Spanning k-arc-strong Subdigraphs with few arcs in k-arc-strong Tournaments

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Abstract

Given a k-arc-strong tournament T, we estimate the minimum number of arcs possible in a k-arc-strong spanning subdigraph of T. We give a construction which shows that for each $k \geq 2$ there are tournaments T on n vertices such that every k-arc-strong spanning subdigraph of T contains at least $nk + \frac{k(k-1)}{2}$ arcs. In fact, the tournaments in our construction have the property that every spanning subdigraph with minimum in- and out-degree at least k has $nk + \frac{k(k-1)}{2}$ arcs. This is best possible since it can be shown that every k-arc-strong tournament contains a spanning subdigraph with minimum in- and out-degree at least k and no more than $nk + \frac{k(k-1)}{2}$ arcs. As our main result we prove that every k-arc-strong tournament contains a spanning k-arc-strong subdigraph with no more than $nk + 136k^2$ arcs. We conjecture that for every k-arc-strong tournament T, the minimum number of arcs in a k-arc-strong spanning subdigraph of T is equal to the minimum number of arcs in a spanning subdigraph of T with the property that every vertex has in- and out-degree at least k. We also discuss the implications of our results on related problems and conjectures.

Keywords: Tournament, connectivity, minimum strong spanning subdigraph, certificates for connectivity, polynomial algorithm, MSSS problem, MEG problem.

1   Introduction

A *tournament* is an orientation of a complete graph. It is well-known and easy to show that every strong tournament has a hamiltonian cycle. Furthermore, it is easy to

find such a cycle in polynomial time. Hence for tournaments the problem of finding a spanning strong subdigraph with the minimum number of arcs (the MSSS problem) is polynomially solvable. For general digraphs this problem is very hard as it generalizes the hamiltonian cycle problem. The MSSS problem forms the most important subproblem of the problem of finding the so-called *minimum equivalent digraph* of a given digraph. That is, given a digraph $D = (V, A)$ find a spanning subdigraph $D' = (V, A')$ such that for every choice of vertices $x, y \in V$ there is a directed path from x to y in D if and only if D' has such a path. The minimum equivalent digraph problem and its generalizations to higher degrees of connectivity has practical applications and has been studied extensively, see e.g. [1, 12, 15, 16, 19]. Furthermore, for a given class of digraphs, which is closed under the operation of taking induced subdigraphs, one can find the minimum equivalent digraph in polynomial time if and only if one can solve the MSSS problem in polynomial time for that class. Hence it is of interest to find classes of digraphs for which one can solve the MSSS problem in polynomial time. Recently it was shown that the MSSS problem is solvable in polynomial time for several classes which contain tournaments as a subclass [4, 6]. In the last section of this paper we also discuss briefly some approximation algorithms for this problem for general digraphs.

Suppose now that we are given a digraph which is k-arc-strong and the goal is to find a spanning subdigraph with the minimum number of arcs which is k-arc-strong. Clearly, this problem contains the MSSS problem as a special case and hence is very hard for general digraphs. For an arbitrary k-arc-strong digraph one can find a spanning k-arc-strong subdigraph with at most twice the optimal number of arcs in polynomial time (see Section 3). However, even for tournaments finding a polynomial algorithm to determine a minimum k-arc-strong spanning subdigraph, seems to be very difficult. One reason for this is that although tournaments have a lot of structure, much of this is either lost, or at least difficult to establish as soon as we delete only a relatively small number of arcs from the tournament in question (for example it is an open problem whether there exists a polynomial algorithm for recognizing those tournaments that have two arc-disjoint hamiltonian cycles). Hence applying an iterative approach which builds up a minimum spanning k-arc-strong subdigraph from certain spanning strong subdigraphs seems doomed to fail.

In this paper we show that every k-arc strong tournament T contains a spanning k-arc-strong subdigraph with at most $nk + 136k^2$ arcs. We show that this is best possible in terms of the exponent on k and we also show that one can find such a subdigraph in polynomial time. Since every k-arc-strong digraph has at least nk arcs this shows that for k-arc-strong tournaments one can get within a function depending only on k of the optimum. The method we use involves iteratively constructing large arc-disjoint strong subdigraphs of T such that the subdigraph D' induced by the union of the arc sets of these subdigraphs contains a large set of vertices X (containing all vertices from T except possibly a linear function of k vertices) and for any two vertices $x, y \in X$, D' contains k arc-disjoint (x, y)-paths.

Our proof uses several new results on digraphs in which the number of non-neighbours of each vertex is bounded by some constant c and digraphs in which some vertices may have more than c non-neighbours but the total number of such vertices is bounded by some other constant.

Finally, we conjecture that for every k-arc-strong tournament T, the minimum num-
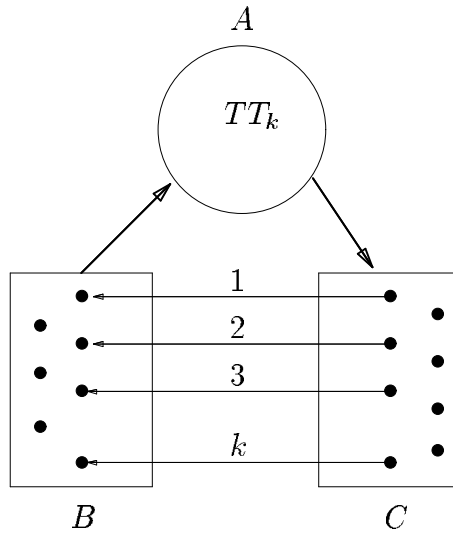
Figure 1: The structure of the tournaments in $\mathcal{T}_{n,k}$. The tournament A is the transitive tournament on k vertices, B and C are arbitrary k-arc-strong tournaments. The bold arcs $B{\to}A, A{\to}C$ indicate that all possible arcs are present in that direction. There are exactly k arcs from C to B and all other arcs go from B to C.

ber of arcs in a k-arc-strong spanning subdigraph of T is equal to the minimum number of arcs in a spanning subdigraph of T with the property that every vertex has in- and out-degree at least k. If true, this would imply that there exists a polynomial algorithm to determine the minimum number of arcs in a spanning k-arc-strong subdigraph of a given k-arc-strong tournament.

Several results from this paper were used in [5].

2 Statement of the Main Result and Conjecture

Let $n \geq 3$ and $k \geq 1$ be two integers. We define $f(n,k)$ to be the smallest integer such that every k-arc-strong tournament on n vertices contains a k-arc-strong spanning subdigraph with at most $f(n,k)$ arcs. Since every strong tournament is hamiltonian we have $f(n,1) = n$. Furthermore, since every vertex in a k-arc-strong digraph has out-degree at least k, $f(n,k) \geq nk$.

For all $k \geq 1$ and $n \geq 5k + 2$ we define $\mathcal{T}_{n,k}$ as the class of tournaments that can be obtained from a transitive tournament A on k vertices and two k-arc-strong tournaments B, C as shown in Figure 1. It is not difficult to show that each tournament in $\mathcal{T}_{n,k}$ is k-arc-strong.

Let T be any member of $\mathcal{T}_{n,k}$. Observe that every k-arc-strong subdigraph D of T must contain at least $k(k+1)/2$ arcs from B to A and exactly k arcs from C to B (there are no more). Hence we have $[\sum_{x\in B} d_D^+(x)] - [\sum_{x\in B} d_D^-(x)] \geq k(k+1)/2 - k$, implying that $\sum_{x\in B} d_D^+(x) \geq k * |B| + k(k-1)/2$. This implies that D has at least $nk + k(k-1)/2$ arcs.

The tournaments in $\mathcal{T}_{n,k}$ show that $f(n,k) \geq nk + ck^2$ for some constant $c > 0$. As our main result we show that this is the right order of magnitude and in fact

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$f(n,k) \leq nk + 136k^2$ holds.

Theorem 2.1 *For any $n \geq 3$ and $k \geq 1$, every k-arc-strong tournament T on n vertices contains a spanning k-arc-strong subdigraph D' with at most $nk + 136k^2$ arcs.*

For any tournament T we denote by $h(k,T)$ the minimum number of arcs in a spanning subdigraph D of T in which has $\delta(D) \geq k$. If $\delta(T) < k$ we let $h(k,T) = \infty$. Here $\delta(D)$ denotes the minimum over all in- and out-degrees of vertices in D.

Proposition 2.2 *Every tournament with $\delta(T) \geq k$ satisfies $h(k,T) \leq nk+k(k+1)/2$. Furthermore, if T is k-arc-strong $h(k,T) \leq nk + k(k-1)/2$.*

Proof: Let $T = (V,A)$ be a tournament on n vertices with $\delta(T) \geq k$. Form a flow network $\mathcal{N}$ with vertex set $U = V^- \cup V^+ \cup \{s,t\}$, where $V^- = \{v^-|v \in V\}$, $V^+ = \{v^+|v \in V\}$, and arc set $\{sv^-|v \in V\} \cup \{v^-w^+|vw \in A\} \cup \{v^+t|v \in V\}$. Every arc a has a capacity $u(a) = 1$ and for each arc a in the set $\{sv^-|v \in V\} \cup \{v^+t|v \in V\}$ there is a lower bound $l(a) = k$ and all other arcs have lower bound $l(a) = 0$. Observe that if x is a feasible integer valued (s,t)-flow in $\mathcal{N}$ of value M (the flow out of s) then the spanning subdigraph D' of T that we obtain by taking those arcs $vw \in A$ for which $x(v^-w^+) = 1$ has precisely M arcs and $\delta(D') \geq k$. The other direction holds as well. Thus the minimum number of arcs in a spanning subdigraph D' of T with $\delta(D') \geq k$ is equal to the minimum value of a feasible integer valued (s,t)-flow in $\mathcal{N}$. It is a standard result in flow theory (see e.g. [2, Theorem 3.9.1]) that this value is equal to the maximum of $l(S,\bar{S}) - u(\bar{S},S)$ over all partitions of U into two sets $S,\bar{S}$ where $s \in S$ and $t \in \bar{S}$.

Let $(S,\bar{S})$ be an arbitrary partition as above and denote by X,Y,Z,W the following sets:

$$X = \{v \in V|v^- \in \bar{S} \text{ and } v^+ \in S\}, \quad Y = \{v \in V|v^- \in \bar{S} \text{ and } v^+ \in \bar{S}\},$$

$$Z = \{v \in V|v^- \in S \text{ and } v^+ \in \bar{S}\}, \quad W = \{v \in V|v^- \in S \text{ and } v^+ \in S\}.$$

Then $U = X \cup Y \cup Z \cup W$ and it is easy to see that we have $l(S,\bar{S}) \leq k(n+|X|-|Z|)$ and $u(\bar{S},S) \geq |X|(|X|-1)/2$ (every arc in X contributes one to $u(\bar{S},S)$). Thus we have

$$l(S,\bar{S}) - u(\bar{S},S) \leq kn + k|X| - k|Z| - |X|(|X|-1)/2. \tag{1}$$

This implies that $l(S,\bar{S}) - u(\bar{S},S) \leq kn + k(k+1)/2$ with equality only if $|Z| = 0$. Furthermore, if T is k-arc-strong then it is easy to see that either $|Z| \geq 1$ or there are at least k arcs from Y to $X \cup W$ in T. In both cases we conclude that $l(S,\bar{S}) - u(\bar{S},S) \leq kn + k(k-1)/2$. Since $(S,\bar{S})$ was chosen arbitrarily the result now follows. $\diamond$

For any tournament T we denote by $g(k,T)$ the minimum number of arcs in a spanning k-arc-strong subdigraph D of T. If T is not k-arc-strong then $g(k,T) = \infty$.

Conjecture 2.3 *For each $k \geq 1$ and for every k-arc-strong tournament T we have $g(k,T) = h(k,T)$.*

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If true, Conjecture 2.3 would imply that the correct value of $f(n,k)$ is $nk+k(k-1)/2$. Furthermore, since a minimum spanning subdigraph D with $\delta(D) \geq k$ can be found in polynomial time, the truth of the conjecture would imply that one can find $g(k,T)$ in polynomial time for every tournament T. We conjecture that finding a minimum k-arc-strong spanning subdigraph of T can also be done in polynomial time.

3   Terminology and Preliminaries

For notation or terminology not discussed here we refer to [2]. We shall always use the number n to denote the number of vertices in the digraph currently under consideration. The digraphs in this paper are finite and have no loops but may have multiple arcs. We use $V(D)$ and $A(D)$ to denote the vertex set and the arc set of a digraph D. The underlying undirected graph of D, denoted $UG(D)$ is the (multi)graph one obtains by suppressing the orientations on each arc. The *complement graph* of an undirected multigraph $G = (V, E)$ is the undirected graph $\bar{G}$ whose vertex set is V and two vertices x, y are joined by an edge in $\bar{G}$ precisely when $xy \notin E$.

The arc from a vertex x to a vertex y will be denoted by xy. Two vertices x and y are *adjacent* if there is at least one arc between them. For disjoint subsets $H, K \subset V(D)$ we use the notation $H \Rightarrow K$ to denote that there are no arcs from K to H. If $X \subseteq V(D)$ then we denote by $D\langle X \rangle$ the subdigraph *induced* by X in D, that is $D\langle X \rangle$ has vertex set X and contains precisely those arcs from $A(D)$ which have both end vertices in X.

The *degree* $d(v)$ of a vertex v is the number of arcs incident with D (i.e the degree in $UG(D)$). If $R \subseteq A(D)$ then we denote by $d_R(v)$ the degree of v in the undirected subgraph of $UG(D)$ induced by the edges of R. We also use the notation $d_R(x, U)$ to denote the number of arcs from R which have one end vertex in U and the other equal to x. A set of vertices S in D is *independent* if no arc of D has both end vertices in S. We denote by $\alpha(D)$ the maximum size of an independent set in D.

By a *cycle (path,* respectively) we mean a directed (simple) cycle (path, respectively). If W is a cycle or a path with two vertices u, v such that u can reach v on W, then $W[u, v]$ denotes the subpath of W from u to v. A cycle (path) of a digraph D is *hamiltonian* if it contains all the vertices of D. A digraph is *hamiltonian* if it has a hamiltonian cycle.

Let U, W be two subsets of $V(D)$. A (U, W)-arc is an arc xy with $x \in U$ and $y \in W$. A (U, W)-path is a path $x_1 x_2 \ldots x_k$ such that $x_1 \in U, x_k \in W$ and $x_i \notin U \cup W$ for $i = 2, 3, \ldots, k - 1$. An (x, y)-*path* is a path from x to y.

A digraph D is *strongly connected* (or just *strong*) if there exists an (x, y)-path and a (y, x)-path for every choice of distinct vertices x, y of D. A digraph D is k-*arc-strong* for some $k \geq 1$ if $D - A'$ is strong for every subset A' of $A(D)$ such that $|A'| \leq k - 1$. Whenever x and y are distinct vertices of D, we denote by $\lambda_D(x, y)$ the maximum number of arc-disjoint (x, y)-paths. By Menger's theorem $\lambda_D(x, y) \geq k$ for all $x, y \in V(D)$ if and only if D is k-arc-strong.

An *out-branching (in-branching)* rooted at r in a digraph D is a tree F in $UG(D)$ which (in D) is oriented in such a way that every vertex except r has precisely one arc coming in to it (going out of it). The following classical result is due to Edmonds:

Theorem 3.1 *[13] Let D be a directed graph and r a vertex of $V(D)$ and let k be a natural number. There exist k arc-disjoint out-branchings (in-branchings) all rooted at r in D if and only if $\lambda_D(r,v) \geq k$ $(\lambda_D(v,r) \geq k)$ for every $v \in V(D) - r$.*

Recall that every k-arc-strong digraph on n vertices has at least nk arcs, since every vertex has at least k arcs out of it. The following easy consequence of Edmonds branching theorem (see Corollary 3.3 below) implies that one can always find a k-arc-strong subdigraph with at most twice that number of arcs.

Corollary 3.2 *Every k-arc-strong digraph contains a spanning k-arc-strong subdigraph with at most $2k(n-1)$ arcs.*

Proof: Let D be k-arc-strong and fix a vertex $r \in V(D)$. Since D is k-arc strong we have in particular that $\lambda_D(r,v) \geq k$ for every $v \in V(D) - r$. Hence by Theorem 3.1, D has k arc-disjoint out-branchings $F_{r,1}^+, F_{r,2}^+, \ldots, F_{r,k}^+$ all of which have root r. Similarly, D has k arc-disjoint in-branchings $F_{r,1}^-, F_{r,2}^-, \ldots, F_{r,k}^-$ all of which have root r. Let D' be the spanning subdigraph of D induced by the arc set of these $2k$ branchings. Then $|A(D')| \leq 2k(n-1)$ and for every $v \in V(D) - r$ we have $\lambda_{D'}(r,v) \geq k$ and $\lambda_{D'}(v,r) \geq k$. Now it is an easy exercise, using Menger's theorem, to prove that D' is k-arc-strong. ◇

Lovász [18] gave a constructive proof of Theorem 3.1, which can easily be turned into a polynomial algorithm to find k arc-disjoint out-branchings with the same root or detect that no such set of branchings exist in a given digraph D with a specified vertex r. Hence we have the following corollary:

Corollary 3.3 *There exist a polynomial algorithm to find, in a given k-arc-strong digraph D, a spanning k-arc-strong subdigraph with at most $2k(n-1)$ arcs.* ◇

The following result due to Camion is well known and easy to prove (see the proof of Lemma 4.1 below).

Theorem 3.4 *[10] Every strongly connected semicomplete digraph contains a hamiltonian cycle.* ◇

We assume throughout this paper that $k \geq 1$ is an integer.

4 Small strong spanning subdigraphs in digraphs with high minimum degree

A cycle C in a digraph D is **non-extendable** if D has no cycle C' such that $C - a$ is a subpath of C' for some arc a of C and $|V(C')| > |V(C)|$. The next two results both generalize Theorem 3.4:

Lemma 4.1 *Let D be a strong digraph and let C be a non-extendable cycle in D. Then no vertex of $D - V(C)$ is adjacent to all vertices in C.*

Proof: Denote $C = c_1 c_2 \ldots c_r c_1$ and let

$$S = \{x \in V(D) - V(C) |\ x \text{ is adjacent to all vertices in } C\}.$$

Then for each vertex $x \in S$ either $x \Rightarrow V(C)$ or $V(C) \Rightarrow x$ as C is non-extendable. Suppose that there exists a vertex x such that $x \Rightarrow V(C)$. Let $c_i v_1 v_2 \ldots v_t x$, $t \geq 1$, be a shortest path from C to x. Then $c_i v_1 v_2 \ldots v_t x C[c_{i+1}, c_i]$ is a cycle of length greater than that of C, contradicting the assumption that C is non-extendable. So there is no vertex x such that $x \Rightarrow V(C)$. A similar argument shows that there is no vertex x such that $V(C) \Rightarrow x$. Hence $S = \emptyset$.   ◇

The next theorem implies that for strong digraphs whose underlying graphs have at most a fixed number c of non-neighbours for every vertex, one can always find a spanning strong subdigraph with at most the same constant c arcs above the number of arcs in a minimum spanning strong subdigraph (take $q = 0$).

Theorem 4.2 *Let D be a strong digraph, let R denote the complement graph of $UG(D)$ and let $c \geq 0$ be an integer. Suppose we have*

$$\sum_{\{u \in V(D): d_R(u) > c\}} [d_R(u) - c] \leq q. \tag{2}$$

Then there exists a strong spanning subdigraph H of D such that $|A(H)| \leq n + c + \sqrt{2q}$.

Proof: Let z_1 be arbitrary, let $D_1 = D$ and let C_1 be a non-extendable cycle containing z_1 in D_1. If C_1 is a hamiltonian cycle, then this can play the role of H. Otherwise contract C_1 into one vertex z_2 and let D_2 denote the resulting digraph (we delete multiple arcs if some are created as well as the loop created at z_2). Let C_2 be a non-extendable cycle containing z_2 in D_2. If C_2 is a hamiltonian cycle in D_2 then stop. Otherwise contract C_2 into one vertex z_3 and let D_3 denote the resulting digraph. Continue this way until the current non-extendable cycle C_j is a hamiltonian cycle in D_j.

Denote by H_i the subdigraph of D induced by the arcs of $C_1, C_2, \ldots, C_i$. It is easy to see that H_i has $|V(H_i)| + i - 1$ arcs and that $|V(H_i)| \geq i + 1$. Hence it suffices to show that the process above can continue for at most $c + \sqrt{2q}$ contraction steps.

Suppose x is a vertex in D_i which does not belong to H_i. We claim that $d_R(x, V(H_i)) \geq i$. The claim holds for $i = 1$ by Lemma 4.1. Suppose the claim does not hold for some step i above and let this i be chosen as small as possible. Then we have $d_R(x, V(H_{i-1})) = i - 1 = d_R(x, V(H_i))$. Since $|V(H_{i-1})| \geq i$ this implies that x has an arc to or from some vertex of $V(H_{i-1})$ in D and thus x is adjacent to z_i in D_i. Furthermore, x must be adjacent to every $y \in V(H_i) - V(H_{i-1})$ since we have $d_R(x, V(H_{i-1})) = d_R(x, V(H_i))$. However, this means that x is adjacent to every vertex of C_i, contradicting the fact that C_i is a non-extendable cycle in the strong digraph D_i. This shows that $d_R(x, V(H_i)) \geq i$ for every step i above and the claim is proved.

Let j be chosen such that C_j is a hamiltonian cycle in D_j. By our remark above H_j is a spanning strong subdigraph of D with $n + j - 1$ arcs. Hence we may assume that $j > c$. Let $p \geq 0$ be chosen such that $j = c + p + 1$. For each $i = c + 1, \ldots, c + p$ we choose a vertex $x_i \in V(H_{i+1}) - V(H_i)$. Using that $d_R(x_i) \geq i - 1$, since x_i has at least $i - 1$ non-neighbours in H_{i-1}, and the definition of q we get

$$q \geq \sum_{i=c+1}^{c+p+1} [d_R(x_i) - c] \geq \sum_{i=0}^{p} i = \frac{p(p+1)}{2}.$$

Thus we have $p(p+1) \leq 2q$ which implies that $p \leq \sqrt{2q}$ (with equality only if $q = 0$). This implies that H_{c+p+1} is a spanning strong subdigraph of D with at most $n + c + \sqrt{2q}$ arcs. ◇

By *subdividing* an arc xy we mean replacing the arc xy by an (x, y)-path $xv_1v_2 \ldots v_r y$ where $r \geq 1$ and each v_i is a new vertex with in-degree and out-degree one. It is easy to derive the following consequence from Theorem 4.2 (simply suppress the subdividing vertices and consider D, see also Section 7):

Corollary 4.3 *Suppose $D = (V, A)$ satisfies the hypothesis of Theorem 4.2 and let $\tilde{D}$ be obtained from D by subdividing some arcs. Then $\tilde{D}$ contains a strong subdigraph $\tilde{H} = (\tilde{V}, \tilde{A})$ such that $V \subseteq \tilde{V}$ and $|\tilde{A}| \leq |\tilde{V}| + c + \sqrt{2q}$.* ◇

5 Reachability in digraphs with high minimum degree

For a given digraph D and each $u \in V(D)$ let $\mathcal{I}_D(u)$ be the set of vertices in D which can reach u by a directed path.

Lemma 5.1 *Let $D = (V, A)$ be a digraph, let R denote the complement graph of $UG(D)$ and let $c \geq 0$ be an integer. Suppose we have*

$$\sum_{\{u \in V : d_R(u) > c\}} [d_R(u) - c] \leq q. \tag{3}$$

Let $a \geq 1$ be a fixed integer and define $\mathcal{I}_{D,a} = \{w : |\mathcal{I}_D(w)| < a\}$. Then $|\mathcal{I}_{D,a}| \leq a + c + \sqrt{2q}$.

Proof: Let $u_1, u_2, \ldots, u_n$ be an ordering of $V(D)$ such that $|\mathcal{I}_D(u_1)| \leq |\mathcal{I}_D(u_2)| \leq \ldots \leq |\mathcal{I}_D(u_n)|$. Observe that if there is an arc between u_i and u_j for some i, j such that $1 \leq i < j \leq n$ then $\mathcal{I}_D(u_i) \subseteq \mathcal{I}_D(u_j)$. This is clear if $u_i \rightarrow u_j$. On the other hand, if $u_j \rightarrow u_i$, then we have $\mathcal{I}_D(u_j) \subseteq \mathcal{I}_D(u_i)$ implying that $\mathcal{I}_D(u_i) = \mathcal{I}_D(u_j)$ because of the way we ordered the vertices.

Choose r such that either $|\mathcal{I}_D(u_r)| < a$ and $|\mathcal{I}_D(u_{r+1})| \geq a$, or $r = n$ holds. We may assume that r exists since otherwise $\mathcal{I}_{D,a} = \emptyset$. Thus we have $\mathcal{I}_{D,a} = \{u_1, u_2, \ldots, u_r\}$.

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If $a + c > r$ then we have $|\mathcal{I}_{D,a}| = r < a + c$ and we are done. So we may assume that $r \geq a + c$. By the remark above, for every i, j such that $1 \leq j < i \leq n$, either u_i and u_j are non-adjacent, or $u_j \in \mathcal{I}_D(u_i)$. Hence, for every i such that $a \leq i \leq r$ we have

$$d_R(u_i) \geq (i - 1) - |\mathcal{I}_D(u_i)| \geq i - 1 - (a - 1) = i - a.$$

Using this we get

$$\sum_{i=a+c}^{r} [d_R(u_i) - c] \geq \sum_{i=0}^{r-a-c} i = \frac{(r - a - c)(r - a - c + 1)}{2}.$$

Combining this with the assumption that $\sum_{\{u \in V: d_R(u) > c\}} [d_R(u) - c] \leq q$ it is easy to see that $r \leq a + c + \sqrt{2q}$ and the proof is complete. $\diamond$

Lemma 5.2 *Let T be a tournament and R a subset of $A(T)$. Suppose that*

$$\sum_{\{u \in V(T): d_R(u) > c\}} [d_R(u) - c] \leq q \tag{4}$$

and that z is a vertex such that

$$d_T^+(u) \leq d_T^+(z) + \gamma \text{ for all } u \in V(T). \tag{5}$$

Let W be the set of vertices which are not reachable from z by a directed path in $D = T - R$. Then

$$|W| \leq 2c + 2d_R(z) + 2\gamma - 1 + \sqrt{2q}.$$

Proof: Let $B = V(T) - W$ and observe that there is no arc from B to W in D. Thus we have

$$
\begin{aligned}
\sum_{u \in W} d_T^+(u) &\geq \frac{|W|(|W| - 1)}{2} + |B||W| - \sum_{v \in W} d_R(v) \\
&= \frac{|W|(|W| - 1)}{2} + |B||W| - \left(\sum_{\{v \in W: d_R(v) \leq c\}} d_R(v) + \sum_{\{v \in W: d_R(v) > c\}} d_R(v) \right).
\end{aligned}
$$

Thus using that $\sum_{\{v \in V(T): d_R(v) > c\}} [d_R(v) - c] \leq q$ we conclude that there exists a vertex $u \in W$ such that

$$d_T^+(u) \geq \frac{|W| - 1}{2} + |B| - c - \frac{q}{|W|}. \tag{6}$$

By the definition of B we have $z \in B$ and

$$d_T^+(z) \leq |B| - 1 + d_R^+(z) \leq |B| - 1 + d_R(z). \tag{7}$$

Combining (5), (6) and (7) we obtain

$$\frac{|W| - 1}{2} + |B| - c - \frac{q}{|W|} \leq d_T^+(u) \leq d_T^+(z) + \gamma \leq |B| - 1 + d_R(z) + \gamma.$$

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This implies that

$$|W|^2 + |W| \leq 2c|W| + 2q + 2|W|d_R(z) + 2\gamma|W|$$

and solving for $|W|$ we get $|W| \leq 2c + 2d_R(z) + 2\gamma - 1 + \sqrt{2q}$. $\diamond$

6 Proof of Theorem 2.1

Let $T = (V, A)$ be a k-arc-strong tournament on n vertices. If $n \leq 64k$ then it follows from Corollary 3.2 that T contains a spanning k-arc-strong subdigraph with at most $2k(n-1) < 128k^2$ vertices and the theorem follows. Hence we may assume below that $n > 64k$.

Let $v_1, v_2, \ldots, v_n$ be an ordering of the vertices of T such that $d^+(v_1) \leq d^+(v_2) \leq \ldots \leq d^+(v_n)$. Note that since T is a tournament this ordering also satisfies $d^-(v_1) \geq d^-(v_2) \geq \ldots \geq d^-(v_n)$. Let $X = \{v_{n-k+1}, v_{n-k+2}, \ldots, v_n\}$ and $Y = \{v_1, v_2, \ldots, v_k\}$.

Since T is k-arc-strong, it follows from Menger's theorem that there are k arc-disjoint paths $P_1, P_2, \ldots, P_k$ from Y to X such that all end vertices of these paths are disjoint (see also Section 7). Let $y_1, y_2, \ldots, y_k$ and $x_1, x_2, \ldots, x_k$ be chosen such that $X = \{x_1, x_2, \ldots, x_k\}$, $Y = \{y_1, y_2, \ldots, y_k\}$ and P_i is a (y_i, x_i)-path for $i = 1, 2, \ldots, k$.

Let $T^* = T - X - Y$ and for each $i = 1, 2, \ldots, k$ define the subtournament T_i of T by $T_i = T\langle V(T^*) \cup \{x_i, y_i\}\rangle$.

Let $c_i = \gamma_i = 2k - 2$, $i = 1, 2, \ldots, k$. Observe that for every $i = 1, 2, \ldots k$ and every $u \in V(T_i)$ we have

$$d^+_{T_i}(u) \leq d^+_{T_i}(x_i) + \gamma_i. \tag{8}$$

This follows from the way we ordered the vertices of T. Similarly, for every $i = 1, 2, \ldots, k$ and every $u \in V(T_i)$ we have

$$d^-_{T_i}(u) \leq d^-_{T_i}(y_i) + \gamma_i. \tag{9}$$

Define $q_1, q_2, \ldots, q_{k+1}$ recursively as follows:

$$
\begin{aligned}
q_1 &= 0 \\
q_i &= q_{i-1} + 2(2k - 2 + \sqrt{2q_{i-1}}), \quad i = 2, 3, \ldots, k+1.
\end{aligned}
\tag{10}
$$

Note that we have $q_1 < q_2 < \ldots < q_{k+1}$ and it is not difficult to show by induction on i that $q_i < 16k(i-1)$. In particular, we have

$$q_{k+1} < 16k^2. \tag{11}$$

This estimate is not sharp, but for convenience we shall use it below.

We first prove the existence of arc-disjoint strong subdigraphs $H_1, H_2, \ldots, H_k$ of T with the following two properties:

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(A) $|A(H_i)| \le |V(H_i)| + 8k$, $i = 1, 2, \ldots, k$.

(B) $|V - (V(H_1) \cap V(H_2) \cap \ldots \cap V(H_k))| \le 64k$.

This is the main step since, as we show at the end of the proof, given this it is rather straightforward to verify the claim of the theorem.

Let $D_1 = (T^* - (A(P_2) \cup A(P_3) \cup \ldots \cup A(P_k))) \cup V(P_1) \cup A(P_1)$. Above and also several times below we abuse the standard notation slightly. By $T^* - (A(P_2) \cup A(P_3) \cup \ldots \cup A(P_k))$ we mean the digraph Z one obtains from T^* by deleting every arc from T^* which belongs to $A(P_2) \cup A(P_3) \cup \ldots \cup A(P_k)$. Thus D_1 is the digraph one obtains from Z by adding all vertices and arcs of P_1 which are not already in Z.

Let $D_1' = D_1 \langle V(T_1) \rangle$. Then D_1' is a spanning subdigraph of T_1 and has $n_1 = n - 2k + 2$ vertices. Since every vertex of T_1 is incident with at most two arcs from each of the paths $P_2, P_3, \ldots, P_k$, every vertex in D_1' has at most $2k - 2$ non-neighbours[1].

Let $R_1 = A(T_1) - A(D_1')$, that is, R_1 consists of those arcs from $A(P_2) \cup A(P_3) \cup \ldots \cup A(P_k)$ that join two vertices in T_1. Applying (8) and Lemma 5.2 to T_1, R_1 and x_1 with the parameters c_1, γ_1, q_1, we see that in D_1' the set of vertices not reachable from x_1 has size at most $2c_1 + 2d_{R_1}(x_1) + 2\gamma_1 - 1$. Analogously, we can prove that in D_1' the set of vertices which cannot reach y_1 has size at most $2c_1 + 2d_{R_1}(y_1) + 2\gamma_1 - 1$. Now using the fact that D_1' is a subdigraph of D_1 and that D_1 contains the (y_1, x_1)-path P_1, we conclude that D_1 has a strong component Q_1 of size at least

$$
\begin{aligned}
n_1 - (4c_1 + 2(d_{R_1}(x_1) + d_{R_1}(y_1)) + 4\gamma_1 - 2) \;\ge\;& n - 2k + 2 - (3(8k - 8) - 2) \\
=\;& n - 26k + 28.
\end{aligned}
$$

Note that since we have assumed $n > 64k$ we do in fact get a strong component containing both x_1 and y_1 in D_1.

Now we apply Corollary 4.3 to Q_1 with $c = c_1$ and $q = q_1$ and conclude that Q_1 contains a spanning strong subdigraph H_1 with at most $|V(Q_1)| + 2k - 2$ arcs (recall that P_1 may contain several vertices from $X \cup Y - \{x_1, y_1\}$ which is why we apply Corollary 4.3 rather than Theorem 4.2 to Q_1).

Next we take $D_2 = (T^* - (A(H_1) \cup A(P_3) \cup A(P_4) \cup \ldots \cup A(P_k)) \cup V(P_2) \cup A(P_2)$ and $D_2' = D_2 \langle V(T_2) \rangle$. Again D_2' is a spanning subdigraph of T_2 and has $n_2 = n - 2k + 2$ vertices. Let $R_2 = A(T_2) - A(D_2')$. As we have removed only arcs from H_1 and $P_3, P_4, \ldots, P_k$ we see that if a vertex u of D_2' has $d_{R_2}(u) > c_2 = 2k - 2$, then $d_{H_1}(u) > 2$. Now using that

$$
\sum_{v \in V(H_1)} (d_{H_1}(v) - 2) = 2(|A(H_1)| - |V(H_1)|) \le 2(2k - 2) = (q_2 - q_1),
$$

we get that

$$
\sum_{\{u \in V(T_2) : d_{R_2}(u) > c_2\}} [d_{R_2}(u) - c_2] \le 2(2k - 2) = q_2. \tag{12}
$$

---

[1]Observe that P_1 may contain several vertices from $X \cup Y - \{x_1, y_1\}$ and each of these vertices have up to $n - 3$ non-neighbours in D_1. This is the reason why we consider D_1' and not D_1 below.

11

Since x_2 and y_2 each are incident with at most two arcs in H_1 (which happens precisely if they are on P_1), we have $d_{R_2}(x_2), d_{R_2}(y_2) \leq 2(k-1)$. Thus, as above we can apply Lemma 5.2 to D'_2 and x_2, y_2 and (using that P_2 is a path in D_2) conclude that D_2 has a strong component Q_2 of size at least

$$
\begin{aligned}
n_2 - (4c_2 + 2d_{R_2}(x_2) + 2d_{R_2}(y_2) + 4\gamma_2 - 2 + 2\sqrt{2q_2}) \;\;&\geq\;\; n - 26k + 28 - 2\sqrt{2q_{k+1}} \\
&>\;\; n - 26k + 28 - 2\sqrt{32k^2} \\
&>\;\; n - 38k + 28.
\end{aligned}
$$

Again it follows from our assumption on n and the fact that P_2 is a path in D_2 that x_2, y_2 do indeed belong to the same strong component of D_2.

Applying Corollary 4.3 to Q_2 with $c = c_2$, $q = q_2$ and using (12), we conclude that Q_2 contains a spanning strong subdigraph H_2 with at most $|V(Q_2)| + 2k - 2 + \sqrt{2q_2}$ arcs. Note that

$$
\sum_{v \in V(H_2)} (d_{H_2}(v) - 2) = 2(|A(H_2)| - |V(H_2)|) \leq 2(2k - 2 + \sqrt{2q_2}) = (q_3 - q_2).
$$

Now the general pattern should be visible: Assume that we have found arc-disjoint strong subdigraphs $H_1, H_2, \ldots, H_{i-1}$ of $D_1, D_2, \ldots, D_{i-1}$ respectively, such that for $j = 1, 2, \ldots, i-1$, $x_j, y_j \in V(H_j)$ and we have

$$
|A(H_j)| \leq |V(H_j)| + 2k - 2 + \sqrt{2q_j} = |V(H_j)| + (q_{j+1} - q_j)/2, \tag{13}
$$

implying that

$$
\sum_{v \in V(H_j)} (d_{H_j}(v) - 2) \leq (q_{j+1} - q_j). \tag{14}
$$

In the ith step we let

$$
D_i = (T^* - (A(H_1) \cup \ldots \cup A(H_{i-1}) \cup A(P_{i+1}) \cup \ldots \cup A(P_k))) \cup V(P_i) \cup A(P_i).
$$

Let $D'_i = D_i \langle V(T_i) \rangle$ and take $R_i = A(T_i) - A(D'_i)$. As x_i, y_i are incident to at most two arcs in each of $H_1, H_2, \ldots, H_{i-1}$ we have $d_{R_i}(x_i), d_{R_i}(y_i) \leq 2k - 2$.

In order to apply Lemma 5.2 to T_i, R_i, x_i we need to estimate the sum $\sum_{\{u \in V(T_i): d_{R_i}(u) > c_i\}} [d_{R_i}(u) - c_i]$. Note that if a vertex $u \in V(T_i)$ has $d_{R_i}(u) > c_i = 2k - 2$, then u is incident with more than 2 arcs in at least one of $H_1, H_2, \ldots, H_{i-1}$. Summing up the possible contributions from degrees above 2 in $H_1, H_2, \ldots, H_{i-1}$ and using (14) we have:

$$
\sum_{\{u \in V(T_i): d_{R_i}(u) > c_i\}} [d_{R_i}(u) - c_i] \leq q_i. \tag{15}
$$

Now we apply Lemma 5.2 to see that D_i contains a strong component Q_i such that x_i, y_i both belongs to Q_i (again we use the assumption on n, the fact that $q_i < 16k^2$ and that D_i contains the path P_i).

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Applying Corollary 4.3 to Q_i with $c = c_i, q = q_i$, we conclude that Q_i contains a spanning strong subdigraph H_i with at most $|V(Q_i)| + (2k - 2) + \sqrt{2q_i} = |V(Q_i)| + (q_{i+1} - q_i)/2$ arcs.

It follows by induction on i that (A) holds (recall that $q_i \leq 16k(i-1)$)

To prove that (B) holds, we need to estimate how many vertices of T belong to at most $k - 1$ of the H_i's.

Let $R^* = A(P_1) \cup A(P_2) \cup \ldots \cup A(P_k) \cup A(H_1) \cup A(H_2) \cup \ldots \cup A(H_k)$ and define the digraph D^* by

$$D^* = T^* - R^*.$$

Thus $V(D^*) = V(T^*)$ and $A(D^*)$ consists of those arcs of T^* that are not in R^*. For each $i = 1, 2, \ldots, k$ we define the following digraphs:

$$
\begin{aligned}
T^*_{y_i} &= T\langle V(T^*) \cup \{y_i\}\rangle \\
T^*_{x_i} &= T\langle V(T^*) \cup \{x_i\}\rangle \\
D^*_{y_i} &= (T^*_{y_i} - R^*) \cup V(H_i) \cup A(H_i) \\
D^*_{x_i} &= (T^*_{x_i} - R^*) \cup V(H_i) \cup A(H_i)
\end{aligned}
$$

Observe that D^* is a subdigraph of each of the digraphs $D_i, D^*_{x_i}$ and $D^*_{y_i}$ for all $i = 1, 2, \ldots, k$.

If a vertex $u \in V(T)$ has $d_{R^*}(u) > 4k$, then u is incident to more than 2 arcs in at least one of $H_1, H_2, \ldots, H_k$. Hence in the same way as we argued above one can prove the following:

$$
\begin{aligned}
\sum_{\{u \in V(T): d_{R^*}(u) > 4k\}} [d_{R^*}(u) - 4k] &\leq \sum_{i=1}^{k} 2(|A(H_i)| - |V(H_i)|) \\
&\leq \sum_{i=1}^{k} (q_{i+1} - q_i) \qquad\qquad (16) \\
&\leq q_{k+1} < 16k^2
\end{aligned}
$$

Below we always use the parameters $c = 4k, \gamma = 2k - 1$ and $q = 16k^2$.

Let

$$R^*_{x_i} = A(T^*_{x_i}) \cap (R^* - A(H_i)) \text{ and } R^*_{y_i} = A(T^*_{y_i}) \cap (R^* - A(H_i))$$

Observe that
$$d_{R^*_{x_i}}(x_i) \leq 2k - 1 \text{ and } d_{R^*_{y_i}}(y_i) \leq 2k - 1. \qquad\qquad (17)$$

This follows from that fact that $R^*_{x_i}$ ($R^*_{y_i}$) contains no arcs from H_i and for each $j \neq i$ an arc incident with x_i (y_i) belongs to $A(H_j)$ only if it also belongs to P_j (x_i (y_i)

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can only belong to H_j if it lies on P_j). Thus for each $A(P_j) \cup A(H_j)$, $j \neq i$ we have at most two arcs incident with x_i (y_i) in $R^*_{x_i}$ ($R^*_{y_i}$). Finally, we may have that the only arc incident with x_i (y_i) on P_i is also in $R^*_{x_i}$ ($R^*_{y_i}$).

For each $i = 1, 2, \ldots, k$, let X_i be the set of those vertices which cannot be reached from x_i in $D^*_{x_i}$ by a path that uses only arcs from $V(T^*_{x_i})$ and let Y_i be the set of those vertices which cannot reach y_i in $D^*_{y_i}$ by a path that uses only arcs from $V(T^*_{y_i})$. That is, in the paths above we are not allowed to use an arc from $A(H_i)$ unless it is also an arc of $V(T^*_{x_i})$ respectively, $V(T^*_{y_i})$. Applying Lemma 5.2 to $T^*_{x_i}$, $R^*_{x_i}$ and x_i, we see that

$$
\begin{aligned}
|X_i| &\leq & 2c + 2d_{R^*_{x_i}}(x_i) + 2\gamma - 1 + \sqrt{2q} \\
&<& 8k + 4k - 2 + 4k - 2 - 1 + 6k \\
&<& 22k
\end{aligned}
$$

Applying Lemma 5.1 to D^* we see that

$$
|\mathcal{I}_{D^*, 22k+1}| \leq 22k + 1 + 4k + \sqrt{32k^2} \leq 32k. \tag{18}
$$

Now we can bound the number of vertices in $X_1 \cup X_2 \cup \ldots \cup X_k$: Using that D^* is a subdigraph of $D^*_{x_i}$ and the fact that x_i can reach all but at most $22k$ vertices in $D^*_{x_i}$, we see that that every vertex which cannot be reached from x_i in $D^*_{x_i}$ must belong to $\mathcal{I}_{D^*, 22k+1}$. This holds for every $i = 1, 2, \ldots, k$. Hence we conclude that

$$
|X_1 \cup X_2 \cup \ldots \cup X_k| \leq 32k. \tag{19}
$$

By an analogous argument (using $R^*_{y_i}$ instead of $R^*_{x_i}$) we see that

$$
|Y_1 \cup Y_2 \cup \ldots \cup Y_k| \leq 32k. \tag{20}
$$

Let X'_i (Y'_i) be the set of vertices that cannot be reached from x_i (cannot reach y_i) in $D^*_{x_i}$ ($D^*_{y_i}$). Clearly $X'_i \subseteq X_i$ and $Y'_i \subseteq Y_i$, so the bounds above also hold with X_i replaced by X'_i and Y_i replaced by Y'_i.

Now using that no vertex of H_i belongs to $X'_i \cup Y'_i$ we get that

$$
|V(H_1) \cap V(H_2) \cap \ldots \cap V(H_k)| \geq n - 64k.
$$

This proves that (B) holds.

Let $W = V(H_1) \cap V(H_2) \cap \ldots \cap V(H_k)$ and $A' = A(H_1) \cup A(H_2) \cup \ldots \cup A(H_k)$. Note that, since the digraphs H_i's are arc-disjoint, the digraph $D' = (V, A')$ satisfies

$$
\lambda_{D'}(u, v) \geq k \text{ for all } u, v \in W. \tag{21}
$$

Now let Q be the directed multigraph on $|V - W| + 1$ vertices that one obtains from T by contracting the set W to one vertex w (that is we delete any possible loop at W, but keep multiple arcs between w and vertices from $V - W$). Since contraction preserves arc-strong connectivity Q is k-arc-strong. Hence, by Corollary 3.2, Q contains a spanning k-arc-strong subdigraph Q' with at most $2k|V - W|$ arcs.

Let A'' be the arc set which corresponds to $A(Q')$ back in T and define H by $H = (V, A' \cup A'')$. Using (21) it is easy to show that H is a spanning k-arc-strong subdigraph of T. It remains to bound the number of arcs in H. We saw that A'' has size at most $2k|V - W|$. Now, combining (A) and (B) we get

$$
\begin{aligned}
|A(H)| = |A' \cup A''| \ &\leq \ |A'| + |A''| \\
&\leq \ (\sum_{i=1}^{k} |A(H_i)|) + 2k|V - W| \\
&< \ (\sum_{i=1}^{k} |A(H_i)|) + 128k^2 \\
&\leq \ \sum_{i=1}^{k} [|V(H_i)| + 8k] + 128k^2 \\
&\leq \ nk + \sum_{i=1}^{k} 8k + 128k^2 \\
&\leq \ nk + 136k^2.
\end{aligned}
$$

Thus H is the desired small k-arc-strong subdigraph of T and the proof is complete.

$\diamond$

7 Algorithmic Aspects

Theorem 7.1 *There exists a polynomial algorithm which given any k-arc-strong tournament T on n vertices returns a spanning k-arc-strong subdigraph D' of T such that $|A(D')| \leq nk + 136k^2$*

Sketch of proof: Basically our proof above is constructive, but let us give a short sketch of how to find the desired subdigraph in polynomial time. First observe that given any digraph one can find a non-extendable cycle in polynomial time just by applying an obvious greedy strategy: start from an arbitrary cycle C. If some vertex of $V - C$ is adjacent to all vertices of C, then (following the proof of Lemma 4.1) it is easy to find a cycle C' extending C. Now continue from C'.

Now it is easy to see that the proof of Theorem 4.2 can be turned into a polynomial algorithm for finding a spanning strong subdigraph with the desired number of arcs. This again implies that Corollary 4.3 has an algorithmic version: First find a spanning strong subdigraph H of D such that $|A(H)| \leq n + c + \sqrt{2q}$. To obtain $\tilde{H}$ from H we simply replace each arc of H which corresponds to a subdivided arc in $\tilde{D}$ by the corresponding path in $\tilde{D}$.

Suppose T is k-arc-strong and that X, Y are as defined in the proof of Theorem 2.1. We can find paths $P_1, P_2, \ldots, P_k$ such that each P_i starts in Y and ends in X and all end vertices of $P_1, P_2, \ldots, P_k$ are disjoint as follows: add two new vertices s, t to T along with the following arcs $\{sy : y \in Y\} \cup \{xt : x \in X\}$. Now we can find

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$P_1, P_2, \ldots, P_k$ by finding k-arc-disjoint (s,t)-paths in the resulting digraph. This can be done in polynomial time using flows (see e.g. [2]).

For each $i = 1, 2, \ldots, k$, we can find Q_i by finding the strong components of D_i and identifying the strong component which contains both of x_i, y_i. Then using the algorithmic version of Corollary 4.3 we can find the subdigraphs $H_1, H_2, \ldots, H_k$ in polynomial time. Let $A' = \bigcup_{i=1}^k A(H_i)$ and $W = \bigcap_{i=1}^k V(Q_i)$ and let Q be obtained from T by contracting W into one vertex w. Using a polynomial algorithm of Corollary 3.3 we can find k-arc disjoint out-branchings and k-arc-disjoint in-branchings all rooted at w in Q. Let A'' be the union of the arc sets of these $2k$ branchings and take $H = (V, A' \cup A'')$.

In the case when $n \leq 64k$, we simply apply the algorithm of Corollary 3.3 for finding k-arc disjoint out-branchings and k-arc-disjoint in-branchings all rooted at some vertex z in T and take their union. $\diamond$

8 Further Consequences and Future Work

For a given digraph we denote by $\bar{\Delta}(D)$ the maximum degree of a vertex in the complement of $UG(D)$. Clearly, $\bar{\Delta}(D) \geq \alpha(D) - 1$.

A digraph $D = (V, A)$ is *semicomplete multipartite* if V can be partitioned into $r \geq 2$ disjoint sets $V_1, V_2, \ldots, V_r$ such that there are no arcs with both end vertices in some V_i and whenever x, y belong to different sets V_i, V_j there is an arc between x and y. It is easy to see that $\bar{\Delta}(D) = \alpha(D) - 1$ holds for every semicomplete multipartite digraph.

The **cyclomatic** number of a directed graph $D = (V, A)$ is the parameter $|A| - |V| + c(D)$, where $c(D)$ denotes the number of connected components of $UG(D)$. A digraph is **cyclic** if every vertex belongs to a cycle. The following conjecture is implicitly formulated in [11]:

Conjecture 8.1 *[11] Every strong digraph D contains a cyclic spanning subdigraph with cyclomatic number at most $\alpha(D)$.*

An example due to Favaron (see [8]) shows that it is not always true that a strong digraph D contains a strong spanning subdigraph with cyclomatic number at most $\alpha(D)$. However, applying Theorem 4.2 we see that the following holds:

Theorem 8.2 *Every strong digraph D contains a strong spanning subdigraph with cyclomatic number at most $\bar{\Delta} + 1$.* $\diamond$

This implies that Conjecture 8.1 is true (even in the stronger form, requirering a strong subdigraph) for the class of semicomplete multipartite digraphs.

Theorem 8.3 *[7] Every strong digraph D has a spanning strong subdigraph with at most $|V(D)| + 2\alpha(D) - 2$ arcs.*

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As we mentioned in the introduction it is an NP-hard problem to find a spanning strong subdigraph with the minimum possible number of arcs in a given digraph. Using Theorem 4.2 and the fact that there is a polynomial algorithm to find the desired strong subdigraph with few arcs, we see that the following holds:

Theorem 8.4 *Every strong digraph D with $\bar{\Delta}(D) \leq \frac{n}{r}$ contains a spanning strong digraph with at most $(1 + \frac{1}{r})n$ arcs. Furthermore, such a subdigraph can be found in polynomial time.*

Khuller et at [16, 17] proved that for general digraphs, a variant of the algorithm used in the proof of Theorem 4.2 (contracting cycles which are sufficiently long and taking the arcs of the contracted cycles as a spanning subdigraph) results in a spanning strong subdigraph with no more than 1.61 times the number of arcs in an optimal spanning strong subdigraph. It is worth noting that whenever $r \geq 1.64$ the algorithm of Theorem 8.4 above gives a better approximation guarantee and furthermore is simple to implement.

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