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A CONSTRUCTIVE THEORY
OF COUNTERFACTUALITY AND
OTHER MODALITIES

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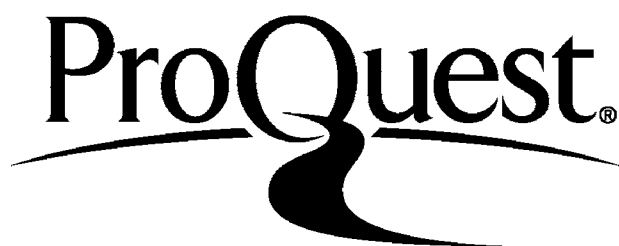
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ABSTRACT

We develop a theory of counterfactuality and other modalities which provides an explanation of how speakers may be able to decide (in other than the completely trivial cases) as to whether a particular counterfactual or modal sentence is true or false. We provide an explication of the ability which native speakers have to evaluate particular counterfactuals or modal sentences in given situations of use.

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A CONSTRUCTIVE THEORY
OF COUNTERFACTUALITY AND
OTHER MODALITIES

It is often maintained that modalities are to be analysed in terms of certain relations between possible worlds, or in terms of functions that take possible worlds as arguments and return possible worlds as results. For example, Kripke semantics for the modal logic of 'necessarily' and 'possibly' appeals to a relation of "accessibility" between possible worlds. Stalnaker analyses counterfactuals with the help of a function which from a world w and a proposition P yields the nearest world to w in which P is true; David Lewis accounts for this construction with the help of a ternary relation of "comparative similarity" between possible worlds.

Our objective has been to develop an approach to counterfactuals and modals which might provide an account of how speakers may be able to decide (in other than the completely trivial cases) as to whether a particular counterfactual or modal sentence is true or false. Our theories differ from those of the above authors in providing an explication of the ability which native speakers have to evaluate particular counterfactuals and other modal sentences in given situations of use.

As one consequence of this, in the theories developed here, the primary relations (of "accessibility" and "comparative similarity") will not connect individual possible worlds but rather "states of affairs", "situations" or "identifiable portions of such worlds".

COUNTERFACTUALS - GOODMAN,STALNAKER AND LEWIS

In his 'Fact, Fiction and Forecast' Nelson Goodman proposed that a counterfactual

- (1) 'If it had been the case that P then it would have been the case that Q'

is true exactly when Q is derivable from P together with a "suitable" set of sentences S. But what are these "suitable sentences" which are meant to be taken in conjunction with the antecedent as a basis for inferring the consequent? A compact but lucid argument (in which he introduces a series of restrictions on the membership of this class) leads Goodman to the following tentative rule:

.... a counterfactual is true if and only if there is some set S of true sentences such that S is compatible with Q (the consequent) and with $\sim Q$ and such that $S \cup \{P\}$ is self-compatible (P the antecedent) and leads by law to Q; while there is no set S' compatible with Q and with $\sim Q$, and such that $S' \cup \{P\}$ is self-compatible and leads by law to $\sim Q$.

Unfortunately the requirement of self-compatibility is too weak; it might be the case that S contains true sentences which though compatible with P would nevertheless not be true if P was true. Consequently, the stated criterion would determine as true many conditionals which we would intuitively regard

as false. To exclude such sentences from the set of relevant conditions Goodman introduces a stronger criterion: S must be 'cotenable' with P. But this requirement is a disastrous one. In order to determine whether or not S is cotenable with P, we have to determine whether or not the counterfactual 'If P were true, then S would not be true' is itself true. But this means that cotenability is defined in terms of counterfactuals which are defined in terms of cotenability; which in turn renders the account circular - a circularity which seems irrevocable as there does not appear to be a simple alternative characterisation of cotenability which does not make some direct or indirect reference to counterfactual constructions. Goodman himself thought that the only possible way to escape from this circularity would be to account for the truth-conditions of counterfactuals in terms of underlying natural laws; and to develop a theory of lawlikeness which would include a solution of his so-called projection problem. Unfortunately, as Goodman himself emphasises, this would destroy his original claim to explicate the concept of law by reference to counterfactuals. Worse still, it now appears that the projection problem, which Goodman subsequently came to see as the kernel of the problem of lawlikeness, can in its turn only be solved if we appeal to certain counterfactuals.

Bob Stalnaker and David Lewis have each developed theories which, although they do not solve completely the problem of truth-conditions, say enough about it to clarify to a large extent the logic of conditionals. Stalnaker supposes that the circumstances in which we use a conditional (1) determine a function (f, say) which maps sentences onto alternative situations in which the sentences are true. The conditional (1) is true in

the situation given, exactly when in the antecedent situation determined by the function the consequent is true. Very little is said in Stalnaker's first statement of his theory about how the function f is determined. In this respect the theory of Lewis is a considerable improvement. This theory rests upon the idea that when we evaluate (1) in the given situation S we are led to consider just those alternative situations or situation in which the antecedent is true while yet as many aspects of S are retained as possible.

Like Goodman's notion of cotenability this idea derives directly from the incontrovertible intuition that, in scientific as well as ordinary discourse, the truth-conditions of counterfactuals involve a *ceteris paribus* rider: here the *cetera paria* are what the situation of evaluation and the relevant alternatives have in common. Lewis determines the situations relevant to the evaluation of (1) in S in terms of a relation of comparative similarity: the relevant situations are, roughly speaking, those which are as similar to S as possible given the truth of the antecedent.

Apart from being conceptually more satisfying than Stalnaker's theory, Lewis' theory also has the laudable advantage of allowing for the distinction between (1) and

- (2) If it had been the case that P it might have
been the case that Q .

In Stalnaker's original theory there is no room for this distinction since

the function f always determines a unique alternative. Lewis, however, refrains from assuming that the (ceteris paribus) problem admits of a unique solution. In other words, given S and P , there may be several situations in which P holds and which are equally similar to S . If in some of these Q is true and in some others Q is false then both (1) and

- (3) If it had been the case that P then it would have been the case that $\sim Q$

are false but (2) is true.

It has been observed in the meantime, in particular by Stalnaker himself, that the difference between Lewis' theory and his own account, becomes much narrower if one allows that the function f may not always be fully determined by the context of utterance. It would seem that in case $f(S, P)$ is not determined (1) would automatically be without truth-value. However, in certain such cases it is still possible to attribute a truth-value to (1), viz when for all functions f compatible with the context, Q has the same truth-value. If on the other hand Q is true in some of these while false in some others then both (1) and (3) would be regarded as truth-valueless. This still does not seem fully satisfactory - in some cases of this kind (3), say, would intuitively seem false rather than undetermined, and (2) true.

The concept of comparative similarity does not lead to a unique concept of validity for sentences containing counterfactual conditionals. But as with the notion of an accessibility relation between possible worlds in modal

logic, it provides us with a family of clearly articulable concepts of validity, distinguished from each other by the sets of conditions that are imposed upon the alternativeness relation. Lewis' book [24], as well as his subsequent writings [23], [25], show how fruitful his theory is in this regard.

Perhaps the single most convincing aspect of Lewis' theory (and Stalnaker's for that matter) is the treatment of the so-called 'paradoxes of conditionals'. These inference patterns were marked as valid by earlier theories despite the fact that there are obvious counter-examples. For example, consider the following pattern:

$$\frac{P \supset Q}{(P \ \& \ R) \supset Q}$$

In the following examples it is surely possible to hold (4) true and (4') false. In particular, one could claim that if the USA dumped its weapons there would be war but this would be avoided if the other nations followed suit.

(4) If the USA threw all its weapons into the sea
tomorrow there would be war.

(4') If the USA threw all its weapons into the sea
tomorrow and the other nations did so also there
would be war.

Lewis explains the possibility of such counter-examples by observing that some worlds where the USA disposes of its weapons are more similar to the actual world than those where all the nations do so.

COMPARATIVE SIMILARITY

AND POSSIBLE WORLDS

Broadly speaking there are two types of objection to the theory of David Lewis. Firstly, there are those who reject the very concept of 'possible world' as Lewis uses it; and then there are those philosophers who, while not too disturbed by the use of possible worlds as such, cannot assign a clear meaning to the relation of comparative similarity. Plainly, for one who rejects possible worlds altogether there is not even a framework within which the second objection could be phrased.

I believe that there is a rather deeper connection between the notion of a possible world and the comparative similarity relation than this way of stating the dichotomy suggests. Careful reflection on the concept of possible world itself, and on its role in the analysis of counterfactual sentences, will also lead to a better understanding of the similarity relation. Indeed, it will lead us back to a perspective of counterfactuals that in some ways resembles the original intuitions of Goodman which led him to the notion of cotenability.

Imagine a situation in which I hold a standard, dry match surrounded by air with the usual percentage of oxygen. Assume further that the match has not

been scratched or ignited. I take it, that in such circumstances, there would be almost universal consent to (5) as opposed to (6).

(5) If the match had been scratched, it would have
lighted

(6) If the match had been scratched, it would not
(could not) have been dry.

The question is: why does everyone hold constant the dryness and conclude that the match would have lighted? Lewis apparently does not provide us with an answer to this question. For he does not tell us enough about what it is for one possible world to be closer than a second to the actual world. Actually, it is difficult to be more explicit on this point than Lewis has permitted himself to be. The difficulty relates to the range of applicability of the theory: it is to be expected that a general theory of the logic of counterfactual conditionals should refrain from greater explicitness - as any further specification of the comparative similarity relation would almost certainly restrict its range of applicability. Nonetheless, it does seem important to enquire how we actually interpret the relation in particular cases such as the one considered. Indeed, a little reflection on this question cannot fail to lead one to the following observations. In those situations where we are able to say something more definite about the comparative similarity relation our statements will have the following form: given that our world is such and such, a world with certain characteristics will be closer to our world than a world with certain other characteristics. That is, any positive assertion of this sort

claims something about the relative closeness of groups of worlds of various types. Furthermore, it seems reasonable to hold that, where a speaker has a definite opinion about the truth-value of a (non-trivial) counterfactual in a specific situation of use, this intuition is based on comparative similarity judgements of this very sort; judgements, that is, which always involve 'identifiable clusters of worlds'.

This suggests that the relation of comparative similarity that figures in the theory of Lewis is not itself to be regarded as sui-generis but rather as if it were definable in terms of a conceptually more fundamental relation; one which does not connect individual possible worlds, but rather sets of them, or perhaps definitions of these sets (ie partial descriptions of possible worlds) or even 'portions of worlds'.

The terms 'partial description' and 'portion' are a little vague. What exactly do we have in mind? The partial-description interpretation is rather closely related to an idea of Hintikka's. In his so-called "Surface Semantics" the interpretation of a sentence S relates to what can happen in a step-by-step investigation of a world in which S is true. A surface model specifies all the different kinds of individuals which one can find in the kind of possible world one is characterising. Presumably, this is to be seen in contrast to the normal approach where we characterise the different models in which expressions may or may not be true in such global terms as quantification over its domain of individuals. The 'portion' interpretation is closely linked to some recent work of Barwise and Perry [4], [5]. They introduce the notion of a 'situation', where a situation consists of certain objects and certain properties and relations which hold

between them. Situations are part of the actual world; events and scenes are examples of situations. According to these authors it is situations that we perceive and talk about - statements designate situations. As the authors point out this conception is not new nor unproblematic. But their presentation is rather convincing. Certainly, the idea of a situation leads one to a view of semantics that is radically different from that developed in possible world theories and one that may well contribute to the solution of some of the problems concerned with intensionality.

In carrying out our analysis we shall primarily be guided by the first interpretation. However, all of what is involved can be understood in terms of 'portions' rather than 'partial descriptions'. In fact, we take the view that partial descriptions delineate such portions or situations.

COMPARATIVE

PLAUSIBILITY

As I have said the point of this approach is that it obviates the need for such dubious entities as possible worlds and a comparative similarity relation between them. It will help us, however, in our explanation of what formal characteristics the comparative similarity relation between partial descriptions should have, to begin by assuming that, possible worlds and a comparative similarity relation in Lewis' sense, is given, and ask what connection there ought to be between this relation and an analogous relation between the so-called partial descriptions or definable portions of worlds.

Consider the following 'might' counterfactual.

- (7) Had Healey become the last Labour leader then the party might have won the last election.

According to Lewis this is true in so far as it is possible to match any world in which Healey won and Labour lost, by a world which is at least as similar to the actual world, but where both Healey and Labour were victorious. Not all worlds where Healey and Labour won will do, of course; some such worlds are so bizarre that they must be considered vastly more dissimilar from the actual world than some others where Healey won and Labour lost. But as a rule these bizarre aspects have nothing to do with Labour leadership, trade unions, or politics; they are irrelevant to the conditional in question.

But what does it mean to say that a world in which Healey and Labour won is at least as similar as one in which Healey won but Labour lost? To answer this question we must take the talk of 'possible worlds' as no more than a metaphor; what then is this last claim a metaphor for? Consider the descriptions:

- (8) 'Is a world in which Healey won the Labour leadership and Labour were victorious at the last general election',

- (9) 'Is a world in which Healey won the leadership
but Labour lost the election'.

Each of these descriptions can be elaborated in innumerable ways; there is no end to the various details - most of which are irrelevant to the conditional under consideration - that we could consistently add to them. Now (8) is at least as close to the actual world as (9) in so far as it is possible to match any consistent elaboration (9') of (9) by a consistent elaboration (8') of (8) which again is at least as close to the actual world. Admittedly, this principle looks circular. It does not explain the 'at least as close as' relation in other terms; it only states a regularity that the notion must obey. Yet, I hope, the intuition behind the principle is clear: no matter how we extend the description (9) (so as to eventually arrive at a complete world description) the successive steps in the process can always be matched by successive elaborations of (8) in such a manner, that at each stage, if the elaboration of (9) is true of a certain world, then the elaboration of (8) is true of some second world at least as similar to the actual world as the first. Moreover, and this is important, in concrete examples such as the one under consideration, our recognition that a world where Healey and Labour were victorious is at least as similar as some other where Healey won and Labour lost is based on precisely this intuition: that whatever further refinements are demanded of our presentations of these various worlds, we can always counter an elaboration (9') of (9) by an elaboration (8') of (8), which is of the same degree of detail as (9'), and which is at least as close to our world as that described by the elaboration of (9). Of course, 'to describe a world which is at least as close to ours...' must itself be understood in terms

of matching still more detailed elaborations of (9) by more detailed elaborations of (8). We shall refer to this as our main principle.

How does the idea of degree of detail that we have implicitly introduced here get realized in our notion of a partial description of a possible world? Hintikka has supplied us with one explication of the term which works only for first order languages. He observed that the sentences of any first-order predicate language can be naturally ordered in terms of the complexity of their quantifier structure: the greater that complexity the greater the variety of propositions expressible. I have already noted that this interpretation of the notion of detail (though perfectly precise), is limited in its applicability, even so, it will enable us to give a formal statement of the constraints on comparative similarity that we have discussed informally.

The notion of comparative similarity has been criticised as too vague to serve as a basis for an account of meaning with clear applications to situations of actual use. Consider the phrase 'u is at least as similar (to w) as v'. We are prompted to ask 'but similar in what respects?' If we look at counterfactuals which people actually use we cannot help but conclude what respects are important. Indeed, it is clear that the content of the antecedent and the consequent of the counterfactual play an important role in determining what these relevant respects actually are. It is questionable whether there can be an objective notion of comparative similarity which is independent of the counterfactual which it is supposed to help evaluate; and the term 'similarity' appears rather as a misnomer in this context. Instead of 'at least as similar as' and 'more similar

than' I shall, therefore, use the terms 'at least as' (and 'more') 'plausible', to denote the intended relations between partial descriptions or situations. This term has its own disadvantages. It seems to suggest a purely epistemic relation, whereas I do not want to preclude a metaphysical interpretation. But at least the word 'plausible' does not imply the false uniformity which 'similar' appears to connote. For reasons of convenience and exposition I shall use the latter term to refer to the comparative similarity relation between worlds - this matching Lewis' own terminology in the strict sense of the word.

What then has become of the relation of comparative similarity between the worlds themselves? Before we can address this question we must get clear about the nature of 'possible worlds' within the present context. Here we are to view 'possible worlds' as maximally consistent sets of partial descriptions; they represent the best descriptions of the worlds available in the underlying language of the partial descriptions.

According to our original insight decisions of comparative similarity between possible worlds are to be taken with respect to partial descriptions of such worlds: one possible world should be considered at least as similar as a second, to the actual world, if we can recognise this on the basis of some finite partial descriptions of these worlds. More precisely, one possible world is to be considered "at least as similar" as a second to the actual world just in case no matter which degree of detail we choose we can find partial descriptions of the worlds, of at least this specified degree of detail, such that the description of the first is at least as plausible as the description of the second, with respect to the actual world.

We began our discussion with an informal connection between these two relations - a partial description a is at least as plausible (from the perspective of w) as a description b if and only if no matter which world v we choose (of which b is a correct description) we can match it by a world u (of which a is a correct description) and which is at least as similar as v to w . Fortunately, this principle is a formal consequence of our theory where the relation of relative plausibility is taken as sui-generis. Furthermore, the standard properties of a relation of comparative similarity (transitivity and connectedness) follow from the corresponding properties for the relation between the partial descriptions.

If you are content to remain with the original relation between the worlds as primitive then the latter principle can be used as a definition of the relation of comparative plausibility between the partial descriptions. Our main principle is a direct consequence of such a definition. Indeed, all other properties of our relation of comparative plausibility follow from the corresponding properties of the relation of comparative similarity. The two theories are equivalent. This is rather satisfactory since it enables one to choose the theory one finds the philosophically/semantically more acceptable without affecting the underlying logic of conditionals.

So far in this section we have said nothing about the role of the actual world. In the previous section we hinted at the following: if we are able to recognise that a is at least as plausible as b , from the perspective of the actual world, then we must be able to achieve this recognition on

the basis of some finite/partial description of the actual world. We could build such a condition into our theory of comparative plausibility, however, this observation points to a three place relation between partial descriptions as playing the fundamental role in our analysis of counterfactuals: one partial description is considered at least as plausible as a second, from the perspective of a third (which represents our description of the actual world). Unfortunately, things are not quite as straightforward as this since one such relation will not do. Intuitively, it need not be definite either that a is at least as plausible as c or c strictly more plausible than a. The information contained in our partial world description may be insufficient to decide the matter one way or the other. This prevents us from defining the strict relation (a is strictly more plausible than c given b) in terms of the non-strict (a is at least as plausible as c given b). The two relations are not so simply interdefinable.

However, eventually matters ought to be decided one way or the other: there ought, given some partial descriptions a and c, to be some degree of detail, such that for every partial description b of at least this specified degree of detail, either a is strictly more plausible than c (with respect to b) or c is at least as plausible as a (with respect to b). In other words, for partial descriptions of the actual world of sufficient degree of detail, the two relations become interdefinable. Such a principle is certainly in keeping with the spirit of the present analysis and in particular with the main principle which governs our relations of relative plausibility.

Our notion of relative plausibility seems closely allied to the Goodman notion of "cotenability". I believe the intuitions behind Goodman's notion can be captured as follows: P is cotenable with Q just in case P & Q is at least as plausible as P & \sim Q. This is to be seen in contrast to the Lewis account of cotenability. His exposition is given in terms of propositions which are sets of possible worlds and where the relation of comparative similarity connects individual worlds; this seems far removed from the original intuitions of Nelson Goodman.

In summary, our concept of comparative plausibility owes much to the theory of Lewis, the only principal difference being that we take the basic relation of comparison to be a relation between partial descriptions of possible worlds rather than between the worlds themselves. So although our theory much resembles Lewis' in appearance it constitutes a return to the original Goodman account to which the Lewis-Stalnaker theory was meant to be an antidote.

TRUTH CONDITIONS

FOR COUNTERFACTUALS

How might we use this idea of comparative plausibility to provide the truth conditions for counterfactuals? According to Goodman we select those propositions "cotenable" with the antecedent and which together with the latter enable us to derive (in some appropriate sense) the consequent. Using this account for inspiration we suggest the following analysis for counterfactuals.

We think of a number of alternative circumstances in which the antecedent could be (or could have been as the case maybe) true. Some of these (in most cases only partially specified) we may regard as more plausible than certain others. Suppose for instance that of the circumstances s_1, \dots, s_n considered, each of $s_1, \dots, s_k, k < n$ is more plausible than each of s_{k+1}, \dots, s_n ; if each of the $s_i, i \leq k$, together with the antecedent, imply the consequent then we may regard the conditional as true; if one of the s_i implies the negation of the antecedent then we may certainly regard the conditional as false. If neither of these cases apply we must partition the antecedent situations more finely and repeat the process.

Such a decision process for counterfactuals yields what might be called a DECISION TREE for the counterfactual. We shall say that a partial description b "entails" a counterfactual $A \square \rightarrow B$ if there exists a decision tree for the antecedent A where each of the most plausible branches (with respect to b) 'entails' B . Unfortunately, we cannot assume that b "refutes" a counterfactual just in case it is false that b entails it. The partial description b may not contain sufficient "information" to decide the matter one way or the other. To combat this problem we shall say that a partial description b "refutes" a counterfactual $A \square \rightarrow B$ if and only if there exists a decision tree for the antecedent A where one of the most plausible branches (with respect to b) entails the negation of the consequent B . Each such decision tree - the most plausible branches - yields a class of possible worlds which generates a 'sphere' of worlds (in the sense of Lewis) through out which the consequent is true providing the antecedent

is true; on the other hand, each such sphere can be used to construct a decision tree for the conditional in question. The two approaches are formally equivalent.

The relationship between our truth conditions and the approach of Goodman should also be clear. Each branch in a decision process for the conditional guarantees a derivation of the consequent from the antecedent; furthermore each such branch is "cotenable" with the antecedent according to our gloss on the Goodman notion of cotenability. There is one more part that is central to Goodman's account: there must not be a similar derivation for the negation of the antecedent. We need to establish that if a counterfactual is true with respect to one such decision tree it is true for all "appropriate" such decision trees. Fortunately, this is also a formal consequence of our theory.

I must stress that this is not offered as a complete solution to Goodman's problem. I do not believe that this can be achieved in such a general setting as this discussion presupposes. In order to say something more substantial about the extension of the plausibility relation one must, it seems, restrict one's attention to particular languages (eg the language of a particular branch of physics) and/or specific contexts of use.

RELATIVE

POSSIBILITY

Kripke semantics for modal logic presupposes a relation of "accessibility"

between possible worlds. A sentence is necessarily true just in case it is true in all "accessible" worlds; a sentence is possibly true if and only if it is true in some such accessible world. Our critique of Lewis' theory of counterfactuals has its obvious counterpart here.

We develop an approach to modal logic which provides some account of how we may decide the truth-value of sentences involving modal notions. In the approach we advocate the primary relation of accessibility will operate between partial descriptions of possible worlds. The nearest approach to ours is due to Hintikka. It deploys the notions of a model set and of a relation of accessibility which operates between such sets. The actual details of Hintikka's development are, however, very different to ours. We suggest that one description is "Accessible" from the perspective of a second exactly when the first is judged to be "possible" from the perspective of the second; in other words, our relation of accessibility is to be identified with some notion of Relative Possibility. This is of course not a particularly surprising interpretation. But it is not instantly clear which principles governing the relation are implied by it. We are interested in those aspects of possibility which are in some sense Recognisable. This new dimension to the notion of possibility is introduced because we are interested in developing a theory of relative possibility which is close to the actual use of such modal notions in natural language.

So what is involved in this process of recognition? One observation seems fairly crucial. Let a be an arbitrary description. From the perspective of a we may be unable to recognise that a second

description is possible or to recognise that it is impossible; there may not be sufficient information available (in a) to decide the matter. It is, therefore, not intuitively reasonable to define the impossibility of one partial description (from the perspective of another) as the negation of its possibility. For this reason we introduce two relations - a positive one and a negative one. The positive one indicates that we are able to recognise that a second description is possible from the perspective of the first; the negative one informs us that we are able to recognise that a second description is impossible from the perspective of the first.

Although the two components of our relation of relative possibility are not logically interdefinable, one certainly excludes the other. This is, however, not the only relationship which holds between the positive and negative components of our relation. The word recognition involves some commitment to the belief that the process involved in deciding between the positive and negative components of our relation must eventually terminate. We can capture this intuition as follows. Let b be some description, then there exists some degree of detail such that for each description a, of at least this specified degree, we can either recognise b to be possible from the perspective of a or we can recognise b as impossible from the perspective of a. This principle is a direct consequence of restricting our attention to those aspects of possibility/impossibility which are decidable or recognisable.

Suppose that we have been able to decide that b is possible (impossible) given the information contained in a. To have taken this decision we must

have reached a point in our deliberations where further refinements of the information contained in a seemed irrelevant to the outcome. In other words, the very fact that we can recognise that b is possible (impossible) from the viewpoint of a must mean that we are able to recognise that b is possible (impossible) from the perspective of all refinements of the information contained in a. On the other hand, suppose that we are able to recognise that b is possible (impossible) from the perspective of each of the refinements of a of some specified degree of detail; then we must already be able to recognise that b is possible (impossible) from the perspective of a itself.

Finally consider the relationship between the inclusion relation between partial descriptions and our relation of relative possibility. Presumably, if some description 'refines' a second then obviously the second can be recognised as possible from the perspective of the first - at least this will be true provided that the second description can be recognised as a refinement of the first.

This concludes our brief survey of what we have in mind by the concept of relative possibility. This account should be sufficient for the reader to follow the truth-theory for modal sentences which we shall shortly present.

We can recover a relation of "accessibility" between possible worlds as a derived relation. Briefly, a possible world v is to be considered Accessible from the perspective of a second u exactly when there exists some degree of detail such that all chosen partial descriptions of v of at least this specified degree of detail, can be 'matched' by a partial

description of u – matched in the sense that it renders the partial description of v possible.

The obvious properties of the derived relation follow directly from the corresponding properties of the primitive relation. So, in particular, if the positive relation of relative possibility is reflexive then so is the derived relation of accessibility; if the positive relation of relative possibility is transitive then so is the derived relation of accessibility; and, finally, if the negative relation of relative possibility is symmetric then so is the derived relation of accessibility. Under such a definition of accessibility a partial description b is deemed to be possible from the perspective of a second description a if and only if every world v , of which b is a correct description can be matched by a world u of which a is a correct description – here ‘matched’ means v is accessible from u .

Alternatively, we can use this connection between the two relations as a definition of relative possibility in a theory where accessibility is taken as primitive. Under certain natural constraints on the relation of accessibility the two theories are equivalent in that the properties of the derived relation of relative possibility formally follow. Furthermore, under these constraints reflexivity, transitivity and the symmetric property for accessibility yield the analogous properties for relative possibility.

TRUTH THEORY FOR

MODAL SENTENCES

To motivate our approach to the truth theory of modal sentences we focus our

attention on a context where the notion of a possible world seems very natural. Suppose that we are deliberating about the future. For example, imagine that we are trying to decide whether or not it is possible for our university to get, in the next academic year, exactly the number of students required to stay solvent. As part of our background knowledge we have information about the university entry system: the number of points required for entry and the general policy of the entry committee. We perhaps also have access to information concerning the numbers applying in the past, their academic standards etc. To evaluate the truth of the sentence "It is possible for us to obtain the required number of students" we need to consider the various possible continuations of the present state of affairs. These continuations will of course only be partially defined. If in one of these possible continuations we get the required number of students then we mark our sentence as true. If in all such possible continuations we don't get the required number we must mark the sentence false. If, however, neither of these situations arises we need to refine our characterisations of the 'possible futures' a little more and repeat the above strategy. We may, for example, need to consider the effect of the Dean's power to lower the entry requirement by one point on the scale: will we get too many students? Might such a decision have other unfortunate consequences?

In general, our deliberation process for modal sentences has the following form. We first consider the various possible alternatives to the actual world - or rather the description of it available. These alternatives will, of course, themselves be only partially described. If one such entails the truth of the sentence A then we may mark the sentence 'It is possible that A'

as true; if they all entail the negation of A then we mark the sentence 'It is impossible that A' as true. It may turn out, however, that these various possibilities are too crudely defined to arbitrate in the matter. Our tactic must then be to refine our description of these various alternatives: we consider alternatives which are refinements of those already obtained and which are possible alternatives to the actual world.

Our approach is linked to the Kripke account in as much as the two theories are in agreement. Consequently every notion of validity yielded by Kripke's approach is matched by an extensionally equivalent notion yielded by ours. For example, validity relative to all contexts where the relation of relative possibility is reflexive corresponds to all models in which the relation of accessibility between partial descriptions is reflexive.

2. SITUATIONS, PARTIAL
DESCRIPTIONS AND
POSSIBLE WORLDS

In our introduction we have argued that possible worlds are not the appropriate objects on which to base theories of counterfactuality and modality; we have argued that the relations of comparative similarity and accessibility must operate between objects which are more like "portions" or partial descriptions of possible worlds. The time has come to be a little more precise about the nature of these entities.

Our notion of a partial description is due to Hintikka and is based upon his concepts of model set and Distributive Normal Form. Our concept of "portion" is more recent in origin and stems from the work of Barwise and Perry on "situations". The Hintikka account is more linguistic in nature - portions of possible worlds are delineated by partial descriptions of those worlds in some underlying description language. On the other hand, 'situations' are actual chunks of the world - situations are collections or arrangements of objects and properties in the world.

Our alternative ontologies are obviously closely related though the notion of 'situation' seems the more fundamental. For this reason we consider the notion of 'situation' first. We then provide an account of Distributive Normal Form for first-order logic. Finally, we develop a more abstract framework which maintains the main structural properties of the two approaches.

SITUATIONS

A situation is a clearly recognisable chunk of the world. The world consists of objects having properties and standing in relation one to another; situations are complexes of objects, properties and relations. Events are situations in time and scenes are visually perceived situations.

More precisely, a situation induces a function (partial)

$$f : \prod_n (R_n \times A^n) \rightarrow \{1,0\}$$

where A is the set of objects and $R = R_1 \cup R_2 \dots$ is the set of relations (R_1 is the set of predicates; R_2 is the set of binary relations etc). So a situation tells us: (1) of an object a and property p whether or not a has p , (2) of a pair $\langle a_1, a_2 \rangle$ of objects and a binary relation r , whether or not a_1 stands in relation r to a_2 etc. Our formal notion of situation will differ slightly from this but nothing we shall now say will be affected by this change. Notice that situations induce partial functions and so not all tuples of objects and relations will be decided, one way or the other, in all situations. It would of course be wrong to identify situations with such partial functions just as it would be wrong to identify possible worlds with total functions, from $\prod_n (R_n \times A^n) \rightarrow \{1,0\}$.

According to this interpretation our relations of comparative plausibility

(and accessibility) will operate between situations: one situation is at least as plausible as a second exactly when we can "match" any "refinement" of the first by a "refinement" of the second which maintains the relation of relative plausibility. We need to be more explicit about the terms "refinement" and "match".

One clear notion of refinement refers to the amount of information a situation contains. One situation may tell us more about certain objects than another situation does. Indeed, it may inform us about objects - their properties and their relationships one to another - of which the other situation makes no reference. Under this notion of refinement one situation refines or extends another exactly when the first agrees with the second, on all points in the second's domain, but possibly tells us more either about objects already referred to by the second, or about objects of which the second makes no reference whatever. In other words, the partial function representing one situation extends the partial function representing the other.

The second term of our pair is a little more elusive. Imagine that we are engaged in deliberations concerning the comparative plausibility of various situations. Apparently, we need to "match" any refinement of one situation by a refinement of the second. To be realistic, each such stage of the deliberation process must only refer to a finite number of possible refinements of the various situations under consideration. This requirement is forced upon us by the demands that we must be able to recognise that each refinement of one situation can be matched by a refinement of the second. Of course, subsequent stages in the

deliberation process may refer to refinements of the newly selected situations; we only insist that there are clearly marked stages in the deliberation process and that each stage only contains a finite number of situations. We do not, for example, impose any upper bound on the number of possible stages. To summarise, we demand only that the class of possible situations be divided into stages, where each stage contains only a finite number of situations, and where situations in subsequent stages refine those in earlier stages.

Matching situations must be chosen from the same stage. This assumes that situations from the same stage contain comparable amounts of information. We seem to have reasonably clear intuitions concerning the amount of detail a situation contains - or at least clear intuitions about one situation containing at least as much detail as a second. These intuitions have something to do with the number of tuples, made up from objects and relations, which the situation decides. We shall assume that our stages reflect such intuitions in that situations from the same stage contain comparable amounts of detail.

DISTRIBUTIVENORMAL FORMS

Our next interpretation is due to Hintikka and concerns the notion of "Distributive Normal form". These distributive normal forms are generalisations of the "complete" normal forms of propositional logic. For pedagogical reasons we consider these first.

Each consistent formula P of propositional logic has a complete disjunctive normal form which is a disjunction of certain conjunctions called (by Hintikka) constituents. Suppose P contains atomic formulae P_1, \dots, P_k . A constituent of P is a conjunction of the form

$$P_1 \ \& \ \sim P_2 \ \& \ P_3 \ \& \ \sim P_4 \ \& \ \dots \ \& \ P_k$$

ie the i^{th} element of the conjunction is either P_i or $\sim P_i$.

Clearly, any consistent formulae of the propositional calculus can be expressed as a disjunction of such constituents.

We have introduced these complete normal forms of propositional logic partly as a means of introducing some notation. Let $S = \{P_1, \dots, P_k\}$ be any set of well-formed formulae of the propositional (or predicate calculus). Then the set

$$G(S) = \{Q_1 \ \& \ \dots \ \& \ Q_k : Q_i = P_i \ \text{or} \ Q_i = \sim P_i, \ 1 \leq i \leq k\}$$

is the "set of constituents generated by S ".

For monadic logic the analysis is a little more involved . In monadic logic a constituent depends upon the predicates which occur in it; so let P_1x, \dots, P_kx be some such predicates. An arbitrary constituent of monadic logic is of the form

$$\exists xA_1 \ \& \ \sim\exists xA_2 \ \& \ \exists xA_3 \ \& \ \dots \ \& \ \sim\exists xA_{2^k}$$

where the A_i are the 2^k distinct elements of $G(\{P_1x, \dots, P_kx\})$. So the constituents of monadic logic are the elements of the set

$$G(\{\exists xA : A \in G(\{P_1x, \dots, P_kx\})\}).$$

This notation is an improvement on Hintikka's but it still tends to obscure some simple ideas. Hopefully, an example will illuminate any further dark corners. Suppose that we have only two predicates P_1 and P_2 . The following are two possible constituents of monadic first-order logic (with respect to P_1x and P_2x):

$$(i) \ \exists x(P_1x \ \& \ P_2x) \ \& \ \exists x(P_1x \ \& \ \sim P_2x) \ \& \ \sim\exists x(\sim P_1x \ \& \ P_2x) \ \& \ \sim\exists x(\sim P_1x \ \& \ \sim P_2x)$$

$$(ii) \ \exists x(P_1x \ \& \ P_2x) \ \& \ \sim\exists x(P_1x \ \& \ \sim P_2x) \ \& \ \sim\exists x(\sim P_1x \ \& \ P_2x) \ \& \ \sim\exists x(\sim P_1x \ \& \ \sim P_2x).$$

In all there are sixteen such constituents which can be described using P_1 and P_2 (plus quantifiers and propositional connectives).

The constituents of first-order logic are more complicated and depend on

three factors:

- (P1) the set of all relations and predicates which occur in the well-formed formula;
- (P2) the set of all free individual symbols which occur in the formula;
- (P3) the maximal length of sequences of nested quantifiers in the formula.

The parameter (P3) will be called the DEPTH of the formula. This will play the role of the "degree of detail" for the constituents.

The atomic formulae which can be formed from the set P1 and P2 (= $\{a_1, \dots, a_k\}$ say) can be partitioned into two classes: those that involve a_k and those that do not. Let $B_i(a_1, \dots, a_k)$ be those atomic formulae which involve a_k ($1 \leq i \leq n$) (at least one occurrence) and $A_i(a_1, \dots, a_{k-1})$, ($1 \leq i \leq m$) be those that do not.

We proceed to the definition of a constituent for first-order logic in two stages. First we introduce the notion of an ATTRIBUTIVE CONSTITUENT (a-constituent) with fixed parameter sets (P1), (P2) (= $\{a_1, \dots, a_k\}$) and depth d. Let

$$S^d(a_1, \dots, a_k)$$

denote the set of a -constituents of depth d ; this is defined recursively in terms of S^{d-1} as follows:

$$S^d(a_1, \dots, a_k) = \{B \ \& \ A : B \in G(\{B_i(a_1, \dots, a_k) : 1 \leq i \leq n\})\}$$

and

$$A \in G(\{\exists x C : C \in G(S^{d-1}(a_1, \dots, a_k, x))\}).$$

We are now in a position to define the set of Constituents of first-order logic. The set of Constituents of first-order logic of depth d , parameters $P1$ and $P2$ ($= \{a_1, \dots, a_k\}$) is the set:

$$C^d(a_1, \dots, a_k) = \{A \ \& \ B : A \in G(\{A_i(a_1, \dots, a_{k-1}) : 1 \leq i \leq m\})\}$$

and

$$B \in S^d(a_1, \dots, a_k).$$

By way of example suppose R is a single binary relation and a_1, a_2 individual symbols. Then a typical element of $S^0(a_1, a_2)$ has the form:

$$R(a_1, a_2) \ \& \ \sim R(a_2, a_2) \ \& \ \sim R(a_2, a_1)$$

More variety is obtained by allowing one level of quantification; elements of $S^1(a_1, a_2)$ have the form:

$$\begin{aligned}
& R(a_1, a_2) \ \& \ \neg R(a_2, a_2) \ \& \ \neg R(a_2, a_1) \\
& \quad \& \\
& \exists x \{ R(a_1, x) \ \& \ R(a_2, x) \ \& \ R(x, x) \ \& \ R(x, a_1) \ \& \ R(x, a_2) \} \\
& \quad \& \\
& \neg \exists x \{ R(a_1, x) \ \& \ R(a_2, x) \ \& \ R(x, x) \ \& \ R(x, a_1) \ \& \ R(x, a_2) \} \\
& \quad \& \\
& \exists x \{ \neg R(a_1, x) \ \& \ R(a_2, x) \ \& \ R(x, x) \ \& \ R(x, a_1) \ \& \ R(x, a_2) \} \\
& \quad \& \\
& \neg \exists x \{ R(a_1, x) \ \& \ \neg R(a_2, x) \ \& \ R(x, x) \ \& \ R(x, a) \ \& \ R(x, a_2) \} \\
& \quad \& \\
& \neg \exists x \{ \neg R(a_1, x) \ \& \ \neg R(a_2, x) \ \& \ \neg R(x, x) \ \& \ \neg R(x, a) \ \& \ \neg R(x, a_2) \}
\end{aligned}$$

The existential well-formed formulae form the elements of $S^0(a_1, a_2, x)$; there are thirty-two in total.

A typical element of $C^1(a_1, a_2)$ has the form $R(a_1, a_1) \ \& \ D$ where D is in $S^1(a_1, a_2)$.

An important result of Hintikka's (at least for us) is the following: each formula F with fixed parameters $P_1 - P_3$ is equivalent to a disjunction of constituents with the same fixed parameters. Moreover, each constituent with depth d and certain given parameters ($P_1 - P_2$) can be converted into a disjunction of constituents with the same parameters ($P_1 - P_2$) but with greater depth ($d + e$ for $e = 1, 2, \dots$).

Suppose that we are dealing with an interpreted language: the individual

symbols are interpreted as objects and the relation symbols are interpreted as actual relations between such objects. The attributive constituents then describe kinds of individuals - they can be considered as complex attributes of the individual referred to by a_k . They may be said to list all the different kinds of individuals that can be specified by means of (i) a given fixed set of predicates and relations (P1); (ii) the 'reference point' individuals specified by a_1, \dots, a_k ; (iii) at most d layers of quantifiers; (iv) propositional connectives. The constituents describe different kinds of situations - Hintikka says different kinds of 'worlds' but we shall cast some doubt on this identification.

POSSIBLE

WORLDS

The reader might feel that much of what has been said is beside the point. Surely, situations are just collections of possible worlds (those portions which each member of the given collection has in common) and partial descriptions represent the best description of the world available in the description language. In other words, the notion of possible worlds can be taken as fundamental and situations and partial descriptions can then be taken as derived notions.

We have already seen one reason why this reduction of situations (partial descriptions) to possible worlds might be misleading. It will not hurt to

spell out the main point of the analysis once more. In our proposed analysis of counterfactuality we appealed to some notion of comparative plausibility; this was identified with a relation whose domain consists of portions of possible worlds. We claimed that it was counter-intuitive to give an account of the truth-conditions for non-trivial counterfactuals, in actual situations of use, with a notion of comparative similarity which connected individual possible worlds. This at least points to situations/partial descriptions as being the fundamental objects in our theory - and possible worlds being the derived notion.

A second argument is due to Barwise and Perry. Suppose we identify a situation with the set of worlds of which it is a part; suppose a situation s is identified with a set of worlds W_s . In each world $w \in W_s$ $R(a_1, \dots, a_n) \vee \sim R(a_1, \dots, a_n)$ is true; but then $R(a_1, \dots, a_n) \vee \sim R(a_1, \dots, a_n)$ is true at s . This implies that $R(a_1, \dots, a_n)$ is true at s or $\sim R(a_1, \dots, a_n)$ is true at s - but this cannot be right since situations only partially specify states of affairs.

We shall not identify situations with sets of worlds; and we shall not view partial descriptions as describing sets of such worlds but rather as delineating situations. Our notion of situation will be taken as fundamental and possible worlds described in terms of it.

CONSTRUCTIVEALGEBRAS

In this section we provide a more formal framework in which to develop our theories of counterfactuality and modality. A convenient framework will be that of Boolean Algebras. Let $B = \langle B, \sqcup, \cap, 0, 1, * \rangle$ be a Boolean algebra: $\sqcup, \cap, 0, 1, *$ are respectively the union, intersection, zero, identity and complementation operations of the algebra. We shall first show how situations and partial descriptions form examples of Boolean algebras. We then show how the notion of "degree of detail" can be formalised within this context.

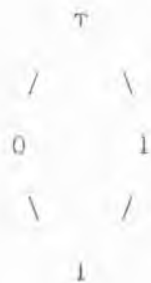
Working with total functions is certainly more convenient than working with partial functions but the obvious way of achieving this - identifying situations with a class of possible worlds - has already been rejected. There is, however, a different route, essentially due to Scott [35]. We first introduce two new elements into our codomain ($= \{1,0\}$). These elements correspond to where the function is not defined or underdetermined (\perp) and to where the function is overdefined or overdetermined (\top). Let the set

$$\text{BOOL} = \{\perp, 1, 0, \top\}$$

be the codomain of situations so that situations now induce total functions

$$f : \left(\prod_n xA^n \right) \rightarrow \text{BOOL}.$$

The set of **BOOL** can be given the structure of a Boolean algebra in a rather natural way as follows: $T \cap 0 = 0$, $T \cap 1 = 1$, $T \cup 0 = T$, $T \cup 1 = T$, $1^* = T$, $T^* = 1$, $0 \cap 1 = 1$, $0 \cup 1 = T$, $0^* = 1$, $1^* = 0$, $1 \cap 1 = 1$, $1 \cup 1 = 1$, $T \cap 1 = 1$, $T \cup 1 = T$, $0 \cap 1 = 1$, $0 \cup 1 = 0$. We can represent this pictorially as follows:



The set of functions (which we shall misleadingly refer to as **SIT**)

$$SIT = \bigcup_n (R_n \times A^n) \rightarrow \text{BOOL}$$

inherits the structure of a Boolean algebra from **BOOL**. We define the union, intersection, zero, identity and complementation operations for **SIT** as follows:

$$f \cap g = \lambda x. \{f(x) \cap g(x)\}$$

$$f \cup g = \lambda x. \{f(x) \cup g(x)\}$$

$$f^* = \lambda x. \{f(x)\}^*$$

$$0 = \lambda x. 1$$

$$1 = \lambda x. T$$

Intuitively, the 'union' of two situations is the situation which results from grafting one situation smoothly onto the other. Of course, these

situations do not have to be adjacent (temporally in the case of events; spatially in the case of scenes) but when they are certain complications may arise. In particular, the situations may be spatially incompatible at certain points (certain elements of $U_n(R_n \times A^n)$); one situation may yield the value 1 and the other 0. The 'union' situation is overdetermined at this point (contradictory information is available) and this is indicated by the value T.

The intersection operation on situations also requires some explanation: the intersection of two situations is the situation which has exactly those features common to both. If the two situations differ at some point then the new situation (the intersection) is undefined (1) at this point. The complement operation requires no explanation. The zero element is the function which is undefined everywhere; the identity element is the function which is overdefined everywhere.

A second example is provided by any first order language. Consider some first order language L with fixed parameter sets of individual constants and relation symbols. Let \vdash be the provability predicate for any complete axiomatisation of first order logic. We define an equivalence relation on L as follows:

$$[P] = \{Q \in L : \vdash P \leftrightarrow Q\}$$

It is well known that these equivalence classes form a Boolean algebra (the Lindenbaum Algebra)

$$\text{LIND} = \langle L/\sim, \cup, \cap, *, 0, 1 \rangle$$

where L/\sim is the set of defined equivalence classes and

$$\begin{aligned} [P] \cap [Q] &= [P \& Q] \\ [P] \cup [Q] &= [P \vee Q] \\ [P]^* &= [\sim P] \\ 0 &= [P \& \sim P] \\ 1 &= [P \vee \sim P]. \end{aligned}$$

A second component in our analysis concerns the phrase "degree of detail". We can summarise our constraints, in the context of Boolean algebras, as follows. We have insisted that the class of situations be divided into stages where each stage is divided into a finite number of situations; situations in subsequent stages refine those in earlier stages and situations in the same stage contain equal amounts of information. In the case of Hintikka Normal Forms the number of nested quantifiers provides a measure of the degree of detail of a partial description.

Definition Let B be a Boolean algebra. A measure for B is any function

$$d : B \rightarrow \omega$$

such that

- (i) $B_n = \{a \in B : d(a) \leq n\}$ is a finite subalgebra of B
- (ii) If a is an atom in B_{n+1} and b is an atom in B_n then a decides b , ie, $a \leq b \vee a \leq b^*$.

This definition requires a word of explanation. An element a of an algebra is an atom if $\neg(a = 0)$ and if b is the algebra with $b \leq a$ then $b = 0$ or $b = a$; an algebra is atomic provided that corresponding to every element $\neg(b = 0)$ there is an atom $a \leq b$ - the atoms play the role constituents played in our example. Notice that condition (ii) makes sense since B_n is finite and finite algebras are atomic. We shall refer to the set of atoms of B_n as T_n ; $T = \bigcup T_n$. We shall call algebras for which such measures exist constructive. Every finite algebra B is trivially constructive - put $B_0 = B_1 = B_2 = \dots = B$.

The algebra LIND affords us a non-trivial example of a constructive algebra.

$$D_n = \{[P] : P \text{ is equivalent (modulo } \vdash) \text{ to a} \\ \text{disjunction of wffs in distributive} \\ \text{normal form of depth at most } n\}$$

The result which states that every constituent of depth d can be seen as a disjunction of constituents of depth $d + e$ ($e = 1, 2, \dots$) guarantees clause (2) of the definition.

In the case of SIT we have already indicated how the notion "degree of

detail" is to be interpreted. One of the consequences of the constraints we imposed amounts to the condition that the class of situations be at most denumerably infinite.

Theorem An algebra is constructive if and only if it is at most denumerably infinite.

Proof

If it is constructive it is obviously at most denumerable infinite.

Conversely, if B is finite it is constructive. If B is denumerably infinite choose an enumeration $\{b_n\}_{n \geq 0}$ of B with 1 as the initial element - assume 0 is absent from the enumeration. Let $B_0 = \{0, 1\}$ and B_{n+1} the algebra generated by $T_n \cup \{b_{n+1}\}$.

In order to compare our approach to those based upon possible worlds we need to decide how possible worlds are to be represented in our theory.

Consider our two example algebras LIND and SIT. Let u be any possible world and D_u be the set of elements b of LIND which represent the sentences of L that are true in u . It follows that D_u is an ultrafilter of LIND.

Conversely, every ultrafilter D of LIND determines a class of possible worlds where for each u in the class satisfies $D = D_u$ - every consistent set of sentences determines at least one world in which every sentence in the set is true. Indeed we could, with Carnap, identify the set of possible worlds with the set of ultrafilters of LIND; but this identification has

been criticised enough: it would render the notion of possible worlds intolerably linguistic in nature. But even if we reject this identification it remains true that the ultrafilters of LIND represent the totalities of information that can be expressed about possible worlds in the language which LIND represents.

In the case of SIT every possible world u determines a class of situations S_u - those situations which u considered as a function $\prod_n (R_n \times A^n) \rightarrow \{1, 0\}$, extends. S_u determines a class of ultrafilters in SIT. On the other hand every ultrafilter S in SIT determines a class of possible worlds - those worlds u such that $S_u \subseteq S$.

In formal contexts there is no objection to using ultrafilters as 'pseudo-worlds' and this is what I shall do here. In fact to make explicit that they are only stand-ins for possible worlds I shall refer to them as 'possible-worlds'.

I shall also frequently write $u \in b$ instead of $b \in u$ where $b \in B$ and u is an ultrafilter in B . This is purely for notational convenience and should cause no confusion. Let $\underline{U(B)}$ denote the ultrafilters of B .

It will be necessary in the sequel to approximate ultrafilters w by ever more detailed partial descriptions. To this end we introduce the following notion.

Definition Let B be a constructive algebra with measure d . A sequence of elements from B , $\{s_n\}_{n \geq 0}$ where $s_n \in B_n$,

is called an ultrafilter sequence (uf-sequence) iff for each n , s_n is an atom in B_n and $s_{n+1} \leq s_n$. A set $D \subseteq B$ is generated by a uf-sequence $\{s_n\}_{n \geq 0}$ exactly when $(\forall n)(D \cap B_n = \{b \in B_n : s_n \leq b\})$.

Our next result relates these uf-sequences to the ultrafilters of a constructive algebra. In fact we prove that each ultrafilter u corresponds to a unique uf-sequence which generates it. The successive members s_n of this sequence may be regarded as representing ever more detailed descriptions of u , which approximate u in the sense that for any finite description b which is entailed by u there is an n such that b is entailed by s_n . We shall make constant use of the existence of these unique approximation sequences for possible-worlds.

Theorem Let B be a constructive algebra with measure d . Then the ultrafilters of B are exactly those sets generated by the ultrafilter sequences.

Proof

Let w be an ultrafilter in B . Let $\hat{w}_n = B_n \cap w$ and $w_n = \bigcap \{b : b \in \hat{w}_n\}$ - this is defined as w_n is finite. Moreover, $w_n \neq 0$ as \hat{w}_n is non-empty and w is an ultrafilter in B . We claim that w_n is an atom in B_n . Suppose it is not; then there exists a $b \in B_n$, $\sim(b = 0)$ and $b < w_n$. By definition $\sim(b \in \hat{w}_n)$ and so $\sim(b \in w)$. Since w is an ultrafilter $b^* \in w$ and as $b^* \in B_n$ we have $b^* \in B_n \cap w$ and so $b^* \geq w_n$ which contradicts $b < w_n$ - one cannot have

$b < w_n \leq b^*$. Moreover any $b \in \hat{w}_n$ satisfies $w_n \leq b$ and so w_n 'generates' \hat{w}_n . It follows that w is generated by the sequence $\{w_n\}_{n \geq 0}$.

Let w be generated by an ultrafilter sequence: assume $w = \bigcup_{n=0}^{\infty} \hat{w}_n$ where \hat{w}_n is generated by w_n ($\hat{w}_n = \{b \in B_n : w_n \leq b\}$) and where $\{w_n\}_{n \geq 0}$ is a uf-sequence. Obviously \hat{w}_n is an ultrafilter in B_n ; it follows that w is an ultrafilter in B since each $b \in B$ occurs in some B_n and so $w_n \leq b$ or $w_n \leq b^*$.

We shall follow the convention used in this proof of denoting the uf-sequence of a ultrafilter for w by $\{w_n\}_{n \geq 0}$.

This completes our discussion of the basic tools for our theory. We shall use the abstract framework developed in this section in chapters three and four to provide a semantic theory for counterfactuality and modality. The reader is urged to chose either of the interpretations offered here in order to make the discussion which follows more concrete. We shall certainly follow our own advice. We shall talk more often than not of partial descriptions but with the understanding that such descriptions delineate situations not sets of possible worlds.

3. COUNTERFACTUALS WITHOUT POSSIBLE WORLDS

In this chapter we present our theory of counterfactuality. It is in three parts. The first deals with a theory of comparative plausibility. As you will recall this relation of comparative plausibility is to operate between partial descriptions rather than between possible worlds. We shall, in fact, develop an axiomatic theory of comparative plausibility. This notion is then used in the second part to provide the truth-conditions for counterfactual sentences; the truth-conditions for counterfactual conditionals are based on the concept of a decision tree where a decision tree is a formal counterpart of the intuitive decision process for conditionals developed in our introductory chapter. Our theory then, appeals to two fundamental ideas: a concept of comparative plausibility and the notion of a decision tree. The final part of the chapter explores the relationships between the theory developed here and those of other authors. In particular, the relationships between our theory and those of David Lewis and Nelson Goodman are explored. Given that our motivation was largely drawn from the intuitions underlying Goodman's discussion of counterfactual conditionals, and that our formal theory has been inspired by the work of David Lewis, we would expect these relationships to be rather close.

COMPARATIVEPLAUSIBILITY

Our analysis of counterfactuals is to be based upon a relation of comparative plausibility on partial descriptions. To gain inspiration about the exact nature of this relation we are to be guided by the principle which informed our original discussion: a partial description a is at least as plausible as a description b just in case we can match refinements of b by refinements of a (of equal degree of detail), and which preserve the relation.

To aid our presentation of the axioms we introduce the notation

$$a \stackrel{w}{\sqsubseteq} b \quad \text{where} \quad a, b \in B, w \in U(B)$$

to mean that a is at least as plausible as b from the perspective of w .

Our first axiom just states that $\stackrel{w}{\sqsubseteq}$ is a transitive relation.

$$(B1) \quad a \stackrel{w}{\sqsubseteq} b \quad \& \quad b \stackrel{w}{\sqsubseteq} c \quad \Rightarrow \quad a \stackrel{w}{\sqsubseteq} c$$

I take this axiom to be reasonably self evident: if a is at least as plausible as b and b is at least as plausible as c then a is at least as plausible as c .

Our second axiom is a principle of linearity: either a is at least as

plausible as b or b is at least as plausible as a.

$$(B2) \quad a \underline{\underline{w}} b \vee b \underline{\underline{w}} a$$

This condition stands or falls with the corresponding relation of comparative similarity on possible-worlds; a little reflection on our informal principle should convince the reader of this. If the relation of comparative similarity on possible-worlds is connected, and we make ever more detailed refinements of the partial descriptions, then the decision must be made one way or the other. This remark will eventually be shown to be a formal consequence of our theory. In any case the addition of (B2) greatly simplifies the technical development of the theory - although this in itself would not be a sufficient reason for its inclusion. I tend to agree with D. Lewis that linearity is an essential part of our understanding of the notion of comparative similarity.

Principle (B3) is related to Lewis's condition of centering. If $b \ni w$ and a , b are disjoint then b should be strictly more plausible than a (where b is strictly more plausible than a just in case it is false that a is at least as plausible as b).

$$(B3) \quad (b \ni w \ \& \ a \cap b = \emptyset) \Rightarrow b \underline{\underline{w}} a$$

$$\text{where } b \underline{\underline{w}} a \langle \Rightarrow \rangle_{\text{def}} \sim (a \underline{\underline{w}} b).$$

Once again this condition is strongly related to the corresponding principle for possible worlds (ie centering). I do, however, believe the

principle (B3) to be more obvious than its analogue for possible worlds.

Our next two axioms are very closely related to our informal principle: a is at least as plausible as b exactly when we can match refinements of b by refinements of a , which are of equal degree of detail, and which preserve the relation of comparative plausibility.

$$(B4) \quad a \sqsubseteq^w b \ \& \ a' \succeq a \ \Rightarrow \ a' \sqsubseteq^w b$$

$$(B5) \quad a \sqsubseteq^w b \ \& \ a \sqsubseteq^w b' \Leftrightarrow a \sqsubseteq^w b \sqcup b'$$

To observe this relationship we indicate that (B4) and (B5) are valid under the interpretation of \sqsubseteq induced by our informal principle. Suppose that $a \sqsubseteq^w b$. Then since any refinement of a is a refinement of a' with $a' \succeq a$, we have $a' \sqsubseteq^w b$. Similarly, (B5) is valid under this interpretation: we need to observe only that a refinement of $b \sqcup b'$, of sufficient degree of detail, is a refinement of b or a refinement of b' ; and conversely any refinement of b is a refinement of $b \sqcup b'$. We shall shortly make these arguments more precise.

Notice that the converse of (B4) actually follows from (B1) - (B5).

$$(B6) \quad a \sqcup a' \sqsubseteq^w b \ \Rightarrow \ a \sqsubseteq^w b \ \vee \ a' \sqsubseteq^w b$$

To see this suppose $a \sqcup a' \sqsubseteq^w b$. By (B2) $a \sqsubseteq^w a'$ or $a' \sqsubseteq^w a$.

Without loss of generality suppose the former. Since $a \sqsubseteq^w a$, by (B5)

$a \sqsubseteq^w a \sqcup a'$. By (B1) we have $a \sqsubseteq^w b$.

Our next axiom relates to the possibility that one of our partial descriptions describes no possible situations - or rather one of our partial descriptions contains contradictory information.

$$(B7) \quad a \stackrel{w}{\sqsubseteq} b \Rightarrow b = 0 \vee \sim(a = 0)$$

This seems very plausible; it states that if a is at least as plausible as b , and a is contradictory, then b must be contradictory also. Notice that, from (B7) (and (B2)), the following useful principle follows:

$$b = 0 \Rightarrow a \stackrel{w}{\sqsubseteq} b$$

We shall make use of this later.

To make some of our informal remarks more precise we need to do the same for our informal principle. We have to exercise some care here because, presumably, the refinements we are interested in, when we claim that we can match any refinement of b by a corresponding refinement of the description a , have to be of a rather special type. They ought to be the "best approximations" (to the possible-worlds) we can get, at the degree of detail specified. Otherwise, we have no guarantee that the refinement of a contains the same "amount of information" as the refinement of b . Recall that T_k is the set of atoms of B_k . These atoms represent the best information available at the degree k . This brings us to a second modification of our principle. Surely, once we are in a position to recognise that $a \stackrel{w}{\sqsubseteq} b$ will be verified within a finite time, then we may regard it as already verified. On

the other hand, given our understanding of these relations of relative plausibility, if we are able to recognise that $a \stackrel{w}{\sqsupseteq} b$ then surely we must have achieved this via some such verification process. With these remarks in mind we can state the formal version of our principle.

$$(B8) \quad a \stackrel{w}{\sqsupseteq} b$$

$\langle \Rightarrow \rangle$

$$\{ (b = 0) \vee (a \neq 0 \ \& \ (\exists k)(\forall m \geq k)(\forall b' \leq b, b' \in T_m)(\exists a' \leq a, a' \in T_m)(a' \stackrel{w}{\sqsupseteq} b')) \}$$

The qualifications $b = 0$, $a \neq 0$, which occur, are to allow for the possibility of contradictory statements.

At last we are in a position to prove that (B8) is formally equivalent to our principles (B4), (B5) and (B7).

Theorem The Theories based upon axiom systems (B1), (B2), (B3), (B8) and (B1) through to (B7) are equivalent.

Proof

Assume (B1) - (B7). Assume $a \stackrel{w}{\sqsupseteq} b$. We can assume by (B7) that $\neg(b = 0)$ and $\neg(a = 0)$. Let $k \geq d(a)$, $d(b)$ and suppose $b' \leq b$ with $b' \in T_k$. By (B5) $a \stackrel{w}{\sqsupseteq} b'$. Let a_1, \dots, a_m be the atoms in T_k such that $a = a_1 \sqcup \dots \sqcup a_m$. By (B6) at least one of these a_i satisfies $a_i \stackrel{w}{\sqsupseteq} b'$. Conversely, assume the right hand side of (B8). Let b_1, \dots, b_m be the atoms of T_k (where k is larger than the guaranteed number for right hand side to hold and also larger than $d(a)$ and $d(b)$) such that

$b = b_1 \sqcup \dots \sqcup b_m$. By hypothesis for each b_i , $1 \leq i \leq m$, there exists $a_i \in T_k$ such that $a_i \stackrel{w}{\sqsubseteq} b_i$. By (B4) for $1 \leq i \leq m$, $a \stackrel{w}{\sqsubseteq} b_i$ and by (B5) $a \stackrel{w}{\sqsubseteq} b$.

Assume (B1) - (B3) and (B8). (B7) follows immediately from (B8). We tackle (B4). If $b = 0$ the result is immediate.

Assume $a \stackrel{w}{\sqsubseteq} b$ and $b \neq 0$; since $b \neq 0$ by (B8) $a \neq 0$.

Then by (B8) there exists a k such that for each $m \geq k$, $b' \leq b$, $b' \in T_m$

there exists $a' \in T_m$, $a' \leq a$ with $a' \stackrel{w}{\sqsubseteq} b'$. But notice that

$a' \leq a$ implies $a' \leq a \sqcup c$ for any c and so by (B8) $a \sqcup c \stackrel{w}{\sqsubseteq} b$.

For (B5) the cases where b or $b' = 0$ are trivial. First assume

$a \stackrel{w}{\sqsubseteq} b$ and $a \stackrel{w}{\sqsubseteq} b'$. Let k be large enough to decide b

and b' (ie $k \geq d(b), d(b')$) and bigger than the numbers guaranteed

by the application of (B8) to $a \stackrel{w}{\sqsubseteq} b$ and $a \stackrel{w}{\sqsubseteq} b'$. Let

$c \leq b \sqcup b'$ and $c \in T_k$. Then $c \leq b$ or $c \leq b'$. Without loss of

generality assume the former. Then application of (B8) to

$a \stackrel{w}{\sqsubseteq} b$ guarantees an $a' \leq a$, $a' \in T_k$ with $a' \stackrel{w}{\sqsubseteq} c$. Then

(B8) gives $a \stackrel{w}{\sqsubseteq} b \sqcup b'$. Conversely assume $a \stackrel{w}{\sqsubseteq} b \sqcup b'$. Let

$c \leq b$ with $c \in T_k$, where k is greater than the number guaranteed

by application of (B8) to $a \stackrel{w}{\sqsubseteq} b \sqcup b'$. Then since $c \leq b \sqcup b'$

there exists $a' \leq a$, $a' \in T_k$ with $a' \stackrel{w}{\sqsubseteq} c$. It follows by (B8)

that $a \stackrel{w}{\sqsubseteq} b$. Similarly $a \stackrel{w}{\sqsubseteq} b'$.

This result confirms our original intuitions; it further informs us that we might have taken the principle (B8) itself as fundamental. I cannot see that it matters much which we do except that (B4), (B5) make no mention of the measure function and so may be somewhat more acceptable if only for reasons

of parsimony.

So far in this discussion of comparative plausibility we have paid little or no attention to the role of the actual world. In particular we might ask if the recognition that $\langle a, b \rangle$ belongs to the extension of $\underline{\underline{w}}$, can be achieved only on the basis of some finite partial description of w . One must address this question with some care for the answer will largely depend upon the notion of comparative plausibility in question. If one is interested in a more metaphysical notion of comparative plausibility the answer to our question is probably no. But it should be clear by now that this is not our primary interest; we are more interested in a notion which will bring us closer to the discussion of counterfactuals given by Goodman. In which case it is unrealistic to insist that decisions of comparative plausibility are to be taken from the perspective of a complete description of the actual world.

To mitigate matters we begin by introducing the following definition:

$$a \underline{\underline{b}} c \Leftrightarrow_{\text{def}} (\forall w \supset b)(a \underline{\underline{w}} c).$$

Our beliefs about the role of the actual world can then be forced by the addition of a new axiom

$$(B9) \quad a \underline{\underline{w}} b \Leftrightarrow (\exists n)(a \underline{\underline{n}}^w b).$$

In words: if we are able to recognise that a is at least as plausible as b from the perspective of w , then we must be able to achieve this recognition on

the basis of some finite partial description of w .

Identical considerations apply to the strict relations and lead to the following definition and axiom:

$$a \overset{b}{\supset} c \Leftrightarrow_{\text{def}} (\forall w \exists b)(a \overset{w}{\supset} c)$$

$$(B10) \quad a \overset{w}{\supset} c \Leftrightarrow (\exists n)(a \overset{w}{\supset}^n c).$$

This may seem rather unsatisfactory for principles (B9) and (B10) look rather circular; (B9), for example, can be rephrased in the following way:

$$a \overset{w}{\supset} b \Leftrightarrow (\exists n)(\forall v \exists w_n)(a \overset{v}{\supset} b).$$

This is not too serious an objection, however, for (B9) only states a regularity that any notion of comparative plausibility must obey. The intuition behind the principle is clear enough: we are able to recognise that a is at least as plausible as b given w exactly when there is a partial description of w of sufficient detail to decide the matter.

A Boolean Frame will be a constructive algebra B (together with a measure d) and a relation of relative plausibility which satisfies (B1) - (B10).

COMPARATIVE PLAUSIBILITY- TERNARY RELATIONS

There may, however, be some force to the objection of circularity in axioms (B9) and (B10). If anything it indicates that we should try to develop the theory directly in terms of a relation which makes no mention of possible-worlds.

Let the notation

$$a \underset{\square}{\overset{b}{\sqsupset}} c$$

indicate that a is at least as plausible as c given the information b (about the actual world). Of course, the information contained in any partial description b may not be sufficient to enable us to decide between $a \underset{\square}{\overset{b}{\sqsupset}} c$ and $c \underset{\square}{\overset{b}{\sqsupset}} a$; we cannot just define the strict relation $a \underset{\square}{\overset{b}{\sqsupset}} c$ in terms of the negation of $a \underset{\square}{\overset{b}{\sqsupset}} c$. It seems necessary, therefore, to develop the theory in terms two relations of relative plausibility. The relation \square is to consist of all triples $\langle a, b, c \rangle$ for which it is definite that a is strictly more plausible than c given b; while $\underset{\square}{\overset{b}{\sqsupset}}$ is to consist of those triples for which it is only definite that a is at least as plausible as c given b. We collect together the properties of these relations in the following axiom system.

AXIOMS For $\underline{\subseteq}, \underline{\supseteq}$

For $a, a', b, b', c, c' \in B$ and in (P1) - (P12), $b, b' \neq 0$.

$$(P1) \quad a \underline{\subseteq} b \text{ c} \ \& \ c \underline{\subseteq} b \text{ d} \quad \Rightarrow \quad a \underline{\subseteq} b \text{ d}$$

$$(P2) \quad a \underline{\subseteq} b \text{ c} \ \& \ a \underline{\subseteq} b' \text{ c} \quad \Leftrightarrow \quad a \underline{\subseteq} b \cup b' \text{ c}$$

$$(P3) \quad a \underline{\subseteq} b \text{ c} \ \& \ a' \geq a \quad \Rightarrow \quad a' \underline{\subseteq} b \text{ c}$$

$$(P4) \quad a \underline{\subseteq} b \text{ c} \ \& \ a \underline{\subseteq} b \text{ c}' \quad \Leftrightarrow \quad a \underline{\subseteq} b \text{ c} \cup c'$$

$$(P5) \quad a \underline{\supseteq} b \text{ c} \ \& \ c \underline{\supseteq} b \text{ d} \quad \Rightarrow \quad a \underline{\supseteq} b \text{ d}$$

$$(P6) \quad a \underline{\supseteq} b \text{ c} \ \& \ a \underline{\supseteq} b' \text{ c} \quad \Leftrightarrow \quad a \underline{\supseteq} b \cup b' \text{ c}$$

$$(P7) \quad a \underline{\supseteq} b \text{ c} \ \& \ a' \geq a \quad \Rightarrow \quad a' \underline{\supseteq} b \text{ c}$$

$$(P8) \quad a \underline{\supseteq} b \text{ c} \ \& \ a \underline{\supseteq} b \text{ c}' \quad \Leftrightarrow \quad a \underline{\supseteq} b \text{ c} \cup c'$$

$$(P9) \quad a \wedge b = 0 \quad \Rightarrow \quad b \underline{\supseteq} a$$

$$(P10) \quad (\exists k)(\forall b \in T_k)(a \underline{\supseteq} b \text{ c} \vee c \underline{\supseteq} a \text{ b} \vee a \underline{\equiv} b \text{ c})$$

$$(P11) \quad a \underline{\supseteq} b \text{ c} \quad \Rightarrow \quad a \underline{\subseteq} b \text{ c} \ \& \ \neg(c \underline{\supseteq} a)$$

$$(P12) \quad c = 0 \ \& \ a \neq 0 \quad \Rightarrow \quad a \underline{\supseteq} b \text{ c}$$

$$(P13) \quad a \underline{\supseteq} 0 \text{ c} \ \& \ a \underline{\supseteq} c.$$

$$\text{where in (P10) } a \underline{\equiv} b \text{ c} \quad \Leftrightarrow_{\text{def}} \quad (a \underline{\supseteq} b \text{ c} \ \& \ c \underline{\supseteq} a)$$

Most of the axioms are direct analogues of the axioms of $\underline{\overset{w}{\subseteq}}$: (P1), (P5) correspond to (B1); (P3), (P7) correspond to (B4); (P4), (P8) to (B5) and (P9) to (B3). Axiom (P2) (and (P6)) incorporates two basic insights. If we are in a position to recognise that $a \underline{\subseteq} b \text{ c}$ then we must be in a position to recognise $a \underline{\subseteq} b' \text{ c}$ for all extensions b' of b . This is the principle incorporated in (P2) from right to left. The converse embodies the idea that

if we are in a position to recognise that the relation will be verified within a finite time, then we may regard it as already verified. In fact, if $(\exists k)(\forall b' \leq b)(d(b') \geq k)(a \stackrel{b'}{\sqsubset} c)$ then using (P2) we can establish that $a \stackrel{b}{\sqsubset} c$. Principle (P10) corresponds to (E2). Because \sqsupset , \sqsubset are not interdefinable we cannot simplify this axiom to

$$(\exists k)(\forall b \in T_k)(a \stackrel{b}{\sqsubset} c \vee c \stackrel{b}{\sqsupset} a).$$

The only relationship between \sqsupset , \sqsubset is given by (P11) and this is not sufficient to guarantee the equivalence between the above and (P10). One more observation about (P10) seems to be in order. Implicitly, it makes reference to possible-worlds through the sets T_k . There seems no obvious way of removing this reference; we need to quantify with respect to finite sets from the algebra. Axiom, (B7) corresponds to (P12). Axiom (P12) may appear quite strange; it embodies the principle that from a contradiction anything follows.

As an afterthought, one might enquire whether or not something like the following principle is valid.

$$a \stackrel{b}{\sqsubset} c \Leftrightarrow (\exists k)(\forall m \geq k)(\forall c' \leq c, c' \in T_m)(\exists a' \leq a, a' \in T_m)(a' \stackrel{b}{\sqsupset} c')$$

After all this is the analogue of our original guiding principle. It is at least as strong as our original principle; one direction is straightforward and the other follows from an easy application of König's lemma. Rather than discuss the proof of this remark I wish to consider the principle itself. I believe it to be intuitively unsound. I have no objection to the inference from right to left; this much follows from our theory and in particular (P3)

and (P4). It is the converse I find controversial. Suppose for arguments sake that $a = b$. The reason why we are prepared to assert $a \stackrel{a}{\sqsubset} c$ is clear: no matter how we extend c (to c' say) and a (to a' say) we can find an extension of a (a' itself) which preserves the relation. But the converse claims more. According to it we must be able to find, for each extension c' of c , an extension a' of a which will maintain the relation for all extensions a'' of a . This follows from (P2). This seems intuitively unsound.

The next few results prove that the two theories are in complete agreement. The one based totally on partial descriptions ((P1) - (P13)) seems the more satisfactory. If our original claim that we can never command more than a partial description of any possible world is correct, then presumably this applies to the actual world. The awkwardness of this theory counts against it in that we require a more elaborate axiom system. For technical convenience, therefore, I shall mostly use the system based upon (B1) - (B10).

Theorem Let $\stackrel{w}{\sqsubset}$ satisfy (B1) - (B10). Then define,

$$a \stackrel{b}{\sqsubset} c \Leftrightarrow (\forall w \supset b)(a \stackrel{w}{\sqsubset} c)$$

$$a \stackrel{b}{\sqsupset} c \Leftrightarrow (\forall w \supset b)(a \stackrel{w}{\sqsupset} c)$$

then these relations satisfy axioms (P1) - (P13)

Proof

(P1) and (P5) follow from (B1); (P3) from (B4) and (P7) from (B5); (P4) follows from (B5) and (P8) from (B6) and (B4). For

axiom (P9) assume $a \wedge b = 0$ then we need to show $a \stackrel{a}{\sqsupset} b$ i.e.
 $(\forall w \exists a)(a \stackrel{w}{\sqsupset} b)$ but this follows immediately from principle (B3).
 Lets turn our attention to (P2) ((P6)). This follows immediately
 from the observation that for every w , $w \exists b \cup b'$ iff $w \exists b$ or $w \exists b'$.
 For (P10) we begin with (B2) for each w , $a \stackrel{w}{\sqsupset} b \vee b \stackrel{w}{\sqsupset} a$.
 Hence $\forall w(a \stackrel{w}{\sqsupset} b \vee b \stackrel{w}{\sqsupset} a \vee b \stackrel{w}{\equiv} a)$. By (B9) and
 (B10) we have $(\forall w)(\exists n)(a \stackrel{w}{\sqsupset}^n b \vee b \stackrel{w}{\sqsupset}^n a \vee b \stackrel{w}{\equiv}^n a)$.
 Hence by König's lemma $(\exists n)(\forall w)(a \stackrel{w}{\sqsupset}^n b \vee b \stackrel{w}{\sqsupset}^n a \vee b \stackrel{w}{\equiv}^n a)$.
 Axiom (P11) follows by the definitions and the observation that
 $a \stackrel{b}{\sqsupset} c$ implies $(\forall w \exists b)(a \stackrel{w}{\sqsupset} c) \& (\forall w \exists b)(\sim(c \stackrel{w}{\sqsupset} a))$. (P12)
 follows directly from (B7). (P13) follows by definition.

Theorem Let $\stackrel{w}{\sqsupset}$ be the relation, derived from a theory based on
 axioms (P1) - (P13), and which is defined by

$$a \stackrel{w}{\sqsupset} b \Leftrightarrow (\exists n)(a \stackrel{w}{\sqsupset}^n b),$$

Then $\stackrel{w}{\sqsupset}$ satisfies (B1) - (B10).

Proof

Once again much, if not all, is tedious but straightforward. (B1)
 follows from (P1) & (P2); (B4) from (P3), and (B5) from (P4) & (P2).
 For (B2) we need to show $(\exists n)(a \stackrel{w}{\sqsupset}^n b) \vee (\exists n)(b \stackrel{w}{\sqsupset}^n a)$.
 But (P10) implies $(\exists n)(\forall w)(a \stackrel{w}{\sqsupset}^n b \vee b \stackrel{w}{\sqsupset}^n a)$.
 Hence $(\forall w)(\exists n(a \stackrel{w}{\sqsupset}^n b) \vee \exists n(b \stackrel{w}{\sqsupset}^n a))$. So (B2)
 holds. Assume $a \wedge b = 0$ and $w \exists a$. We know from (P9) that
 $a \stackrel{a}{\sqsupset} b$. Hence, since $(\exists k)(w_k \leq a)$, we have $(\exists k)(a \stackrel{w}{\sqsupset}^k b)$

by (P6). Hence (B3) is true. We now need to check (B9) and (B10). Assume $a \stackrel{w}{\sqsupset} b$. Then by definition we have $\sim(b \stackrel{w}{\sqsubset} a)$ i.e. $(\forall k)(\sim(b \stackrel{w}{\sqsubset}^k a))$. By (P10) $(\exists k)(a \stackrel{w}{\sqsupset}^k b \vee b \stackrel{w}{\sqsupset}^k a \vee b \stackrel{w}{\equiv}^k a)$. The second and third possibilities are ruled out by (P2) and the fact that $(\forall k)(\sim(b \stackrel{w}{\sqsubset}^k a))$. Hence $(\exists k)(a \stackrel{w}{\sqsupset}^k b)$. Conversely, assume $(\exists k)(a \stackrel{w}{\sqsupset}^k b)$. Suppose $b \stackrel{w}{\sqsubset} a$. Then by definition $(\exists k)(b \stackrel{w}{\sqsupset}^k a)$. Hence by (P2) $(\exists k)(\forall m \geq k)(b \stackrel{w}{\sqsupset}^m a)$. So for k large enough we have (this time by (P6)) $(\forall m \geq k)(b \stackrel{w}{\sqsupset}^m a \ \& \ a \stackrel{w}{\sqsupset}^m b)$ and this cannot be (by P11). Thus (B10) holds - (B9) is just a definition. (B7) follows from (P12).

As a bonus, the way the three place relations are defined in the original theory, is a formal consequence of (P1) through (P13).

Theorem For a, b, c in B we have:

$$(1) \ a \stackrel{b}{\sqsubset} c \Leftrightarrow (\forall w \ni b)(a \stackrel{w}{\sqsubset} c)$$

$$(2) \ a \stackrel{b}{\sqsupset} c \Leftrightarrow (\forall w \ni b)(a \stackrel{w}{\sqsupset} c)$$

where $\stackrel{w}{\sqsubset}$ is derived relation of the theory (P1) - (P13).

Proof

Assume $a \stackrel{b}{\sqsupset} c$ then the right hand side follows from (P2) and the definition $a \stackrel{w}{\sqsupset} c \Leftrightarrow (\exists n)(a \stackrel{w}{\sqsupset}^n c)$. Conversely, assume right hand side. By definition and König's lemma

$(\exists k)(\forall w \supset b)(a \stackrel{w}{\underset{k}{\supset}} c)$. (P2) gives immediately that $a \stackrel{b}{\underset{b}{\supset}} c$.

For $b=0$ use (P13). Part (2) is proven similarly.

TRUTH CONDITIONS FOR
COUNTERFACTUAL SENTENCES

We shall provide the truth conditions for a simple propositional language containing counterfactual operators. Let L be any countable language, then L_{\supset} is the smallest superset of L that is closed under the truth functional connectives \neg , $\&$ and the counterfactual operator \supset . The other logical connectives \vee , \rightarrow , \Leftrightarrow and the intensional \supset (if - then it might be the case that -) are given by definition in the normal way.

Our truth conditions are given in a rather unorthodox way. Briefly, we define two semantic relations $b \models P$ and $b \not\models P$ by (simultaneous) induction on P . Intuitively, $b \models P$ expresses the fact that b contains sufficient information to assert P ; and $b \not\models P$ expresses the fact that b contains enough information to refute P . There is of course no reason to expect these relations to be interdefinable. We may not have sufficient information to decide the matter one way or the other so, in particular, we cannot infer that $b \not\models P$ holds from the fact that $b \models P$ does not hold; we cannot define $b \not\models P$ as $\neg(b \models P)$.

We are going to give the truth-conditions for L_{\supset} in terms of some Boolean frame B . You will recall that our algebras can be thought of as induced by some underlying description language DL . What exactly is the relationship

between L , L_{\rightarrow} , and DL ? Presumably $L \subseteq DL$. A BOOLEAN MODEL $M = \langle B, f \rangle$ consists of a Boolean frame B together with a function $f: L \rightarrow B$. This function might be seen as just reflecting this inclusion. We might, however, demand more than this namely that L (or L_{\rightarrow}) and DL actually coincide. This is much more problematic. To argue for this view is to take what might be described as a 'Nominalist position'. We are demanding that any distinctions that we are able to draw between different possible situations can be articulated in the object language. But surely this is not the only possible and perhaps not even the most plausible view. We might well have the capacity to draw finer situational distinctions than those expressible in the object language.

We provide the truth-conditions for L_{\rightarrow} gradually. Initially we consider simple counterfactuals, namely those whose antecedents and consequents are not themselves counterfactuals. Let L' be the smallest superset of L which is closed under the truth-functional connectives and let $L_{\rightarrow}(1)$ be the smallest superset of L' which contains $P \rightarrow Q$ for P, Q in L' . The reason for this is entirely pedagogical. In order to provide the truth-conditions for more complex counterfactuals we need to associate with each simple counterfactual an element of the algebra. The fact that we are able to do this will follow from our truth-conditions for simple counterfactual conditions.

Let $M = \langle B, f \rangle$ be a Boolean model. We define $b \models P$ and $b \models P$ where b is in B and P is in $L_{\rightarrow}(1)$. The definition is by (simultaneous) induction on P .

(i) $b \models P$ iff $b \leq f(P)$; P in L

$b \models P$ iff $b \leq f(P)^*$; P in L .

(ii) $b \models P \ \& \ Q$ iff $b \models P$ and $b \models Q$

$b \models P \ \& \ Q$ iff $b \models P$ or $b \models Q$.

(iii) $b \models \sim P$ iff $b \models P$

$b \models \sim P$ iff $b \models P$.

This completes the truth conditions for L' . Before we give the clause for the counterfactual we indicate how to extend the function f to L' :

$$f(P \ \& \ Q) = f(P) \cap f(Q)$$

$$f(\sim P) = f(P)^*.$$

Lemma For each b in B and P in L' the following holds.

(1) $b \models P$ and $b' \leq b$ implies $b' \models P$

(2) $b \models P$ and $b' \leq b$ implies $b' \models P$

The result follows by a simple inductive argument. The next result links the function f with the relations \models , \models .

Lemma

Let $P \in L'$. Then there exists an n such that for each $b \in T_m$
 $m \geq n$,

$$(1) \quad b \models P \quad \text{iff} \quad b \leq f(P)$$

$$(2) \quad b \not\models P \quad \text{iff} \quad b \leq f(P)^*$$

Proof

First define $d : L' \rightarrow \omega$ by recursion as follows:

$$(1) \quad d(P) = d(f(P)) \quad \text{for } P \in L$$

$$(2) \quad d(\sim P) = d(P)$$

$$(3) \quad d(P \& Q) = \max(d(P), d(Q))$$

Notice that $d(P) \geq d(f(P))$. The number $d(P)$ is the number we want; we prove this by induction on P .

The case $P \in L$ is trivial as indeed is the case for negation. We concentrate on the conjunction. Suppose $b \models Q \& R$. This is equivalent to $b \models Q$ and $b \models R$. Choose b such that $b \in T_m$, $m \geq d(Q \& R)$. Then $b \models Q$ and $b \models R$ is equivalent to $b \leq f(Q)$ and $b \leq f(R)$ - by induction hypothesis. Next suppose $b \not\models Q \& R$. This is equivalent to $b \not\models Q$ or $b \not\models R$. Once again let $b \in T_m$, $m \geq d(Q \& R)$. This is then equivalent to $b \leq f(Q)^*$ or $b \leq f(R)^*$. But b is an atom in T_n where

$n \geq d(Q \& R) \geq d(Q), d(R) \geq d(f(Q)), d(f(R))$; hence $b \leq f(Q)^*$
or $b \leq f(R)^*$ is equivalent to $b \leq f(Q)^* \sqcup f(R)^*$.

We shall refer to $d(P)$ as the Degree of P.

Theorem For each P in L' we have: $(\exists k)(\forall t \in T_k)(t \models P \vee t \not\models P)$.

Proof

Put $k = d(P)$.

Since $k \geq d(f(P))$ we know that for any $t \in T_k$, $t \leq f(P)$

or $t \leq f(P)^*$. By the previous theorem $t \models P$ or $t \not\models P$.

To provide the truth conditions for the counterfactual itself our strategy will be to give an account of our informal procedure for the assessment of counterfactuals. If you recall it was of the following form. First we must think of a number of alternative circumstances in which the antecedent could be (would be) true. Next we select the most plausible branches (with respect to our relations of relative plausibility). We then check to see if these most plausible alternatives, together with the antecedent, imply the consequent; otherwise we partition the antecedent situations more finely. Let M be a Boolean model and let $P \in L'$. A P-decision tree p , is any finite tree p , labelled with elements from B with $f(P)$ at the vertex, and which satisfies the following conditions.

- (i) For each branch t of p , $\hat{t} \neq 0$ (the intersection of the elements of t - where it is clear we shall often just write t rather than \hat{t}).

- (ii) If a is a node of p with immediate ancestors a_1, \dots, a_m then $a_1 \cup \dots \cup a_m = a$ and $a_i \cap a_j = 0, i \neq j$.

Let $b \in B$. We shall call p a $\langle b, P \rangle$ -decision tree if p is a P -decision tree which satisfies condition (iii) below. Where P is unimportant we shall refer to a b -decision tree.

- (iii) Let $B(p)$ denote the branches of p . Then $s, t \in B(p)$ implies

$$s \stackrel{b}{\supseteq} t \vee t \stackrel{b}{\supseteq} s \vee t \stackrel{b}{\equiv} s.$$

The set of "most plausible" branches of p , with respect to b , is the set $Np(b) = \{s \in B(p) : \forall t \in B(p), s \stackrel{b}{\supseteq} t\}$.

The notion of a $\langle b, P \rangle$ -decision tree is meant to be a precise counterpart of the informal account; the various layers in the decision tree are meant to reflect the finer and finer partitions of the antecedent. I take it that when we refine the antecedent we wish to do this in a consistent way: we induce the finer partitions by the addition of information which is neither contradicted nor entailed by that already accumulated. This is the content of condition (i). The second condition is included to ensure that our refinements are genuine partitions of the antecedent, ie, we do not lose any antecedent situations. Condition (iii) is somewhat less obvious; it is included to ensure that b contains sufficient information to decide the most plausible branches. We shall shortly say a little more about this condition.

This brings us to the clause in the truth conditions for the counterfactual conditional. The idea should be obvious by now: a counterfactual is true just in case there is a decision tree which guarantees it to be.

(iv) $b \models P \Box \rightarrow Q$ iff there exists a $\langle b, P \rangle$ -decision tree p , where each branch c decides Q and where for each branch c in $N_p(b)$, $\hat{c} \models Q$

$b \models P \Diamond \rightarrow Q$ iff there exists a $\langle b, P \rangle$ -decision tree p , where each branch c decides Q and for some branch c in $N_p(b)$, $\hat{c} \models Q$

where in (iv) a branch c decides Q if and only if $\hat{c} \models Q$ or $\hat{c} \models \neg Q$.

The truth-conditions for $P \Diamond \rightarrow Q$, the 'might' conditional, can be deduced from the definition $P \Diamond \rightarrow Q \stackrel{\text{def}}{\iff} \neg(P \Box \rightarrow \neg Q)$.

Theorem Let B be a Boolean frame. Then for any $b \neq 0$, $b' \neq 0$ in B with $b' \leq b$ we have:

(1) $b \models P \Box \rightarrow Q$ implies $b' \models P \Box \rightarrow Q$

(2) $b \models P \Diamond \rightarrow Q$ implies $b' \models P \Diamond \rightarrow Q$

Proof

Let p be a tree which guarantees the truth of $b \models P \Box \rightarrow Q$. Claim $N_p(b') \subseteq N_p(b)$. Let $s \in N_p(b')$. Then for each $t \in B(p)$,

$s \stackrel{b'}{\sqsubseteq} t$. Suppose $s \notin N_p(b)$. Then by the following lemma there is some $t \in B(p)$, $t \stackrel{b}{\sqsupset} s$. By (P6) $t \stackrel{b'}{\sqsupset} s$ and so we have a contradiction (by P11). On the other hand (P2) yields $N_p(b) \subseteq N_p(b')$. Hence $N_p(b) = N_p(b')$. (1) and (2) now follow. \square

Lemma

For each decision tree p

$$N_p(b) = \bigcap \{A : 0 \neq A \subseteq B(p) : x \in B(p) - A \ \& \ y \in A \Rightarrow y \stackrel{b}{\sqsupset} x\}$$

Proof

Let $N = \{A : 0 \neq A \subseteq B(p) \text{ and } x \in B(p) - A \text{ and } y \in A \Rightarrow y \stackrel{b}{\sqsupset} x\}$.

First claim $N_p(b) \in N$. Now $N_p(b) \subseteq B(p)$, and if $\neg(x \in N_p(b))$

and $y \in N_p(b)$ then $y \stackrel{b}{\sqsupset} x$ and $\neg(x \stackrel{b}{\sqsupset} y)$. By

condition (iii) in our definition of decision trees $y \stackrel{b}{\sqsupset} x$.

Hence $N_p(b) \in N$. Hence $\bigcap N \subseteq N_p(b)$. Conversely, if $z \in N_p(b)$

then if $A \in N$ and $z \notin A$ we know that $y \stackrel{b}{\sqsupset} z$ for each $y \in A$. But $z \stackrel{b}{\sqsupset} y$

as $z \in N_p(b)$ - contradiction. Hence $N = \emptyset$. But this

is impossible, since $N_p(b) \in N$.

This last theorem is in keeping with our intuitive understanding of the relations \vdash and \dashv : the relations should be preserved under possible increases in information. Once a decision concerning a counterfactual has been taken further refinements will not alter matters. But how can we be sure that the truth value of a counterfactual is independent of the particular way that we partition the antecedent? The following result should relieve any anxiety over this matter.

Theorem

For any Boolean frame B , with $b \neq 0$ in B , we have:

(1) $b \models P \rightarrow Q$ iff for each $\langle b, P \rangle$ -decision tree p ,
 (where each branch decides Q),
each $c \in N_p(b)$ satisfies $c \models Q$.

(2) $b \not\models P \rightarrow Q$ iff for each $\langle b, P \rangle$ -decision tree p ,
 (where each branch decides Q) some
 $c \in N_p(b)$ satisfies $c \not\models Q$.

Proof

A little notation will facilitate the presentation of the proof. Suppose $A \subseteq B(p)$ - some set of branches in p . For example $A = \{a_1, \dots, a_n\}$. Let \hat{a}_i be the last element on the branch a_i and let $\bar{A} = \hat{a}_1 \cup \dots \cup \hat{a}_n$. Suppose contrary to the theorem $b \models P \rightarrow Q$ and for some $\langle b, P \rangle$ -decision tree q , (where each b branch decides Q) some $c \in N_q(b)$ satisfies $\sim(c \models Q)$ (and, therefore, $c \not\models Q$ as each branch decides Q). We shall deduce a contradiction

Let p be the decision tree which guarantees the truth of $b \models P \rightarrow Q$.

First notice

$$(i) \quad \overline{N_p(b)} \stackrel{b}{\supseteq} \overline{B(p) - N_p(b)} \quad \text{and}$$

$$(ii) \quad \hat{c}_q \stackrel{b}{\supseteq} \overline{B(q) - \{c_q\}}.$$

Presumably, $\hat{a} \models Q$ for each $a \in N_p(b)$. Now either some such \hat{a} satisfies $\hat{a} \cap \hat{c}_q \neq 0$ or $\hat{c}_q \cap \overline{N_p(b)} = 0$; if there is such an $a \in N_p(b)$ such that $\hat{a} \cap \hat{c}_q \neq 0$ then $\hat{a} \cap \hat{c}_q \models Q \ \& \ \sim Q$. But this is a contradiction (since a simple inductive argument shows that if $b \models Q \ \& \ \sim Q$ then $b = 0$ in B). Hence $\hat{c}_q \cap \overline{N_p(b)} = 0$. So $\hat{c}_q \leq \overline{B(p) - N_p(b)}$ - as $\hat{c}_q \leq f(p)$ and $f(p) = \overline{B(p)}$. Similarly, $\overline{N_p(b)} \leq \overline{B(q) - \{c_q\}}$. So, by (i) and (P8), $\overline{N_p(b)} \stackrel{b}{\supseteq} \hat{c}_q$ and, by (ii) and (P4), $\hat{c}_q \stackrel{b}{\supseteq} \overline{N_p(b)}$. This is a contradiction by (P11).

Theorem

For any boolean frame B with $b \in B$, if $b \models P \supset Q$ and $b \not\models P \supset Q$ then $b = 0$.

Proof

By the theorem if $b \neq 0$ then any decision tree p for P must have some $c \in N_p(b)$ such that $\hat{c} \models P$ and $\hat{c} \not\models P$. This cannot be (induction of $P \in L'$) unless $c = 0$. But then, by the definition (i) of decision trees, this cannot be. Hence $b = 0$.

The reader can perhaps now see the necessity for condition (iii) in our

definition of decision trees: it ensures that a decision concerning the truth or falsity of a counterfactual does not need to be withdrawn in the light of new information.

We can connect our account with that of Goodman. On his view (roughly) $P \leftrightarrow Q$ is backed by an argument or derivation $P, P_1, \dots, P_n \vdash Q$, where P_i are "cotenable" with P and where there is no such derivation for $\neg Q$. Our notion of comparative plausibility is obviously closely allied to Goodman's notion of cotenability; indeed if we translate Goodman's criteria into the terms of comparative plausibility we assert P is "cotenable" with Q just in case $P \& Q$ is at least as plausible as $P \& \neg Q$. If this is a correct interpretation of Goodman's notion of cotenability then our truth conditions provide the appropriate derivation of the consequent from the antecedent: the most plausible branches are cotenable with the antecedent and imply the consequent. The previous theorem provides the second half of Goodman's criteria in that it ensures that no such derivation for the negation of the consequent can exist. This is of course not a total solution to Goodman's problem, but it is the best we can do in such a general framework.

Our discussion highlights a problem which has played a central part in the theorising about counterfactuals; and not just about counterfactuals but about conditionals in general; indeed it crops up at almost every point in the philosophy of language. The problem concerns the relationship between the truth-conditions of sentences and the means that we have available for verifying them. Do the evaluation procedures we have just described capture the truth-conditions of counterfactuals exhaustively, or should they rather be seen as modelling the accessible means of assessing truth?

Questions of this sort are notoriously difficult. It is of course always dangerous to claim that no more of interest can be said about a certain question but nevertheless some such questions do seem unanswerable. Indeed, the insight that there really is no reason to expect such questions to be conclusively answered is itself a valuable change of attitude in language theory; just as it was a helpful change of attitude when people gave up the idea that a semantic analysis would be inadequate unless it attributes to the sentence analysed a truth-value in all circumstances or contexts of use. The reason why we should not automatically expect there to be answers to such questions is this: as long as there is a fair measure of agreement in the means of verification (for proposition of a certain type) that users have in common, so that they come in the majority of cases to the same conclusion regarding the truth-value of such a proposition when they apply the means of evaluation, that each has at his disposal, the sentences in question will serve adequately as vehicles of communication. The question whether the methods of evaluation used do themselves constitute the truth-conditions for these sentences need never be settled; there may be a gap in truth-conditions here just as truth-gaps arise in so many areas of language. It is always possible to adopt truth-conditions which transcend the procedures of evaluation that we recognise as tied to the sentence type in its actual use. This is perfectly acceptable as long as these new truth-conditions do not significantly conflict with the procedures in all those cases where the latter yield a definite result; as long, that is, as the imposition of truth-conditions respect the instances where truth-conditions are decidable. Indeed, in many instances, there may be valid theoretical or technical reasons for such impositions. We should, however, be fully aware of what it is that we are

doing. We are reforming old (or even introducing new) vernacular and not merely spelling out what was already there to be discovered.

On the face of it our relations \models , \models admit truth-value gaps for as we have previously stated it may not be the case that either $b \models P$ or $b \models P$ for some particular description b . We may not know enough to decide the matter. Nevertheless, we may reach some point where for a particular P we may be able to choose one way or the other. In other words, for a description b , of sufficient degree of detail, either b entails P or b entails $\neg P$.

Theorem For any model M if $f(P) \neq \emptyset$
then

$$(\exists k)(\forall b \in T_k)(b \models P \rightarrow Q \vee b \models P \rightarrow \neg Q)$$

Proof

Consider the decision tree p with $f(P)$ at the vertex and its ancestors those atoms t of B_n ($n \geq d(Q), d(P)$) such that $t \leq f(P)$. By (P10) we can choose k large enough to "decide" modulo \sqsupset, \sqsubset the branches of p i.e. for each pair of branches of p , x, y say, and for each b in T_k

$$x \sqsupset^b y \vee y \sqsupset^b x \vee x \equiv^b y.$$

But then for each $b \in T_k$ and each $x \in N_p(b)$ either $x \models Q$ or $x \models \neg Q$. Hence $b \models P \rightarrow Q$ or $b \models P \rightarrow \neg Q$.

We shall refer to the k uncovered in this result as the degree of $P \multimap Q$ and write $d(P \multimap Q) = k$.

Despite this result, our theory of counterfactuals may well appear as a square refusal to accept such transcendental truth-conditions. If anything, the very fact that, according to our theory, any counterfactual can apparently be decided within a finite amount of time strengthens this view. This is to be seen in contrast to the semantic analysis offered by Lewis and Stalnaker. This, however, is not the only possible interpretation of our theory.

The truth of a counterfactual is said, by the present theory, to consist in the existence of a certain decision tree. Unlike possible worlds the relevant portion of the tree can be known completely by the language user; and this is, of course, part of the motivation behind the present theory. Despite this fact, the statement that the partition tree has the required property may well transcend what is directly verifiable. The main problem concerns the consistency checks involved in the construction of the decision tree. So far it has been implicitly assumed that these verifications of consistency are instantaneous. A much more realistic approach would require these verifications to be carried out as part of the partition process itself. The important observation here concerns the termination of such a verification process: if DL contains predicate logic then such a process can not in general be completed within a finite number of steps.

At long last we can provide the promised extension of the function f to the whole of $L_{\multimap}(l)$.

Theorem Let M be some Boolean model. Then f can be extended to a function $f : L_{0 \rightarrow}(1) \rightarrow B$ such that for each $b \in T_{d(P \rightarrow Q)}$,
if $f(P) \neq 0$, then

$$(1) \quad b \leq f(P \rightarrow Q) \quad \text{iff} \quad b \models P \rightarrow Q$$

$$(2) \quad b \leq f(P \rightarrow Q)^* \quad \text{iff} \quad b \models P \rightarrow Q$$

(If $f(P) = 0$ put $f(P \rightarrow Q) = 1$)

Proof

Let $k = d(P \rightarrow Q)$. Let b_1, \dots, b_m be the atoms of B_k .

Let b_1, \dots, b_n be the atoms such that $b_i \models P \rightarrow Q$, $1 \leq i \leq n$
and b_{n+1}, \dots, b_m the atoms such that $b_j \not\models P \rightarrow Q$, $n+1 \leq j \leq m$.

Let $f(P \rightarrow Q) = b_1 \cup \dots \cup b_n$. If there are no atoms such
that $b_i \models P \rightarrow Q$ then of course $f(P \rightarrow Q)$ is put equal to 0.

This result can be strengthened a little since any $b \in T_k$ for $k \geq d(P \rightarrow Q)$

satisfies the above conditions. To see this just notice that

$$b \leq b_i \leq f(P \rightarrow Q) \quad \text{iff} \quad b \leq b_i \models P \rightarrow Q \text{ etc.}$$

This result is the key to the treatment of more complex counterfactuals - those which have antecedents and/or consequents which are themselves counterfactuals. Our definition of a model provides a function $f : L \rightarrow B$; we were then able to give a truth theory for sentences in $L_{0 \rightarrow}$ which involves only one level of counterfactual conditionals. Our theorem guarantees the

existence of a function $\hat{f} : L_{\rightarrow}(1) \rightarrow B$. In its turn this enables us to provide truth conditions for sentences in L_{\rightarrow} which involve two levels of counterfactuals - $L_{\rightarrow}(2)$. So far we have said little about the problem of counterfactuals in $L_{\rightarrow}(1)$ which have impossible antecedents (in terms of the model $f(P) = 0$ where P is the antecedent of the conditional). Lewis argues for the truth of such counterfactuals, and to facilitate comparison with his theory we have followed suit by putting $f(P \rightarrow Q) = 1$ for $f(P) = 0$. Proceeding in this way we can define a sequence of functions $\{f_n\}_{n \geq 0}$ where

$$f_0 = f$$

$$f_{n+1} = \hat{f}_n$$

where $f_n : L_{\rightarrow}(n) \rightarrow B$, $L_{\rightarrow}(0) = L$ and $L_{\rightarrow}(n+1)$ is obtained from $L_{\rightarrow}(n)$ as follows. First close $L_{\rightarrow}(n)$ under the truth-functional operators ($\&$, \sim) to obtain $L'_{\rightarrow}(n)$. $L_{\rightarrow}(n+1)$ is the smallest superset of $L'_{\rightarrow}(n)$ containing $P \rightarrow Q$ for $P, Q \in L'_{\rightarrow}(n)$. Let $L_{\rightarrow} = \bigcup_{n=0} L_{\rightarrow}(n)$. We can define

$$f_{\infty} : L_{\rightarrow} \rightarrow B$$

by $f_{\infty} = \bigcup_{n=0} f_n$ where in the obvious sense $f_n \subset f_{n+1}$ (f_{n+1} is an extension of f_n). We shall write f for f_{∞} to avoid cumbersome notation.

Our last result is bound to provoke certain misgivings. It ascertains that each counterfactual sentence P "coincides" with an element b_p of the Boolean frame. But in the case where the frame represents the language DL in

which the possible situations are described this means that P is equivalent with the DL-sentences which correspond to the element b_p .

Have we then not reduced the counterfactual concept to a purely extensional one? Of course, whether this is so or not depends upon the language DL. Our original discussion of the decision process for counterfactuals may well have suggested that we indeed think of DL as purely extensional. But is this a reasonable assumption? In a narrow sense we might be asking whether or not DL contains such apparently intensional devices such as the counterfactual itself or other modal terms or constructions. If so, I believe it to be in keeping with the spirit of the present analysis to assume that DL does contain some such devices, but for the sake of argument we shall not make such an assumption, for there is still another sense in which DL may not be purely extensional. The basic predicates of L may contain certain intensional notions. For among these predicates there might be such predicates as 'soluble' or 'solid' which, according to some at least, are essentially counterfactual in meaning. To say that x is soluble, for example is to assert a battery of counterfactuals such as 'If x were submerged it would dissolve', etc. I find it rather difficult to assess this theory of dispositional predicates. It might be argued for instance that the meaning of such predicates is bound up with the fundamental nature of substance. In which case such counterfactuals as 'If a were submerged it would dissolve' would be derivable from statements such as 'a is soluble'. Let us, however, assume for the moment that DL must contain some such predicates. The result of the theorem might still appear rather surprising. For it claims that all counterfactuals are eliminable except those which are concealed beneath the facades of some of the primitive predicates of the description language.

There are, however, other reasons why this reduction of counterfactuals to complicated extensional constructions is largely illusory. In the first place the reduction depends on a particular choice of the relation $\underline{\supset}$. This relation however may well be said to incorporate much, if not all, that has always been considered intractable in counterfactual constructions. What for example makes a situation in which I strike the match and in which there is just as much oxygen as there is in the actual world nearer to the actual world than a situation in which I strike the match and in which there is insufficient oxygen for ignition? This, you might say, is Goodman's cotenability problem all over again. This is, of course, a criticism which affects not only this analysis but those of Lewis and Stalnaker. Indeed, I do not think such 'circularity' can be avoided in any analysis at such level of generality as the present one. I believe it possible to say something more definite about the relation $\underline{\supset}$ within the narrower context of particular languages in which the counterfactual construction interacts with other than just truth-functional operators. But this is a topic for another occasion.

A second intensional component in our analysis might arise by consideration of the Boolean relation $\underline{\leq}$. So far we have said next to nothing about these relations. When b_1 and b_2 represent (equivalence classes of) sentences of DL then $b_1 \underline{\leq} b_2$ is to mean b_1 entails b_2 . But "entails" in what sense precisely? The only notion of entailment which is both relevant and which we can clearly articulate is DL's notion of logical consequence. However, it is by no means clear that this is the correct notion of "entails". Indeed there seems good reason to include in the extension of $\underline{\leq}$ pairs of the form $\langle b_1, b_2 \rangle$ where b_2 follows from b_1 with the

help of certain natural laws. For surely when we consider the various alternative situations in which a given hypothesis could have been true we surely leave out all those alternatives which are physically impossible, and equally we ignore when contemplating how further details of an already partially described situation could be filled in, those additional specifications of detail which would render the existing specifications physically (though not logically) incoherent. If we opt for such an interpretation of \leq we thereby introduce a further intensional component into our analysis.

We thus find ourselves in the following situation. Once we settle on a particular interpretation of \sqsupset and \sqsupseteq we thereby assent, according to my theory, to an equivalence between counterfactual conditionals and corresponding constructions which contain only extensional operators. This still does not automatically make the counterfactual conditional into an extensional construction. In the first place, it is plausible that the exact extensions of \sqsupset and \leq are not fully determined by the meaning of the counterfactual conditional. If this is indeed so then it will be impossible to pinpoint any particular extensional sentence to which a given counterfactual is equivalent. Secondly, the question for the correct interpretation of \leq and \sqsupset is complicated by its apparently presupposing a previous understanding of certain counterfactuals or cognate constructions such as lawlike generalisations. It is an interesting problem whether something more specific could be said concerning which counterfactuals should be fed into the characterisations of \sqsupset and \sqsupseteq . Given that this impact has been spelled out we would then be faced with the potentially quite interesting problem of whether the analysis of counterfactuals so produced is coherent: is

there a function T from conditional sentences to extensional sentences so that when we determine the relations \sqsubset and \sqsupseteq in terms of the T -counterparts relevant to those relations then the analysis determines as the extensional equivalents of counterfactuals precisely their values under T .

We have here an example of a construction which would be extensional on the assumption that all the components of our theory are fully defined but whose intensionality resides precisely therein that not all the components are fully defined, so that a number of different non-equivalent, extensional sentences may be different paraphrases for the same sentence in which the construction figures.

COMPARATIVE PLAUSIBILITY

AND COMPARATIVE SIMILARITY

As stated at the outset of our discussion we have opted to take the relation of comparative plausibility on partial descriptions of possible-worlds as primitive; the relation of comparative similarity on possible-worlds is then taken as the derived relation. But how exactly are we to define it? It will be helpful at this point to recall that corresponding to every possible-world there is a unique (with respect to a given Boolean frame) approximation sequence which generates it; the successive members of this sequence may be regarded as representing ever more detailed descriptions of the possible-worlds. With this in mind the following definition seems correct

$$u \stackrel{w}{\leq} v \stackrel{\text{def}}{\iff} (\forall k)(\exists m, n \geq k)(u \stackrel{w}{\leq}_m v_n).$$

The content of the definition is reasonably easy to grasp: for every degree of detail chosen we can find partial descriptions of u and v , of at least the chosen degree of detail, which satisfy the original relation of relative plausibility. The usual definition of the strict $\sim(v \stackrel{w}{\leq} u)$ leads to the following equivalence

$$u \stackrel{w}{>} v \iff (\exists k)(\forall m, n \geq k)(u \stackrel{w}{\not\leq}_m v_n).$$

This may be reason for some concern. Under this definition to establish that $u \stackrel{w}{>} v$ it is sufficient to establish that for all finite levels of description u is closer to w than v . But surely it is possible for one world to be closer than a second (to the actual world), at all finite levels of description, and yet not be so in the limit. I believe this objection would be more telling if our notion of comparative similarity was, like that of Lewis, a purely metaphysical one; but ours is clearly not. Our development has been motivated by a desire to come closer to the way we evaluate counterfactuals; and it is exactly this constraint on our theory which demands that decisions concerning comparative similarity be based on finite amounts of information about the worlds in question.

Theorem Let $u, v, z, w \in U(B)$ for some Boolean frame B . The derived relation \leq satisfies:

$$(L1) \quad u \stackrel{w}{\leq} z \quad \& \quad z \stackrel{w}{\leq} v \quad \Rightarrow \quad u \stackrel{w}{\leq} v$$

$$(L2) \quad u \stackrel{w}{\leq} v \quad \vee \quad v \stackrel{w}{\leq} u$$

$$(L3) \quad \neg(u = v) \Rightarrow u \succ v$$

Proof

Assume $u \succ z$ and $z \succ v$. Choose some k . By $u \succ z$ there exists $m, n \geq k$ with $u_n \succeq z_m$. Choose $k' \geq m, n$. By $z \succ v$ there exists $r, m' \geq k'$ such that $z_{m'} \succeq v_r$. Now since $z_{m'} \leq z_m$ ($m' \geq m$) it follows from (B5) that $u_n \succeq z_{m'}$ (as $u_n \succeq z_m$). By the transitivity of \succeq we have $u_n \succeq v_r$ and so by definition $u \succ v$. So \succ is transitive. For (L2) assume $\neg(u \succeq v)$. By definition $(\exists k)(\forall m, n \geq k)(\neg(u_n \succeq v_m))$. By linearity of \succeq we have $(\exists k)(\forall m, n \geq k)(v_m \succeq u_n)$. But then obviously, $v \succeq u$ - for every k' chosen ensure the m, n required are larger than the k for which $v_m \succeq u_n$, for $m, n \geq k$. For (L3) assume $\neg(u = v)$. Let k be large enough so that $u_k \cap v_k = 0$. Since for each $m, n \geq k$, $u_m \cap v_n = 0$ we have by (B3) $u_m \succeq v_n$. Hence $u \succ v$.

These properties are exactly those which are deemed necessary by Lewis for any such relation of comparative similarity. We shall say more about such relations a little later.

We can thus characterise the relation of relative similarity in terms of the relation of relative plausibility but can we characterise the latter in terms of the former? According to our informal discussion we were inclined to assert that a is at least as plausible as b , exactly when we can match any world v , of which b is a correct description, by a world in which a is a correct

description and which is at least as similar as v . The following result guarantees that this intuition is preserved in our theory.

Theorem For each $a, b \in B$

$$a \stackrel{w}{\leq} b \Leftrightarrow (\forall v \exists b)(\exists u \exists a)(u \stackrel{w}{\leq} v)$$

Proof

The cases where $b = 0$ and $a = 0$ is straightforward - use B7 and B8.

So assume $a \neq 0$ and $b \neq 0$. Assume $a \stackrel{w}{\leq} b$.

Let $\exists v \exists b$. Choose $k \geq d(a), d(b)$. By (B5) $a \stackrel{w}{\leq} v_k$ since $v_k \leq b$. By (B6) we can find an atom $u_k \leq a$ of B_k such that $u_k \stackrel{w}{\leq} v_k$. To see this notice that $a = u_1 u \dots u u_n$

for atoms $u_k \in B_k$. Applying (B5) and (B6), once again,

we can find an atom u_{k+1} of B_{k+1} , $u_{k+1} \leq u_k$ such

that $u_{k+1} \stackrel{w}{\leq} v_{k+1}$. Proceeding in this way we can

construct a sequence $u_k \geq u_{k+1} \geq u_{k+2} > \dots$ such that

$u_{k+i} \stackrel{w}{\leq} v_{k+i}$, $i \geq 0$. Let $u \{u_{k+i}, \forall i \geq 0$.

Clearly $u \stackrel{w}{\leq} v$. Conversely assume $\neg(a \stackrel{w}{\leq} b)$. Let

$k \geq d(a), d(b)$. By (B5) there exists $v_k \leq b$, v_k an atom in

B_k , such that $\neg(a \stackrel{w}{\leq} v_k)$. By repeated application of

(B5) we can construct a sequence $v_k \geq v_{k+1} \geq v_{k+2} \geq \dots$

such that v_{k+i} is an atom in B_{k+i} and $\neg(a \stackrel{w}{\leq} v_{k+i})$

for $i \geq 0$. Let $\exists v_{k+i}$, $i \geq 0$. By (B4) we have for each $u \exists a$

and each $m \geq k$, $i \geq 0$, $\neg(u_m \stackrel{w}{\leq} v_{k+i})$. It follows

that for each $u \exists a$ there exists a k such that $(\forall m, n \geq k)(\neg(u_m \stackrel{w}{\leq} v_n))$;

hence for each $u \exists a$, $\neg(u \stackrel{w}{\leq} v)$. This implies that

$$(\exists v \exists b)(\forall u \exists a)(v \succ u).$$

This state of affairs is quite satisfactory. It leads one to believe that the formalization chosen has captured the appropriate intuitions. The relationship elaborated in the theorem is the relationship which guided our intuitions in the development of principle (B8); if you recall (B8) emerged as the outcome of our informal investigation into the relationship between the relations of comparative plausibility and comparative similarity. The formal theory, which is the result of these informal insights, has exactly this relationship as a formal consequence.

We have take the first step towards a comparison between our theory and that based on comparative similarity. We ought now to go further and relate the truth-conditions. To do this we need some notions from the writings of D. Lewis. A non-empty set $U \subseteq U(B)$, is called a w-sphere just in case $u \in U$ and $v \prec u$ implies $v \in U$. Lewis has shown that the totality of w-spheres forms a nested system of sets with the singleton set $\{w\}$ at the centre; the smaller the sphere containing the world the more similar the world is to w .

Theorem For any Boolean model M the following hold:

$$(A1) \quad [P \ \& \ Q] = [P] \cap [Q]$$

$$(A2) \quad [\sim P] = U(B) - [P]$$

$$(A3) \quad [P \Rightarrow Q] = \{w \in U(B) : \text{either } [P] \text{ is empty or} \\ U \cap [P] \subseteq [Q] \text{ for some } w\text{-sphere} \\ \text{with } U \cap [P] \text{ non-empty}\}$$

$$\text{where } [P] = \{w \in U(B) : f(P) \in w\}$$

Proof

Clause (A3) is the only non-trivial case; the rest follow directly from the obvious properties of f : $f(P \ \& \ Q) = f(P) \cap f(Q)$ and $f(\sim P) = f(P)^*$. So we concentrate on (A3). The vacuous case is easy: $f(P) = 0$ just in case $[P]$ is empty. More generally assume $w \in [P \Rightarrow Q]$. By definition $f(P \Rightarrow Q) \in w$ and so there exists an n such that $w_n \subseteq f(P \Rightarrow Q)$. Hence there exists an n (choose $n \geq d(P \Rightarrow Q)$) such that $w_n \models P \Rightarrow Q$. Hence, there exists a $\langle w_n, P \rangle$ -decision tree p such that for each $c \in N_p(w_n)$, $\hat{c} \models Q$. In fact we can ensure that the tree is decorated with atoms T_r where $r \geq d(Q)$; hence $\hat{c} \models Q$ iff $\hat{c} \subseteq f(Q)$ for each branch $c \in N_p(w_n)$. Furthermore (see last section for notation),

$$\overline{N_p(w_n)} \stackrel{w_n}{\supseteq} \overline{B(p) - N_p(w_n)};$$

and so by (B10)

$$\overline{N_p(w_n)} \stackrel{w}{\supseteq} \overline{B(p) - N_p(w_n)}.$$

By the relationship between $\stackrel{w}{\supseteq}$ and $\stackrel{w}{\not\supseteq} \text{---} (a \stackrel{w}{\supseteq} b \Leftrightarrow$

$(\exists u \exists a)(\forall v \exists b)(u \stackrel{w}{\not\supseteq} v))$ there is some $u \in \overline{N_p(w_n)}$ such

that $u \stackrel{w}{\not\supseteq} v$ for each $v \in \overline{B(p) - N_p(w_n)}$. Clearly

$U = \{v \in U(B) : v \stackrel{w}{\supseteq} u\}$ is a w -sphere. Claim

$U \cap [P] \subseteq \{v : \overline{N_p(w_n)} \ni v\}$ - otherwise there is $v \in U \cap [P]$ and

$\neg(v \in \overline{N_p(w_n)})$ and so $v \stackrel{w}{\supseteq} u$ and $u \stackrel{w}{\not\supseteq} v$ which

is a contradiction. Furthermore, for each $c \in \overline{N_p(w_n)}$, $c \leq f(Q)$

and so $\{u : \overline{N_p(w_n)} \ni u\} \subseteq [Q]$. It follows that $U \cap [P] \subseteq [Q]$.

Conversely, suppose $U \cap [P] \subseteq [Q]$ for some w -sphere U . Since

$U \cap [P] \subseteq [Q] \cap [P]$ and $[P] \cap [\sim Q] \subseteq ([P] - U)$ it follows that

$f(P) \cap f(Q) \stackrel{w}{\supseteq} f(P) \cap f(\sim P)$. This follows because U is a

w -sphere and so for some $u \in U$, $u \stackrel{w}{\supseteq} v$ for each $v \in [P] - U$; and

hence the claim follows by the relationship between $\stackrel{w}{\supseteq}$ and

$\stackrel{w}{\not\supseteq}$ and the definition of $[\]$.

Build a P -decision tree p as follows. Put $f(P)$ at the vertex and as

ancestors place all the atoms t in T_m with $t \leq f(P)$ and where

$m \geq d(P), d(Q)$. Choose n large enough so that for each pair $s,$

$t \in B(p)$ either $s \stackrel{w}{\supseteq} t$ or $t \stackrel{w}{\supseteq} s$ or $s \stackrel{w}{\equiv} t$; and (by B10)

large enough so that $f(P) \cap f(Q) \stackrel{w}{\supseteq} f(P) \cap f(\sim Q)$. Now suppose

$t \in \overline{N_p(w_n)}$. Then $t \leq f(P) \cap f(Q)$ for otherwise $t \leq f(P) \cap f(\sim Q)$.

But there is a $u \in f(P) \cap f(Q)$ with $u \ni s$, for some $s \in B(p)$ and $u \stackrel{w}{\supseteq} v$

for each $v \ni t$. This contradicts $t \in \overline{N_p(w_n)}$. But then each

$t \in N_p(w_n)$ satisfies $t \leq f(Q)$ and hence because of the choice of m , $t \models Q$.

The proof of this result is quite interesting in its own right. It displays a correspondence between decision trees and spheres of possible worlds. The more you refine the decision tree the smaller the corresponding sphere. This seems intuitively correct: the more information you have about the alternative worlds the smaller the sphere of possible worlds under scrutiny.

POSSIBLE WORLDS AND COMPARATIVE SIMILARITY

You may find all this rather unconvincing; you may along with Lewis believe that there are possible worlds other than the one we inhabit and moreover that a relation of similarity between such alternative worlds provides the correct analysis of counterfactuals. Rather than enter into this dispute I will instead discuss some relationships between the two theories. In any case, I have said enough about why I believe the Lewis theory to be unsatisfactory.

We shall show that the two notions of validity coincide: for every sentence P of L_{\Box} , P is valid with respect to all Boolean models exactly when it is valid with respect to all Lewis models. To justify this remark we first remind the reader of the basic framework of Lewis. We shall use the formulation based upon his relations of comparative similarity and introduce the notion of

a nested system of spheres as a derived concept. Lewis has taught us that the two formulations are entirely equivalent.

For possible worlds u, v, w we introduce the notation

$$u \stackrel{w}{\leq} v$$

to mean u is at least as similar as v to w . A structure $L = \langle W, \leq \rangle$, where W is a non-empty set (of possible worlds) and \leq is a relation (of comparative similarity) which satisfies (L1), (L2), (L3) we call a LEWIS FRAME.

How exactly are we to recover the structure of a Boolean frame from that of a Lewis frame?

According to our original intuitions we should regard a partial description a as at least as plausible as a partial description b just in case for any possible world v in b there exists a possible world u in a which is at least as similar as v to the actual world. I take it that propositions (or sets of possible worlds) are, in this context at least, to play the role of partial descriptions. The above then suggest the following definition

$$a \stackrel{w}{\geq} b \stackrel{\text{def}}{\iff} (\forall v \in b)(\exists u \in a)(u \stackrel{w}{\leq} v).$$

We shall use a, b, c etc to refer to propositions.

Theorem The relation $\stackrel{w}{\geq}$ so defined satisfies the axioms (B1) - (B7) of a Boolean frame.

Proof

For (B1) assume $a \stackrel{w}{\supseteq} b$ and $b \stackrel{w}{\supseteq} c$. Then for each $z \in c$ we can find $v \in b$ such that $v \stackrel{w}{\supseteq} z$. But we can find $u \in a$ such $u \stackrel{w}{\supseteq} v$ so by (L1) $u \stackrel{w}{\supseteq} z$.

In (B2) assume $\neg(a \stackrel{w}{\supseteq} b)$. By definition and (L2) there exists $u \in b$ such that $u \not\stackrel{w}{\supseteq} v$ for each $v \in a$. It follows that $b \stackrel{w}{\supseteq} a$. For (B3) we have to show that $w \in b$ and $a \wedge b = 0$ implies $b \stackrel{w}{\supseteq} a$ i.e. $(\exists u \in b)(\forall v \in a)(u \stackrel{w}{\supseteq} v)$. We choose $u = w$. Since $w \in b$ and $a \wedge b = 0$ it follows that $\neg(w \in a)$ for each $v \in a$. By (L3) $w \stackrel{w}{\supseteq} v$ for each $v \in a$. (B4) follows immediately since $v \in a$ implies $u \in a \cup a'$ - hence $a \stackrel{w}{\supseteq} b$ implies $a \cup a' \stackrel{w}{\supseteq} b$. (B5) from right to left is also trivial since $v \in b$ implies $v \in b \cup b'$. Conversely, $v \in b \cup b'$ implies $v \in b$ or $v \in b'$ and so (B5), from right to left, follows. Axiom (B7) is trivially true.

The closest Lewis comes to our relation of relative plausibility is the relation he refers to as a relation of "comparative possibility". We shall follow Lewis and write

$$a \stackrel{w}{\supseteq} b$$

to mean that the proposition a is at least as possible, at the world w , as the proposition b . The strict relation is defined in the normal way by

$$a \stackrel{w}{\succ} b \Leftrightarrow \neg(b \stackrel{w}{\supseteq} a)$$

These relations are meant to satisfy the following five conditions.

- (C1) $a \succ_w b \ \& \ b \succ_w c \Rightarrow a \succ_w c$
 (C2) $a \succ_w b \vee b \succ_w a$
 (C3) $(\forall b \in J)(a \succ_w b) \Leftrightarrow a \succ_w \bigcup J$
 (C4) $w \notin b \Leftrightarrow (\forall a)(b \succ_w a)$
 (C5) $(\forall b \in J)(\{u\} \succ_w b) \Rightarrow \{u\} \succ_w \bigcup J$

where J is a set of propositions and $\bigcup J = \bigcup \{b : b \in J\}$.

We know from Lewis that axioms (C1) - (C5) are equivalent to (L1) - (L3).

Notice also that axioms (B1) - (B3), restricted to singleton propositions, are just (L1) - (L3). It follows that, where propositions are to be represented as sets of possible worlds, the three axiom sets (L1) - (L3), (B1) - (B7), (C1) - (C5) are equivalent. Of course, in contexts where propositions are to be thought of as partial descriptions of possible worlds, certain of the axioms in (C1) - (C5) are rather obscure. What, for example, are we to make of (C3) in such contexts?

Despite all these obvious equivalences the relation $\overset{w}{\underset{J}{\succ}}$ do not satisfy all the axioms of a Boolean frame; (B9) and (B10) are not even obviously expressible in such a general context. These axioms crucially rely on propositions as being partial descriptions rather than sets of possible worlds. To obtain a structure which has these additional properties we need to be more devious in our construction. We need to use the notion of a Lewis model rather than just the idea of a Lewis frame.

A Lewis model N consists of a Lewis frame $L = \langle W, \leq \rangle$ together with a function $[] : L_{\rightarrow} \rightarrow P(W)$ (the power set of W) which satisfies (A1), (A2) and (A3).

To obtain a Boolean frame from a Lewis model N we proceed much the same as before. The major change concerns the actual Boolean algebra. Instead of using the full power set of W we select just those propositions which can be expressed in L_{\rightarrow} . More exactly, let $B = \{[P] : P \in L_{\rightarrow}\}$ and the operations of the algebra be the usual set-theoretic ones. Since B is denumerable we know from a previous result that B is constructive with some measure d . The ultrafilters of B do not necessarily exhaust the ultrafilters in the full power set algebra $P(W)$. Furthermore, we have a $\underline{1}^W$ defined only for $w \in W$; we need to extend the definition to $U(B)$ - ultrafilters of B . Presumably, a $\underline{1}^X$ is true just in case a $\underline{1}^W$ holds for each possible world w in X :

$$a \underline{1}^X b \quad \langle = \rangle_{\text{def}} \quad (\forall w \in \hat{X})(a \underline{1}^W c)$$

where $\hat{X} = \bigcap \{[P] : [P] \in X\}$ and $a, b \in B$.

Theorem The structure $\mathbb{B} = \langle B, \underline{1}, d \rangle$, so defined, satisfies the axioms of a Boolean frame.

Proof

(B1), (B4) and (B5) follow from the corresponding properties for $\underline{1}^W$. For the rest we need a couple of preliminary observations. Lewis has shown that there is some sentence 'a $\underline{1} b$ ' of L_{\rightarrow} such that

$$w \in [a \sqsubseteq b] \quad \text{iff} \quad a \sqsubseteq^w b$$

(In fact $a \sqsubseteq b$ is just $(avb \rightarrow avb) \supset (avb \rightarrow a)$. The antecedent of the material conditional is there to allow for impossible antecedents). Next observe that if $\hat{X} \cap [P]$ is non-empty then $\hat{X} \subseteq [P]$ - this much is clear since X is an ultrafilter in B and so X contains $[P]$ or $[\neg P]$ but not both. But then $(\exists w \in \hat{X})(a \sqsubseteq^w b) \Leftrightarrow (\exists w \in \hat{X})(w \in [a \sqsubseteq b]) \Leftrightarrow \hat{X} \cap [a \sqsubseteq b] \text{ non-empty} \Leftrightarrow \hat{X} \subseteq [a \sqsubseteq b] \Leftrightarrow (\forall w \in \hat{X})(a \sqsubseteq^w b)$. The same applies to the strict relation: $(\exists w \in \hat{X})(b \sqsupset^w a) \Leftrightarrow (\forall w \in \hat{X})(b \sqsupset^w a)$ where $b \sqsupset^w a \Leftrightarrow_{\text{def}} \neg(a \sqsubseteq^w b)$.

We return to the audit of our properties. (B2) follows immediately given the above observation. For (B3) suppose $a \wedge b = 0$ in B and $a \in X$. Then presumably $a \wedge b = 0$ in $P(W)$. Moreover, $u \in \hat{X}$ implies $u \in a$ and so $\neg(u = v)$ for each $v \in b$; and so by (B3) for \sqsupset^u we have $a \sqsupset^u b$. But then we have $(\exists u \in \hat{X})(a \sqsupset^u b)$. Which is sufficient to guarantee $a \sqsupset^X b$. Next, we establish (B9) ((B10) is similar). Notice that $a \sqsupset^X b \Leftrightarrow (\forall w \in \hat{X})(a \sqsupset^w b) \Leftrightarrow \hat{X} \subseteq [a \sqsupset b]$. But the latter is equivalent to $[a \sqsupset b] \in X$. Let X_1, X_2, \dots be the uf-sequence of X ; then there exists k such that $X_k \subseteq [a \sqsupset b]$ i.e.

$$[a \sqsupset b] \in X \Leftrightarrow (\exists k)(X_k \subseteq [a \sqsupset b])$$

(The measure is obtained from some enumeration of B). Hence $a \sqsupset^X b \Leftrightarrow (\exists k)(X_k \subseteq [a \sqsupset b])$; which is equivalent to $(\exists k)(\forall w \in X_k)(a \sqsupset^w b)$ ie $(\exists k)(a \sqsupset^{X_k} b)$.

For purposes of identification we refer to this derived Boolean frame as B_N where N is the Lewis model from whence it came. In fact we can obtain a Boolean model from N , $B_N = \langle B_N, f \rangle$ where $f : L \rightarrow B$ is just $f(P) = \{P\}$. Our next result informs us that the two models are equivalent. The proof is almost identical to the corresponding proof in the previous section and so we omit it.

Theorem Let N be a Lewis model and M_N the corresponding Boolean model. Then for each $P \in L_{\text{prop}}$, $\{w : f(P) \in w\} = \{P\}$.

What information does this result give us about the underlying notions of validity? Let Val_B be the set of all sentences P in L such that for each Boolean model $f(P) = 1$; and let Val_L be the set of P such that for each Lewis model $\{P\} = W$. This result, together with that of the previous section, gives us the $\text{Val}_L = \text{Val}_B$; or in words, validity in Lewis models corresponds exactly to validity in Boolean models. The equivalence of the two theories allows one to choose the theory which one finds the philosophically/semantically more acceptable without affecting the underlying concept of validity.

The theory based on Boolean models is clearly motivated by more pragmatic considerations than the account of Lewis. It tries to relate to the way we might actually evaluate counterfactuals whereas Lewis is motivated by more metaphysical considerations.

4. RELATIVE POSSIBILITYANDMODAL LOGIC

In this chapter we develop an approach to modal logic analogous to that developed for counterfactuals. Our theory has much in common with that of Kripke but, as explained in the first chapter, our aim has been to develop a theory of modality which provides an explication of the native speakers' ability to evaluate sentences containing modal operators in given situations of use. As in the case of counterfactuals the primary relation of accessibility will not connect individual possible worlds but rather identifiable portions of such worlds.

We first provide an axiomatic theory of Relative Possibility. Once this much has been achieved we use this notion to provide the truth conditions for a language involving the modal operators of possibility and necessity. This account is based on the informal notion of a decision process for modal sentences sketched in chapter one. The final sections of this chapter relate our theory to that of Kripke.

RELATIVEPOSSIBILITY

Our relation of "comparative possibility" is to operate between partial descriptions of possible worlds. We are interested in the relation between

partial descriptions which asserts that one such description contains sufficient information to conclude that the other is possible. Of course, we may not always be able to assert either that some partial description is possible relative to another or that it is impossible from the perspective of the other; there may not be sufficient information to decide the matter. It is, therefore, not sensible to define the impossibility of one partial description (from the perspective of second) as the negation of its being possible. The two notions cannot be so easily interdefinable. To do justice to this complication we separate our relation of relative possibility into a positive and a negative 'component'.

We shall use the notation

$$aR^+b$$

to indicate that a contains sufficient information to conclude that b is possible and the notation

$$aR^-b$$

to convey the fact that a contains sufficient information to conclude that b is impossible.

Our previous point can now be expressed by saying that a may not contain enough information to deduce either aR^+b or aR^-b . However, partial descriptions of sufficient degree of detail should decide the matter one way or the other. It is this observation which leads to our first principle.

$$(A1) \quad (\forall b)(\exists n)(\forall a \in T_n)(aR^+b \text{ or } aR^-b).$$

The relations R^+ , R^- are meant to reflect our ability to decide on matters of relative possibility. It is with this understanding that we are to interpret the axiom (A1), and under this reading it is surely sound.

Next, consider the notion of entailment in our underlying description language. Presumably, this is sufficient to guarantee relative possibility: if a entails b then b should be deemed possible from the perspective of a . We put matters more formally as follows.

$$(A2) \quad a \neq 0 \ \& \ a \leq b \Rightarrow aR^+b$$

Now let a be some partial description. If either b is possible from the perspective of a or c is possible from the same perspective then, I take it that the join of the information in b and c ($b \sqcup c$) is also possible; on the other hand, presumably $b \sqcup c$ is impossible from the standpoint of a if b and c both are. We summarise these remarks as follows.

For $a \neq 0$:

$$(A3) \quad \begin{cases} (aR^+b \text{ or } aR^+c) \Rightarrow aR^+(b \sqcup c) \\ (aR^-b \ \& \ aR^-c) \Rightarrow aR^-(b \sqcup c) \end{cases}$$

We now examine the stability of our relation under any increases in the information content of the relation's components. The first term of our

relation relates to the environment or background information from which we evaluate the possibility of the second. If we increase the information content of this term then anything previously possible (or impossible) remains so. This justifies principle (A4) from left to right - the fact that b is possible, or impossible, relative to a guarantees that b is possible, or impossible, relative to any description that extends a . On the other hand, if we are able to recognise that b is possible from the perspective of each extension of a (of some specified degree of detail), then this ought to be sufficient for us to recognise that b is possible from the perspective of a itself.

For $a \neq 0$:

$$(A4) \begin{cases} aR^+b \iff \exists k(\forall m \geq k)(\forall d \leq a)(d \in T_m \rightarrow dR^+b) \\ aR^-b \iff \exists k(\forall m \geq k)(\forall d \leq a)(d \in T_m \rightarrow dR^-b). \end{cases}$$

If we combine A1 and A4 we obtain a slightly more useful form of A1 namely:

$$(A1') (\forall b)(b \neq 0)(\exists n)(\forall m \geq n)(\forall a \in T_m)(aR^+b \text{ or } aR^-b).$$

This implies that, once we have decided between aR^+b and aR^-b , one way or the other there, is no going back. In the discussion which follows it is this form of A1 we shall more often refer to.

A second connection exists between R^+ and R^- : if a guarantees the impossibility of b then it is surely false that b is possible given a .

$$(A5) \quad a \neq 0 \ \& \ aR^-b \Rightarrow \sim(aR^+b)$$

We require two more principles to complete our description of relative possibility. Provided that a partial description a is not contradictory then, from the perspective of a , no contradiction is possible.

$$(A6) \quad a \neq 0 \Rightarrow aR^-0$$

The next principle just encodes the idea that a contradictory statement renders everything both possible and impossible. Contradictory statements contain too much information.

For each $b \in B$

$$(A7) \quad \left\{ \begin{array}{l} aR^+b \\ aR^-b \end{array} \right.$$

We summarise this discussion in the form of the concept of a Boolean frame (for modal logic). This consists of a constructive algebra B (with measure d) together with relations of relative possibility and impossibility satisfying A1 through to A7. Although this notion seems far removed from the relation of accessibility between possible worlds introduced by Kripke the two theories will turn out to be equivalent - at least with respect to the language of propositional modal logic.

RELATIVE POSSIBILITY
AND ACCESSIBILITY

We have argued that such relations between partial descriptions should be the primary ones in our analysis of modality. But what is the connection between such relations and the relation of accessibility between possible-worlds? From our relations R^+ and R^- we can define a relation of accessibility between possible-worlds. We achieve this in two stages. Initially, we introduce a relation

$$wRb$$

between possible-worlds and partial descriptions. The above is to convey the information that b is entertainable or possible from the perspective of the world w . The idea behind our definition is simple enough: b is possible from the perspective of w just in case we can recognise this on the basis of some finite information about w . In other words:

$$(M1) \quad wRb \Leftrightarrow_{\text{def}} (\exists n)(w_n R^+ b)$$

If we can decide that wRb then we must do this on the basis of some finite partial description of w , and conversely, if we can find some partial description of w for which the relation holds then this is sufficient to guarantee the relation holds for the possible-world itself (A4 guarantees this).

Since we are no longer plagued by lack of information concerning, the point of

perspective we, can introduce the relation of necessity by definition. Presumably, b is necessary, from the stance of w , exactly when the negation of b is impossible.

$$(M2) \quad wSb \Leftrightarrow_{\text{def}} \sim(wRb^*)$$

Theorem The following properties follow from the axioms A1 - A7.

$$(M3) \quad b \rightarrow w \text{ implies } wRb$$

$$(M4) \quad wR(b \cup c) \text{ iff } wRb \text{ or } wRc$$

$$(M5) \quad wSb \text{ iff } (\exists k)(w_k R^- b^*)$$

$$(M6) \quad wS1$$

Proof

M3. Let $b \rightarrow w$. Then $(\exists n)(w_n \leq b)$. By A2

$$(\exists n)(w_n R^+ b).$$

M4. The direction from right to left follows from A3 and A4.

For the opposite direction assume $wR(b \cup c)$. Assume also $\sim wRb$ and $\sim wRc$; by definition we have $(\forall m)(\sim(w_m R^+ b))$ and

$$(\forall m)(\sim(w_m R^+ c)).$$

By A4 and A1 $(\exists k)(\forall m \geq k)(w_m R^- b$ and

$$w_m R^- c).$$

By A3 $(\exists k)(\forall m \geq k)(w_m R^- b \cup c)$. By A5

$$(\exists k)(\forall m \geq k)(\sim(w_m R^+ b \cup c)).$$

Application of A4 to our original

$$\text{assumption yields a contradiction - for by the assumption}$$

$$(\exists k)(w_k R^+(b \cup c)).$$

M5. Assume wSb ; by definition $(\forall k)\sim(w_k R^+ b^*)$. By

$$A1 (\exists k)(w_k R^- b^*).$$

Conversely, $(\exists k)(w_k R^- b^*)$ implies,

$$\text{by A4 } (\exists k)(\forall m \geq k)(w_m R^- b^*).$$

By A5 and A4 $(\forall k)(\sim(w_k R^+ b^*))$.

M6. This follows from M5 and A6.

Theorem For each a, b in B we have:

$$(1) \quad aR^+b \text{ iff } (\forall u\exists a)(uRb)$$

$$(2) \quad aR^-b \text{ iff } (\forall u\exists a)(uSb^*)$$

Proof

Use A4 and König's lemma together with the definition and M5.

Perhaps we ought to be a little more explicit, at least for (1).

Assume aR^+b . Let $u\exists a$. Then $(\exists k)(u_k \leq a)$. By A4

$(\exists k)(u_k R^+b)$. Conversely, assume $(\forall u\exists a)(uRb)$. Then

$(\forall u\exists a)(\exists k)(u_k R^+b)$. By König's lemma $(\exists k)(\forall u\exists a)(u_k R^+b)$.

A4 yields the result. For $a = 0$ the result follows from A7.

On the other hand, we could take the relation wRb , between possible-worlds and partial descriptions, as sui-generis. Our original relations could then be gleaned from the following definitions.

$$(M7) \quad aR^+b \Leftrightarrow_{\text{def}} (\forall w\exists a)(wRb)$$

$$(M8) \quad aR^-b \Leftrightarrow_{\text{def}} (\forall w\exists a)(wSb^*)$$

The following result informs us that M1 through M8 are sufficient to recover the original postulates.

Theorem The postulates M1, M3, M4, M5 and M6 together with definitions M2, M7 and M8 guarantee the truth of the axiom set A.

Proof

- A1. By definition wRb or wSb^* . Hence by M1 and M5
 $(\forall w)(\exists n)(w_n R^+ b \text{ or } w_n R^- b)$. By König's lemma
 $(\exists n)(\forall w)(w_n R^+ b \text{ or } w_n R^- b)$ since for each
 $a \in T_n$ there is $w, a \in w$ such that $a = w_n$.
- A2. Assume $0 < a \leq b$. Then $w \leq a$ implies $w \leq b$ and so by M3 wRb ; and hence by definition $aR^+ b$.
- A3. Follows immediately from M4 - for the second part notice that M4 can be rephrased as $(wSb \text{ and } wSc) \text{ iff } wSb \cup c$.
- A4. Follows from the definitions M7, M1 and König's lemma. The second half follows from M8 and M5 and König's lemma.
- A5. This follows from application of M8 followed by M2 and finally M7.
- A6. Follows from M8 and M6.
- A7. Follows from M7 and M8 - for $a = 0$ the right hand side is vacuously true.

The upshot of all this is clear enough: we could have taken the relation R as primitive and treat the relations R^+ and R^- as derived. From a technical point of view it makes no difference which we choose as primitive, R or $(R^+$ and $R^-)$, since the two systems are equivalent. It should be clear, however, that our original choice, namely the system based on R^+ and R^- , is to be preferred if we are interested in more practical

considerations relating to our assessment of sentences containing modal operators in actual situations of use.

Indeed, once we have adopted R^+ and R^- , and the relations R and S as defined in terms of them, we are in a much better position to introduce a relation of accessibility between possible-worlds. Intuitively, v should be "accessible" from w just in case we can decide this on the basis of some finite partial description of v . This constraint is clearly in keeping with our overall aim, namely, to give some more pragmatic account of our use of the accessibility relation between possible-worlds. We can put this insight in a more direct way as follows. The world v is to be considered "accessible" from w if and only if we can find some degree of detail such that every description of v , of at least this degree of detail, is deemed to be possible from the perspective of w . This leads to the following definition.

$$wRv \iff_{\text{def}} (\exists k)(\forall m \geq k)(wRv_m)$$

But what is the exact connection between R^+ , R^- and r so defined?

Intuitively, a partial description b , should be possible given a if and only if, no matter which world we choose in a (i.e. of which a is a correct description) (w say), we can match it by a world in b (v say) which is accessible from w ; similarly, b should be impossible from a iff any world accessible from a world w in a is in b^* . Fortunately, these connections are a formal consequence of our theory.

Theorem For a, b in B we have:

- (1) aR^+b iff $(\forall w \exists a)(\exists v \exists b)(wrv)$
- (2) aR^-b iff $(\forall w \exists a)(wrv \text{ implies } v \exists b^*)$.

Proof

We derive 1 and 2 as corollaries to: wRb iff $(\exists v \exists b)(wrv)$. The actual derivation of 1 and 2 then follows, and so we concentrate on the proof of this equivalence. Since $wS1 \Leftrightarrow \neg(wR0)$ we may safely assume that $b \neq 0$. Assume wRb and $b \neq 0$. Choose $m \geq d(b)$.

Let t_1, \dots, t_k be the atoms of T_m such that $b = t_1 \cup \dots \cup t_k$.

By (M4) wRt_i for some $i, 1 \leq i \leq k$. By a second application of (M4) we can find a $t \leq t_i$ in T_{m+1} with wRt . By repeating

this process we can construct a possible world v (with b in v) such that wrv . Conversely, if the right hand side holds, then

$(\exists v \exists b)(\exists k)(\forall m \geq k)(wRv_m)$. By (M4) wRb .

What of the properties of relation r ? Is it reflexive, transitive or what?

Well, as one might expect, much depends upon the attributes of the relations R^+, R^- .

Theorem The relation r is reflexive. Moreover, if R^+ is transitive so is r , and if R^- is symmetric then r is symmetric.

Proof

We first prove r is reflexive. Obviously, $(\forall k)(w_k \in w)$ and so

by M3 $(\forall k)(wRw_k)$ and hence by definition wrw. Next suppose that R^+ is transitive. By definition, wrv and vru implies $(\exists k)(\forall m \geq k)(\exists n)(w_n R^+ v_m)$ and $(\exists k)(\forall m \geq k)(\exists n)(v_n R^+ u_m)$. Choose the maximum of these guaranteed k's. Then by (A4) $(\exists k)(\forall m \geq k)(\exists n)(w_n R^+ v_m \& v_n R^+ u_m)$. Actually, by (A4), we can choose $n \geq$ this k and obtain $(\exists k)(\forall m \geq k)(\exists n)(w_n R^+ v_n \& v_n R^+ u_m)$. Hence, by the assumption $(\exists k)(\forall m \geq k)(\exists n)(w_n R^+ u_m)$.

Assume that R^- is symmetric. By definition wrv implies $(\exists k)(\forall m \geq k)(wRv_m)$. By (M2) this is equivalent to $(\exists k)(\forall m \geq k)(\neg(wSv_m^*))$. By (M5) we obtain $(\exists k)(\forall m \geq k)(\neg(\exists n)(w_n R^- v_m))$. By the symmetric nature of R^- we have $(\exists k)(\forall m \geq k)(\neg(\exists n)(v_m R^- w_n))$. Hence, $(\exists k)(\forall m \geq k)(\forall n)(\neg(v_m R^- w_n))$. This implies $(\forall n)(\exists k)(\forall m \geq k)(\neg(v_m R^- w_n))$. By (A1) $(\forall n)(\exists k)(\forall m \geq k)(v_m R^+ w_n)$ i.e. $(\forall n)(\exists k)(v_k R^+ w_n)$. But this gives wrw.

This result has some consequences . If we restrict our system to the logic based on A1-A6 then we obtain the modal logic T. If we add the transitivity of R^+ we obtain S4 and if we add the symmetry of R^- we obtain S5.

TRUTH CONDITIONS

FOR MODAL LOGIC

Modal propositional calculus (PML) is obtained from the propositional calculus

by the addition of the modal operator M to read 'it is possible that'. Our truth-conditions will be given in terms of a relation between the elements of a modal frame \mathfrak{B} and sentences of the language of the propositional modal calculus. The relation

$$b \models P$$

is to be understood as expressing the fact that b contains sufficient information to justify the assertion of P and

$$b \not\models P$$

is used to indicate that b justifies the refutation or denial of P . The reason that both these relations are required should be obvious: there is no reason to expect that $b \not\models P$ be definable in terms of the negation of $b \models P$ for one may be unable both to assert P and to refute it; there may just be too little information at hand.

A Model for PML will consist of a Boolean Frame $\mathfrak{B} = (B, R^+, R^-)$ together with a function $f : L \rightarrow B$ where L is some denumerable language (PML is obtained from L by closing under M and the truth-functional connectives). This last function just reflects the inclusion of L in the underlying language of the algebra. At this point we are making no further assumptions about the language DL . In particular, we are not assuming that the whole of PML is included in the language of B or indeed that the language of B contains any modal operators.

To begin with we give only the truth-conditions for the propositional part of PML. More precisely, we provide the truth-conditions for L' which is the smallest superset of L closed under the logical operators $\&$, \sim . The reason for this will become clearer as we proceed.

For $b \in B$ and $P \in L'$ we define \models, \models by (simultaneous) induction on P .

$$\begin{array}{l} (1) \quad b \models P \quad \text{iff} \quad b \leq f(P) \\ \quad \quad b \models P \quad \text{iff} \quad b \leq f(P)^* \end{array} \left. \vphantom{\begin{array}{l} (1) \\ \quad \quad \end{array}} \right\} \text{for } P \in L$$

$$(2) \quad b \models \sim P \quad \text{iff} \quad b \not\models P$$

$$b \not\models \sim P \quad \text{iff} \quad b \models P$$

$$(3) \quad b \models P \& Q \quad \text{iff} \quad b \models P \text{ and } b \models Q$$

$$b \not\models P \& Q \quad \text{iff} \quad b \not\models P \text{ or } b \not\models Q$$

We can extend the function f to the whole of L' in the most obvious way:

$$f(P \& Q) = f(P) \wedge f(Q)$$

$$f(\sim P) = f(P)^*.$$

It is our objective to extend the function f to the whole of PML and as a first step we relate f to our truth-conditions.

Theorem Let $P \in L'$. There exists an n such that for each $b \in T_m$, $m \geq n$:

$$(1) \quad b \models P \quad \text{iff} \quad b \leq f(P)$$

$$(2) \quad b \not\models P \quad \text{iff} \quad b \leq f(P)^*,$$

Proof

By induction on P .

For $P \in L$ the result is automatic. Suppose $P = \sim Q$. Then

$$b \models \sim Q \quad \text{iff} \quad b \not\models Q \quad \text{iff} \quad b \leq f(Q)^* \quad \text{iff} \quad b \leq f(\sim Q) \quad \text{and}$$

$$b \not\models \sim Q \quad \text{iff} \quad b \models Q \quad \text{iff} \quad b \leq f(Q) \quad \text{iff} \quad b \leq f(\sim Q)^*.$$

Suppose $P \models Q \ \& \ R$. Then choose the number demanded for P to be any number larger than those demanded for both Q and R . Then

$$b \models Q \ \& \ R \quad \text{iff} \quad b \models Q \quad \text{and} \quad b \models R. \quad \text{by induction this is equivalent to}$$

$$b \leq f(Q) \quad \text{and} \quad b \leq f(R) \quad \text{ie} \quad b \leq f(Q) \wedge f(R). \quad \text{Now suppose } b \not\models Q \ \& \ R.$$

$$\text{This is true exactly when } b \not\models Q \quad \text{or} \quad b \not\models R \quad \text{ie (by induction) } b \leq f(Q)^*$$

$$\text{or } b \leq f(R)^*. \quad \text{Inductively, it is easy to see that the number}$$

$$\text{required for } Q \text{ is at least as big as } d(f(Q)); \text{ and similarly for } R.$$

$$\text{Hence } b \leq f(Q)^* \text{ or } b \leq f(R)^* \text{ is equivalent to } b \leq f(Q)^* \cup f(R)^*.$$

Let $d(P)$ be the number guaranteed by this result. We shall call it the Degree of P .

Theorem For any $P \in L'$:

$$(\exists k)(\forall t \in T_k)(t \models P \vee t \models \neg P)$$

Proof

Follows from the previous theorem given the observation that $d(P) \geq d(\neg P)$.

To complete our truth-conditions for PML we must supply the clause(s) for the modal operator itself. To do this we appeal to our informal procedure for the evaluation of modal sentences. This had the following form. Suppose that the partial description b represents our knowledge about the actual world. We first consider all the possible alternatives to b (at some level of detail) and which themselves may only be partially specified. If one such partial description entails the sentence P then we mark the sentence 'It is possible that P ' true; if they all refute P then we mark the sentence 'It is impossible that P ' true. Chances are, neither of these cases will arise, and so we need to consider more precisely described possible alternatives to P and repeat the process. Of course, at each level of refinement we need to ensure that we have not ignored any possible alternatives and so we must be able to decide that any of the situations not considered is impossible relative to the information already accumulated (cf (i) below). In other words, we must be able to recognise which situations are possible, relative to the current body of information, and also to recognise that these are all the possible ones. The following notion seems to capture our intuitions regarding such a decision process.

A b-DECISION TREE is any finite tree labelled with elements from B (b at the vertex) such that:

- (i) If, at any level in the tree, b has ancestors a_1, \dots, a_k then $bR^+ a_i, 1 \leq i \leq k$ and $bR^-(a_1 \sqcup \dots \sqcup a_k)^*$. Moreover, $a_i \neq 0, 1 \leq i \leq k$.
- (ii) If a (not the vertex) has ancestors a_1, \dots, a_n then $a = a_1 \sqcup \dots \sqcup a_n$ and $a_i \cap a_j = 0, i \neq j$.

We can now introduce the truth-conditions for the modal operator using this notion of a decision tree.

- (4) $b \models MP$ iff there exists a b -decision tree where each leaf decides P and some leaf c satisfies $c \models P$.

- $b \not\models MP$ iff there exists a b -decision tree where each leaf c decides P and satisfies $c \not\models P$.

Our first result establishes the stability of our truth-conditions under increases in the background information.

Theorem For $b, b' \neq 0$ we have

$$(1) \quad b \models MP \ \& \ b' \leq b \Rightarrow b' \models MP$$

$$(2) \quad b \not\models MP \ \& \ b' \leq b \Rightarrow b' \not\models MP$$

Proof

Follows from the definitions given that bR^+c & bR^-c implies (for $b, b' \neq 0$), by A4, $b'R^+c$ & $b'R^-c$ by A4.

We next establish that our decision concerning the truth-value of modal sentences does not depend upon the particular way that we choose to partition the alternative situations. We shall establish that, provided decision trees contain sufficient information, they will agree on the truth value of the sentence in question.

Theorem (a) For any P in PML if $b \models P$ and $b \not\models P$ then $b = 0$.

(b) For each $b \in B$, $b \neq 0$, and P in PML we have

(1) $b \models MP$ iff for each b -decision tree P , for which each leaf c decides P , there exists some leaf c such that $c \models P$.

(2) $b \not\models MP$ iff for each b -decision tree, P for which each leaf c decides P , each leaf c satisfies $c \not\models P$.

Proof

We prove (a) and (b) by simultaneous induction on P . Actually, the only difficult clause for (a) is the case where $P = MQ$ and this follows by (b). We concentrate on the proof of (b).

Suppose P is a b -decision tree such that for each leaf (say a_1 , a_2 for the sake of argument) $a_1 \Vdash P$ and $a_2 \nVdash P$. Let q be a b -decision tree such for some leaf d , $d \Vdash P$. Claim that $d \cap a_1 \neq 0$ or $d \cap a_2 \neq 0$. Since $bR^-(a_1 \sqcup a_2)^*$ we have $(\forall u \exists b)(\forall v)(urv \rightarrow v \exists (a_1 \sqcup a_2))$. But $bR^+d \Leftrightarrow (\forall u \exists b)(\exists v \exists d)(urv)$. So $d \cap (a_1 \sqcup a_2) \neq 0$. Hence $d \cap a_1 \neq 0$ (say). But then $d \cap a_1 \Vdash P \ \& \ \neg P$ - contradiction as this would require $d \cap a_1 = 0$ by the induction assumption of the previous theorem.

Theorem For each P in PML

$$(\exists k)(\forall b \in T_k)(d(b) \geq k)(b \Vdash P \text{ or } b \nVdash P)$$

Proof

By induction on P . We concentrate on the case MP - we have already done the rest. Let t_1, \dots, t_k be the atoms of T_r , where r big enough (by induction) so that for each t_i , $t_i \Vdash P$ or $t_i \nVdash P$. Choose $b \in T_s$ where s large enough so that bR^+t_i or bR^-t_i for each i , $1 \leq i \leq k$. Then consider the tree with b at the vertex and those t_j , ($1 \leq j \leq k$) such that bR^+t_j , as antecedents - suppose t_1, \dots, t_n , $n \leq k$ are the appropriate atoms. By (A3) $bR^-(t_1 \sqcup \dots \sqcup t_n)^*$. Since $t_i \in T_r$ we know $t_i \Vdash P$ or $t_i \nVdash P$. If each t_i on the tree satisfies

$t_i \Rightarrow P$ then $b \Rightarrow MP$; if one satisfies $t_i \models P$ then $b \models MP$.

We shall refer to the (minimum such) k guaranteed by the last result as the degree of P (written $d(P)$).

Theorem Let M be some Boolean Model. Then f can be extended from L' to the whole of PML such that for each $b \in T_m$ with $m \geq d(MP)$:

$$(1) \quad b \models MP \quad \text{iff} \quad b \leq f(P)$$

$$(2) \quad b \Rightarrow MP \quad \text{iff} \quad b \leq f(P)^*$$

Proof

Let b_1, \dots, b_n be the atoms of B_m such that

$b_i \models MP$ where $m = d(MP)$. Put $f(MP) = b_1 \sqcup \dots \sqcup b_n$.

If there are no atoms such that $b_i \models MP$ put $f(MP) = 0$. The result is now clear.

According to the last result we are able to associate with each element of our Modal language an element of the algebra. This is not to say that we have eliminated all intensionality from our analysis. The language underlying our algebra may contain modal operators and so its notion of entailment will not necessarily be a purely extensional one. A second place where some intensionality may reside is in the relations \models and \Rightarrow since these involve the notions R^+ and R^- . In order to determine the exact element of B , that a given element of our language is equivalent to, we involve ourselves in a process of deliberation which appeals to some intensional notions. We should not, therefore, be lulled into a false sense of extensionality. These points

have been made in more detail in our discussion of the counterfactual conditional.

KRIPKE MODELS
AND VALIDITY

In this section we compare our approach to the more traditional one based on possible worlds. Our approach is linked to the Kripke account in as much as the two theories are in agreement. Consequently, every notion of validity yielded by Kripke's approach is matched by an extensionally equivalent notion yielded by ours.

To carry out the comparison we need to introduce the notion of a Kripke Model which consists of a non-empty set of possible worlds W , a relation of "accessibility" on W (which is at least reflexive) and a function $f : L \rightarrow P(W)$ (the power set of W). Let $K = \langle W, r, f \rangle$ be such a model. We extend the function f to a function $[] : PML \rightarrow P(W)$ to produce the truth-conditions of PML.

$$\begin{array}{ll}
 [P \ \& \ Q] = [P] \cap [Q] & k1 \\
 [\sim P] & = W - [P] & k2 \\
 [MP] & = \{w \in W : (\exists u \in W)(wru \ \& \ u \in [P])\} & k3
 \end{array}$$

We shall show that the underlying logics of the two systems are identical. Let Val_K be the set of all sentences of PML true in all possible worlds, throughout all Kripke models; let Val_B be the set of all sentences P such that for all Boolean models $f(P) = 1$. We shall prove

$$\text{Val}_K = \text{Val}_B,$$

that is, the two classes of universally valid sentences coincide.

As a first step we prove that $\text{Val}_K \subseteq \text{Val}_B$. Suppose that $M = \langle B, R^+, R^-, f \rangle$ is a supplied Boolean model. We construct a derived Kripke Model $K_M = (U(B), r, g)$ in the following manner: the set $U(B)$ is the set of ultrafilters of B , r is the derived relation of accessibility and the function $g : L \rightarrow P(U(B))$ is given by $g(P) = \{u \mid f(P)\}$.

Theorem Let M be a Boolean model and K_M the derived Kripke model. Then for each Q in PML

$$w \in [Q] \iff (\exists n)(w_n \Vdash Q)$$

Proof

By induction Q . We consider the case where $Q = MZ$. Then

$$w \in [MZ] \iff (\exists v)(wrv \ \& \ v \in [Z]).$$

Let wrv .

$$(\exists k)(\forall m \geq k)(\exists n)(w_n R^+ v_m) \text{ and } v \in [Z].$$

By induction $v \in [Z]$ implies $(\exists t)(v_t \Vdash Z)$. Choose k larger than $t_d(Z)$.

$$\text{Hence } (\exists k)(\exists n)(w_n R^+ v_k \ \& \ v_k \Vdash Z).$$

So let p be the tree with w_n (n guaranteed above) at the vertex and all atoms t_1, \dots, t_m of $T_{d(v_k)}$ (guaranteed above), such that $w_n R^+ t_i$ as ancestors. Observe that we can choose n as large as we like (by A4); and, in particular, large enough so that $w_n R^+ t_i$ or $w_n R^- t_i$ for each atom t_i in $T_{d(v_k)}$. Since $w_n R^+ v_k$ and $v_k \models Z$, $(\exists n)(w_n \models MZ)$.

Conversely, suppose $(\exists n)(w_n \models MZ)$. Then for such an n there exists a decision tree p with vertex w_n and a leaf c on p such that $c \models Z$. Moreover, $w_n R^+ c$. Hence, by the established relationship between R^+ and r , there exists some $v \in c$ such that $w_n r v$. But $v \in c$ implies $(\exists n)(v_n \leq c)$ and so $v_n \models Z$ which, by induction, implies $v \in [Z]$. Hence there exists $v \in [Z]$ such that $w_n r v$. Hence $w_n \in [MZ]$.

Corollary $\text{Val}_K \subseteq \text{Val}_B$.

This supplies half of our equivalence proof; the other direction requires the construction of a Boolean model from a provided Kripke model $K = \langle w, r, g \rangle$. The derived Boolean model is the structure $M_K = \langle B_M, R^+, R^-, f \rangle$ where $B_M = \text{PML}/\sim$ is the set of equivalence of classes PML induced by the following relation:

$$P \sim Q \iff [P] = [Q].$$

The relations R^+ and R^- are given in the following way:

$$\begin{aligned}
 [P]R^+[Q] & \text{ iff } (\forall w \in [P])(\exists u)(u \in [Q] \text{ and } wru) \\
 \text{and } [P]R^-[Q] & \text{ iff } (\forall w \notin [P])(wru \rightarrow u \in (W - [Q])).
 \end{aligned}$$

The function f is defined as $f(Q) = g(Q)$. We first establish that this structure satisfies the axioms (A1) - (A6).

Theorem The structure $\langle PML/\sim, R^+, R^- \rangle$ satisfies the axioms of a Boolean frame.

Proof

Since PML/\sim is denumerable it is a constructive algebra.

Axiom (A1) follows immediately, since at some point in the enumeration Mx is decided. Axiom (A2) follows since r is reflexive whereas (A3) follows from the definitions of R^+ and R^- .

(A4) is clear since any extension of $[P]$ maintains the relation; and if all extensions satisfy the relations $[P]$ must also. Axiom (A5) is a direct consequence of the definitions; (A6) is vacuously true.

The following result shows that the truth-conditions agree for the two models. Because the proof is identical to the analogous one for Boolean models we omit it.

Theorem Let K be a Kripke model and M_K the derived Boolean model.

Then for each Q in PML we have:

$$w \in [Q] \quad \text{iff} \quad (\exists k)(w_k \models Q).$$

Corollary $\text{Val}_M \subseteq \text{Val}_K$.

We have shown that $\text{Val}_B = \text{Val}_K$. In other words, for PML, the two notions of validity coincide. This is rather satisfactory. It gives some credibility to the claim that we have uncovered the appropriate notions of relative possibility.

We have only considered the case where no other restrictions are imposed upon r other than reflexivity. What about the remaining cases? The following result follows from the definitions of R^+ , R^- in terms of r .

Theorem If r is transitive then so is R^+ and
if r is symmetric then so is R^- .

We should stress once more that our objective has been to give an account of certain aspects of modal notions in actual situations of use. This is exactly what we have tried to reflect in our formulation of the truth-conditions.

5. BOOLEAN ALGEBRAS,
MODALITY, AND
COUNTERFACTUALITY

The literature on modal and intensional logic contains an approach to modality which deploys Boolean algebras with an additional operator. As examples we might cite the so-called neighbourhood semantics of Scott and Lemmon as well as the semantics for modality based on Boolean algebras due originally to McKinsey. In such approaches the semantics of an intensional language is given with respect to a Boolean algebra with an additional operator which corresponds to the intensional construct of the language.

Our approach to the semantics of counterfactuals and modality has made much use of the notion of a Boolean algebra but in a way that is quite different to the above accounts. Instead of an additional operator or function we added certain relations which connect elements of the algebra. What, if any, is the relationship between these two approaches? In this chapter we provide some answers to this question. In the case of the counterfactual conditional there is actually no account (that I know of) based on Boolean algebras with an operator corresponding to the conditional itself. In this chapter we suggest one. The analysis we shall offer arises very naturally from our original account and indeed reflects certain intuitions about counterfactuals which we have so far not discussed.

BOOLEAN ALGEBRASAND MODAL LOGIC

Following Hughes and Cresswell [17] we add to a Boolean algebra $B = \langle B, \cup, \cap, *, 0, 1 \rangle$ a monadic operator $@$ with the following additional postulates:

- (F1) If $b \in B$ then $@b \in B$
- (F2) If $b \in B$ then $b \leq @b$
- (F3) If $b, c \in B$ then $@(b \cup c) = @b \cup @c$
- (F4) $@0 = 0$

We shall in addition assume, throughout this chapter, that B is constructive. We shall call any algebra which satisfies these conditions a Modal algebra.

We have already gone some way towards the introduction of Modal algebras into our theory of modality. In the previous chapter we ascertained that to each sentence of the language of propositional logic there corresponds an element of the algebra. Intuitively, the element which corresponds to a sentence of the form MP is the "minimal" element (modulo \leq) of B which renders P (or rather that element of the algebra which corresponds to P) possible.

We can be more systematic about this. Let B be a Boolean frame for Modal logic. We construct a Modal algebra as follows. The carrier of the algebra is B but

we augment B by the addition of the operator $@$ defined by:

for each $b \in B$

$@b =$ the unique a such that
 aR^+b and a^*R^-b .

We need to check that, for each b in B , such an element exists and is unique.

The existence follows from the axiom which states that for each $b \in B$

$$(\exists n)(\forall a \in T_n)(aR^+b \text{ or } aR^-b).$$

This axiom provides us with the following way of constructing the required

$@b$. Let n be the integer provided for b . Let t_1, \dots, t_k be the elements of T_n ; and let t_1, \dots, t_m be those atoms such that $t_j R^+ b$ for $1 \leq j \leq m$ and t_i be those such that $t_i R^- b$ for $m < i \leq k$.

Put $a = t_1 \cup \dots \cup t_m$.

The property A4 gives the result. We need to be a little careful here.

If there are no atoms t such that tR^+b then we put $a=0$. By A7 aR^+b .

Also, by supposition and A4 $1R^-b$. So things work out here also.

The uniqueness is also clear: if cR^+b and c^*R^-b then

$c \leq a$; for otherwise $c \cap a^* \neq 0$ and so by the property A3

$c \cap a^* R^+ b$ and $c \cap a^* R^- b$ which is a contradiction. Similarly, $a \leq c$ and so $c=a$.

Theorem Let B be a Boolean frame for modal logic. Then if

$\textcircled{}$ is introduced as a monadic operator on B by the definition

$$\textcircled{b} = \text{the unique } a \text{ such that} \\ aR^+b \text{ and } a^*R^-b$$

then $\textcircled{}$ satisfies the axioms of a Modal algebra.

Proof

Axiom F4 follows since $0R^+0$ by A7 and $1R^-0$ by A6.

Hence $\textcircled{0} = 0$. Axiom F2 is a little more delicate.

Notice that bR^+b and so $b \leq \textcircled{b}$ - for otherwise

$$b \wedge (\textcircled{b})^* \neq 0 \text{ and so } b \wedge (\textcircled{b})^* R^+b \text{ and } b \wedge (\textcircled{b})^* R^-b$$

which is a contradiction.

Axiom F3 follows from A3, A4 and A7 as follows : first observe

$\textcircled{d} = 0$ implies $d = 0$ by F2 and so we may safely assume that $\textcircled{b}, \textcircled{c} \neq 0$.

By definition $\textcircled{b}R^+b$ and $\textcircled{c}R^+c$ and so by A3 and

$$A4 \textcircled{b} \wedge \textcircled{c} R^+b \wedge c.$$

Similarly, $(\textcircled{b} \wedge \textcircled{c})^* R^-b \wedge c$. Hence, by uniqueness $\textcircled{(b \wedge c)} =$

$$\textcircled{b} \wedge \textcircled{c}.$$

Axiom F1 is automatic.

Theorem For each a in B ,

$$[1] \quad a \leq \textcircled{b} \quad \text{iff} \quad aR^+b$$

$$[2] \quad a \leq (\textcircled{b})^* \quad \text{iff} \quad aR^-b$$

Proof

The cases where $a = 0$ follow from A7.

The direction from left to right follows from A4 directly.

The other direction follows because if $a \wedge (@b)^* \neq 0$ then

by A4 $a \wedge (@b)^* R^+ b$ and $a \wedge (@b)^* R^- b$ which is a contradiction.

Hence $a \leq @b$. Similarly, [2] can be established.

We have shown how to construct, for each Boolean frame an equivalent modal algebra. We now show how we can construct Boolean frames from such Modal algebras.

Let $B = \langle B, \cup, \cap, *, 0, 1, @ \rangle$ be a Modal algebra. We then define

$$wRb \iff w @ b$$

We need to check that this definition satisfies M3, M4, M5 and M6.

M3 follows from F2 and M4 follows from F3. For M5 we proceed as follows.

By definition $wSb \iff \neg(wRb^*) \iff w @ (b^*)^*$. Hence

$wSb \iff (\exists n)(w_n \leq (@b^*)^*)$ which is equivalent to $(\exists n)(w_n R^- b)$ -

by the definition of R^- in terms of R . M6 follows because $@0=0$ by F4.

In summary we have:

Theorem Let $B = \langle B, \cup, \cap, *, 0, 1, @ \rangle$ be a Modal algebra. Then R defined

by

$$wRb \iff w @ b$$

satisfies the axioms M3, M4, M5 and M6.

Theorem Let $\mathbb{B} = \langle B, \cup, \cap, *, 0, 1, @ \rangle$ be a modal algebra. Let R, R^+, R^- be the derived relations. Then $@b$ is the unique a in B such that aR^+b and $a*R^-b$.

Proof

By definition $aR^+b \Leftrightarrow a \leq @b$ and $aR^-b \Leftrightarrow a \leq (@b)*$.
Hence $@bR^+b$ and $(@b)*R^-b$.

So far we have been successful in relating Boolean frames and Modal algebras but we have said nothing, in the context of Modal algebras, about the truth conditions for PML, the language of propositional modal logic. Let \mathbb{B} be a modal algebra. We first introduce the notion of a valuation function. This is just a function

$$h: \text{PML} \rightarrow B$$

such that

$$h(P \& Q) = h(P) \cap h(Q)$$

$$h(\neg P) = h(P)*$$

$$h(MP) = @h(P).$$

Is there any intuitively clear interpretation for this account of Modality. According to the interpretation offered $@h(P)$ is that element of b which renders $h(P)$ possible; and is the minimal one that does in the sense that any other element which satisfies $aR^+h(P)$ also satisfies $a \leq @h(P)$. In other words, it represents the minimal amount of information necessary for us to

decide that P is possible. To put matters rather differently, $@h(P)$ represents the "join" of the information sufficient to render P possible - or the set of all situations in which P is possible. This does appear to be quite an attractive account and is much in keeping with the spirit of the theory developed in the previous chapter. If you recall the purpose of that theory was to provide some account of how we may decide the truth-value of sentences, which involve the modal notions of possibility and necessity, in given situations of use. The account offered here is complementary to that theory.

In the account of modal logic developed in the previous chapter we introduced a function $f:PML \rightarrow B$ where B was some Boolean frame. Call this function the DERIVED function. We now establish that f and h agree - provided, that is, they agree on the atomic sentences of PML.

Theorem Let $M = \langle B, f \rangle$ be a Boolean model for PML. Let $@$ be the derived monadic operator. We define a valuation function for PML as follows:

$$\begin{aligned} h(P) &= f(P) \text{ for } P \text{ atomic} \\ h(P \& Q) &= h(P) \wedge h(Q) \\ h(\sim P) &= h(P)^* \\ h(MP) &= @h(P). \end{aligned}$$

Then for each P in PML, $f(P) = h(P)$.

Proof

We establish the result by induction on P . The only non-trivial

case is where $P=MQ$. It is sufficient to show that

$$f(MQ) = \text{the unique } b \text{ in } B \text{ such that} \\ bR^+f(Q) \text{ and } bR^-f(Q).$$

The rest will then follow by the induction hypothesis.

First claim that $b \models MQ$ implies $bR^+f(Q)$ and $b \not\models Q$ implies $bR^-f(Q)$. Assume $b \models MQ$. Then there exists a decision tree p with branches which are atoms in T_m where $m \geq d(Q)$ and with some branch t which is an atom which satisfies bR^+t and $t \models Q$.

But by the choice of m , $t \leq f(Q)$. Hence by A3 $bR^+f(Q)$. The

second half of the claim can be established similarly -

observe that $bR^-(t_1 \dots t_k)^*$ for the branches

of p, t_1, \dots, t_k .

Now let t_1, \dots, t_r be the elements of T_n , where n is greater than $d(MQ)$, such that $t_i \models MQ, 1 \leq i \leq r$. From the claim

$t_i R^+f(Q)$, hence by A4 $f(MQ)R^+f(Q)$. Note, in this

regard, that if there are no such atoms such that $tR^+f(Q)$

then the result is automatic by A7 if we put $f(MQ)=0$.

In a similar manner we can show $f(MQ)R^-f(Q)$.

The following result can be established in a similar manner. This time we start with a Modal algebra,

Theorem Let B be a modal algebra and B' the derived Boolean frame. Furthermore, let h be some valuation function for B and let f be its restriction to the atomic elements of PML. If f is

extended to the whole of PML, as the derived function, then
h and f agree on the whole of PML.

COUNTERFACTUALS AND

BOOLEAN ALGEBRAS

We now turn our attention to the counterfactual conditional. As far as I know there is no treatment for counterfactuals based on Boolean algebras with additional operators. In this section we propose one. We shall add to Boolean algebras a binary operator whose interpretation is based on our notion of comparative plausibility. The operator O is to be understood as follows: $O(a,b)$ is to be that element of B which contains just enough information for us to decide that a is at least as plausible as b . We introduce O as a primitive operator with the following postulates:

AXIOMS FOR ' O '

- (E1) $O(a,b) \wedge O(b,c) \leq O(a,c)$
- (E2) $O(a,b) \cup O(b,a) = 1$
- (E3) $c \leq b$ and $a \wedge b = 0 \Rightarrow c \leq O(a,b)^*$
- (E4) $O(a,b) \leq O(a \cup a', b)$
- (E5) $O(a,b) \wedge O(a,b') = O(a, b \cup b')$
- (E6) $a=0$ & $b \neq 0 \Rightarrow O(a,b)=0.$

We shall call an algebra which satisfies E1 - E7 a COUNTERFACTUAL ALGEBRA (CF-Algebra for short).

So far so good. But how exactly do such algebras arise? Our main insight arises from the actual truth conditions for the counterfactual conditional - or rather from a certain consequence of them. In chapter three we ascertained that to each element of L_{\rightarrow} there corresponds an element of the algebra. In other words, we defined a function (what we shall call the DERIVED function) from L_{\rightarrow} into B . It is this observation which provides the clue to the definition of the operator O . The construction of the derived function leads to the following tentative definition: for $a, b \in B$ define

$O(a, b) =$ the unique c such that

$$a \stackrel{c}{\supset} b \quad \& \quad b \stackrel{c^*}{\supset} a$$

We establish the existence and the uniqueness of such a proposed element. First recall that one of our axioms for Boolean frames (for the counterfactual) states:

$$\exists k (\forall t \in T_k) (a \stackrel{t}{\supset} b \vee b \stackrel{t}{\supset} a \vee b \stackrel{t}{\equiv} a).$$

Let $c = t_1 \mu \dots \mu t_m$ where $a \stackrel{t_1}{\supset} b$ and $t_i \in T_k$ for $1 \leq i \leq m$ and where k is the smallest integer guaranteed by the forementioned axiom. Clearly c satisfies the requirement. If there are no t such that $a \stackrel{t}{\supset} b$ then put $c = 0$. Then by P13, $a \stackrel{c}{\supset} b$. By supposition and P6, $b \stackrel{1}{\supset} a$. Hence things work out here also. Uniqueness is also clear and is proven as follows: if $a \stackrel{d}{\supset} b$ and $b \stackrel{d^*}{\supset} a$ then either $d \leq c$ or $d \wedge c \neq 0$. In the latter case $a \stackrel{d \wedge c}{\supset} b$ & $b \stackrel{d \wedge c}{\supset} a$ - contradiction. So $d \leq c$ and similarly $c \leq d$ and therefore $c = d$.

We have thus established the existence of the required element of B ; we now need to prove that the element so defined satisfies the axioms E1-E7.

Theorem The operator '0' so defined satisfies the axioms E1-E7.

Proof

We first establish E1. Let $b=0(a,c)$ and $g=0(c,d)$.

We may assume that $b,g \neq 0$. By definition

$a \frac{b}{\frac{1}{2}} c$ and $c \frac{g}{\frac{1}{2}} d$. Suppose that $e=0(a,d)$.

By the axioms for $\frac{1}{2}$ we have $a \frac{b \wedge g}{\frac{1}{2}} d$. Claim $b \wedge g \leq e$.

Suppose otherwise. Then $x = b \wedge g \wedge e^* \neq 0$, in which case $a \frac{x}{\frac{1}{2}} d$ and $d \frac{x}{\frac{1}{2}} a$ -contradiction. Hence $b \wedge g \leq e$.

Axiom E2 follows immediately from the way that '0' was constructed.

For axiom E3 assume $c \leq b$ and $a \wedge b = 0$. By axiom B3 $b \frac{c}{\frac{1}{2}} a$. Then $c \leq 0(a,b)^*$.

Axiom E4 follows directly from P3: let $e=0(a,b)$ then $a \frac{e}{\frac{1}{2}} b$ and so by P3 $a \wedge a' \frac{e}{\frac{1}{2}} b$. Hence $e \leq 0(a \wedge a', b)$.

Axiom E5 demands a little more work. Let $b=0(a,c)$ and $e=0(a,c')$. (We may assume $b,e \neq 0$ for otherwise, the result follows by P13. Similarly, we can assume $b,e \neq 1$).

Then $a \frac{b}{\frac{1}{2}} c$ & $c \frac{b^*}{\frac{1}{2}} a$ and $a \frac{e}{\frac{1}{2}} c'$ & $c' \frac{e^*}{\frac{1}{2}} a$.

Hence by the properties P2, P6, P4 and P3

$a \frac{b \wedge e}{\frac{1}{2}} c \wedge c'$ and $c \wedge c' \frac{b^* \wedge e^*}{\frac{1}{2}} a$ i.e.

$c \wedge c' \frac{(b \wedge e)^*}{\frac{1}{2}} a$. It follows that $b \wedge e = 0(a, c \wedge c')$.

Axiom E6 follows from P12.

Theorem For each b in B ,

$$[1] \quad b \leq_0(a, c) \Leftrightarrow a \leq_c^b c$$

$$[2] \quad b \leq_0(a, c)^* \Leftrightarrow c \leq_a^b a.$$

Proof

We prove [1] as [2] is similar. If $b \leq_0(a, c)$ then by the property P2, $a \leq_c^b c$. Conversely, if $a \leq_c^b c$ then since $a \leq_c^{0(a, c)} c$ and $c \leq_a^{0(a, c)^*} a$ we have $b \leq_0(a, c)$.

We next show how to recover the structure of a Boolean frame from such a CF-Algebra. To do this we have, of course, to introduce a notion of comparative plausibility. We in fact recover the structure of a Boolean frame with the aid of the following definition:

$$(E7) \quad a \leq_c^w b \Leftrightarrow w \leq_0(a, b).$$

We first establish that axioms E1-E6 (together with definition E7) give us the structure of a Boolean frame.

Theorem If B is a CF algebra and " \leq_c " defined in terms of \leq_0 as in E7 then "the structure with \leq_c but without \leq_0 " is a Boolean Frame.

Proof

Axioms B1, B2, B4, B5, B7 follow directly from E1, E2, E4, E5, and E6 directly. Axiom B3 requires a little more work. Obviously $b \in w$ implies that there exists some n such that $w \leq_n b$. By E3 $\exists n (w \leq_n O(a, b)^*)$ ie $O(a, b)^* \in w$. By definition this gives $b \leq_n^w a$. Axioms B9 and B10 follow because $a \leq_n^w b \Leftrightarrow O(a, b) \in w$ which is equivalent to $\exists n (w \leq_n O(a, b))$.

Theorem

Let B be a CF-algebra and " $\underline{\quad}$ " defined in terms of O as in E7. Then

$O(a, b) =$ the unique c in B such that
 $a \leq_n^c b$ and $b \leq_n^{c^*} a$.

Proof

By definition E7 and the definition of \leq_n^c in terms of \leq_n^w , $a \leq_n^c b \Leftrightarrow c \leq_n O(a, b)$ also, $b \leq_n^{c^*} a \Leftrightarrow (\forall w) (c^* \in w \Rightarrow (a \leq_n^w b)) \Leftrightarrow c^* \leq_n O(a, b)^*$. Hence, $a \leq_n^{O(a, b)} b$ and $b \leq_n^{(O(a, b))^*} a$. Uniqueness is established as follows: suppose $a \leq_n^d b$ and $b \leq_n^{d^*} a$ then by the properties of \leq_n, \geq_n (guaranteed by the way they are defined in terms of \leq_n^w together with the derived properties of \leq_n^w): if $d \wedge O(a, b)^* \neq 0$ then $a \leq_n^{d \wedge O(a, b)^*} b$ and $b \leq_n^{d \wedge O(a, b)^*} a$ - contradiction. Hence, $d \leq_n O(a, b)$ and similarly, $O(a, b) \leq_n d$.

We have thus established that given any CF-Algebra we can construct a

Boolean frame ; and conversely, given any Boolean frame we can construct a CF-algebra. Our objective now is to compare these structures with respect to our language $L_{\square\rightarrow}$. In order to do this we must first define the notion of a VALUATION function from $L_{\square\rightarrow}$ into a CF-Algebra.

A VALUATION is any function $h : L_{\square\rightarrow} \rightarrow B$ which satisfies:

$$h(A \& B) = h(A) \wedge h(B)$$

$$h(\sim A) = h(A)^*$$

$$h(A \square\rightarrow B) = \begin{cases} 1 & \text{if } h(A) = 0_B \\ 0(h(A \& \sim B), h(A \& B))^* & \text{otherwise} \end{cases}$$

So in particular, h assigns to the counterfactual $A \square\rightarrow B$ an element of the algebra. But which element exactly? Generally, whenever a statement $A \square\rightarrow B$ is uttered the speaker has in mind a certain set of statements which entail the material conditional 'if A then B '. One interpretation of the valuation function h is that it assigns to each such conditional the element of the algebra which corresponds to this set of statements. According to our original intuitions h assigns to $A \square\rightarrow B$ the element of B which contains just enough information to judge that $h(A \& B)$ is strictly more plausible than $h(A \& \sim B)$. This is quite an interesting interpretation and seems rather close to those theories of counterfactual conditionals which Lewis refers to as Metalinguistic theories; such a theory attempts to spell out those parts

of the antecedent and the consequent that were originally left implicit.

This leads us to a second aspect of this account which is rather interesting. The element of **B** corresponding to $A \rightarrow B$ depends on both the antecedent **A** and consequent **B**. The following examples provide good evidence for this.

- (1) If I were the Pope, I would have allowed the use
of the pill in India
- (2) If I were the Pope, I would have dressed
more humbly.

Presumably, in the first statement, we must assume that India remains poor in resources and greatly overpopulated. In the second we need to assume nothing of the sort.

A second pair of examples are rather well known:

- (3) If New York were in Georgia, then New York would
be in the South
- (4) If New York were in Georgia, then Georgia
would be in the North.

Clearly, in (3) "Georgia is in the South" must remain true and in (4) "New York is in the North" must retain its truth-value.

The idea that, in order to evaluate a counterfactual conditional, we consider

that set of sentences which are cotenable with the antecedent and the consequent and which together with the latter entail (in some appropriate sense) the former is, of course, due to Goodman. It is exactly this intuition which lies at the heart of the account given here; the element of the Boolean algebra selected contains just enough information to decide the truth of the corresponding material conditional while being cotenable with the antecedent and consequent. This seems quite satisfactory. In fact, as we shall now see, this approach to counterfactual truth is implicit in our original approach.

Theorem Let \mathbf{B} be a counterfactual algebra. Let h be a valuation. Let \mathbf{B}' be the derived Boolean frame. Let f be the restriction of h to the atomic sentences of $L_{\mathbf{B}'}$. Then the DERIVED function f and the valuation h agree on the whole of $L_{\mathbf{B}'}$.

Proof

By induction on A . We concentrate on the case $A = P \Rightarrow Q$. First recall that $O(a, b)$ is the unique $c \in \mathbf{B}$ such that

$$a \stackrel{c}{\supset} b \text{ and } b \stackrel{c^*}{\supset} a.$$

Next claim for $f(P) \neq 0$ that $f(P \Rightarrow Q)$ is the unique b such that

$$\begin{aligned} f(P \& Q) \stackrel{b}{\supset} f(P \& \neg Q) \text{ and} \\ f(P \& \neg Q) \stackrel{b^*}{\supset} f(P \& Q) \end{aligned}$$

Let $f(P \leftrightarrow Q) = t_1 \dot{\vee} \dots \dot{\vee} t_k, t_i \in T_m$, where \uparrow is larger than the degree of $P \leftrightarrow Q$ for $1 \leq i \leq k$. We distinguish the case where $f(P \leftrightarrow Q) \neq 0$. For $t_i \in T_r$ with $r \geq d(P \leftrightarrow Q)$, $t_i \leq f(P \leftrightarrow Q)$ iff $t_i \models P \leftrightarrow Q$ which implies by the lemma, $f(P \& Q) \stackrel{f}{\supseteq} f(P \& \sim Q)$. So by P6 $f(P \& Q) \stackrel{f(P \leftrightarrow Q)}{\supseteq} f(P \& \sim Q)$. If $f(P \leftrightarrow Q) = 0$ then by P13 $f(P \& Q) \stackrel{f(P \leftrightarrow Q)}{\supseteq} f(P \& \sim Q)$. In a similar manner, using part [2] of the lemma we can prove that $f(P \& \sim Q) \stackrel{f(P \leftrightarrow Q)^*}{\supseteq} f(P \& Q)$ - this time one has to distinguish the case where $f(P \leftrightarrow Q)^* = 0$ and appeal to P13. Hence, $f(P \leftrightarrow Q) = 0(f(P \& \sim Q), f(P \& Q))^*$. The induction hypothesis now yields the result.

Lemma For $b \neq 0$ we have:

[1] If $b \models P \leftrightarrow Q$ then $f(P \& Q) \stackrel{b}{\supseteq} f(P \& \sim Q)$

[2] If $b \not\models P \leftrightarrow Q$ then $f(P \& \sim Q) \stackrel{b}{\supseteq} f(P \& Q)$.

Proof

We establish [1]; [2] is entirely similar.

Let $b \models P \leftrightarrow Q$ and suppose p is a decision tree which guarantees this: assume p is decorated with elements of T_m where m is at least as big as the degree of $P \& Q$.

Then

$$\overline{N_p(b)} \stackrel{b}{\supseteq} \overline{B(p) - N_p(b)}$$

Moreover, each $t \in N_p(b)$ satisfies $t \models P \& Q$.

Since we can ensure $d(t) \geq d(P \& Q)$ we know $t \leq f(P \& Q)$.

Hence, $\overline{N_p(b)} \leq f(P \& Q)$. It follows from the property

P7 that $f(P \& Q) \stackrel{b}{\leq} \overline{B(p) - N_p(b)}$. On the other hand,
 if $t \models P \& -Q$ then $t \in B(p) - N_p(b)$. Moreover,
 $t \models P \& -Q$ iff $t \leq f(P \& -Q)$ and so $f(P \& -Q) \stackrel{b}{\leq} \overline{B(p) - N_p(b)}$.
 Hence, by the property P4, $f(P \& Q) \stackrel{b}{\leq} f(P \& -Q)$.

We have thus established one half of the equivalence, but what if we start with Boolean Frames rather than CF-Algebras? The following result has an almost identical proof and establishes the other half of the equivalence.

Theorem Let $M = \langle B, f \rangle$ be a Boolean model for $L_{\Box, \neg}$. Let B' be the derived CF-Algebra and h a valuation function for $L_{\Box, \neg}$ (with respect to B') which agrees with f on the atomic sentences of $L_{\Box, \neg}$. Then h agrees with f (the derived function) on the whole of $L_{\Box, \neg}$.

I believe that the intuitions which led to our original approach to counterfactuality are enhanced by the account developed in this chapter: the interpretation of $O(a, b)$, and the consequent interpretation of the counterfactual conditional, seem to get at the intuitions which lie at the heart of the Goodman account.

In the previous chapters we were much concerned with the question of whether or not we had reduced certain aspects of intensional logic to purely extensional notions. As you will recall, the concern arose because we were able to associate with each sentence of the intensional language under study a corresponding element of the Boolean algebra. This observation, in conjunction with the fact that our algebra could be seen as arising from some underlying description language DL, was the cause for the concern. On the face of it then, the constructions we have carried out in this chapter should be cause for some mild neurosis. How were we able to placate matters previously? The anxiety was a consequence of the assumption that the language DL was itself a purely extensional language. But as we have already pointed out, both in the case of the counterfactual conditional and the modal operator of possibility, this is not perhaps the most plausible view. It is in keeping with the spirit of the present study to allow DL to make reference to certain intensional notions - even perhaps to constructions not themselves available in the object language.

In previous chapters we developed theories of counterfactuality and other modalities which were an attempt to give an account of how speakers might decide, in other than the completely trivial cases, as to whether a particular counterfactual or modal sentence was true or false. In this chapter we have offered a supplement to this theory: a supplement which adds to the accounts in terms of decision trees or processes an extension which isolates those aspects of the theory which result from the decision processes. In the case of the counterfactual conditional the decision trees yield a set of conditions or situations which render the conditional

true or false. The theory developed here concentrates on the "minimal" set of such conditions or situations which are sufficient to guarantee the truth of the conditional.

APPENDIX
AXIOMS SYSTEMS

THE AXIOM SYSTEM B

$$(B1) \quad a \underline{\underline{w}} b \quad \& \quad b \underline{\underline{w}} c \quad \Rightarrow \quad a \underline{\underline{w}} c$$

$$(B2) \quad a \underline{\underline{w}} b \quad \vee \quad b \underline{\underline{w}} a$$

$$(B3) \quad (b \in w \quad \& \quad a \cap b = 0) \quad \Rightarrow \quad b \underline{\underline{w}} a$$

$$\text{where } b \underline{\underline{w}} a \quad \Leftrightarrow_{\text{def}} \quad \sim(a \underline{\underline{w}} b).$$

$$(B4) \quad a \underline{\underline{w}} b \quad \Rightarrow \quad a \cup a' \underline{\underline{w}} b$$

$$(B5) \quad a \underline{\underline{w}} b \quad \& \quad a \underline{\underline{w}} b' \quad \Leftrightarrow \quad a \underline{\underline{w}} b \cup b'$$

$$(B6) \quad a \cup a' \underline{\underline{w}} b \quad \Rightarrow \quad a \underline{\underline{w}} b \quad \vee \quad a' \underline{\underline{w}} b$$

$$(B7) \quad a \underline{\underline{w}} b \quad \Rightarrow \quad b = 0 \quad \vee \quad \sim(a = 0)$$

$$(B8) \quad a \underline{\underline{w}} b$$

$$\Leftrightarrow$$

$$\{(b = 0) \vee$$

$$(a \neq 0 \quad \& \quad (\exists k)(\forall m \geq k)(\forall b' \leq b, b' \in T_m)(\exists a' \leq a, a' \in T_m)(a' \underline{\underline{w}} b'))\}$$

$$(B9) \quad a \underline{\underline{w}} b \quad \Leftrightarrow \quad (\exists n)(a \underline{\underline{w}}^n b).$$

where

$$a \underline{\underline{w}}^n b \quad \Leftrightarrow_{\text{def}} \quad (\forall w \ni b)(a \underline{\underline{w}}^n c)$$

$$(B10) \quad a \underline{\underline{w}} c \quad \Leftrightarrow \quad (\exists n)(a \underline{\underline{w}}^n c)$$

where

$$a \underline{\underline{w}}^n c \quad \Leftrightarrow_{\text{def}} \quad (\forall w \ni b)(a \underline{\underline{w}}^n c)$$

THE AXIOM SYSTEM P

For $b \neq 0_B$

$$(P1) \quad a \underline{\underline{b}} c \ \& \ c \underline{\underline{b}} d \quad \Rightarrow \quad a \underline{\underline{b}} d$$

$$(P2) \quad a \underline{\underline{b}} c \ \& \ a \underline{\underline{b'}} c \quad \Leftrightarrow \quad a \underline{\underline{b \cup b'}} c$$

$$(P3) \quad a \underline{\underline{b}} c \quad \Rightarrow \quad a \cup a' \underline{\underline{b}} c$$

$$(P4) \quad a \underline{\underline{b}} c \ \& \ a \underline{\underline{b}} c' \quad \Leftrightarrow \quad a \underline{\underline{b}} c \cup c'$$

$$(P5) \quad a \underline{\underline{b}} c \ \& \ c \underline{\underline{b}} d \quad \Rightarrow \quad a \underline{\underline{b}} d$$

$$(P6) \quad a \underline{\underline{b}} c \ \& \ a \underline{\underline{b'}} c \quad \Leftrightarrow \quad a \underline{\underline{b \cup b'}} c$$

$$(P7) \quad a \underline{\underline{b}} c \quad \Rightarrow \quad a \cup a' \underline{\underline{b}} c$$

$$(P8) \quad a \underline{\underline{b}} c \ \& \ a \underline{\underline{b}} c' \quad \Leftrightarrow \quad a \underline{\underline{b}} c \cup c'$$

$$(P9) \quad a \cap b = 0 \quad \Rightarrow \quad a \underline{\underline{a}} b$$

$$(P10) \quad (\exists k)(\forall b \in T_k)(a \underline{\underline{b}} c \vee c \underline{\underline{b}} a \vee a \underline{\underline{b}} c)$$

$$(P11) \quad a \underline{\underline{b}} c \Rightarrow a \underline{\underline{b}} c \ \& \ \neg(c \underline{\underline{b}} a)$$

$$(P12) \quad a \underline{\underline{b}} c \Rightarrow c = 0 \vee \neg(a = 0)$$

$$(P13) \quad a \underline{\underline{0}} b \ \& \ a \underline{\underline{0}} b$$

where in (P10) $a \underline{\underline{b}} c \Leftrightarrow_{\text{def}} (a \underline{\underline{b}} c \ \& \ c \underline{\underline{b}} a)$

THE AXIOM SYSTEM A

For $a \neq 0$ in B

$$(A1) \quad (\forall b) \quad (\exists n)(\forall a \in T_n)(aR^+b \text{ or } aR^-b)$$

$$(A2) \quad a \leq b \text{ implies } aR^+b$$

$$(A3) \quad \begin{cases} aR^+b \text{ or } aR^+c \text{ implies } aR^+b \sqcup c \\ aR^-b \ \& \ aR^-c \text{ implies } aR^-b \sqcup c \end{cases}$$

$$(A4) \quad \begin{cases} aR^+b \text{ iff } (\exists k)(\forall m \geq k)(\forall d \leq a)(d \in T_m \rightarrow dR^+b) \\ aR^-b \text{ iff } (\exists k)(\forall m \geq k)(\forall d \leq a)(d \in T_m \rightarrow dR^-b) \end{cases}$$

$$(A5) \quad aR^-b \text{ implies } \sim(aR^+b)$$

$$(A6) \quad a \neq 0 \text{ implies } aR^-0$$

$$(A7) \quad 0R^+b \text{ and } 0R^-b$$

THE AXIOM SYSTEM M

$$(M1) \quad wRb \Leftrightarrow (\exists n)(w_n R^+ b)$$

$$(M2) \quad wSb \Leftrightarrow \sim(wRb^*)$$

$$(M3) \quad b \ni w \text{ implies } wRb$$

$$(M4) \quad wR(b \cup c) \text{ iff } wRb \text{ or } wRc$$

$$(M5) \quad wSb \text{ iff } (\exists k)(w_k R^- b^*)$$

$$(M6) \quad wS1$$

where

$$(M7) \quad aR^+ b \Leftrightarrow_{\text{def}} (\forall w \ni a)(wRb)$$

$$(M8) \quad aR^- b \Leftrightarrow_{\text{def}} (\forall w \ni a)(wSb^*)$$

AXIOM SYSTEM F

- (F1) If $b \in B$ then $\exists b \in B$
 (F2) If $b \in B$ then $b \leq \exists b$
 (F3) If $b, c \in B$ then $\exists(b \cup c) = \exists b \cup \exists c$
 (F4) $\exists 0 = 0$

AXIOM SYSTEM E

- (E1) $0(a, b) \wedge 0(b, c) \leq 0(a, c)$
 (E2) $0(a, b) \cup 0(b, a) = 1$
 (E3) $c \leq b$ and $a \wedge b = 0 \Rightarrow c \leq 0(a, b)^*$
 (E4) $0(a, b) \leq 0(a \cup a', b)$
 (E5) $0(a, b) \wedge 0(a, b') = 0(a, b \cup b')$
 (E6) $a = 0 \& b \neq 0 \Rightarrow 0(a, b) = 0.$

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