, if

$$\alpha_n = \frac{(2d_3-1)_n(d_3+\frac{1}{2})_n(a_1)_n(a_2)_n(-1)^n}{n!\,(d_3-\frac{1}{2})_n(2d_3-a_1)_n(2d_3-a_2)_n}$$

then $\beta_n = \dfrac{(2d_3-a_1-a_2)_n}{n!(2d_3-a_1)_n(2d_3-a_2)_n}$. Hence,

$$\frac{\Gamma(\tfrac12)\Gamma(d_3+\tfrac12)\Gamma(\tfrac12(1+d_1+d_2))\Gamma(\tfrac12(1-d_1-d_2)+d_3)}{\Gamma(\tfrac12(1+d_1))\Gamma(\tfrac12(1+d_2))\Gamma(\tfrac12(1-d_1)+d_3)\Gamma(\tfrac12(1-d_2)+d_3)} \times$$

$$ {}_9F_8\left[\begin{matrix} d_3-\tfrac12, d_3, \tfrac14(2d_3+3), \tfrac12(1+a_1), \tfrac12 a_1, \tfrac12(1+a_2), \tfrac12 a_2, \\ \tfrac12, \tfrac14(2d_3-1), d_3-\tfrac12 a_1, d_3+\tfrac12(1-a_1), d_3-\tfrac12 a_2, d_3+\tfrac12(1-a_2), \end{matrix}\right.$$

$$\left.\begin{matrix} \tfrac12 d_1, \tfrac12 d_2 ; 1 \\ \tfrac12(1-d_1)+d_3, \tfrac12(1-d_2)+d_3 ; \end{matrix}\right]$$

$$-\frac{(2d_3+1)a_1a_2d_1d_2\Gamma(\tfrac12)\Gamma(d_3+\tfrac12)\Gamma(\tfrac12(1+d_1+d_2))\Gamma(\tfrac12(1-d_1-d_2)+d_3)}{(2d_3-a_1)(2d_3-a_2)\Gamma(1+\tfrac12 d_1)\Gamma(1+\tfrac12 d_2)\Gamma(1-\tfrac12 d_2+d_3)\Gamma(1-\tfrac12 d_1+d_3)} \times$$

$$ {}_9F_8\left[\begin{matrix} d_3+\tfrac12, d_3, \tfrac14(5+2d_3), \tfrac12(1+a_1), 1+\tfrac12 a_1, \tfrac12(1+a_2), 1+\tfrac12 a_2, \\ 3/2, \tfrac14(1+2d_3), 1+d_3-\tfrac12 a_1, d_3-\tfrac12(a_1-1), 1+d_3-\tfrac12 a_2, d_3-\tfrac12(a_2-1), \end{matrix}\right.$$

$$\left.\begin{matrix} \tfrac12(1+d_1), \tfrac12(1+d_2); 1 \\ 1-\tfrac12 d_1+d_3, 1-\tfrac12 d_2+d_3 ; \end{matrix}\right] \qquad \ldots(1)$$

$$= {}_4F_3\left[\begin{matrix} 2d_3-a_1-a_2, d_1, d_2, d_3 ; 1 \\ 2d_3-a_1, 2d_3-a_2, \tfrac12(1+d_1+d_2) ; \end{matrix}\right]$$

If $\alpha_{2n+1} = 0$, and

$$\alpha_{2n} = \frac{(d_3-\tfrac12)_n(\tfrac14(3+2d_3))_n(a)_n}{n!(\tfrac14(2d_3-1))_n(d_3-a+\tfrac12)_n}$$

then $\beta_n = \dfrac{(d_3-a)_n}{n!(2d_3-2a)_n(d_3)_n}$.

Hence,
$$\frac{\Gamma(\tfrac12)\Gamma(d_3+\tfrac12)\Gamma(\tfrac12(1+d_1+d_2))\Gamma(\tfrac12(1-d_1-d_2)+d_3)}{\Gamma(\tfrac12(1+d_1))\Gamma(\tfrac12(1+d_2))\Gamma(\tfrac12(1-d_1)+d_3)\Gamma(\tfrac12(1-d_2)+d_3)} \times$$

$$ {}_5F_4\left[\begin{matrix} d_3-\tfrac12, \tfrac14(3+2d_3), \tfrac12 d_1, \tfrac12 d_2, a ; 1 \\ \tfrac14(2d_3-1), \tfrac12(1-d_1)+d_3, \tfrac12(1-d_2)+d_3, d_3-a+\tfrac12 ; \end{matrix}\right]$$

$$= {}_3F_2\left[\begin{matrix} d_3-a, d_1, d_2 ; 1 \\ 2d_3-2a, \tfrac12(1+d_1+d_2) ; \end{matrix}\right] \qquad \ldots(2)$$

§ (18.6) Applications of theorems twenty-two and twenty-three

All the transformations due to summation of F_γ by theorem twenty-three will be contained in those due to summation by theorem twenty-two. There are seven possible ways of summing F_γ by theorem twenty-two, of which only two give summable series for F_β. These are:-

(1) $$F_\gamma = {}_4F_3\left[\begin{matrix} f+2n, 1+\tfrac{1}{2}f+n, & d+n, & e; 1 \\ \tfrac{1}{2}f+n, 1+f-d+n, 1+f+2n-e; \end{matrix}\right]$$

$$F_\beta = \sum_{r=0}^{n} \frac{(-n)_r (f+n)_r}{(1+f-e+n)_r (1-e-n)_r} \alpha_r$$

and (2) $$F_\gamma = {}_4F_3\left[\begin{matrix} f+2n, 1+\tfrac{1}{2}f+n, & d_1+n, & d_2+n; 1 \\ \tfrac{1}{2}f+n, 1+f-d_1+n, 1+f-d_2+n; \end{matrix}\right]$$

$$F_\beta = \sum_{r=0}^{n} (-n)_r (f+n)_r (-1)^r \alpha_r.$$

§(18.7) Case one.

If $\delta_n = \dfrac{(1+\frac{1}{2}f)_n (d)_n}{(\frac{1}{2}f)_n (1+f-d)_n}$, then

$$\gamma_n = \frac{\Gamma(\frac{1}{2}(1+f))\Gamma(1+f-d)\Gamma(1+f-e)\Gamma(\frac{1}{2}(1+f)-d-e)}{\Gamma(1+f)\Gamma(\frac{1}{2}(1+f)-d)\Gamma(\frac{1}{2}(1+f)-e)\Gamma(1+f-d-e)} \frac{(\frac{1}{2}(1+f))_n}{(\frac{1}{2}(1+f)-e)_n} \times \frac{(d)_n}{(1+f-d-e)_n}$$

Summing F_β by Saalschutz's theorem, if $\alpha_n = \dfrac{(1-2e)_n}{n!}$

then $$\beta_n = \frac{(f)_n (\frac{1}{2}(f+e))_n (1-e)_n (\frac{1}{2}(1+f+e))_n}{n!\,(1+\frac{1}{2}(f-e))_n (\frac{1}{2}(1+f-e))_n (f+e)_n}$$

and we have

$$\frac{\Gamma(\frac{1}{2}(1+f))\Gamma(1+f-d)\Gamma(1+f-e)\Gamma(\frac{1}{2}(1+f)-d-e)}{\Gamma(1+f)\Gamma(\frac{1}{2}(1+f)-d)\Gamma(\frac{1}{2}(1+f)-e)\Gamma(1+f-d-e)} \, {}_3F_2\left[\begin{matrix} 1-2e, & \frac{1}{2}(1+f), d; 1 \\ \frac{1}{2}(1+f)-e, 1+f-d-e; \end{matrix}\right]$$

$$= {}_6F_5\left[\begin{matrix} f, 1+\frac{1}{2}f, & d, & \frac{1}{2}(1+f+e), & \frac{1}{2}(f+e), 1-e; 1 \\ \frac{1}{2}f, 1+f-d, \frac{1}{2}(1+f-e), 1+\frac{1}{2}(f-e), & f+e; \end{matrix}\right] \qquad \ldots(1)$$

Summing F_β by Dougall's theorem, if $1+f-e = a_1+a_2+a_3+e$,

and $$\alpha_n = \frac{(f-e)_n (1+\frac{1}{2}(f-e))_n (a_1)_n (a_2)_n (a_3)_n}{n!\,(\frac{1}{2}(f-e))_n (1+f-e-a_1)_n (1+f-e-a_2)_n (1+f-e-a_3)_n}$$

then $\beta_n = \dfrac{(f)_n(e+a_1)_n(e+a_2)_n(e+a_3)_n}{n!(1+f-e-a_1)_n(1+f-e-a_2)_n(1+f-e-a_3)_n}$

Hence, $\dfrac{\Gamma(\frac{1}{2}(1+f))\Gamma(1+f-d)\Gamma(1+f-e)\Gamma(\frac{1}{2}(1+f)-d-e)}{\Gamma(1+f)\Gamma(\frac{1}{2}(1+f)-d)\Gamma(\frac{1}{2}(1+f)-e)\Gamma(1+f-d-e)}$

$$\times\ {}_7F_6\left[\begin{matrix} f-e, 1+\frac{1}{2}(f-e), & a_1, & a_2, & a_3, & d, & \frac{1}{2}(1+f); 1 \\ \frac{1}{2}(f-e), & 1+f-e-a_1, & 1+f-e-a_2, & 1+f-e-a_3, & 1+f-e-d, & \frac{1}{2}(1+f)-e; \end{matrix}\right]$$

$$= {}_6F_5\left[\begin{matrix} f, 1+\frac{1}{2}f, & e+a_1, & e+a_2, & e+a_3, & d; 1 \\ \frac{1}{2}f, & 1+f-e-a_1, & 1+f-e-a_2, & 1+f-e-a_3, & 1+f-d; \end{matrix}\right] \quad \ldots(2)$$

provided $Rl(1+f-2d)>0$, and $Rl(e)<0$.

Summing F_β as a well-poised ${}_4F_3(1)$ series,

if $\alpha_n = \dfrac{(f-e)_n(1+\frac{1}{2}(f-e))_n}{n!(\frac{1}{2}(f-e))_n}$ then

$$\beta_n = \frac{(f)_n(\frac{1}{2}(1+f+e))_n}{n!(\frac{1}{2}(1+f-e))_n}.$$

Hence, $\dfrac{\Gamma(\frac{1}{2}(1+f))\Gamma(1+f-d)\Gamma(1+f-e)\Gamma(\frac{1}{2}(1+f)-d-e)}{\Gamma(1+f)\Gamma(\frac{1}{2}(1+f)-d)\Gamma(\frac{1}{2}(1+f)-e)\Gamma(1+f-d-e)}$

$$\times\ {}_4F_3\left[\begin{matrix} f-e, 1+\frac{1}{2}(f-e), & d, & \frac{1}{2}(1+f); 1 \\ \frac{1}{2}(f-e), & 1+f-d-e, & \frac{1}{2}(1+f)-e; \end{matrix}\right]$$

$$= {}_4F_3\left[\begin{matrix} f, 1+\frac{1}{2}f, & d, & \frac{1}{2}(1+f+e); 1 \\ \frac{1}{2}f, & 1+f-d, & \frac{1}{2}(1+f-e); \end{matrix}\right] \quad \ldots(3)$$

This is a transformation between two well-poised ${}_4F_3(1)$ series which are convergent if $Rl(d)<0$ and $Rl(e)<0$.

Iff $\alpha_{2n+1} = 0$, and $\alpha_{2n} = \dfrac{(\frac{1}{2}(f-e))_n(1+\frac{1}{4}(f-e))_n(-e)_n}{n!(\frac{1}{4}(f-e))_n(1+\frac{1}{2}(f+e))_n}$

then $\beta_n = \dfrac{(f)_n(-e)_n(\frac{1}{2}(1+f+e))_n(-1)^n}{n!(1+f+e)_n(\frac{1}{2}(1+f-e))_n}$.

Hence, $\dfrac{\Gamma(\frac{1}{2}(1+f))\Gamma(1+f-d)\Gamma(1+f-e)\Gamma(\frac{1}{2}(1+f)-d-e)}{\Gamma(1+f)\Gamma(\frac{1}{2}(1+f)-d)\Gamma(\frac{1}{2}(1+f)-e)\Gamma(1+f-d-e)}$

$$\times\ {}_7F_6\left[\begin{matrix} \frac{1}{2}(f-e), 1+\frac{1}{4}(f-e), & -e, & \frac{1}{4}(1+f), & \frac{1}{4}(3+f), & \frac{1}{2}d, & \frac{1}{2}(1+d); 1 \\ \frac{1}{4}(f-e), & 1+\frac{1}{2}(f+e), & \frac{1}{4}(3+f)-\frac{1}{2}e, & \frac{1}{4}(1+f)-\frac{1}{2}e, & 1+\frac{1}{2}(f-d-e), & \frac{1}{2}(1+f-d-e); \end{matrix}\right]$$

$$= {}_5F_4\left[\begin{matrix} f, 1+\tfrac{1}{2}f, & -e, & d, \tfrac{1}{2}(1+f+e); & -1 \\ \tfrac{1}{2}f, & 1+f+e, & 1+f-d, \tfrac{1}{2}(1+f-e); & \end{matrix}\right] \qquad \ldots(4)$$

provided that $Rl(1+f-2d) > 0$, and $Rl(1+e) > 0$.

§ (18.8) The second case.

If $\delta_n = \dfrac{(1+\tfrac{1}{2}f)_n (d)_n (-N)_n}{n!\,(\tfrac{1}{2}f)_n (1+f-d)_n (1+f+N)_n}$, then

$$\gamma_n = \frac{(1+f)_N (\tfrac{1}{2}(1+f)-d)_N (\tfrac{1}{2}(1+f))_n (d)_n (-N)_n (-1)^n}{(\tfrac{1}{2}(1+f))_N (1+f-d)_N (\tfrac{1}{2}(1-f)+d-N)_n}.$$

Summing F_β by Saalschutz's theorem, if

$\alpha_n = \dfrac{(a)_n (-1)^n}{n!\,(A)_n (1+f+a-A)_n}$, then

$$\beta_n = \frac{(f)_n (A-a)_n (1+f-A)_n}{n!(A)_n (1+f+a-A)_n}.$$

Hence, $\dfrac{(1+f)_N (\tfrac{1}{2}(1+f)-d)_N}{(\tfrac{1}{2}(1+f))_N (1+f-d)_N}\, {}_4F_3\left[\begin{matrix} a, \tfrac{1}{2}(1+f), d, & -N; & 1 \\ 1+f+a-A, A, \tfrac{1}{2}(1-f)+d-N; & & \end{matrix}\right]$

$$= {}_6F_5\left[\begin{matrix} f, 1+\tfrac{1}{2}f, & d, & A-a, 1+f-A; & -N; & 1 \\ \tfrac{1}{2}f, & 1+f-d, & 1+f+a-A, & A, 1+f+N; & \end{matrix}\right]$$

§ (19.1) Summation of F_β by theorem twenty-two.

Suppose $\beta_n = \dfrac{1}{n!(1+f)_n} \displaystyle\sum_{r=0}^{n} \dfrac{(-n)_r (-1)^r}{(1+f+n)_r} \alpha_r$.

Let $\alpha_n = \dfrac{(f)_n (1+\tfrac{1}{2}f)_n (a)_n (-1)^n}{n!(\tfrac{1}{2}f)_n (1+f-a)_n}$, and sum F_β by theorem (22).

Then $\beta_n = \dfrac{(\tfrac{1}{2}(1+f)-a)_n}{n!(\tfrac{1}{2}(1+f))_n (1+f-a)_n}$. If $\delta_n = (d_1)_n (d_2)_n$, then

$\gamma_n = \dfrac{(d_1)_n (d_2)_n (-1)^n}{(1+f-d_1)_n (1+f-d_2)_n}$, summing F_γ by Gauss's theorem

(as in § 4.3).

Hence,

$$\frac{\Gamma(1+f)\Gamma(1+f-d_1-d_2)}{\Gamma(1+f-d_1)\Gamma(1+f-d_2)}\,{}_5F_4\left[\begin{matrix} f, 1+\tfrac{1}{2}f, & a, & d_1, & d_2; & -1 \\ & \tfrac{1}{2}f, 1+f-a, & 1+f-d_1, & 1+f-d_2; & \end{matrix}\right]$$

$$= {}_3F_2\left[\begin{matrix} \tfrac{1}{2}(1+f)-a, & d_1, & d_2; & 1 \\ & \tfrac{1}{2}(1+f), 1+f-a & ; & \end{matrix}\right] \qquad \ldots(1)$$

provided that $Rl(1+f-d_1-d_2)>0$ and that $Rl(a)<0$.

If $\delta_n = \dfrac{(d_1)_n(d_2)_n(-N)_n}{(f+d_1+d_2-N)_n}$, summing F_γ by Saalschutz's theorem, then $\gamma_n = \dfrac{(1+f-d_1)_N(1+f-d_2)_N(d_1)_n(d_2)_n(-N)_n(-1)^n}{(1+f)_N(1+f-d_1-d_2)_N(1+f-d_1)_n(1+f-d_2)_n(1+f+N)_n}$

(as in § 5.2).

Hence,
$$\frac{(1+f-d_1)_N(1+f-d_2)_N}{(1+f)_N(1+f-d_1-d_2)_N}\cdot{}_6F_5\left[\begin{matrix} f, 1+\tfrac{1}{2}f, & a, & d_1, & d_2, & -N; & 1 \\ & \tfrac{1}{2}f, 1+f-a, & 1+f-d_1, & 1+f-d_2, & 1+f+N; & \end{matrix}\right]$$

$$= {}_4F_3\left[\begin{matrix} \tfrac{1}{2}(1+f)-a, & d_1, & d_2, & -N; & 1 \\ & \tfrac{1}{2}(1+f), 1+f-a, & d_1+d_2+f-N & ; & \end{matrix}\right] \qquad \ldots(2)$$

If $\delta_n = \dfrac{(\tfrac{1}{2}f_1)_n(\tfrac{1}{2}(1+f_1))_n(f_2+N)_n(-N)_n}{(\tfrac{1}{2}f_2)_n(\tfrac{1}{2}(1+f_2))_n}$

summing F_γ by theorem (19), (as in § 8.5),

$$\gamma_n = \frac{(\tfrac{1}{2}f_1)_n(-N)_n(-1)^n(f_2-f_1)_N}{(1+\tfrac{1}{2}f_1)_n(1+f_1-f_2-N)_n(f_2)_N}.$$

Hence,
$$\frac{(f_2-f_1)_N}{(f_2)_N}\,{}_3F_2\left[\begin{matrix} f_1, & a, & -N; & 1 \\ & 1+f_1-a, & 1+f_1-f_2-N; & \end{matrix}\right]$$

$$= {}_4F_3\left[\begin{matrix} \tfrac{1}{2}(1+f_1)-a, & \tfrac{1}{2}f_1, f_2+N, & -N; & 1 \\ & 1+f_1-a, \tfrac{1}{2}f_2, \tfrac{1}{2}(1+f_2); & & \end{matrix}\right] \qquad \ldots(3)$$

If $\delta_n = \dfrac{(d_1)_n(d_2)_n(d_3)_n}{(\tfrac{1}{2}(1+d_1+d_2))_n}$, summing F_γ by theorem (21), Watson's theorem, (as in § 18.5), then

$$\gamma_{2n} = \frac{\Gamma(\tfrac{1}{2})\Gamma(d_3+\tfrac{1}{2})\Gamma(\tfrac{1}{2}(1+d_1+d_2))\Gamma(\tfrac{1}{2}(1-d_1-d_2)+d_3)}{\Gamma(\tfrac{1}{2}(1+d_1))\Gamma(\tfrac{1}{2}(1+d_2))\Gamma(\tfrac{1}{2}(1-d_1)+d_3)\Gamma(\tfrac{1}{2}(1-d_2)+d_3)} \times \frac{(\tfrac{1}{2}d_1)_n(\tfrac{1}{2}d_2)_n}{(\tfrac{1}{2}(1-d_1)+d_3)_n(\tfrac{1}{2}(1-d_2)+d_3)_n}$$

and $$\gamma_{2n+1} = \frac{\Gamma(\tfrac{1}{2})\Gamma(d_3+\tfrac{1}{2})\Gamma(\tfrac{1}{2}(1+d_1+d_2))\Gamma(\tfrac{1}{2}(1-d_1-d_2)+d_3)}{\Gamma(1+\tfrac{1}{2}d_1)\Gamma(1+\tfrac{1}{2}d_2)\Gamma(1-\tfrac{1}{2}d_1+d_3)\Gamma(1-\tfrac{1}{2}d_2+d_3)}$$
$$\times \frac{d_1 d_2 (\tfrac{1}{2}(1+d_1))_n (\tfrac{1}{2}(1+d_2))_n}{(1-\tfrac{1}{2}d_1+d_3)_n (1-\tfrac{1}{2}d_2+d_3)_n}$$

(putting $2d_3-1$ for f).

Hence, $$\frac{\Gamma(\tfrac{1}{2})\Gamma(d_3+\tfrac{1}{2})\Gamma(\tfrac{1}{2}(1+d_1+d_2))\Gamma(\tfrac{1}{2}(1-d_1-d_2)+d_3)}{\Gamma(\tfrac{1}{2}(1+d_1))\Gamma(\tfrac{1}{2}(1+d_2))\Gamma(\tfrac{1}{2}(1-d_1)+d_3)\Gamma(\tfrac{1}{2}(1-d_2)+d_3)}$$
$$\times {}_7F_6\left[\begin{matrix} d_3-\tfrac{1}{2}, d_3, \tfrac{1}{2}a, \tfrac{1}{2}(1+a), \tfrac{1}{2}d_1, \tfrac{1}{2}d_2, \tfrac{1}{4}(2d_3+3); 1 \\ \tfrac{1}{2}, d_3-\tfrac{1}{2}(a-1), d_3-\tfrac{1}{2}a, \tfrac{1}{2}(1-d_1)+d_3, \tfrac{1}{2}(1-d_2)+d_3, \tfrac{1}{4}(2d_3-1); \end{matrix}\right]$$
$$- \frac{d_1 d_2 (2d_3-1)\, a\, \Gamma(\tfrac{1}{2})\Gamma(d_3+\tfrac{1}{2})\Gamma(\tfrac{1}{2}(1+d_1+d_2))\Gamma(\tfrac{1}{2}(1-d_1-d_2)+d_3)}{(2d_3-a)\Gamma(1+\tfrac{1}{2}d_1)\Gamma(1+\tfrac{1}{2}d_2)\Gamma(1-\tfrac{1}{2}d_1+d_3)\Gamma(1-\tfrac{1}{2}d_2+d_3)}$$
$$\times {}_7F_6\left[\begin{matrix} d_3, d_3+\tfrac{1}{2}, \tfrac{1}{2}(1+a), 1+\tfrac{1}{2}a, \tfrac{1}{2}(1+d_1), \tfrac{1}{2}(1+d_2), \tfrac{1}{4}(2d_3+5); 1 \\ 3/2, d_3-\tfrac{1}{2}a+1, d_3+\tfrac{1}{2}(1-a), 1-\tfrac{1}{2}d_1+d_3, 1-\tfrac{1}{2}d_2+d_3, \tfrac{1}{4}(2d_3+1); \end{matrix}\right] = {}_3F_2\left[\begin{matrix} d_3-a, d_1, d_2; 1 \\ 2d_3-a, \tfrac{1}{2}(1+d_1+d_2); \end{matrix}\right] \quad \ldots(4),$$

provided that $Rl(1+2d_3-d_1-d_2) > 0$ and that $Rl(a) < 0$.

If $a = 0$, this transformation reduces to Watson's theorem.

§ (19.2) The same cases when $\alpha_{2n+1} = 0$.

If $\alpha_{2n+1} = 0$ and $\alpha_{2n} = \dfrac{(\tfrac{1}{2}f)_n (1+\tfrac{1}{4}f)_n}{n!(\tfrac{1}{4}f)_n}$

then, summing the same series for F_β by theorem (22),

$\beta_n = \dfrac{(\tfrac{1}{4}f)_n}{n!(\tfrac{1}{2}f)_n(\tfrac{1}{2}(1+f))_n}$. Hence, with the same values of

δ_n and γ_n as in the previous paragraph, we have

$$\frac{\Gamma(1+f)\Gamma(1+f-d_1-d_2)}{\Gamma(1+f-d_1)\Gamma(1+f-d_2)}\,{}_6F_5\left[\begin{matrix}\tfrac{1}{2}f, 1+\tfrac{1}{4}f, & \tfrac{1}{2}d_1, & \tfrac{1}{2}(1+d_1), & \tfrac{1}{2}d_2, & \tfrac{1}{2}(1+d_2); 1\\ \tfrac{1}{4}f, & 1+\tfrac{1}{2}(f-d_1), & \tfrac{1}{2}(1+f-d_1), & 1+\tfrac{1}{2}(f-d_2), & \tfrac{1}{2}(1+f-d_2);\end{matrix}\right] = {}_3F_2\left[\begin{matrix}\tfrac{1}{4}f, & d_1, & d_2; 1\\ \tfrac{1}{2}f, & \tfrac{1}{2}(1+f);\end{matrix}\right] \quad \ldots(1)$$

provided that $Rl(1+f-d_1-d_2)>0$ and that $Rl(f)>0$.

$$\frac{(1+f-d_1)_N(1+f-d_2)_N}{(1+f)_N(1+f-d_1-d_2)_N}\,{}_8F_7\left[\begin{matrix}\tfrac{1}{2}f, 1+\tfrac{1}{4}f, & \tfrac{1}{2}d_1, & \tfrac{1}{2}(1+d_1), & \tfrac{1}{2}d_2, & \tfrac{1}{2}(1+d_2), & -\tfrac{1}{2}N, & \tfrac{1}{2}(1-N); 1\\ \tfrac{1}{4}f, & 1+\tfrac{1}{2}f-\tfrac{1}{2}d_1, & \tfrac{1}{2}(1+f-d_1), & 1+\tfrac{1}{2}f-\tfrac{1}{2}d_2, & \tfrac{1}{2}(1+f-d_2), & 1+\tfrac{1}{2}f+\tfrac{1}{2}N, & \tfrac{1}{2}(1+f+N);\end{matrix}\right]$$

$$= {}_4F_3\left[\begin{matrix}\tfrac{1}{4}f, & d_1, & d_2, & -N; 1\\ \tfrac{1}{2}f, & \tfrac{1}{2}(1+f), & f+d_1+d_2-N;\end{matrix}\right] \quad \ldots(2)$$

$$\frac{(f_2-f_1)_N}{(f_2)_N}\,{}_3F_2\left[\begin{matrix}\tfrac{1}{2}f_1, & -\tfrac{1}{2}N, & \tfrac{1}{2}(1-N); 1\\ \tfrac{1}{2}(1+f_1-f_2-N), & 1+\tfrac{1}{2}(f_1-f_2-N);\end{matrix}\right]$$

$$= {}_3F_2\left[\begin{matrix}\tfrac{1}{4}f_1, & f_2+N, & -N; 1\\ \tfrac{1}{2}f_2, & \tfrac{1}{2}(1+f_2);\end{matrix}\right] \quad \ldots(3)$$

and

$$\frac{\Gamma(\tfrac{1}{2})\Gamma(d_3+\tfrac{1}{2})\Gamma(\tfrac{1}{2}(1+d_1+d_2))\Gamma(\tfrac{1}{2}(1-d_1-d_2)+d_3)}{\Gamma(\tfrac{1}{2}(1+d_1))\Gamma(\tfrac{1}{2}(1+d_2))\Gamma(\tfrac{1}{2}(1-d_1)+d_3)\Gamma(\tfrac{1}{2}(1-d_2)+d_3)}$$

$$\times\ {}_4F_3\left[\begin{matrix}\tfrac{1}{2}d_1, & \tfrac{1}{2}d_2, & d_3-\tfrac{1}{2}, & \tfrac{1}{4}(3+2d_3); 1\\ \tfrac{1}{2}(1-d_1)+d_3, & \tfrac{1}{2}(1-d_2)+d_3, & \tfrac{1}{4}(2d_3-1);\end{matrix}\right]$$

$$= {}_3F_2\left[\begin{matrix}\tfrac{1}{4}(2d_3-1), & d_1, & d_2; 1\\ d_3-\tfrac{1}{2}, & \tfrac{1}{2}(1+d_1+d_2);\end{matrix}\right] \quad \ldots(4)$$

provided that $Rl(1+2d_3-2d_1-2d_2)>0$,

which transforms a well-poised ${}_4F_3(+1)$ series into Watson's ${}_3F_2(1)$ series.

§(19.3) <u>Another group of F_p sums using theorem twenty-two.</u>

If $\alpha_n = \dfrac{(f-e)_n(1+\frac{1}{2}(f-e))_n}{n!\,(\frac{1}{2}(f-e))_n}$, and

$$\beta_n = \frac{(e)_n(f)_n}{n!\,(F)_n}\sum_{r=0}^{n}\frac{(-n)_r(f+n)_r}{(1-e-n)_r(F+n)_r}\alpha_r$$

putting F = 1+f−e, and summing F_β by theorem (22), then

$$\beta_n = \frac{(f)_n(\tfrac{1}{2}(1+f+e))_n}{n!\,(\tfrac{1}{2}(1+f-e))_n}.$$

If $\delta_n = 1$, summing F_γ by Gauss's theorem (as in § 4.42)

then $\gamma_n = \dfrac{\Gamma(1+f-e)\Gamma(1-2e)\,(f)_{2n}}{\Gamma(1-e)\Gamma(1+f-2e)(1+f-2e)_{2n}}$.

Hence, $\dfrac{\Gamma(1+f-e)\Gamma(1-2e)}{\Gamma(1-e)\Gamma(1+f-2e)}\,{}_4F_3\left[\begin{matrix}f-e,1+\tfrac{1}{2}(f-e), & \tfrac{1}{2}f, & \tfrac{1}{2}(1+f); & 1\\ \tfrac{1}{2}(f-e), & 1+\tfrac{1}{2}f-e, & \tfrac{1}{2}(1+f)-e; & \end{matrix}\right]$

$$= {}_2F_1\left[\begin{matrix}f,\tfrac{1}{2}(1+f+e); & 1\\ \tfrac{1}{2}(1+f-e); & \end{matrix}\right] \qquad \ldots(1)$$

provided that $Rl(f+e)<0$ and $Rl(1-2e)>0$.

If $\delta_n = \dfrac{(-N)_n}{(-N+2e)_n}$, summing F_γ by Saalschutz's theorem,

(as in § 5.3), then

$$\gamma_n = \frac{(1+f-2e)_N(1-e)_N(1+f+N-2e)_n(\tfrac{1}{2}f)_n(\tfrac{1}{2}(1+f))_n(-N)_n}{(1+f-e)_N(1-2e)_N(e-N)_n(\tfrac{1}{2}f+1-e)_n(\tfrac{1}{2}(1+f)-e)_n(1+f-e+N)_n}$$

Hence,

$$\frac{(1+f-2e)_N(1-e)_N}{(1+f-e)_N(1-2e)_N}\,{}_6F_5\left[\begin{matrix}f-e,1+\tfrac{1}{2}(f-e), & 1+f+N-2e, & \tfrac{1}{2}(1+f), & \tfrac{1}{2}f; & 1\\ \tfrac{1}{2}(f-e), & e-N, & \tfrac{1}{2}(1+f)-e, & 1+\tfrac{1}{2}f-e; & \end{matrix}\right]$$

$$= {}_3F_2\left[\begin{matrix}f,\tfrac{1}{2}(1+f+e), & -N; & 1\\ \tfrac{1}{2}(1+f-e), & 2e-N; & \end{matrix}\right] \qquad \ldots(2)$$

If $\delta_n = \dfrac{(1+\tfrac{1}{2}f)_n(d_1)_n(d_2)_n(d_3)_n(-N)_n}{(\tfrac{1}{2}f)_n(1+f-d_1)_n(1+f-d_2)_n(1+f-d_3)_n(1+f+N)_n}$

summing F_γ by Dougall's theorem, (as in § 6.3),

$$\gamma_n = \frac{(1+f)_N(1+f-e-d_1)_N(1+f-e-d_2)_N(1+f-d_1-d_2)_N(d_1)_n(d_2)_n}{(1+f-e)_N(1+f-d_1)_N(1+f-d_2)_N(1+f-e-d_1-d_2)_N(1+f-e-d_1)_n}$$
$$\times\frac{(d_3)_n(-N)_n}{(1+f-e-d_2)_n(1+f-e-d_3)_n(1+f-e+N)_n}$$

Hence,

$$\frac{(1+f)_N(1+f-e-d_1)_N(1+f-e-d_2)_N(1+f-d_1-d_2)_N}{(1+f-e)_N(1+f-d_1)_N(1+f-d_2)_N(1+f-e-d_1-d_2)_N}$$

$$\times\ {}_6F_5\left[\begin{matrix} f-e,\ 1+\tfrac{1}{2}(f-e), & d_1, & d_2, & d_3, & -N;\ 1 \\ \tfrac{1}{2}(f-e), & 1+f-e-d_1, & 1+f-e-d_2, & 1+f-e-d_3, & 1+f-e+N; \end{matrix}\right]$$

$$= {}_7F_6\left[\begin{matrix} f,\ 1+\tfrac{1}{2}f,\ \tfrac{1}{2}(1+f+e), & d_1, & d_2, & d_3, & -N;\ 1 \\ \tfrac{1}{2}f,\ \tfrac{1}{2}(1+f-e), & 1+f-d_1, & 1+f-d_2, & 1+f-d_3, & 1+f+N; \end{matrix}\right] \quad \ldots(3)$$

where $1+2f = e+d_1+d_2+d_3-N$.

If $\delta_n = \dfrac{(-N)_n}{(1+2e-N)_n}$, summing F_γ by theorem (13),

(as in § 7.45) then

$$\gamma_n = \frac{(f-2e)_N(1+\tfrac{1}{2}f-e)_N(-e)_N(f)_{2n}(f-2e+N)_n(-N)_n}{(1+f-e)_N(\tfrac{1}{2}f-e)_N(-2e)_N(1+f-2e)_{2n}(1+f-e+N)_n(1+e-N)_n}$$

Hence, $\dfrac{(f-2e)_N(1+\tfrac{1}{2}f-e)_N(-e)_N}{(1+f-e)_N(\tfrac{1}{2}f-e)_N(-2e)_N}$

$$\times\ {}_6F_5\left[\begin{matrix} f-e,\ 1+\tfrac{1}{2}(f-e), & \tfrac{1}{2}(1+f), & \tfrac{1}{2}f,\ f-2e+N, & -N;\ 1 \\ \tfrac{1}{2}(f-e), & \tfrac{1}{2}(1+f)-e, & 1+\tfrac{1}{2}f-e,\ 1+e-N, & 1+f-e+N; \end{matrix}\right]$$

$$= {}_3F_2\left[\begin{matrix} f,\ \tfrac{1}{2}(1+f+e), & -N;\ 1 \\ \tfrac{1}{2}(1+f-e), & 1+2e-N; \end{matrix}\right] \quad \ldots(4)$$

If $\delta_n = \dfrac{(1+\tfrac{1}{2}f)_n(-N)_n}{(\tfrac{1}{2}f)_n(1+2e-N)_n}$, summing F_γ by theorem (14)

(as in § 7.61) then

$$\gamma_n = \frac{(f-2e)_N(-e)_N(1+\tfrac{1}{2}f)_n(\tfrac{1}{2}(1+f))_n(f-2e+N)_n(-N)_n}{(1+f-e)_N(-2e)_N(\tfrac{1}{2}f-e)_n(1+e-N)_n(1+f-e+N)_n(\tfrac{1}{2}(1+f)-e)_n}$$

Hence, $\dfrac{(f-2e)_N(-e)_N}{(1+f-e)_N(-2e)_N}\ {}_6F_5\left[\begin{matrix} f-e,\ 1+\tfrac{1}{2}(f-e),\ 1+\tfrac{1}{2}f,\ \tfrac{1}{2}(1+f), \\ \tfrac{1}{2}(f-e),\ \tfrac{1}{2}f-e,\ \tfrac{1}{2}(1+f)-e, \end{matrix}\right.$

$$\left.\begin{matrix} f-2e+N, & -N;\ 1 \\ 1+e-N, & 1+f-e+N; \end{matrix}\right]$$

$$= {}_4F_3\left[\begin{matrix} f,\ 1+\tfrac{1}{2}f,\ \tfrac{1}{2}(1+f+e), & -N;\ 1 \\ \tfrac{1}{2}f,\ \tfrac{1}{2}(1+f-e), & 1+2e-N; \end{matrix}\right] \quad \ldots(5)$$

If $\delta_n = \dfrac{(1+\tfrac{1}{2}f)_n(d)_n}{(\tfrac{1}{2}f)_n(1+f-d)_n}$, summing F_γ by theorem (22)

(as in § 18.7) then

$$\Upsilon_m = \frac{\Gamma(\frac{1}{2}(1+f))\Gamma(1+f-d)\Gamma(1+f-e)\Gamma(\frac{1}{2}(1+f)-d-e)(\frac{1}{2}(1+f))_n(d)_n}{\Gamma(1+f)\Gamma(\frac{1}{2}(1+f)-d)\Gamma(\frac{1}{2}(1+f)-e)\Gamma(1+f-d-e)(\frac{1}{2}(1+f)-e)_n} \times \frac{1}{(1+f-e-d)_n}$$

Hence,

$$\frac{\Gamma(\frac{1}{2}(1+f))\Gamma(1+f-d)\Gamma(1+f-e)\Gamma(\frac{1}{2}(1+f)-d-e)}{\Gamma(1+f)\Gamma(\frac{1}{2}(1+f)-d)\Gamma(\frac{1}{2}(1+f)-e)\Gamma(1+f-d-e)}$$

$$\times\ {}_4F_3\left[\begin{matrix} f-e, 1+\frac{1}{2}(f-e), \frac{1}{2}(1+f), d;\ 1 \\ \frac{1}{2}(f-e), \frac{1}{2}(1+f)-e, 1+f-d-e; \end{matrix}\right]$$

$$= {}_4F_3\left[\begin{matrix} f, 1+\frac{1}{2}f, d, \frac{1}{2}(1+f+e);\ 1 \\ \frac{1}{2}f, 1+f-d, \frac{1}{2}(1+f-e); \end{matrix}\right] \qquad \ldots(6)$$

provided that $Rl(2d+e) < 0$.

This concludes the examples (which can be deduced from theorem (1)) of transformations between ordinary hypergeometric series.

§ (19.4) <u>The convergence of a certain double series.</u>

When the series involved in the above transformations are infinite, to justify the change in the order of summation of the double series (which is involved in theorem one) it is necessary to consider the convergence of this double series.

It has already been shown (§ 3.1, eq. 4) that the general form of the series is $S =$

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{(a_1)_n \cdots \cdots (d_1)_{n+r} \cdots \cdots (f_1)_{2n+r} \cdots \cdots (e_1)_r \cdots}{(A_1)_n \cdots \cdots (D_1)_{n+r} \cdots \cdots (F_1)_{2n+r} \cdots \cdots (E_1)_r \cdots}$$

($A_1 = 1$, and $E_1 = 1$)

where, for the moment, we suppose that the parameters are all real, and that none of the denominator parameters are negative integers.

Let n_a = the number of 'a' parameters,

n_A = the number of 'A' parameters, (including $A_1=1$)

and so on.

Then, for S to be a hypergeometric series of the type ${}_{k+1}F_k(1)$, it is necessary that

$$n_a+n_d+n_f+n_e = n_A+n_D+n_F+n_E. \qquad \ldots(1)$$

Since none of the parameters are infinite, there exists a positive integer K, such that

$a_1,\ldots\ \ldots d_1,\ldots\ \ldots f_1,\ldots\ \ldots e_1,\ldots\ \ldots A_1,\ldots\ \ldots D_1,\ldots\ \ldots F_1,\ldots$ $\ldots E_1,\ldots$ are all less than K.

The series S can then be split up, thus,

$$S = \left[\sum_{n=0}^{K} \frac{(a_1)_n\ldots\ \ldots(d_1)_n\ldots\ \ldots(f_1)_{2n}\ldots}{(A_1)_n\ldots\ \ldots(D_1)_n\ldots\ \ldots(F_1)_{2n}\ldots} + \frac{(a_1)_K\ldots\ \ldots(d_1)_K\ldots\ \ldots(f_1)_{2K}\ldots}{(A_1)_K\ldots\ \ldots(D_1)_K\ldots\ \ldots(F_1)_{2K}\ldots}\right.$$

$$\left.\times \sum_{m=0}^{\infty} \frac{(a_1+K)_m\ldots\ \ldots(d_1+K)_m\ldots\ \ldots(f_1+2K)_{2m}\ldots}{(A_1+K)_m\ldots\ \ldots(D_1+K)_m\ldots\ \ldots(F_1+2K)_{2m}\ldots}\right]$$

$$\times \left[\sum_{r=0}^{K} \frac{(d_1+n)_r\ldots\ \ldots(f_1+2n)_r\ldots\ \ldots(e_1)_r\ldots}{(D_1+n)_r\ldots\ \ldots(F_1+2n)_r\ldots\ \ldots(E_1)_r\ldots} + \frac{(d_1+n)_K\ldots\ \ldots(f_1+2n)_K\ldots\ \ldots(e_1)_K\ldots}{(D_1+n)_K\ldots\ \ldots(F_1+2n)_K\ldots\ \ldots(E_1)_K\ldots}\right.$$

$$\left.\times \sum_{s=0}^{\infty} \frac{(d_1+n+K)_s\ldots\ \ldots(f_1+2n+K)_s\ldots\ \ldots(e_1+K)_s\ldots}{(D_1+n+K)_s\ldots\ \ldots(F_1+2n+K)_s\ldots\ \ldots(E_1+K)_s\ldots}\right]$$

i.e. $S = (S_1+P_1S_2)(S_3+P_2S_4)$, say.

S_1 is a finite series, and P_1 a finite product. S_3 is a finite series and P_2 a finite product, for all values of n, provided that

$$2n_F+n_D \geqslant 2n_f+n_d. \qquad ...(2)$$

The problem is thus reduced to the consideration of the double series

$$S_2S_4 = \sum_{m=0}^{\infty} \frac{(a_1+K)_m\ldots\ \ldots(d_1+K)_m\ldots\ \ldots(f_1+2K)_{2m}\ldots}{(A_1+K)_m\ldots\ \ldots(D_1+K)_m\ldots\ \ldots(F_1+2K)_{2m}\ldots}$$
$$\times \sum_{s=0}^{\infty} \frac{(d_1+2K+m)_s\ldots\ \ldots(f_1+3K+2m)_s\ldots\ \ldots(e_1+K)_s\ldots}{(D_1+2K+m)_s\ldots\ \ldots(F_1+3K+2m)_s\ldots\ \ldots(E_1+K)_s\ldots}$$

all the terms of which are positive.

S_4 converges if condition (2) is satisfied, and

$$D_1+\ldots\ \ldots+F_1+\ldots\ \ldots+E_1+\ldots\ \ldots-d_1-\ldots\ \ldots-f_1-\ldots\ \ldots-e_1-\ldots$$
$$+K(2n_D+3n_F+n_E-2n_d-3n_f-n_e)>1,$$

i.e.

$$D_1+\ldots\ \ldots+F_1+\ldots\ \ldots+E_2+\ldots\ \ldots-d_1-\ldots\ \ldots-f_1-\ldots\ \ldots-e_1-\ldots$$
$$+K(n_A-n_a)>0, \quad \text{(by (1) above)} \qquad ...(3)$$

(under conditions two and three, S_4 is a positive decreasing function of m) and S_2 converges if

$$A_2+\ldots\ \ldots+D_1+\ldots\ \ldots+F_1+\ldots\ \ldots-a_1-\ldots\ \ldots-d_1-\ldots\ \ldots-f_1-\ldots$$
$$+K(n_A-n_a)>0 \qquad ...(4)$$

Hence the double series S converges under conditions (2), (3) and (4).

By analytic continuation this result can be extended to complex values of the parameters, provided that none of the denominator parameters are negative integers.

Conditions (3) and (4) then become

$$Rl(D_1+\ldots\ldots+F_1+\ldots\ldots+E_2+\ldots\ldots-d_1-\ldots\ldots-f_1-\ldots\ldots-e_1-\ldots) +K(n_A-n_a)>0 \qquad \ldots(5)$$

and

$$Rl(A_2+\ldots\ldots+D_1+\ldots\ldots+F_1+\ldots\ldots-a_1-\ldots\ldots-d_1-\ldots\ldots-f_1-\ldots) +K(n_A-n_a)>0 \qquad \ldots(6)$$

For example, in § 4.2 condition (2) is not satisfied and summation is only possible in terms of finite series. In § 4.3, however, condition two is satisfied, and conditions (5) and (6) are the given conditions for convergence.

§ (20.1) Basic series; notation and preliminary definitions.

We turn now to a consideration of the generalised basic hypergeometric series

$$1+\frac{(1-a_1)(1-a_2)\ldots\ldots(1-a_p)}{(1-q)(1-b_1)(1-b_2)\ldots\ldots(1-b_{p-1})}z$$
$$+\frac{(1-a_1)(1-a_1q)(1-a_2)(1-a_2q)\ldots\ldots(1-a_p)(1-a_pq)}{(1-q)(1-q^2)(1-b_1)(1-b_1q)(1-b_2)\ldots\ldots(1-b_{p-1})(1-b_{p-1}q)}z^2$$
$$+\ldots$$

where $a_1,a_2,\ldots a_p$, $b_1,b_2,\ldots\ldots b_{p-1}$, q and z are all complex numbers, and, for convergence, $|q|<1$, $|z|<1$.

This series will now be investigated by the previous method, using theorem (1). Following Watson's notation ("A new proof of the R.-R. identities" Journal Lond. Maths. Soc. 4 (1929)), we shall denote this series by

$${}_{p}\Phi_{p-1}\left[\begin{matrix} a_1, \ldots \ldots a_p ; z \\ b_1 \ldots b_{p-1} ; \end{matrix}\right]$$

This notation differs from that originally used by Heine, in that a_r is written for q^{a_r}, and b_r for q^{b_r}.

Similarly, we shall denote the series

$$1 + \frac{(1-a_1)(1-a_2)\ldots\ldots(1-a_p)}{(1-q^2)(1-b_1)(1-b_2)\ldots\ldots(1-b_{p-1})} z$$

$$+ \frac{(1-a_1)(1-a_1 q^2)(1-a_2)(1-a_2 q^2)\ldots\ldots(1-a_p)(1-a_p q^2) z^2}{(1-q^2)(1-q^4)(1-b_1)(1-b_1 q^2)\ldots\ldots(1-b_{p-1} q^2)}$$

by ${}_{p}\Phi_{p-1}^{q^2}\left[\begin{matrix} a_1, a_2, \ldots \ldots a_p ; z \\ b_1, b_2, \ldots . b_{p-1} ; \end{matrix}\right]$

Let $(a)_n \equiv (a)_{q,n} \equiv (1-a)(1-aq)(1-aq^2)\ldots\ldots(1-aq^{n-1})$, where $(a)_{q,o} \equiv 1$.

Also let

$(a)_{q^k,n} \equiv (1-a)(1-aq^k)(1-aq^{2k})\ldots\ldots(1-aq^{k(n-1)})$,

$\Pi(a) \equiv \prod_{n=o}(1-aq^n) = \prod_q(a)$,

and, similarly,

$$\prod_{q^k}(a) = \prod_{n=o}(1-aq^{kn}).$$

With these definitions,

$$\frac{\Pi(a)}{\Pi(aq^N)} = (a)_N$$

$$(aq^{-n})_n = (-a)^n q^{-\frac{1}{2}n(n+1)}(q/a)_n$$

$$(a)_{N-n} = \frac{(a)_N}{(aq^{N-n})_n}$$

Also, $(a)_{2n} = (\sqrt{a})_n(-\sqrt{a})_n(\sqrt{aq})_n(-\sqrt{aq})_n$

$(a)_{q,2n} = (a)_{q^2,n}(aq)_{q^2,n}$.

(For further relations among such products, see appendix 2).

We can extend the definitions, as before, by defining $\dfrac{\prod(a)}{\prod(aq^b)}$ as $(a)_{q,b}$ where b is not necessarily an integer. All the relations, given in appendix (2), between such products, are true with this extended definition. Also, in particular, $(a)_{q,2n} = (a)_{q^2,n}\,(aq)_{q^2,n}$.

It is obvious that the product $1/\prod(a)$ will play very much the same kind of role in the study of basic series, as the gamma function $\Gamma(a)$ played in the study of the ordinary hypergeometric series. Thus, analogous to the Lagrange duplication formula, and the Gauss multiplication formula, we have the simple results,

$$\prod_q(a)\prod_q(a\sqrt{q}) = \prod_{\sqrt{q}}(a) \quad \text{and, in general,}$$

$$\prod_q(a)\prod_q(aq^{1/k})\prod_q(aq^{2/k}) \ldots \ldots \prod_q(aq^{(k-1)/k})$$
$$= \prod_{q^{1/k}}(a).$$

§ (20.2) Various summation theorems for basic series.

We first prove the basic analogue of Dougall's theorem. This is

Theorem (24) (Jackson's theorem)

$${}_8\Phi_7\left[\begin{matrix}a, q\sqrt{a}, -q\sqrt{a}, & b, & c, & d, & e, & f; q\\ \sqrt{a}, -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f;\end{matrix}\right]$$

$$= \frac{\Pi(aq)\Pi(aq/bc)\Pi(aq/cd)\Pi(aq/bd)\Pi(aq/be)\Pi(aq/ce)}{\Pi(aq/b)\Pi(aq/c)\Pi(aq/d)\Pi(aq/e)\Pi(aq/bcd)\Pi(aq/cde)}$$

$$\times \frac{\Pi(aq/de)\Pi(aq/bcde)}{\Pi(aq/bde)\Pi(aq/bce)}$$

provided that $a^2q = bcdef$ and one of the parameters b,c,d, e or f is of the form q^{-N} where N is a positive integer. This ensures that the series terminates.

This result was given by F.H.Jackson in his paper "Summation of q-hypergeometric series", (Messenger of Mathematics, Vol. 50 (1921)). The proof follows the lines of the original proof of Dougall's theorem.

Proof. Suppose the theorem is true when f, c or d has one of the values $1, q^{-1}, \ldots, q^{-(N-1)}$, that is, if c or $a^2q/bcef$ has one of these values. It is therefore true in particular, when $f = q^{-N}$ and c has one of these 2N values, by symmetry. But when $f = q^{-N}$, on multiplying by $(aq/c)_{q,N}(aq/bcd)_{q,N}$, we see that the formula now asserts the equality of two polynomials of degree 2N in c. Also, when $c = aq^N$, which is a pole of the last term only of the series, the formula is still true. Hence two polynomials of degree 2N in c are equal for 2N+1 values of c, and hence, equal for all values of c.

Thus, the result is true when $f = q^{-N}$, if it is true when $f = 1, q^{-1}, q^{-2}, \ldots, q^{-(N-1)}$. Hence the result is true in general by induction.

Now let $f = q^{-N}$, substitute for e and let N tend to infinity in Jackson's theorem. Then we obtain

Theorem (25)

$$ {}_6\Phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, & b, & c, & d; & aq/bcd \\ \sqrt{a}, -\sqrt{a}, & aq/b, & aq/c, & aq/d; & \end{matrix}\right] $$

$$ = \frac{\Pi(aq)\,\Pi(aq/bc)\,\Pi(aq/bd)\,\Pi(aq/cd)}{\Pi(aq/b)\,\Pi(aq/c)\,\Pi(aq/d)\,\Pi(aq/bcd)} $$

We next obtain the analogue of Saalschutz's theorem,

Theorem (26)

$$ {}_3\Phi_2\left[\begin{matrix} b, c, & q^{-N}; & q \\ d, & bcq^{1-N}/d; & \end{matrix}\right] = \frac{(d/b)_{q,N}(d/c)_{q,N}}{(d)_{q,N}(d/bc)_{q,N}} $$

by substituting for e in Jackson's theorem, replacing d by aq/d, and then letting a tend to ~~infinity~~ zero.

Next, let N tend to infinity, in theorem (26). This gives

Theorem (27)

$$ {}_2\Phi_1\left[\begin{matrix} b, c; & d/bc \\ d; & \end{matrix}\right] = \frac{\Pi(d/b)\,\Pi(d/c)}{\Pi(d)\,\Pi(d/bc)} $$

This is the analogue of Gauss's theorem.

Now, let c tend to zero, in theorem (26). This gives an analogue of Vandermonde's theorem,

Theorem (28)

$$ {}_2\Phi_1\left[\begin{matrix} a, q^{-N}; & q \\ b; & \end{matrix}\right] = \frac{(b/a)_N a^N}{(b)_N} $$

Finally, let d = bcx, and let c tend to zero in theorem (27). Then we have

Theorem (29)

$$ {}_1\Phi_0\left[a;\,;\,x\right] = \frac{\Pi(ax)}{\Pi(x)} $$

These last three theorems were known to Heine.

§ (20.3) The general transformation of basic series, deduced from theorem (1)

Let $u_n = \dfrac{(e_1)_n (e_2)_n \cdots}{(q)_n (E_2)_n \cdots}$,

$$v_n = \frac{(f_1)_n (f_2)_n \cdots}{(F_1)_n (F_2)_n \cdots}$$

and $\delta_n = \dfrac{(d_1)_n (d_2)_n \cdots}{(D_1)_n (D_2)_n \cdots} t^n$.

Then $\gamma_n = \sum_{r=n}^{\infty} \delta_n \, u_{r-n} v_{r+n}$.

$$= \sum_{s=0}^{\infty} \frac{(d_1)_{s+n} \cdots}{(D_1)_{s+n} \cdots} \frac{(e_1)_s (e_2)_s \cdots}{(q)_s (E_2)_s \cdots} \frac{(f_1)_{s+2n} \cdots}{(F_1)_{s+2n} \cdots} t^{s+n}$$

(putting $s+n = r$)

Hence,

$$\gamma_n = \frac{(d_1)_n \cdots \; (f_1)_{2n} \cdots \; t^n}{(D_1)_n \cdots \; (F_1)_{2n} \cdots} \Phi_{k,k-1} \left[\begin{matrix} e_1, d_1 q^n, \cdots \; f_1 q^{2n}, \cdots ; \\ E_2, D_1 q^n, \cdots F_1 q^{2n}, \cdots ; \end{matrix} \; t \right]$$

This series will be denoted by Φ_γ.

Also, $\beta_n = \sum_{r=0}^{\infty} \alpha_r \, u_{n-r} v_{n+r}$

$$= \sum_{r=0}^{n} \frac{(e_1)_{n-r} (e_2)_{n-r} (f_1)_{n+r} \cdots\cdots}{(q)_{n-r} (E_2)_{n-r} \cdots \; (F_1)_{n+r} \cdots} \alpha_r$$

Hence,

$$\beta_n = \frac{(e_1)_n (e_2)_n \cdots \; (f_1)_n \cdots}{(q)_n (E_2)_n \cdots \; (F_1)_n \cdots}$$

$$\times \sum_{r=0}^{\infty} \alpha_r \frac{(q^{-n})_r \, (q^{1-n}/E_2)_r \cdots \; (f_1 q^n)_r \cdots (qE_2 \cdots)^r}{(q^{1-n}/e_1)_r (q^{1-n}/e_2)_r \cdots \; (F_1 q^n)_r \cdots \; (e_1 e_2 \cdots)^r}$$

$$\times \left(-q^{n-\frac{1}{2}(r+1)}\right)^{rp}$$

where p is the difference between one plus the number of E parameters and the number of e parameters (excluding $E_1 = 1$).

The function α_n must be of the general form

$$\frac{(a_1)_n(a_2)_n\cdots}{(q)_n(A_2)_n\cdots}t'^n$$

where the numbers $a_1, a_2, \ldots\ A_2, \ldots$ and q are all independent of r. t' however, can be a function of r; for example, we might take $t' = q^{\frac{1}{2}(r+1)}$. The series occurring in β_n will be denoted by Φ_β.

The general transformation resulting from theorem (1) is then

$$\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots\ (d_1)_n\cdots\ (f_1)_{2n}\cdots}{(q)_n(A_2)_n\cdots\ (D_1)_n\cdots\ (F_1)_{2n}\cdots}t'^n t^n$$

$$\times {}_k\Phi_{k-1}\left[\begin{matrix} e_1, e_2, \ldots\ d_1q^n, \ldots\ f_1q^{2n}, \ldots\ ; t \\ E_2, \ldots\ D_1q^n, \ldots\ F_1q^{2n}, \ldots\ ; \end{matrix}\right]$$

$$= \sum_{n=0}^{\infty}\frac{(e_1)_n(e_2)_n\cdots\ (d_1)_n\cdots\ (f_1)_n\cdots}{(q)_n(E_2)_n\cdots\ (D_1)_n\cdots\ (F_1)_n\cdots}t^n$$

$$\times\sum_{r=0}^{n}\frac{(q^{-n})_r(q^{1-n}/E_2)_r\cdots\ (f_1q^n)_r\cdots\ (a_1)_r(a_2)_r\cdots}{(q^{1-n}/e_1)_r(q^{1-n}/e_2)_r\cdots\ (F_1q^n)_r\cdots\ (q)_r(A_2)_r\cdots}$$

$$\times t'^r\frac{(qE_2\cdots)^r(-q^{n-\frac{1}{2}(r+1)})^{rp}}{(e_1e_2\cdots)^r}$$

provided that either both the double series terminate, or that the change in the order of summation can be justified.

§ (20.4) Summation of Φ_γ by theorem twenty-nine.

There are three possible sums for Φ_γ by using theorem tweny-nine, of which only two give rise to summable series for Φ_β. These are:-

(1) $\Phi_\gamma = {}_1\Phi_0\left[a;\ ;\ z\right]$ whence

$$\beta_n = \frac{(e)_n}{(q)_n}\sum_{r=0}^{n}\frac{(q^{-n})_r q^r}{(q^{1-n}/e)_r e^r}\alpha_r$$

and (2) $\Phi_\gamma = {}_1\Phi_0\left[dq^n;\ ;\ z\right]$ whence

$$\beta_n = \frac{1}{(q)_n}\sum_{r=0}^{n}(q^{-n})_r(-q)^r q^{nr-\frac{1}{2}r(r+1)}\alpha_r$$

§ (20.5) The first case.

If $\delta_n = z^n$, then $\gamma_n = \frac{\Pi(ez)}{\Pi(z)} z^n$.

Summing Φ_β by Vandermonde's analogue, if $\alpha_n = \frac{(a)_n e^n}{(q)_n}$

then $\beta_n = \frac{(ea)_n}{(q)_n a^n}$

Hence, $\frac{\Pi(ez)}{\Pi(z)}\ {}_1\Phi_0\left[a;\ ;\ ez\right] = {}_1\Phi_0\left[ea;\ ;\ z/a\right]$ ...(1)

Summing Φ_β by Saalschutz's analogue, if

$\alpha_n = \frac{(a)_n(A/ae)_n e^n}{(q)_n(A)_n}$ then, $\beta_n = \frac{(A/a)_n(ae)_n}{(A)_n(q)_n}$.

Hence $\frac{\Pi(ez)}{\Pi(z)}\ {}_2\Phi_1\left[\begin{matrix}a, A/ea;\ ez\\ A\ ;\end{matrix}\right] = {}_2\Phi_1\left[\begin{matrix}ea, A/a;\ z\\ A\ ;\end{matrix}\right]$ ...(2)

(Heine, Theorie der Kugelfunctionen (1878)).

§ (20.6) The second case.

If $\delta_n = (d)_n z^n$ then $\gamma_n = \frac{\Pi(dz)(d)_n z^n}{\Pi(z)\ (dz)_n}$

Summing Φ_β by Gauss's analogue, if $\alpha_n = \frac{(a)_n A^n q^{\frac{1}{2}n(n+1)}}{(q)_n(A)_n a^n\ (-q)^n}$

then $\beta_n = \dfrac{(A/a)_n}{(q)_n(A)_n}$. Hence,

$$\frac{\Pi(dz)}{\Pi(z)}\,{}_2\Phi_2\left[\begin{matrix}a,\ d;\ -Azq^{\frac{1}{2}(n-1)}/a\\ A,dz;\end{matrix}\right] = {}_2\Phi_1\left[\begin{matrix}d,A/a;\ z\\ A;\end{matrix}\right] \qquad \ldots(1)$$

Summing the ${}_2\Phi_1$ by Gauss's analogue, if $z = a/d$, we get

$$ {}_1\Phi_1\left[\begin{matrix}d;\ -Aq^{\frac{1}{2}(n-1)}/d\\ A;\end{matrix}\right] = \frac{\Pi(A/d)}{\Pi(A)} \qquad \ldots(2)$$

Next suppose that $\alpha_{2n+1} = 0$, and that

$\alpha_{2m} = \dfrac{q^{2n^2-2n}A^n}{(q^2)_{q^2,n}(A)_{q^2,n}}$, and sum Φ_β by Gauss's analogue;

Then $\beta_n = \dfrac{(A/q)_{q^2,n}}{(q)_{q,n}(A/q)_{q,n}}$, and hence

$$\frac{\Pi(dz)}{\Pi(z)}\,{}_2\Phi_{q^2\,4}\left[\begin{matrix}d,dq;\ Az^2q^{2(n-1)}\\ A,dz,dqz\ ;\end{matrix}\right]$$

$$= \sum_{n=0}^{\infty}\frac{(A/q)_{q^2,n}(d)_{q,n}z^n}{(q)_{q,n}(A/q)_{q,n}} \qquad \ldots(3)$$

§ (21.1) Summation of Φ_γ by Vandermonde's analogue.

There are, in all, twenty-seven possible cases in which Φ_γ can be summed by Vandermonde's analogue. Of these, however, only fourteen lead to summable series for Φ_β. These are:-

(1) $\Phi_\gamma = {}_2\Phi_1\left[\begin{matrix}e,q^{-N};\ q\\ E\ ;\end{matrix}\right]$ $\quad \Phi_\beta = \displaystyle\sum_{r=0}^{n}\frac{(q^{-n})_r(q^{1-n}/E)_r(Eq^{N+1})^r}{(q^{1-n}/e)_r(q^{1+N-n})_r\,e^r}\alpha_r$

(2) $\Phi_\gamma = {}_2\Phi_1\left[\begin{matrix}e,q^{n-N};\ q\\ E\ ;\end{matrix}\right]$

$$\Phi_\beta = \sum_{r=0}^{n}\frac{(q^{-n})_r(q^{1-n}/E)_r(-Eq)^r\,q^{nr-\frac{1}{2}r(r+1)}}{(q^{1-n}/e)_r\,e^r}\alpha_r$$

(3) $\Phi_\gamma = {}_2\Phi_1\left[\begin{matrix} e, q^{n-N}; & q \\ Dq^{n}; & \end{matrix}\right]$

$$\Phi_\beta = \sum_{r=0}^{n} \frac{(q^{-n})_r \; q^r}{(q^{1-n}/e)_r \; e^r} \alpha_r$$

(4) $\Phi_\gamma = {}_2\Phi_1\left[\begin{matrix} e, q^{2n-N}; & q \\ Dq^{n}; & \end{matrix}\right]$

$$\Phi_\beta = \sum_{r=0}^{n} \frac{(q^{-n})_r (q^{n-N})_r q^r}{(q^{1-n}/e)_r \; e^r} \alpha_r$$

(5) $\Phi_\gamma = {}_2\Phi_1\left[\begin{matrix} e, q^{2n-N}; & q \\ Fq^{2n}; & \end{matrix}\right]$

$$\Phi_\beta = \sum_{r=0}^{n} \frac{(q^{-n})_r (q^{n-N})_r q^r}{(q^{1-n}/e)_r (Fq^{n})_r \; e^r} \alpha_r ,$$

(6) $\Phi_\gamma = {}_2\Phi_1\left[\begin{matrix} dq^{n}, q^{-N}; & q \\ E; & \end{matrix}\right]$

$$\Phi_\beta = \sum_{r=0}^{n} \frac{(q^{-n})_r (q^{1-n}/E)_r (-q^{N+1}E)^r q^{nr-\frac{1}{2}(r+1)r}}{(q^{1+N-n})_r} \alpha_r$$

(7) $\Phi_\gamma = {}_2\Phi_1\left[\begin{matrix} dq^{m}, q^{n-N}; & q \\ E; & \end{matrix}\right]$

$$\Phi_\beta = \sum_{r=0}^{n} (q^{-n})_r (q^{1-n}/E)_r (qE)^r q^{2nr-r(r+1)} \alpha_r$$

(8) $\Phi_\gamma = {}_2\Phi_1\left[\begin{matrix} dq^{n}, q^{-N}; & q \\ Dq^{n}; & \end{matrix}\right]$

$$\Phi_\beta = \sum_{r=0}^{n} \frac{(q^{-n})_r q^{r(1+N)}}{(q^{1+N-n})_r} \alpha_r$$

(9) $\Phi_\gamma = {}_2\Phi_1\left[\begin{matrix} dq^{n}, q^{n-N}; & q \\ Dq^{n}; & \end{matrix}\right]$

$$\Phi_\beta = \sum_{r=0}^{n} (q^{-n})_r (-q)^r \; q^{nr-\frac{1}{2}r(r+1)} \alpha_r$$

(10) $\Phi_\gamma = {}_2\Phi_1\left[\begin{matrix} dq^n, q^{n-N}; q \\ Fq^{2n}; \end{matrix}\right]$

$$\Phi_\beta = \sum_{r=0}^{n} \frac{(q^{-n})_r q^{r+nr}(-1)^r}{(Fq^n)_r q^{\frac{1}{2}r(r+1)}} \alpha_r$$

(11) $\Phi_\gamma = {}_2\Phi_1\left[\begin{matrix} dq^n, q^{2n-N}; q \\ Fq^{2n} \end{matrix}\right]$

$$\Phi_\beta = \sum_{r=0}^{n} \frac{(q^{-n})_r (q^{n-N})_r (-q)^r q^{nr-\frac{1}{2}r(r+1)}}{(Fq^n)_r} \alpha_r$$

(12) $\Phi_\gamma = {}_2\Phi_1\left[\begin{matrix} fq^{2n}, q^{-N}; q \\ Dq^n; \end{matrix}\right]$

$$\Phi_\beta = \sum_{r=0}^{n} \frac{(q^{-n})_r (fq^n)_r q^{r(N+1)}}{(q^{1+N-r})_r} \alpha_r$$

(13) $\Phi_\gamma = {}_2\Phi_1\left[\begin{matrix} fq^{2n}, q^{-N}; q \\ Fq^{2n}; \end{matrix}\right]$

$$\Phi_\beta = \sum_{r=0}^{n} \frac{(q^{-n})_r (fq^n)_r q^{r(N+1)}}{(Fq^n)_r (q^{1+N-n})_r} \alpha_r$$

(14) $\Phi_\gamma = {}_2\Phi_1\left[\begin{matrix} fq^{2n}, q^{n-N}; q \\ Fq^{2n}; \end{matrix}\right]$

$$\Phi_\beta = \sum_{r=0}^{n} \frac{(q^{-n})_r (fq^n)_r (-q)^r q^{nr-\frac{1}{2}r(r+1)}}{(Fq^n)_r} \alpha_r$$

§ (21.2) Cases one and two.

In case one, if $\delta_n = q^n$, then

$\gamma_n = \dfrac{(E/e)_N e^N}{(E)_N} q^n$. Summing Φ_β by Saalschutz's analogue,

if $\alpha_n = \dfrac{(Eq^N/e)_n e^n}{(q)_n E^n q^{Nn}}$ then

$$\beta_n = \frac{(E/e)_n (Eq^N)_n e^n}{(q)_n (E)_n q^{Nn} E^n}$$. Hence,

$$\frac{(E/e)_N e^N}{(E)_N}\ {}_1\Phi_0\left[E/eq^{1-N};\ ;e/Eq^{N-1}\right] = {}_2\Phi_1\left[\begin{matrix}E/e, Eq^N; & eq^{1-N}/E \\ E\ ; & \end{matrix}\right] \qquad \ldots(1)$$

in case two, if $\delta'_n = (q^{-N})_n q^n$, then

$$\gamma_n = \frac{e^N (E/e)_N (q^{-N})_n (q^{1-N}/E)_n q^n}{(E)_N \quad (q^{1-N}e/E)_n}$$

Let $\alpha_n = \dfrac{(-1)^n q^{\frac{1}{2}n(n-1)}}{(q)_n}$. Then, summing Φ_β by Gauss's analogue, $\beta_n = \dfrac{(E/e)_n (-1)^n q^{\frac{1}{2}n(n-1)} e^n}{(q)_n\ (E)_n}$. Hence,

$$\frac{e^N (E/e)_N}{(E)_N}\ {}_2\Phi_1\left[\begin{matrix}q^{1-N}/E, q^{-N}; & -q^{\frac{1}{2}(n+1)} \\ q^{1-N}e/E\ ; & \end{matrix}\right]$$

$$= {}_2\Phi_1\left[\begin{matrix}E/e, q^{-N}; & -eq^{\frac{1}{2}(n+1)} \\ E\ ; & \end{matrix}\right] \qquad \ldots(2)$$

§ (21.3) Cases three and four.

In case three, if $\delta_n = \dfrac{(q^{-N})_n q^n}{(D)_n}$ then

$$\gamma_n = \frac{(D/e)_N e^N q^n (q^{-N})_n}{(D)_N (D/e)_n e^n}$$. Let $\alpha_n = \dfrac{(a)_n (A/ae)_n e^n}{(q)_n (A)_n}$.

and sum Φ_β by Saalschutz's analogue.

Then $\beta_n = \dfrac{(A/a)_n (ae)_n}{(q)_n (A)_n}$. Hence,

$$\frac{(D/e)_N e^N}{(D)_N}\ {}_3\Phi_2\left[\begin{matrix}a, A/ae, q^{-N}; & q \\ A,\ D/e\ ; & \end{matrix}\right] = {}_3\Phi_2\left[\begin{matrix}A/a, ae, q^{-N}; & q \\ A,\ D; & \end{matrix}\right] \qquad \ldots(1)$$

Now suppose that $\alpha_{2n+1} = 0$, and that

$\alpha_{2n} = \dfrac{(1/e^2)_{q^2,n} e^{2n}}{(q^2)_{q^2,n}}$. Then, summing Φ_β by

Saalschutz's analogue, $\beta_n = \dfrac{(e)_{q,n}(q^{1-n}e)_{q^2,\frac{1}{2}n}(1/e)_{q^2,\frac{1}{2}n}}{(q)_{q,n}(q^{1-n}/e)_{q^2,\frac{1}{2}n}(e)_{q^2,\frac{1}{2}n}}$

$$\therefore \quad \beta_n = \frac{(1/e)_{q,n} e^n (-1)^n}{(q)_{q,n}}$$

Hence, $\dfrac{(D/e)_{q,N} e^N}{(D)_{q,N}} \; {}_3\Phi_2{}_{q^2}\left[\begin{matrix} 1/e^2, q^{-N}, q^{1-N}; & q^2 \\ D/e, Dq/e & ; \end{matrix}\right]$

$$= {}_2\Phi_1{}_q\left[\begin{matrix} 1/e, q^{-N}; & -qe \\ D & ; \end{matrix}\right] \qquad \ldots(2)$$

In case four, if $\delta_n = \dfrac{q^n}{(D)_n}$ then

$$\gamma_m = \frac{(D/e)_N e^N (q^{1-N}/D)_n (q^{-N})_{2n} q^m}{(D)_N (D/e)_n (q^{1-N}e/D)_n e^m}$$

Let $\alpha_n = \dfrac{e^n}{(q)_n}$ and sum Φ_β by Vandermonde's analogue.

Then $\beta_n = \dfrac{(eq^{-N})_{2n}(q^{-N})_n}{(q)_n(eq^{-N})_n}$.

Hence, $\dfrac{(D/e)_N e^N}{(D)_N} {}_5\Phi_2\left[\begin{matrix} q^{1-N}/D, q^{-\frac{1}{2}N}, -q^{-\frac{1}{2}N}, q^{\frac{1}{2}(1-N)}, -q^{\frac{1}{2}(1-N)}; & q \\ D/e, \quad q^{1-N}e/D & ; \end{matrix}\right]$

$$= {}_5\Phi_2\left[\begin{matrix} \sqrt{eq^{-N}}, -\sqrt{eq^{-N}}, \sqrt{eq^{1-N}}, -\sqrt{eq^{1-N}}; q^{-N} & ; q \\ eq^{-N}, \quad D & ; \end{matrix}\right] \qquad \ldots(3)$$

<u>§(21.4)</u> <u>Case five.</u>

If $\delta_n = q^n$, then $\gamma_n = \dfrac{(F/e)_N e^N (q^{-N})_{2n} q^n}{(F)_N (F/e)_{2n} e^{2n}}$

Let $\alpha_n = \dfrac{(Fq^N/e)_n e^n}{(q)_n}$. Then, summing Φ_β by Saalschutz's analogue,

$$\beta_n = \frac{(Fq^N)_n (eq^{-N})_{2n} (q^{-N})_n}{(q)_n (F)_{2n} (eq^{-N})_n} .$$

Hence,

$$\frac{(F/e)_N e^N}{(F)_N} {}_5\Phi_4\left[\begin{matrix} Fq^N/e, q^{-\frac{1}{2}N}, q^{\frac{1}{2}(1-N)}, -q^{-\frac{1}{2}N}, -q^{\frac{1}{2}(1-N)}; q/e \\ \sqrt{F/e}, -\sqrt{F/e}, \sqrt{Fq/e}, -\sqrt{Fq/e} ; \end{matrix}\right]$$

$$= {}_6\Phi_5\left[\begin{matrix} Fq^N, \sqrt{eq^{-N}}, -\sqrt{eq^{-N}}, \sqrt{eq^{1-N}}, -\sqrt{eq^{1-N}}, q^{-N}; q \\ \sqrt{F}, -\sqrt{F}, \sqrt{Fq}, -\sqrt{Fq}, eq^{-N} ; \end{matrix}\right] \qquad \ldots(1)$$

(compare with eq.(1) § 4.42)

Next, let

$$\alpha_n = \frac{(q^{-N}/e)_n (q\sqrt{q^{-N}/e})_n (-q\sqrt{q^{-N}/e})_n (a_1)_n (a_2)_n}{(q)_n (\sqrt{q^{-N}/e})_n (-\sqrt{q^{-N}/e})_n (q^{1-N}/a_1 e)_n} \times \frac{(a_3)_n e^n}{(q^{1-N}/a_2 e)_n (q^{1-N}/a_3 e)_n}$$

where $q^{1-N}/e = a_1 a_2 a_3 e$, then, summing Φ_β by Jackson's theorem,

$$\beta_n = \frac{(a_1 e)_n (a_2 e)_n (a_3 e)_n (q^{-N})_n}{(q)_n (q^{1-N}/a_1 e)_n (q^{1-N}/a_2 e)_n (q^{1-N}/a_3 e)_n}$$

Hence,

$$\frac{e^N (q^{1-N}/e^2)_N}{(q^{1-N}/e)_N} {}_{10}\Phi_9\left[\begin{matrix} q^{-N}/e, q\sqrt{q^{-N}/e}, -q\sqrt{q^{-N}/e}, a_1, a_2, a_3, q^{-\frac{1}{2}N}, -q^{-\frac{1}{2}N}, q^{\frac{1}{2}-\frac{1}{2}N}, -q^{\frac{1}{2}-\frac{1}{2}N}; q/e \\ \sqrt{q^{-N}/e}, -\sqrt{q^{-N}/e}, q^{1-N}/a_1 e, q^{1-N}/a_2 e, q^{1-N}/a_3 e, q^{\frac{1}{2}(1-N)}/e, -q^{\frac{1}{2}(1-N)}/e, q^{1-\frac{1}{2}N}/e, -q^{1-\frac{1}{2}N}/e ; \end{matrix}\right]$$

$$= {}_4\Phi_3\left[\begin{matrix} a_1 e, a_2 e, a_3 e, q^{-N}; q \\ q^{1-N}/a_1 e, q^{1-N}/a_2 e, q^{1-N}/a_3 e; \end{matrix}\right] \qquad \ldots(2)$$

Now suppose that $\alpha_{2n+1} = 0$, and that

$$\alpha_{2m} = \frac{(q^{-N}/e)_{q^2,n}(q^2\sqrt{q^{-N}/e})_{q^2,n}(-q^2\sqrt{q^{-N}/e})_{q^2,n}(1/e^2)_{q^2,m}e^{2n}}{(q^2)_{q^2,n}(\sqrt{q^{-N}/e})_{q^2,n}(-\sqrt{q^{-N}/e})_{q^2,n}(q^{2-N}e)_{q^2,m}}$$

Then, summing Φ_β by Jackson's theorem,

$$\beta_m = \frac{(eq^{1-N})_{q^2,m}(1/e)_{q,n}(q^{-N})_{q,n}(-eq)^n}{(q)_{q,n}(q^{1-N}/e)_{q^2,n}(eq^{1-N})_{q,n}}$$

Hence,

$$\frac{e^N(q^{1-N}/e^2)_{q,N}}{(q^{1-N}/e)_{q,N}}\ {}_{12}\Phi_{11}{}_{q^2}\left[\begin{matrix} q^{-N}/e, q^2\sqrt{q^{-N}/e}, -q^2\sqrt{q^{-N}/e}, & 1/e^2, \\ \sqrt{q^{-N}/e}, -\sqrt{q^{-N}/e}, eq^{2-N}, & \end{matrix}\right.$$

$$q^{-\frac{1}{2}N},\ -q^{-\frac{1}{2}N},\ q^{1-\frac{1}{2}N},\ -q^{1-\frac{1}{2}N},\ q^{\frac{1}{2}(1-N)},\ -q^{\frac{1}{2}(1-N)},$$

$$q^{2-\frac{1}{2}N}/e, -q^{2-\frac{1}{2}N}/e, q^{1-\frac{1}{2}N}/e, -q^{1-\frac{1}{2}N}/e, q^{\frac{1}{2}(3-N)}/e, -q^{\frac{1}{2}(3-N)}/e,$$

$$\left.\begin{matrix} q^{\frac{1}{2}(3-N)},\ -q^{\frac{1}{2}(3-N)};\ q^2/e^2 \\ q^{\frac{1}{2}(1-N)}/e, -q^{\frac{1}{2}(1-N)}/e; \end{matrix}\right]$$

$$= {}_4\Phi_3{}_q\left[\begin{matrix} q^{-N}, \sqrt{eq^{1-N}}, -\sqrt{eq^{1-N}}, & 1/e; -eq \\ \sqrt{q^{1-N}/e}, -\sqrt{q^{1-N}/e}, eq^{1-N}; & \end{matrix}\right] \qquad \ldots(3)$$

§ (21.5) Cases six and seven.

In case six, if $\delta_m = (d)_m q^m$, then

$$\gamma_n = \frac{d^N(E/d)_N(d)_m(qd/E)_n q^n}{(E)_N(q^{1-N}d/E)_m}.$$

Let $\alpha_m = \dfrac{(-1)^m q^{\frac{1}{2}n(n-1)}}{(q)_m}$, and sum Φ_β by Gauss's analogue,

Then, $\beta_m = \dfrac{(Eq^N)_n(-q^{\frac{1}{2}(n-1)-N})^m}{(q)_n(E)_n}$.

Hence,
$$\frac{d^N(E/d)_N}{(E)_N}\ {}_2\Phi_1\left[\begin{matrix} d, qd/E & ; -q^{\frac{1}{2}(n+1)} \\ q^{1-N}d/E; & \end{matrix}\right]$$

$$= {}_2\Phi_1\left[\begin{matrix} d, Eq^N; & -q^{\frac{1}{2}(n+1)-N} \\ E & ; \end{matrix}\right]$$

(compare § 4.82, eq. (1))

In case seven, if $\delta_n = (d)_n(q^{-N})_n q^n$ then

$$\gamma_n = \frac{(E/d)_N d^N (d)_n (q^{-N})_n (dq/E)_n (q^{1-N}/E)_n q^n}{(E)_N (dq^{1-N}/E)_{2n}}$$

Let $\alpha_n = \dfrac{q^{n(n-1)}A^n}{(q)_n (A)_n}$ and sum Φ_β by Gauss's analogue. Then,

$$\beta_n = \frac{(AE/q)_{2n}}{(q)_n (E)_n (A)_n (AE/q)_n} .$$

Hence,

$$\frac{(E/d)_N d^N}{(E)_N}\, {}_4\Phi_5\left[\begin{matrix} d, & dq/E, & q^{1-N}/E, & q^{-N}; & Aq^n \\ A, \sqrt{dq^{1-N}/E}, & -\sqrt{dq^{1-N}/E}, & q\sqrt{dq^{-N}/E}, & -q\sqrt{dq^{-N}/E}; & \end{matrix}\right]$$

$$= {}_6\Phi_3\left[\begin{matrix} \sqrt{AE/q}, -\sqrt{AE/q}, \sqrt{AE}, -\sqrt{AE}, d, q^{-N}; & q \\ A,\ E,\ AE/q \quad ; & \end{matrix}\right]$$

(compare with § 4.62 eq.(1))

§ (21.6) Cases eight and nine.

In case eight, if $\delta_n = \dfrac{(d)_n q^n}{(D)_n}$, then

$$\gamma_n = \frac{(D/d)_N d^N q^{Nn+n} (d)_n}{(D)_N \ (Dq^N)_n}$$

Let $\alpha_n = \dfrac{(a)_n (Aq^N/a)_n}{(q)_n \ (A)_n}\, q^{-Nn}$ and sum Φ_β by Saalschutz's analogue. Then $\beta_n = \dfrac{(A/a)_n (aq^{-N})_n}{(q)_n (A)_n}$.

Hence,

$$\frac{(D/d)_N d^N}{(D)_N}\, {}_3\Phi_2\left[\begin{matrix} a, d, Aq^N/a; & q \\ A, Dq^N \ ; & \end{matrix}\right]$$

$$= {}_3\Phi_2\left[\begin{matrix} d, A/a, aq^{-N}; & q \\ D, \ A; & \end{matrix}\right] \qquad \ldots(1)$$

(compare with § 4.7 eq.(1))

Next, suppose that $\alpha_{2n+1} = 0$, and that

$\alpha_{2m} = \dfrac{(q^{2N})_{q^2,m}}{(q^2)_{q^2,m}} q^{-2Nm}$. Then, summing Φ_β by Saalschutz's analogue, $\beta_m = \dfrac{(q^N)_{q,m}(-q^{-N})^m}{(q)_{q,n}}$.

Hence, $$\frac{(D/d)_{q,N}d^N}{(D)_{q,N}} {}_3\Phi_2{}_{q^2}\left[\begin{matrix} d, & qd, & q^{2N}; & q^2 \\ & Dq^N, & Dq^{N+1}; & \end{matrix}\right]$$

$$= {}_2\Phi_1{}_q\left[\begin{matrix} d, q^N; & -q^{1-N} \\ D; & \end{matrix}\right] \qquad \ldots(2)$$

In case nine, if $\delta_m = \dfrac{(d)_n(q^{-N})_m q^m}{(D)_m}$, then

$$\gamma_m = \frac{(D/d)_N d^N (d)_m (q^{-N})_n (-1)^m}{(D)_N\ (q^{1-N}d/D)_n D^m q^{\frac{1}{2}(n^2-3n)}}$$

Let $\alpha_m = \dfrac{(a)_m\ A^m(-q^{\frac{1}{2}(n-1)})^m}{(q)_m(A)_n a^m}$. Then, summing Φ_β by Saalschutz's analogue, $\beta_m = \dfrac{(A/a)_m}{(q)_m(A)_m}$.

Hence, $$\frac{(D/d)_N d^N}{(D)_N} {}_3\Phi_2\left[\begin{matrix} d, a, & q^{-N}; & Aq/aD \\ A, & q^{1-N}d/D; & \end{matrix}\right]$$

$$= {}_3\Phi_2\left[\begin{matrix} d, A/a, & q^{-N}; & q \\ A, & D & ; \end{matrix}\right] \qquad \ldots(3)$$

(compare with eq. (1) § 4.52)

Next, suppose that $\alpha_{2n+1} = 0$, and that $\alpha_{2m} = \dfrac{q^{2n^2-2n}A^m}{(q^2)_{q^2,n}(A)_{q^2,n}}$, and sum Φ_β by Gauss's analogue.

Then, $\beta_m = \dfrac{(A/q)_{q^2,m}}{(q)_{q,n}(A/q)_{q,n}}$.

Hence, $$\frac{(D/d)_N d^N}{(D)_N}\,{}_4\Phi_3\left[\begin{matrix} d,\ qd,\ q^{-N},\ q^{1-N}\ ;\ Aq/D^2 \\ A,\ q^{1-N}d/D,\ q^{2-N}d/D\ ; \end{matrix}\right]_{q^2}$$
$$= {}_4\Phi_2\left[\begin{matrix} \sqrt{A/q},\ -\sqrt{A/q},\ d,\ q^{-N}\ ;\ q \\ A/q,\ D\ ; \end{matrix}\right]_q \qquad \ldots(4)$$

§ (21.7) Cases ten and eleven.

In case ten, if $\delta_m = (d)_n(q^{-N})_n q^n$, then

$$\gamma_m = \frac{(F/d)_N d^N (d)_n (q^{-N})_n q^{m+Nn-n^2}}{(F)_N (F/d)_n (Fq^N)_n d^m}$$

Let $$\alpha_m = \frac{(F/q)_n(\sqrt{Fq})_n(-\sqrt{Fq})_n(a_1)_n(a_2)_m F^n q^{\frac{1}{2}n(n-1)}}{(q)_n(\sqrt{F/q})_m(-\sqrt{F/q})_n(F/a_1)_n(F/a_2)_m a_1^n a_2^n}(-1)^m$$

and sum Φ_β as a well-poised ${}_6\Phi_5$ series. Then

$$\beta_m = \frac{(F/a_1a_2)_n}{(q)_m(F/a_1)_m(F/a_2)_m}$$

Hence, $$\frac{(F/d)_N d^N}{(F)_N}\,{}_7\Phi_6\left[\begin{matrix} F/q,\ \sqrt{Fq},\ -\sqrt{Fq},\ a_1,\ a_2,\ d,\ q^{-N}\ ;\ \frac{-Fq^{\frac{1}{2}(1-m)}}{a_1a_2d\ q^{-N}} \\ \sqrt{F/q},\ -\sqrt{F/q},\ F/a_1,\ F/a_2,\ F/d,\ Fq^N\ ; \end{matrix}\right]$$
$$= {}_3\Phi_2\left[\begin{matrix} F/a_1a_2,\ d,\ q^{-N}\ ;\ q \\ F/a_1,\ F/a_2\ ; \end{matrix}\right] \qquad \ldots(1)$$

(compare with eq. (a) § 4.3)

Next, suppose that $\alpha_{2n+1} = 0$, and that

$$\alpha_{2m} = \frac{(F/q)_{q^2,n}(q^2\sqrt{F/q})_{q^2,n}(-q^2\sqrt{F/q})_{q^2,n}(a)_{q^2,n}\,q^{(2n-1)n}F^m}{(q^2)_{q^2,n}(\sqrt{F/q})_{q^2,n}(-\sqrt{F/q})_{q^2,n}(Fq/a)_{q^2,n}\,a^m}$$

Then, summing Φ_β as a well-poised ${}_6\Phi_5$,

$$\beta_m = \frac{(F/a)_{q^2,n}}{(q)_{q,n}(F/a)_{q,n}(F)_{q^2,n}}.$$

Hence,

$$\frac{(F/d)_{q,N}d^N}{(F)_{q,N}}\,{}_8\Phi_7{}_{q^2}\left[\begin{matrix}F/q,q^2\sqrt{F/q},-q^2\sqrt{F/q}, & a, & d, & dq, & q^{-N},q^{1-N}; & \frac{Fq^{1+2N}}{ad^2q^2} \\ \sqrt{F/q},-\sqrt{F/q},Fq/a,Fq/d,F/d,Fq^{1+N},Fq^N & & & & & ;\end{matrix}\right]$$

$$= {}_4\Phi_3{}_{q}\left[\begin{matrix}\sqrt{F/a},-\sqrt{F/a},d, & q^{-N}; & q \\ F/a,\sqrt{F},-\sqrt{F}, & ; & \end{matrix}\right] \qquad \ldots(2)$$

If $\alpha_{3n+1} = \alpha_{3n-1} = 0$, and

$$\alpha_{3m} = \frac{(F/q)_{q^3,n}(q^3\sqrt{F/q})_{q^3,n}(-q^3\sqrt{F/q})_{q^3,n}F^m q^{\frac{1}{2}(3n-1)3n}(-1)^m}{(q^3)_{q^3,m}(\sqrt{F/q})_{q^3,n}(-\sqrt{F/q})_{q^3,m}\,q^m}$$

then, summing Φ_β as a well-poised ${}_6\Phi_5$ series,

$$\beta_m = \frac{(F/q)_{q^3,n}(Fq^2)_{q^3,2n/3}}{(q)_{q,n}(F)_{q,2n}(F/q)_{q^3,n/3}}\;.$$

Hence, $\frac{(F/d)_{q,N}d^N}{(F)_{q,N}}\,{}_9\Phi_8{}_{q^3}\left[\begin{matrix}F/q,q^3\sqrt{F/q},-q^3\sqrt{F/q}, & d, & dq, \\ \sqrt{F/q}, & -\sqrt{F/q},Fq^2/d,Fq/d, & \end{matrix}\right.$

$$\left.\begin{matrix}dq^2, & q^{-N}, & q^{1-N},q^{2-N}; & \frac{-Fq^{3N-\frac{1}{2}(9n+1)}}{d^3} \\ Fq^{2+N},F/d, & Fq^{1+N}, & Fq^N\;; & \end{matrix}\right]$$

$$= \sum_{n=o}^{N}\frac{(F/q)_{q^3,n}(Fq^2)_{q^3,2n/3}(d)_{q,n}(q^{-N})_{q,n}q^m}{(q)_{q,n}(F)_{q,2n}(F/q)_{q^3,n/3}} \qquad \ldots(3)$$

In case eleven, if $\delta_n = (d)_n q^n$, then, summing Φ_β by Gauss's analogue, if $\alpha_n = \frac{(-Fq^{N+\frac{1}{2}(n-1)})^n}{(q)_n}$,

$$\beta_m = \frac{(q^{-N})_m(Fq^N)_m}{(q)_n(F)_{2m}},\ \text{and}\ \gamma_n = \frac{(F/d)_N d^N(d)_n(q^{-N})_{2n}(-q^{\frac{1}{2}(3-3m)})^m}{(F)_N(F/d)_n(q^{1-N}d/F)_m d^nF^m}$$

Hence,

$$\frac{(F/d)_N d^N}{(F)_N}\ {}_5\Phi_2\left[\begin{matrix} d, q^{-\frac{1}{2}N}, -q^{-\frac{1}{2}N}, q^{\frac{1}{2}(1-N)}, -q^{\frac{1}{2}(1-N)}; \frac{q^{1+N-n}}{d} \\ F/d, q^{1-N}d/F \quad ; \end{matrix}\right]$$

$$= {}_3\Phi_4\left[\begin{matrix} Fq^N, d, q^{-N} ; q \\ \sqrt{F}, -\sqrt{F}, \sqrt{qF}, -\sqrt{qF}; \end{matrix}\right] \qquad \ldots(4)$$

(compare with eq (2), § 4.2);

§ (21.8) Case twelve.

If $\delta_n = \frac{q^n}{(D)_n}$, then

$$\gamma_m = \frac{(D/f)_N f^N (f)_{2n} (qf/D)_n q^{n(1+N)}}{(D)_N (q^{1-N}f/D)_n (Dq^N)_n}$$

Let $\alpha_n = \frac{q^{-Nn}}{(q)_n}$. Then, summing Φ_β by Vandermonde's analogue, $\beta_n = \frac{(f)_n (fq^{-N})_{2n} q^m}{(q)_n (fq^{-N})_n}$.

Hence,

$$\frac{(D/f)_N f^N}{(D)_N}\ {}_5\Phi_2\left[\begin{matrix} qf/D, \sqrt{f}, -\sqrt{f}, \sqrt{qf}, -\sqrt{qf}; q \\ q^{1-N}f/D, q^N D \quad ; \end{matrix}\right]$$

$$= {}_5\Phi_2\left[\begin{matrix} f, \sqrt{fq^{-N}}, -\sqrt{fq^{-N}}, \sqrt{fq^{1-N}}, -\sqrt{fq^{1-N}}; q \\ D, fq^{-N} \quad ; \end{matrix}\right]$$

(cpmpare with § 4.51)

§ (21.9) Cases thirteen and fourteen.

If $\delta_m = q^m$ then

$$\gamma_m = \frac{(F/f)_N f^N (f)_{2n} q^{n+2nN}}{(F)_N (Fq^N)_{2n}}.$$

Let $\alpha_m = \frac{(Fq^N/f)_n}{(q)_m q^{Nn}}$, and sum Φ_β by Saalschutz's analogue.

Then, $\beta_n = \dfrac{(f)_n(F/f)_n(fq^{-N})_{2n}}{(q)_n(F)_{2n}(fq^{-N})_n}$.

Hence, $$\frac{(F/f)_N f^N}{(F)_N}\,{}_5\Phi_4\left[\begin{matrix} Fq^N/f, \sqrt{f}, -\sqrt{f}, \sqrt{qf}, -\sqrt{qf};\ q^{1+N} \\ \sqrt{Fq^N}, -\sqrt{Fq^N}, \sqrt{Fq^{1+N}}, -\sqrt{Fq^{1+N}}; \end{matrix}\right]$$

$$= {}_6\Phi_5\left[\begin{matrix} f, F/f, \sqrt{fq^{-N}}, -\sqrt{fq^{-N}}, \sqrt{fq^{1-N}}, -\sqrt{fq^{1-N}};\ q \\ fq^{-N}, \sqrt{F}, -\sqrt{F}, \sqrt{qF}, -\sqrt{qF}; \end{matrix}\right] \qquad \ldots(1)$$

(compare with eq. (1) § 4.42).

Now, let $F = q^{1+N}f$, and

$$\alpha_n = \frac{(fq^N)_n(q\sqrt{fq^N})_n(-q\sqrt{fq^N})_n(a_1)_n(a_2)_n(a_3)_n q^{-Nn}}{(q)_n(\sqrt{fq^N})_n(-\sqrt{fq^N})_n(fq^{1+N}/a_1)_n(fq^{1+N}/a_2)_n(fq^{1+N}/a_3)_n}$$

Then, summing Φ_β by Jackson's theorem, if $fq^{1+2N} = a_1a_2a_3$,

$$\beta_n = \frac{(f)_n(a_1q^{-N})_n(a_2q^{-N})_n(a_3q^{-N})_n}{(q)_n(fq^{1+N}/a_1)_n(fq^{1+N}/a_2)_n(fq^{1+N}/a_3)_n}$$

Hence, $$\frac{(q^{1+N})_N f^N}{(fq^{1+N})_N}\,{}_{10}\Phi_9\left[\begin{matrix} fq^N, q\sqrt{fq^N}, -q\sqrt{fq^N}, a_1, a_2, a_3, \sqrt{f}, -\sqrt{f}, \sqrt{qf}, -\sqrt{qf};\ q^{1+N} \\ \sqrt{fq^N}, -\sqrt{fq^N}, fq^{1+N}/a_1, fq^{1+N}/a_2, fq^{1+N}/a_3, \sqrt{fq^{1+N}}, -\sqrt{fq^{1+N}}, \sqrt{fq^{1+2N}}, -\sqrt{fq^{1+2N}}; \end{matrix}\right]$$

$$= {}_4\Phi_3\left[\begin{matrix} f, a_1q^{-N}, a_2q^{-N}, a_3q^{-N};\ q \\ fq^{1+N}/a_1, fq^{1+N}/a_2, fq^{1+N}/a_3; \end{matrix}\right] \qquad \ldots(2)$$

provided that $fq^{1+2N} = a_1a_2a_3$.

Next, suppose that $\alpha_{2n+1} = 0$, and that

$$\alpha_{2n} = \frac{(fq^N)_{q^2,n}(q^2\sqrt{fq^N})_{q^2,n}(-q^2\sqrt{fq^N})_{q^2,n}(q^{2N})_{q^2,n}q^{-2nN}}{(q^2)_{q^2,n}(\sqrt{fq^N})_{q^2,n}(-\sqrt{fq^N})_{q^2,n}(fq^{2-N})_{q^2,n}}$$

Then, summing Φ_β as a well-poised ${}_8\Phi_7$ series, by Jackson's theorem,

$$\beta_m = \frac{(fq^{1-N})_{q^2,n}(q^N)_{q,n}(f)_{q,n}(-q^{-N})^m}{(q)_{q,n}(q^{1+N}f)_{q^2,n}(q^{1-N}f)_{q,m}}$$

Hence,

$$\frac{(q^{1+N})_{q,N}f^N}{(q^{1+N}f)_{q,N}}\ {}_{12}\Phi_{11}\left[\begin{matrix} fq^N, q^2\sqrt{fq^N}, -q^2\sqrt{fq^N}, \sqrt{f}, -\sqrt{f}, \sqrt{fq}, -\sqrt{fq}, q\sqrt{f}, -q\sqrt{f}, q\sqrt{fq}, -q\sqrt{fq}, q^{2N}; q^{2N+2} \\ \sqrt{fq^N}, -\sqrt{fq^N}, q^{2+N}\sqrt{f}, -q^{2+N}\sqrt{f}, q^{1+N}\sqrt{qf}, -q^{1+N}\sqrt{qf}, q^{1+N}\sqrt{f}, -q^{1+N}\sqrt{f}, q^N\sqrt{qf}, -q^N\sqrt{qf}, fq^{2-N}; \end{matrix}\right]_{q^2}$$

$$= {}_4\Phi_3\left[\begin{matrix} f, \sqrt{fq^{1-N}}, -\sqrt{fq^{1-N}}, q^N; -q^{1-N} \\ \sqrt{fq^{1+N}}, -\sqrt{fq^{1+N}}, fq^{1-N}; \end{matrix}\right] \qquad \ldots(3)$$

In case fourteen, if $\delta_n = (q^{-N})_n q^n$, then

$$\gamma_m = \frac{(F/f)_N f^N (f)_{2m}(q^{-N})_n(-q^{N+\frac{1}{2}(1-3n)})^m}{(F)_N\,(q^{1-N}f/F)_n(Fq^N)_n\,F^n}$$

Let $\alpha_m = \dfrac{F^n(-q^{\frac{1}{2}(n+1)})^n}{(q)_n f^n}$, and sum Φ_β by Gauss's analogue, then

$$\beta_n = \frac{(f)_n(F/f)_n f^n q^{n^2}}{(q)_n(F)_{2n}}.$$

Hence,

$$\frac{(F/f)_N f^N}{(F)_N}\ {}_5\Phi_2\left[\begin{matrix} \sqrt{f}, -\sqrt{f}, \sqrt{qf}, -\sqrt{qf}, q^{-N}; \frac{q^{N-n+1}}{f} \\ q^{1-N}f/F,\ Fq^N ; \end{matrix}\right]$$

$$= {}_3\Phi_4\left[\begin{matrix} f, F/f, q^{-N}; q \\ \sqrt{F}, -\sqrt{F}, \sqrt{qF}, -\sqrt{qF}; \end{matrix}\right] \qquad \ldots(4)$$

(compare with eq. (2) § 4.2)

§ (22.1) Application of Gauss's analogue to the summation of Φ_γ.

There are eighteen possible cases of summation of Φ_γ by

Gauss's analogue, of which only four give summable series for Φ_β. These are:-

(1) $\Phi_\gamma = {}_2\Phi_1\left[\begin{matrix} e_1, e_2 ; & \dfrac{E}{e_1 e_2} \\ E ; & \end{matrix}\right]$

$$\Phi_\beta = \sum_{r=0}^{n} \frac{(q^{-n})_r (q^{1-n}/E)_r \, q^r E^r}{(q^{1-n}/e_1)_r (q^{1-n}/e_2)_r \, e_1^r e_2^r} \alpha_r$$

(2) $\Phi_\gamma = {}_2\Phi_1\left[\begin{matrix} dq^n, & e ; \; D/de \\ Dq^n ; & \end{matrix}\right]$

$$\Phi_\beta = \sum_{r=0}^{n} \frac{(q^{-n})_r \, q^r}{(q^{1-n}/e)_r e^r} \alpha_r$$

(3) $\Phi_\gamma = {}_2\Phi_1\left[\begin{matrix} d_1 q^n, d_2 q^n ; & \dfrac{F}{d_1 d_2} \\ Fq^{2n} ; & \end{matrix}\right]$

$$\Phi_\beta = \sum_{r=0}^{n} \frac{(q^{-n})_r (-q^{n-\frac{1}{2}(r-1)})^r}{(Fq^n)_r} \alpha_r$$

(4) $\Phi_\gamma = {}_2\Phi_1\left[\begin{matrix} e, fq^{2n} ; & F/ef \\ Fq^{2n} ; & \end{matrix}\right]$

$$\Phi_\beta = \sum_{r=0}^{n} \frac{(q^{-n})_r (fq^n)_r \, q^r}{(q^{1-n}/e)_r (Fq^n)_r \, e^r} \alpha_r$$

These series for Φ_γ are convergent if $|q| < 1$ and if "$|z|$" < 1.

§ (22.2) Case one.

If $\delta_n = \dfrac{E^n}{e_1^n e_2^n}$ then

$$\gamma_n = \frac{\Pi(E/e_1)\Pi(E/e_2)}{\Pi(E)\Pi(E/e_1 e_2)} \frac{E^n}{e_1^n \, e_2^n} .$$

Let $\alpha_n = \dfrac{(E/e_1e_2)_n e_1^n e_2^n}{(q)_n E^n}$. Then, summing Φ_β by Saalschutz's analogue, $\beta_n = \dfrac{(E/e_1)_n (E/e_2)_n e_1^n e_2^n}{(q)_n (E)_n E^n}$

Hence, $\dfrac{\Pi(E/e_1)\Pi(E/e_2)}{\Pi(E)\Pi(E/e_1e_2)} {}_1\Phi_0\,(E/e_1e_2;\ ;\ 1)$

$$= {}_2\Phi_1\left[\begin{matrix}E/e_1, E/e_2;\ 1\\ E;\end{matrix}\right]$$

provided both series terminate.

§ (22.3) Case two.

If $\delta_n = \dfrac{(d)_n D^n}{(D)_n d^n e^n}$ then

$$\gamma_n = \frac{\Pi(D/d)\Pi(D/e)\ D^n\ (d)_n}{\Pi(D)\Pi(D/de)\ d^n e^n (D/e)_n}.$$

Let $\alpha_n = \dfrac{(a)_n (A/ae)_n e^n}{(q)_n (A)_n}$ and sum Φ_β by Saalschutz's analogue. Then $\beta_n = \dfrac{(ae)_n (A/a)_n}{(q)_n (A)_n}$.

Hence, $\dfrac{\Pi(D/d)\Pi(D/e)}{\Pi(D)\Pi(D/de)} {}_3\Phi_2\left[\begin{matrix}a, A/ae, d;\ D/d\\ D/e, A;\end{matrix}\right]$

$$= {}_3\Phi_2\left[\begin{matrix}ae, A/a, d;\ D/de\\ A,\ D\ ;\end{matrix}\right] \qquad \ldots(1)$$

provided $|D/de|<1$, (the analogue of eq. (1) § 4.7).

Next, let $\alpha_{2n+1} = 0$, and $\alpha_{2n} = \dfrac{(1/e^2)_{q^2,n} e^{2n}}{(q^2)_{q^2,n}}$

then, summing Φ_β by Vandermonde's analogue,

$$\beta_n = \frac{(1/e)_{q,n}(-e)^n}{(q)_{q,n}}.$$

Hence,

$$\frac{\prod_q(D/d)\prod_q(D/e)}{\prod_q(D)\prod_q(D/de)}\;{}_3\Phi_2\left[\begin{matrix}d,\ qd,1/e^2;\ D^2/d^2\\ D/e,qD/e;\end{matrix}\right]_{q^2}$$

$$= {}_2\Phi_1\left[\begin{matrix}1/e,d;\ -D/d\\ D;\end{matrix}\right]_q \qquad \ldots(2)$$

provided that $|D/d|<1$, (the analogue of § 16.4).

§ (22.4) Case three.

If $\delta_n = \frac{(d_1)_n(d_2)_n F^n}{d_1^n d_2^n}$, then

$$\gamma_m = \frac{\prod(F/d_1)\prod(F/d_2)(d_1)_m(d_2)_m F^m}{\prod(F)\prod(F/d_1d_2)\,(F/d_1)_m(F/d_2)_m d_1^m d_2^m}$$

Let

$$\alpha_n = \frac{(F/q)_n(q\sqrt{F/q})_n(-q\sqrt{F/q})_n(a_1)_m(a_2)_m(-Fq^{\frac{1}{2}(n-1)})^m}{(q)_n(\sqrt{F/q})_n(-\sqrt{F/q})_m(F/a_1)_n(F/a_2)_m\ a_1^m a_2^m}$$

and sum Φ_β as a well-poised ${}_6\Phi_5$. Then,

$$\beta_m = \frac{(F/a_1a_2)_m}{(q)_m(F/a_1)_n(F/a_2)_m}\ .$$

Hence,

$$\frac{\prod(F/d_1)\prod(F/d_2)}{\prod(F)\prod(F/d_1d_2)}\;{}_7\Phi_6\left[\begin{matrix}F/q,\sqrt{Fq},-\sqrt{Fq}, & a_1, & a_2, & d_1, & d_2; & \frac{-F^2q^{\frac{1}{2}(n-1)}}{a_1a_2d_1d_2}\\ \sqrt{F/q},-\sqrt{F/q}, & F/a_1, & F/a_2, & F/d_1, & F/d_2 &;\end{matrix}\right]$$

$$= {}_3\Phi_2\left[\begin{matrix}F/a_1a_2, & d_1, & d_2;\ F/d_1d_2\\ & F/a_1,F/a_2 &;\end{matrix}\right] \qquad \ldots(1)$$

provided that $|F/d_1d_2|< 1$, (an analogue of eq. (8), § 4.3)

Now suppose that $\alpha_{2n+1} = 0$, and that

$$\alpha_{2m} = \frac{(F/q)_{q^2,n}(q^2\sqrt{F/q})_{q^2,n}(-q^2\sqrt{F/q})_{q^2,n}(a)_{q^2,n}q^{(2n-1)n}F^m}{(q^2)_{q^2,n}(\sqrt{F/q})_{q^2,n}(-\sqrt{F/q})_{q^2,n}(Fq/a)_{q^2,n}a^n}$$

Then, summing Φ_β as a well-poised ${}_6\Phi_5$.

$$\beta_m = \frac{(F/a)_{q^2,n}}{(q)_{q,n}(F/a)_{q,n}(F)_{q^2,n}} .$$

Hence,
$$\frac{\Pi(F/d_1)\Pi(F/d_2)}{\Pi(F)\Pi(F/d_1d_2)}\,{}_8\Phi_7\left[\begin{matrix}F/q, q^2\sqrt{F/q}, -q^2\sqrt{F/q}, & a, & d_1, & qd_1, & d_2, & qd_2; & \frac{F^3q^{(2n-1)}}{ad_1^2d_2^2} \\ \sqrt{F/q}, \quad -\sqrt{F/q}, Fq/a, & & Fq/d_1, & F/d_1; & Fq/d_2, & F/d_2 ; & \end{matrix}\right]_{q^2}$$

$$= {}_4\Phi_3\left[\begin{matrix}\sqrt{F/a}, -\sqrt{F/a}, d_1, d_2; F/d_1d_2 \\ F/a, \sqrt{F}, -\sqrt{F};\end{matrix}\right]_q \qquad \ldots(2)$$

provided that $|F/d_1d_2|<1$ (an analogue of eq. (1) § 16.1)

Similarly, if $\alpha_{3n+1} = \alpha_{3n-1} = 0$, and

$$\alpha_{3m} = \frac{(F/q)_{q^3,n}(q^3\sqrt{F/q})_{q^3,n}(-q^3\sqrt{F/q})_{q^3,n}F^nq^{(3n-1)3n/2}(-1)^n}{(q^3)_{q^3,n}(\sqrt{F/q})_{q^3,n}(-\sqrt{F/q})_{q^3,n}q^n}$$

Then, summing Φ_β as a well-poised ${}_6\Phi_5$,

$$\beta_m = \frac{(F/q)_{q^3,n}(Fq^2)_{q^3,2n/3}}{(q)_{q,n}(F)_{q,2n}(F/q)_{q^3,n/3}} .$$

Hence,
$$\frac{\Pi_q(F/d_1)\Pi_q(F/d_2)}{\Pi_q(F)\Pi_q(F/d_1d_2)}\,{}_9\Phi_8\left[\begin{matrix}F/q, q^3\sqrt{F/q}, -q^3\sqrt{F/q}, & d_1, & d_1q, d_1q^2, & d_2, & d_2q, d_2q^2; & \frac{-F^4q^{\frac{1}{2}(9n-5)}}{d_1^3d_2^3} \\ \sqrt{F/q}, \quad -\sqrt{F/q}, Fq^2/d_1, & & Fq/d_1, F/d_1, & Fq^2/d_2, & Fq/d_2, F/d_2; & \end{matrix}\right]_{q^3}$$

$$= \sum_{n=0}^{\infty} \frac{(F/q)_{q^3,n}(Fq^2)_{q^3,2n/3}(d_1)_{q,n}(d_2)_{q,n}F^n}{(q)_{q,n}(F)_{q,2n}(F/q)_{q^3,n/3}\, d_1^{\,n} d_2^{\,n}} \qquad \ldots(3)$$

provided that $|F/d_1d_2| < 1$ (compare with eq.(2) § 16.1).

§ (22.5) Case four.

If $\delta_n = \dfrac{F^n}{e^n f^n}$, then

$$\gamma_n = \frac{\Pi(F/e)\Pi(F/f)}{\Pi(F)\Pi(F/ef)} \frac{(f)_{2n}F^n}{(F/e)_{2n}\, e^n f^n} .$$

Let $\alpha_n = \dfrac{(F/ef)_n e^n}{(q)_n}$, and sum Φ_β by Saalschutz's analogue. Then, $\beta_n = \dfrac{(f)_n(F/f)_n(ef)_{2n}}{(q)_n(F)_{2n}(ef)_n}$.

Hence, $\dfrac{\Pi(F/e)\Pi(F/f)}{\Pi(F)\Pi(F/ef)}\, {}_5\Phi_4\left[\begin{matrix} F/ef,\sqrt{f},-\sqrt{f},\sqrt{qf},-\sqrt{qf}; \\ \sqrt{F/e},-\sqrt{F/e},\sqrt{qF/e},-\sqrt{qF/e}; \end{matrix} F/f\right]$

$$= {}_6\Phi_5\left[\begin{matrix} f,F/f,\sqrt{ef},-\sqrt{ef},\sqrt{qef},-\sqrt{qef}; \\ ef,\sqrt{F},-\sqrt{F},\sqrt{qF},-\sqrt{qF}; \end{matrix} F/ef\right] \qquad \ldots(1)$$

provided that $|F/ef|<1$, (compare with eq. (1) § 4.42).

Now let $F = qf/e$, and

$$\alpha_n = \frac{(f/e)_n(q\sqrt{f/e})_n(-q\sqrt{f/e})_n(a_1)_n(a_2)_n(a_3)_n e^n}{(q)_n(\sqrt{f/e})_n(-\sqrt{f/e})_n(qf/ea_1)_n(qf/ea_2)_n(qf/ea_3)_n}$$

where $fq = a_1a_2a_3e^2$. Then, summing Φ_β by Jackson's theorem, $\beta_n = \dfrac{(f)_n(ea_1)_n(ea_2)_n(ea_3)_n}{(q)_n(fq/ea_1)_n(fq/ea_2)_n(fq/ea_3)_n}$.

Hence, $\dfrac{\Pi(qf/e^2)\Pi(q/e)}{\Pi(qf/e)\Pi(q/e^2)}\, {}_{10}\Phi_9\left[\begin{matrix} f/e,q\sqrt{f/e},-q\sqrt{f/e}, & a_1, \\ \sqrt{f/e},-\sqrt{f/e}, & fq/ea_1, \end{matrix}\right.$

$$\left.\begin{matrix} a_2, & a_3, & \sqrt{f}, & -\sqrt{f}, & \sqrt{qf}, & -\sqrt{qf}; \\ fq/ea_2, & fq/ea_3, & q\sqrt{f/e}, & -q\sqrt{f/e}, & \sqrt{qf/e}, & -\sqrt{qf/e}; \end{matrix} q/e\right]$$

$$= {}_4\Phi_3\left[\begin{matrix} f,ea_1, & ea_2, & ea_3; \\ fq/ea_1, & fq/ea_2, & fq/ea_3; \end{matrix} q/e^2\right] \qquad \ldots(2)$$

provided that $|q/e|<1$.

Next, let $\alpha_{2n+1} = 0$ and

$$\alpha_{2m} = \frac{(f/e)_{q^2,n}(q\sqrt{f/e})_{q^2,n}(-q\sqrt{f/e})_{q^2,n}(1/e^2)_{q^2,m}e^{2m}}{(q^2)_{q^2,m}(\sqrt{f/e})_{q^2,n}(-\sqrt{f/e})_{q^2,n}(q^2ef)_{q^2,m}}$$

Then, summing Φ_β by Jackson's theorem,

$$\beta_m = \frac{(feq)_{q^2,n}(1/e)_{q,n}(-e)^n(f)_{q,n}}{(q)_{q,n}(qf/e)_{q^2,n}(qef)_{q,m}}$$

Hence, $$\frac{\Pi(qf/e^2)\Pi(q/e)}{\Pi(qf/e)\Pi(q/e^2)}\ {}_{12}\Phi_{11}{}_{q^2}\left[\begin{matrix}f/e, q^2\sqrt{f/e}, -q^2\sqrt{f/e}, 1/e^2, \sqrt{f}, -\sqrt{f}, \sqrt{qf}, -\sqrt{qf}, q\sqrt{f}, -q\sqrt{f}, q\sqrt{qf}, -q\sqrt{qf}; \\ \sqrt{f/e}, -\sqrt{f/e}, q^2ef, q^2\sqrt{f/e}, -q^2\sqrt{f/e}, q\sqrt{qf/e}, -q\sqrt{qf/e}, q\sqrt{f/e}, -q\sqrt{f/e}, \sqrt{qf/e}, -\sqrt{qf/e};\end{matrix}\ q^2/e^2\right]$$

$$= {}_4\Phi_3{}_q\left[\begin{matrix}f, 1/e, \sqrt{qef}, -\sqrt{qef}; \\ qef, \sqrt{qf/e}, -\sqrt{qf/e};\end{matrix}\ -q/e\right] \qquad \ldots(3)$$

provided that $|q/e|<1$, (an analogue of § 16 .2)

§ (23.1) Applications of Saalschutz's ~~theorem~~ analogue to the summation of Φ_γ.

If Φ_γ is summable by Saalschutz's analogue, then, there are in all ten possibilities for Φ_γ. Of these, only four lead to summable series for Φ_β. These are:-

(1) $$\Phi_\gamma = {}_3\Phi_2\left[\begin{matrix}fq^{2n}, q^{n-N}, e; \\ efq^{1+n-N}/F, Fq^{2n};\end{matrix}\ q\right]$$

$$\Phi_\beta = \sum_{r=0}^{n}\frac{(q^{-n})_r(fq^n)_r q^r}{(q^{1-n}/e)_r(Fq^n)_r\, e^r}\alpha_r$$

(2) $$\Phi_\gamma = {}_3\Phi_2\left[\begin{matrix}d_1q^n, d_2q^n, q^{n-N}; q \\ Fq^{2n}, d_1d_2q^{1+n-N}/F;\end{matrix}\right]$$

$$\Phi_\beta = \sum_{r=0}^{n} \frac{(q^{-n})_r(-1)^r q^{nr-\frac{1}{2}r(r-1)}}{(Fq^n)_r} \alpha_r$$

(3) $$\Phi_\gamma = {}_3\Phi_2\left[\begin{matrix} dq^n, q^{n-N}, e; q \\ Dq^n, deq^{1+n-N}/D; \end{matrix}\right]$$

$$\Phi_\beta = \sum_{r=0}^{n} \frac{(q^{-n})_r q^r}{(q^{1-n}/e)_r e^r} \alpha_r$$

(4) $$\Phi_\gamma = {}_3\Phi_2\left[\begin{matrix} e_1, e_2, q^{n-N}; q \\ e_1 e_2 q^{1+n-N}/E, E; \end{matrix}\right]$$

$$\Phi_\beta = \sum_{r=0}^{n} \frac{(q^{-n})_r(q^{1-n}/E)_r q^r E^r}{(q^{1-n}/e_1)_r(q^{1-n}/e_2)_r e_1^r e_2^r} \alpha_r$$

<u>§ (23.2)</u> <u>Case one.</u>

If $\delta_n = \dfrac{(q^{-N})_n q^n}{(efq^{1-N}/F)_n}$ then

$$\gamma_n = \frac{(F/f)_N(F/e)_N(f)_{2n}(Fq^N/e)_n(q^{-N})_n q^n}{(F)_N(F/ef)_N(Fq^{-N})_n(fq^{1-N}/F)_n(F/e)_{2n}e^n}.$$

The general transformation is

$$\frac{(F/f)_N(F/e)_N}{(F)_N(F/ef)_N}\sum_{n=0}^{N}\frac{(f)_{2n}(Fq^N/e)_n(q^{-N})_n q^n}{(Fq^{-N})_n(F/e)_{2n}(fq^{1-N}/F)_n e^n}\alpha_n$$

$$= \sum_{n=0}^{N}\frac{(q^{-N})_n q^n}{(q^{1-N}ef/F)_n}\beta_n$$

Let $\alpha_n = \dfrac{(F/ef)_n e^n}{(q)_n}$, and sum Φ_β by Saalschutz's analogue,

then, $\beta_n = \dfrac{(f)_n(F/f)_n(ef)_{2n}}{(q)_n(F)_{2n}(ef)_n}$.

Hence,

$$\frac{(F/f)_N(F/e)_N}{(F)_N(F/ef)_N}\,{}_7\Phi_6\left[\begin{matrix}F/ef,q^NF/e,\ \sqrt{f},\ -\sqrt{f},\ \sqrt{qf},\ -\sqrt{qf},\ q^{-N};\ q\\ q^{1-N}f/F,\sqrt{qF/e},-\sqrt{qF/e},\sqrt{F/e},-\sqrt{F/e},Fq^N\ ;\end{matrix}\right]$$

$$={}_7\Phi_6\left[\begin{matrix}f,F/f,\sqrt{ef},-\sqrt{ef},\sqrt{qef},-\sqrt{qef},\ q^{-N};\ q\\ ef,\sqrt{qF},-\sqrt{qF},\sqrt{F},-\sqrt{F},q^{1-N}ef/F\ ;\end{matrix}\right]\qquad\ldots(1)$$

(an analogue of eq. (1) § 5.3)

Next, summing Φ_β as a well-poised ${}_8\Phi_7$, if

$fq = a_1a_2a_3e^2$, and if

$$\alpha_n=\frac{(f/e)_n(q\sqrt{f/e})_n(-q\sqrt{f/e})_n(a_1)_n(a_2)_n(a_3)_ne^n}{(q)_n(\sqrt{f/e})_n(-\sqrt{f/e})_n(qf/ea_1)_n(qf/ea_2)_n(qf/ea_3)_n}$$

then $\beta_n=\dfrac{(f)_n(ea_1)_n(ea_2)_n(ea_3)_n}{(qf/ea_1)_n(qf/ea_2)_n(qf/ea_3)_n(q)_n}$.

Hence,

$$\frac{(q/e)_N(qf/e^2)_N}{(qf/e)_N(q/e^2)_N}\,{}_{12}\Phi_{11}\left[\begin{matrix}f/e,q\sqrt{f/e},-q\sqrt{f/e},\ a_1,\ a_2,\ a_3,\ \sqrt{f},\ -\sqrt{f},\ \sqrt{qf},\ -\sqrt{qf},\ q^{1+N}f/e^2,\ q^{-N};\ q\\ \sqrt{f/e},\ -\sqrt{f/e},qf/ea_1,qf/ea_2,qf/ea_3,q\sqrt{f}/e,-q\sqrt{f}/e,\sqrt{qf}/e,-\sqrt{qf}/e,\ q^{-N}e,q^{1+N}f/e;\end{matrix}\right]$$

$$={}_5\Phi_4\left[\begin{matrix}f,\ ea_1,\ ea_2,\ ea_3,\ q^{-N};\ q\\ qf/ea_1,qf/ea_2,qf/ea_3,q^{-N}e^2\ ;\end{matrix}\right]\qquad\ldots(2)$$

where $fq=a_1a_2a_3e^2$, (an analogue of eq. (2) § 5.3).

This is a transformation between a finite well-poised ${}_{12}\Phi_{11}$ and a Saalschutzian nearly-poised finite ${}_5\Phi_4$. (Bailey, Proc. Lond. Maths. Soc. 1943, p. 431. eq. 9.5)

Now let $\alpha_{2n+1}=0$ and

$$\alpha_{2n}=\frac{(f/e)_{q^2,n}(q^2\sqrt{f/e})_{q^2,n}(-q^2\sqrt{f/e})_{q^2,n}(1/e^2)_{q^2,n}q^{2n}}{(q^2)_{q^2,n}(\sqrt{f/e})_{q^2,n}(-\sqrt{f/e})_{q^2,n}(q^2ef)_{q^2,n}}$$

Then, summing Φ_β by Jackson's theorem,

$$\beta_n = \frac{(fqe)_{q^2,n}(1/e)_{q,n}(f)_{q,n}(-e)^n}{(q)_{q,n}(qf/e)_{q^2,n}(qef)_{q,n}}$$

Hence,
$$\frac{(q/e)_N(qf/e^2)_N}{(qf/e)_N(q/e^2)_N}\,{}_{16}\Phi_{15}\left[\begin{matrix} f/e,\ q^2\sqrt{f/e},\ -q^2\sqrt{f/e},\ 1/e^2,\ q^{1+N}f/e^2,\ q^{2+N}f/e^2,\ \sqrt{f},\ -\sqrt{f},\ \sqrt{qf},\ -\sqrt{qf},\ q\sqrt{f},\ -q\sqrt{f},\ q\sqrt{qf},\ -q\sqrt{qf},\ q^{1-N},\ q^{-N};\ q^2 \\ \sqrt{f/e},\ -\sqrt{f/e},\ q^2ef,\ q^{1-N}e,\ q^{-N}e,\ q^2\sqrt{f/e},\ -q^2\sqrt{f/e},\ q\sqrt{qf/e},\ -q\sqrt{qf/e},\ q\sqrt{f/e},\ -q\sqrt{f/e},\ \sqrt{qf/e},\ -\sqrt{qf/e},\ q^{1+N}f/e,\ q^{2+N}f/e\ ; \end{matrix}\right]_{q^2}$$

$$= {}_{5}\Phi_{4}\left[\begin{matrix} f,\ 1/e,\ \sqrt{qef},\ -\sqrt{qef},\ q^{-N};\ -eq \\ qef,\ \sqrt{qf/e},\ -\sqrt{qf/e},\ q^{-N}e^2; \end{matrix}\right]_q \qquad \ldots(3)$$

(an analogue of § 16.6)

This transforms a well-poised terminating special ${}_{16}\Phi_{15}$ into a nearly-poised terminating ${}_5\Phi_4$.

§(23.3) Case two.

If $\delta_n = \dfrac{(d_1)_n(d_2)_n(q^{-N})_n q^n}{(d_1d_2q^{1-N}/F)_n}$, then

$$\gamma_n = \frac{(F/d_1)_N(F/d_2)_N(d_1)_n(d_2)_n(q^{-N})_n(-1)^n q^{Nn}F^n}{(F)_N(F/d_1d_2)_N(F/d_1)_n(F/d_2)_n(Fq^N)_n q^{\frac{1}{2}n(n-1)}d_1^n d_2^n}$$

The general transformation is

$$\frac{(F/d_1)_N(F/d_2)_N}{(F)_N(F/d_1d_2)_N}\sum_{n=0}^{N}\frac{(d_1)_n(d_2)_n(q^{-N})_n q^{Nn}(-1)^nF^n}{(F/d_1)_n(F/d_2)_n(Fq^N)_n d_1^n d_2^n q^{\frac{1}{2}n(n-1)}}\,\alpha_n$$
$$= \sum_{n=0}^{N}\frac{(d_1)_n(d_2)_n(q^{-N})_n q^n}{(d_1d_2q^{1-N}/F)_n}\,\beta_n\,.$$

Let $\alpha_n = \dfrac{(F/q)_n(q\sqrt{Fq})_n(-q\sqrt{Fq})_n(a_1)_n(a_2)_n(-1)^n q^{\frac{1}{2}n(n-1)}F^n}{(q)_n(\sqrt{F/q})_n(-\sqrt{F/q})_n(F/a_1)_n(F/a_2)_n a_1^n a_2^n}$.

then, summing Φ_β as a well-poised ${}_6\Phi_5$,

$$\beta_n = \frac{(F/a_1a_2)_n}{(F/a_1)_n(F/a_2)_n(q)_n}.$$

Hence,

$$\frac{(F/d_1)_N(F/d_2)_N}{(F)_N(F/d_1d_2)_N}\,{}_8\Phi_7\left[\begin{matrix}F/q, q\sqrt{F/q}, -q\sqrt{F/q}, & a_1, & a_2, & d_1, & d_2, & q^{-N} & ; \dfrac{F^2q^N}{a_1a_2d_1d_2} \\ \sqrt{F/q}, -\sqrt{F/q}, & F/a_1, & F/a_2, & F/d_1, & F/d_2, & Fq^N & ;\end{matrix}\right]$$

$$= {}_4\Phi_3\left[\begin{matrix}F/a_1a_2, & d_1, & d_2, & q^{-N} & ; q \\ & F/a_1, & F/a_2, & d_1d_2q^{1-N}/F & ;\end{matrix}\right] \qquad \ldots(1)$$

(an analogue of eq. (1) § 5.2).

This is a transformation between a well-poised ${}_8\Phi_7$ and a Saalschutzian ${}_4\Phi_3$. (Watson, "A new proof of the Rogers-Ramanujan identities" Journal, Lond. Maths. Soc. (4) 1929). From this transformation, the Rogers-Ramanujan identities are usually deduced, by letting a_1, a_2, d_1, d_2, and N tend to infinity, and then putting $F = aq$, and $a = 1$, or $a = q$, in the result.

If $\alpha_{2n+1} = 0$, and

$$\alpha_{2n} = \frac{(F/q)_{q^2,n}(q^2\sqrt{F/q})_{q^2,n}(-q^2\sqrt{F/q})_{q^2,n}(a)_{q^2,n}q^{n(2n-1)}F^n}{(q^2)_{q^2,n}(\sqrt{F/q})_{q^2,n}(-\sqrt{F/q})_{q^2,n}(Fq/a)_{q^2,n}a^n}$$

then, summing Φ_β as a well-poised ${}_6\Phi_5$,

$$\beta_n = \frac{(F/a)_{q^2,n}}{(q)_{q,n}(F/a)_{q,n}(F)_{q^2,n}}$$

Hence,

$$\frac{(F/d_1)_N(F/d_2)_N}{(F)_N(F/d_1d_2)_N}\,{}_{10}\Phi_9\left[\begin{matrix}F/q,\ q^2\sqrt{F/q},\ -q^2\sqrt{F/q},\ a,\ d_1,\\ \sqrt{F/q},\ -\sqrt{F/q},\ Fq/a,\ Fq/d_1,\end{matrix}\right.$$

$$\left.\begin{matrix}d_1q,\ d_2,\ d_2q,\ q^{-N},\ q^{1-N};\ \frac{F^3q^{2Nn}}{ad_1^2d_2^2}\\ F/d_1,\ Fq/d_2,\ F/d_2,\ Fq^{1+N},\ Fq^N;\end{matrix}\right]_{q^2}$$

$$= {}_5\Phi_4\left[\begin{matrix}\sqrt{F/a},\ -\sqrt{F/a},\ d_1,\ d_2,\ q^{-N};\ q\\ F/a,\ \sqrt{F},\ -\sqrt{F},\ d_1d_2q^{1-N}/F;\end{matrix}\right] \qquad \ldots(2)$$

Similarly, if $\alpha_{3n+1} = \alpha_{3n-1} = 0$, and

$$\alpha_{3n} = \frac{(F/q)_{q^3,n}(q^3\sqrt{F/q})_{q^3,n}(-q^3\sqrt{F/q})_{q^3,n}F^nq^{(3n-1)3n/2}}{(q^3)_{q^3,n}(\sqrt{F/q})_{q^3,n}(-\sqrt{F/q})_{q^3,n}q^n}$$

then, summing Φ_β as a well-poised ${}_6\Phi_5$,

$$\beta_n = \frac{(F/q)_{q^3,n}(Fq^2)_{q^3,2n/3}}{(q)_{q,n}(F)_{q,2n}(F/q)_{q^3,n/3}}$$

Hence,

$$\frac{(F/d_1)_{q,N}(F/d_2)_{q,N}}{(F)_{q,N}(F/d_1d_2)_{q,N}}\,{}_{12}\Phi_{11}\left[\begin{matrix}F/q,\ q^3\sqrt{F/q},\ -q^3\sqrt{F/q},\ d_1,\\ \sqrt{F/q},\ -\sqrt{F/q},\ Fq^2/d_1,\end{matrix}\right.$$

$$\left.\begin{matrix}d_1q,\ d_1q^2,\ d_2,\ d_2q,\ d_2q^2,\ q^{-N},\ q^{1-N},\ q^{2-N};\ \frac{-F^4q^{3N}}{d_1^3d_2^3}\\ Fq/d_1,\ F/d_1,\ Fq^2/d_2,\ Fq/d_2,\ F/d_2,\ Fq^{2+N},\ Fq^{1+N},\ Fq^N;\end{matrix}\right]_{q^3}$$

$$= \sum_{n=0}^{N}\frac{(F/q)_{q^3,n}(Fq^2)_{q^3,2n/3}(d_1)_{q,n}(d_2)_{q,n}(q^{-N})_{q,n}q^n}{(q)_{q,n}(F)_{q,2n}(F/q)_{q^3,n/3}(d_1d_2q^{1-N}/F)_{q,n}} \qquad \ldots(3)$$

§ (23.4) Case three.

If $\delta_n = \dfrac{(d)_n(q^{-N})_nq^n}{(D)_n(deq^{1-N}/D)_m}$, then

$$\gamma_n = \frac{(D/d)_N (D/e)_N (d)_n (q^{-N})_n q^n}{(D)_N (D/de)_N (D/e)_n (dq^{1-N}/D)_n \, e^n}$$

The general transformation is

$$\frac{(D/d)_N (D/e)_N}{(D)_N (D/de)_N} \sum_{n=o}^{N} \frac{(d)_n (q^{-N})_n q^n}{(D/e)_n (q^{1-N} d/D)_n e^n} \alpha_n$$

$$= \sum_{n=o}^{N} \frac{(d)_n (q^{-N})_n q^n}{(D)_n (q^{1-N} de/D)_n} \beta_n$$

If $\alpha_n = \dfrac{(a)_n (A/ae)_n e^n}{(q)_n (A)_n}$, then, summing Φ_β by Saalschutz's analogue, $\beta_n = \dfrac{(ea)_n (A/e)_n}{(q)_n (A)_n}$.

Hence, $$\frac{(D/d)_N (D/e)_N}{(D)_N (D/de)_N} {}_4\Phi_3\left[\begin{matrix} a, A/ae, d, & q^{-N}; \; q \\ D/e, \; A, q^{1-N} d/D \; ; & \end{matrix}\right]$$

$$= {}_4\Phi_3\left[\begin{matrix} ea, A/a, d, & q^{-N}; \; q \\ A, D, q^{1-N} de/D; & \end{matrix}\right] \qquad \ldots(1)$$

(an analogue of § 5.4 eq.(1)).

If $\alpha_{2n+1} = 0$, and $\alpha_{2n} = \dfrac{(1/e)_{q^2,n} e^{2n}}{(q^2)_{q^2,n}}$, then,

summing Φ_β by Saalschutz's analogue, $\beta_n = \dfrac{(1/e)_{q,n} (-e)^n}{(q)_{q,n}}$

Hence, $$\frac{(D/d)_N (D/e)_N}{(D)_N (D/de)_N} {}_5\Phi_4{}_{q^2}\left[\begin{matrix} 1/e, \; d, \; dq, \; q^{-N}, & q^{1-N}; \; q^2 \\ D/e, Dq/e, q^{1-N} d/D, q^{2-N} d/D; & \end{matrix}\right]$$

$$= {}_3\Phi_2\left[\begin{matrix} 1/e, d, & q^{-N}; \; -eq \\ D, q^{1-N} de/D; & \end{matrix}\right] \qquad \ldots(2)$$

(an analogue of § 16.7).

§ (23.5) Case four.

If $\delta_n = \dfrac{(q^{-N})_n q^n}{(e_1 e_2 q^{1-N}/E)_n}$, then

$$\gamma_n = \frac{(E/e_1)_N (E/e_2)_N (q^{1-N}/E)_n (q^{-N})_n q^n}{(E/e_1e_2)_N (E)_N (q^{1-N}e_1/E)_n (q^{1-N}e_2/E)_n}$$

Here the general transformation is

$$\frac{(E/e_1)_N (E/e_2)_N}{(E)_N (E/e_1e_2)_N} \sum_{n=0}^{N} \frac{(q^{1-N}/E)_n (q^{-N})_n q^n}{(q^{1-N}e_1/E)_n (q^{1-N}e_2/E)_n} \alpha_n$$

$$= \sum_{n=0}^{N} \frac{(q^{-N})_n q^n}{(q^{1-N}e_1e_2/E)_n} \beta_n$$

If $\alpha_n = \dfrac{(E/e_1e_2)_n e_1^n e_2^n}{(q)_m E^n}$, then, summing Φ_β by

Saalschutz's analogue, $\beta_n = \dfrac{(E/e_1)_n (E/e_2)_n E^m}{(E)_n e_1^n e_2^m}$

Hence, $\dfrac{(E/e_1)_N (E/e_2)_N}{(E)_N (E/e_1e_2)_N} {}_3\Phi_2 \left[\begin{matrix} q^{1-N}/E, E/e_1e_2, q^{-N} ; e_1e_2q/E \\ e_1q^{1-N}/E, e_2q^{1-N}/E ; \end{matrix} \right]$

$$= {}_3\Phi_2 \left[\begin{matrix} E/e_1, E/e_2, q^{-N} ; qe_1e_2/E \\ E, e_1e_2q^{1-N}/E ; \end{matrix} \right]$$

(an analogue of § 5.5).

§ (24.1) Application of Jackson's theorem to the summation of Φ_γ.

There are, in all, four possible ways of summing Φ_γ by Jackson's theorem, of which only two lead to summable series for Φ_β. These are:-

(1) $$\Phi_\gamma = {}_8\Phi_7\left[\begin{matrix} fq^{2n}, q\sqrt{f}q^{n}, -q\sqrt{f}q^{n}, f_1q^{2n}, dq^{n}, q^{n-N}, \quad e_1, \quad e_2 ; q \\ \sqrt{f}q^{n}, -\sqrt{f}q^{n}, fq/f_1, fq^{1+n}/d, fq^{1+n+N}, fq^{1+2n}/e_1, fq^{1+2n}/e_2 \end{matrix}\right]$$

$$\Phi_\beta = \sum_{r=0}^{n} \frac{(q^{-n})_r (fq^{n})_r (f_1q^{n})_r (f_1q^{-n}/f)_r q^{2r} f^{r} \, \alpha_r}{(q^{1-n}/e_1)_r (q^{1-n}/e_2)_r (fq^{1+n}/e_1)_r (fq^{1+n}/e_2)_r \, f_1^{r} e_1^{r} e_2^{r}}$$

where $f^2q = f_1dq^{-N}e_1e_2$.

(2) $$\Phi_\gamma = {}_8\Phi_7\left[\begin{matrix} fq^{2n}, q\sqrt{f}q^{n}, -q\sqrt{f}q^{n}, d_1q^{m}, d_2q^{n}, d_3q^{n}, q^{n-N}, \quad e ; q \\ \sqrt{f}q^{n}, -\sqrt{f}q^{n}, fq^{1+n}/d_1, fq^{1+n}/d_2, fq^{1+n}/d_3, fq^{1+n+N}, fq^{1+2n}/e ; \end{matrix}\right]$$

$$\Phi_\beta = \sum_{r=0}^{n} \frac{(q^{-n})_r (fq^{n})_r q^{r}}{(q^{1-n}/e)_r (fq^{1+n}/e)_r \, e^{r}} \alpha_r$$

where $f^2q = d_1d_2d_3q^{-N}e$.

The cases in which summation of Φ_γ is possible, either as a well-poised ${}_4\Phi_3$ or as a well-poised ${}_6\Phi_5$, are only special cases of the summation of Φ_γ as a well-poised ${}_8\Phi_7$, by Jackson's theorem.

§ (24.2) The first case.

Let $f_1 = fe_2/e_1$, so that the series for Φ_β can be made well-poised.

If $\delta_n = \dfrac{(q\sqrt{f})_n(-q\sqrt{f})_n(d)_n(q^{-N})_n q^{n}}{(\sqrt{f})_n(-\sqrt{f})_n(fq/d)_n(fq^{1+N})_n}$, then,

$$\gamma_n = \frac{(fq/de_1e_2)_n (fq^{1+N}/e_1e_2)_n (fe_2/e_1)_{2n} (d)_n (q^{-N})_n q^{n}}{(fq/de_1)_n (fq^{1+N}/e_1)_n (fq^{1+N}/e_2)_n (fq/de_2)_n (fq/e_1e_2)_n} \cdot K_1$$

where $K_1 \equiv \dfrac{(fq)_N (fq/de_1)_N (fq/de_2)_N (fq/e_1e_2)_N}{(fq/d)_N (fq/e_1)_N (fq/e_2)_N (fq/de_1e_2)_N}$

and $f\,q = dq^{-N}e_2^{2}$.

The general transformation is

$$K_1 \cdot \sum_{n=0}^{N} \frac{(de_2/e_1)_n (q^{-N}e_2/e_1)_n (fe_2/e_1)_{2n} (d)_n (q^{-N})_n q^n \alpha_n}{(fq/de_1)_n (fq^{1+N}/e_1)_n (fq^{1+N}/e_2)_n (fq/e_1e_2)_n (fq/de_2)_n}$$

$$= \sum_{n=0}^{N} \frac{(q\sqrt{f})_n (-q\sqrt{f})_n (d)_n (q^{-N})_n q^n}{(\sqrt{f})_n (-\sqrt{f})_n (fq/d)_n (fq^{1+N})_n} \beta_n,$$

where $fq = dq^{-N}e_2^2$.

$$\text{If } \alpha_n = \frac{(f/e_1)_n (q\sqrt{f/e_1})_n (-q\sqrt{f/e_1})_n (q/e_2^2)_n e_2^{2n}}{(q)_n (\sqrt{f/e_1})_n (-\sqrt{f/e_1})_n (fe_2^2/e_1)_n}$$

then, summing Φ_8 as a well-poised ${}_8\Phi_7$, then

$$\beta_n = \frac{(f)_n (fe_2/e_1)_n (fe_2)_{2n} (q/e_2)_n (qe_1/e_2^2)_n \, q^{n(n+1)}}{(qe_1/e_2)_n (q)_n (fq/e_2)_{2n} (fe_2)_n (fe_2^2/e_1)_n \, q^{2n} e_1^{2n}}.$$

Hence,

$$K_1 \cdot {}_{12}\Phi_{11}\left[\begin{matrix} f/e_1, q\sqrt{f/e_1}, -q\sqrt{f/e_1}, \; q/e_2^2, fq/de_1e_2, \\ \sqrt{f/e_1}, -\sqrt{f/e_1}, fe_2^2/e_1, fq^{1+N}/e_2, \end{matrix}\right.$$

$$\begin{matrix} fq^{1+N}/e_1e_2, \sqrt{fe_2/e_1}, -\sqrt{fe_2/e_1}, \sqrt{qfe_2/e_1}, -\sqrt{qfe_2/e_1}, \quad d, \\ fq/de_2, q\sqrt{f/e_1e_2}, -q\sqrt{f/e_1e_2}, \sqrt{qf/e_1e_2}, -\sqrt{qf/e_1e_2}, fq/de_1, \end{matrix}$$

$$\left.\begin{matrix} q^{-N}; \; qe_2^2 \\ fq^{1+N}/e_1; \end{matrix}\right] = {}_{12}\Phi_{11}\left[\begin{matrix} f, q\sqrt{f}, -q\sqrt{f}, qe_1/e_2^2, \; q/e_2, \\ \sqrt{f}, -\sqrt{f}, fe_2^2/e_1, \; fe_2, \end{matrix}\right.$$

$$\left.\begin{matrix} \sqrt{fe_2}, -\sqrt{fe_2}, \sqrt{qfe_2}, -\sqrt{qfe_2}, \; fe_2/e_1, \; d, \; q^{-N} \; ; \; q/e_1^2 \\ q\sqrt{f/e_2}, -q\sqrt{f/e_2}, \sqrt{qf/e_2}, -\sqrt{qf/e_2}, e_1q/e_2, fq/d, fq^{1+N}; \end{matrix}\right]$$

provided that $fq = dq^{-N}e_2^2$, (an analogue of § 6.2).

This is a transformation between two special well-poised terminating ${}_{12}\Phi_{11}$ series.

§ (24.3) The second case.

If $\delta_n = \dfrac{(q\sqrt{f})_n(-q\sqrt{f})_n(d_1)_n(d_2)_n(d_3)_n(q^{-N})_n q^n}{(\sqrt{f})_n(-\sqrt{f})_n(fq/d_1)_n(fq/d_2)_n(fq/d_3)_n(fq^{1+N})_n}$

then $\gamma_n = K_2 . \dfrac{(d_1)_n(d_2)_n(d_3)_n(q^{-N})_n q^n}{(qf/d_1e)_n(qf/d_2e)_n(qf/d_3e)_n(q^{1+N}f/e)_n}$.

where $K_2 \equiv \dfrac{(fq)_N(fq/d_1d_2)_N(fq/d_1d_3)_N(fq/d_2d_3)_N}{(fq/d_1)_N(fq/d_2)_N(fq/d_3)_N(fq/d_1d_2d_3)_N}$

The general transformation is

$$K_2 . \sum_{n=0}^{N} \frac{(d_1)_n(d_2)_n(d_3)_n(q^{-N})_n q^n}{(qf/d_1e)_n(qf/d_2e)_n(qf/d_3e)_n(q^{1+N}f/e)_n} \alpha_n$$

$$= \sum_{n=0}^{N} \frac{(q\sqrt{f})_n(-q\sqrt{f})_n(d_1)_n(d_2)_n(d_3)_n(q^{-N})_n q^n}{(\sqrt{f})_n(-\sqrt{f})_n(qf/d_1)_n(qf/d_2)_n(qf/d_3)_n(q^{1+N}f)_n} \beta_n$$

where $f^2q^{1+N} = d_1d_2d_3$.

Let $\alpha_n = \dfrac{(q/e^2)_n e^n}{(q)_n}$, and sum Φ_β by Saalschutz's analogue, then $\beta_n = \dfrac{(f)_n(q/e)_n(ef)_{2n}}{(qf/e)_{2n}(q)_n(ef)_n}$.

Hence, $K_2 . {}_5\Phi_4\left[\begin{matrix} q/e^2, & d_1, & d_2, & d_3, & q^{-N}; q \\ & qf/d_1e, & qf/d_2e, & qf/d_3e, & q^{1+N}f/e ; \end{matrix}\right]$

$$= {}_{12}\Phi_{11}\left[\begin{matrix} f, q\sqrt{f}, -q\sqrt{f}, q/e, & \sqrt{ef}, & -\sqrt{ef}, \sqrt{qef}, & -\sqrt{qef}, & d_1, \\ \sqrt{f}, & -\sqrt{f}, ef, q\sqrt{f/e}, -q\sqrt{f/e}, \sqrt{qf/e}, -\sqrt{qf/e}, qf/d_1, \end{matrix}\right.$$

$$\left.\begin{matrix} d_2, & d_3, & q^{-N}; q \\ qf/d_2, & qf/d_3, & q^{1+N}f ; \end{matrix}\right] \qquad \ldots(1)$$

This transforms a nearly-poised ${}_5\Phi_4$ into a well-poised ${}_{12}\Phi_{11}$. (Bailey, Proc. Lond. Maths. Soc. July, 1947).

Next, if $\alpha_n = \dfrac{(f/e)_n(q\sqrt{f/e})_n(-q\sqrt{f/e})_n(a_1)_n(a_2)_n(a_3)_n}{(q)_n(\sqrt{f/e})_n(-\sqrt{f/e})_n(qf/ea_1)_n(qf/ea_2)_n(qf/ea_3)_n}$

then, summing Φ_8 as a well-poised ${}_8\Phi_7$,

$$\beta_n = \frac{(f)_n(ea_1)_n(ea_2)_n(ea_3)_n}{(q)_n(fq/ea_1)_n(fq/ea_2)_n(fq/ea_3)_n}.$$

Hence, K_2, $${}_{10}\Phi_9\left[\begin{matrix} f/e, q\sqrt{f/e}, -q\sqrt{f/e}, & a_1, & a_2, & a_3, & d_1, & d_2, & d_3, & q^{-N}; q \\ \sqrt{f/e}, -\sqrt{f/e}, & qf/ea_1, & qf/ea_2, & qf/ea_3, & qf/ed_1, & qf/ed_2, & qf/ed_3, & q^{1+N}f/e; \end{matrix}\right]$$

$$= {}_{10}\Phi_9\left[\begin{matrix} f, q\sqrt{f}, -q\sqrt{f}, & ea_1, & ea_2, & ea_3, & d_1, & d_2, & d_3, & q^{-N}; q \\ \sqrt{f}, -\sqrt{f}, & qf/ea_1, & qf/ea_2, & qf/ea_3, & qf/d_1, & qf/d_2, & fq/d_3, & fq^{1+N}; \end{matrix}\right] \quad \ldots(2)$$

provided that $f^2q = d_1d_2d_3q^{-N}e = a_1a_2a_3fe^2$, (an analogue of § 6.3).

(Bailey, "An identity involving Heine's series". Journal Lond. Maths. Soc. (4) 1929).

Finally, if $\alpha_{2n+1} = 0$, and

$$\alpha_{2n} = \frac{(f/e)_{q^2,n}(q^2\sqrt{f/e})_{q^2,n}(-q^2\sqrt{f/e})_{q^2,n}(1/e^2)_{q^2,n}}{(q^2)_{q^2,n}(\sqrt{f/e})_{q^2,n}(-\sqrt{f/e})_{q^2,n}(fq^2e)_{q^2,n}}$$

then, summing Φ_8 as a well-poised ${}_8\Phi_7$,

$$\beta_n = \frac{(f)_{q,n}(1/e)_{q,n}(fq/e)_{q^2,n}(-1)^n}{(q)_{q,n}(qfe)_{q^2,n}(qef)_{q,n}\,e^n}.$$

Hence, K_2. $${}_{12}\Phi_{11}{}_{q^2}\left[\begin{matrix} f/e, q^2\sqrt{f/e}, -q^2\sqrt{f/e}, 1/e^2, & d_1, & d_1q, \\ \sqrt{f/e}, -\sqrt{f/e}, q^2ef, & fq^2/ed_1, & fq/ed_1, \end{matrix}\right.$$

$$\left.\begin{matrix} d_2, & d_2q, & d_3q, & d_3, & q^{-N}, & q^{1-N}; & q^2 \\ fq^2/ed_2, & fq/ed_2, & fq\ /ed_3, & fq^2/ed_3, & fq^{2+N}/e, & fq^{1+N}/e; & \end{matrix}\right]$$

$$= {}_{10}\Phi_9\left[\begin{matrix} f, q\sqrt{f}, -q\sqrt{f}, \sqrt{fq/e}, -\sqrt{fq/e}, 1/e, & d_1, & d_2, & d_3, & q^{-N}; -\frac{q}{e} \\ \sqrt{f}, -\sqrt{f}, \sqrt{qef}, -\sqrt{qef}, qef, fq/d_1, & fq/d_2, & fq/d_3, & fq^{1+N} & ; \end{matrix}\right]$$

where $f^2q = d_1d_2d_3q^{-N}e$. ...(3)

(an analogue of § 17.1).

This transforms a well-poised special terminating ${}_{12}\Phi_{11}$ into a special terminating well-poised ${}_{10}\Phi_9$.

§ (25.1) Five particular transformations.

Let us now return to a consideration of the work of § (22.4). Here, summing Φ_β by Gauss's analogue, the general transformation involved is

$$\frac{\Pi(F/d_1)\Pi(F/d_2)}{\Pi(F)\Pi(F/d_1d_2)}\sum_{n=0}^{\infty}\frac{(d_1)_n(d_2)_nF^n}{(F/d_1)_n(F/d_2)_nd_1^nd_2^n}\alpha_n$$

$$= \sum_{n=0}^{\infty}\frac{(d_1)_n(d_2)_nF^n}{d_1^nd_2^n}\beta_n,$$

where $\beta_n = \frac{1}{(q)_n(F)_n}\sum_{r=0}^{n}\frac{(q^{-n})_r(-q^{n-\frac{1}{2}(r+1)})^r}{(Fq^n)_r}\alpha_r$.

Putting x for q and ax for F, this equation becomes

$$\sum_{n=0}^{\infty}(d_1)_n(d_2)_n\frac{a^nx^n}{d_1^nd_2^n}\beta_n$$

$$= \prod_{n=1}^{\infty}\frac{(1-ax^n/d_1)(1-ax^n/d_2)}{(1-ax^n)(1-ax^n/d_1d_2)}\sum_{n=0}^{\infty}\frac{(d_1)_n(d_2)_na^nx^n}{(ax/d_1)_n(ax/d_2)_nd_1^nd_2^n}\alpha_n$$

...(1)

(Equation (3.1) "Identities of the Rogers-Ramanujan type" W.N.Bailey, Proc.Lond. Maths.Soc. June 1944).

If we sum Φ_β as a well-poised ${}_6\Phi_5$ and let d_1 and d_2 tend to infinity, we get the Rogers-Ramanujan identities, for $a = 1$ and $a = x$, respectively. If however, we put a^2x^2 for F, and x^2 for q, in the general transformation, and sum as a well-poised ${}_6\Phi_5$, we get

$$\sum_{n=0}^{\infty}\frac{(d_1)_{x^2,n}(d_2)_{x^2,n}(-ax/b)_{x,2n}a^{2n}x^{2n}}{(x^2)_{x^2,n}(a^2x^2/b^2)_{x^2,n}(-ax)_{x,2n}\,d_1^n d_2^n}$$

$$=\prod_{n=1}^{\infty}\frac{(1-a^2x^{2n}/d_1)(1-a^2x^{2n}/d_2)}{(1-a^2x^{2n})(1-a^2x^{2n}/d_1d_2)}$$

$$\times\left\{1+\sum_{n=1}^{\infty}\frac{(ax)_{x,n-1}(b)_{x,n}(d_1)_{x^2,n}(d_2)_{x^2,n}(1-ax^{2n})a^{3n}x^{n^2+2n}}{(x)_{x,n}(ax/b)_{x,n}(a^2x^2/d_1)_{x^2,n}(a^2x^2/d_2)_{x^2,n}d_1^n d_2^n b^n}\right\}$$

(Loc.cit.eq.6.1). ...(2)

Similarly, putting a^3x^3 for F, and x^3 for q, and summing Φ_β as a well-poised ${}_6\Phi_5$, we get

$$\sum_{n=0}^{\infty}\frac{(d_1)_{x^3,n}(d_2)_{x^3,n}(ax)_{x,3n}a^{3n}x^{3n}}{(x^3)_{x^3,n}(a^3x^3)_{x^3,2n}d_1^n d_2^n}$$

$$=\prod_{n=1}^{\infty}\frac{(1-a^3x^{3n}/d_1)(1-a^3x^{3n}/d_2)}{(1-a^3x^{3n})(1-a^3x^{3n}/d_1d_2)}$$

$$\times\left\{1+\sum_{n=1}^{\infty}\frac{(-1)^n(ax)_{x,n-1}(d_1)_{x^3,n}(d_2)_{x^3,n}(1-ax^{2n})a^{3n}x^{3n+\frac{1}{2}(3n^2-n)}a^n}{(x)_{x,n}(a^3x^3/d_1)_{x^3,n}(a^3x^3/d_2)_{x^3,n}\,d_1^n d_2^n}\right\}$$

(Loc.cit.eq. 6.2) ...(3)

Next, putting ax for F, x for q, and f for a, in eq.(2) § 22.4,

$$\sum_{n=0}^{\infty}\frac{(d_1)_{x,n}(d_2)_{x,n}(ax/f)_{x^2,n}a^n x^n}{(x)_{x,n}(ax)_{x^2,n}(ax/f)_{x,n}\, d_1^n d_2^n}$$

$$=\prod_{n=1}^{\infty}\frac{(1-ax^n/d_1)(1-ax^n/d_2)}{(1-ax^n)(1-ax^n/d_1d_2)}\times$$

$$\left\{1+\sum_{n=0}^{\infty}\frac{(ax^2)_{x^2,n-1}(f)_{x^2,n}(1-ax^{4n})(d_1)_{x,2n}(d_2)_{x,2n}a^{3n}x^{2n^2+2n}}{(x^2)_{x^2,n}(ax^2/f)_{x^2,n}(ax/d_1)_{x,2n}(ax/d_2)_{x,2n}d_1^{2n}d_2^{2n}f^n}\right\}$$

(Loc.cit.eq. 6.3). ...(4)

and finally, putting ax for F, in eq. (3), § 22.4,

$$1+\sum_{n=1}^{\infty}\frac{(d_1)_{x,n}(d_2)_{x,n}(ax^3)_{x^3,n-1}a^n x^n}{(x)_{x,n}(ax)_{x,2n-1}\; d_1^n d_2^n}$$

$$=\prod_{n=1}^{\infty}\frac{(1-ax^n/d_1)(1-ax^n/d_2)}{(1-ax^n)(1-ax^n/d_1d_2)}\times$$

$$\left\{1+\sum_{n=1}^{\infty}\frac{(-1)^n(ax^3)_{x^3,n-1}(1-ax^{6n})(d_1)_{x,3n}(d_2)_{x,3n}a^{4n}x^{3n(3n+1)/2}}{(x^3)_{x^3,n}(ax/d_1)_{x,3n}(ax/d_2)_{x,3n}\,d_1^{3n}\,d_2^{3n}}\right\}$$

(Loc.cit. eq. 6.4) ...(5)

Now let us consider the second and fourth transformations (eq. (2) and (4) above) in detail.

§ (25.2) The second transformation.

If $a = x^p$ where p is a positive integer, from eq. (2) above, we have,

$$\sum_{n=0}^{\infty}\frac{(d_1)_{x^2,n}(d_2)_{x^2,n}(-x^{p+1}/b)_{x,2n}x^{2n(p+1)}}{(x^2)_{x^2,n}(x^{2p+2}/b^2)_{x^2,n}(-x^{p+1})_{x,2n}d_1^n d_2^n}$$

$$=\prod_{n=1}^{\infty}\frac{(1-x^{2p+2n}/d_1)(1-x^{2p+2n}/d_2)}{(1-x^{2p+2n})(1-x^{2p+2n}/d_1d_2)}\times$$

$$\left\{1+\sum_{n=1}^{\infty}\frac{(x^{p+1})_{x,n-1}(b)_{x,n}(d_1)_{x^2,n}(d_2)_{x^2,n}(1-x^{p+2n})\,x^{3pn+n^2+2n}}{(x)_{x,n}(x^{p+1}/b)_{x,n}(x^{2p+2}/d_1)_{x^2,n}(x^{2p+2}/d_2)_{x^2,n}d_1^{\,n}d_2^{\,n}b^n}\right\} \quad \ldots(1)$$

Here there are three free parameters, d_1, d_2, and b.

The following possibilities arise,

(1) d_1, d_2 and b tend to infinity,

(2) $d_1 = \pm x^r$, d_2 and b tend to infinity,

(3) $d_1 = \pm x^r$, $d_2 = \pm x^s$, and $b = \pm x^q$, or b tends to infinity, or b tends to zero,

(4) d_1 and d_2 tend to infinity, and $b = \pm x^q$,

(5) $d_1 = \pm x^r$, d_2 tends to infinity, and $b = \pm x^q$,

(6) d_1 and d_2 tend to infinity, and b tends to zero.,

(7) $d_1 = \pm x^r$, d_2 tends to infinity, and b tends to zero.

In every case summation of one of the series will be attempted by using the following theorem,

$$\prod_{n=1}^{\infty}(1-x^{a(2n-1)}z^b)(1-x^{a(2n-1)}/z^b)(1-x^{2an}) = \sum_{n=-\infty}^{\infty}(-1)^n x^{an^2} z^{bn}.$$

This result was originally due to Jacobi, (Fundamenta Nova, 1829).

Now
$$\sum_{n=-\infty}^{\infty}(-1)^n x^{an^2} z^{bn} = \sum_{n=1}^{\infty}(-1)^n x^{an^2} z^{bn} + \sum_{n=-\infty}^{-k-1}(-1)^n x^{an^2} z^{bn} + \sum_{n=-k}^{0}(-1)^n x^{an^2} z^{bn}$$

$$= \sum_{n=1}^{\infty}(-1)^n x^{an^2} z^{bn} + \sum_{n=1}^{\infty}(-1)^{n+k} x^{a(n+k)^2}/z^{b(n+k)} + \sum_{n=0}^{k}(-1)^{n+k}\frac{x^{a(-n+k)^2}}{z^{b(-n+k)}}$$

by changing $-n-k$ into n in the second ~~and third series~~.

series and n+k into n in the third series.

Hence,

$$1+\frac{\sum_{n=1}^{\infty}(-1)^n\left\{x^{an^2}z^{bn}+(-1)^k x^{a(k+n)^2}z^{-b(k+n)}\right\}}{\sum_{n=0}^{k}(-1)^{n+k}x^{a(k-n)^2}/z^{b(k-n)}}$$

$$=\frac{\prod_{n=1}^{\infty}(1-x^{a(2n-1)}z^b)(1-x^{a(2n-1)}/z^b)(1-x^{2an})}{\sum_{n=0}^{k}(-1)^{n+k}x^{a(k-n)^2}/z^{b(k-n)}} \quad \ldots(2)$$

This is the form in which we shall use the result.

In particular, if $k = 0$, $\sum_{n=0}^{k}(-1)^{n+k}x^{a(k-n)^2}/z^{b(k-n)} = 1$,

and if $k = 1$, $\sum_{n=0}^{k}(-1)^{n+k}x^{a(k-n)^2}/z^{b(k-n)} = 1 - x^a/z^b$.

§ (25.3) Case one of the second transformation.

Let d_1, d_2, and b tend to infinity, in eq. (1) § 25.2. Then, since $\lim_{a\to\infty}\frac{(a)_{x,n}}{a^n} = (-1)^n x^{\frac{1}{2}(n-1)n}$,

$$\sum_{n=0}^{\infty}\frac{x^{2n(p+n)}}{(x^2)_{x,n}(-x^{p+1})_{x,2n}} = \frac{1}{\prod_{n=1}^{\infty}(1-x^{2p+2n})}\times$$

$$\left\{1+\sum_{n=1}^{\infty}\frac{(x^{p+1})_{x,n-1}(1-x^{p+2n})(-1)^n x^{3pn+7n^2/2-n/2}}{(x)_{x,n}}\right\} \quad \ldots(1)$$

When $p = 0$, this equation becomes

$$\sum_{n=0}^{\infty}\frac{x^{2n^2}}{(x^2)_{x,n}(-x)_{x,2n}} = \left\{1+\sum_{n=1}^{\infty}(-1)^n(1+x^n)x^{7n^2/2-n/2}\right\}\times\frac{1}{\prod_{n=1}^{\infty}(1-x^{2n})}$$

$$=\prod_{n=1}^{\infty}\frac{(1-x^{7n})(1-x^{7n-3})(1-x^{7n-4})}{(1-x^{2n})} \quad \ldots(2)$$

and when p = 1,

$$\sum_{n=0}^{\infty}\frac{x^{2n(n+1)}}{(x^2)_{x^2,n}(-x^2)_{x,2n}} = \frac{1}{\prod_{n=1}^{\infty}(1-x^{2+2n})}\left\{1+\sum_{n=1}^{\infty}\frac{(x^2)_{x,n-1}\,x^{\frac{7n^2}{2}+\frac{5n}{2}}(1-x^{2n+1})(-1)^n}{(x)_{x,n}}\right\}$$

$$= \prod_{n=1}^{\infty}\frac{(1-x^{7n-1})(1-x^{7n-6})(1-x^{7n})}{(1-x)(1-x^{2+2n})} \quad \ldots(3)$$

(Bailey, "Some identities in combinatory analysis" (Proc. Lond.Maths.Soc. (2) 49. 1947) eq. (1.3), and (1.5)).

If p = 2, no such summation by Jacobi's theorem, is possible.

§ (25.4) Case two of the second transformation.

Let d_1 and b tend to infinity. In order to sum by Jacobi's theorem, we must take $d_2 = -x^{p+1}$. Then eq. (1) § 25.2, becomes

$$\sum_{n=0}^{\infty}\frac{(-x^{p+1})_{x^2,n}\,x^{n(p+n)}}{(x^2)_{x^2,n}(-x^{p+1})_{x,2n}}$$

$$= \prod_{n=1}^{\infty}\frac{(1+x^{p+2n-1})}{(1-x^{2p+2n})}\left\{1+\sum_{n=1}^{\infty}\frac{(x^{p+1})_{x,n-1}(1-x^{2n+p})(-1)^n x^{2pn+5n^2/2-n/2}}{(x)_{x,n}}\right\}$$

$$\ldots(1)$$

If p = o,

$$\sum_{n=0}^{\infty}\frac{(-x)_{x^2,n}\,x^{n^2}}{(x^2)_{x^2,n}(-x)_{x,2n}} = \prod_{n=1}^{\infty}\frac{(1+x^{2n-1})(1-x^{5n-2})(1-x^{5n-3})(1-x^{5n})}{(1-x^{2n})}$$

i.e.

$$\sum_{n=0}^{\infty}\frac{x^{n^2}}{(x^4)_{x^4,n}} = \prod_{n=1}^{\infty}\frac{1}{(1+x^{2n})(1-x^{5n-1})(1-x^{5n-4})} \quad \ldots(2)$$

("Identities of the R-R. Type" Bailey, Proc.Lond.Maths.Soc.1948 equs. (6.6) and (6.7)).

If p = 1,

$$\sum_{n=0}^{\infty}\frac{(-x^2)_{x^2,n}\,x^{n(n+1)}}{(x^2)_{x^2,n}(-x^2)_{x,2n}} = \prod_{n=1}^{\infty}\frac{(1+x^{2n})(1-x^{5n-1})(1-x^{5n-4})(1-x^{5n})}{(1-x^{2n+2})(1-x)}$$

i.e.
$$\sum_{n=0}^{\infty}\frac{x^{n(n+1)}}{(x^2)_{x^2,n}(-x^3)_{x^2,n}} = \prod_{n=1}^{\infty}\frac{1}{(1+x^{2n+1})(1-x^{5n-2})(1-x^{5n-3})} \quad \ldots(3)$$

§ (25.5) Case three of the second transformation.

Let $d_1 = \pm x^r$, $d_2 = \pm x^s$, and b tend to infinity. Then, in eq. (1) § 25.2, the product becomes

$$\prod_{n=1}^{\infty}\frac{(1\pm x^{2p+2n-r})(1\pm x^{2p+2n-s})}{(1-x^{2p+2n})(1\pm x^{2p+2n-r-s})}$$. The only possibility

for summation by Jacobi's theorem, is $r = s = p+1$. But these values give a product of the form $\prod_{n=1}^{\infty}(1-x^{2n-2})$ in the denominator. Hence the process gives no result in this case. The same thing happens when $b = \pm x^q$, and also when b tends to zero.

§ (25.6) Case four of the second transformation.

Let d_1 and d_2 tend to infinity, and let $b = -x^{\frac{1}{2}(p+1)}$, in eq. (1) § 25.2. Then

$$\sum_{n=0}^{\infty}\frac{(x^{\frac{1}{2}(p+1)})_{x,2n}\,x^{2(p+n)n}}{(x^2)_{x^2,n}(x^{p+1})_{x^2,n}(-x^{p+1})_{x,2n}}$$
$$= \frac{1}{\prod_{n=1}^{\infty}(1-x^{2p+2n})}\left\{1+\sum_{n=1}^{\infty}\frac{(-1)^n(x^{p+1})_{x,n-1}(1-x^{2n+p})x^{3n^2+\frac{1}{2}n(5p-1)}}{(x)_{x,n}}\right\} \quad \ldots(1)$$

If $p = o$, this gives,

$$\sum_{n=0}^{\infty}\frac{(x^{\frac{1}{2}})_{x,2n}\,x^{2n^2}}{(x^2)_{x^2,n}(x)_{x^2,n}(-x)_{x,2n}} = \prod_{n=1}^{\infty}\frac{(1-x^{6n-5/2})(1-x^{6n-7/2})(1-x^{6n})}{(1-x^{2n})}$$

Putting x^2 for x in this result, we get,

$$\sum_{n=0}^{\infty}\frac{x^{4n^2}}{(x^4)_{x^4,2n}} = \prod_{n=1}^{\infty}\frac{(1-x^{12n-5})(1-x^{12n-7})}{(1-x^{12n-4})(1-x^{12n-8})} \qquad \ldots(2)$$

If $p = 1$,

$$\sum_{n=0}^{\infty}\frac{(x)_{x^2,n}\,x^{2\,n(n+1)}}{(x^2)_{x^2,n}(-x)_{x,2n}} = \prod_{n=1}^{\infty}\frac{(1-x^{2n-1})}{(1-x^{6n-2})(1-x^{6n-3})(1-x^{6n-4})} \qquad \ldots(3)$$

§ (25.7) Case five of the second transformation.

Let d_1 tend to infinity, $d_2 = \pm x^r$, and $b = \pm x^q$

Then, eq. (1) § 25.2 becomes

$$\sum_{n=0}^{\infty}\frac{(-1)^n(\pm x^r)_{x^2,n}(\mp x^{p+1-q})_{x,2n}\,x^{2pn+2n}}{(x^2)_{x^2,n}(x^{2+2p-2q})_{x^2,n}(\pm x^r)^n(-x^{p+1})_{x,2n}}$$

$$= \prod_{n=1}^{\infty}\frac{(1\mp x^{2p+2n-r})}{(1-x^{2p+2n})}\times$$

$$\left\{1+\sum_{n=1}^{\infty}\frac{x^{3pn+n^2+2n}(x^{p+1})_{x,n-1}(\pm x^q)_{x,n}(-1)^n(\pm x^r)_{x^2,n}(1-x^{p+2n})\,x^{n(n-1)}}{(x)_{x,n}(\pm x^{p+1-q})_{x,n}(\pm x^{2p+2-r})_{x^2,n}(\pm x^r)^n(\pm x^q)^n}\right\} \qquad \ldots(1)$$

Here $p+1 = r$, and both signs must be positive or negative together. Either $q = \frac{1}{2}(p+1)$ or $q = 1$.

If $q = \frac{1}{2}(p+1)$, and $r = p+1$, and both signs are positive, we have,

$$\sum_{n=0}^{\infty}\frac{(-1)^n(-x^{\frac{1}{2}(p+1)})_{x,2n}\,x^{pn+n^2}}{(x^2)_{x^2,n}(-x^{p+1})_{x,2n}} = \prod_{n=1}^{\infty}\frac{(1-x^{p+2n-1})}{(1-x^{2p+2n})}\times$$

$$\left\{1+\sum_{n=1}^{\infty}\frac{(-1)^n(x^{p+1})_{x,n-1}(1-x^{p+2n})\,x^{n^2+\frac{1}{2}(3p+1)n}}{(x)_{x,n}}\right\} \qquad \ldots(2)$$

If $p = o$, this gives,

putting x^2 for x,

$$\sum_{n=0}^{\infty} \frac{(-1)^n(-x)_{x^2,2n}\, x^{2n^2}}{(x^4)_{x^4,n}(-x^2)_{x^2,2n}} = \prod_{n=1}^{\infty} (1-x^{8n})(1-x^{8n-3})(1-x^{8n-5}) \qquad \ldots(3)$$

If $p = 1$, we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n^2+n}}{(x^2)_{x^2,n}} = \prod_{n=1}^{\infty} \frac{(1-x^n)}{(1-x^{4n-2})} \qquad \ldots(4)$$

Next, suppose that both signs are negative together. If $p = o$, this gives equation (3) again, and if $p = 1$, we get equation (4). Now let $q = 1$, $r = p+1$, and take both signs positive. If $p = 1$, this gives equation (4), and if $p = o$ both series are undefined.

§ (25.8) Case six of the second transformation.

If $d_1 = -x^{p+1}$, d_2 tends to infinity, and b tends to zero, then, since

$$\lim_{b\to o} \left\{(1-x^{p+1}/b)\ldots\ldots(1-x^{p+n}/b)\ .\ b^n\right\} = (-1)^n x^{np+\frac{1}{2}n(n+1)},$$

eq. (1) § 25.2, becomes

$$\sum_{n=0}^{\infty} \frac{(-x^{p+1})_{x^2,n}\, x^{\frac{1}{2}(3n+1)n+pn}}{(x^2)_{x^2,n}(-x^{p+1})_{x,2n}} = \prod_{n=1}^{\infty} \frac{(1+x^{p+2n-1})}{(1-x^{2p+2n})} \times \left\{1 + \sum_{n=1}^{\infty} \frac{(x^{p+1})_{x,n-1}(1-x^{p+2n})(-1)^n x^{pn+\frac{1}{2}(3n-1)n}}{(x)_{x,n}}\right\} \qquad \ldots(1)$$

If $p = o$, this gives,

$$\sum_{n=0}^{\infty} \frac{x^{\frac{1}{2}(3n+1)n}}{(x^4)_{x^4,n}} = \prod_{n=1}^{\infty}(1+x^{4n-2}) \qquad \ldots(2)$$

If $p = 1$, we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{\frac{1}{2}(3n+3)n}}{(x^2)_{x^2,n}(-x^3)_{x^2,n}} = \prod_{n=1}^{\infty} \frac{1}{(1+x^{2n+1})} \qquad \ldots(3)$$

§ (25.9) Case seven of the second transformation.

If d_1 and d_2 tend to infinity, and b tends to zero, then eq. (1) § 25.2 becomes

$$\sum_{n=0}^{\infty} \frac{x^{3n^2+2pn}(-1)^n}{(x^2)_{x^2,n}(-x^{p+1})_{x,2n}} = \frac{1}{\prod_{n=1}^{\infty}(1-x^{2p+2n})} \times$$

$$\left\{ 1 + \sum_{n=1}^{\infty} \frac{(x^{p+1})_{x,n-1}(1-x^{p+2n})x^{\frac{1}{2}(5n^2-n)+2pn}(-1)^n}{(x)_{x,n}} \right\} \qquad \ldots(1)$$

If p = o, we get

$$\sum_{n=0}^{\infty} \frac{x^{3n^2}(-1)^n}{(x^4)_{x^4,n}(-x)_{x^2,n}} = \prod_{n=1}^{\infty} \frac{(1-x^{2n-1})}{(1-x^{5n-1})(1-x^{5n-4})} \qquad \ldots(2)$$

(Bailey, Some identities in combinatory analysis, Proc.Lond. Maths. Soc. (2) 49, (1947), eq. 4.3)

and if p = 1,

$$\sum_{n=0}^{\infty} \frac{x^{3n^2+2n}(-1)^n}{(x^4)_{x^4,n}(-x)_{x^2,n}} = \prod_{n=1}^{\infty} \frac{(1-x^{2n-1})}{(1-x^{5n-3})(1-x^{5n-2})} \qquad \ldots(3)$$

(Loc.cit.eq.5.3)

§ (26.1) The fourth transformation.

If a = x^p, where p is a positive integer, the fourth transformation (eq.(4), § 25.1) becomes

$$\sum_{n=0}^{\infty} \frac{(d_1)_{x,n}(d_2)_{x,n}(x^{p+1}/f)_{x^2,n}x^{(p+1)n}}{(x)_{x,n}(x^{p+1})_{x^2,n}(x^{p+1}/f)_{x,n}d_1^n d_2^n} =$$

$$\prod_{n=1}^{\infty}\frac{(1-x^{p+n}/d_1)(1-x^{p+n}/d_2)}{(1-x^{p+n})(1-x^{p+n}/d_1d_2)}\times$$

$$\left\{1+\sum_{n=1}^{\infty}\frac{(x^{p+2})_{x^2,n-1}(f)_{x^2,n}(1-x^{p+4n})(d_1)_{x,2n}(d_2)_{x,2n}x^{3pn+2n^2+2n}}{(x^2)_{x^2,n}(x^{p+2}/f)_{x^2,n}(x^{p+1}/d_1)_{x,2n}(x^{p+1}/d_2)_{x,2n}d_1^{2n}d_2^{2n}f^n}\right\}$$

Here there are three free parameters d_1, d_2 and f. The following possibilities arise:-

(1) d_1, d_2 and f all tend to infinity.

(2) $d_1 = \pm x^r$, d_2 and f tend to infinity.

(3) $d_1 = \pm x^r$, $d_2 = \pm x^s$, and f tends to infinity, or

$f = \pm x^q$, or

f tends to zero.

(4) d_1 and d_2 tend to infinity, and $f = \pm x^q$.

(5) $d_1 = \pm x^r$, d_2 tends to infinity, and $f = \pm x^q$.

(6) d_1 and d_2 tend to infinity, and f tends to zero.

(7) $d_1 = \pm x^r$, d_2 tends to infinity, and f tends to zero.

§ (26.2) Case one of the ~~second~~ fourth transformation.

If d_1, d_2 and f all tend to infinity, then when $p = 0$ we have,

$$\sum_{n=0}^{\infty}\frac{x^{n^2}}{(x)_{x,n}(x)_{x^2,n}} = \prod_{n=1}^{\infty}\frac{(1-x^{14n-6})(1-x^{14n-8})(1-x^{14n})}{(1-x^n)} \quad \ldots(1)$$

When $p = 1$, no summation by Jacobi's theorem, is possible.

When $p = 2$, we have

$$\sum_{n=0}^{\infty}\frac{x^{n^2+2n}}{(x)_{x,n}(x)_{x^2,n+1}} = \prod_{n=1}^{\infty}\frac{(1-x^{14n-2})(1-x^{14n-12})(1-x^{14n})}{(1-x^n)} \quad \ldots(2)$$

(Bailey, "Some identities in combinatory analysis", equs. (10.2) and (10.3)).

§ (26.3) Case two of the ~~second~~ fourth transformation.

Let d_2 and f tend to infinity, and $d_1 = -x^{\frac{1}{2}(p+1)}$, in eq. (1) § 26.1. Then, putting x^2 for x, if $p = 0$, we have

$$\sum_{n=0}^{\infty} \frac{(-x)_{x^2,n} x^{n^2}}{(x^2)_{x^2,n}(x^2)_{x^4,n}} = \prod_{n=1}^{\infty} \frac{(1+x^{2n-1})(1-x^{20n-12})(1-x^{20n-8})(1-x^{20n})}{(1-x^{2n})} \qquad \ldots(1)$$

and if $p = 2$,

$$\sum_{n=0}^{\infty} \frac{(-x^3)_{x^2,n} x^{2n+n^2}}{(x^2)_{x^2,n}(x^2)_{x^4,n+1}} = \prod_{n=1}^{\infty} \frac{(1+x^{2n+1})(1-x^{20n-4})(1-x^{20n-16})(1-x^{20n})}{(1-x^{2n})} \qquad \ldots(2)$$

§ (26.4) Case three of the fourth transformation.

If $d_1 = \pm x^r$ and $d_2 = \pm x^s$, eq. (1) § 26.1, involves a product of the form

$$\prod_{n=1}^{\infty} \frac{(1\mp x^{p+n-r})(1\mp x^{p+n-s})}{(1-x^{p+n})(1-x^{p+n-r-s})}$$

In this case, the only possibility for summation by Jacobi's theorem is $r = s = \frac{1}{2}(p+1)$. But, when r and s have this value, a product of the form $\prod_{n=1}^{\infty}(1-x^{n-1})$ arises in the denominator, and so this process gives no result.

§ (26.5) Case four of the fourth transformation.

Let d_1 and d_2 tend to infinity, and $f = -x^{1+\frac{1}{2}p}$, in the fourth transformation. Then, if $p = 0$, we have

$$\sum_{n=0}^{\infty} \frac{(-x^2)_{x^2,n-1} x^{n^2}}{(x)_{x,n}(x)_{x^2,n}(-x)_{x,n-1}} = \prod_{n=1}^{\infty} \frac{(1-x^{12n-5})(1-x^{12n-7})(1-x^{12n})}{(1-x^n)} \qquad \ldots(1)$$

If p = 2,

$$\sum_{n=0}^{\infty}\frac{(-x)_{x^2,n}\,x^{n^2+2n}}{(x)_{x,n}(x)_{x^2,n+1}(-x)_{x,n}}$$

$$=\prod_{n=1}^{\infty}\frac{(1-x^{12n-2})(1-x^{12n-10})(1-x^{12n})}{(1-x^{n})}\qquad\ldots(2)$$

§ (26.6) Case five of the ~~second~~ fourth transformation.

Let $d_1 = \pm x^{\frac{1}{2}(p+1)}$, d_2 tend to infinity, and $f = x^{1+\frac{1}{2}p}$. Then, putting x^2 for x and taking the positive sign, if p = o,

$$\sum_{n=0}^{\infty}\frac{(x)_{x^2,n}(x^4)_{x^4,n-1}\,x^{n^2}}{(x^2)_{x^2,n}(x^2)_{x^4,n}(x^2)_{x^2,n-1}}$$

$$=\prod_{n=1}^{\infty}\frac{(1+x^{2n-1})(1-x^{16n-6})(1-x^{16n-10})(1-x^{16n})}{(1-x^{2n})}\qquad\ldots(1)$$

If p = 2,

$$\sum_{n=0}^{\infty}\frac{(x^3)_{x^2,n}(x^2)_{x^4,n}\,x^{2n+n^2}}{(x^2)_{x^2,n}(x^2)_{x^4,n+1}(x^2)_{x^2,n}}$$

$$=\prod_{n=1}^{\infty}\frac{(1+x^{2n+1})(1-x^{16n-4})(1-x^{16n-12})(1-x^{16n})}{(1-x^{2n})}\qquad\ldots(2)$$

Taking the negative sign, if p = 0,

$$\sum_{n=0}^{\infty}\frac{(-x)_{x^2,n}(-1)^n(x^4)_{x^4,n-1}\,x^{n^2}}{(x^2)_{x^2,n}(x^2)_{x^4,n}(x^2)_{x^2,n-1}}$$

$$=\prod_{n=1}^{\infty}\frac{(1-x^{2n-1})(1-x^{16n-10})(1-x^{16n-6})(1-x^{16n})}{(1-x^{2n})}\qquad\ldots(3)$$

and if p = 2,

$$\sum_{n=0}^{\infty}\frac{(-x^3)_{x,n}(-1)^n(x^2)_{x^4,n}\,x^{2n+n^2}}{(x^2)_{x^2,n}(x^2)_{x^4,n+1}(x^2)_{x^2,n}}$$

$$=\prod_{n=1}^{\infty}\frac{(1-x^{2n+1})(1-x^{16n-4})(1-x^{16n-12})(1-x^{16n})}{(1-x^{2n})}\qquad\ldots(4)$$

§ (26.7) Case six of the fourth transformation.

If d_1 and d_2 tend to infinity, and f tends to zero, since $\lim_{f\to 0} \dfrac{(x^{p+1}/f)_{x,2n}}{(x^p/f)_{x,n}} = x^{\frac{1}{2}n(n-1)}$,

eq. (1) § 26.1, becomes

$$\sum_{n=0}^{\infty} \frac{x^{\frac{1}{2}(3n-1)n+pn}}{(x)_{x,n}(x^{p+1})_{x^2,n}} = \prod_{n=1}^{\infty} \frac{1}{(1-x^{p+n})} \times$$

$$\left\{ 1 + \sum_{n=1}^{\infty} \frac{(x^{p+2})_{x^2,n-1}(1-x^{p+4n})(-1)^n x^{2pn+5n^2-n}}{(x^2)_{x^2,n}} \right\} \quad \ldots(1)$$

If p = o,

$$\sum_{n=0}^{\infty} \frac{x^{\frac{1}{2}(3n-1)n}}{(x)_{x,n}(x)_{x^2,n}} = \prod_{n=1}^{\infty} \frac{(1-x^{10n-6})(1-x^{10n-4})(1-x^{10n})}{(1-x^n)} \quad \ldots(2)$$

If p = 2,

$$\sum_{n=0}^{\infty} \frac{x^{\frac{1}{2}(n+1)3n}}{(x)_{x,n}(x)_{x^2,n+1}} = \prod_{n=1}^{\infty} \frac{(1-x^{10n-2})(1-x^{10n-8})(1-x^{10n})}{(1-x^n)} \quad \ldots(3)$$

("Some identities in combinatory analysis" equs. 10.4, and 10.5)

§ (26.8) Case seven of the fourth transformation.

If d_1 tends to infinity, $d_2 = x^{\frac{1}{2}(p+1)}$, and f tends to zero, when p = 0,

$$\sum_{n=0}^{\infty} \frac{(x)_{x^2,n}(-1)^n x^{2n^2-n}}{(x^2)_{x^2,n}(x^2)_{x^4,n}} = \prod_{n=1}^{\infty} \frac{(1-x^{2n-1})(1-x^{12n-4})(1-x^{12n-8})(1-x^{12n})}{(1-x^{2n})} \quad \ldots(1)$$

and, when p = 2,

$$\sum_{n=0}^{\infty} \frac{(x^3)_{x^2,n}(-1)^n x^{2n^2+n}}{(x^2)_{x^2,n}(x^2)_{x^4,n+1}} = \prod_{n=1}^{\infty} \frac{(1-x^{2n+1})(1-x^{12n-4})(1-x^{12n-8})(1-x^{12n})}{(1-x^{2n})} \qquad \ldots(2)$$

§ (27.1) Another transformation.

From each general transformation, involving two infinite basic series, we can obtain identities of the Rogers-Ramanujan type. For example, summing both Φ_β and Φ_γ as infinite ${}_6\Phi_5$ series (or by letting N tend to infinity in eq.(2) § 24.3) we have

$$\frac{\Pi(fq)\Pi(fq/d_1d_2)\Pi(fq/d_1e)\Pi(fq/d_2e)}{\Pi(fq/d_1)\Pi(fq/d_2)\Pi(fq/e)\Pi(fq/d_1d_2e)} \times$$

$${}_6\Phi_5\left[\begin{matrix} f/e, q\sqrt{f/e}, -q\sqrt{f/e}, & a, & d_1, & d_2; \ fq/d_1d_2ea \\ \sqrt{f/e}, & -\sqrt{f/e}, fq/ae, fq/d_1e, fq/d_2e; & & \end{matrix}\right] =$$

$${}_6\Phi_5\left[\begin{matrix} f, q\sqrt{f}, -q\sqrt{f}, & ae, & d_1, & d_2; \ fq/d_1d_2ea \\ \sqrt{f}, & -\sqrt{f}, fq/ae, fq/d_1, fq/d_2; & & \end{matrix}\right] \qquad \ldots(1)$$

If we put x for q, a for f, b for 1/e, c for a, d for d_1, and e for d_2, equation one above becomes

$$\prod_{n=1}^{\infty} \frac{(1-ax^n)(1-ax^n/de)(1-ax^n b/d)(1-abx^n/e)}{(1-ax^n/d)(1-ax^n/e)(1-abx^n)(1-abx^n/de)} \times$$

$$\left\{1 + \sum_{n=1}^{\infty} \frac{(abx)_{x,n-1}(1-abx^{2n})(c)_{x,n}(d)_{x,n}(e)_{x,n} a^n b^n x^n}{(x)_{x,n}(abx/c)_{x,n} \text{[illegible]} (abx/d)_{x,n}(abx/e)_{x,n} c^n d^n e^n}\right\}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(ax)_{x,n-1}(1-ax^{2n})(c/b)_{x,n}(d)_{x,n}(e)_{x,n} a^n b^n x^n}{(x)_{x,n}(abx/c)_{x,n}(ax/d)_{x,n}(ax/e)_{x,n} c^n d^n e^n} \qquad \ldots(2)$$

Let c, d and e tend to infinity. Then, since

$$\lim_{c\to\infty} \frac{(c/b)_{x,n} b^n}{c^n} = (-1)^n x^{\frac{1}{2}(n-1)n},$$

$$\prod_{n=1}^{\infty}\frac{(1-ax^n)}{(1-abx^n)}\left\{1+\sum_{n=1}^{\infty}\frac{(abx)_{x,n-1}(1-abx^{2n})(-ab)^n x^{\frac{1}{2}(3n-1)n}}{(x)_{x,n}}\right\}$$

$$=1+\sum_{n=1}^{\infty}\frac{(ax)_{x,n-1}(1-ax^{2n})(-a)^n x^{\frac{1}{2}(3n-1)n}}{(x)_{x,n}} \qquad \ldots(3)$$

If a = 1, this gives

$$1+\sum_{n=1}^{\infty}\frac{(bx)_{x,n-1}(1-bx^{2n})(-b)^n x^{\frac{1}{2}(3n-1)n}}{(x)_{x,n}} = \prod_{n=1}^{\infty}(1-bx^n). \qquad \ldots(4)$$

Next, let c and d tend to infinity, and $e = -\sqrt{ax}$, then

$$\prod_{n=1}^{\infty}\frac{(1-ax^n)(1+\sqrt{ax}\,bx^n)}{(1-abx^n)(1+\sqrt{ax}.x^n)}\left\{1+\sum_{n=1}^{\infty}\frac{(abx)_{x,n-1}(1-abx^{2n})(-\sqrt{ax})_{x,n}(-\sqrt{ab})^n}{(x)_{x,n}(-b\sqrt{ax})_{x,n}\; x^{\frac{1}{2}n-n^2}}\right\}$$

$$=1+\sum_{n=1}^{\infty}\frac{(ax)_{x,n-1}(1-ax^{2n})(-1)^n a^{\frac{1}{2}n} x^{n^2-\frac{1}{2}n}}{(x)_{x,n}} \qquad \ldots(5)$$

If a = 1, then, putting x^2 for x, we have

$$1+\sum_{n=1}^{\infty}\frac{(bx^2)_{x^2,n-1}(1-bx^{4n})(-x)_{x^2,n}(-b)^n x^{2n^2-n}}{(x^2)_{x^2,n}(-bx)_{x^2,n}}$$

$$=\prod_{n=1}^{\infty}\frac{(1-bx^{2n})}{(1+bx^{2n-1})} \qquad \ldots(6)$$

If a = x,

$$1+\sum_{n=1}^{\infty}\frac{(bx^2)_{x,n-1}(1-bx^{2n+1})(-x)_{x,n}(-b)^n x^{n^2}}{(x)_{x,n}(-bx)_{x,n}}$$

$$=\prod_{n=1}^{\infty}\frac{(1-bx^{n+1})(1+x^{n+1})(1-x^{2n+1})}{(1+bx^{n+1})} \qquad \ldots(7)$$

Further similar results are given

(1) by letting d tend to infinity, $c = -b\sqrt{ax}$ and $e = -\sqrt{ax}$,

(2) by letting d and e tend to infinity, and $c = -b\sqrt{ax}$,

(3) by letting d and e tend to infinity, and c tend to zero,

(4) by letting $d = -\sqrt{ax}$, e tend to infinity, and c tend to zero.

§ (27.2) List of results for ordinary hypergeometric series.

Several of the results of § 3.2–9.6 have not been given before, but they are perhaps too trivial to be of interest. The results of § 10.1–14.3 are new, but in the form in which they arise here, they are rather cumbrous. In § 15.1–17.5, substantially the same results arise in more elegant forms. The most interesting of these results are:—

(1) $${}_7F_6\left[\begin{matrix} a, 1+\tfrac12 a, & b, & b+\tfrac12, & c, & c+\tfrac12, & d; & 1 \\ \tfrac12 a, & 1+a-b, & \tfrac12+a-b, & 1+a-c, & \tfrac12+a-c, & 1+a-d; & \end{matrix}\right]$$

$$= \frac{\Gamma(1+2a-2b)\Gamma(1+2a-2c)}{\Gamma(1+2a)\Gamma(1+2a-2b-2c)}\, {}_3F_2\left[\begin{matrix} a+\tfrac12-d, & 2b, & 2c; & 1 \\ 1+2a-2d, & \tfrac12+a; & \end{matrix}\right]$$

This transforms a well-poised ${}_7F_6(1)$ into a ${}_3F_2(1)$.

(§ 16.1, eq (1)).

(2) $${}_8F_7\left[\begin{matrix} a, 1+\tfrac12 a, & b, & b+1/3, & b+2/3, & c, & c+1/3, & c+2/3; & -1 \\ \tfrac12 a, & 1+a-b, & 2/3+a-b, & 1/3+a-b, & 1+a-c, & 2/3+a-c, & 1/3+a-c; & \end{matrix}\right]$$

$$= \frac{\Gamma(1+3a-3b)\Gamma(1+3a-3c)}{\Gamma(1+3a)\Gamma(1+3a-3b-3c)}\, {}_3F_2\left[\begin{matrix} a, & 3b, & 3c; & 3/4 \\ 3/2\,a, & \tfrac12(1+3a); & \end{matrix}\right]$$

This transforms a well-poised ${}_8F_7(-1)$ into a ${}_3F_2(\tfrac34)$.

(§ 16.1, eq. (2)).

(3) $${}_7F_6\left[\begin{matrix} a, 1+\tfrac12 a, & b, & \tfrac14(2a-b), & \tfrac14(1+2a-b), & \tfrac14(2+2a-b), & \tfrac14(3+2a-b); & 1 \\ \tfrac12 a, & 1+a-b, & 1+\tfrac14(2a+b), & \tfrac14(3+2a+b), & \tfrac14(2+2a+b), & \tfrac14(1+2a+b); & \end{matrix}\right]$$

$$= \frac{\Gamma(1+b)\Gamma(1+2a+b)}{\Gamma(1+2a-b)\Gamma(1+2b)}\, {}_3F_2\left[\begin{matrix} 2a-b, & b, & \tfrac12+a-b; & -1 \\ 1+2a-2b, & \tfrac12+a; & \end{matrix}\right]$$

This transforms a special well-poised ${}_7F_6(1)$ into a well-poised ${}_3F_2(-1)$. (§ 16.2)

(4) $$ {}_9F_8\left[\begin{matrix} a, 1+\tfrac12 a, & b, & \tfrac12+b, & c, & \tfrac12+c, & d, & -\tfrac12 N, & \tfrac12(1-N); & 1 \\ & \tfrac12 a, & 1+a-b, & \tfrac12+a-b, & 1+a-c, & \tfrac12+a-c, & 1+a-d, & 1+a+\tfrac12 N, & \tfrac12+a+\tfrac12 N; & \end{matrix}\right] $$

$$ = \frac{(1+2a)_N(1+2a-2b-2c)_N}{(1+2a-2b)_N(1+2a-2c)_N}\; {}_4F_3\left[\begin{matrix} \tfrac12+a-d, & 2b, & 2c, & -N; & 1 \\ \tfrac12+a, & 1+a-2d, & 2b+2c-2a-N & ; & \end{matrix}\right] $$

This transforms a well-poised terminating ${}_9F_8(1)$ into a nearly-poised terminating ${}_4F_3(1)$. (§ 16.5, eq. (1)).

(5). $$ {}_{11}F_{10}\left[\begin{matrix} a, 1+\tfrac12 a, & b, & b+1/3, & b+2/3, & c, & c+1/3, & c+2/3, & -N/3, & (1-N)/3, & (2-N)/3 & ; & 1 \\ & \tfrac12 a, & 1+a-b, & a-b+2/3, & a-b+1/3, & 1+a-c, & a-c+2/3, & a-c+1/3, & 1+a+N/3, & a+(2+N)/3, & a+(1+N)/3; & \end{matrix}\right] $$

$$ = \frac{(1+3a)_N(1+3a-3b-3c)_N}{(1+3a-3b)_N(1+3a-3c)_N}\; {}_4F_3\left[\begin{matrix} a, & 3b, & 3c, & -N; & 3/4 \\ 1+3a/2, & (1+3a)/2, & 3b+3c-3a-N; & & \end{matrix}\right] $$

This transforms a well-poised terminating ${}_{11}F_{10}(1)$ into a ${}_4F_3(3/4)$. (§ 16.5, eq. (2)).

(6). $$ {}_{11}F_{10}\left[\begin{matrix} a, 1+\tfrac12 a, & b, & \tfrac14(2a-b), & \tfrac14(1+2a-b), & \tfrac14(2+2a-b), & \tfrac14(3+2a-b), & a+\tfrac12(1+b+N), & a+1+\tfrac12(b+N), & -\tfrac12 N, & \tfrac12(1-N); & 1 \\ & \tfrac12 a, & 1+a-b, & 1+\tfrac14(2a+b), & \tfrac14(3+2a+b), & \tfrac14(2+2a+b), & \tfrac14(1+2a+b), & \tfrac12(1-b-N), & -\tfrac12(b+N), & 1+a+\tfrac12 N, & a+\tfrac12(1+N); & \end{matrix}\right] $$

$$ = \frac{(1+2b)_N(1+2a)_N}{(1+b)_N\,(1+2a+b)_N}\; {}_4F_3\left[\begin{matrix} 2a-b, & b, & \tfrac12+a-b, & -N; & -1 \\ & 1+2a-2b, & \tfrac12+a, & -N-2b; & \end{matrix}\right] $$

This transforms a well-poised terminating ${}_{11}F_{10}(1)$ into a nearly-poised terminating ${}_4F_3(-1)$. (§ 16.6)

(7) $$ {}_{11}F_{10}\left[\begin{matrix} a, 1+\tfrac12 a, & b, & b+\tfrac12, & c, & c+\tfrac12, & d, & d+\tfrac12, & e, & -\tfrac12 N, & \tfrac12(1-N); & 1 \\ & \tfrac12 a, & 1+a-b, & \tfrac12+a-b, & 1+a-c, & \tfrac12+a-c, & 1+a-d, & \tfrac12+a-d, & 1+a-e, & 1+a+\tfrac12 N, & \tfrac12+a+\tfrac12 N; & \end{matrix}\right] $$

$$ = \frac{(1+2a-2b-e)_N(1+2a-2c-e)_N(1+2a-2e)_N(1+2a-2b-2c)_N}{(1+2a-e)_N(1+2a-2b)_N(1+2a-2c)_N(1+2a-2b-2c-e)_N} \times $$

$$ {}_8F_7\left[\begin{matrix} 2a-e, 1+a-\tfrac12 e, & b, & c, & d, & e, & \tfrac12+a-e, & -N; & -1 \\ & a-\tfrac12 e, & 1+2a-e-b, & 1+2a-e-c, & 1+2a-e-d, & 1+2a-2e, & \tfrac12+a, & 1+2a-e+N; & \end{matrix}\right] $$

This transforms a well-poised ${}_{11}F_{10}(1)$ into a well-poised terminating ${}_8F_7(-1)$. (§ 17.1)

(8)
$$
{}_{11}F_{10}\left[\begin{matrix} a, 1+\tfrac12 a, b, \tfrac14(2a-b), \tfrac14(1+2a-b), \tfrac14(2+2a-b), \tfrac14(3+2a-b), a+\tfrac12(b+N), a+\tfrac12(1+b+N), -\tfrac12 N, \tfrac12(1-N); 1 \\ \tfrac12 a, 1+a-b, 1+\tfrac14(2a+b), \tfrac14(3+2a+b), \tfrac14(2+2a+b), \tfrac14(1+2a+b), 1-\tfrac12(b+N), \tfrac12(1-b-N), 1+a+\tfrac12 N, \tfrac12(1+a+N); \end{matrix}\right]
$$
$$
= \frac{(1+2a)_N (a+\tfrac12 b)_N (2b)_N}{(2a+b)_N (1+a+\tfrac12 b)_N (b)_N}\ {}_4F_3\left[\begin{matrix} 2a-b, b, \tfrac12+a-b, -N; -1 \\ 1+2a-2b, \tfrac12+a, 1-2b-N; \end{matrix}\right].
$$

This transforms a terminating well-poised ${}_{11}F_{10}(1)$ into a nearly-poised ${}_4F_3(-1)$. (§ 17.3).

(9)
$$
{}_{11}F_{10}\left[\begin{matrix} a, 1+\tfrac12 a, b, \tfrac14(2a-b+1), \tfrac14(2a-b+2), \tfrac14(2a-b+3), 1+\tfrac14(2a-b), a+\tfrac12(1+b+N), a+\tfrac12(b+N), -\tfrac12 N, \tfrac12(1-N); 1 \\ \tfrac12 a, 1+a-b, \tfrac14(2a+b+3), \tfrac14(2a+b+2), \tfrac14(2a+b+1), \tfrac14(2a+b), \tfrac12(1-b-N), 1-\tfrac12(b+N), 1+a+\tfrac12 N, \tfrac12+a+\tfrac12 N; \end{matrix}\right]
$$
$$
= \frac{(1+2a)_N (2b)_N}{(2a+b)_N (b)_N}\ {}_5F_4\left[\begin{matrix} 2a-b, 1+a-\tfrac12 b, b, \tfrac12+a-b, -N; -1 \\ a-\tfrac12 b, 1+2a-2b, \tfrac12+a, 1-2b-N; \end{matrix}\right]
$$

This transforms a well-poised terminating ${}_{11}F_{10}(1)$ into a nearly-poised terminating ${}_5F_4(-1)$. (§ 17.4)

(10)
$$
{}_4F_3\left[\begin{matrix} a, -N/3, (1-N)/3, (2-N)/3; 1 \\ 1+a-b-N/3, a-b+(2-N)/3, a-b+(1-N)/3; \end{matrix}\right]
$$
$$
= \frac{(3b)_N}{(3b-3a)_N}\ {}_3F_2\left[\begin{matrix} a, 3b+N, -N; 3/4 \\ 3b/2, \tfrac12(1+3b); \end{matrix}\right]
$$

This transforms a nearly-poised ${}_4F_3(1)$ into a special ${}_3F_2(\tfrac34)$. (§ 17.5, eq. (2)).

§ (27.3) List of results for basic series.

In § 20.5-24.3, most of the results are already known. The following are of the greatest interest:-

(1) $${}_{12}\Phi_{11}\left[\begin{matrix} aq^{-N}, q^2\sqrt{aq^{-N}}, -q^2\sqrt{aq^{-N}}, & a^2, & q^{-\frac12 N}, & -q^{-\frac12 N}, \\ \sqrt{aq^{-N}}, & -\sqrt{aq^{-N}}, q^{2-N}/a, aq^{2-\frac12 N}, -aq^{2-\frac12 N}, \end{matrix}\right.$$
$$\left.\begin{matrix} q^{\frac12(1-N)}, & -q^{\frac12(1-N)}, & q^{1-\frac12 N}, & -q^{1-\frac12 N}, & q^{\frac12(3-N)}, & -q^{\frac12(3-N)}; & a^2q^2 \\ aq^{\frac12(3-N)}, & -aq^{\frac12(3-N)}, & aq^{1-\frac12 N}, & -aq^{1-\frac12 N}, & aq^{\frac12(1-N)}, & -aq^{\frac12(1-N)}; & \end{matrix}\right]_{q^2}$$

$$\cancel{= \frac{(aq^{1-N})_{q,N}\, a^N}{(a^2q^{1-N})_{q,N}}}$$

$$= \frac{(1/a)_{q,N}}{(1/a^2)_{q,N}}\ {}_4\Phi_3\left[\begin{matrix} q^{-N}, \sqrt{q^{1-N}/a}, -\sqrt{q^{1-N}/a}, & a; & -q/a \\ \sqrt{aq^{1-N}}, & -\sqrt{aq^{1-N}}, q^{1-N}/a\ ; & \end{matrix}\right]_q$$

This transforms a well-poised ${}_{12}\Phi_{11}$ into a well-poised ${}_4\Phi_3$. (§ 21.4, eq. (3)).

(2) $${}_8\Phi_7\left[\begin{matrix} a, q^2\sqrt{a}, -q^2\sqrt{a}, & b, & c, & cq, & q^{-N}, & q^{1-N}; & \frac{aq^{2N+2-2n}}{bc^2} \\ \sqrt{a}, & -\sqrt{a}, & aq^2/b, & aq^2/c, & aq/c, & aq^{2+N}, aq^{1+N}; & \end{matrix}\right]_{q^2}$$

$$= \frac{(aq)_{q,N}}{(aq/c)_{q,N}\, c^N}\ {}_4\Phi_3\left[\begin{matrix} c, \sqrt{aq/b}, -\sqrt{aq/b}, q^{-N}; & q \\ \sqrt{aq}, -\sqrt{aq}, aq/b\ ; & \end{matrix}\right]_q$$

This transforms a well-poised ${}_8\Phi_7$ into a ${}_4\Phi_3$. (§ 21.7. eq(2))

(3) $${}_9\Phi_8\left[\begin{matrix} a, q^3\sqrt{a}, -q^3\sqrt{a}, & b, & bq, & bq^2, & q^{-N}, & q^{1-N}, & q^{2-N}; & \frac{-aq^{3N}}{b^3q^{\frac12(9n-1)}} \\ \sqrt{a}, & -\sqrt{a}, & aq^3/b, & aq^2/b, & aq/b, & aq^{3+N}, & aq^{2+N}, aq^{1+N}; & \end{matrix}\right]_{q^3}$$
$$= \frac{(aq)_{q,N}}{(aq/b)_{q,N}\, b^N}\sum_{n=o}^{N}\frac{(a)_{q^3,n}(aq^3)_{q^3,2n/3}(b)_{q,n}(-q^{-N})_{q,n}\, q^n}{(q)_{q,n}(aq)_{q,2n}(a)_{q^3,n/3}}$$

(§ 21.7 eq. (3)).

(4) $${}_{12}\Phi_{11}\left[\begin{matrix} fq^N, q^2\sqrt{fq^N}, -q^2\sqrt{fq^N}, & \sqrt{f}, & -\sqrt{f}, & \sqrt{qf}, \\ \sqrt{fq^N}, & -\sqrt{fq^N}, q^{2+N}\sqrt{f}, -q^{2+N}\sqrt{f}, q^{1+N}\sqrt{qf}, & & \end{matrix}\right.$$

$$\left.\begin{matrix} -\sqrt{qf}, & -q\sqrt{f}, & q\sqrt{f}, & q\sqrt{qf}, & -q\sqrt{qf}, & q^{2N}; & q^{2N+2} \\ -q^{1+N}\sqrt{qf}, & -q^{1+N}\sqrt{f}, & q^{1+N}\sqrt{f}, & q^N\sqrt{qf}, & -q^N\sqrt{qf}, & fq^{2-N}; & \end{matrix}\right]_{q^2} =$$

$$\frac{(q^{1+N}f)_{q,N}}{(q^{1+N})_{q,N}f^N}\;{}_4\Phi_3{}_{q}\left[\begin{matrix} f, \sqrt{fq^{1-N}}, -\sqrt{fq^{1-N}}, & q^N & ; -q^{1-N} \\ \sqrt{fq^{1+N}}, -\sqrt{fq^{1+N}}, & fq^{1-N} & ; \end{matrix}\right]$$

This transforms a well-poised ${}_{12}\Phi_{11}$ into a well-poised ${}_4\Phi_3$. (§ 21.9.eq.(3)).

(5) $${}_8\Phi_7{}_{q^2}\left[\begin{matrix} a, q^2\sqrt{a}, -q^2\sqrt{a}, & b, & c, & cq, & d, & dq & ; \frac{a^3q^{2n+2}}{bc^2d^2} \\ \sqrt{a}, & -\sqrt{a}, & aq^2/b, & aq^2/c, & aq/c, & aq^2/d, aq/d & ; \end{matrix}\right]$$

$$= \frac{\Pi(aq)\Pi(aq/cd)}{\Pi(aq/c)\Pi(aq/d)}\;{}_4\Phi_3{}_{q}\left[\begin{matrix} \sqrt{aq/b}, -\sqrt{aq/b}, & c, & d; & aq/dc \\ aq/b, \sqrt{aq}, -\sqrt{aq} & & ; & \end{matrix}\right]$$

This transforms a well-poised ${}_8\Phi_7$ into a ${}_4\Phi_3$.

(§ 22.4.eq.(2)).

(6) $${}_9\Phi_8{}_{q^3}\left[\begin{matrix} a, q^3\sqrt{a}, -q^3\sqrt{a}, & b, & bq, & bq^2, & c, & cq, & cq^2 & ; \frac{a^4q^{\frac{1}{2}(3n-1)3n}}{b^3c^3} \\ \sqrt{a}, & -\sqrt{a}, & aq^3/b, & aq^2/b, & aq/b, & aq^3/c, & aq^2/c, & aq/c & ; \end{matrix}\right]$$

$$= \frac{\Pi(aq)\Pi(aq/bc)}{\Pi(aq/b)\Pi(aq/c)}\sum_{n=0}^{\infty}\frac{(a)_{q^3,n}(aq^3)_{q^3,2n/3}(b)_{q,n}(c)_{q,n}}{(q)_{q,n}(aq)_{q,2n}(a)_{q^3,n/3}}\frac{a^nq^{m}}{b^nc^{m}}$$

(§ 22.4. eq.(3)).

(7) $${}_{12}\Phi_{11}{}_{q^2}\left[\begin{matrix} a^2b, q^2a\sqrt{b}, -q^2a\sqrt{b}, & b^2, & a, & -a, & a\sqrt{q}, \\ a\sqrt{b}, & -a\sqrt{b}, & a^2q^2/b, & aq^2b, & -aq^2b, & abq\sqrt{q}, \end{matrix}\right.$$
$$\left.\begin{matrix} -a\sqrt{q}, \; aq, \; -aq, aq\sqrt{q}, -aq\sqrt{q}; & b^2q^2 \\ -abq\sqrt{q}, abq, -abq, ab\sqrt{q}, -ab\sqrt{q}; & \end{matrix}\right]$$

$$= \frac{\Pi(a^2bq)\Pi(b^2q)}{\Pi(a^2b^2q)\Pi(bq)}\;{}_4\Phi_3{}_{q}\left[\begin{matrix} a^2, & b, & a\sqrt{q/b}, & -a\sqrt{q/b}; & -bq \\ a^2q/b, & a\sqrt{qb}, & -a\sqrt{qb}; & & \end{matrix}\right]$$

This transforms a well-poised ${}_{12}\Phi_{11}$ into a well-poised ${}_4\Phi_3$. (§ 22.5.eq.(3)).

(8) $${}_{16}\Phi_{15}{}_{q^2}\left[\begin{matrix} a^2b, q^2a\sqrt{b}, -q^2a\sqrt{b}, & b^2, & q^{1+N}a^2b^2, & q^{2+N}a^2b^2, \\ a\sqrt{b}, & -a\sqrt{b}, & q^2a^2/b, & q^{1-N}/b, & q^{-N}/b, \end{matrix}\right.$$

$$\left.\begin{matrix} a, & -a, & a\sqrt{q}, & -a\sqrt{q}, & aq, & -aq, & aq\sqrt{q}, & -aq\sqrt{q}, & q^{1-N}, \\ abq^2, & -abq^2, & abq\sqrt{q}, & -abq\sqrt{q}, & abq, & -abq, & ab\sqrt{q}, & -ab\sqrt{q}, & a^2bq^{1+N}, \end{matrix}\right.$$

$$\left.\begin{matrix} q^{-N}; & q^2 \\ a^2bq^{2+N}; & \end{matrix}\right] = \frac{(a^2bq)_N(b^2q)_N}{(qb)_N(qa^2b^2)_N} \times$$

$${}_5\Phi_4\left[\begin{matrix} a^2, & b, & a\sqrt{q/b}, & -a\sqrt{q/b}, & q^{-N}; & -q/b \\ & a^2q/b, & a\sqrt{qb}, & -a\sqrt{qb}, & q^{-N}/b^2 & ; \end{matrix}\right]$$

This transforms a well-poised ${}_{16}\Phi_{15}$ into a nearly-poised ${}_5\Phi_4$. (§ 23.2.eq.(3)).

(9) $${}_{10}\Phi_9\left[\begin{matrix} a, q^2\sqrt{a}, -q^2\sqrt{a}, & b, & c, & cq, & d, & dq, \\ \sqrt{a}, -\sqrt{a}, & aq^2/b, & aq^2/c, & aq/c, & aq^2/d, & aq/d, \end{matrix}\right.$$
base q^2

$$\left.\begin{matrix} q^{-N}, & q^{1-N}; & \dfrac{a^3q^{2N+3}}{bc^2d^2} \\ aq^{2+N}, & aq^{1+N}; & \end{matrix}\right] = \frac{(aq)_N(aq/cd)_N}{(aq/c)_N(aq/d)_N} \times$$

$${}_5\Phi_4\left[\begin{matrix} \sqrt{aq/b}, -\sqrt{aq/b}, & c, & d, & q^{-N}; & q \\ & aq/b, \sqrt{aq}, -\sqrt{aq}, cdq^{-N}/a; & & & & \end{matrix}\right]$$
base q

This transforms a well-poised ${}_{10}\Phi_9$ into a Saalschutziam ${}_5\Phi_4$. (§ 23.3.eq.(2)).

(10) $${}_{12}\Phi_{11}\left[\begin{matrix} a, q^3\sqrt{a}, -q^3\sqrt{a}, & b, & bq, & bq^2, & c, & cq, & cq^2, \\ \sqrt{a}, -\sqrt{a}, & aq^3/b, & aq^2/b, & aq/b, & aq^3/c, & aq^2/c, & aq/c, \end{matrix}\right.$$
base q^3

$$\left.\begin{matrix} q^{-N}, & q^{1-N}, & q^{2-N}; & \dfrac{-a^4q^{3N+4}}{b^3c^3} \\ aq^{3+N}, & aq^{2+N}, & aq^{1+N}; & \end{matrix}\right] = \frac{(aq)_N(aq/bc)_N}{(aq/b)_N(aq/c)_N} \times$$

$$\sum_{n=0}^{N} \frac{(a)_{q^3,n}(aq^3)_{q^3,2n/3}(b)_{q,n}(c)_{q,n}(q^{-N})_{q,n}q^n}{(q)_{q,n}(aq)_{q,2n}(a)_{q^3,n/3}(bcq^{-N}/a)_{q,n}}$$

(§ 23.3.eq.(3)).

(11)

$${}_{12}\Phi_{11}\left[\begin{matrix} a/b,\ q\sqrt{a/b},\ -q\sqrt{a/b},\ q/c^2,\ aq/bcd,\ cd/b,\ \sqrt{ac/b},\ -\sqrt{ac/b},\ \sqrt{qac/b},\ -\sqrt{qac/b},\ d,\ q^{-N};\ qc^2 \\ \sqrt{a/b},\ -\sqrt{a/b},\ ac^2/b,\ cd,\ aq/cd,\ q\sqrt{a/bc},\ -q\sqrt{a/bc},\ \sqrt{qa/bc},\ -\sqrt{qa/bc},\ aq/bd,\ aq^{1+N}/b; \end{matrix}\right]$$

$$= \frac{(aq/b)_N(aq/c)_N(aq/d)_N(aq/bcd)_N}{(aq)_N(aq/bc)_N(aq/bd)_N(aq/cd)_N}\ {}_{12}\Phi_{11}\left[\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ qb/c^2,\ q/c,\ \sqrt{ac},\ -\sqrt{ac},\ \sqrt{qac},\ -\sqrt{qac},\ ac/b,\ d,\ q^{-N};\ q/b^2 \\ \sqrt{a},\ -\sqrt{a},\ ac^2/b,\ ac,\ q\sqrt{a/c},\ -q\sqrt{a/c},\ \sqrt{qa/c},\ -\sqrt{qa/c},\ qb/c,\ aq/d,\ aq^{1+N}; \end{matrix}\right]$$

where $aq = c^2dq^{-N}$.

This transforms a well-poised ${}_{12}\Phi_{11}$ into another well-poised ${}_{12}\Phi_{11}$. (§ 24.2)

(12)

$${}_{12}\Phi_{11}{}_{q^2}\left[\begin{matrix} ae,\ q^2\sqrt{ae},\ -q^2\sqrt{ae},\ e^2,\ b,\ bq,\ c,\ cq,\ d,\ dq,\ q^{-N},\ q^{1-N};\ q^2 \\ \sqrt{ae},\ -\sqrt{ae},\ q^2a/e,\ aeq^2/b,\ aeq/b,\ aeq^2/c,\ aeq/c,\ aeq^2/d,\ aeq/d,\ aeq^{2+N},\ aeq^{1+N}; \end{matrix}\right] =$$

$$\frac{(aq/b)_N(aq/c)_N(aq/d)_N(aq/bcd)_N}{(aq)_N(aq/bc)_N(aq/bd)_N(aq/cd)_N}\ {}_{10}\Phi_9\left[\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ \sqrt{aeq},\ -\sqrt{aeq},\ b,\ c,\ d,\ e,\ q^{-N};\ -eq \\ \sqrt{a},\ -\sqrt{a},\ \sqrt{aq/e},\ -\sqrt{aq/e},\ aq/b,\ aq/c,\ aq/d,\ aq/e,\ aq^{1+N}; \end{matrix}\right]$$

where $a^2qe = bcdq^{-N}$.

This transforms a well-poised ${}_{12}\Phi_{11}$ into a well-poised ${}_{10}\Phi_9$. (§ 24.3.eq.(3)).

Appendix one.

Relations between products of the type $(a)_n$.

$$(a)_n = a(a+1)(a+2)\ldots\ldots(a+n-1)$$

(1) $$\frac{\Gamma(a+n)}{\Gamma(a)} = (a)_n.$$

(2) $$(a+n)_n = \frac{(a)_{2n}}{(a)_n}, \text{ and in general, } (a+kn)_n = \frac{(a)_{(k+1)n}}{(a)_{kn}}.$$

(3) $$(a-n)_n = (-1)^n(1-a)_n, \text{ and in general,}$$

$$(a-kn)_n = (-1)^n\frac{(1-a)_{kn}}{(1-a)_{(k-1)n}}.$$

(4) $$\frac{\Gamma(a-n)}{\Gamma(a)} = \frac{(-1)^n}{(1-a)_n}$$

(5) $$(a)_{N-n} = \frac{(-1)^n(a)_N}{(1-a-N)_n}$$

(6) $$(a+n)_{N-n} = \frac{(a)_N}{(a)_n} \text{ and in general,}$$

$$(a+kn)_{N-n} = \frac{(a)_N(a+N)_{(k-1)n}}{(a)_{kn}}$$

(7) $$(a-n)_{N-n} = \frac{(a)_N(-1)^n(1-a)_n}{(1-a-N)_{2n}}, \text{ and in general}$$

$$(a-kn)_{N-n} = \frac{(-1)^n(a)_N(1-a)_{kn}}{(1-a-N)_{(k+1)n}}$$

(8) $$(a-n)_N = \frac{(1-a)_n(a)_N}{(1-a-N)_n}$$

(9) $$(a+n)_N = \frac{(a)_N(a+N)_n}{(a)_n}$$

(10) $$(a-n)_{N-2n} = \frac{(a)_N(1-a)_n}{(1-a-N)_{3n}}, \text{ and in general,}$$

$$(a-kn)_{N-\ell n} = \frac{(a)_N(-1)^{(k+\ell)n}(1-a)_{kn}}{(1-a-N)_{\ell n}}$$

(11) $$(a+n)_{N-2n} = \frac{(-1)^n(a)_N}{(1-a-N)_n(a)_n}, \text{ and in general,}$$

$$(a+kn)_{N-\ell n} = \frac{(a)_N(a+N)_{kn-\ell n}}{(a)_{kn}}$$

(12) $(a+n)_{2n} = \frac{(a)_{3n}}{(a)_n}$, and in general

$$(a+kn)_{\ell n} = \frac{(a)_{(k+\ell)n}}{(a)_{kn}}$$

(13) $(a-n)_{2n} = (-1)^n (a)_n (1-a)_n$, and in general

$$(a-kn)_{\ell n} = \frac{(a)_{(\ell-k)n}(-1)^{kn}(1-a)_{kn}}{} \quad \text{if } \ell > k$$

$$= \frac{(1-a)_{kn}(-1)^{\ell n}}{(1-a)_{(k-\ell)n}} \cdot \quad \text{if } \ell < k$$

(14) $(a)_{-n} = \frac{(-1)^n}{(1-a)_n}$

(15) $(a)_{2n} = (a/2)_n(\frac{1}{2}(a+1))_n 2^{2n}$, and in general,

$$(a)_{kn} = (a/k)_n((a+1)/k)_n \ldots \ldots((a+k-1)/k)_n k^{kn}.$$

Appendix two.

Relations between products of the type $(a)_{q,n}$.

$$(a)_{q,n} = (1-a)(1-aq)(1-aq^2)\ldots\ldots(1-aq^{n-1}).$$

(1) $$\frac{\Pi(a)}{\Pi(aq^N)} = (a)_{q,N}.$$

(2) $$(aq^n)_{q,n} = \frac{(a)_{q,2n}}{(a)_{q,n}}, \text{ and in general,}$$

$$(aq^{kn})_{q,n} = \frac{(a)_{q,(k+1)n}}{(a)_{q,kn}}$$

(3) $$(aq^{-n})_{q,n} = (-a)^n q^{-\frac{1}{2}(n+1)n}(q/a)_{q,n}, \text{ and in general,}$$

$$(aq^{-kn})_{q,n} = (-a)^n q^{\frac{1}{2}(n-1)n-kn^2}\frac{(q/a)_{kn}}{(q/a)_{(k-1)n}}.$$

(4) $$\frac{\Pi(aq^{-N})}{\Pi(a)} = (-a)^n q^{-\frac{1}{2}(n+1)n}(q/a)_{q,n}.$$

(5) $$(a)_{q,N-n} = \frac{(a)_N q^{\frac{1}{2}(n+1)n}}{(q^{1-N}/a)_n(-a)^n q^{Nn}}.$$

(6) $$(aq^n)_{q,N-n} = \frac{(a)_{q,N}}{(a)_{q,n}}, \text{ and in general,}$$

$$(aq^{kn})_{q,N-n} = \frac{(a)_{q,N}(aq^N)_{(k-1)n}}{(a)_{kn}}$$

(7) $$(aq^{-n})_{q,N-n} = \frac{(a)_{q,N}(q/a)_{q,n}(-1)^n q^{\frac{1}{2}(3n+1)n-2Nn}}{(q^{1-N}/a)_{q,2n}\, a^n}.$$

(8) $$(aq^{-n})_{q,N} = \frac{(a)_{q,N}(q/a)_{q,n}q^{-Nn}}{(q^{1-N}/a)_{q,n}}$$

(9) $$(aq^n)_{q,N} = \frac{(a)_{q,N}(aq^N)_{q,n}}{(a)_{q,n}}$$

(10) $$(aq^n)_{N-2n} = \frac{(a)_N(-1)^n q^{\frac{1}{2}(n+1)n}}{(a)_n(q^{1-N}/a)_n\, a^n q^{Nn}}$$

(11) $(a)_{q^2,m} = (\sqrt{a})_{q,n}(-\sqrt{a})_{q,m}$

(12) $(a)_{q,2n} = (a)_{q^2,n}(aq)_{q^2,n}$

(13) $(a)_{q,3n} = (a)_{q^3,n}(aq)_{q^3,n}(aq^2)_{q^3,n}.$

Appendix three.

Summation Theorems for Ordinary Hypergeometric Series.

(1) The Binomial theorem.

$$ {}_1F_0\left[a; \; ; x\right] = \frac{1}{(1-x)^a} $$

(2) Saalschutz's theorem.

$$ {}_3F_2\left[\begin{matrix} a, b, -n; & 1 \\ c, d\ ; & \end{matrix}\right] = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n} $$

provided that c+d = a+b-n+1.

(3) Gauss's theorem.

$$ {}_2F_1\left[\begin{matrix} a, b; & 1 \\ c; & \end{matrix}\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} $$

or, if b = -n,

Vandermonde's theorem.

$$ {}_2F_1\left[\begin{matrix} a, -n; & 1 \\ c; & \end{matrix}\right] = \frac{(c-a)_n}{(c)_n} $$

(4) Kummer's theorem.

$$ {}_2F_1\left[\begin{matrix} a, & b; & -1 \\ & 1+a-b; & \end{matrix}\right] = \frac{\Gamma(1+a-b)\Gamma(1+\frac{1}{2}a)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)} $$

(5) Gauss's second theorem.

$$ {}_2F_1\left[\begin{matrix} a, & b; & \frac{1}{2} \\ & \frac{1}{2}(1+a+b); & \end{matrix}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(1+a+b))}{\Gamma(\frac{1}{2}(1+a))\Gamma(\frac{1}{2}(1+b))} $$

(6)

$$ {}_2F_1\left[\begin{matrix} a, 1-a; & \frac{1}{2} \\ c; & \end{matrix}\right] = \frac{\Gamma(\frac{1}{2}c)\Gamma(\frac{1}{2}(1+c))}{\Gamma(\frac{1}{2}(c+a))\Gamma(\frac{1}{2}(1+c-a))} $$

(7) Dixon's theorem.

$$ {}_3F_2\left[\begin{matrix} a, & b, & c; & 1 \\ & 1+a-b, & 1+a-c; & \end{matrix}\right] = \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)} $$

or, if c = -n,

$$ {}_3F_2\left[\begin{matrix} a, & b, & -n; & 1 \\ & 1+a-b, & 1+a+n; & \end{matrix}\right] = \frac{(1+a)_n (1+\frac{1}{2}a-b)_n}{(1+\frac{1}{2}a)_n (1+a-b)_n} $$

(8) $${}_4F_3\left[\begin{matrix} a, 1+\tfrac{1}{2}a, & b, & c; & -1 \\ \tfrac{1}{2}a, & 1+a-b, & 1+a-c; & \end{matrix}\right] = \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(1+a-b-c)}$$

or, if $c = -n$,

$${}_4F_3\left[\begin{matrix} a, 1+\tfrac{1}{2}a, & b, & -n; & -1 \\ \tfrac{1}{2}a, & 1+a-b, & 1+a+n; & \end{matrix}\right] = \frac{(1+a)_n}{(1+a-b)_n}$$

(9) $${}_5F_4\left[\begin{matrix} a, 1+\tfrac{1}{2}a, & b, & c, & d; & 1 \\ \tfrac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d; & \end{matrix}\right]$$

$$= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)}$$

or, if $d = -n$,

$${}_5F_4\left[\begin{matrix} a, 1+\tfrac{1}{2}a, & b, & c, & -n; & 1 \\ \tfrac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a+n; & \end{matrix}\right]$$

$$= \frac{(1+a)_n(1+a-b-c)_n}{(1+a-b)_n(1+a-c)_n}$$

(10) Dougall's theorem. If $1+2a = b+c+d+e-n$,

$${}_7F_6\left[\begin{matrix} a, 1+\tfrac{1}{2}a, & b, & c, & d, & e, & -n; & 1 \\ \tfrac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a+n; & \end{matrix}\right]$$

$$= \frac{(1+a)_n(1+a-b-c)_n(1+a-b-d)_n(1+a-c-d)_n}{(1+a-b)_n(1+a-c)_n(1+a-d)_n(1+a-b-c-d)_n}$$

(11) $${}_3F_2\left[\begin{matrix} a, 1+\tfrac{1}{2}a, -n; & 1 \\ \tfrac{1}{2}a, b; & \end{matrix}\right] = \frac{(b-a-1-n)(b-a)_{n-1}}{(b)_n}$$

(12) $${}_3F_2\left[\begin{matrix} a, & b, & -n; & 1 \\ & 1+a-b, & 1+2b-n; & \end{matrix}\right] = \frac{(a-2b)_n(1+\tfrac{1}{2}a-b)_n(-b)_n}{(1+a-b)_n(\tfrac{1}{2}a-b)_n(-2b)_n}$$

(13) $${}_4F_3\left[\begin{matrix} a, 1+\tfrac{1}{2}a, & b, & -n; & 1 \\ \tfrac{1}{2}a, & 1+a-b, & 1+2b-n; & \end{matrix}\right] = \frac{(a-2b)_n(-b)_n}{(1+a-b)_n(-2b)_n}$$

(14) $${}_4F_3\left[\begin{matrix} a, 1+\tfrac{1}{2}a, & b, & -n; & 1 \\ \tfrac{1}{2}a, & 1+a-b, & 2+2b-n; & \end{matrix}\right]$$

$$= \frac{(a-2b-1)_n(\tfrac{1}{2}(a+1)-b)_n(-b-1)_n}{(1+a-b)_n(\tfrac{1}{2}(a-1)-b)_n(-2b-1)_n}$$

(15) $${}_7F_6\left[\begin{matrix} a, 1+\tfrac{1}{2}a, & \tfrac{1}{2}d, \tfrac{1}{2}(1+d), a-d, 1+2a-d+n, & -n; 1 \\ & \tfrac{1}{2}a, 1+a-\tfrac{1}{2}d, a+\tfrac{1}{2}(1-d), 1+d, d-a-n, 1+a+n; & \end{matrix}\right] = \frac{(1+a)_n(1+2a-2d)_n}{(1+a-d)_n(1+2a-d)_n}$$

(16) $${}_4F_3\left[\begin{matrix} \tfrac{1}{2}a, \tfrac{1}{2}(1+a), & b+n, -n; 1 \\ & \tfrac{1}{2}b, \tfrac{1}{2}(b+1), 1+a; \end{matrix}\right] = \frac{(b-a)_n}{(b)_n}$$

(17) $${}_3F_2\left[\begin{matrix} a, 1+\tfrac{1}{2}a, & b; -1 \\ & \tfrac{1}{2}a, 1+a-b; \end{matrix}\right] = \frac{\Gamma(\tfrac{1}{2}(1+a))\Gamma(1+a-b)}{\Gamma(1+a)\Gamma(\tfrac{1}{2}(1+a)-b)}$$

(18) $${}_4F_3\left[\begin{matrix} a, 1+\tfrac{1}{2}a, & b, & c; 1 \\ & \tfrac{1}{2}a, 1+a-b, 1+a-c; & \end{matrix}\right] = \frac{\Gamma(\tfrac{1}{2}(1+a))\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(1+a)\Gamma(\tfrac{1}{2}(1+a)-b)\Gamma(\tfrac{1}{2}(1+a)-c)} \times \frac{\Gamma(\tfrac{1}{2}(1+a)-b-c)}{\Gamma(1+a-b-c)}$$

(19) Watson's theorem,

$${}_3F_2\left[\begin{matrix} a, & b, c; 1 \\ & \tfrac{1}{2}(1+a+b), 2c; \end{matrix}\right] = \frac{\Gamma(\tfrac{1}{2})\Gamma(c+\tfrac{1}{2})\Gamma(\tfrac{1}{2}(1+a+b))\Gamma(\tfrac{1}{2}(1-a-b)+c)}{\Gamma(\tfrac{1}{2}(1+a))\Gamma(\tfrac{1}{2}(1+b))\Gamma(\tfrac{1}{2}(1-a)+c)\Gamma(\tfrac{1}{2}(1-b)+c)}$$

(20) Whipple's theorem, if $a+b = 1$ and $e+f = 2c+1$,

$${}_3F_2\left[\begin{matrix} a, b, c; 1 \\ e, f; \end{matrix}\right] = \frac{\pi\Gamma(e)\Gamma(f)}{2^{2c-1}\Gamma(\tfrac{1}{2}(a+e))\Gamma(\tfrac{1}{2}(a+f))\Gamma(\tfrac{1}{2}(f+e))\Gamma(\tfrac{1}{2}(b+f))}$$

Appendix four.

Summation theorems for Basic Series.

(1) Vandermonde's Analogue.

$$ {}_2\Phi_1\left[\begin{matrix} a, q^{-N}; & q \\ c\ ; & \end{matrix}\right] = \frac{(c/a)_N\, a^N}{(c)_N} $$

(2) Gauss's Analogue.

$$ {}_2\Phi_1\left[\begin{matrix} a, b; & c/ab \\ c\ ; & \end{matrix}\right] = \frac{\Pi(c/a)\,\Pi(c/b)}{\Pi(c)\,\Pi(c/ab)} $$

or, if $b = q^{-N}$,

$$ {}_2\Phi_1\left[\begin{matrix} a, q^{-N}; & cq^N/a \\ c\ ; & \end{matrix}\right] = \frac{(c/a)_N}{(c)_N} $$

(3) Saalschutz's Analogue.

$$ {}_3\Phi_2\left[\begin{matrix} a, b, q^{-N}; & q \\ c, d\ ; & \end{matrix}\right] = \frac{(c/a)_N (c/b)_N}{(c)_N (c/ab)_N} $$

provided that $cd = abq^{1-N}$.

(4) Dixon's Analogue.

$$ {}_4\Phi_3\left[\begin{matrix} a, -q\sqrt{a}, & b, & c; & q\sqrt{a}/bc \\ & -\sqrt{a}, & aq/b, & aq/c; \end{matrix}\right] $$

$$ = \frac{\Pi(aq)\,\Pi(\sqrt{a}q/b)\,\Pi(\sqrt{a}q/c)\,\Pi(aq/bc)}{\Pi(aq/b)\,\Pi(aq/c)\,\Pi(\sqrt{a}q)\,\Pi(\sqrt{a}q/bc)} $$

or, if $c = q^{-N}$,

$$ {}_4\Phi_3\left[\begin{matrix} a, -q\sqrt{a}, & b, & q^{-N}; & q^{1+N}\sqrt{a}/b \\ & -\sqrt{a}, & aq/b, & aq^{1+N}; \end{matrix}\right] = \frac{(aq)_N (\sqrt{a}q/b)_N}{(\sqrt{a}q)_N (aq/b)_N} $$

(5)

$$ {}_6\Phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, & b, & c, & d; & aq/bcd \\ \sqrt{a}, -\sqrt{a}, & aq/b, & aq/c, & aq/d; \end{matrix}\right] $$

$$ = \frac{\Pi(aq)\,\Pi(aq/cd)\,\Pi(aq/bd)\,\Pi(aq/bc)}{\Pi(aq/b)\,\Pi(aq/c)\,\Pi(aq/d)\,\Pi(aq/bcd)} $$

(6) Jackson's theorem. If $a^2q = bcdeq^{-N}$,

$$ {}_8\Phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, & b, & c, & d, & e, & q^{-N}; & q \\ \sqrt{a}, -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{1+N}; \end{matrix}\right] $$

$$ = \frac{(aq)_N\,(aq/bc)_N\,(aq/cd)_N\,(aq/bd)_N}{(aq/b)_N\,(aq/c)_N\,(aq/d)_N\,(aq/bcd)_N} $$