INVOLUTIONS ON COMPACT 3-MANIFOLDS

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ABSTRACT

Let G be a finite group and X a closed fixed-point free G-manifold of odd dimension, that is G acts on X preserving the orientation. We have associated to (G,X) for each $g \in G$, $g \neq 1$, an invariant (α), as follows.

According to the free cobordism theory of Conner and Floyd (4), the disjoint union kX of k copies bounds a free G-manifold Y, for some k. α is defined by

$$\alpha(g,X) = 1/k \ \text{Sign}\ (g,Y)\ ,\ g \neq 1.$$

When G has order two, $G = \{1, T : T^2 = 1\}$, we have a fixed-point free involution $T : X \longrightarrow X$ and it turns out that α coincides with the Browder-Livesay invariant (β) of (T,X).

In this thesis we develop the proof by F.Hirzebruch and K.Jänich that $\alpha = \beta$, when $H_{2m+2}(X,Q) = 0$, where $\dim X = 4m + 3$.

We also compute the Browder-Livesay invariant of involutions derived from free actions of the generalized quaternion groups

$$Q_{4t} = \{x,y : x^{2t} = 1\ ,\ x^t = y^2,\ y^{-1}xy = x^{-1}\}$$

on the spheres S^{4m-1}.

Furthermore, Lopez de Medrano constructs involutions on homology 3-spheres, as follows.

<u>Theorem</u> (Medrano). For every $i \in \mathbb{Z}$, there is a fixed-point free involution (T, Σ^3) of a homology 3-sphere Σ^3 such that $\beta(T, \Sigma^3) = 8i$.

We work with the examples above and prove the following theorem.

<u>Theorem</u> . If $\beta(T,\Sigma^3)/8$ is odd, where (T,Σ^3) is one of Medrano's examples, then Σ^3 cannot be h-cobordant to S^3. Also, Σ^3 does not imbed in R^4 .

For this, we compute first the signature of a suitable 4-manifold with Σ^3 as boundary, and compute the μ-invariant of Σ^3 .

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INTRODUCTION

Several authors have been concerned with involutions on odd dimensional compact manifolds.

Browder and Livesay have defined an invariant, β, for fixed point free involutions on $(4m - 1)$ - homotopy spheres $(m \geq 1)$ and Lopez de Medrano[13] has used this invariant for their classification. If (T, Σ^3) denotes such an involution, there is an invariant $(2m - 2)$ - connected submanifold W of Σ given by $W = A \cap T A$, such that $\Sigma = A \cup T A$, where A is a compact submanifold of Σ with boundary $\partial A = W$. Then, β is defined as the signature of the form

$$f(x,y) = x \circ T_* y,$$

on

$$\text{Ker } \{H_{2m-1}(W) \longrightarrow H_{2m-1}(A)\}$$

(see I.1.3).

A manifold X with a fixed point free involution T can be considered as a free G-manifold with $G = \{1, T : T^2 = 1\}$.

C. T. Wall and others have used an invariant, α, for the classification of free G-manifolds. The invariant α of (T,X) is given by the number

$$\frac{1}{k} \text{ sign } (T,X) ,$$

where Y is as free G-manifold with boundary $\partial Y = k X$ and sign (T,Y) is the equivariant signature of Atiyah and Singer (1).

The invariants β, on the other hand, are also defined for a class larger than that of homotopy spheres and it turned out that $\alpha = \beta$, in the case of involutions.

Part of this dissertation deals with the exposition of the proof of $\alpha = \beta$, due to Hirzebruch-Janich, and which we use a tool for a subsequent result.

Lopez de Medrano, in his above mentioned work, gives examples of involutions on homology 3-spheres with values of β being of the form 8i, for any integer number i. The examples with $i \neq 0$ cannot be homeomorphic to the standard sphere, as any involution of S^3 is equivalent to the antipodal map (12). If it happens that $\pi_1(\Sigma^3)=0$, where Σ^3 is one of those examples, a counter-example to the Poincaré conjecture in dimension 3 will have been found. This is unlikely to be the case. Using the result $\alpha = \beta$ and the μ - invariant for homology spheres defined by Hirzebruch in (9) , we prove the following theorem.

<u>Theorem</u> : Let (T, Σ^3_i) be a Medrano's example with $\beta(T, \Sigma^3_i) = 8i (i \in \mathbb{Z})$. Then, if i is odd, we have:

(i) Σ^3_i is not h - cobordant to S^3 ,

(ii) Σ^3_i does not imbed in R^4 .

To prove this we compute first the signature of a suitable 4-manifold with Σ^3 as boundary and we compute the μ-invariant of Σ^3. More precisely, we prove first the following proposition:

<u>Proposition</u> . Let (T, Σ^3_i) be as in the theorem. Then Σ^3_i bounds a π - manifold M_i such that sign $(M_i) \equiv \beta(T, \Sigma^3_i)$ (mod.16). Furthermore, the μ - invariants of Σ^3_i are

$$\mu(\Sigma^3_i) = 1/2 \text{ , for i odd,}$$
$$= 0 \text{ , for i even.}$$

Another problem of interest is the computation of the invariants themselves. Hirzebruch has calculated the Browder-Livesay invariants of involutions on lens speces (7). Following his procedure, we compute the β - invariants of involutions T derived from free actions of the generalized quaternion groups Q_{4t}, of order 4t, on the spheres S^{4m-1}.

This exposition is divided into two chapters.

In chapter I we define the invariants α and β , as well as other invariants needed in the sequel. We give the first part of the proof of α = β . We end this chapter with the computation of the β - invariants of $(T,[S^{4m-1}, Q_{4t}])$.

Chapter II starts with the construction of Lopez de Medrano's examples. We define the μ - invariants of homology spheres. Next we end the proof of α = β , and finally we prove our proposition and theorem above.

Chapter I

INVOLUTIONS ON SPACES OF CONSTANT CURVATURE

I.0 Notation.

Throughout this chapter, an m-manifold Y is a compact oriented differentiable manifold without boundary, of dimension m. When we want to consider manifolds with boundary, we specify it. In this case we denote ∂X for the boundary of X. $-X$ denotes the manifold X with orientation reversed. If $\partial X_1 = Y$ and $\partial X_2 = -Y$, we write $X_1 \cup_Y X_2$ for the manifold obtained by pasting X_1 and X_2 along the boundary and smoothing if necessary.The technique for this can be found in Milnor (17,19)

A symmetric bilinear form over $\mathbb{Z}$ is a symmetric bilinear map $\varphi : V \times V \longrightarrow \mathbb{Z}$, defined on a finitely generated free $\mathbb{Z}$ - module V. We say that φ is non-degenerate when it satisfies the condition

$$\varphi(x,y) = 0 \text{ for all } y \in V \Longrightarrow x = 0.$$

Following the notation of Hirzebruch (7) we can consider φ as a form over the real numbers and by sign(φ) we mean the difference between the number of positive diagonal entries and that of the negative ones in the matrix of φ (after being diagonalized in its decomposition in to unary forms).

If Y is a 4n- manifold,the cup-product defines a non-degenerate symmetric bilinear form on $H^{2n}(Y ; \mathbb{R})$ over $\mathbb{R}$, that is, a form

$$\varphi : H^{2n}(Y;\mathbb{R}) \otimes H^{2n}(Y;\mathbb{R}) \to \mathbb{R} ,$$

given by

$$\varphi(x,y) = (x \cup y)\,[Y], \quad x, y \in H^{2n}(Y\,;\mathbb{R}),$$

where [Y] denotes the fundamental class of Y. The signature of this form is then called the signature (or index) of Y, and denoted by sign (Y).

Most of the invariants next defined are known and involve a series of extensive works. For this reason, some related theorems and propositions in this chapter are announced without proof. Our intention is to make use of them either for a subsequent theoretic exposition or for application in the calculation of invariants.

I.1 Definition of invariants.

I.1.1 G- Signatures.

Let Y be a manifold (possibly with boundary), and suppose that a compact Lie group G acts differentiably on Y preserving the orientation. We call Y a G-manifold.

If Y is a 4n-manifold and a G-manifold, we can associate to it the element sign (G,Y) of the representation ring R(G)of G. This is done as follows. Let $\varphi(x,y)$ denote the symmetric bilinear form on $H^{2n}(Y;\mathbb{R})$ given by cup-product, that is

$$\varphi(x,y) = (x \cup y)\,[Y], \quad x,y \in H^{2n}(Y;\mathbb{R})$$

This form is G-invariant. If we choose any definite inner-product $\langle\ ,\ \rangle$ on H^{2n}, invariant under G, then the operator A defined by

$$\varphi(x,y) = \langle x, A\,y \rangle$$

commutes with the action of G. Notice that A is self-adjoint, so that we get a decomposition $H^{2n} = H^{2n}_{+} \oplus H^{2n}_{-}$, invariant under G, given by the positive and negative eigenspaces of A. We have two real representations of G, ρ^+ and ρ^-, say. The G-signature is now defined as

$$\text{sign}\ (G,Y) = \rho^+ - \rho^- \in R\,O\,(G) \subset R(G),$$

where R O(G) denotes the real representation ring of G. This is independent of the choice of inner-product, as discussed in (1).

Evaluating the character of sign (G,Y) on $g \in G$, we obtain a real number sign(g,Y). This number is determined by the action of g on the real cohomology of Y .

If now Y is a G-manifold with boundary $\partial Y = X$, then sign(g,Y) is defined by the action of G on $\hat{H}^n(Y;\mathbb{R})$ (the image of $H^n(Y,X;\mathbb{R}) \longrightarrow H^n(Y;\mathbb{R})$).

Atiyah and others, in their series of papers on the index of elliptic operators ( ) , defined sign (G,Y) for a 2n-dimensional G - manifold (as a generalization of sign (Y), when Y is 4n-dimensional), and proved the following theorem.

G-signature theorem . sign (g,Y) = L(g,Y), for a 2n-manifold Y(without boundary).
Here L(g,Y) is a number involving the evaluation of certain characteristic classes on the fixed point set F ix(g,Y) of g and it depends only on the action of g in the neighbourhood of Fix(g,Y),

Remarks . a) sign (g,Z) = 0 ($g \in G$) for an even dimensional free G-manifold Z.

b) Novikov gives the following additivity property of the equivariant signature

$$\text{sign}\ (G,Z) = \text{sign}\ (G,Y) + \text{sign}\ (G,Y'),$$

when $Z = Y \cup_X Y'$ and Y, Y' are even dimensional G-manifolds with boundaries X and $-X$, respectively.

Proof. Let $f : H^n(Y,X) \longrightarrow H^n(Y)$ (with real coefficients) be the natural homomorphism above. We have the bilinear form φ on $\hat{H}^n(Y)$ given by

$$\varphi(f(x), f(y)) = (x \cup y)[Y],$$

which is symmetric for n even and skew symmetric for n odd and defined sign (G,Y). (Note that we have Poincaré duality isomorphism $H^n(Y) \cong H_n(Y,X)$, so that $H^n(Y)$ is dual to $H^n(Y,X)$, hence φ is non-degenerate). Consider the dual cohomology sequences

$$H^n(Y',X) \xrightarrow{\alpha'} H^n(Z) \xrightarrow{\beta} H^n(Y)$$
$$H^n(Y') \xleftarrow{\beta'} H^n(Z) \xleftarrow{\alpha} H^n(Y,X)$$

of (Z,Y) and (Z,Y') where $H^n(Y',X) \cong H^n(Z,Y)$ and $H^n(Y,X) \cong H^n(Z,Y')$.

Then $\mathrm{Im}(\alpha)$ and $\mathrm{Im}\,\alpha'$ (the images of α and α' in $H^n(Z)$) are mutual anihilators for the bilinear form $\varphi(Z)$ given by cup-product on $H^n(Z)$. $\mathrm{Im}\,\alpha \cap \mathrm{Im}\,\alpha'$ anihilates $\mathrm{Im}\,\alpha + \mathrm{Im}\,\alpha'$ and so

$$H^n(Z)/\ \mathrm{Im}\,\alpha + \mathrm{Im}\,\alpha' \cong (\mathrm{Im}\,\alpha \cap \mathrm{Im}\,\alpha').$$

We also have the isomorphisms

$$(\mathrm{Im}\,\alpha + \mathrm{Im}\,\alpha')/(\mathrm{Im}\,\alpha \cap \mathrm{Im}\,\alpha') \cong \mathrm{Im}\,\alpha/(\mathrm{Im}\,\alpha \cap \mathrm{Im}\,\alpha') \oplus$$
$$\mathrm{Im}\,\alpha'/(\mathrm{Im}\,\alpha \cap \mathrm{Im}\,\alpha') \cong \mathrm{Im}\,\beta\alpha \oplus \mathrm{Im}\,\beta'\alpha' \cong \hat{H}^n(Y) \oplus \hat{H}^n(Y')$$

We get a decomposition of G-modules

$$H^n(Z) \cong (\mathrm{Im}\,\alpha \cap \mathrm{Im}\,\alpha') \oplus \hat{H}^n(Y) \oplus \hat{H}^n(Y') \oplus (\mathrm{Im}\,\alpha \cap \mathrm{Im}\,\alpha')^*.$$

Then the bilinear form $\wp(Z)$ on $H^n(Z)$ is represented by a matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \wp(Y) & 0 & * \\ 0 & 0 & \wp(Y') & * \\ (-1)^n & * & * & * \end{pmatrix}$$

All the terms * can be eliminated by a transformation of the form T'( ) T, so that

$$\wp(Z) \cong \wp(Y) \oplus \wp(Y') \oplus C.$$

C is here the natural bilinear form on

$$(\mathrm{Im}\,\alpha + \mathrm{Im}\,\alpha') \oplus (\mathrm{Im}\,\alpha + \mathrm{Im}\,\alpha')^*$$

and has signature egual to zero (1, § 5) . Hence

$$\mathrm{sign}\,(G,Z) = \mathrm{sign}\,(G,Y) + \mathrm{sign}\,(G,Y'),$$

as required.

I.1.2 Definition of α-invariants. Computation formulae.

a. Let G be a finite group and X be a fixed - point free G-manifold of odd dimension. Then the disjoint union kX of k copies of X bounds a free G-manifold Y, for some integer k, according to equivariant cobordism theory (4; see also 25).

<u>Definition</u> . $\alpha\,(g, X) = \frac{1}{k}\,\mathrm{sign}\,(g,Y)$, for $g \neq 1$.

This is well defined, for if Y_1 and Y_2 satisfy the conditions above, we take $Z = Y_1 \cup_{kX} -Y_2$ and apply the G-signature theorem (remark a)) and the Novikov additivity property, to obtain $\mathrm{sign}\,(g, Y_1) = \mathrm{sign}\,(g, Y_2)$. We get a function

$$\alpha\,(g,X) : G - \{1\} \longrightarrow \mathbb{C}\,.$$

When X is an integral homology sphere of dimension 4k + 3 and $G = \{1, T : T^2 = 1\}$, $\alpha(T,X)$ is an integer, since it is the Browder-Livesay invariant of the involution T (see definition of α and β and proof of $\alpha = \beta$).

A different expression can be used for the computation of $\alpha(g,X)$ in some cases. If X bounds a G-manifold Y, not necessarily free, then

$$\alpha(g,X) = \text{sign}(g,Y) - L(g,Y),$$

for $g \neq 1$. This also follows from the G-signature theorem. Let Y be as above and let Y_1 be a free G-manifold as in the definition of α. Then $\partial Y_1 = kX$, $\partial Y = X$. Let Y_2 be the disjoint union of k copies of -Y, and consider the manifold $Z = Y_1 \cup_{kX} Y_2$, obtained by identifying each copy of $-X = \partial(-Y)$ with one of $X \subset \partial Y_1$. Then Z is also a G-manifold. We have

$$\text{sign}(g,Z) = \text{sign}(g,Y_1) + \text{sign}(g,Y_2)$$
$$= \text{sign}(g,Y_1) - k\,\text{sign}(g,Y)$$

We also have

$$L(g,Z) = L(g,Y_2) = -k\,L(g,Y),$$

because the fixed - point set Fix(g,Z) is the same as $\text{Fix}(g,Y_2)$, as Y_1 is G-free. Applying the G-signature theorem to Z, the result follows.

b. Atiyah - Bott (2) gives a simpler version of the theorem, for the case of isolated fixed-points.

<u>Theorem</u> . Let $f : Y \longrightarrow Y$ be an isometry of the even dimensional Riemann manifold Y. Assume that f has only isolated fixed points P, and let $\{\theta_k^P\}$ be a system of coherent angles for $df_p : T_p^* \longrightarrow T_p^*$.

Then the signature of f is given by

$$\text{sign}\ (f,Y) = \sum_P i^{-m}\ \pi_k \cot\ (\theta_k^P/2),\ \text{where}\ \dim Y = 2m.$$

A coherent system for df_P is a set of angles $\{\theta_k\}$ obtained in the following way. Let $\lambda^k T\ Y$ denote the bundle of $k^{\underline{th}}$ exterior powers of T Y. We have a basic 2m-form $\nu \in \Gamma\ (\lambda^{2m}\ T\ Y)$, which arises from the orientation and riemannian structure in T Y, and which is characterized by the requirement that, at every point P,

$$\nu_P = \theta_1 \wedge \dots \wedge \theta_{2m}\ ,$$

for each orthonormal frame $(\theta_1,\dots,\theta_{2m})$ for T_PY, in the orientation of Y.Now f is an isometry of Y, df_P is an isometry of the cotangent space T^*_PY , and so this decomposes into a direct sum of orthogonal 2-planes, invariant under df_P ,

$$T^*_PY = V_1 \oplus V_2 \oplus \dots \oplus V_m\ ,$$

say, where dim Y = 2m. We can choose an orthogonal base of V_k, (e_k, e'_k), say, satisfying

$$\nu_P(e_1 \wedge e'_1 \wedge \dots \wedge e_m \wedge e'_m) = 1.$$

It follows that

$$df_P \cdot e_k = \cos\theta_k\ e_k + \sin\theta_k\ e'_k$$

$$df_P . e'_k = \sin\theta_k\ e_k + \cos\theta_k\ e'_k\ ,$$

that is, df_P is given by rotations through angles θ_k in V_k .

For the case of only one fixed point P, we get the formula

$$\text{sign}\ (f,Y)\ .\ \pi\ (1 - \xi_k) = \pi\ (1 + \xi_k)\ ,$$

where $\xi_k = e^{-i\ \theta_k}$.

I.1.3 The Browder-Livesay invariants.

These invariants were originally defined for homotopy spheres. So, we start with this case.

Denote (T, Σ^m) for an involution T on a homotopy sphere Σ^m. A characteristic submanifold for (T, Σ^m) is an $(m - 1)$-submanifold W such that $W = A \cap TA$ and $\Sigma = A \cup TA$, where A is a compact submanifold of Σ with boundary $\partial A = W$, together with the involution $T \mid W$.

It follows that every (T, Σ) has a characteristic submanifold,(see 13) . Given two characteristic submanifolds W_o and W_1, let $W_o = W_o \times \{0\}$ and $W_1 = W_1 \times \{1\}$ in $\Sigma \times I$. Then $\Sigma \times I = B \cup (T \times 1)$ B, where $B \cap (T \times 1) B = V$ and V is transverse to W_o and W_1; B is a compact submanifold of $\Sigma \times I$ with boundary $A_o \times \{0\} \cup V \cup A_1 \times \{1\}$ (and corners along W_o and W_1). We call $(T \times 1 \mid V, V)$ a characteristic cobordism (13).

Now, let W be a characteristic submanifold for (T, Σ^m). If W is $(q - 1)$-connected, for $q \leq [\frac{m-3}{2}]$, we use the process of equivariant surgery, described in the introduction of Chapter II, on an element $\alpha \in K = \mathrm{Ker}\, \{H_q(W) \longrightarrow H_q(A)\}$, to kill α, that is, to obtain a new characteristic submanifold W' with $\mathrm{Ker}\, \{H_q(W') \longrightarrow H_q(A)\} \cong K/_{(\alpha)}$. (see, for example, 17). Repeating this process a finite number of times, we get an $[\frac{m-3}{3}]$-connected characteristic submanifold for (T, Σ^m).

Suppose now that $m = 4n + 3$ and let W be a 2n-connected-2n characteristic submanifold for (T, Σ). The intersection form on $H_{2n+1}(W)$ is skew-symmetric and unimodular,by Poincaré duality.

We consider the form

$$f(x,y) = x \circ T_* y ,$$

defined on $K = \mathrm{Ker}\,\{H_{2n+1(W)} \longrightarrow H_{2n+1}(A)\}$, where $W \longrightarrow A$ is the inclusion. This form is now symmetric. It is also even : let the imbedded sphere S^{2n+1} represent $x \in K$; by general position , S^{2n+1} meets $T\,S^{2n+1}$ transversely, so, the form is even as $S^{2n+1} \cap T\,S^{2n+1}$ is invariant under T. From the Mayer-Vietoris sequence of (Σ, A, TA) it follows that $H_{2n+1(W)} = K \oplus T_*K$ and so f is unimodular. According to Milnor (18) sign (f) is a multiple of 8.

Definition. $\beta\,(T, \Sigma) = \mathrm{sign}\,(f)$.

β is independent of the choice of W, since there al - ways exists a characteristic cobordism joining two of them. So, β is a well-defined invariant of (T, Σ).

The Browder-Livesay invariant is also defined when $m = 4n + 1$ and in this case as the Arf invariant of a quadratic form associated to f, as f is now skew-symmetric.

General case.

Denote by $(T, M^n, \partial M)$ a fixed-point free involution of a manifold M with boundary ∂M. A characteristic submanifold is an im - bedded submanifold $(W^{m-1}, \partial W) \subset (M, \partial M)$ satisfying

i) $\partial M = A \cup TA$, $\partial W = A \cap TA$, where A is a compact submanifold of ∂M with boundary ∂W,

ii) $M = B \cup TB$, $W = B \cap TB$, where B is a compact submanifold of M with boundary $A \cup W$ (and corner along ∂W).

Similar results to the previous ones hold for $(T, M^m, \partial M)$.

There is always a characteristic submanifold for $(T,M,\partial M)$, any two of them can be joined by a characteristics cobordism, etc. Also, / any characteristic submanifold for $(T|\partial M, \partial M)$ is the boundary of one of $(T, M, \partial M)$.

Suppose now that $\partial M = \emptyset$. Define the following invariants (Browder-Livesay) for $m = 4k + 3$ and T orientation preserving, or $m = 4k + 1$ and T orientation reversing. Let $N = A \cap TA$ be a characteristic submanifold for (T,M) and

$\partial : H_{\frac{m+1}{2}}(M) \longrightarrow H_{\frac{m+1}{2}}(N)$ be the Mayer-Vietories boundary homomorphism.

Write

$K = \mathrm{Ker}\left(H_{\frac{m-1}{2}}(N) \longrightarrow H_{\frac{m-1}{2}}(A) \right)$. Then the form

$f(x,y) = x \circ T_* y$ is defined on $K/\mathrm{im}\,\partial$. It is symmetric and unimodular, and so its signature is a multiple of 8. We put

$$\beta\,(T,M) = \mathrm{sign}\,(f).$$

I.1.4 $\alpha = \beta$. First part of the proof.

Let (T,X) be an orientation preserving fixed-point free involution of a compact oriented differentiable $(4k + 3)$-manifold. The invariant $\alpha(T,X)$ is defined. Suppose that some multiple mX of X bounds a compact oriented differentiable manifold Y and / $T : Y \longrightarrow Y$ is an orientation preserving involution extending T, possibly with fixed points. We have

$$\alpha\,(T,X) = \frac{1}{m}\left(\tau\,(T',Y) - \mathrm{sign}\,(\mathrm{Fix}\,T' \circ \mathrm{Fix}\,T')\right)$$

(Hirzebruch-Jänich's formula (10,8)). In this formula we denote by $\mathcal{T}(T',X)$ the signature of the bilinear form f on $H_{2k+2}(Y; \mathbb{Q})$;

given by

$$f(x,y) = x \circ T_* y ,$$

and Fix T' o Fix T' is the oriented self intersection cobordism / class (1) of the fixed point set Fix T' (even dimensional) of T'.

We want to study a manifold $\mathcal{D}$ with an involution $\mathcal{J}$ such that $\partial \mathcal{D} = (T,X) - 2X/T$ and $\mathcal{J}$ is the trivial involution on 2X/T. This manifold was first constructed by Dold in (5), and we show that

$$\alpha (T,X) = \mathcal{T}(\mathcal{J}, \mathcal{D}) = - \operatorname{sign}(\mathcal{D}).$$

The Browder-Livesay invariant is defined for (T,X). It turns out that, when $H_{2k+2}(X; \mathbb{Q}) = 0$, then

$$\beta (t,X) = -\operatorname{sign}(\mathcal{D}),$$

so that we obtain

$$\alpha (T,X) = \beta(T,X) \quad , \text{ if } H_{2k+2}(X; \mathbb{Q}) = 0.$$

The proof of $\alpha = \beta$ (in this case), which is to be shown is due to Hirzebruch and Jänich (8).

<u>The Dold construction</u> . Let M be a compact differentiable manifold without boundary and V a closed sub-manifold without boundary of M and of codimension 1. Let $Y = M \times [0,1]$ and $Z = V \times [0,1/_2]$. We construct a double covering $\mathcal{D}$ of Y, branched at $V \times \{1/_2\}$ such that the covering transformation is an involution $\mathcal{T}$ on $\mathcal{D}$.

Let $\tilde{V}$ be the $\mathbb{Z}/_{2\mathbb{Z}}$ - principal bundle over V, defined by the normal bundle of V in M. If we "cut" M along V, we obtain a compact differentiable manifold $A = (M - V) \cup \tilde{V}$ (disjoint union) , with boundary $\partial A = V$. In the same way, let B be the disjoint union

$$(Y - V \times [0, 1/2)) \cup (V \times [0, 1/2)),$$

with the topology given in a canonical way. Let B*another copy of B and $B \cup B^*$ be the disjoint union. Identify each $x \in V \times \{1/2\} \subset B$ with $x^* \in V \times \{1/2\} \subset B^*$ and, for $0 \leq t < 1/2$, each point $x \in \tilde{V} \times \{t\} \subset B$ with $-x^* \in \tilde{V} \times \{t\} \subset B^*$. To the topological space $\mathcal{D}$ thus obtained, it can be given a differentiable structure such that it coincides with the canonical structure in $\mathcal{D} - V \times \{1/2\}$ (as in (8)).

The involution $\mathcal{J} : \mathcal{D} \longrightarrow \mathcal{D}$ can now be defined by

$\mathcal{J}(x) = x^*$

$\mathcal{J}(x^*) = x$, for $x \in B$, $x^* \in B^*$.

The fixed-point set of $\mathcal{J}$ is then Fix $\mathcal{J} = V \times \{\frac{1}{2}\}$.

The given differentiable structure is such that Fix $\mathcal{J}$ has a non-zero normal section and is regular, in the following sense : Fix $\mathcal{J}$ is a submanifold in the interior of M and has a linear normal bundle in M,which can be imbedded as a tubular neigh-bourhood of Fix $\mathcal{J}$, such that $\mathcal{J}$ is the antipodal map on each fibre. It follows that Fix $\mathcal{J}$ o Fix $\mathcal{J}$ = 0.

Let $\pi : \mathcal{D} \longrightarrow M \times [0,1]$ be the projection.

Then $\pi^{-1}(M \times \{1\}) = 2M$, the disjoint union of two copies of M, and $\pi^{-1}(M \times \{0\}) = \tilde{M}$ is a differentiable manifold obtained from the disjoint union $A \cup A^*$, of two copies of A, by identifying each point $x \in \tilde{V} \subset A$ with its opposite point $-x^* \in \tilde{V} \subset A^*$.

Now, if (T,X) is a fixed-point free involution on a compact differentiable manifold without boundary X and $X/_T \cong M$, then, for a suitable submanifold V of M, (T,X) will be equivariantly diffeomorphic to $(\tilde{M} ; \mathcal{J}/\tilde{M})$. So, we are interested in the case when $\tilde{M}$ is oriented and ($\mathcal{J}/\tilde{M}, \tilde{M}$) is orientation preserving.

Orientation . We make the following convention: if Y is a orientable manifold with boundary ∂Y, then the orientation of Y is obtained from that of ∂Y, followed by the inwards pointing normal vector.

With this assumption, the orientation of $\tilde{M}$ induces an orientation on M and on $\partial \mathcal{D}$ such that

$$\partial \mathcal{D} = \tilde{M} - 2M.$$

$\alpha(T,X) = \tau(\mathcal{J}, \mathcal{D})$. Applying Hirzebruch-Jänich's formula , we have

$$\alpha(\mathcal{J} | \partial \mathcal{D}, \partial \mathcal{D}) = \tau(\mathcal{J}, \mathcal{D}).$$

The involution $\mathcal{J}$ on 2X/T is the trivial one. Let N be an oriented cobordism between the two copies of X/T. This always exists, since the elements of Ω_{4k+3} are of order two. Let T_1 be the trivial involution on 2M. Then

$$\tau(T_1, 2M) = 0,$$

and

$$\alpha(\mathcal{J} | 2X_{/T}, 2X_{/T}) = \frac{1}{2} \tau(T_1, 2M) = 0,$$

so that

$$\alpha(T,X) = \alpha(\mathcal{J} | \partial \mathcal{D}, \partial \mathcal{D}) = \tau(\mathcal{J}, \mathcal{D}).$$

$\tau(\mathcal{J}, \mathcal{D}) = -\operatorname{sign}(\mathcal{D})$. Let U be the normal closed disc bundle of Fix $\mathcal{J}$. In U all intersection numbers are zero, because Fix $\mathcal{J}$ has a non-zero normal section, hence / $\tau(\mathcal{J} | U, U) = 0$. Now, U is invariant under $\mathcal{J}$ and ($\mathcal{J} | \mathcal{D} - U, \mathcal{D} - U$) is a fixed-point free involution. Also

$$\mathcal{T}(\mathcal{f}, \mathcal{D}) = \mathcal{T}(\mathcal{f} \mid \mathcal{D} - U, \mathcal{D} - U).$$

Therefore,

$$\mathcal{T}(\mathcal{f} \mid \mathcal{D} - U, \mathcal{D} - U) = 2\operatorname{sign}(Y - \pi(U)) - \operatorname{sign}(\mathcal{D} - U)$$

$$= 2\operatorname{sign}(Y) - \operatorname{sign}(\mathcal{D} - U)$$

$$= - \operatorname{sign}(\mathcal{D} - U)$$

(We have applied above the formula which relates the signatures sign (T,M), sign (M) and sign(M/T); its proof is to be shown in Chapter II)

As sign $(\mathcal{D} - U) = $ sign $(\mathcal{D})$, it follows that

$$\mathcal{T}(\mathcal{f}, \mathcal{D}) = - \operatorname{sign}(\mathcal{D}),$$

and so

$$\alpha(T,X) = \mathcal{T}(\mathcal{f}, \mathcal{D}) = - \operatorname{sign}(\mathcal{D}).$$

I.2 Hirzebruch's results on lens spaces.

Le $\mathbb{Z}/_{q}\mathbb{Z}$ be the cyclic group of order q, and / $p_1,\ldots,p_n$ be natural numbers prime to q. $\mathbb{Z}/q\mathbb{Z}$ is isomorphic to the group of $q^{\underline{th}}$ roots of unity in C . Then

$$\zeta(z_1,\ldots,z_n) = (\zeta^{P1} z_1,\ldots,\zeta^{P_n} z_n), \quad \in \mathbb{Z}/q\mathbb{Z}$$

is a free action of $\mathbb{Z}/q\mathbb{Z}$ on the sphere

$$S^{2n-1} = \{z_1,\ldots,z_n) \in C^n : z_1\overline{z_1} + \ldots + z_n\overline{z_n} = 1\}.$$

The orbit space is the (2n - 1)-dimensional lens space $\mathcal{L}(q\,;\,P_1,\ldots,P_n)$.

The inclusion $\begin{array}{c} Z/qZ \\ \downarrow \\ Z/2qZ \end{array}$ gives a natural map $\begin{array}{c} \mathcal{L}(q:P_1,\ldots,P_n), \\ \downarrow \\ \mathcal{L}(2q:P_1,\ldots,P_n) \end{array}$

when the p_j's are prime to 2q. This is a double covering and the covering translation in an involution T of $\mathfrak{b}(q;\, P_1,\ldots,P_n)$.

The Browder-Livesay invariant for the involution T is then given by the formula

$$\alpha(T, \mathfrak{b}(q;\, P_1,\ldots,P_n)) = -\frac{1}{q} \sum_{\substack{\zeta^{2q}=1 \\ \zeta^{q}\neq 1}} \prod_{j=1}^{n} \frac{(\zeta^{P_j}+1)}{(\zeta^{P_j}-1)} .$$

For n even, it is also given by

$$\alpha(T, \mathfrak{b}(q;p_1,\ldots,p_n)) = \frac{(-1)^{\frac{n}{2}+1}}{q} \sum_{\substack{j=1 \\ j \text{ odd}}}^{2q-1} \cot(\frac{jp_1}{2q}\pi).\cot(\frac{jp_2}{2q}\pi)\ldots$$

$$\ldots \cot(\frac{jp_n}{2q}\pi)$$

(Hirzebruch (7).

He introduced the integer

$$t(a_1,\ldots,a_m;b_1,\ldots,b_m) = \#\{x \in \mathbb{Z}^m : 0<x_k<a_k \text{ and } 0 < \sum_{k=1}^{m} \frac{x_k b_k}{a_k} < 1 \bmod 2\}$$

$$- \#\{x \in \mathbb{Z}^m : 0<x_k<a_k \text{ and } 1 < \sum_{k=1}^{m} \frac{x_k b_k}{a_k} < 2 \bmod 2\},$$

for any 2m-row of natural numbers $(a_1,\ldots,a_m\, ;\, b_1,\ldots,b_m)$ with b_j / and $2a_j$ coprime, and proved

$$t(a_1,\ldots,a_m;b_1,\ldots,b_m) = \frac{(-1)^{\frac{m-1}{2}}}{N} \sum_{\substack{j=1 \\ j \text{ odd}}}^{2N-1} \cot(\frac{J\pi}{2N})\cot(\frac{jb_1}{2a_1}\pi)\ \ldots$$

$$\ldots\cot(\frac{j\, b_m}{2a_m}\pi),$$

for m odd, where N is any common multiple of $a_1, \ldots, a_m$,

and

$$\alpha(T, \wp (q;1,p_2,\ldots,p_n) = t(q,\ldots,q;p_2,\ldots,p_n).$$

The proof of the first formula was done in this way: first Hirzebruch used Atiyah-Bott formula to get $\alpha(\zeta, S^{2n-1})$, $\zeta \in \mathbb{Z}/q\mathbb{Z}$ and then he applied the following proposition.

Proposition. Let X be a free G-manifold (without boundary) of odd dimension. Let U be a normal subgroup of G. Then X/U is a free G/U manifold. If $p : G \longrightarrow G/U$ is the natural homomorphism, then for / $\xi \in G/U$ ($\xi \neq 1$),

$$\alpha(\xi, X/U) = \frac{1}{|U|} \sum_{g \in p^{-1}(\xi)}{}' \alpha(g,X).$$

Proof. Let W be a representation space of G, where W is a real (or a complex) finite dimensional vector space. Then the group G/U acts on

$$W^U = \{x \in W \mid u\,x = x \text{ for all } u \in U\}$$

Let R(G) denote the representation ring of G. We get a map

$$\rho : R(G) \longrightarrow R(G/_U) .$$

Now, if Y is a free G-manifold with boundary $\partial Y = k\,X$, then $\partial(Y/U) = k(X/U)$ and Y/U is a free G-manifold. The real (or complex) cohomology of Y/U can be identified with the U-invariant part of the cohomology of Y. Hence,

$$\rho(\text{sign}(G,Y)) = \text{sign}(G, Y/U).$$

On the other hand, the image of

$$\frac{1}{|U|} \sum_{u \in U} g\,u : W \longrightarrow W$$

is W^U and

$$\frac{1}{|U|} \sum_{u \in U} g\,u\,(x) = g(x) \quad , \text{ for } x \in W^U .$$

Hence the trace of this endomorphism is the trace of g on W^U .Therefore, if $h \in R(G)$, the character of h and $\rho\, h$ satisfy

$$\chi_{\rho\, h}(\xi) = \frac{1}{|U|} \sum_{g \in p^{-1}(\xi)} \chi_h(g) \quad , \text{ for } \xi \in G/U,$$

and the result follows.

I.3 The generalized quaternion group case.

Let Q_{4t} denote the generalized quaternion group

$$Q_{4t} = \{x,y : x^{2t} = 1,\ x^t = y^2,\ y^{-1}xy = x^{-1}\},$$

of order 4t, for a natural number t. Elements of Q_{4t} are of the form

$$x^r y^s \ , \qquad 0 \leq r \leq 2t - 1, \qquad s = 0 \text{ or } 1.$$

Q_{4t} acts freely on

$$S^3 = \{(z_1,z_2) \in C^2 : z_1\overline{z_1} + z_2\overline{z_2} = 1\} ,$$

via the representation T_q given by

$$Q_{4t} \longrightarrow U(2)$$

$$x \longmapsto \begin{pmatrix} \zeta^q & 0 \\ 0 & \zeta^{-q} \end{pmatrix}, \quad y \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

for q a natural number prime to 2t and $\zeta = e^{\frac{2\pi i}{2t}}$.

As $S^{4n-1} = S^3 * \ldots * S^3$ (join of n copies of S^3), then for $q_1,\ldots,q_n$ prime to 2t, Q_{4t} acts freely on S^{4n-1} via the sum of these actions

$$T_{q_1} + T_{q_2} + \dots + T_{q_n} .$$

Now, for $q_1,\dots,q_n$ prime to 4t, the inclusion

$$\begin{array}{ccc} Q_{4t} & \text{gives us a map} & [\,S^{4n-1}, Q_{4t}] \\ \downarrow & & \downarrow \\ Q_{8t} & & [\,S^{4n-1}, Q_{8t}\,] \end{array}$$ which is a double covering.

The covering translation is a fixed-point involution T on $[S^{4n-1},Q_{4t}]$.

We follow the metod used by Hirzebruch, to prove the

<u>Proposition</u> . The Browder-Livesay invariant of the involution T on $[S^{4n-1}, Q_{4t}]$ is given by the formula

$$\alpha(T,[S^{4n-1},Q_{4t}] = \frac{-1}{4t} \sum_{\substack{\xi^{4t}=1 \\ \xi^{2t}\neq 1}} \prod_{j=1}^{n} \frac{(\xi^{q_j}+1)(\xi^{-q_j}+1)}{(\xi^{q_j}-1)(\xi^{-q_j}-1)} - \frac{(-1)^n}{2}$$

Consider the case n = 1 first, so that we have action on S^3.

$$T_q : x \longmapsto \begin{pmatrix} \zeta^q & 0 \\ 0 & \zeta^{-q} \end{pmatrix} , \quad \zeta = e^{2\pi i/4t}$$

$$y \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

gives

$$x^r \longmapsto \begin{pmatrix} \xi^q & 0 \\ 0 & \xi^{-q} \end{pmatrix}$$

$$x^r y \longmapsto \begin{pmatrix} 0 & -\xi^q \\ \xi^{-q} & 0 \end{pmatrix} , \quad \xi = \zeta^r .$$

Taking $G = Q_{8t}$, $U = Q_{4t}$ in Hirzebruch's proposition, we see that the elements that belong to $p^{-1}(T)$ are of the form

$$x^r y^s \ , \ 0 \leq r \leq 4t - 1 \ , \ r \text{ odd}, \ s = 0 \text{ or } 1.$$

Now S^3 bounds the disc D^4, on which $g \in G$ operates and we have

$$\alpha(g,S^3) = \text{sign } (g,D^4) - L(g, D^4) \ , \ g \neq 1.$$

The origin is the only fixed point, sign $(g,D^4) = 0$, and so

$$\alpha(g,S^3) = - L(g,D^4).$$

$L(g,D^4)$ can be calculated by using the G-signature theorem, in its special case given by Atiyah-Bott's theorem.

As a matter of notation, as $T_p^* D^4 \cong C^2 \cong (R^4)^*$, we fix as basis for $T_p^* D^4$ the set of vectors

$$\{(1000), (0100), (0010), (0001)\} .$$

Take f to be the operation of x^r on S^3 and P to be the origin. Then $T_p^* D^4$ decomposes as $T_p^* D^4 = \mathbb{R}^{2*} \oplus \mathbb{R}^{2*}$, with basis

$$\{\overset{e_1}{(1000)}, \overset{e_1'}{(0100)}\} , \{\overset{e_2}{(0010)}, \overset{e_2'}{(0\ 0\ 0\ 1)}\}, \text{ respectively,}$$

which satisfies the requirements of Atiyah-Bott's theorem.

With this notation, $df_p : T_p Y \longrightarrow T_p Y$, where $T_p Y$ is the tangent space of Y at P, is given by

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & & \\ \sin\theta & \cos\theta & & 0 \\ & 0 & \cos(-\theta) & -\sin(-\theta) \\ & & \sin(-\theta) & \cos(-\theta) \end{pmatrix} , \ \theta = \frac{2\pi\ i\ r\ q}{4t}.$$

We have

$$e_1 \cdot A = (\cos\theta \quad -\sin\theta \quad 0 \quad 0) = \cos\theta\, e_1 - \sin\theta\, e_1'$$

$$e_1' \cdot A = (\sin\theta \quad \cos\theta \quad 0 \quad 0) = \sin\theta\, e_1 + \cos\theta\, e_1'$$

$$e_2 \cdot A = (0 \quad 0 \quad \cos(-\theta) - \sin(-\theta)) = \cos(-\theta)\, e_2 - \sin(-\theta)\, e_2'$$

$$e_2' \cdot A = (0 \quad 0 \quad \sin(-\theta) \quad \cos(-\theta)) = \sin(-\theta)\, e_2 + \cos(-\theta)\, e_2'$$

So $\{\theta_1, \theta_2\} = \{-\theta, +\theta\}$ is a coherent system for df^*p .

Let $\xi_k = e^{-i\,\theta_k}$.

$$L(x^r, D^4) = \frac{(1+\xi_1)(1+\xi_2)}{(1-\xi_1)(1-\xi_2)} = \frac{(1+\xi^q)(1+\xi^{-q})}{(1-\xi^q)(1-\xi^{-q})} .$$

Take f to be, now, the action of $x^r y$. We have the decomposition

$$T_p^* D^4 = V_1 \oplus V_2$$

with basis $\{e_1, e_1'\}$, $\{e_2, e_2'\}$, respectively, satisfying the required conditions , given by

$$e_1 = (1 \ 0 \ 0 \ 0)$$

$$e_1' = (0 \ 0 \ \cos\theta \ -\sin\theta)$$

$$e_2 = (0 \ 1 \ 0 \ 0)$$

$$e_2' = (0 \ 0 \ \sin\theta \ \cos\theta) .$$

$d f_p : T_p X \longrightarrow T_p X$ has matrix

$$A = \begin{pmatrix} 0 & & -\cos\theta & \sin\theta \\ & & -\sin\theta & -\cos\theta \\ \cos(-\theta) & -\sin(-\theta) & & 0 \\ \sin(-\theta) & \cos(-\theta) & & \end{pmatrix}$$

so

$$e_1 A = (0 \quad 0 \quad -\cos\theta \quad \sin\theta) = - e_1'$$

$$e_1' A = (1 \quad 0 \quad 0 \quad 0) = e_1$$

$$e_2 A = (0 \quad 0 \quad -\sin\theta \quad -\cos\theta) = - e_2'$$

$$e_2' A = (0 \quad 1 \quad 0 \quad 0) = e_2$$

Then $\{\theta_1 , \theta_2\} = \{- \pi/_2 , - \pi/_2\}$ is a coherent system for $df * p$.

$$L (x^r y, D^4) = \frac{(1 + i)\ (1 + i)}{(1 - i)\ (1 - i)} = -1 .$$

Finally, we aply Hirzebruch's proposition to obtain

$$\alpha (T,[S^3,Q_{4t}] = - \frac{1}{4t} \sum_{\substack{\xi^{4t} = 1 \\ \xi^{2t} \neq 1}} \left(\frac{(\xi^q + 1)(\xi^{-q} + 1)}{(\xi^q - 1)(\xi^{-q} - 1)} - 1 \right)$$

$$= - \frac{1}{4t} \sum_{\substack{\xi^{4t} = 1 \\ \xi^{2t} \neq 1}} \frac{(\xi^q + 1)\ (\xi^{-q} + 1)}{(\xi^q - 1)\ (\xi^{-q} - 1)} + 1/_2$$

For an arbitrary n, we have the action of $T_{q_1} + T_{q_2} + \ldots + T_{q_n}$ on S^{4n-1}. According to C.T.C. Wall (25) the value of α for the join action is the product of the values of α for the separate actions on S^3. So, we have

$$- \alpha\ (x^r, D^{4n}) = L(x^r, D^{4n}) = \prod_{j=1}^{n} \frac{(\xi^{q_j} + 1)(\xi^{-q_j} + 1)}{(\xi^{q_j} + 1)(\xi^{-q_j} - 1)}$$

$$- \alpha\ (x^r y, D^{4n}) = L(x^r y, D^{4n}) = \prod_{j=1}^{n} \frac{(1 - i)\ (1 - i)}{(1 + i)\ (1 + 1)} = (-1)^n$$

As a result,

$$\alpha\ (T,[S^{4n-1}, Q_{4t}]\ ) = \frac{-1}{4t} \sum_{\substack{\xi^{4t} = 1 \\ \xi^{2t} \neq 1}} \left(\prod_{j=1}^{n} \frac{(\xi^{q_j} + 1)(\xi^{-q_j} + 1)}{(\xi^{q_j} - 1)(\xi^{-q_j} - 1)} + (-1)^n \right)$$

$$= \frac{-1}{4t} \sum_{\substack{\xi^{4t} = 1 \\ \xi^{2t} \neq 1}} \left(\prod_{j=1}^{n} \frac{(\xi^{q_j} + 1)(\xi^{-q_j} + 1)}{(\xi^{q_j} - 1)(\xi^{-q_j} - 1)} - \frac{(-1)^n}{2} \right.$$

Now, taking $q_1 = 1$, we can rewrite the above formula as

$$\alpha(T,[S^{4n-1}, Q_{4t}]\ ) = - \frac{1}{4t} \sum_{\substack{r=1 \\ r \text{ odd}}}^{4t-1} i^{-2n} \cot \frac{-r}{4t}\pi \cot \frac{-r}{4t}\pi \cot \frac{-r\ q_2}{4t}\pi$$

$$\cot \frac{r\ q_2}{4t}\pi \ldots \cot \frac{-r\ q_n}{4t}\pi \quad \cot \frac{r\ q_n}{4t}\pi - \frac{(-1)^n}{2} =$$

$$= \frac{(-1)^n}{2} \cdot \frac{(-1)^{n+1}}{2t} \sum_{\substack{r=1 \\ r \text{ odd}}}^{4t-1} \cot \frac{r}{4t}\pi \cot \frac{r}{4t}\pi \cot \frac{r\ q_2}{4t}\pi \cot \frac{r\ q_2}{4t}\pi \ldots$$

$$\cot \frac{r\ q_n}{4t}\pi \quad \cot \frac{r\ q_n}{4t}\pi - \frac{(-1)^n}{2} \quad ,$$

so that we have the

Proposition . The Browder-Livesay invariant of the involution T on $[\ S^{4n-1}, Q_{4t}]$ is given by the formula

$$\alpha(T,[S^{4n-1},Q_{4t}]) = \frac{(-1)^n}{2}\,(t(2t,\ldots,2t;1,q_2,q_2,\ldots,q_n,q_n)-1).$$

Remark. If we look at the finite groups which act orthogonally on S^3 (20), we see that there is also an exotic case which could be considered, that is, the case of the action on S^3 by the binary tetrahedral and octahedral groups, T_* and O_*, of order 24 and 48, respectively. These groups are obtained from the tetrahedral and octahedral subgroups T and 0 of SO(3), which can be lifted to S^3 by means of

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow S^3 \longrightarrow SO(3) \longrightarrow 1,$$

giving T_* and O_*. The inclusion $\begin{array}{c} T_* \\ \downarrow \\ O_* \end{array}$ leads to an involution on orbit spaces $\begin{array}{c} [S^3,T_*] \\ \downarrow \\ [S^3,O_*] \end{array}$.

The invariant $\alpha(T, [S^3,T_*])$ can be calculated in a similar way.

Chapter II

INVOLUTIONS ON HOMOLOGY SPHERES

II.0 Introduction. Definitions e notation.

This chapter is devoted to prove the proposition announced in the introduction. We work with the involutions on homology 3-spheres given by Lopez de Medrano in (13) and (14). Other references for the invariants for involutions are (3), (8) and (4).

For the theory of quadratic forms and their signatures, as well as the intersection theory, see (3) and (9).

As a background of homology and cohomology theories, duality theorems, we refer to (24).

We recall the following definitions which will be used in this chapter.

A G-homology n-sphere is a connected compact oriented differentiable manifold X of dimension n, with $\partial X = \emptyset$ and such that $H_o(X) = H_n(X) = G$ and $H_i(X) = 0$ otherwise.

Let D^n denote the unit discin euclidean n-space and S^{n-1} be its boundary.

The connected sum $M_1 \# M_2$ of two connected compact oriented differentiable manifolds M_1 and M_2 is defined as follows. Let

$$i_1 : D^n \longrightarrow M_1$$

$$i_2 : D^n \longrightarrow M_2$$

be imbeddings, i_1 orientation preserving and i_2 orientation reversing. $M_1 \# M_2$ is obtained from the disjoint union $(M_1 - i_1(0)) \cup$ $\cup (M_2 - i_2(0))$ by identifying $i_1(tu)$ with $i_2((1-t)_u)$ for each $u \in S^{n-1}$ and each $0 < t < 1$. Choose for $M_1 \# M_2$ the orientation compatible with that of M_1 and M_2.

Similarly the connected sum along the boundary of the two compact oriented differentiable (n + 1) -manifolds W_1 and W_2 with connected boundaries is obtained in the same way by taking now two half-discs neighbourhoods of points in W_1 and W_2. The resulting manifold W, with boundary $\partial W = \partial W_1 \# \partial W_2$ has the homotopy type of $W_1 \vee W_2$, the union with a single point in commom (11).

Disjoint equivariant surgery : Let (T,N) be an involution on a simply-connected compact differentiable manifold N and let $X = A \cap TA$ be a characteristic submanifold for (T,N). Suppose that $f : (D^{q+1}, S^q) \longrightarrow (A,X)$ is an imbedding such that $f(D^{q+1} - S^q) \subset$ $\subset A - X$, $f(D^{q+1})$ intersects X transversely and $f(S^q) \cap Tf(S^q) = \emptyset$. Then, there is a tubular neighbourhood V of $(f(D^{q+1}), f(S^q))$ in (A,X) such that $V \cap TV = \emptyset$. Let $A' = \overline{A - N \cup TN}$ (after smoothing). We get a new characteristic submanifold for (T,N), which is $X' =$ $= A' \cap TA'$.

Let M be a compact differentiable manifold with boundary $\partial M = A \cup B$, $A \cap B = \partial A = \partial B$ and suppose that (T,A) is an involution on A. We can form a new manifold M', with an involution $T' : M' \longrightarrow M'$ as follows. Let M* be another copy of M. Then $M' = M \cup_{(T,A)} M^*$ is obtained from the disjoint union of M and M* by identifying x* and T(x) for each $x \in A$, where x* is the point in M* corresponding to $x \in M$. Define $T' : M' \longrightarrow M'$ by $T'(x) = x^*$ and $T'(x^*) = x$, for each $x \in M$. This is compatible with the identifications because T is an involution. An example of this construction is the Dold manifold in I.1.4. .

II.1. Construction of Lopez de Medrano's examples.

<u>Theorem</u> . For every $i \in \mathbb{Z}$ there is a fixed-point free involution (T, Σ^3) of a homology 3-sphere Σ^3 such that $\beta(T,\Sigma^3) = 8i$.

(S.L. de Medrano (13),(14))

We give a detailed construction of such involutions, which we divide into two parts. First, we start with the antipodal / map on the sphere S^3 to obtain (T, Σ^3) and then we consider the algebraic conditions in order to have $\beta(T,\Sigma^3) = 8i$.

a) Let $T_o : S^3 \longrightarrow S^3$ be the antipodal map. Any invariant $S^2 \subset S^3$ is a characteristic submanifold for (T_o,S^3). Perform k stardard disjoint equivariant surgeries on S^2 to obtain a new characteristic submanifold $W = \#_{2k} (S^1 \times S^1)$ (the connected sum of 2k copies of $S^1 \times S^1$). We want to construct a homology sphere Σ^3 with an involution T such that W is a characteristic submanifold for (T,Σ^3).

Let V be connected sum along the boundary of 2k copies of $S^1 \times D^2$ and V' be the closure of its complement in S^3, so that $W = \partial V$. Let $\{\alpha_1,\dots,\alpha_{2k},\beta_1,\dots,\beta_{2k}\}$ be a standard basis for $H_1(W)$ such that $\{\alpha_1,\dots,\alpha_{2k}\}$ generates $K = \ker \{i_* : H_1(W) \longrightarrow H_1(V)\}$ and $\{\beta_1,\dots,\beta_{2k}\}$ generates $\ker \{i'_*: H_1(W) \longrightarrow H_1(V')\}$, where i, i' denote the inclusions $W \subset V$ and $W \subset V'$ respectively, so that the matrix of intersections numbers with respect to it is

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} .$$

Now, choose another basis for K, $\alpha_1', \ldots, \alpha_{2k}'$, say and elements $\beta_1', \ldots, \beta_{2k}' \in H_1(W)$ satisfying $\alpha_i' \circ \alpha_j' = 0$, $\alpha_i' \circ \beta_j' = \delta_{ij}$ and $\beta_i' \circ \beta_j' = 0$. Then $\{\alpha_1', \ldots, \alpha_{2k}', \beta_1', \ldots, \beta_{2k}'\}$ is a basis for $H_1(W)$ and the matrix of intersection numbers with respect to it is again

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

This imply that the automorphism of $H_1(W)$ given by $\left.\begin{array}{l} \alpha_i' \longrightarrow \alpha i \\ \beta_i' \longrightarrow \beta i \end{array}\right.$ can be lifted to an automorphism of π_1 (W) (see(16-pp-177 178 and 355-356))and so, it can be realized by a homeo - morphism $f : W \longrightarrow W$ (According to Nielsen's theorem (21-(p.266))

Let A be the mapping cilinder of f, union V. Then $\partial A = W$, A is homeomorphic to V and the map $H_1(W) \longrightarrow H_1(A)$ induced by the inclusion $W \subset A$ is given by

$$\alpha_i' \longrightarrow 0$$

$$\beta_i' \longrightarrow bi ,$$

where $\{b_1, \ldots, b_{2k}\}$ is a basis for $H_1(A)$.

Consider now another copy A* of A. Let $\Sigma^3 = A \cup_{T_0} A^*$ and define $T : \Sigma^3 \longrightarrow \Sigma^3$ by $T(x) = x^*$, $T(x^*) = x$. T is a fixed point free involution on Σ^3 and $T|W = T_0$. Σ^3 is a homology sphere, as fol - lows easily from the Mayer-Vietories sequence of the triad (Σ^3, A, A^*), if we only observe that

$$H_1(W) \longrightarrow H_1(A) \oplus H_1(A^*)$$

is an isomorphism.

b) In a) we have obtained, for each choice of /
$\{\alpha_1',\ldots,\alpha_{2k}', \beta_1',\ldots,\beta_{2k}'\}$ satisfying the required conditions, an involution $(T, \sum^3)$ of a homology sphere $\sum^3$. Now, we want to make such a choice more explicitly in order to determine $\beta(T,\sum^3)$. For this, we must look at the matrices giving the change of basis. /
First, observe that the matrix U of $T_{o*} : H_1(W) \longrightarrow H_1(W)$, with respect to $\{\alpha_i, \beta_i\}$ consists of 1's the non-principal diagonal and 0's / elsewhere.

Let $\alpha_i' = \sum p_{ij} \alpha_j + \sum q_{ij}\beta_j$ and the (2k x 2k) -matrices $P = (p_{ij})$, $Q = (q_{ij})$ and $H = PUP^t - QUQ^t$ be such that PQ^t is even and symmetric and H is unimodular, that is, det (H) = $\pm$ 1. /
$\beta_i' = \sum r_{ij} \alpha_j + \sum s_{ij} \beta_j$, where $(r_{ij}) = R = H^{-1} Q U$ and $(s_{ij}) = S = H^{-1} P U$.

Then

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix}^t = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

and $\{\alpha_1',\ldots,\alpha_{2k}', \beta_1',\ldots,\beta_{2k}'\}$ satisfies the conditions in a).

The matrix of T_o is then given by

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} 0 & U \\ U & 0 \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix}^{-1} = \begin{pmatrix} 0 & H \\ H^{-1} & 0 \end{pmatrix}$$

and it follows that the matrix of $(\alpha_i' \circ T_{o*} \alpha_j')$ is H.

Associated to this choice of basis we get an involution $(T, \sum^3)$ with $\beta(T, \sum^3)$ = Sign (H). Then, the following proposition completes the proof of the theorem. (Its proof can be found explicitly in (15).

<u>Proposition</u> . For every $i \in \mathbb{Z}$, there is an integral $(2k \times 2k)$ - matrix H for some k , with sign (H) = 8i and such that

(I) H is unimodular

(II) there exist integral matrices P,Q such that PQ^t is even and symmetric and $H = PUP^t - QUQ^t$, where U denotes the / $(2k \times 2k)$ - matrix with 1's in the non principal diagonal and 0's elsewhere.

We wish to show that when i is odd, $\sum^3$ cannot be h-cobordant to S^3.

We first construct a suitable 4-manifold with $\sum^3$ as boundary and compute its signature, as well as the μ-invariant of $\sum^3$. In the following section we deal with the definition of μ.

<u>Remark</u> . In the higher dimensional case, Lopez de Medrano also cons_ tructs involutions of (4m + 3) - homotopy spheres $(m \geq 1)$, with / values of β of the form 8i, for each $i \in \mathbb{Z}$.

Here we start with W^{4m+2} $(m \geq 1)$, being the connected sum of 2k copies of $S^{2m+1} \times S^{2m+1}$ and $T_o : W \longrightarrow W$ being the restric_ tion to W of the antipodal map $T_o : S^{4m+3} \longrightarrow S^{4m+3}$.

If $\{\alpha_1,\ldots,\alpha_{2k},\beta_1,\ldots,\beta_{2k}\}$ is the standard basis of $H_{2m+1}(W)$, then we consider a new basis $\{\alpha'_1,\ldots,\alpha'_{2k},\beta'_1,\ldots,\beta'_{2k}\}$, satisfying the same conditions as before, so that the matrix of $(\alpha'_i \circ T_* \alpha'_j) = H$. We can perform framed surgery on the elements α'_i of $H_{2m+1}(W)$ and we get a framed cobordism between W and homotopy sphere. This is now diffeomorphic to a proper sphere, since it is of dimension 4m + 2, with $m \geq 1$, and bounds a π-manifold(11). We attach a disc to this sphere to get a differentiable manifold A, with $\partial A = W$. Then, if A^* is another copy of A, we form the differentiable manifold $\Sigma^{4m+3} = A \cup_T A^*$ and define $T' : \Sigma \longrightarrow \Sigma$ by $T'(x) = X^*$ and $T'(x^*) = x$, for each $x \in \Sigma'$. It follows that T' is an involution and Σ^{4m+3} is a homotopy sphere, with $\beta(T,\Sigma) = \text{sign}(H)$.

II.2 Definition of the μ-invariant (Hirzebruch (9)).

Let X be a Z/2Z - homology 3-sphere. The μ-invariant of X is defined to be

<u>Definition</u> . $\mu(X) = \frac{-\text{sign}(N)}{16}$ (reduced mod. 1 over Q), where M is given by the following lemma.

<u>Lemma</u> . Let X be as above. Then X bounds a compact connected oriented differentiable 4-manifold M such that $H_1(M)$ has no 2-torsion and the symmetric bilinear form on M (that is, the form given by cup product on $H^2(M,\partial M)$) is even. Furthermore, μ(X) is independent of the choice of M, so that μ(X) is a well-defined invariant of the diffeomorphism type of X.

<u>Proof</u>. To prove that μ(X) is well defined, suppose that X boundes M_1 and M_2 which satisfy the conditions above. Let $N = M_1 \cup_X - M_2$. Consider the Mayer-Vietoris sequence

$$H_1(X) \longrightarrow H_1(M_1) \oplus H_1(M_2) \longrightarrow H_1(N) \longrightarrow 0$$

Now, M_1 and M_2 have no 2-torsion and $H_1(X)$ is a torsion group. By Poincaré duality of homology groups, N has no 2-torsion. As the bilinear form on M_1 and M_2 are even, it follows that the bilinear form on N is also even (see the proof of the Novikov additivity property, / chapter I). Let $w_2(N)$ denote the 2^{nd} Stiefel-Whitney class of N(recall that $w_2(N)$ is the unique characteristic element in $H_2(N;\mathbb{Z}/2\mathbb{Z})$ of the form over $\mathbb{Z} / 2\mathbb{Z}$ given by cup product and such that $(x \cup x)$ $[N] = (x \cup w_2(N))$ $[N]$ for each $x \in H^2$ $(N;\mathbb{Z}/2\mathbb{Z})$. As N has no 2-torsion, the short exact sequence of coefficients,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{(x\,2)} \mathbb{Z} \longrightarrow \mathbb{Z} / 2\mathbb{Z} \longrightarrow 0$$

leads to the short exact sequence,

$$0 \longrightarrow H^2(N;\ \mathbb{Z}) \xrightarrow{(x2)} H^2(N;\mathbb{Z}) \longrightarrow H^2(N,\mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

so, in this case, the bilinear form on N being even is equivalent to $w_2(N) = 0$. Thus, we can apply Rohlin's theorem to have

$$\text{Sign}\ (N) \equiv 0 \quad (\text{mod}.16).$$

Remark . For a proof of Rohlin's theorem, see (23):

If M is a 4-manifold with $\partial M = \emptyset$ and $w_2 = 0$, then Sign (M) $\equiv 0$ (mod.16).

Again, by Novikov's additivity property, we have

$$\text{Sign}(N) \equiv \text{Sign}(M_1) - \text{Sign}\ (M_2), \text{ so}$$

$$\text{Sign}\ (M_1) \equiv \text{Sign}\ (M_2) \quad (\text{mod}.16)$$

This proves that μ is well defined.

Now, according to Milnor (19) (and Hirsch(6)) X bounds a compact simply-connected π-manifold M, in particular, $H_1(M) = 0$ and $w_2(M) = 0$. Let $N = M \cup_X -M$. From the Mayer-Vietoris sequence

$$0 \longrightarrow H^2(N,\mathbb{Z}/2\mathbb{Z}) \longrightarrow H^2(M;\ \mathbb{Z}/2\mathbb{Z}) \oplus H^2(M;\mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

and the naturality of the Stiefel-Whitney classes, we have $w_2(N) = 0$ and hence the bilinear form on N is even. As before, the bilinear / form on M is also even and the lemma is proved.

II.3 Second part of the proof of $\alpha = \beta$.

Following the notation in I.1.4 , let $(T,\tilde{M})$ be an involution on the compact oriented differentiable (4k + 3) - manifold $\tilde{M}$, and $(\mathcal{J}, \mathcal{D})$ be the Dold construction with boundary $\partial\mathcal{D} = (T,\tilde{M}) - 2(\tilde{M}/T)$. Then $\beta(T,\tilde{M})$, defined in chapter I, is the signature of the form

$$(x,y) \longrightarrow x \circ T_* y$$

defined on

$$L = \mathrm{Ker}\ \{ H_{2k+1}(\tilde{V};\mathbb{Q}) \longrightarrow H_{2k+1}(A,\mathbb{Q}) \},$$

where $H_{2k+1}(\tilde{V};\mathbb{Q}) \longrightarrow H_{2k+1}(A;\mathbb{Q})$ is induced by the inclusion /
$\tilde{V} = \partial A \subset A$.

Suppose further, that $H_{2k+2}(\tilde{M};\mathbb{Q}) = 0$.

<u>Proof of $\beta(T,\tilde{M}) = -\mathrm{sign}\ (\mathcal{D})$</u> All homology groups considered are with coefficients in $\mathbb{Q}$. Let /

$F = \pi^{-1}(M \times \{\frac{1}{2}\})$. F is a topological space, obtained from the disjoint union $A \cup A^*$ of two copies of A, by identifying all four points $x \in \tilde{V} \subset A$, $-x \in \tilde{V} \subset A$, $x^* \in \tilde{V} \subset A^*$ and $-x^* \in \tilde{V} \subset A^*$. We start on studying $H_{2k+2}(F)$, as F is a deformation retract of $\mathcal{D}$ (because $M \times \{\frac{1}{2}\}$ is a deformation retract of $M \times [0,1]$).

Let $V = V \times \{\frac{1}{2}\} \subset F$ and $A \cup A^* \subset F$ be the inclusions; they are homotopic in $\mathcal{D}$ to maps into $\mathcal{D} - F$. Let $\tilde{V} \cup \tilde{V}^*$ be the disjoint union of two copies of $\tilde{V}$ and consider the Mayer-Vietories sequence

$$H_{2k+2}(V) \oplus H_{2k+2}(A \cup A^*) \xrightarrow{\varphi} H_{2k+2}(F) \xrightarrow{\chi} H_{2k+1}(\tilde{V} \cup \tilde{V}^*)$$

$$\xrightarrow{\psi} H_{2k+1}(V) \oplus H_{2k+1}(A \cup A^*) .$$

Then $x \circ y = 0$, for $x \in \mathrm{Im}\,\varphi$ and $y \in H_{2k+2}(F)$ and hence the intersection form $(x,y) \longrightarrow x \circ y$ is well defined on

$$L' = H_{2k+2}(F) \,/\, \mathrm{Im}\, \wp \quad .$$

Its signature is then sign ($\mathcal{D}$).

Let G denote the manifold obtained from $A \cup A^*$ by identifying $x \in \tilde{V} \subset A$ with $x^* \in \tilde{V}^* \subset A^*$ and consider the following Mayer-Vietoris sequences

$$H_{2k+2}(\tilde{M}) \longrightarrow H_{2k+1}(\tilde{V} \cup \tilde{V}^*) \xrightarrow{\tilde{\psi}} H_{2k+1}(\tilde{V}) \oplus H_{2k+1}(A \cup A^*)$$

$$H_{2k+2}(G) \xrightarrow{\chi_G} H_{2k+1}(\tilde{V} \cup \tilde{V}^*) \xrightarrow{\psi_G} H_{2k+1}(\tilde{V}) \oplus H_{2k+1}(A \cup A^*)$$

In the first sequence, the homomorphism $H_{2k+1}(\tilde{V} \cup \tilde{V}^*)$ $H_{2k+1}(\tilde{V})$ is induced by the identity $\tilde{V} \longrightarrow \tilde{V}$ and by $T : \tilde{V}^* \longrightarrow \tilde{V}$, while in the second one by the identities $\tilde{V} \longrightarrow \tilde{V}$ and $\tilde{V}^* \longrightarrow \tilde{V}$.

Now

$$H_{2k+1}(\tilde{V} \cup \tilde{V}^*) \cong H_{2k+1}(\tilde{V}) \oplus H_{2k+1}(\tilde{V})$$

and so the kernel of

$$H_{2k+1}(\tilde{V} \cup \tilde{V}^*) \longrightarrow H_{2k+1}(A \cup A^*)$$

is $L \oplus L$.

Thus

$$\mathrm{Ker}\ \tilde{\psi} = \{ (a,b) \in L \oplus L \mid a + Tb = 0\}$$

$$\mathrm{Ker}\ \psi_G = \{ (a,b) \in L \oplus L \mid a + b = 0\}$$

and so

$$\mathrm{Ker}\ \psi = \{(a,b \in L \oplus L \mid a + Ta + b + T b = 0\} .$$

As $H_{2k+2}(\tilde{M} ; \mathbb{Q}) = 0$, it follows that

$$\mathrm{Ker}\ \tilde{\psi} = 0 ,$$

and

$$\text{Ker } \psi = \text{Ker } \psi_{\mathfrak{G}} = \{(a, -a) \mid a \in L\}$$

This implies that $L \cong L'$, as $L' \cong \text{Ker } \psi$.

If $f : G \longrightarrow F \subset \mathfrak{D}$ denotes the canonical map, then as $\text{Ker } \psi = \text{Ker } \psi_{\mathfrak{G}}$, any element of L' is represented by $f_*(x)$ for some $x \in H_{2k+2}(G)$.

Let $x,y \in H_{2k+2}(G)$. Then $\chi_{\mathfrak{G}}(x) =$

$= \chi(f_*(x)) = (a, -a)$ and $\chi_{\mathfrak{G}}(y) =$

$= \chi(f_*(y)) = (b, -b)$, for some $a,b \in L$.

We prove next that

$$-f_*(x) \circ f_*(y) = a \circ Tb,$$

and the proof of the theorem will be completed.

We may assume that x and y are represented by oriented $(2k-2)$- sub manifolds of G, transversal at $\tilde{V} \subset G$. $\tilde{V} \cap x$ and $\tilde{V} \cap y$ can be oriented as the boundary of $A \cap x$ and $A \cap y$ and $\tilde{V} \cap x$ and $\tilde{V} \cap y$ represent a and b, respectively. So, denote / $a = \tilde{V} \cap x$ and $b = \tilde{V} \cap y$. We may assume that a is transversal to b and Tb.

The immersion $f : G \longrightarrow \mathfrak{D}$ induces an immersion $f : x \longrightarrow \mathfrak{D}$, which represents $f_*(x) \in H_{2k+2}(\mathfrak{D})$. We can now substitute f by an isotopic immersion $f' : G \longrightarrow \mathfrak{D}$ (see (8)), so that $f' : y \longrightarrow \mathfrak{D}$ represents $f_*(y) \in H_{2k+2}(\mathfrak{D})$ and is transversal to $f : x \longrightarrow \mathfrak{D}$.

Let $p \in x$ and $q \in y$. Then

$$f(p) = f'(q) \Longleftrightarrow p = q \in a \cap b \text{ or } p = Tq \in a \cap Tb.$$

Considering the orientations involved, we get

$$-f_*(x) \circ f_X(y) = a \circ Tb + a \circ b.$$

Now $\tilde{V}$ is the boundary of the oriented manifold A and a and b are in the kernel of

$$H_{2k+1}(\tilde{V}) \longrightarrow H_{2k+1}(A) \;,$$

so that

$$a \circ b = 0.$$

Hence

$$-f_X(x) \circ f_*(y) = a \circ Tb,$$

and the result follows.

II.4 In this section we prove the following proposition

<u>Proposition</u> . Let $(T, \sum_i^3)$ be an involution of a Medrano's homology sphere $\sum_i^3$ such that $\beta(T, \sum_i^3) = 8i$ $(i \in \mathbb{Z})$.

Then $\sum_i^3$ bounds a π-manifold M_i such that Sign $(M_i) \equiv \beta(T,\sum_i^3)$ (mod.16) Furthermore, the μ-invariants of $\sum_i^3$ are

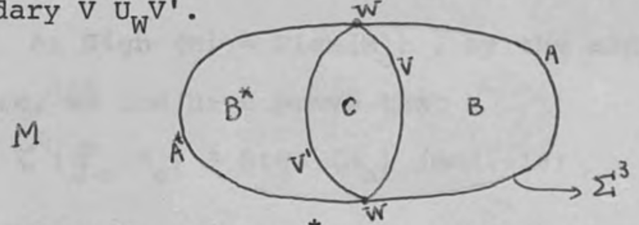
$$\mu\,(\textstyle\sum_i^3) = 1/2 \;, \text{ for i odd}$$
$$= 0 \;, \text{ for i even.}$$

<u>Remark</u>. Orlik-Rourke proved in (22) that the analgously constructed homotopy-sphere $\sum_i^{4m+3}$, $m \geq 1$, with an involution T such that $\beta(T, \sum_i) = 8i$, bounds a π-manifold M_i such that $\text{Sign}(M_i) = \beta(T, \sum_i)$.

<u>Proof of the proposition</u> . Let $(T, \sum^3)$ denote a Lopez de Medrano's example. Recall that $\sum^3 = A \cup_{T_o} A^*, \partial A = W$, where A is the mapping cilinder of $f : W \longrightarrow W$, as before(see II.1),

and $W = \partial V$, if V denotes the solid torus of genus 2k, for some integer k; $T_o : S^3 \longrightarrow S^3$ is the antipodal map.

Now, $V \cup_W A$ is a compact 3-dimensional manifold, hence it bounds a compact simply-connected π-manifold B (Milnor-Hirsch). Let V' be the closure of the complement of V in S^3 and let B^* be another copy of B glued to V' by T_o/V'. Let C be the standard 4-disc with boundary $V \cup_W V'$.



Then $M = B \cup_V C \cup_{V'} B^*$ is a compact symply-connected π-manifold with boundary $\partial M = \sum^3$.

Define $\mathcal{T} : M \longrightarrow M$ by

$\mathcal{T}(x) = x^*$, if $x \in B$,

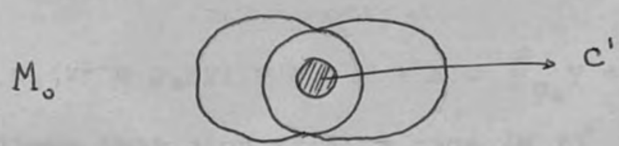
$\mathcal{T}(x^*) = x$, if $x^* \in B^*$,

$\mathcal{T}(x) = T_o(x)$, if $x \in C$.

$\mathcal{T}$ is an orientation preserving involution on M such that $\mathcal{T} \mid C$ is the antipodal map and $\mathcal{T}|\sum^3 = T$. Furthermore, $\text{Fix}\,\mathcal{T} = \{P\}$, where P is the origin of C.

Let C' be an invariant 4-disc about P contained in the interior of C and $S'^3 = \partial C'$.

Let M_o be the closure of M - C'.



$(\mathcal{T}_o, M_o)$ is a free involution on M_o, where $\mathcal{T}_o = \mathcal{T} | M_o : M_o \longrightarrow M_o$, and $\partial M_o = \Sigma^3 \cup S'^3$.

We get

$\alpha(T, \Sigma^3) = \alpha(\mathcal{T} | \partial M_o, \partial M_o) - \alpha(T_o | S'^3, S'^3) = \alpha(\mathcal{T} | \partial M_o, \partial M_o)$.

Now, Hirzebruch-Janich's formula(see 8) gives $\alpha(T, \Sigma^3) = \mathcal{C}(\mathcal{T}_o, M_o)$.

As Sign (M) = Sign(M_o) , by the additivity property of the signature, we now have prove that

$\mathcal{C}(\mathcal{T}_o, M_o) \equiv \text{Sign}(M_o) \pmod{16}$

and the result will follow.

At this point, we need a result obtained by Lopez de Medrano (13). As Fix $\mathcal{T}_o = \emptyset$, then

$$\mathcal{C}(\mathcal{T}_o, M_o) = 2\ \text{sign}\ (M_o/\mathcal{T}_o) - \text{sign}\ (M_o)\ .$$

To prove this, consider homology with real coefficients. Let $H = H_2(M_o;R) = A \oplus B$, where $A = (1+\mathcal{T}_{o_*})H$, $B = (1-\mathcal{T}_{o_*})H$. A and B are orthogonal for the intersetion form as well as for the form $f(x,y) = x \circ \mathcal{T}_{o_*} y$, and

$$f(x,y) = x \circ y \text{ on } A,$$
$$-x \circ y \text{ on } B.$$

Let $p: M_o \longrightarrow M_o/\mathcal{T}_o$ be the projection. Then $p_* | B = 0$ and $p_* | A ; A \longrightarrow H_2(M_o/\mathcal{T}_o;R)$ is an isomorphism. For $x,y \in A$, we have

$$p_*(x) \circ p_*(y) = x \circ y + x \circ \mathcal{T}_{o_*} y = 2\ x \circ y.$$

It follows that sign $(f|A)$ = sign $(M_o/\mathcal{T}_o)$, and so

$$\tau(\rho_o, M_o) = \text{sign}\,(f|A) + \text{sign}\,(f|B),$$

$$\text{sign}(M_o) = \text{sign}\,(f|A) - \text{sign}\,(f|B),$$

that is,

$$\tau(\rho_o, M_o) = 2\,\text{sign}\,(M_o/\rho_o) - \text{sign}\,(M_o).$$

Now, let $\langle\ ,\ \rangle$ denote the intersection form on M_o, that is

$$\langle x,y\rangle = x \circ y \text{ , for } x,y \in H_2(M_o) \text{ .}$$

Considering the results above , it is enough to prove that

$$\text{sign}\,(\langle\ ,\ \rangle|B) \equiv 0 \pmod{8}.$$

For this, consider the Mayer-Vietoris sequence of $(M_o;B \cup C_o, C_o \cup B^*)$, where $\varphi : B \cup C_o \to M_o$, $\varphi' : C_o \cup B^* \to M_o$ are the inclusions:

$$\begin{array}{ccccccc} o=H_2(C_o) \to & H_2(B \cup C_o) \oplus H_2(C_o \cup B^*) & \xrightarrow[\cong]{\varphi_* + \varphi'_*} & H_2(M_o) & \to H_1(C_o) = o \\ & \downarrow \rho_{o*} & & \downarrow \rho_{o*} & \\ & H_2(B \cup C_o) \oplus H_2(C_o \cup B^*) & \xrightarrow{\varphi_* + \varphi'_*} & H_2(M_o) & \end{array}$$

This diagram commutes. Note that ρ_{o*} interchanges $H_2(B \cup C_o)$ and $H_2(C_o \cup B^*)$, that is, $\rho_{o*}(u,v) = (\rho_{o*}u, \rho_{o*}v)$, for $(u,v) \in H_2(B \cup C_o) \oplus H_2(C_o \cup B^*)$. It follows that

$$\varphi'_* \rho_{o*} = \rho_{o*} \varphi_*.$$

As M_o is simply-connected, $H_1(M_o) = 0$ and $H^2(M_o)$ is free abelian (by the universal coefficient theorem); as $H_2(M_o)$ is isomorphic to $H_2(M_o, \partial M_o)$, $H_2(M_o)$ is also free abelian, by Poincaré duality. Also

$$H_2(M_o;\mathbb{Q}) \cong H_2(M_o)\otimes\mathbb{Q} .$$

Note that the same conclusions hold for $B \cup C_o$, as $H_1(\partial(B \cup C_o)) = H_2(\partial(B \cup C_o)) = 0$ (it follows from the Mayer-Vietoris sequence of $(B \cup C;B,C)$ that $\partial(B \cup C)$ is a homology 3-sphere.)

Let

$$\Delta_- = \{x \in H_2(M_o);\ \int_{o_*}(x) = -x\} .$$

If $u \in H_2(B \cup C_o)$,

$$(\varphi_* + \varphi_*^!)(u, -\int_{o_*}u) = \varphi_* u - \int_{o_*}\varphi_* u \in \Delta_- .$$

Conversely, suppose $x \in \Delta_-$ and write $x = \varphi_* u + \varphi_*' v$, for some $(u,v) \in H_2(B \cup C_o)\oplus H_2(C_o \cup B^*)$. From the diagram above, $\int_{o_*} x = -x$ implies

$$(\varphi_* + \varphi_*^!)(\int_{o_*}v, \int_{o_*}u) = (\varphi_* + \varphi_*^!)(-u,-v)$$

and thus

$$v = -\int_{o_*}u .$$

Therefore

$$\Delta_- = \{\varphi_* u - \int_{o_*}\varphi_* u \in H_2(M_o) : u \in H_2(B \cup C_o)\} .$$

We get the same result for rational coefficients, that is, if

$$\Delta_-^{\mathbb{Q}} = \{x \in H_2(M_o;\mathbb{Q}) : \int_{o_*}x = -x\}$$

then

$$\Delta_-^{\mathbb{Q}} = \{\varphi_* u - \int_{o_*}\varphi_* u \in H_2(M_o;\mathbb{Q}) : u \in H_2(B \cup C_o;\mathbb{Q})\} .$$

Let

$$\Delta_-\otimes\mathbb{Q} \to \Delta_-^{\mathbb{Q}}$$

be the inclusion. It is an isomorphism: let $x = \varphi_* u - \int_{o_*}\varphi_* u \in \Delta_-^{\mathbb{Q}}$, $u \in H_2(B \cup C_o;\mathbb{Q})$. Then

$u = \sum u_i \otimes \lambda_i$, $u_i \in H_2(B \cup C_o)$, $\lambda_i \in \mathbb{Q}$.

Thus

$$x = \sum(\rho_* u_i - \int_{o*} \rho_* u_i) \otimes \lambda_i \in \Delta_- \otimes \mathbb{Q} .$$

Therefore, we have a well defined form on Δ_-, given by $< \; , \; >|\Delta_-$.

Let $x, y \in \Delta_-$. Write $x = \rho_* u - \int_{o*} \rho_* u$, $y = \rho_* v - \int_{o*} \rho_* v$. As $\int_o$ is orientation preserving, $<x,y> = <\int_{o*} x, \int_{o*} y>$, for $x, y \in H_2(M_o)$.

Thus,

$$<x,y> = <\rho_* u - \int_{o*} \rho_* u, \rho_* v - \int_{o*} \rho_* v> =$$
$$= 2 <\rho_* u, \rho_* v> - 2 <\rho_* u, \int_{o*} \rho_* v> .$$

We can consider the form $[\; , \;]$ on Δ_- given by $[x,y] = \frac{1}{2} <x,y>$.

As Sign $([\; , \;]) =$ Sign $(< \; , \; >|\Delta_-)$, we have now to show that $[\; , \;]$ is even and unimodular. Then by Milnor (18) (see also (9, p.30)), Sign $([\; , \;]) \equiv 0$ (mod.8) as required.

To show it is unimodular, we must prove that the homomorphism dual to $[\; , \;]$,

$$\psi : \Delta_- \longrightarrow \mathrm{Hom}(\Delta_-; \mathbb{Z})$$

given by

$$\psi x \cdot y = \tfrac{1}{2} <x,y> , \text{ for } x, y \in \Delta_- ,$$

is an isomorphism.

Let

$$\partial : \Delta_- \longrightarrow \mathrm{Hom}(\Delta_-; \mathbb{Z})$$

denote the homomorphism dual to $<\ ,\ >|\Delta_-$,

$$\partial x \cdot y = <x,y> \text{ , for } x,y \in \Delta_- .$$

(similarly for $x,y \in \Delta_-^{\mathbb{Q}}$).

Consider the diagram

$$\begin{array}{ccccc} & \Delta_- & \xrightarrow{\partial} & \mathrm{Hom}(\Delta_-;\mathbb{Z}) & \\ & \cap & & \cap & \\ \Delta_-^{\mathbb{Q}} \cong & \Delta_-\otimes\mathbb{Q} & \xrightarrow{\partial} & \mathrm{Hom}(\Delta_-\otimes\mathbb{Q};\mathbb{Q}) & \cong \mathrm{Hom}(\Delta_-;\mathbb{Z})\otimes\mathbb{Q} . \end{array}$$

$\partial : \Delta_-\otimes\mathbb{Q} \to \mathrm{Hom}(\Delta_-\otimes\mathbb{Q};\mathbb{Q})$ is an isomorphism, by Poincaré duality. The diagram commutes and hence, ψ is a monomorphism.

To prove it is also an epimorphism, consider the following commutative diagram:

$$\begin{array}{ccc} H_2(M_o) & \xrightarrow{\partial} & \mathrm{Hom}(H_2(M_o);\mathbb{Z}) \\ \cup & \searrow & \downarrow \text{ restriction} \\ \Delta_- & \xrightarrow{\partial} & \mathrm{Hom}(\Delta_-;\mathbb{Z}) \end{array}$$

Since any element of $H_2\ (M_o)$ is of the form $\varphi_*u - \int_{o*}\varphi_*v$, $u,v \in H_2(B\cup C_o)$, for each $f \in \mathrm{Hom}(\Delta_-;\mathbb{Z})$, define $\tilde{f} \in \mathrm{Hom}(H_2(M_o);\mathbb{Z})$ by

$$\tilde{f}(\varphi_*u - \int_{o*}\varphi_*v) = f(\varphi_*u - \int_{o*}\varphi_*u) .$$

Thus, the restriction is onto, hence ---> is also onto.

So, given $f \in \mathrm{Hom}(\Delta_-;\mathbb{Z})$, there exists $x \in H_2(M_o)$, such that

$$f(y) = <x,y> \text{ , for all } y \in \Delta_- .$$

Now

$$<x,y> = <\int_{o*}x, \int_{o*}y> = <\int_{o*}x,-y> = -<\int_{o*}x,y> .$$

Hence

$$<x - \int_{o*}x, y> = 2<x,y>,$$

where $x - \int_{o*}x \in \Delta_-$. We get

$$\psi(x - \int_{o*}x)\cdot y = \frac{1}{2} <x - \int_{o*}x, y> = f(y),$$

for any $y \in \Delta_-$, and $x - \int_{o*}x$ maps onto f, under ψ. Therefore ψ is onto and hence it is an isomorphism.

To prove [,] is even, let $x = \int_* u - \int_{o*}\int_* u \in \Delta_-$, where $u \in H_2(B \cup C_o)$.

$$[x,x] = \frac{1}{2} < \int_* u - \int_{o*}\int_* u, \int_* u - \int_{o*}\int_* u> =$$

$$= < \int_* u, \int_* u> - < \int_* u, \int_{o*}\int_* u>.$$

Now, let $x_1,\ldots,x_e$ be a basis for $H_2(M_o)$ represented by immersed spheres S_i^2. Then $\int_o(S_i^2)$ represents $\int_{o*}x_i$. By general position, we can arrange such that S_i^2 and $\int_o(S_i^2)$ meet transversely in a finite set of points. $S_i^2 \cap \int_o(S_i^2)$ is invariant under $\int_o$ and $\int_o$ is fixed-point free, so S_i^2 and $\int_o(S_i^2)$ must intersect in an even number of points. Therefore $<x_i, \int_{o*}x_i>$ is even. On the other hand, as the form < , > is even, as in II.2, it follows that $<x_i,x_i>$ is even.

Therefore [x,x] is also even, as required.

This proves that Sign ([,]) and hence Sign (< , >|Δ_-) is a multiple of 8, and thus

$$\tau(\int_o, M_o) \equiv \text{sign}(M_o) (\text{mod.}16) .$$

Therefore

$$\beta(T,\Sigma^3) = \alpha(T,\Sigma^3) \equiv \text{Sign}(M) (\text{mod. } 16),$$

and the first part of the proposition is proved.

Now, the μ-invariants of Σ_i^3 are given by

$$\mu(\Sigma_i^3) = \frac{-\text{Sign}(M_i)}{16} (\text{mod.}1) = \frac{-\beta(T,\Sigma_i^3)}{16} (\text{mod.}1)$$

$$= 1/2 \text{ , for } i \text{ odd}$$

$$0 \text{ , for } i \text{ even.}$$

This completes the proof of the proposition.

II.5 Now, we are able to prove the following theorem.

Theorem. If $\beta(\tau, \Sigma^3)/8$ is odd, where (τ, Σ^3) is one of Medrano's example, then Σ^3 cannot be h-cobordant to S^3. In particular, Σ does not imbed in R^4.

Remark. We say that X_1 is h-cobordant to X_2, where X_1 and X_2 are n-manifolds (compact oriented differentiable) without boundary, if there is an (n+1)-manifold W such that ∂W is the disjoint union of X_1 and $-X_2$ and the inclusions $X_1 \subset W$, $X_2 \subset W$ are both homotopy equivalences. In this case, we say that $(W; X_1, X_2)$ is an h-cobordism.

Lemma. (Hirzebruch) Let X_1 and X_2 be h-cobordant $\mathbb{Z}/2\mathbb{Z}$-homology 3-spheres. Then $\mu(X_1) = \mu(X_2)$.

Proof. Let $(W; X_1, X_2)$ be an h-cobordism and let $X_2 = \partial M_2$, where M_2 is given by the Lemma in II.2, that is, $H_1(M_2)$ has no 2-torsion and the symmetric bilinear form on M_2 is even. Let $M_1 = M_2 \cup_{X_2} W$.

Then $X_1 = \partial M_1$ and M_2 is a deformation retract of M_1.
This implies that $H_q(M_2) \cong H_q(M_1)$ and the isomorphism preserves the bilinear form, so that M_1 satisfies the required conditions in II.2 and

$$\mu(X_1) = -\frac{\text{sign}(M_1)}{16} \pmod{1} = -\frac{\text{sign}(M_2)}{16} \pmod{1} = \mu(X_2) .$$

Proof of the theorem. The first part now follows from the proposition in section II.3 and the Lemma above, if we only note that $\mu(S^3) = 0$. Suppose now that $\sum^3$ denotes a Medrano's example which is embeddable in $\mathbb{R}^4$. By Alexander duality, $\mathbb{R}^4 - \sum^3$ has two components. Let V be the bounded component, then $W = V \cup \sum^3 \subset \mathbb{R}^4$ is a compact manifold with boundary $\partial W = \sum^3$. Let V' be the umbounded component and let $W' = V' \cup \sum^3$. Then $(W,\sum^3)$ is a strong deformation retract of $(W \cup U, U)$, where $\sum^3$ is a deformation retract of an open neighbourhood $U \subset W'$. This implies that

$$H_q(W,\textstyle\sum^3) \cong H_q(W \cup U,U) \cong H_q(\mathbb{R}^4,W') ,$$

and we get the Mayer-Vietoris sequence of $(\mathbb{R}^4;W,W')$,

$$H_q(\textstyle\sum^3) \to H_q(W) \oplus H_q(W') \to H_q(\mathbb{R}^4) .$$

Hence, $H_2(W) = H_1(W) = 0$, W satisfies the conditions in II.2 and, as $\mathrm{Sign}(W) = 0$, it follows that $\mu(\sum^3) = 0$. This completes the proof of the theorem.

We have shown that when $\beta(T,\sum^3)/8$ is odd, then the homology sphere $\sum^3$ is not h-cobordant to S^3. Now, $\pi_1(\sum^3) = 0$ would imply that $\sum^3$ were a homotopy sphere. Hence, the interest of the theorem relies in the fact that, if it could be shown that all homotopy spheres were h-cobordant to S^3, then $\pi_1(\sum^3) \neq 0$. The problem is that it may be fake 3-spheres which do not bound contractible 4-manifolds. Kervaire and Milnor (11) have proved that the groups θ_n $(n \neq 3)$ of h-cobordism classes of homotopy spheres are all finite. The case n = 3 is open, but if we assume, conversely, the Poincaré hypothesis, we would have $\theta_3 = 0$.

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