HAMILTONIANS , DERIVATIONS AND OPERATOR ALGEBRAS

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ABSTRACT

This thesis falls into two parts. In Part A, Tomita-Takesaki theory is extended to the unbounded CCR-algebra $\mathcal{A}$ in infinitely many degrees of freedom. A particular Hamiltonian with polynomially bounded spectrum defines the Gibbs state ω_β on $\mathcal{A}$. It is shown that $\mathcal{A}$ admits a modular operator and that ω_β is a KMS-state with respect to the modular automorphism. In the GNS-representation induced by ω_β, the commutant $\pi_\beta(\mathcal{A})'$ is shown to satisfy the conclusion of Tomita's theorem. This is done by constructing another representation of $\mathcal{A}$ - on the Hilbert-Schmidt operators - for which Tomita's result is known.

In Part B, perturbations of dynamical systems are considered. For a C*-dynamical system $(\mathcal{A},\alpha)$ with generator δ and $\mathcal{A}\subset\mathcal{B}(\mathcal{H})$, δ is perturbed by a derivation Δ on $\mathcal{A}$ and it is shown that $\delta+\Delta$ generates an automorphism group of $\mathcal{A}$ if Δ is inner, and an automorphism group of $\mathcal{A}''$ if Δ is polynomially relatively bounded. Finally, a result by Buchholz and Roberts on bounded perturbations is generalised. For two W*-dynamical systems $(\mathcal{M},\alpha)$, $(\mathcal{M},\beta)$ with generators δ_α and δ_β, respectively, it is shown that, under a local commutativity condition on α and β, the norm-proximity of α and β on $\mathcal{D}(\delta_\alpha)\cap\mathcal{D}(\delta_\beta)$ is described in terms of the operator $\delta_\beta\Gamma-\Gamma\delta_\alpha$ on $\mathcal{D}(\delta_\alpha)\cap\mathcal{D}(\delta_\beta)$, where Γ is a linear operator mapping $\mathcal{D}(\delta_\alpha)$ into $\mathcal{D}(\delta_\beta)$.

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Zur Erinnerung an meinen Vater,

der das Ende dieser Arbeit

gerne miterlebt hätte.

INTRODUCTION

This thesis deals with some aspects of operator algebra, a field which is growing very rapidly at present.

The thesis is organised in the following way. In Chapter 1 we follow standard textbooks to establish notation and review basic results. Theorems, propositions etc. taken from other authors are clearly marked by a specific reference to the source; all other theorems, propositions etc. are due to the author. The references are listed at the end of the thesis. The thesis is then divided into two parts which deal with different aspects of operator algebras.

In Part A (consisting of Chapters 2 and 3) we consider the CCR-algebra - an algebra which consists of unbounded operators - and show that Tomita's theorem can be generalised to such an algebra in infinitely-many degrees of freedom. This work is a generalisation of results obtained by Gudder and Hudson [G&H] . The results described in Part A are published (see [K&K]) and represent joint work with my fellow student Aristides Katavalos; they formed a major part of his Ph.D. thesis, which was accepted by the University of London in 1977.

In Part B we are concerned with C*- and W*-dynamical systems and consider questions of stability of such systems when the dynamics are perturbed by derivations and Hamiltonians. The results presented in this Part I obtained by myself in 1977-78 and 1983-85. I arrived at the proofs presented in section 4.2 independently of Buchholz Roberts [BuR] in 1977-78 while I was a student at Bedford College. Although their approach to the problem was somewhat different from mine, the results and methods of proof are so similar - both using cocycles - that I could not publish my results. I was interested in showing when a perturbed derivation is the generator of an automorphism group of a given C*-algebra, while Buchholz and Roberts characterised the closeness in norm of two automorphism groups of a W*-algebra. Their paper is reviewed briefly in section 4.4. In section 4.3 we present an alternative way - using the Trotter product formula - of dealing with the problems raised above, and we show that this method can be applied to polynomially relatively bounded as well as bounded perturbations of generators. The last section of Part B is a generalisation of the results of Buchholz and Roberts which I proved in 1983-5.

Finally, I would like to express my thanks to my supervisor Ray Streater for suggesting this research and for his help and guidance. I also want to thank Aristides Katavalos, Françoise Debacker-Mathot and Larry Landau for many useful discussions; and I want to thank the British Council, who supported me as a fellow during 1975-78. Furthermore, I would like to thank the Department of Mathematics of the Australian National University, and in particular Gert Pedersen and Derek Robinson, for their generous hospitality in 1983. In addition I want to thank Bob Anderssen and Terry Speed of CSIRO, Division of Mathematics and Statistics, for their encouragement and support.

Lastly, I want to thank my parents and Alun for their understanding and help throughout this thesis.

Chapter 1

PRELIMINARIES AND BASIC RESULTS

1.1 OPERATORS ON A HILBERT SPACE

Let H denote a complex Hilbert space with inner product $<.,.>$ which is linear in the first component. A *linear operator* x on H is a linear map from a subspace $\mathcal{D}(x)$ in H, called its *domain*, into H. In general we shall be concerned with those linear operators whose domain is dense in H. Let $B(H)$ denote the set of bounded linear operators whose domain equals H. Then $x \in B(H)$ if and only if x is continuous. For a densely defined linear operator x on H, its *adjoint* x^* is a linear operator on H with domain

$$\mathcal{D}(x^*) = \{ \eta \in H : \xi \longmapsto <x\xi,\eta> \text{ is continuous on } \mathcal{D}(x) \}$$

and such that $<x\xi,\eta> = <\xi,x^*\eta>$ for $\xi \in \mathcal{D}(x)$, $\eta \in \mathcal{D}(x^*)$.

The *graph* $G(x)$ of a linear operator x on H is the subspace in $H \times H$ of all ordered pairs $(\xi,x\xi)$, for $\xi \in \mathcal{D}(x)$. A linear operator x on H is *closed* if its graph is a closed subspace in $H\times H$. This is equivalent to the following: if (ξ_n) is a sequence in $\mathcal{D}(x)$ which converges in H, and if $(x\xi_n)$ also converges in H, then

$$\lim_{n\to\infty} \xi_n \in \mathcal{D}(x) \quad \text{and} \quad x(\lim_{n\to\infty} \xi_n) = \lim_{n\to\infty} x\xi_n . \tag{1.1}$$

By the closed graph theorem, $x \in B(H)$ if and only if $\mathcal{D}(x) = H$ and x is closed.

Let x, y be two linear operators on H with domains $\mathcal{D}(x)$ and $\mathcal{D}(y)$ respectively. We say, y is an *extension* of x, denoted by $x \subseteq y$ if $\mathcal{D}(x) \subseteq \mathcal{D}(y)$ and if $y\xi = x\xi$ for $\xi \in \mathcal{D}(x)$. Furthermore, x is called *closable* if it can be extended to a closed operator, which is then denoted by $\bar{x}$. One can add and multiply linear operators, but care has to be taken with the domains involved:

1. $x + y$ is the operator defined on $\mathcal{D}(x) \cap \mathcal{D}(y)$.
2. xy is the operator with domain $\mathcal{D}(xy) = \{ \xi \in \mathcal{D}(y) : y\xi \in \mathcal{D}(x) \}$.

3. $x^*y^* \subseteq (yx)^*$ and equality holds if $y \in B(H)$.

In the sequel we shall encounter the following linear operators on H. We say x is *symmetric* if $x \subseteq x^*$, and x is *selfadjoint* if $x = x^*$. Furthermore, a symmetric operator x is *essentially selfadjoint* if $\bar{x}$ is selfadjoint. If $x \in B(H)$, then the three concept coincide. A selfadjoint operator x is *positive* if there exists a selfadjoint operator y on H such that $x = y^2$. An operator u is *isometric* if $\|u\xi\| = \|\xi\|$ for $\xi \in \mathcal{D}(u)$, and u is *unitary* if $u \in B(H)$ and $u^*u = uu^* = 1$. Finally, a bounded operator p is called a *projection* if $p^2 = p$.

Amongst the bounded operators we shall also meet two particular subclasses of operators. If $x \in B(H)$ is positive, the *trace* of x is

$$\mathrm{tr}(x) = \sum_{i\geqslant 1} \langle x\xi_i, \xi_i\rangle \tag{1.2}$$

where $\{\xi_i\}$ are an orthonormal basis for H. If $x \in B(H)$ is not positive, the element $|x| = |xx^*|^{1/2}$ is positive and the trace can be defined for $|x|$. The set of *trace class operators*, $T(H)$, consists of those $x \in B(H)$ for which $\mathrm{tr}(|x|) < \infty$; and $x \in B(H)$ belongs to $HS(H)$, the set of *Hilbert-Schmidt operators*, if $\mathrm{tr}(x^*x) < \infty$. Some of the important properties of the trace are: the trace is independent of the basis; $\mathrm{tr}(x^*x) = \mathrm{tr}(xx^*)$ and tr also defines an inner product on the space $HS(H)$ by

$$\langle x, y\rangle_{\mathrm{HS}} = \mathrm{tr}(y^*x) \tag{1.3}$$

which makes $HS(H)$ into a Hilbert space. Furthermore, the following relationship holds for trace class and Hilbert-Schmidt operators.

THEOREM 1.1 ([G&V] Ch. I.2.2 thm.3)

1. $x \in HS(H)$ *if and only if* $\sum_{n\geq 1} \lambda_n^2 < \infty$ *where the* λ_n *are the eigen-*

values of x.

2. $x \in T(H)$ *if and only if* $\sum_{n\geq 1} \lambda_n < \infty$, *where the* λ_n *are the eigenvalues of* x.

3. $x \in HS(H)$ *if and only if* $x\xi = \sum_{n\geq 1} \lambda_n \langle \xi, \xi_n \rangle \eta_n$ (1.4)

   where $\{\xi_n\}$ *and* $\{\eta_n\}$ *are orthonormal sets in* H, *and* $\lambda_n \in l^2(\mathbb{R}_+)$.

4. *For every* $x, y \in HS(H)$, *the product* xy *is trace class, and every* $x \in T(H)$ *can be written as the product of two* $y, z \in HS(H)$.

5. $T(H)$ *is a two-sided ideal in* $B(H)$.

6. $T(H) \subseteq HS(H) \subseteq B(H)$.

Now we return to unbounded selfadjoint linear operators which possess some important properties. Let X be a linear operator on H, and denote by $sp(X)$ its *spectrum*, that is, the set of $\lambda \in \mathbb{C}$ for which $(X - \lambda 1)$ is not invertible. Note, if X is selfadjoint, then $sp(X) \subseteq \mathbb{R}$.

Let $\mathcal{M}$ denote a σ-algebra in $\mathbb{R}$, then we call a map $E: \mathcal{M} \to B(H)$ a *projection-valued measure* if

1. $E(\emptyset) = 0$ and $E(\mathbb{R}) = 1$
2. For $\omega \in \mathcal{M}$, $E(\omega)$ is a selfadjoint projection
3. For $\omega_1, \omega_2 \in \mathcal{M}$, $E(\omega_1 \cap \omega_2) = E(\omega_1)E(\omega_2)$
4. If $\omega_1 \cap \omega_2 = \emptyset$, $\omega_1, \omega_2 \in \mathcal{M}$, then $E(\omega_1 \cup \omega_2) = E(\omega_1) + E(\omega_2)$
5. For $\xi, \eta \in H$, the function $E_{\xi,\eta}$ defined by

   $$E_{\xi,\eta}(\omega) = \langle E(\omega)\xi, \eta \rangle$$

   (for $\omega \in \mathcal{M}$) is a complex measure on $\mathcal{M}$.

Using these measures, we get the famous spectral theorem whose origin goes back as far as Hilbert for bounded operators.

THEOREM 1.2 ([Rud] thm. 13.30)
For each selfadjoint operator x *on* H *there exists a unique projection valued measure* E *such that*

$$\langle x\xi,\eta\rangle = \int_{\mathbb{R}} t \, dE_{\xi,\eta}(t) \tag{1.5}$$

for $\eta \in H$, $\xi \in \mathcal{D}(x)$. *E is concentrated on* $sp(x)$. *Furthermore, if f is a real-valued E-measurable function then*

$$f(x) = \int_{\mathbb{R}} f(t)dE(t) . \tag{1.6}$$

If the operator x is also bounded, then its spectrum lies in the interval $[-\|x\|,\|x\|]$.

We conclude this section with another relationship between selfadjoint and unitary operators. Let $\{u_t : t \in \mathbb{R}\}$ denote a family of unitary operators on H. We call $u = \{u_t\}$ a *strongly continuous one-parameter group of unitaries* or a *unitary representation of* $\mathbb{R}$ *into the unitary group* $U(H)$ *of* $B(H)$ if

1. $u_o = 1$
2. $u_{t+s} = u_t u_s$ and $u_t^* = u_{-t}$ $(s, t \in \mathbb{R})$
3. the map $u: \mathbb{R} \to U(H)$, $t \mapsto u_t$, is continuous in the strong topology

of $B(H)$. (See section 1.2 for the definition of the strong topology.) The group u is *norm continuous* or *uniformly continuous* if requirements 1 and 2 above hold and 3 is replaced by

3: the map $t \mapsto u_t$ is continuous in the topology of $B(H)$ which is induced by the sup norm.

THEOREM 1.3 (Stone) ([Rud] thm. 13.37 , [B&RI] props. 3.1.1, 3.1.6)
Let $\{u_t\}$ *be a strongly continuous one-parameter group of unitaries on* H . *Then there exists a unique selfadjoint operator* x *on* H *such that*

$$u_t = e^{itx} \quad \textit{for } t \in \mathbb{R}, \tag{1.7}$$

and $\mathcal{D}(x)$ *is dense in* H *.*

Furthermore, if u_t *is norm continuous, then the selfadjoint operator* x *corresponding to* u *via (1.7) is bounded and defined everywhere on* H*.*

The selfadjoint operator x is often called the *generator* of the unitary group u.

1.2 *-ALGEBRAS, FUNCTIONALS AND REPRESENTATIONS

Since we shall be concerned with algebras of bounded as well as unbounded operators, we begin with Power's [Pow] definition of a *-algebra.

A **-algebra* A is an algebra over the complex field with a map $x \longmapsto x^* \in A$, called the *involution*, such that

1. $(x^*)^* = x$
2. $(\alpha x + y)^* = \bar{\alpha}x^* + y^*$ for $x,y \in A$, $\alpha \in \mathbb{C}$
3. $(xy)^* = y^*x^*$

In part A of this thesis, we shall be interested in a special class of *-algebras, the algebra of the canonical commutation relations (CCR) which is described here for a finite number, n, of degrees of freedom. Let $H = L^2(\mathbb{R}^n)$; on this Hilbert space, we consider two actions.

1. The *position operators* q_i are defined by $(q_i f)x = x_i f(x)$ for $f \in L^2(\mathbb{R}^n)$ such that $x_i f \in L^2(\mathbb{R}^n)$, where $x = (x_1,\ldots,x_n)$.
2. The *momentum operators* p_i are defined by $(p_i f)x = -i\frac{d}{dx_i} f(x)$ for $f \in L^2(\mathbb{R}^n)$ such that the partial derivatives are in $L^2(\mathbb{R}^n)$ $(i = 1,.,n)$.

The *CCR-algebra* A *in n degrees of freedom* is the free non-commutative algebra of all polynomials in p_i and q_i $(i=1,\ldots,n)$ subject to the relations:

$$[p_i,p_j] = [q_i,q_j] = 0 \qquad \text{and} \qquad [q_i,p_j] = i\delta_{ij}1 . \tag{2.1}$$

The set of bounded operators on H, $B(H)$, can be equipped with a *-algebra structure as well as topologies different from the norm or uniform topology referred to in the last section. For $\xi \in H$, $x \mapsto \|x\xi\|$ is a seminorm on $B(H)$. The *strong topology* is the locally convex topology on $B(H)$ defined by these seminorms. For $\xi, \eta \in H$, $x \mapsto |\langle x\xi,\eta\rangle|$ is a seminorm on $B(H)$. The *weak topology* is the locally convex topology which is defined by these seminorms. For sequences (ξ_n) , (η_n) in H which satisfy $\sum \|\xi_n\|^2 < \infty$ and $\sum \|\eta_n\|^2 < \infty$, $x \mapsto \sum |\langle x\xi_n , \eta_n \rangle|$ defines a seminorm on $B(H)$. The *σ-weak or ultraweak topology* on $B(H)$ is the locally convex topology which is defined by these seminorms. On the unit ball of $B(H)$ these three topologies coincide; in general, however, they are not equivalent, and they are all weaker than the uniform topology on $B(H)$. The weak topology is weaker than the strong and σ-weak topology, but the latter two are not usually comparable. There exists a number of other locally convex topologies on $B(H)$, but we shall not be concerned with them here.

Let N be a subset of $B(H)$. The *commutant* N' of N is defined by $N' = \{ x \in B(H) : \forall y \in N \quad xy = yx \}$. The *double commutant* N'' is analogously defined as the commutant of N' , and $N \subseteq N''$.

A *von Neumann algebra* or *W*-algebra* on H is a *-subalgebra M in $B(H)$ such that $M = M''$.

THEOREM 2.1 (Bicommutant theorem) ([Ped] thm. 2.2.2)
*For a *-subalgebra M in $B(H)$ the following are equivalent:*

1. $M = M''$
2. *M is σ-weakly closed*
3. *M is weakly closed*
4. *M is strongly closed.*

Without reference to an underlying Hilbert space, one can define a *-algebra. A *C*-algebra* A is a Banach space which is also a *-algebra such that multiplication and involution of A are compatible with the Banach space norm of A , that is, $\|xy\| \leq \|x\|\,\|y\|$ and $\|x^*x\| = \|x\|\,\|x^*\| = \|x\|^2$ for $x, y \in A$.

Amongst the large class of linear functionals $\phi: A \to \mathbb{C}$, an interesting subclass consists of the 'states' which represent the states of a system or the dynamics in quantum mechanics. Let A be a *-algebra with unit 1. A linear functional ϕ is a *state* on A if ϕ is positive (i.e. $\phi(x^*x) \geqslant 0$) and $\phi(1) = 1$. A state ϕ on A is *faithful* if $\phi(x) \neq 0$ for every non-zero $x \in A$, and a state ϕ is *pure* if it cannot be written as a convex combination of other states.

Other physically relevant states such as the ground state or the equilibrium states of a system are usually density matrices, that is, "statistical mixtures" of pure states. For a von Neumann algebra M, these states can be characterised explicitly: put

$$M_* = \{ \phi : \phi = \sum_{n\geqslant 1} <.\xi_n, \eta_n> , \xi_n, \eta_n \in H, \sum_{n\geqslant 1} \|\xi_n\|^2 < \infty \sum_{n\geqslant 1} \|\eta_n\|^2 < \infty \}.$$

M_* is the space of all *σ-weakly continuous linear functionals* on M. Let M_+ denote the set of positive elements in M. A state ϕ on M is *normal* if for each increasing net (x_i) in M_+ with upper bound x, $\phi(x_i)$ converges to $\phi(x)$.

By [B&RI] prop. 2.4.18, M_* is a closed subspace of M^*, the Banach space dual of M, and M is isomorphic to the Banach space dual of M_*. For this reason, M_* is called the *predual* of M and it can be characterised in the following way.

THEOREM 2.2 ([B&RI] thm. 2.4.21)

Let ϕ be a state on the von Neumann algebra $M \subseteq B(H)$. The following are equivalent:

1. *ϕ is normal*
2. *ϕ is σ-weakly continuous*
3. *there exists a positive trace class operator $y \in T(H)$ with $tr(y) = 1$ such that $\phi(x) = tr(yx)$ for every $x \in M$.*

With every *-algebra A and every state ϕ on it, one can associate a representation of A on some Hilbert space (see thm. 2.3 below), but representations (that is, particular realisations of A) can also be defined abstractly without reference to a state on A.

A **-representation* $(\pi, H, \mathcal{D}_\pi)$ of a *-algebra A with unit is a *-algebra homomorphism π of A into linear operators on H which are defined on a common domain $\mathcal{D}_\pi$ that is dense in H such that

1. $\pi(1) = 1$,
2. $<\pi(x)\xi, \eta> = <\xi, \pi(x^*)\eta>$ for $\xi, \eta \in \mathcal{D}_\pi$, $x \in A$
3. $\pi(x)\mathcal{D}_\pi \subseteq \mathcal{D}_\pi$ for every $x \in A$.

If A is a C*-algebra or a von Neumann algebra, $\mathcal{D}_\pi = H$ (so abridge $(\pi, H, \mathcal{D}_\pi)$ to (π, H)) , and $\pi(A)$ is a *-subalgebra of $B(H)$ for which $\pi(x)^* = \pi(x^*)$. Note that 2. above implies that $\pi(x)^* \supseteq \pi(x^*)$. Since we are only interested in those representations which preserve the involution as in 2., it suffices to refer to the *-representations used here simply as *representations*.

Because of the added complications involved in the domain of general *-algebras, we concentrate on C*-algebras and von Neumann algebras first. Unless specified differently, A will denote a C*- or a W*-algebra. A representation (π, H) of A is *cyclic* if there is a $\xi \in H$ such that $\pi(A)\xi$ is dense in H. The vector ξ is called *cyclic* and the cyclic representation will be denoted by (π, H, ξ).

THEOREM 2.3 (Gelfand-Naimark-Segal) ([B&RI] thm. 2.3.16)
Let A be a C-algebra with 1 or a W*-algebra. For every positive linear functional ϕ on A there exists a cyclic representation $(\pi_\phi, H_\phi, \xi_\phi)$ of A such that*

$$\phi(x) = <\pi_\phi(x)\xi_\phi , \xi_\phi> \qquad \textit{for } x \in A$$

The cyclic representation corresponding to ϕ *is unique up to unitary equivalence, that is, if* (π_i, H_i, ξ_i) *i=1,2 are two cyclic representations of* A *such that*

$$\langle\pi_1(x)\xi_1,\xi_1\rangle = \phi(x) = \langle\pi_2(x)\xi_2,\xi_2\rangle \qquad (x \in A),$$

then there exists a unitary $u : H_1 \to H_2$ *such that* $u\xi_1 = \xi_2$ *and* $u\pi_1(x) = \pi_2(x)u$ *for* $x \in A$.

Such a cyclic representation is called the *GNS-representation* of A associated with ϕ. A vector $\xi \in H$ is *separating* for $\pi(A) \subseteq B(H)$, if $\pi(x)\xi \neq 0$ for $\pi(x) \neq 0$. If ϕ is a faithful state, then A is isomorphic to $\pi(A)$, and the cyclic vector associated with the GNS-representation is separating for $\pi(A)$.

THEOREM 2.4 ([B&R1] prop. 2.5.3)
Let $M \subseteq B(H)$ *be a von Neumann algebra. A vector* $\xi \in H$ *is cyclic for* M *if and only if* ξ *is separating for the commutant* M'.

The pure states played a special role amongst the linear functionals and they will also give rise to a special class of representations.

THEOREM 2.5 ([B&R1] thm. 2.3.19)
Let A *be a* C^**-algebra.* φ *is a pure state on* A *if and only if the GNS-representation* (π_φ, H_φ) *induced by* φ *is irreducible, that is, the commutant* $\pi_\varphi(A)'$ *consists of multiples of the identity only.*

We now return to representations of *-algebras. If $(\pi, H, \mathcal{D}_\pi)$ is a representation of a *-algebra A , the *bounded commutant* $\pi(A)'$ of $\pi(A)$ is the set of all $y \in B(H)$ such that

$$\langle y\pi(x)\xi, \eta\rangle = \langle y\xi, \pi(x^*)\eta\rangle \tag{2.2}$$

for $\xi, \eta \in \mathcal{D}_\pi$ and $x \in A$.

As in the case of unbounded operators where the concept of a closed operator was introduced in order to compensate partially for the lack in continuity, for *-algebras the concept of a closed representation is required for a GNS-type theorem. Let $(\pi, H, \mathcal{D}_\pi)$ be a representation of A. There is a natural induced topology on $\mathcal{D}_\pi$ which is defined as follows. Let S be a finite set of elements in A, and define a seminorm $\|.\|_S$ on $\mathcal{D}_\pi$ by

$$\|\xi\|_S = \sum \|\pi(x)\xi\| \tag{2.3}$$

where the sum is taken over the elements $x \in S$. The *SN -topology* or the *induced topology* on $\mathcal{D}_\pi$ is the topology which is induced by these seminorms. Note that $\pi(x)$ becomes a continuous operator from $\mathcal{D}_\pi$ into $\mathcal{D}_\pi$ in the SN-topology for every $x \in A$. Furthermore, a representation $(\pi, H, \mathcal{D}_\pi)$ of A is *closed* if $\mathcal{D}_\pi$ is complete in the SN-topology, and a vector $\xi \in \mathcal{D}_\pi$ is *strongly cyclic* for $\pi(A)$ if $\pi(A)\xi$ is dense in $\mathcal{D}_\pi$ in the SN-topology. A representation $(\pi, H, \mathcal{D}_\pi)$ of A is then called *strongly cyclic* if there is a vector $\xi \in \mathcal{D}_\pi$ which is strongly cyclic for $\pi(A)$.

The concepts of faithful and pure states are the same for *-algebras as for C*- and W*-algebras, since they do not depend on topologies. The GNS-theorem can now be stated for *-algebras in the following way.

THEOREM 2.6 ([Pow] thm. 6.3)
*For each state ϕ of a *-algebra A with unit there is a closed strongly cyclic representation $(\pi, H, \mathcal{D}_\pi)$ of A with strongly cyclic vector $\xi \in \mathcal{D}_\pi$ such that $\phi(x) = \langle \pi(x)\xi, \xi \rangle$ for $x \in A$. This representation is determined by ϕ up to unitary equivalence, and it is irreducible if and only if ϕ is a pure state.*

1.3 DERIVATIONS, AUTOMORPHISM GROUPS AND THEIR GENERATORS

Unless otherwise specified, A will denote a C*-algebra or a von Neumann algebra in this section.

A **-derivation* (or for short a *derivation*) δ is a linear operator from a *-subalgebra $\mathcal{D}(\delta)$ in A, its domain, into A which satisfies

$$\begin{aligned} &1.\quad \delta(x^*) = \delta(x)^* \\ &2.\quad \delta(xy) = \delta(x)y + x\delta(y) \qquad \text{for } x,y \in \mathcal{D}(\delta) \\ &3.\quad \delta(1) = 0 \quad \text{if } 1 \in \mathcal{D}(\delta). \end{aligned} \tag{3.1}$$

In general we shall assume that $1 \in \mathcal{D}(\delta)$. In the case of a C*-algebra with unit, this follows by [B&RI] p.238.

Derivations are linear operators, and usually we shall assume that they are densely defined, that is, if δ is a derivation on a C*-algebra A then $\mathcal{D}(\delta)$ is norm-dense in A, and if δ is a derivation on a von Neumann algebra, we shall assume that $\mathcal{D}(\delta)$ is σ-weakly-dense in A. Corresponding to the bounded operators on a Hilbert space, we say a derivation δ is *bounded* if $\mathcal{D}(\delta) = A$ and δ is a bounded linear operator on A or equivalently if δ is continuous and defined everywhere on A.
Since derivations are linear operators, the notions of graph, closedness and closability apply to derivations in an analogous way, where H is replaced by A in the definition, and the σ-weak topology is used everywhere instead of the norm topology if one is dealing with derivations on von Neumann algebras.

A derivation δ on $A \subseteq B(H)$ is called *spatial* if there exists a symmetric operator H on H with domain $\mathcal{D}(H)$ such that

$$\begin{aligned} &1.\quad \text{for } x \in \mathcal{D}(\delta),\quad x\mathcal{D}(H) \subseteq \mathcal{D}(H) \\ &2.\quad \delta(x) = i[H,x] \qquad \text{for } x \in \mathcal{D}(\delta) \text{ on } \mathcal{D}(H). \end{aligned} \tag{3.2}$$

If H and δ satisfy the above relationship, we say H *implements* δ.

If a derivation is bounded, then it is always spatial (see [B&RI] cor. 3.2.47), while there exist unbounded derivations which are not spatial. One criterion for a derivation to be spatial is the existence of a δ-*invariant* state ϕ , that is, $\phi(\delta(x)) = 0$ for $x \in \mathcal{D}(\delta)$. Via the GNS-representation $(\pi, \mathcal{H}, \xi)$ induced by ϕ this implies (by [B&RI] prop. 3.2.28) that there exists a symmetric operator H on $\mathcal{H}$ which satisfies

$$
\begin{aligned}
&1.\quad \pi(\delta(x)) = i[H, \pi(x)] \quad \text{on } \mathcal{D}(H) \qquad \text{for } x \in \mathcal{D}(\delta)\,. \\
&2.\quad H\xi = 0.
\end{aligned}
\tag{3.3}
$$

As we shall see in thm. 3.2 below, derivations also play an important role as generators of automorphism groups, to which we turn now.

Let aut(A) denote the set of *-automorphisms of A. A map $\alpha: \mathbb{R} \to \text{aut}(A)$, $t \mapsto \alpha_t$, is called a *one-parameter group of *-automorphisms of* A if

$$
\begin{aligned}
&1.\quad \alpha_{t+s} = \alpha_t \alpha_s \quad \text{for } t, s \in \mathbb{R} \\
&2.\quad \alpha_0 = \text{id} \quad \text{the identity automorphism.}
\end{aligned}
\tag{3.4}
$$

For brevity, a one-parameter group of *-automorphisms of A will often be referred to as an *automorphism group* of A and the pair (A, α) is called a *dynamical system*.

Often one will require that α satisfies some continuity assumptions. We shall be concerned with the following concepts. The automorphism group α on A is *norm continuous* if the map $t \mapsto \alpha_t(x)$ is continuous in the norm topology of A for every $x \in A$. The pair (A, α) is called a *C*-dynamical system* if A is a C*-algebra and α is pointwise norm continuous, that is, for every $x \in A$, the map $\alpha(x): \mathbb{R} \to A$, $t \mapsto \alpha_t(x)$, is continuous in the norm topology. Lastly, a pair (A, α) is a *W*-dynamical system* if A is a von Neumann algebra and α is σ-weakly continuous, that is, for $x \in A$, the map $\alpha(x): \mathbb{R} \to A$ is σ-weakly continuous.

A *covariant representation* of a C*-(respectively W*-) dynamical system (A, α) is a triple $(\pi, \mathcal{H}, \mathcal{U})$ where $(\pi, \mathcal{H})$ is a representation of A and $\mathcal{U}$

is a strongly continuous (respectively σ-weakly continuous) unitary representation of $\mathbb{R}$ into $U(H)$ such that

$$\pi(\alpha_t(x)) = u_t \pi(x) u_t^* \qquad \text{for } x \in A. \tag{3.5}$$

Covariant representations are related to GNS-representations via invariant states. We call a state ϕ on (A,α) *α-invariant* if $\phi(\alpha_t(x)) = \phi(x)$ for $t \in \mathbb{R}$ and $x \in A$.

THEOREM 3.1 ([B&RI] p. 235)

Let (A,α) denote a C^*-dynamical system. Let (π, H) be the GNS-representation of A induced by an α-invariant state ϕ. Then there exists a strongly continuous unitary representation u such that

$$\begin{aligned} &1. \quad \pi(\alpha_t(x)) = u_t \pi(x) u_t^* \\ &2. \quad u_t \xi = \xi \end{aligned} \tag{3.6}$$

for $x \in A$, and ξ is the cyclic vector which arises in the GNS-representation induced by ϕ.

For a dynamical system (A,α), the automorphism group α is called *implemented* if there exists a group of unitaries which satisfies (3.6), and α is called *inner* if the group u_t belongs to A. Not every automorphism is implemented (see [B&RI] p.306). However, if $A = B(H)$ or if A is simple and contains a unit, then α is implemented, by [Sak] thm. 4.1.19.

Let (A,α) be a C^*- (respectively W^*-) dynamical system. The *(infinitesimal) generator* Δ of α is the linear operator on A whose domain $\mathcal{D}(\Delta)$ consists of all $x \in A$ for which the limit

$$\Delta(x) = \lim_{t \to 0} \frac{1}{t} (\alpha_t(x) - x) \tag{3.7}$$

exists in the pointwise norm topology (respectively in the σ-weak topology). If α is norm continuous, then its generator is defined analogously. We write $\alpha_t = e^{t\Delta}$ if Δ is the generator of α. From the properties of automorphism groups it follows that Δ satisfies all the properties of a derivation.

THEOREM 3.2 ([B&RI] cor. 3.2.49, thm. 3.2.50)
Let A be a C-algebra, and α an automorphism group with generator δ. The following holds.*

1. *α is norm continuous if and only if δ is a bounded derivation (with domain $\mathcal{D}[\delta] = A$).*

2. *α is pointwise norm continuous if and only if δ is a norm-densely defined norm-closed derivation.*

3. *(A,α) is a W*-dynamical system if and only if δ is a σ-weakly-closed σ-weakly-densely-defined derivation.*

Since we are concerned with *-derivations only and not with the more general concept of a derivation, we shall give the next theorem in a form adjusted to our situation.
In analogy to Stone's theorem (thm. 1.3), which related selfadjoint operators and unitary groups, one has:

THEOREM 3.3 ([B&R1] thm. 3.2.50)
Let A be a C-algebra. A norm-densely defined, norm-closed *-derivation δ generates a group of *-automorphisms α of A if and only if*

1. *The set of analytic elements of δ is norm-dense in A.*
2. $\|\lambda\delta(x) + x\| \geq \|x\|$ *for* $\lambda \in \mathbb{R}$, $x \in \mathcal{D}(\delta)$.

Note if we replace "norm-dense" by "σ-weakly-dense" each time it occurs and "norm-closed" by "σ-weakly-closed" then the above theorem holds for von Neumann algebras, and δ is the generator of a σ-weakly continuous group of *-automorphisms.

1.4 TOMITA-TAKESAKI-THEORY

Let (A,α) denote a C*- or a W*-dynamical system. A state ϕ on A satisfies the *Kubo-Martin-Schwinger condition wrt* α (or is a *KMS-state wrt* α) *at* $\beta > 0$ if for $x, y \in A$ there exists a function $F_{x,y}$ which is continuous and uniformly bounded in the strip $\{ z \in \mathbb{C} : 0 \leqslant \mathrm{im} z \leqslant \beta \}$ and analytic inside the strip and satisfies the boundary conditions

$$F_{x,y}(t) = \phi(\alpha_t(x) y) \qquad F_{x,y}(t+i\beta) = \phi(y\alpha_t(x)) \qquad (4.1).$$

Statements equivalent to this can be found in [B&RII] section 5.3.1. The limiting case for $\beta = \infty$ (which is interpreted as the zero temperature case) is usually referred to as the ground state of a system.

KMS-states are considered to be good candidates for the physically important equilibrium states. The latter are represented by the Gibbs states in quantum statistical mechanics. Starting with a finite system Λ and a specific Hamiltonian H_Λ on the system, one defines the Gibbs equilibrium state $\omega_{\beta,\Lambda}$ by

$$\omega_{\beta,\Lambda}(x) = \mathrm{tr}((e^{-\beta H_\Lambda})x)/\mathrm{tr}(e^{-\beta H_\Lambda}) \qquad (4.2)$$

where x is an observable, β can be taken to be the inverse temperature or the chemical potential. And one is then interested in taking the thermodynamic limit as $\Lambda \to \infty$. For this definition of a Gibbs state to make sense - the equilibrium state is required to be a density matrix - the Hamiltonian H_Λ has to be a selfadjoint operator such that $e^{-\beta H_\Lambda} \in T(H)$. Furthermore, putting $\alpha_t = e^{itH_\Lambda} \cdot e^{-itH_\Lambda}$, we see that

$$\omega_{\beta,\Lambda}(\alpha_t(x) y) = \omega_{\beta,\Lambda}(y\, \alpha_{t+i\beta}(x)) \qquad (\text{for } x,y \in A) \qquad (4.3)$$

since $\mathrm{tr}(xy) = \mathrm{tr}(yx)$. For more general states, this behaviour is no longer observed in general. However, it can be shown that the KMS-states satisfy some of the characteristics associated with equilibrium states.

THEOREM 4.1 ([B&R2] prop. 5.3.3 and cor. 5.3.9)

1. *If φ is a KMS-state on (A,α), then φ is α-invariant.*
2. *Let φ be a KMS-state on (A,α) with GNS-representation (π,H,ξ), then ξ is separating for $\pi(A)''$.*

Tomita-Takesaki theory has become part of the repertoire of every standard textbook on operator algebras, and we thus restrict our attention here to the main theorem. The proofs have been much simplified over the years, and a particularly nice account is given in [Ped] in which a bounded operator approach due to Rieffel and van Daele [R&D] is presented.

Let $M \subseteq B(H)$ be a von Neumann algebra with cyclic and separating vector $\xi \in H$. By thm. 2.4, ξ is also cyclic and separating for M', thus $M\xi$ and $M'\xi$ are both dense in H. We now define two antilinear operators S_o with $\mathcal{D}(S_o) = M\xi$, F_o with domain $\mathcal{D}(F_o) = M'\xi$:

$$S_o x\xi = x^*\xi \qquad (x \in M) \qquad F_o x'\xi = x'^*\xi \quad (x' \in M') \tag{4.4}$$

Then $S = \overline{S_o} = F_o^*$ and $F = \overline{F_o} = S_o^*$, where S (respectively F) is the closure of S_o (respectively F_o). Let $S = J\Delta^{1/2}$ denote the polar decomposition of S where J is the antiunitary part and $\Delta = S^*S$ the positive selfadjoint part. Δ is called the *modular operator*, and J is called the *modular conjugation* associated with (M,ξ). The following relations hold:

$$\begin{array}{llll} 1. & \Delta = FS & \Delta^{-1} = SF & \\ 2. & S = J\Delta^{1/2} & F = J\Delta^{-1/2} & \\ 3. & J = J^* & J^2 = 1 & J\xi = \xi \\ 4. & \Delta^{-1/2} = J\Delta^{1/2}J & & \end{array} \tag{4.5}$$

If ξ corresponds to a tracial state ϕ on A via the GNS-representation - ϕ could be the equilibrium state of (4.2)- then $S = F$, $\Delta = 1$ and $J = S$.

Furthermore, it then follows that $JMJ = M'$. For general faithful normal states the unbounded modular operator Δ in some sense makes up for the trace property:

THEOREM 4.2 ([B&R1] thm. 2.5.14)

Let M be a von Neumann algebra with cyclic and separating vector ξ. Let Δ and J denote the modular operator and the modular conjugation respectively which are associated with (M,ξ). Then

1. $JMJ = M'$
2. $\Delta^{it} M \Delta^{-it} = M \qquad$ *for* $t \in \mathbb{R}$ $\qquad$ (4.6)

Takesaki made the connection between Tomita's theorem and the KMS-states, which demonstrates the close link between tracial states and KMS-states, by making use of the information contained in the modular group. Let $\sigma_t = \Delta^{it}.\Delta^{-it}$ in the notation of thm. 4.2. Then σ is a one-parameter group of *-automorphisms, called the *modular automorphism group* (associated with (M,ξ)). Starting with a faithful state $\varphi \in M_*$, we get the following extension of Tomita's result:

THEOREM 4.3 ([Ped] thm. 8.14.5)

Let M be a von Neumann algebra with a faithful normal state ϕ. Then there exists a unique W-dynamical system $(\pi_\phi(M),\sigma)$ such that ϕ satisfies the KMS condition for $\beta = -1$ with respect to σ.*

PART A

TOMITA-TAKESAKI THEORY FOR THE CCR-ALGEBRA

Chapter 2

A KMS-STATE ON THE CLOSABLE HILBERT ALGEBRA OF THE CCR

2.1 INTRODUCTION

In this chapter, the CCR-algebra A in infinitely many degrees of freedom and a "Gibbs" state ω_β on it are introduced, and in the following chapter, the algebra $\pi_\beta(A)$ is shown to satisfy Tomita's theorem.

The research presented in this and the following chapter shows how Tomita-Takesaki-theory can be extended to a class of unbounded algebras. Although Tomita-Takesaki-theory has undergone refinements and the proofs have been simplified (see section 1.4), in our approach we follow the original proof of Tomita: we define analogues of the left Hilbert algebra and the modular Hilbert algebra, and then show the commutation theorem of the algebra via that of our "modular" algebra.

A triple $(A, *, <\,,\,>)$ consisting of a $*$-algebra A together with an inner product which makes A into a pre-Hilbert space (whose completion is denoted by H) is a *Hilbert algebra* if

1. $\langle x, y\rangle = \langle y^*, x^*\rangle$ for $x,\ y,\ z \in A$
2. $\langle xy, z\rangle = \langle y, x^*z\rangle$
3. For $x \in A$, the map $y \longmapsto xy$ is continuous
4. A^2 is dense in A (with respect to the topology induced by the inner product).

For Hilbert algebras, Dixmier showed (see [Dix] Ch.1.6) that

$$L(A)' = J\,L(A)\,J = R(A) \tag{1.1}$$

where $L(A)$ consists of elements $\ell_x \in B(H)$ such that ℓ_x extends the continuous map $y \to xy$, for $x,\ y \in A$; and $R(A)$ is defined analogously using multiplication operators on the right. The operator J is an extension of the involution $*$ on A to an antiunitary involution on H. An important example of Hilbert algebras which we consider in the next chapter is the algebra of Hilbert Schmidt operators.

A triple $(A,{}^*,<\,,\,>)$ is a *left Hilbert algebra* if it satisfies conditions 2 to 4 of Hilbert algebras and 1 is replaced by

1a. the map $S: x \mapsto x^*$ is closable in H. Furthermore, the triple $(A,{}^*,<\,,\,>)$ is a *modular Hilbert algebra* if there exists a complex one-parameter group Δ of homomorphisms of A such that conditions 2 to 4 of Hilbert algebras hold together with

5. $(\Delta_z x)^* = \Delta_{-\bar{z}} x^*$
6. $\langle \Delta_z x, y\rangle = \langle x, \Delta_{\bar{z}} y\rangle$ for $x, y, z \in A$
7. $\langle \Delta_1 x^*, y^*\rangle = \langle y, x\rangle$ $z \in \mathbb{C}$
8. $z \longmapsto \langle \Delta_z x, y\rangle$ is entire
9. for $t \in \mathbb{R}$ $A_t = \{(1 + \Delta_t)x : x \in A\}$ is dense in A.

Tomita proved (1.1) for modular Hilbert algebras, where the existence of the "modular" automorphism is used instead of condition 1. He then showed the following:

THEOREM. *For every left Hilbert algebra* $(A,{}^*,<.,.>)$ *there exists a dense subalgebra* $\tilde{A}$ *which is a modular Hilbert algebra such that* $L(A) = L(\tilde{A})$.

If M is a von Neumann algebra with cyclic and separating vector ξ as in section 1.4, then $A = M\xi$ is a left Hilbert algebra where multiplication and involution of A are induced by the corresponding operations on M (i.e. if $a = a_1\xi$, $b = b_1\xi \in A$, then $ab = a_1 b_1 \xi$ and $a^* = a_1{}^*\xi$).

We shall pursue the same direction, with the main difference that we shall deal with an unbounded algebra.

2.2 THE CCR-ALGEBRA AND THE FOCK REPRESENTATION

To define the CCR-algebra for infinitely many degrees of freedom, we use the *annihilation operator* a and the *creation operator* $a^{\#}$ as generators which are defined from the position and momentum operators:

$$a = \frac{1}{\sqrt{2}}(p - iq) \qquad a^{\#} = \frac{1}{\sqrt{2}}(p + iq) \tag{2.1}$$

Let A denote the *CCR-algebra in infinitely many dimensions* i.e. A is the *-algebra with generators $a(f)$ and $a^{\#}(g)$ ($f, g \in S(\mathbb{R})$) and relations

$$\begin{aligned} [a(f), a(g)] &= [a^{\#}(f), a^{\#}(g)] = 0 \\ [a^{\#}(g), a(\bar{f})] &= \langle g,f\rangle 1 \end{aligned} \tag{2.2}$$

General elements of A are denoted by x, y or by x_f, $x(f)$ if necessary.

We now turn to the Fock representation of A. For the CCR-algebra in n degrees of freedom, the appropriate Hilbert space is $H = L^2(\mathbb{R}^n)$. Since we work with infinitely many degrees of freedom, an appropriate Hilbert space is the *Fock space* $F(H)$ which is defined by

$$F(H) = \bigoplus_{n \geq 0} H^n = \bigoplus_{n \geq 0} L^2(\mathbb{R}^n) \tag{2.3}$$

where $H^o = \mathbb{C}$, $H = L^2(\mathbb{R})$, and H^n denotes the tensor product $H \otimes \ldots \otimes H$ (n factors). Note that H^n is isometrically isomorphic to $L^2(\mathbb{R}^n)$, and elements $\xi_1 \otimes \ldots \otimes \xi_n$ in H^n with $\xi_i \in H$ can be represented as functions f^n in $L^2(\mathbb{R}^n)$. For more details on tensor product spaces see [R & S] p. 49 ff.

An element $\Psi \in F(H)$ is a sequence of vectors $\Psi = \{\psi^n\}_{n > o}$ with $\psi^n \in H^n$. The norm of $F(H)$ is derived from the norms of the direct summands and uses the isomorphism between H^n and $L^2(\mathbb{R}^n)$. Let $\Psi = \{\psi^n\}$ $\in F(H)$; then

$$\|\Psi\|^2 = \sum_{n \geq 0} \|\Psi^n\|^2 \quad \text{and}$$

$$\|\Psi^n\|^2 = \int |\Psi^n(t_1,\ldots,t_n)|^2 \, dt_1 \ldots dt_n < \infty , \tag{2.4}$$
$$\|\Psi^o\|^2 = |\Psi^o|^2 .$$

Unfortunately, Fock space is not quite the right Hilbert space on which to represent A. Since A denotes the CCR-algebra which corresponds to the Bose-Einstein statistics, the appropriate choice of Hilbert space is the symmetrised or Bose Fock space, a subspace of $F(H)$ which we now turn to. But we first define the *symmetrisation operator* Π_n for the space H^n. For $\Psi^n = \xi_1 \otimes \ldots \otimes \xi_n$ in H^n we put

$$\Pi_n(\Psi^n) = \frac{1}{n!} \sum_{\pi \in P_n} \xi_{\pi(1)} \otimes \ldots \otimes \xi_{\pi(n)} \tag{2.5}$$

where the sum is taken over all elements π in the group P_n of permutations on n elements. By linearity, Π_n extends to a projection operator on H^n, that is, Π_n has norm one and $\Pi_n^2 = \Pi_n$. The subspace $H^n_+ = \Pi_n H^n$ is called the *symmetrised* part of H^n and consists of those elements in H^n which are invariant under permutation of the components $\xi_1,\ldots,\xi_n$.

Similarly, we can define a symmetrisation operator Π on $F(H)$. We do this in such a way that Π restricted to H^n agrees with Π_n (see e.g. [B & R2] section 5.2.1). The *symmetrised* or *Bose Fock space* will here be denoted by H_+ and equals

$$\begin{aligned} H_+ &= F(H)_+ = \Pi F(H) \\ &= \bigoplus_{n \geq 0} \Pi_n H^n = \bigoplus_{n \geq 0} H^n_+ \end{aligned} \tag{2.6}$$

Elements in H_+ are sequences of vectors $\Psi = \{\psi^n\}$ where $\psi^n \in H_+^n$, and the norm on H_+ is inherited from the norm of $F(H)$ in the natural way. Using this Hilbert space H_+, we show the following.

THEOREM 2.1 *There exists a closed *-representation* $(\pi_o, H_+, \mathcal{D}_\pi)$ *of* A *such that for* $x \in A$, $\pi_o(x)$ *is a continuous linear operator on* $\mathcal{D}_\pi$ *with respect to the SN-topology.*

For an analogue of this theorem in the case of the CCR-algebra in n degrees of freedom see [Pow] section V example 2, where it is shown that for the Schrödinger representation π of the algebra $\mathcal{D}_\pi = S(\mathbb{R}^n)$ and the SN-topology is equivalent to the Schwartz space topology on $S(\mathbb{R}^n)$. We shall proceed in a similar fashion and first concentrate on the space $\mathcal{D}_\pi$.

We first consider a space $\mathcal{D}$. We define the Fock representation $(\pi_o, H_+, \mathcal{D})$ and equip the domain $\mathcal{D}$ with the SN-topology. We show that $\pi_o(x)$ is continuous on $\mathcal{D}$ in the SN-topology and then extend $\pi_o(x)$ to $\mathcal{D}_\pi = \overline{\mathcal{D}}^{SN}$ by continuity.

Let $\mathcal{D}$ denote the algebraic sum

$$\mathcal{D} = \oplus S_+^n \qquad \text{with} \tag{2.7}$$

$$S_+^o = \mathbb{C}\ ,\ S_+^1 = S(\mathbb{R})\ ,\ \text{and}\ \ S_+^n = \Pi_n S^n\ . \tag{2.8}$$

The space S_+^n is the symmetrised Schwartz space in n dimensions, and Π_n denotes the symmetrisation operator (see (2.5)).

LEMMA 2.2 *$\mathcal{D}$ is dense in H_+.*

Proof. First note that Π_n is a projection operator on $L^2(\mathbb{R}^n)$ which maps S^n into itself. But S^n is dense in $L^2(\mathbb{R}^n)$ and therefore

$\Pi_n S^n$ is dense in $\Pi_n L^2(\mathbb{R}^n)$ for every $n \in \mathbb{N}$. In complete analogy to the L^2-case, one can form the direct sum of these symmetrised Schwartz spaces. In general, $\mathcal{D}$ is not a complete space, but $\mathcal{D}$ is dense in H_+, since S^n_+ is dense in H^n_+ for $n \in \mathbb{N}$ and the topologies of H_+ and $\mathcal{D}$ are derived from the direct summands respectively. ///.

We now define a *-representation $(\pi_o, H_+, \mathcal{D})$ on elements $\Psi^n \in S^n_+$. For $\psi^o \in \mathbb{C}$, $\psi^n = \psi_1 \otimes \ldots \otimes \psi_n \in S^n$, and $f \in S(\mathbb{R})$, put

$$\begin{aligned}
&\pi_o(a(\overline{f})\Psi^o) = 0 \qquad \pi_o(a^{\#}(f)\Psi^o) = f \\
&\pi_o(a(\overline{f})\Psi^n) = \sqrt{n}\, \langle\psi_1, f\rangle\, (\psi_2 \otimes \ldots \otimes \psi_n) \\
&\pi_o(a^{\#}(f)\Psi^n) = \sqrt{n+1}\, (f \otimes \psi_1 \otimes \ldots \otimes \psi_n)
\end{aligned} \tag{2.9}$$

To guarantee that $\pi_o(a)$ and $\pi_o(a^{\#})$ map symmetrised spaces into symmetrised spaces, put $\pi_o(a_s(f)) = \pi_o(\Pi a(f)\Pi)$ and $\pi_o(a^{\#}_s(f)) = \pi_o(\Pi a^{\#}(f)\Pi)$. From p.9 in [B&R2] it now follows that $\pi_o(a_s(f)) = \pi_o(a(f)\Pi)$ and $\pi_o(a^{\#}_s(f)) = \pi_o(\Pi a^{\#}(f))$. We are only interested in the symmetrised spaces and the symmetrised operators, and by abuse of notation will write $\pi_o(a)$ and $\pi_o(a^{\#})$ instead of $\pi_o(a_s)$ and $\pi_o(a^{\#}_s)$ respectively.

LEMMA 2.3 *The operators $\pi_o(a(\overline{f}))$ and $\pi_o(a^{\#}(f))$ are mutually adjoint on H_+, that is, $\langle\pi_o(a^{\#}(f))\psi, \phi\rangle = \langle\psi, \pi_o(a(\overline{f}))\phi\rangle$ for $\psi, \phi \in H_+$.*

Proof. To see this, consider $\Psi^n = \Psi_1 \otimes \ldots \otimes \Psi_n$ in S^n_+ and $\phi^{n+1} = \phi_1 \otimes \ldots \otimes \phi_{n+1}$ in S^{n+1}_+. Then for f in $S(\mathbb{R})$ we have

$$\begin{aligned}
\langle\pi_o(a^{\#}_s(f))\Psi^n, \phi^{n+1}\rangle &= \langle\pi_o(\Pi a^{\#}(f))\Psi^n, \phi^{n+1}\rangle \\
&= (n+1)^{\frac{1}{2}} \langle\Pi(f \otimes \psi_1 \ldots \otimes \psi_n), \phi_1 \otimes \ldots \otimes \phi_{n+1}\rangle \\
&= (n+1)^{\frac{1}{2}} \langle f, \phi_1\rangle\langle\psi_1 \otimes \ldots \otimes \psi_n, \Pi(\phi_2 \otimes \ldots \otimes \phi_{n+1})\rangle \\
&= (n+1)^{\frac{1}{2}} \langle\psi_1 \otimes \ldots \otimes \psi_n, \langle\phi_1, f\rangle\Pi(\phi_2 \otimes \ldots \otimes \phi_{n+1})\rangle
\end{aligned}$$

$$= \langle \Psi^n, \pi_o(a(\bar{f})\Pi)\Phi^{n+1}\rangle \quad . \tag{2.10}$$

///.

Starting with $\Psi^o \in \mathbb{C}$, which corresponds physically to the vacuum or the zero-particle state, we can "create" particles in S^n , $S^n \subset H^n$ $n = 1, 2, \ldots$ by successively applying $a^\#$ as outlined in (2.9). Let $f_i \in S(\mathbb{R})$ $i = 1,\ldots,n$, then

$$\begin{aligned} \phi^n &= (n!)^{-\frac{1}{2}} a^\#(f_n)a^\#(f_{n-1})\ldots a^\#(f_1)\Psi^o \\ &= f_n \otimes f_{n-1} \otimes\ldots\otimes f_1 \end{aligned} \tag{2.11}$$

Thus $\phi^n \; \varepsilon \; S^n$, and $\Pi\phi^n \; \varepsilon \; S^n_+$. Elements obtained in this way are total for S^n if the f_i $(i = 1,\ldots,n)$ are taken to be a basis in $S(\mathbb{R})$ (see e.g. section 2.1 in [Gui]), a fact which we shall make use of in later sections.

The elements Φ^n can be regarded as natural embeddings from S^n_+ into $\mathcal{D}$; they are denoted by $\Phi^n = \Phi(\phi^n)$. The n-th component of Φ^n is ϕ^n and all other components are zero. With this notation, the following holds (for a proof see [KMT] thm 3.14)

LEMMA 2.4. *Every ϕ in $\mathcal{D}$ has a unique representation as a finite orthogonal sum*

$$\phi = \sum_{i=0}^{n} \phi^i_\varphi = \sum_{i\geq 0}{}' \phi^i[\varphi^{(i)}] = \sum_{i\geq 0}{}' \phi^i(\varphi_1\ldots\varphi_i) \tag{2.12}$$

and $\sum'$ denotes that the sum is actually finite, and $\varphi^{(i)} = \varphi_1 \otimes\ldots\otimes \varphi_i$ in S^i_+ .

Since we are aiming at a closed representation, we now consider a natural induced topology on $\mathcal{D}$ which is given by the family of seminorms:

$$\|\phi\|_r^2 = \sum_{n \geq 0} \|\varphi^{(n)}\|_r^2 = \sum_{n \geq 0} \langle (h_n)^r \varphi^{(n)}, \varphi^{(n)} \rangle \tag{2.13}$$

where ϕ is as in (2.12), $h_0 = \mathrm{id}$ on $\mathbb{C}$, $h_n = \frac{1}{2} \sum_{i=1}^{n} (p_i^2 + q_i^2 + 1)$ (see [R&S] p.142), and $\langle .^{(n)}, .^{(n)} \rangle$ is the inner product in H^n. From section V in [Pow] it follows that this topology is locally convex and equivalent to the SN-topology introduced in section 1.2. For $r = 0$, the seminorm is indeed a norm, namely the usual Hilbert space norm on H_+; and for $r > 0$, the seminorms are increasing.

LEMMA 2.5 *For the representation* $(\pi_o, H_+, \mathcal{D})$ *of* A, $\pi_o(x)\mathcal{D} \subseteq \mathcal{D}$.

Proof. From the definitions and calculations (2.9) and (2.10) it follows that π_o is a #-homomorphism. Hence for general elements $x \in A$, $(\pi_o(x))^{\#} \supseteq \pi_o(x^{\#})$. It remains to be shown that the operators $\pi_o(x)$ map $\mathcal{D}$ into itself. To see this, consider $\pi_o(a_{\bar{f}})$ and $\pi_o(a_f^{\#})$ and then extend by linearity. Note that a total set in S_+^n can be generated using (2.11). But since $f_1, \ldots, f_n \in S(\mathbb{R})$, $\Pi_n(f_1 \otimes \ldots \otimes f_n) \in S_+^n$, and hence a further application of $\pi_o(a_f^{\#})$ maps this element to S_+^{n+1}. ///.

In general, $\mathcal{D}$ is not closed in the SN-topology defined by the seminorms (2.13). Let $\mathcal{D}_\pi$ denote the completion of $\mathcal{D}$ in the SN-topology; clearly, $\mathcal{D}_\pi \subseteq H_+$. It remains to show that $\pi_o(x)$ can be defined on $\mathcal{D}_\pi$ and maps $\mathcal{D}_\pi$ continuously into itself with respect to the SN-topology. We do this by first showing that for $x \in A$, $\pi_o(x)$ is SN-continuous on $\mathcal{D}$. Let $L(\mathcal{D}_\pi)$ denote the space of linear operators from $\mathcal{D}_\pi$ into itself equipped with the topology given by the family of seminorms:

$$\|T\|_r^B = \sup_{\phi \in B} \|T\phi\|_r \tag{2.14}$$

where B is a bounded set in $\mathcal{D}_\pi$. (Note that if $\mathcal{D}_\pi$ is a Hilbert space, this topology reduces to the norm topology of $B(\mathcal{D}_\pi)$.) A similar topology can be defined on the space $L(\mathcal{D})$ where $\mathcal{D}_\pi$ is replaced by $\mathcal{D}$ everywhere in the definition. The next lemma completes the proof of theorem 2.1.

LEMMA 2.6 *For* $x \in A$, $\pi_o(x)$ *is a continuous linear operator on* $\mathcal{D}_\pi$, *and thus* $\pi_o(x) \in L(\mathcal{D}_\pi)$.

Proof. The continuity of $\pi_0(x)$ on $\mathcal{D}_\pi$ follows by referring to some results presented in [KMT]. Note that $\pi_0(x)$ is a continuous operator on $\mathcal{D}_\pi$ means that $\pi_0(x)$ is in $L(\mathcal{D}_\pi)$ endowed with the topology described above, since this topology arises from continuous seminorms on $\mathcal{D}_\pi$. But the latter result follows, that is, $\pi_0(x) \in L(\mathcal{D}_\pi)$, when we have shown that the map $\pi_0 \circ \underline{a} : S(\mathbb{R}) \longrightarrow L(\mathcal{D}_\pi)$ is continuous, where $\underline{a}$ denotes a or $a^\# \in A$. This result is proved in two steps in [KMT]; on p. 193 the map $\pi_0 \circ \underline{a} : S(\mathbb{R}) \longrightarrow L(\mathcal{D})$ is shown to be continuous. Note that $\pi_o(\underline{a}_f)$ can be uniquely defined on $\mathcal{D}_\pi$, as $\pi_o(\underline{a}_f)$ is SN-continuous, and the extension of $\pi_o \circ \underline{a}$ to $L(\mathcal{D}_\pi)$ now follows from lemma 3.42 of [KMT]. Hence $\pi_o(x) \in L(\mathcal{D}_\pi)$, and this concludes the proof of theorem 2.1. ///.

2.3 THE HAMILTONIAN AND THE GIBBS STATE

We begin the description of our Hamiltonian by recalling the definition of second-quantisation operators, since this is the kind of operator on $\mathcal{H}_+$ which we shall use.

Let h be an (unbounded) selfadjoint operator on a Hilbert space $\mathcal{K}$. Define an operator h_n on the symmetrised tensor product space $\mathcal{K}_+^n$ by

$$h_n(f_1 \otimes \ldots \otimes f_n) = \Pi_n\left(\sum_{i=1}^{n} f_i \otimes \ldots \otimes hf_i \otimes \ldots \otimes f_n\right), \qquad (3.1)$$

where $f_i \in \mathcal{D}(h)$, the domain of h in $\mathcal{H}$, $i = 1,\ldots,n$. Then h_n is essentially selfadjoint on $\mathcal{K}_+^n$ (see section 5.2.1 in [B & R2]). Since $\mathcal{K}_+$ is the direct sum of the spaces $\mathcal{K}_+^n$ $(n \geq 0)$, we define the *second-quantisation operator* (associated with the operator h on $\mathcal{K}$) to be the closure of the direct sum of the h_n and denote it by H_Γ:

$$H_\Gamma = \overline{\bigoplus_{n \geq 0} h_n}. \qquad (3.2)$$

Note that the closure of the direct sum operator ensures that H_Γ is selfadjoint on $\mathcal{K}_+$. (For further details see [B & R2], section 5.2.1).

Since we are dealing with an unbounded algebra, we will choose a particularly nice Hamiltonian H, for which we can show that $e^{-\beta H} \in \mathcal{T}(\mathcal{H}_+)$ for positive real β.

Let $\{e_k\}_{k \geq 0}$ denote the basis of $L^2(\mathbb{R})$ consisting of the Hermite functions (see [R & S] p.142). Then $e_k \in \mathcal{S}(\mathbb{R})$ for $k=0,1,\ldots$. Let

$$e^n(n_0.n_1\ldots.) = \Pi_n(e_1^{\;n_1} \otimes e_2^{\;n_2} \;\ldots.) \qquad (3.3)$$

where the vector e_j appears n_j-times on the right hand side of (3.3) (before symmetrisation), and $n = \sum n_j$. Elements of this kind form a basis for H_+^n (see [Gui] Ch. 2.1). We write Ψ_e^n or $\Psi^n(e^n(n_0 n_1 \ldots))$ for the element in H_+ which is obtained from $e^n(n_0 n_1 \ldots)$ by the natural embedding (see also section 2.2).

The Hamiltonian H will be the second-quantisation operator of the linear operator h on $L^2(\mathbb{R})$ which is defined by

$$he_k = w(k)e_k \qquad \text{for } k = 0,1\ldots, \tag{3.4}$$

where w denotes a real-valued non-negative function on the integers such that there exists $t \in \mathbb{N}$ and real $\varepsilon > 0$ with

$$(k + \varepsilon) \leqslant w(k) \leqslant (k + 1)^t \qquad \text{for all } k \in \mathbb{N}.$$

LEMMA 3.1. *The operator* h *is selfadjoint on* $L^2(\mathbb{R})$ *, and maps* $S(\mathbb{R})$ *continuously into itself with respect to the Schwartz space topology.*

Proof. By linearity, h extends to a selfadjoint operator on $L^2(\mathbb{R})$ with domain $\mathcal{D}(h) = S(\mathbb{R})$, since $\{e_k\} \in S(\mathbb{R})$; and h is continuous on $S(\mathbb{R})$, since w is polynomially bounded. For $f \in S(\mathbb{R})$, $f = \sum_{k=0}^{N} a_k e_k$ and fixed m in $\mathbb{N}$ we have

$$\|hf\|_m^2 = \|\sum_k a_k w(k) e_k\|_m^2 = \sum_k |a_k|^2 w(k)^2 (k + 1)^m \tag{3.5}$$

$$\leq \sum_k |a_k|^2 (k + 1)^{m + 2t} = \|f\|_{m + 2t}^2$$

Recall that the topology induced by these seminorms is equivalent to the Schwartz space topology (see [R & S], Appendix to Ch. V.3). ///.

The action of h_n on H_+^n can be written down explicitly on the basis elements of the form $e^n(n_0 n_1 \ldots.)$ (see (3.3)):

$$h_n e^n(n_0 n_1 \dots) = \Pi_n\Big(\sum_{k \geq 0} \Pi_n(e_0 \otimes \dots \otimes he_k \otimes \dots)\Big) \qquad (3.6)$$

$$\sum_{k \geq 0} w(k) n_k \, e^n(n_0 n_1 \dots.)$$

Let H denote the second-quantisation operator of h, and put $H\psi^o = 0$ for $\psi^o \in \mathbb{C}$ and

$$H\Psi^n(e^n(n_0 n_1 \dots.)) = \Big(\sum_k w(k) n_k\Big)\Psi^n(e^n(n_0 n_1 \dots.)). \qquad (3.7)$$

THEOREM 3.2 *The linear operator* H *is selfadjoint on* H_+, *belongs to* $L(\mathcal{D}_\pi)$ *and for* $\beta > 0$, $e^{-\beta H}$ *is positive and* $e^{-\beta H} \in T(H_+)$.

Proof. The selfadjointness of H follows since H is a second-quantisation operator on H_+ (see [B&R2] section 5.2.1). Furthermore, by lemma 3.16 of [KMT], H is SN-continuous on $\mathcal{D}_\pi$, and hence $H \in L(\mathcal{D}_\pi)$.
It remains to show that $e^{-\beta H}$ is a positive trace class operator on H_+, that is, for every orthonormal basis $\{\xi_i\}$ in H_+

$$\sum_{i \geq 0} \langle e^{-\beta H}\xi_i, \xi_i \rangle < \infty \quad . \qquad (3.8)$$

Fix $\beta > 0$ and put $T = e^{-\beta H}$. Then

$$\sum_{i \geq 0} \langle T\Psi^n_{e_i}, \Psi^n_{e_i} \rangle = \sum_{n \geq 0} \sigma_n \qquad (3.9)$$

and for fixed n, one obtains by (3.6) and (3.7)

$$\sigma_n = \sum_{i \geq 0} \langle e^{-\beta H} e^n_i(n_0 n_1 \dots), e^n_i(n_0 \dots)\rangle \qquad (3.10)$$

$$= \sum_{J_n} \langle e^{-\beta \sum w(k) n_k} e^n_i(n_0 n_1 \dots), e^n_i(n_0 n_1 \dots)\rangle = \sum_{J_n} e^{-\beta \sum w(k) n_k}$$

with

$$J_n = \{(n_0 n_1 \dots) : \textstyle\sum n_k = n\} \quad , \qquad (3.11)$$

and the partial sums are taken over all sequences in J_n. Hence (3.8) is satisfied provided $\sum \sigma_n$ converges. This is shown in the next two lemmas. ///.

LEMMA 3.3 *Let* $S_n = \sum_{J_n} exp(-\beta \sum kn_k)$. *Then for* $n \in \mathbb{N}$,

$$S_n = \prod_{k=1}^{n} (1 - e^{-k\beta})^{-1} \tag{3.12}$$

Proof. For $n = 0$, put $S_0 = 1$, and define the empty product to be 1. For $n \geq 1$, the claim is an application of some results in the theory of partitioning. From [And] thm. 1.1, we conclude that

$$\prod_{k=1}^{n} (1 - x^k)^{-1} = \sum p(\text{"}\{1,\ldots,n\}\text{"}, \ell)\, x^{\ell}$$

where $p(\text{"}\{1,\ldots,n\}\text{"}, \ell)$ denotes the number of partitions of ℓ such that no part exceeds n; and by applying [And] thm. 1.4 (which states that the number of partitions of n, in which no part exceeds m, equals the number of partitions of n with at most m parts) we prove the claim for every $n \in \mathbb{N}$. ///.

LEMMA 3.4 $\sum \sigma_n < \infty$.

Proof.

$$\sum_{n \geq 0} \sum_{J_n} e^{-\beta \sum_k w(k) n_k} \leq \sum_{n \geq 0} \sum_{J_n} e^{-\beta \sum_k (k + \varepsilon) n_k}$$

$$= \sum_{n \geq 0} e^{-\beta n \varepsilon} \sum_{J_n} e^{-\beta \sum_k k n_k} = \sum_{n \geq 0} e^{-\beta n \varepsilon} S_n$$

$$= \sum_{n \geq 0} e^{-\beta n \varepsilon} \prod_{k=1}^{n} (1 - e^{-k\beta})^{-1}$$

by lemma 3.3. The ratio test shows that this sum is finite: the ratio of the $(n + 1)$-st and the n-th term is

$$e^{-\beta\varepsilon}(1 - e^{-\beta(n+1)})^{-1} \longrightarrow e^{-\beta\varepsilon} < 1 \text{ as } n \longrightarrow \infty .$$

The trace property of $e^{-\beta H}$ follows now from (3.8) and (3.9). Also, $e^{-\beta H}$ is positive, since

$$\sum_{J_n} \langle e^{-\beta \sum w(k) n_k} \Psi_e^n \ \Psi_e^n \rangle = \sum_{J_n} \| e^{-\frac{1}{2}\beta \sum w(k) n_k} \Psi_e^n \|^2 \qquad ///.$$

We want to use the operator $e^{-\beta H}$ in the definition of the Gibbs state. Since all operators $\pi_o(X)$ with $X \in A$ map $\mathcal{D}_\pi$ into itself, the operator $e^{-\beta H} \pi_o(X)$ makes sense as an element of $L(\mathcal{D}_\pi)$ if we can show that $e^{-\beta H}$ is a trace class operator from H_+ into $\mathcal{D}_\pi$. To do this, we make use of the equivalence (stated below in a form adjusted to our requirements):

THEOREM 3.5. ([Tre], prop. 47.2)

A continuous map T *from* H_+ *into* $\mathcal{D}_\pi$ *is a trace class operator if and only if the following holds:*

There exist

1. *a sequence* $\{\xi_k\}$ *in the closed unit ball of* H_+,
2. *a sequence* $\{\eta_k\}$ *in a convex balanced bounded subset of* $\mathcal{D}_\pi$,
3. *an* l^1*-sequence of complex numbers* $\{\lambda_k\}$

such that T *equals the map*

$$\xi \longmapsto \sum_k \lambda_k \langle \xi, \xi_k \rangle \eta_k \tag{3.13}$$

LEMMA 3.6. *For* $e^n(n_0 n_1 \ldots)$ *in* H_+^n, *let*

$$\phi_n^\beta = e^{-\frac{1}{4}\beta H} \Psi^n(e^n(n_0 n_1 \ldots)) .$$

Then the convex balanced hull of the elements ϕ_n^β *is an* SN*-bounded subset in* $\mathcal{D}_\pi$.

Proof. It is sufficient to show that the ϕ_n^β are bounded with respect to the seminorms defined in (2.13). For $m \in \mathbb{N}$,

$$\|\Psi_e^n\|_m^2 = \|\Psi^n(e^n(n_o n_1 \ldots))\|_m^2 = (\sum_k (k+1)n_k)^m$$

follows from lemma 3.1 and the definition of the m-seminorms: thus

$$\|\phi_n^\beta\|_m^2 = (e^{-\frac{1}{2}\beta\sum w(k)n_k})\|\Psi_e^n\|_m^2 \leq (e^{-\frac{1}{2}\beta\sum(k+\varepsilon)n_k})(\sum_k (k+1)n_k)^m$$

which are bounded, uniformly in $n, n_0, n_1 \ldots$. Clearly, $\lambda\phi_n^\beta$ is in $\mathcal{D}_\pi$ for ϕ_n^β in $\mathcal{D}_\pi$ and $|\lambda| < 1$, and since the elements Ψ_e^n constitute a basis of $\mathcal{H}_+$, the ϕ_n^β define a sequence which is contained in a convex balanced bounded subset of $\mathcal{D}_\pi$. ///.

PROPOSITION 3.7 *The operator $e^{-\beta H}$ is trace class from $\mathcal{H}_+$ into $\mathcal{D}_\pi$.*

Proof. We first show that $e^{-\beta H}$ is continuous from $\mathcal{H}_+$ into $\mathcal{D}_\pi$ where $\mathcal{D}_\pi$ is equipped with the SN-topology. To do this, consider the space of all finite linear combinations of elements Ψ_e^n equipped with the L^2-topology (and the SN-topology respectively) as a dense linear subset of $\mathcal{H}_+$ (and as a dense linear subset of $\mathcal{D}_\pi$ respectively). The unit ball of the first space consists of finite linear combinations $\sum \lambda_n \Psi_e^n$ where $\sum|\lambda_n|^2 \leq 1$. The map $e^{-\beta H}$ sends this unit ball to an SN-bounded set in $\mathcal{D}_\pi$, because for $m \in \mathbb{N}$

$$\|e^{-\beta H}\sum_n \lambda_n \Psi_e^n\|_m^2 \leq \|\sum \lambda_n \phi_n^{4\beta}\|_m^2 \leq \sum|\lambda_n|^2\|\phi_n^{4\beta}\|_m^2 \leq \sup_n\|\phi_n^{4\beta}\|_m^2 < \infty$$

by lemma 3.6. This implies that $e^{-\beta H}$ maps bounded sets into bounded sets. Since $\mathcal{H}_+$ and $\mathcal{D}_\pi$ are metrisable, it follows that $e^{-\beta H}$ is continuous; see [Tre], prop. 14.8. Now extend by linearity to general elements of H_+ and $\mathcal{D}_\pi$.

Having shown the continuity of $e^{-\beta H}$, we next put $\lambda_k = \sum_{J_k} e^{-(3/4)\beta\sum w(k)n_k}$; then by lemma 3.4, $\{\lambda_k\} \in \ell^1$, and for $\xi \in H_+$ we have

$$e^{-\beta H}\xi = \sum_{n \geq 0} \sum_{J_n} e^{-\beta\sum w(k)n_k} \langle \xi, \Psi_e^n \rangle \Psi_e^n = \sum_{n \geq 0} \lambda_n \langle \xi, \Psi_e^n \rangle \phi_n^\beta$$

which is the desired form of $e^{-\beta H}$ by theorem 3.5. ///.

Since the trace class operators from an ideal in $L(\mathcal{D}_\pi)$ (see [Tre] prop. 47.1), the next corollary follows immediately from prop 3.7.

COROLLARY 3.8. *For* $x \in A$, $e^{-\beta H}\pi_0(x)$ *is a trace class operator on* $\mathcal{D}_\pi$.

We now define our *Gibbs state* ω_β on A. Let $\beta > 0$, then

$$\omega_\beta : x \longmapsto \mathrm{tr}(e^{-\beta H}\pi_0(x))/\mathrm{tr}(e^{-\beta H}). \tag{3.14}$$

A comparison with the state $\omega_{\Lambda,\beta}$ for finite systems (see section 1.4) indicates why we are justified in calling ω_β a Gibbs state. We conclude this section with the following observation on ω_β.

THEOREM 3.9 *For* $\beta > 0$, ω_β *is a faithful state on* A, *and the GNS-representation* $(\pi_\beta, H_+, 1_\beta)$ *induced by* ω_β *is faithful.*

Proof. ω_β is well-defined and positive, since $e^{-\beta H}$ is a positive linear operator. The linear functional ω_β is a state, since $\omega_\beta(1) = 1$. It remains to check that ω_β is faithful. This is true because $\omega_\beta(xx^*) = 0$ implies that $\|\pi_0(x)\Psi_e^n\| = 0$, while the vectors of the form Ψ_e^n are total in $\mathcal{D}_\pi$. Hence $x = 0$, and π_β is faithful, since $1_\beta = \pi_\beta(1)$.

///.

2.4 THE ALGEBRA $\tilde{A}$ INSIDE THE CLOSABLE ALGEBRA A AND THE KMS STATE

In this section, we describe the algebra $\tilde{A}$ which sits inside A and for which we shall prove Tomita's theorem first. This algebra $\tilde{A}$ bears the same relationship to A as the modular Hilbert algebra does to the left Hilbert algebra.

The triple $(B, {}^{\#}, <.,.>)$ is a *closable (Hilbert) algebra* if B is a $*$-algebra with unit such that

(1a) the map $x \longmapsto x^{\#}$ is closable

(2) $<xy, z> = <y, x^{\#}z>$ for $x, y, z \in B$ (4.1)

(4) B^2 is dense in B .

The triple $(B, {}^{\#}, <.,.>)$ is an *almost modular (Hilbert) algebra* if B is a $*$-algebra and there exists a complex one-parameter group Δ of $*$-homomorphisms of B such that

(2) $<xy, w> = <y, x^{\#}w>$

(4) B^2 is dense in B

(5) $(\Delta_z x)^{\#} = \Delta_{-\bar{z}} x^{\#}$ $\quad x, y, w \in B$

(6) $<\Delta_z x, y> = <x, \Delta_{\bar{z}} y>$ (4.2)

(7) $<\Delta_1 x^{\#}, y^{\#}> = <y, x>$ $\quad z \in \mathbb{C}$

(8) $z \longmapsto <\Delta_z x, y>$ is entire

(9) for $t \in \mathbb{R}$, $B_t = \{(1 + \Delta_t)x : x \in B\}$ is dense in B

A comparison with left Hilbert algebras (respectively modular Hilbert algebras) (see section 2.1) shows that the maps $y \mapsto xy$ (for $x, y \in B$) are no longer continuous in a closable (respectively almost modular) algebra.

We first concentrate on A and let $(A, \| \ \|)$ denote the algebra A equipped with the topology inherited from H_+, and let (A, τ) denote the algebra A equipped with the topology of uniform convergence on bounded sets inherited from $L(\mathcal{D}_\pi)$.

LEMMA 4.1. *The map* $x \longmapsto \omega_\beta(x^\# x)$ *is a continuous norm on* A *equipped with the subspace topology inherited from* $L(\mathcal{D}_\pi)$.

Proof. First recall that the topology of $L(\mathcal{D}_\pi)$ is given by a family of seminorms (see (2.13)), and that bounded sets in $\mathcal{D}_\pi$ are those for which all these seminorms are bounded. Let B be the closed convex hull of elements ϕ_n^β which were defined in Lemma 3.6. If $\{x_i\}$ is a net in A such that $\pi_0(x_i) \longrightarrow 0$ in the subspace topology, then, for sufficiently large i, we have

$$\|\pi_0(x_i)\phi_n^\beta\|^2 < \varepsilon \quad \text{for all } n, n_0, n_1 \ldots .$$

and hence

$$\begin{aligned} \mathrm{tr}(e^{-\beta H})\|x_i\|^2 &= \mathrm{tr}(e^{-\beta H})\omega_\beta(x_i x_i^\#) \\ &= \sum_{n \geq 0} \sum_{J_n} e^{-\beta \sum w(k) n_k} \|\pi_0(x_i)\Psi_e^n\|^2 \\ &= \sum_{n \geq 0} \sum_{J_n} e^{-\frac{1}{2}\beta \sum w(k) n_k} \|\pi_0(x_i)\phi_n^\beta\|^2 \\ &< (\varepsilon) \sum_{n \geq 0} \sum_{J_n} e^{-\frac{1}{2}\beta \sum w(k) n_k} = \mathrm{tr}(e^{-\frac{1}{2}\beta H})\varepsilon \ . \end{aligned}$$

///.

Now let $\tilde{A}$ denote the $^\#$-algebra which is algebraically generated by all elements $a(e_k)$, $a^\#(e_k)$ (the basis (e_k) is as defined just before (3.3)) and subject to the commutation relations (2.2). Then $\tilde{A} \subseteq A$.

LEMMA 4.2. $\tilde{A}$ *is dense in* A *with respect to the subspace topology inherited from* $L(\mathcal{D}_\pi)$.

Proof. First notice that the map from $S(\mathbb{R}) \times S(\mathbb{R}) \rightarrow A$ which sends $(f,g) \mapsto x_f x_g$ (for $x_f, x_g \in A$) is separately continuous.

But $\mathcal{S}(\mathbb{R})$ is a Frechet space, and, by [Tre] cor 34.1, this implies that the above map is jointly continuous, too. Consider elements $f, g \in \mathcal{S}(\mathbb{R})$ with $f = \sum_{r \geq 0} c_r e_r$, $g = \sum_{r \geq 0} d_r e_r$ and define

$$f_N = \sum_{r=0}^{N} c_r e_r \,, \quad g_N = \sum_{r=0}^{N} d_r e_r \,.$$

Clearly, $a^+(f_N)a^+(g_N)$ approximates $a^+(f)a^+(g)$ by above, where a^+ is either a or $a^\#$. For general elements $x \in A$, where $x = a^+(f_1)\ldots a^+(f_n)a^+(g_1)\ldots a^+(g_m)$, the result now follows by induction.

///.

The next result follows easily from the preceding lemmas.

COROLLARY 4.3. *Consider the topological spaces* (A, τ) *and* $(A, \|\|)$. *The* τ*-topology is the stronger one on* A, *and* $\mathcal{A}$ *is dense in* A *with respect to both topologies.*

One of the important ingredients in the definition of the almost modular Hilbert algebra is the modular operator to which we turn now. For fixed $\beta > 0$, define the *modular operator* $\Delta\colon \mathbb{C} \to \tilde{A}$ by:

$$\begin{aligned}
&1. \quad \Delta(z)1 = 1 \\
&2. \quad \Delta(z)a^\#(e_r) = a^\#(e^{-\beta z h} e_r) \qquad (a(e_r)\,,\ a^\#(e_r) \in \tilde{A}) \qquad (4.3)\\
&3. \quad \Delta(z)a(e_r) = a(e^{\beta z h} e_r)
\end{aligned}$$

$\mathcal{A}$ together with this modular operator Δ has all the desired properties of an almost modular algebra, as the next theorem shows.

THEOREM 4.4. *The modular operator* Δ *defines a group of* **-automorphisms from* $\mathbb{C}$ *into* $\mathrm{aut}(\mathcal{A})$, *which makes* $\mathcal{A}$ *an almost modular algebra.*

Proof. a) From (4.3) it is clear that $\Delta(z)$ is well-defined on

$\tilde{A}$ for every $z \in \mathbb{C}$. Since $\Delta(z)$ preserves the commutation relations, it defines an automorphism on $\tilde{A}$. We now show that $\tilde{A}$ and Δ satisfy the condition given in (4.2): (2) and (4) follow from the definition of $\tilde{A}$ which inherits the inner product of A, and (5) follows from (4.3).

(b) to prove (6): $\langle\Delta(z)x, y\rangle = \langle x, \Delta(\bar{z})y\rangle$, we first claim that

$$\pi_0(\Delta(z)x)\Psi_e^n = e^{-\beta zH}\pi_0(x)e^{z\beta H}\Psi_n \qquad (4.4)$$

which we show on the generating elements $a(e_r)$, $a^{\#}(e_r)$. Note that $e^{z\beta H}\Psi_e^n = e^{z\beta\sum w(k)n_k}\Psi_e^n$. Thus we have:

$$e^{-z\beta H}\pi_0(a_{\bar{e}_r})e^{z\beta H}\Psi_e^n = e^{-z\beta H}\pi_0(a_{\bar{e}_r})e^{z\beta\sum w(k)n_k}\ \Psi_e^n$$

$$= e^{-z\beta H}\sqrt{n}.e^{z\beta\sum w(k)n_k}\ \Psi_e^{n-1}$$

$$= e^{-z\beta\sum w(k)n_k - w(r)}\sqrt{n}\ e^{z\beta\sum w(k)n_k}\ \Psi_e^{n-1} \quad \text{(since } e_r \text{ is deleted)}$$

$$= e^{z\beta w(r)}\ \pi_0(a_{\bar{e}_r})\Psi_e^n = \pi_0(\Delta(z)a_{\bar{e}_r})\Psi_e^n \quad .$$

Similarly, one proves that

$$e^{-\beta zH}\ \pi_0(a^{\#}(e_r))e^{+\beta zH}\ \Psi_e^n = \pi_0(\Delta(z)a^{\#}(e_r))\Psi_e^n \quad .$$

From these equations for the generating elements, we get for y, $x \in \mathcal{X}$:

$$\langle\pi_0(\Delta(z)x\Psi_e^n\ ,\ \pi_0(y)\Psi_e^n\rangle = \langle e^{-\beta zH}\ \pi_0(x)e^{\beta zH}\ \Psi_e^n\ ,\ \pi_0(y)\Psi_e^n\rangle$$

$$= \langle e^{\beta\bar{z}\sum w(k)n_k}\ e^{-\beta zH}\ \pi_0(x)\Psi_e^n\ ,\ \pi_0(y)\Psi_e^n\rangle$$

$$= \langle e^{-\beta zH}\ \pi_0(x)\Psi_e^n\ ,\ \pi_0(y)e^{\beta\bar{z}H}\ \Psi_e^n\rangle$$

$$= \langle\pi_0(x)\Psi_e^n\ ,\ e^{-\beta\bar{z}H}\ \pi_0(y)e^{\beta\bar{z}H}\ \Psi_e^n\rangle = \langle\pi_0(x)\Psi_e^n\ ,\ \pi_0(\Delta(\bar{z})y)\Psi_e^n\rangle\ ,$$

and the result follows using the definition of the inner product on $\mathcal{X}$.

c) The proof of (7) : $\langle \Delta(1)x^{\#}, y^{\#} \rangle = \langle y, x \rangle$

Since $\Delta(1)a_{\bar{e}_r} = a(\overline{e^{\beta h} e_r}) = e^{\beta w(r)} a(\bar{e}_r)$, we have the following for $x \in \tilde{A}$.

$$\langle x^{\#}, \Delta(1)(a^{\#}_{e_r})^{\#} \rangle = \langle x^{\#}, \Delta(1)a_{\bar{e}_r} \rangle$$

$$= (\operatorname{tr} e^{-\beta H})^{-1} \sum_{n \geq 0} \sum_{J_n} e^{-\beta \sum w(k) n_k} \langle \pi_0(x^{\#})\Psi^n_e , \pi_0(\Delta(1)a_{\bar{e}_r})\Psi^n_e \rangle$$

$$= (\operatorname{tr} e^{-\beta H})^{-1} \sum_{n \geq 0} \sum_{J_n} e^{-\beta \sum w(k) n_k - w(r)} \sqrt{n} \langle \Psi^n_e, \pi_0(x)\Psi^{n-1}_e \rangle$$

$$= (\operatorname{tr} e^{-\beta H})^{-1} \sum_{n \geq 0} \sum_{J_n} e^{-\beta \sum w(k) n_k} \sqrt{n} \langle \Psi^{n+1}_e, \pi_0(x)\Psi^n_e \rangle$$

$$= (\operatorname{tr} e^{-\beta H})^{-1} \sum_{n \geq 0} \sum_{J_n} e^{-\beta \sum w(k) n_k} \langle a^{\#}_{e_r} \Psi^n_e , \pi_0(x)\Psi^n_e \rangle = \langle a^{\#}_{e_r}, x \rangle$$

Similarly, $\langle x^{\#}, \Delta(1)(a_{\bar{e}_r})^{\#} \rangle = \langle a_{\bar{e}_r}, x \rangle$. And the general result follows by taking finite linear combinations of the a and $a^{\#}$.

d) The proof of (8) : $z \longmapsto \langle \Delta(z)x, y \rangle$ is entire.

A general element $y \in \tilde{A}$ is of the form

$$y = a^{\#}_{e_{j1}} a^{\#}_{e_{j2}} \ldots . a^{\#}_{e_{jm}} a_{\bar{e}_{k1}} a_{\bar{e}_{k2}} \ldots . a_{\bar{e}_{kn}}$$

Thus the result follows from the following calculation.

$$\langle x, \Delta(z)y \rangle = \langle x, \prod_{r=1}^{m} \Delta(z)a^{\#}_{e_{jr}} \prod_{s=1}^{n} \Delta(z)a_{\bar{e}_{ks}} \rangle$$

$$= \prod_{r=1}^{m} e^{-\beta z w(jr)} \prod_{s=1}^{n} e^{\beta z w(ks)} \langle x, \prod_{r=1}^{m} a^{\#}_{e_{jr}} \prod_{s=1}^{n} a_{\bar{e}_{ks}} \rangle$$

$$= \langle e^{-\beta z \sum w(j_r) - \sum w(k_s)} \rangle \langle x, y \rangle$$

which proves that $z \longmapsto \langle x, \Delta(z)y \rangle$ is analytic.

e) The proof of (9): $\mathcal{X}_t$ is dense in $\mathcal{X}$.

Recall that $\mathcal{X}_t = \{(1 + \Delta_t(t))x : x \in A\}$. Take y as in the proof of d) , and put

$$x = (1 + e^{-\beta t \sum w(j_r) - \sum w(k_s)})^{-1} y \in A.$$

Then

$$y = (1 + \Delta(t))x \text{ which is in } \mathcal{X}_t .$$

From this construction of y by elements $x \in A$, it follows that A_t is dense. This completes the proof of theorem 4.4 ///.

Next we want to extend the modular automorphism Δ from $\mathcal{X}$ to A where this is possible and meaningful. For this, we use the following restrictive form and put

$$\begin{aligned} \Delta(z)a^{\#}(f) &= a^{\#}(e^{-\beta z h} f) && \text{if Re } z \geq 0 \\ \Delta(z)a(\overline{f}) &= a(e^{\beta z h}\, \overline{f}) && \text{if Re } z \leq 0 \end{aligned} \qquad (4.5)$$

PROPOSITION 4.5. *a. For* Re $z \leq 0$, $e^{z\beta h}$ *is a well-defined continuous linear operator on* $S(\mathbb{R})$ *and* $z \longmapsto e^{\beta z h} f$ *is analytic for every* $f \in S(\mathbb{R})$.

b. $\langle \Delta(1)y, x \rangle = \langle x^{\#}, y^{\#} \rangle$ *for* $x, y \in A$, *and* $y \in \mathcal{D}(\Delta(1))$.

c. $\sigma_t = \Delta(it)$ $(t \in \mathbb{R})$ *is a group of *-automorphisms on* A.

Proof. a. The map $z \longmapsto e^{\beta zh} e_r = e^{\beta zw(r)} e_r$ is analytic into $S(\mathbb{R})$. Hence for arbitrary elements $f \in S(\mathbb{R})$ with $f = \sum_{n \geq 0} c_r e_r$ we have

$$\| \sum_{r=N}^{M} c_r e^{\beta zh} e_r \|_m^2 = \sum_{r=N}^{M} |c_r|^2 (r+1)^{2m} e^{2\beta w(r).\mathrm{Re}(z)} .$$

For $N, M \longrightarrow \infty$, the right hand side of the last equation tends to zero, uniformly in z in the strip $\mathrm{Re}(z) \leq 0$. Thus, $\sum_{r \geq 0} c_r (e^{\beta zh}) e_r = e^{\beta zh} f$, converges in $S(\mathbb{R})$ inside the strip $\mathrm{Re}(z) \leq 0$, and defines an analytic function of z for $\mathrm{Re}(z) < 0$ which is continuous on the boundary.

b. The proof is similar to that given in (c) of proof to thm 4.4.

c. Put $z = it$. Since $\mathrm{Re}\, z = 0$, $\Delta(it)$ can be extended to all of H_+. It is now clear that $\Delta(it)$ is isometric on H_+. Thus $\sigma_t = \Delta(it)$ is a group of *-automorphisms on A, since it is defined everywhere. ///.

The involution # which has been used so far corresponds to the operator S_0 of section 1.4. We define the b-involution in the following way. For $x \in A$ put

$$\begin{aligned} x^b &= \Delta(1)x^{\#} \\ x^+ &= \Delta(\tfrac{1}{2})x^{\#} \end{aligned} \tag{4.6}$$

Note that b and + are both involutions on $\mathcal{X}$ which we call the *adjoint* and *unitary involution*, respectively. By theorem 4.4, $^+$ is isometric on $\tilde{A}$: $\langle \Delta(1)x^{\#}, y^{\#} \rangle = \langle x, y \rangle$, hence it can be extended to a conjugate linear isometry J on H_+, the completion of $(A, \| \ \|)$.

THEOREM 4.6. *The triple* $(A, \#, \langle .,. \rangle)$ *is a closable Hilbert algebra.*

Proof. Using prop. 4.5(b), we have for $x \in A$, $y \in \mathcal{X}$ that

$$\langle y^b, x\rangle = \langle \Delta(1)y^{\#}, x\rangle = \langle x^{\#}, y\rangle \quad .$$

And this shows that the map $x \longmapsto \langle x^{\#}, y\rangle$ $(x \in A)$ is contiuous for every $y \in A$. But $\mathcal{X}$ is dense in A, and hence the adjoint of the map $x \longmapsto x^{\#}$ is densely defined, whence $x \longmapsto x^{\#}$ is closable. ///.

As a last point in this chapter, we turn to the Gibbs state ω_β again and show:

THEOREM 4.7 *Let* $\beta > 0$, *and let* $\sigma_t = \Delta(it)$ *be the modular automorphism on* A. *Then* ω_β *is a* σ_t*-KMS state on* A.

Proof. The invariance of ω_β under σ_t follows from the calculation below (for a definition of KMS state see section 1.4):

$$\omega_\beta(\sigma_t(x)) = \langle\sigma_t(x), 1\rangle = \langle x, \sigma_{-t}(1)\rangle = \langle x, 1\rangle = \omega_\beta(x) \ .$$

For a general $x \in A$ which is of the form $x = a^{\#}_{f_1} \dots a^{\#}_{f_n} a_{\bar{g}_1} \dots a_{\bar{g}_m}$ and for $y \in A$, we define the function F by

$$F_{x,y}(z) = \omega_\beta(\Delta(iz)(a_{\bar{g}_1} \dots a_{\bar{g}_m})y\Delta(iz)(a^{\#}_{f_1} \dots a^{\#}_{f_n})) \ . \qquad (4.7)$$

Let $S = \{z \in \mathbb{C} : 0 \leq \operatorname{Im} z \leq \beta\}$; then it follows from thm. 4.4 that F is well-defined on S, that F is continuous on S and analytic inside S. Hence, F must attain its maximum modulus on the boundary and, for $t \in \mathbb{R}$, we thus have

$$F_{x,y}(t) = \langle \Delta(1)\Delta(it)(a^{\#}_{f_1} \ldots a^{\#}_{f_n}) , [\Delta(it)(a_{\bar{g}_1} \ldots a_{\bar{g}_m})y]^{\#} \rangle$$

$$= \langle \Delta(it)(a_{\bar{g}_1} \ldots a_{\bar{g}_m})y , [\Delta(it)(a^{+}_{f_1} \ldots a^{+}_{f_n})]^{\#} \rangle = \omega_{\beta}(\Delta(it)x\, y$$

by theorem 4.4. Similarly, one shows that $F_{x,y}(t + i\beta) = \omega_{\beta}(y\Delta(it)x)$. Finally, F is uniformly bounded on S , since

$$|F_{x,y}(t)| = |\langle y, \sigma_t(x)^{\#} \rangle| = |\langle \sigma_t(x^{\#}), y \rangle| \leq \|x^{\#}\| \|y\|$$

and

$$|F_{x,y}(t + i\beta)| = \langle \sigma_t(x), y^{\#} \rangle \leq \|y^{\#}\| \|x\| \quad .$$

This concludes the proof of the σ_t-KMS-property of the state ω_{β} .

///.

Chapter 3

TOMITA'S THEOREM FOR THE CCR-ALGEBRA

3.1 INTRODUCTION

In this chapter, we prove the commutation theorem for the algebra $\pi_\beta(A)$. Since we do not have any concrete knowledge of $L_\beta(A)$ and $R_\beta(A)$, we first construct another representation π of A on a Hilbert space of Hilbert-Schmidt operators for which the commutant of $L_\pi(A)$ can be given explicitly (see below). By proving that the two representations π_β and π are unitarily equivalent, we can then show the commutation theorem for $\pi_\beta(A)$ - essentially by the uniqueness of the GNS-construction.

This line of proof was adopted by Haag, Hugenholtz and Winnink [HHW] who proved the commutation theorem for a C*-algebra by showing that the GNS-representation of the Gibbs state is unitarily equivalent to the left regular representation of the Hilbert-Schmidt operators. Thereby they reduced their problem to the case of Hilbert algebras, for which the commutation theorem was known before Tomita-Takesaki theory (see also section 2.1).

For details of the following see [Dix] Ch. 1.6.
Let $HS(K)$ denote the Hilbert space of Hilbert-Schmidt operators. We identify $HS(K)$ with the tensor product $K' \otimes K$ where K' denotes the conjugate or opposed Hilbert space of K. Under the isomorphism, $x \in HS(K)$ is mapped to $\sum_i \alpha_i \xi_i \otimes \eta_i$ such that for $\xi \in K$

$$x(\xi) = \sum_i \alpha_i \langle \xi , \xi_i \rangle \eta_i \qquad , \quad \|x\|^2 = \sum_i |\alpha_i|^2 \tag{1.1}$$

where the (ξ_i) and the (η_i) form orthonormal sets in K, $\alpha_i \in \mathbb{C}$.

Since $HS(K)$ is a two-sided ideal in $B(K)$, multiplication on the left and right is continuous. Clearly the left (resp. right) von Neumann algebra associated with $B = HS(K)$ is

$$L(\mathcal{B}) = \mathbb{C}_{K'} \otimes \mathcal{B}(K) \tag{1.2}$$

(resp $\quad R(\mathcal{B}) = \mathcal{B}(K') \otimes \mathbb{C}_K$)

(see prop. 6 of [Dix]). It clearly follows (by prop. 14 of [Dix]) that

$$L(\mathcal{B})' = (\mathbb{C}_{K'} \otimes \mathcal{B}(K))' = \mathcal{B}(K') \otimes \mathbb{C}_K = R(\mathcal{B}) \tag{1.3}$$

In the case of the CCR-algebra (which consists of unbounded operators) we consider a representation on the Hilbert space of Hilbert-Schmidt operators. We show that, in this representation, the elements of the algebra act continuously on the common dense domain.

3.2 THE REPRESENTATION $(\pi, H_{\otimes}, \mathcal{D}_{\otimes})$

The aim of this section is to construct a representation of A on the Hilbert space $HS(H_+)$ which is identified with the tensor product space $H_+' \otimes H_+$. The results presented in this section can easily be generalised to a representation on $K' \otimes H_+$, where K denotes any separable Hilbert space. In the subsequent sections, however, we shall only be interested in the case where $K = H_+$, and we shall thus refrain from this generalisation. The interested reader will find no difficulty in adjusting the results to the general situation. We show the following.

THEOREM 2.1 *There exists a *-representation* $(\pi, H_{\otimes}, \mathcal{D}_{\otimes})$ *of* A *and for* $x \in A$, $\pi(x) \in L(\mathcal{D}_{\otimes})$.

Our candidate for the space $\mathcal{D}_{\otimes}$ is the space $H_+' \otimes \mathcal{D}_\pi$, and we claim;

$$H_+' \otimes \mathcal{D}_\pi \cong HS(H_+, \mathcal{D}_\pi), \tag{2.1}$$

where $HS(H_+, \mathcal{D}_\pi) = \{ x \in HS(H_+) : x(HS(H_+)) \subseteq \mathcal{D}_\pi$, and x is continuous from H_+ into $\mathcal{D}_\pi \}$.

To prove (2.1), recall that $\mathcal{D}_\pi$ is the completion of the space $\mathcal{D}$ with respect to the family of seminorms given in Ch.2 (2.13). For each $r = 0, 1, 2, \ldots$ these seminorms are in fact norms. Let $\mathcal{D}_r$ denote the completion of $\mathcal{D}_\pi$ with respect to the first r norms. Then $\mathcal{D}_0 = H_+$, $\mathcal{D}_r \supseteq \mathcal{D}_{r+1}$ and $\mathcal{D}_r$ is a Hilbert space contained in H_+ for each $r = 0, 1, 2, \ldots$. Furthermore, $\mathcal{D}_\pi = \bigcap \mathcal{D}_r$. We will use the notation $H_{\otimes} = H_+' \otimes H_+$.

LEMMA 2.2. *For* $r = 0, 1, 2, \ldots$, *the spaces* $H_+' \otimes \mathcal{D}_r$ *and* $HS(H_+, \mathcal{D}_r)$ *are isomorphic.*

Proof. Since $\mathcal{D}_0 = H_+$, $H_+' \otimes \mathcal{D}_0 = HS(H_+)$. For $r = 1, 2, \ldots$, we first show that the space of finite rank operators on H_+ with range in

$\mathcal{D}_r$ is isomorphic to the algebraic tensor product $H_+^{'} \odot \mathcal{D}_r$. (Recall that a linear operator χ: $H_+ \to \mathcal{D}_r$ is a *finite rank operator* if χ maps H_+ onto a finite dimensional subspace of $\mathcal{D}_r$.) The result then follows since the two spaces are dense in $HS(H_+, \mathcal{D}_r)$ and $H_+^{'} \otimes \mathcal{D}_r$ respectively and we can extend by continuity.

We now describe the isomorphism and its inverse. The reader can check that the maps are in fact Hilbert space isomorphisms.
Let $r \in \mathbb{N}$ be fixed. Let $F \in H_+^{'} \odot \mathcal{D}_r$, and choose $\xi_i \in H_+$, $\eta_i \in \mathcal{D}_r$, $i=1,\ldots,n$ such that $F = \sum_{i=1}^{N} \xi_i \otimes \eta_i$. Define $\mathfrak{f} \in HS(H_+, \mathcal{D}_r)$ by $\mathfrak{f}(\xi) = \sum_{i=1}^{N} \langle\xi,\xi_i\rangle\eta_i$ for $\xi \in H_+$. The definition of $\mathfrak{f}$ does not depend on the choice of the ξ_i, η_i, and $F \mapsto \mathfrak{f}$ is clearly a Hilbert space homomorphism.

Conversely let $t : H_+ \to \mathcal{D}_r$ be a finite fank operator and let $t = ua$ denote its polar decomposition. So u is an isometry from H_+ into $\mathcal{D}_r$ and a is positive from H_+ into H_+ ($a = t^* t$). Let S denote the orthonormal basis for H_+ which consists of eigenvectors for a. Let ξ_i be those elements in S which correspond to non-zero eigenvalues λ_i, and put $\eta_i = \lambda_i u \xi_i$. Then $t \mapsto \sum \xi_i \otimes \eta_i$ is a Hilbert space homomorphism and is the inverse to $F \mapsto \mathfrak{f}$. ///.

PROPOSITION 2.3 *The topological vector spaces $H_+^{'} \otimes \mathcal{D}_\pi$ and $HS(H_+, \mathcal{D}_\pi)$ are isomorphic.*

Proof. Note that the tensor product $H_+^{'} \otimes \mathcal{D}_\pi$ is not a Hilbert space since $\mathcal{D}_\pi$ is not a Hilbert space. However, it is well-defined as the (π-) completion (see [Tre] Ch. 45) of the algebraic tensor product $H_+^{'} \odot \mathcal{D}_\pi$. The space $HS_\pi = HS(H_+, \mathcal{D}_\pi)$ was defined immediately after (2.1). The isomorphism between $H_+^{'} \odot \mathcal{D}_\pi$ and F_π, the space of finite rank operators form H_+ into $\mathcal{D}_\pi$, follows in a similar fashion to the corresponding proof given in the previous lemma for the spaces $\mathcal{D}_r$.

To show that the topological vector spaces HS_π and $H_+^{\boxtimes} = H_+' \boxtimes \mathcal{D}_\pi$ are isomorphic, we show that

i. HS_π is the inverse limit of the spaces $HS(H_+,\mathcal{D}_r)$ (that is $HS_\pi = \varprojlim HS(H_+,\mathcal{D}_r)$),

ii. $H_+^{\boxtimes} = \varprojlim (H_+' \boxtimes \mathcal{D}_r)$.

The claim then follows since inverse limits are unique up to isomorphism (see [Sem] 11.8.1) and since the Hilbert spaces $HS(H_+, \mathcal{D}_r)$ and $H_+' \boxtimes \mathcal{D}_r$ are isomorphic as Hilbert spaces by lemma 2.2 for r=0,1,2,... .

To prove i. , note that HS_π is endowed with the topology induced by $\mathcal{D}_\pi$, that is, the topology derived from the inner products $\langle .,. \rangle_r$, r=0,1,... (see Ch.2 (2.13)) which make the spaces $HS(H_+,\mathcal{D}_r)$ into Hilbert spaces (see also Ch.1 (1.3)). Since $\mathcal{D}_\pi$ is complete, it follows that HS_π is a complete topological vector space.

Let T be a Hilbert-Schmidt operator on H_+ with range in $\mathcal{D}_\pi$. Then the following chain of equivalences a. - d. demonstrates that HS_π is the intersection of the spaces $HS(H_+, \mathcal{D}_r)$.

a. $T: H_+ \to \mathcal{D}_\pi$ is continuous.

b. $i_r \circ T : H_+ \to \mathcal{D}_r$ is continuous for r=0,1,... , where i_r denotes the inclusion map from $\mathcal{D}_\pi$ into $\mathcal{D}_r$, and r=0,1,...
(this equivalence follows from prop.4 p.30 of [Bou]).

c. $T: H_+ \to \mathcal{D}_r$ is continuous for r=0,1,... (range of $T \subseteq \mathcal{D}_r$, since $\mathcal{D}_\pi \subseteq \mathcal{D}_r$).

d. $T \in HS(H_+, \mathcal{D}_r)$ for every r=0,1,... .

Thus HS_π is the intersection, with the intersection topology, of the spaces $HS(H_+, \mathcal{D}_r)$, that is, the inverse limit.

To show ii. , let $L = \varprojlim (H_+' \boxtimes \mathcal{D}_r)$ denote the inverse limit of the spaces $H_+' \boxtimes \mathcal{D}_r$, and let $C = H_+' \otimes \mathcal{D}_\pi$ denote the algebraic tensor product of

H_+ and $\mathcal{D}_\pi$. Clearly, $\mathcal{C}$ embeds continuously into each space $H'_+ \boxtimes \mathcal{D}_r$, and hence (from the definition of the inverse limit) there is a unique map $c : \mathcal{C} \to L$ such that the following diagram commutes for each r=0,1,...

$$\begin{tikzcd}
L \arrow[r, hook] & H'_+ \boxtimes \mathcal{D}_r \\
\mathcal{C} \arrow[u, "c"] \arrow[r, hook] & H'_+ \odot \mathcal{D}_r \arrow[u, hook]
\end{tikzcd}$$

where $\hookrightarrow$ denotes the natural embedding. It follows that c is also an embedding. Let $\overline{\mathcal{C}}$ denote the closure of $c(\mathcal{C})$ in L. Then by [Bou] Cor.(ii p.49, $\overline{\mathcal{C}}$ is the inverse limit of the closures of $\mathcal{C}$ with respect to the topology induced by $H'_+ \boxtimes \mathcal{D}_r$ (r=0,1,...) . Since the algebraic tensor product $\mathcal{C}$ is dense in $H'_+ \boxtimes \mathcal{D}_r$ for each r=0,1,... , it follows that $\overline{\mathcal{C}} = \varprojlim (H'_+ \boxtimes \mathcal{D}_r) = L$, and that c is a dense embedding. Next observe (see [Bou] cor. p.187) that L is complete since the spaces $H'_+ \boxtimes \mathcal{D}_r$ are complete for r=0,1,... . A further reference to Bourbaki ([Bou] prop.13 p.195) now shows that L is the completion of $\mathcal{C}$, and hence $L = H'_+ \boxtimes \mathcal{D}_\pi$.

///.

Putting $\mathcal{D}_\otimes = H'_+ \otimes \mathcal{D}_\pi = HS(H_+, \mathcal{D}_\pi)$, it is clear that $\mathcal{D}_\otimes$ is dense in $H'_+ \otimes H_+$, since $\mathcal{D}_\pi$ is dense in H_+. $\mathcal{D}_\otimes$ is complete in the tensor product topology on $H'_+ \otimes \mathcal{D}_\pi$ where $\mathcal{D}_\pi$ carries the topology given by the norms of Ch.2 (2.13). We next concentrate on the representation π. For $x \in A$, let $\pi(x)$ denote the linear operator on $H_\boxtimes$ which is defined for $T \in \mathcal{D}_\boxtimes$ by

$$\pi(x) : T \longmapsto \pi_0(x)T \quad . \tag{2.2}$$

From (2.2) it follows immediately that π is a *-homomorphism from A into the linear operators on $H_\boxtimes$.

PROPOSITION 2.4.

1. *For* $x \in A$ *,* $\pi(x)$ *maps* $\mathcal{D}_{\otimes}$ *into itself.*
2. *For* $x \in A$ *,* $\pi(x) \in L(\mathcal{D}_{\otimes})$.

Proof. 1. Let $T \in \mathcal{D}_{\otimes}$, $x \in A$, then $\pi(x)T = \pi_0(x)T$ is in $\mathcal{D}_{\otimes}$. To see this, note that T is a Hilbert-Schmidt operator from H_+ into $\mathcal{D}_\pi$, $\pi_0(x)$ leaves $\mathcal{D}_\pi$ invariant by Ch. 2 thm. 2.1 and hence $\pi_0(x)T$ is a Hilbert-Schmidt operator from H_+ into $\mathcal{D}_\pi$ by [G & V] thm. 3 p. 36.

2. To prove that $\pi(x)$ is continuous from $\mathcal{D}_{\otimes}$ into itself, note that for $x \in A$ $\pi_0(x) \in L(\mathcal{D}_\pi)$; hence for $m \in \mathbb{N}$, there exists a $k > 0$ and $r \in \mathbb{N}$ such that

$$\|\pi_0(x)\eta\|_m \leq k\|\eta\|_r \qquad \text{for each } \eta \in \mathcal{D}_\pi .$$

Put $T = \sum_{i=1}^{N} \overline{\xi}_i \otimes \eta_i$, where ξ_i is an orthonormal basis in H_+ and $\eta_i \in \mathcal{D}_\pi$, then

$$\|\pi_0(x)T\|_m^2 = \sum_{i=1}^{N} \|\pi_0(x)\eta_i\|_m^2 \leq k^2 \sum_{i=1}^{N} \|\eta_i\|_r^2 = k^2\|T\|_r^2 .$$

So $\pi_0(x)T \in H_+' \otimes \mathcal{D}_\pi$, and $\pi(x) \in L(\mathcal{D}_{\otimes})$, by Ch.2 (2.13)&(2.14). ///.

The proof of thm. 2.1 depended on the properties of the space $\mathcal{D}_\pi$ in the sense that we showed that properties of $\mathcal{D}_\pi$ carried over to and were compatible with the structure of the tensor product. This mode of proof is no longer possible when considering the commutant π' of π . We first consider the commutant of π_o.

PROPOSITION 2.5 *$\pi_o(A)$ is irreducible.*

Proof. By Ch.1 (2.2), π_o is irreducible if and only if $\pi_o(A)' = \{\lambda 1\}$ We first show that $\pi_o(\tilde{A})' = \{\lambda 1\}$. The result then follows for $\pi_0(A)'$, since $\pi_o(\tilde{A}) \subseteq \pi_o(A)$ implies that $\pi_o(A)' \subseteq \pi_o(\tilde{A})'$. Put $N(e_r) = a^{\#}(e_r)a(e_r)$. Then $\pi_0(N(e_r))\Psi^n_e = n\Psi^n_e$, and it follows that for $T \in \pi_0(\tilde{A})'$

$$n \langle T\Psi^o, \Psi^n_e\rangle = \langle T\Psi^o, \pi_0(N(e_r))\Psi^n_e\rangle = \langle T\pi_0(N(e_r))\Psi^o, \Psi^n_e\rangle = 0$$

since $\pi_0(N(e_r))\Psi^o = 0$. But this implies that $T\Psi^o$ is orthogonal to all the spaces S^n_+ for $n > 0$, hence $T\Psi^o$ is in the subspace spanned by Ψ^o .

For general $x \in \tilde{A}$ and $\xi \in \mathcal{D}_\pi$ we have thus for some $t \in \mathbb{C}$

$$\langle T\pi_0(x)\Psi^o, \xi\rangle = \langle T\Psi^o, \pi_0(x^{\#})\xi\rangle = t\langle \Psi^o, \pi_0(x^{\#})\xi\rangle = t\langle \pi_0(x)\Psi^o, \xi\rangle .$$

From this it follows that $T\pi_0(x)\Psi^o = t\pi_0(x)\Psi^o$. And from prop. 4.7 in [Thu] we conclude that Ψ^o is cyclic for $\pi_0(A)$. Hence, Ψ^o is cyclic for $\pi_0(\tilde{A})$, too, whence it follows that $T = t1$, since T is bounded. ///.

The next theorem is a generalisation of the one-dimensional results proved in [G & H].

THEOREM 2.6. *The commutant $\pi(A)'$ of $\pi(A)$ consists of all right multiplications by bounded operators on H_+, that is,*

$$\pi(A)' = \{C \in B(H'_+ \otimes H_+) : \exists C_1 \in B(H_+) \text{ with } C(T) = TC_1 \ \forall T \in H'_+ \otimes H_+\} .$$

Proof. Let $C_1 \in B(H_+)$. Define $C \in B(H_+' \otimes H_+)$ by $C(T) = TC_1$ for $T \in H_+' \otimes H_+$. Since $H_+' \otimes H_+$ is a right $B(H)$-module, it is clear that C leaves $H_+' \boxtimes H_+$ invariant. Since

$$<C(\pi(x)T),S> = <C(\pi_o(x)T),S> = <\pi_o(x)TC_1,S> = <C(T),\pi_o(x)^{\#}S> \quad ,$$

for $x \in A$, $S, T \in \mathcal{D}_{\boxtimes}$, it follows that $C \in \pi(A)'$.

Conversely, let $C \in \pi(A)'$. Fix $\xi,\xi' \in H_+$ and consider the map $B_{\xi,\xi'}$ which is defined on $H_+ \times H_+$ by

$$B_{\xi,\xi'}(\eta', \eta) = <C(\overline{\xi}' \otimes \eta'), \overline{\xi} \otimes \eta>$$

Now, since $|B_{\xi,\xi'}(\eta,\eta')| \leq \|C\|\|\xi\|\|\xi'\|\|\eta\|\|\eta'\|$, where $\|C\|$ denotes the operator norm, all the other norms are the L^2-norms, $B_{\xi,\xi'}$ is a bounded sesqui-linear form and thus there exists a unique $K \in B(H_+)$ depending on ξ and ξ' such that

$$B_{\xi,\xi'}(\eta',\eta) = <K\eta', \eta> \quad \text{for all } \eta, \eta' \in H_+ \; .$$

Clearly $\quad \|K\| \leq \|C\|\|\xi\|\|\xi'\| \quad , \qquad (2.3)$

where $\|K\|$ and $\|C\|$ denote operator norms in $B(H_+)$ and $B(H_+' \otimes H_+)$ respectively. We claim that $K \in \pi_0(A)'$.

To see this, let $\zeta, \zeta' \in \mathcal{D}_\pi$, $x \in A$ and consider the expression

$$<K\pi_0(x)\zeta', \zeta> = B_{\xi,\xi'}(\pi_0(x)\zeta', \zeta)$$

$$= <C(\overline{\xi}' \otimes \pi_0(x)\zeta'), \overline{\xi} \otimes \zeta>$$

$$= <C\pi(x)(\overline{\xi}' \otimes \zeta'), \overline{\xi} \otimes \zeta>$$

$$= <C(\overline{\xi}' \otimes \zeta'), \pi(x^*)(\overline{\xi} \otimes \zeta)>$$

$$= B_{\xi,\xi'}(\zeta', \pi_0(x^*)\zeta> \quad = <K\zeta', \pi_0(x^*)\zeta> \quad .$$

This proves the claim above, by prop. 2.5, and hence there is a $k \in \mathbb{C}$, k depending on ξ and ξ', such that $K = kI$. Thus for $\eta, \eta' \in H_+$, we have

$$k = <K\eta', \eta>/<\eta', \eta> = B_{\xi,\xi'}(\eta', \eta)/<\eta', \eta> \tag{2.4}$$

$$= <C(\bar{\xi}' \otimes \eta'), \bar{\xi} \otimes \eta>/<\eta', \eta> \quad .$$

This last equation shows that the map from $H_+ \times H_+ \longrightarrow \mathbb{C}$ which sends $(\bar{\xi}', \bar{\xi})$ to k is sequilinear. It is also bounded, since $|k| = \|K\| \leq \|C\|\|\bar{\xi}'\|\|\bar{\xi}\|$ by (2.4). Hence there exists a unique $C_1 \in B(H_+')$ such that $k = <C_1 \bar{\xi}', \xi>$. By (2.4) it now follows that

$$<C(\bar{\xi}' \otimes \eta'), \bar{\xi} \otimes \eta> = k<\eta', \eta> \quad = <(\bar{\xi}' \otimes \eta')C_1 , \bar{\xi} \otimes \eta> \quad .$$

because of the isomorphism between $H_+' \otimes H_+$ and $HS(H_+, \mathcal{D}_\pi)$.

But the rank one operators are total in $H_+' \otimes H_+$, and it thus follows that $C(\bar{\xi} \otimes \eta) = (\bar{\xi} \otimes \eta)C_1$, and hence $C(T) = T.C_1$ for arbitrary $T \in H_+' \otimes H_+$. ///.

3.3 THE STRONGLY CYCLIC VECTOR FOR $(\pi, \mathcal{H}_{\boxtimes}, \mathcal{D}_{\boxtimes})$

In this section we show that π has a strongly cyclic vector Ω_π such that $\omega_\beta(x) = \langle\pi(x)\Omega_\pi , \Omega_\pi\rangle$ for $x \in A$. To exhibit such a vector for the representation $(\pi, \mathcal{H}_{\boxtimes}, \mathcal{D}_{\boxtimes})$ is the single most important step in proving the commutation theorem for $\pi_\beta(A)$, as it will enable us to construct the unitary operator between the two representations, and then use the uniqueness of the GNS-representation to deduce the result.

In Ch. 2, prop 3.7, we have shown that for $\beta > 0$, $e^{-\beta H}$ maps $\mathcal{H}_+$ continuously into $\mathcal{D}_\pi$ and is trace class. But every trace class operator is a Hilbert-Schmidt operator, hence $e^{-\beta H}$ belongs to $\mathcal{D}_\otimes = \mathcal{H}_+' \otimes \mathcal{D}_\pi = HS(\mathcal{H}_+, \mathcal{D}_\pi)$. Put

$$\Omega = e^{-\frac{1}{2}\beta H} . \tag{3.1}$$

Then $\Omega \in \mathcal{D}_\otimes$, by the last comment, and Ω can be approximated by finite rank operators of the following form:

$$\Omega^n = \sum_{J_n} e^{-\frac{1}{2}\beta H}\, \overline{\Psi}_e^n \otimes \Psi_e^n \tag{3.2}$$

where J_n is as in Ch.2 (3.11) and the Ψ_e^n are as in section 2.3 (see the paragraph following eq.(3.3)).

PROPOSITION 3.1. *Ω^n belongs to $\mathcal{D}_\otimes$. For $n \longrightarrow \infty$, Ω^n converges to Ω in $\mathcal{D}_\otimes$.*

A proof of this follows the next lemma.

Let $J_{n,\rho} = \{(n_0 n_1 \ldots .) \in J_n : n_i = 0 \text{ for } 0 \le i \le \rho - 1\}$. For positive real λ, let

$$S_{n,\rho}(\lambda) = \sum_{J_{n,\rho}} e^{-\beta\lambda \sum_k k n_k} \tag{3.3}$$

LEMMA 3.2. *For* $\varepsilon > 0$, $m \in \mathbb{N}$, *we have*

$$\sum_{J_{n,\rho}} e^{-\beta\sum_k (k+\varepsilon)n_k} \left(\sum_k (k+\varepsilon)n_k\right)^m = \left(\frac{-1}{\beta}\frac{d}{d\lambda}\right)^m S_{n,\rho}(\lambda)e^{-\varepsilon n\lambda\beta}\Big|_{\lambda=1}$$

Proof. Note that the power series in $e^{-\lambda}$ converges absolutely; we can thus differentiate $S_{n,\rho}$ term by term m times. This clearly gives the desired equality. ///.

Proof of Proposition 3.1. Let $\|\ \|_0^2$ denote the Hilbert space norm on $H'_+ \otimes \ell^2(\mathbb{R})$. Then $\|\Omega^n\|_0^2 = \sum_{J_n} e^{-\beta\sum_k w(k)n_k}$. This expression is finite by Ch. 2 lemma 3.4, and thus Ω^n belongs to $H'_+ \otimes \ell^2(\mathbb{N}^n)$. To prove the claim we show that $\Omega^n \in H'_+ \otimes S^n_+$. This means that we have to show that Ω^n converges for all the seminorms which define the topology on S^n_+, and hence on $H'_+ \otimes S^n_+$.
Fix $m \in \mathbb{N}$, and let $\|\ \|_m$ denote the m-seminorm. Then

$$\|\Omega^n\|_m^2 = \sum_{J_n}\left[e^{-\frac{1}{2}\beta\sum_k w(k)n_k}\right]^2 \left(\sum(k+1)n_k\right)^m$$

$$\leq \sum_{J_n} e^{-\beta\sum(k+\varepsilon)n_k}\left(\sum(k+1)n_k\right)^m \qquad \text{since } w(k) \geq k+\varepsilon$$

$$\leq \sum e^{-\beta\sum(k+\varepsilon)n_k}\left(\sum(k+\varepsilon)n_k\right)^m (n+A)^{A/2} \qquad \text{for some } A > 0$$

$$= \left(\frac{-1}{\beta}\frac{d}{d\lambda}\right)^m S_n(\lambda)e^{-\varepsilon n\beta\lambda}\Big|_{\lambda=1} \cdot (n+A)^{A/2} \qquad \text{by Lemma 3.2}$$

with $S_n = S_{n,0}$.

Replacing β by $\beta\lambda$ in Ch.2 lemma 2.3, it follows immediately that the series $\sum_{n>0} e^{-\varepsilon n\beta\lambda} S_n(\lambda)(n+A)^{A/2}$ converges absolutely for $\lambda > 0$ (by Ch.2 lemma 3.3 and lemma 3.4) and thus it can be differentiated term by term, leading to

$$\sum_{n\geq 0}\|\Omega^n\|_m^2 \leq \left(\frac{-1}{\beta}\frac{d}{d\lambda}\right)^m \sum_{n\geq 0} e^{-\varepsilon n\beta\lambda} S_n(\lambda)\Big|_{\lambda=1} \cdot (n+A)^{A/2} < \infty$$

///.

We now define the strongly cyclic vector by

$$\Omega_\pi = \Omega/\mathrm{tr}(e^{-\beta H})^{\frac{1}{2}} \quad . \tag{3.4}$$

LEMMA 3.3. *Ω_π is a unit vector in $H_+' \otimes H_+$ which is contained in $\mathcal{D}_\otimes$. Furthermore, for $x \in A$, we have*

$$\omega_\beta(x) = \langle \pi(x)\Omega_\pi \ , \Omega_\pi \rangle \tag{3.5}$$

Proof. The vector Ω belongs to $HS(H_+)$. Since $\Omega_\pi = e^{-\frac{1}{2}\beta H}/\mathrm{tr}(e^{-\beta H})^{\frac{1}{2}}$, its Hilbert-Schmidt norm is

$$\|\Omega_\pi\|_{HS}^2 = \mathrm{tr}(\Omega_\pi^*\Omega_\pi) = \mathrm{tr}(e^{-\frac{1}{2}\beta H}e^{-\frac{1}{2}\beta H})/\mathrm{tr}\ e^{-\beta H} = 1 \ .$$

Since $\Omega \in \mathcal{D}_\boxtimes$ by Ch.2 prop 3.7, $\Omega_\pi \in \mathcal{D}_\boxtimes$. Using the inner product in $HS(H_+)$, and the definition of π (see eq.(2.2)), for $x \in A$ we have

$$\langle \pi(x)\Omega_\pi, \Omega_\pi \rangle = \mathrm{tr}(\Omega_\pi^*\pi(x)\Omega_\pi) = \mathrm{tr}(e^{-\beta H}\pi(x))/\mathrm{tr}(e^{-\beta H}) = \omega_\beta(x)$$

///.

THEOREM 3.4 *Ω_π is strongly cyclic for $(\pi, H_\boxtimes, \mathcal{D}_\boxtimes)$.*

Outline of Proof. The idea of the proof of theorem 3.4 consists in showing that if C is the closure of $\pi(A)\Omega_\pi$ in the topology of $\mathcal{D}_\otimes$, then $C = \mathcal{D}_\otimes$. This will show that $\pi(A)\Omega_\pi$ is dense in $\mathcal{D}_\boxtimes$ in the t_π-topology which is induced by the family of semi-norms $\{\|.\|_{\pi(x)} = \|\pi(x).\| \ : x \in A\}$, since this topology is weaker than the topology induced on $\pi(A)\Omega_\pi$ by $\mathcal{D}_\otimes$.

Let $r \in \mathbb{N}$, $\underline{k} = (k_0,\ldots, k_r) \in \mathbb{N}^{r+1}$ be fixed but arbitrary and set $k = \sum_{j=0}^{r} k_j$.

For $x \in A$, $\underline{z} = (z_0,\ldots,z_r) \in \mathbb{C}^{r+1}$ with $\mathrm{Im} z_j > \beta\varepsilon/4$, put

$$T_r(x, \underline{z}) = \pi(x \ e^{i\sum_j z_j N(e_j)})\Omega \quad , \tag{3.6}$$

$$N(e_j) = a^{\#}(e_j)a(\bar{e}_j) \qquad . \tag{3.7}$$

We first prove (in lemma 3.5) that $T_r(x, \underline{\theta}) \in C$ for $\theta_j \in [0, 2\pi]$. Next we show (lemma 3.6) that

$$I_r^{\underline{k}}(T) = \left[\frac{1}{2\pi}\right]^{r+1} \int_{[0, 2\pi]^{r+1}} e^{-i\sum_{j=0}^{r} k_j\theta_j} T_r(x, \theta)d\theta \tag{3.8}$$

exists in the weak sense and hence belongs to C.

Finally, we prove in lemma 3.7 that the integral $I_r^{\underline{k}}(T)$ approximates a scalar multiple of $\pi_0(x)\bar{\psi}_e^k \boxtimes \psi_e^k$ as $r \to \infty$. The result follows from this, since vectors of the latter kind are total in $\mathcal{D}_{\boxtimes}$.

LEMMA 3.5. *For* $\underline{\theta} = (\theta_0, \theta_1, \ldots, \theta_r) \in [0, 2\pi]^{r+1}$, $x \in A$, $r \in \mathbb{N}$

$$T_r(x, \underline{\theta}) = \pi(x\, e^{i\sum_{j=0}^{r} \theta_j N(e_j)})\Omega \qquad \textit{belongs to} \quad C\ .$$

Proof. 1. By (3.2) and prop 3.1, we can rewrite $T_r(x, \underline{z})$ in the following way

$$T_r(x, \underline{z}) = \sum_{n \geq 0} \sum_{J_n} e^{-\frac{1}{2}\beta\sum_k w(k)n_k} e^{i\sum_{j=0}^{r} n_j z_j} \pi_0(x)\bar{\Psi}_e^n \otimes \Psi_e^n \tag{3.9}$$

which is the form we shall be using in the proof.

Let $F_{N,q}(x, \underline{z}) = \sum_{n=0}^{N} \sum_{J_n^q} e^{-\frac{1}{2}\beta\sum_k w(k)n_k} e^{i\sum_{j=0}^{r} n_j z_j} \pi_0(x)\bar{\Psi}_e^n \otimes \Psi_e^n$,

where $J_n^q = \{(n_0 n_1 \ldots) \in J_n : n_i = 0 \;\; \forall i > q\}$, $N, q \in \mathbb{N}$, and $\mathrm{Im} z_j > \frac{1}{4}\beta\varepsilon$.

These functions $F_{N,q}$ are (separately) analytic functions of $\underline{z}$ into $\mathcal{D}_{\otimes}$, as they are finite rank operators with analytic coefficients.

Note that $\beta \sum_{k \geq 0} w(k)n_k + 2 \sum_{j=0}^{r} \mathrm{Im} z_j n_j$

$$\geq \beta \sum_{k \geq 0} w(k)n_k - \sum_{j=0}^{r} (\beta\varepsilon/2)n_j \geq \beta \sum_{k \geq 0} (w(k) - \tfrac{\varepsilon}{2})n_k .$$

Thus

$$\|T_r(x, \underline{z}) - F_{N,q}(x, \underline{z})\|_m^2$$

$$= (\sum_{n\geq 0} \sum_{J_n} - \sum_{n=0}^{N} \sum_{J_n^q}) \| e^{-\frac{1}{2}\beta\sum_k w(k)n_k} e^{i\sum_{j=0}^{r} n_j z_j} \pi_0(x)\overline{\Psi}_e^n \otimes \Psi_e^n\|_m^2$$

$$\leq (\sum_{n\geq 0} \sum_{J_n} - \sum_{n=0}^{N} \sum_{J_n^q})(e^{-\beta\sum_k (w(k) - \frac{\varepsilon}{2})n_k})\|\pi_0(x)\Psi_e^n\|_m^2 \longrightarrow 0$$

as $N, q \longrightarrow \infty$, uniformly in $\underline{z}$ in the region $\mathrm{Im} z_j > -\frac{1}{4}\beta\varepsilon$, since

$$\sum_{n \geq 0} \sum_{J_n} e^{-\beta\sum_k w(k)n_k} \|\pi_0(x)\Psi_e^n\|_m^2 < \infty$$

by prop 3.1, where we may replace $w(k)$ by $w'(k) = w(k) - \frac{\varepsilon}{2}$, since $w'(k) \geq k + \frac{\varepsilon}{2}$.

Thus we have shown that

$$\lim_{N,q \longrightarrow \infty} F_{N,q}(x, \underline{z}) = T_r(x, \underline{z})$$ uniformly in $\underline{z}$ in the region

$\mathrm{Im} z_j > \frac{1}{4}\beta\varepsilon$ in the topology of $\mathcal{D}_\otimes$. This clearly shows that $T_r(x, \underline{z})$

is (separately) analytic in $\underline{z}$ in the above region.

2. To show that $T_r(\chi, \underline{\theta}) \in C$, we use the following approach. Let $S \in \mathcal{D}_{\otimes}^*$, the topological dual of $\mathcal{D}_{\otimes}$, be such that $S(\pi(\chi)\Omega_\pi) = 0$ for each $\chi \in A$. We show that $S(T_r(\chi, \underline{\theta})) = 0$ for each $\underline{\theta} \in [0, 2\pi]^{r+1}$. This, then, will prove, by the Hahn-Banach theorem, that $T_r(\chi, \underline{\theta}) \in C$.

Define $f(\underline{z}) = S(T_r(\chi, \underline{z}))$. This function is analytic in the region where T_r is. For $k_0, k_1, \ldots, k_r \in \mathbb{N}$ consider partial derivatives of $F_{N,q}$ and put $\partial_j^{k_j} = \left(\frac{\partial}{\partial z_j}\right)^{k_j}$; then

$$\partial_0^{k_0}\partial_1^{k_1} \ldots \partial_r^{k_r} F_{N,q}(\chi, \underline{z})$$

$$= \sum_{n=0}^{N} \sum_{J_n^q} e^{-\frac{1}{2}\beta\sum_k w(k)n_k} (in_0)^{k_0}(in_1)^{k_1}\ldots(in_r)^{k_r} e^{i\sum_{j=0}^{r} n_j z_j} \pi_0(\chi)\bar{\Psi}_e^n \otimes \Psi_e^n$$

$$= \sum_{n=0}^{N} \sum_{J_n^q} e^{-\frac{1}{2}\beta\sum_k w(k)n_k + i\sum_{j=0}^{r} n_j z_j} \pi_0(\chi(iN(e_0))^{k_0}\ldots(iN(e_r))^{k_r})\bar{\Psi}_e^n \otimes \Psi_e^n$$

$$\xrightarrow[N,q\to\infty]{} T_r(\chi(iN(e_0))^{k_0}\ldots(iN(e_r))^{k_r}, \underline{z}) \text{ as in part 1 of the proof.}$$

Hence

$$\lim_{N,q\longrightarrow\infty} S(\partial_0^{k_0}\ldots\partial_r^{k_r}F_{N,q}(\chi,\underline{z})) = S(T_r(\chi(iN(e_0))^{k_0}\ldots(iN(e_r))^{k_r},\underline{z}))$$

and

$$\partial_0^{k_0}\ldots\partial_r^{k_r} f(\underline{\theta}) = S(T_r(x(iN(e_0))^{k_0}\ldots(iN(e_r))^{k_r}, \underline{\theta}))$$

as can be seen by repeated application of the dominated convergence theorem. Thus

$$\partial_0^{k_0}\ldots\partial_r^{k_r} f(\underline{\theta})\Big|_{\underline{\theta}=0} = S(\pi(x(iN(e_0))^{k_0}\ldots(iN(e_r))^{k_r})\Omega) = 0$$

since $x(iN(e_0))^{k_0}\ldots(iN(e_r))^{k_r} \in A$ and $T_r(y, \underline{0}) = \pi(y)\Omega$. Thus, the $(r + 1)$-fold Taylor expansion for f which converges for $\underline{\theta} \in [0, 2\pi]^{r+1}$ by the analyticity of f , gives

$$f(\underline{\theta}) = \sum_{k\geq 0} (k!)^{-1}\Big(\sum_{j=0}^{r} \theta_j \frac{\partial}{\partial\varphi_j}\Big)^k f(\underline{\varphi})\Big|_{\underline{\varphi}=0} = 0 \quad ,$$

that is, $S(T_r(x, \underline{\theta})) = 0$ which shows that $T_r(x, \underline{\theta}) \in C$. ///.

LEMMA 3.6 *The integral* $I_r^{\underline{k}}(T)$ *exists and belongs to* C. *For* $s \geq r$ *and* $\tilde{k} = (k_0,\ldots,k_r,k_{r+1},\ldots,k_s) \in \mathbb{N}^{s+1}$ *with* $k_j = 0$ *for* $r<j\leq s$, *we have*

$$I_s^{\tilde{k}}(T) = e^{-\frac{1}{2}\beta\sum_{j=0}^{s} w(j)k_j}\,\pi_0(x)\overline{\Psi}^k(e(k_0,\ldots k_s,0\ldots)\otimes\Psi^k(e(k_0\ldots,k_s,0\ldots)$$

$$+\sum_{n\geq k+1}\sum_{I_n^s} e^{-\frac{1}{2}\beta\sum_i w(i)n_i}\,\pi_0(x)\overline{\Psi}_e^n\otimes\Psi_e^n \quad . \tag{3.10}$$

where $I_n^s = \{(n_0 n_1\ldots) \in J_n : \sum n_i = n,\ n_i = k_i \text{ for } 0\leq i\leq s\}$.

Proof. The function $\underline{\theta} \longmapsto e^{-i\sum_{j=0}^{r} k_j\theta_j}\, T_r(x, \underline{\theta})$ from $[0, 2\pi]^{r+1}$ into C is continuous, hence integrable in the weak sense ([Rud] thm. 3.27) and its integral $I_r^{\underline{k}}(T)$ which belongs to C, has the following form:

$$I_r^{\underline{k}}(T) = \left[\frac{1}{2\pi}\right]^{r+1} \int_{[0,\, 2\pi]^{r+1}} e^{-i\sum_{j=0}^{r} k_j\theta_j}\, T_r(x, \underline{\theta})\, d\underline{\theta}$$

$$= \sum_{n\geq 0} \sum_{J_n} e^{-\frac{1}{2}\beta\sum_k w(k)n_k} \cdot \left[\frac{1}{2\pi}\right]^{r+1} \int_{[0,2\pi]^{r+1}} e^{i\sum_{j=0}^{r} (n_j - k_j)\theta_j}\, \pi_0(x)\overline{\Psi}_e^n \otimes \Psi_e^n d$$

$$= \sum_{n\geq 0} \sum_{J_n} e^{-\frac{1}{2}\beta\sum_k w(k)n_k} \prod_{j=0}^{r} \delta(k_j, n_j)\, \pi_0(x)\overline{\Psi}_e^n \otimes \Psi_e^n$$

$$= e^{-\frac{1}{2}\beta\sum_{j=0}^{r} w(j)k_j}\, \pi_0(x)\overline{\Psi}^k(e(k_0,\ldots,k_r,0\ldots) \otimes \Psi^k(e(k_0,\ldots,k_r,0\ldots$$

$$+ \sum_{n\geq k+1} \sum_{L_n^r} e^{-\frac{1}{2}\beta\sum_i w(i)n_i}\, \pi_0(x)\overline{\Psi}_e^n \otimes \Psi_e^n$$

For $s \geq r$ and for $\tilde{k} \in \mathbb{N}^{s+1}$ as in the statement of the lemma instead of $\underline{k} \in \mathbb{N}^{r+1}$, we now estimate the integral $I_s^{\tilde{k}}(T)$.

$$I_s^{\tilde{k}}(T) = \left[\frac{1}{2\pi}\right]^{s+1} \int_{[0,\, 2\pi]^{s+1}} e^{-i\sum_{j=0}^{s} k_j\theta_j}\, T_s(x, \underline{\theta})\, d\underline{\theta}$$

$$= e^{-\frac{\beta}{2}\sum_{j=0}^{s} w(j)k_j}\, \pi_0(x)\overline{\Psi}^k(e(k_0,\ldots k_r,0\ldots) \otimes \Psi^k(e(k_0,\ldots k_r,0,\ldots)$$

$$+ \sum_{n\geq k+1} \sum_{L_n^s} e^{-\frac{\beta}{2}\sum_i w(i)n_i} \pi_0(x)\bar{\Psi}^n \otimes \Psi^n \quad .$$

This is the required result. ///.

LEMMA 3.7 *For $s \to \infty$, $\tilde{I}_s^k(T)$ approximates $\lambda \pi_0(x)\bar{\psi}_e^k \otimes \psi_e^k$ $(\lambda \in \mathbb{C})$.*

Proof. If we consider the expression for $\tilde{I}_s^k(T)$ exhibited at the end of the proof of lemma 3.6, we notice that the first term remains unchanged for any $s \geq r$, since $k_j = 0$ for $r < j \leq s$.

We prove the lemma by showing that the second term in (3.10) goes to zero as $s \to \infty$. The result then follows for $\lambda = \exp(\frac{-\beta}{2} \sum_{j=0}^{r} w(j)k_j)$.

Let E_r be the orthogonal projection onto the subspace of H_+ spanned by $\{\Psi_e^n : (n_0,\dots) \in I_n^s, n \in \mathbb{N}\}$. Then $E_s\Psi^k(e(k_0,..,k_r,0\dots))$ $= \Psi^k(e(k_0,..,k_r,0\dots))$ for all $s \geq r$, while $E_s \longrightarrow 0$ strongly in the orthogonal complement of the $\Psi^k(e(k_0,\dots,k_r,0,\dots))$.

Let F_k denote the projection onto the orthogonal complement of the subspace spanned by $\{\Psi_e^n : 0 \leq n \leq k\}$, and put $T_k = e^{-\beta H}F_k$. Then T_k is a trace class operator by Ch. 2 cor. 3.8. For $m \in \mathbb{N}$, $x \in A$, $\eta \in \mathcal{D}_\pi$ consider

$$\|B^m\pi_0(x)T_kE_s\eta\|_0^2 = \|\pi_0(x)T_kE_s\eta\|_2^{2m} \leq C^2\|T_kE_s\eta\|_{2p}^2$$

where B denotes the second-quantisation operator of h (Ch.2 (2.13)), $p \in \mathbb{N}$ and C constant. The last inequality follows since $\pi_0(x) \in L(\mathcal{D}_\pi)$. Thus the above equals

$$c^2\|B^pT_kE_s\eta\|_0^2 \le c^2\|B^pT_k\|^2\|\eta\|_0^2 \;;$$ B^pT_k is trace class, since T_k is. Thus

$$\|B^m\pi_0(x)T_kE_s\| \le c\|B^pT_k\|$$

and hence the sequence $(B^m\pi_0(x)T_kE_s)_r$ is norm bounded, and as it converges to zero strongly, it also converges ultrastrongly. Thus

$$\|\sum_{n\ge k+1}\sum_{I_n^s} e^{-\frac{\beta}{2}\sum_i w(i)n_i}\pi_0(x)\overline{\Psi}_e^n \otimes \Psi_e^n\|_{2m}^2$$

$$= \sum_{n\ge k+1}\sum_{I_n^s} e^{-\beta\sum_i w(i)n_i}\|\pi_0(x)\Psi_e^n\|_{2m}^2$$

$$= \sum_{n\ge k+1}\sum_{I_n^s} e^{-\beta\sum_i w(i)n_i}\|B^m\pi_0(x)\Psi_e^n\|_0^2$$

$$= \sum_{n\ge k+1}\sum_{I_n^s} \|B^m\pi_0(x)T_kE_s\Psi_e^n\|_0^2 \longrightarrow 0 \quad \text{as} \quad s\longrightarrow\infty$$

by the definition of the ultraweak topology. Thus $\tilde{I}_s^k(T) \to \lambda\pi_0(x)\overline{\Psi}_e^k \otimes \Psi_e^k$ as $s \longrightarrow \infty$ $(\lambda \in \mathbb{C})$.

///.

So far, we have shown that for $r, k_0,\dots,k_r \in \mathbb{N}$ with $\sum_i k_i = k$ and for $x \in A$, the element $\pi_0(x)\overline{\Psi}_e^k \otimes \Psi_e^k$ belongs to C. To complete the proof of theorem 3.4, we now observe that these elements are total in $\mathcal{D}_\otimes$. To see this, let $\eta \in \mathcal{D}_\pi$. Since $\{\pi_0(x)\Psi^\circ : x \in A\}$ is dense in $\mathcal{D}_\pi$, given $\varepsilon > 0$, $m \in \mathbb{N}$ there exists $x \in A$ such that

$\|\eta - \pi_0(x)\Psi^\circ\|_m < \varepsilon$. Now, given $k_0,\dots,k_r \in \mathbb{N}$, let $y \in A$ be such that $\pi_0(y)\Psi^k_e = \Psi^\circ$.

Then $\|\Psi^k_e \otimes \eta - \Psi^k_e \otimes \pi_0(xy)\Psi^k_e\|_m = \|\eta - \pi_0(x)\Psi^\circ\|_m < \varepsilon$

thus $\Psi^k_e \otimes \eta \in C$. This concludes the proof, since such elements are clearly total in $\mathcal{D}_\otimes$.

///.

The analogous theorem for $\tilde{A}$ is now almost a corollary, but we will state it here, since it will be needed in the next section.

THEOREM 3.8. *Ω_π is a strongly cyclic vector for the representation $(\pi, H_\otimes, \mathcal{D}_\otimes)$ of $\tilde{A}$.*

Proof. We claim that $\{\pi_0(x)\Psi^\circ : x \in \tilde{A}\}$ is dense in $\mathcal{D}_\pi$. By Ch. 2 cor. 4.3, $\tilde{A}$ is dense in $(A, \|\ \|)$, and we may thus approximate an element $x \in A$ which is of the form $x = a^\#(f_1)\dots a^\#(f_n)a(\overline{g}_1)\dots a(\overline{g}_n)$ by the corresponding elements of $\tilde{A}$. The rest follows in the same way as in the case of A .

///.

3.4 THE COMMUTATION THEOREM

In this section, we finally prove the commutation theorem for the representation $(\pi_\beta, H_+, \mathcal{D}_\pi)$ of A. As in the case of Tomita's original proof we show the commutation theorem for the almost modular algebra $\tilde{A}$ first and then deduce the corresponding result for A.

Let $(\bar{\pi}_\beta, H_+, \mathcal{D}_{\bar{\pi}_\beta})$ denote the closure of the *-representation $(\pi_\beta, H_+, \mathcal{D}_\pi)$ with respect to the t_{π_β}-topology on $\mathcal{D}_\pi$ which is induced by the family of seminorms $\{ \|\cdot\|_{\pi_\beta(x)} = \|\pi_\beta(x)\cdot\| \ : \ x \in A\}$. In Ch.2 (4.6), the isometric involution $^+$ and the ^{b}-involution on $\tilde{A}$ were defined. The former extends to an anti-unitary involution J on H_+, while the latter is used in the definition of the right regular representation. Let $(\rho_\beta, H_+, \mathcal{D}_{\rho_\beta})$ denote the closure of the right regular representation of $\tilde{A}$: ρ_β is a ^{b}-homomorphism of $\tilde{A}$ into the continuous linear operators on $\mathcal{D}_{\rho_\beta}$ (where $\mathcal{D}_{\rho_\beta}$ is equipped with the t_{ρ_β} -topology which is induced by the family of seminorms $\{ {}_{\rho_\beta(x)}\|\cdot\| = \|\cdot\rho_\beta(x)\| \ : \ x \in \tilde{A} \} $) and ρ_β acts by multiplication on the right and preserves the ^{b}-involution.

LEMMA 4.1 *J maps* $\mathcal{D}_{\bar{\pi}_\beta}$ *onto* $\mathcal{D}_{\rho_\beta}$ *and vice versa, and for* $x \in \tilde{A}$
$\bar{\pi}_\beta(x) = J\rho_\beta(x^+)J$.

Proof. Since, for $x, y \in \tilde{\mathcal{A}}$, we have ${}_y\|x^+\| = \|yx^+\| = \|J(yx^+)\| = \|xy^+\| = \|x\|_{y^+}$, J is a topological ismorphism from $(\tilde{\mathcal{A}}, {}_x\|\ \|)$ onto $(\tilde{\mathcal{A}}, \|\ \|_x)$ and hence extends to the completions. For $x, y \in \tilde{\mathcal{A}}$ we now have $\xi = \pi_\beta(y)1_\beta \in \mathcal{D}_{\rho_\beta}$ and

$$J\pi_\beta(x^+)J\xi = J\pi_\beta(x^+)\pi_\beta(y^+)1_\beta = J(\pi_\beta(x^+y^+))1_\beta = \pi_\beta(y)\pi_\beta(x)1_\beta = \rho_\beta(x)\xi.$$

Thus, $J\pi_\beta(x^+)J\xi = \rho_\beta(x)\xi$ for each $\xi \in \mathcal{D}_{\rho_\beta}$, by the continuity of all the operators involved. Since $\mathcal{D}_{\rho_\beta}$ is dense in H_+, the result follows.

- ///.

To show that J intertwines the left and right von Neumann algebras $\bar{\pi}_\beta(A)'$ and $\rho_\beta(A)'$, we have to make use of the representation $(\pi, H_\boxtimes, \mathcal{D}_\boxtimes)$. Similarly to the above, let $(\bar{\pi}, H_\boxtimes, \mathcal{D}_{\bar{\pi}})$ denote the closure of $(\pi, H_\boxtimes, \mathcal{D}_\boxtimes)$ with respect to the t_π-topology (see outline of proof of thm. 3.4). Note that $(\bar{\pi}_\beta, H_+, 1_\beta)$ and $(\bar{\pi}, H_\boxtimes, \Omega_\pi)$ are two GNS-triples for $\tilde{A}$ with respect to the state ω_β. This implies that for $x \in \tilde{A}$

$$\omega_\beta(x) = \langle \bar{\pi}_\beta(x) 1_\beta, 1_\beta \rangle = \langle \bar{\pi}(x)\Omega_\pi, \Omega_\pi \rangle \quad . \tag{4.1}$$

By Ch.1 thm. 2.6 it follows that there exists a unitary operator U from H_+ onto $H_\boxtimes$ which maps $\mathcal{D}_{\bar{\pi}_\beta}$ continuously onto $\mathcal{D}_{\bar{\pi}}$ and is such that

$$U\bar{\pi}_\beta(x) 1_\beta = \bar{\pi}(x)\Omega_\pi \qquad \text{for } x \in \tilde{A}. \tag{4.2}$$

Using this unitary U, we define the right regular representation $(\rho, H_\boxtimes, \mathcal{D}_\rho)$ of $\tilde{A}$ by

$$\rho(x) = U\rho_\beta(x)U^{-1} \qquad \text{for } x \in \tilde{A}, \tag{4.3}$$

and the anti-unitary operator $J_\boxtimes$ on $H_\boxtimes$ by

$$J_\boxtimes = UJU^{-1} \quad . \tag{4.4}$$

From these definitions, it follows that ρ is a ^{b}-anti-representation on $H_\boxtimes$ with domain $\mathcal{D}_\rho = U\mathcal{D}_{\rho_\beta}$. To see how $J_\otimes$ acts on $H_\boxtimes$ we show;

LEMMA 4.2 *For* $T \in H_\boxtimes$, $J_\boxtimes$ *maps* T *to its operator adjoint* T^*.

Proof. Both $T \longmapsto T^*$, and $J_\otimes$ are antilinear isometries on $H_\boxtimes$. It is thus enough to prove the result on a dense set. Let $T = \pi(x)\Omega_\pi$, $x \in \mathcal{X}$. Then

$$J_\boxtimes T = UJU^{-1}\pi(x)\Omega_\pi = UJ\pi_\beta(x)1_\beta = U\pi_\beta(x^+)1_\beta = \pi(x^+)\Omega_\pi.$$

Thus we must show that $(\pi(x)\Omega_\pi)^+ = \pi(x^+)\Omega_\pi$ for all $x \in \mathcal{X}$. For $\eta \in \mathcal{D}_\pi$, $r \in \mathbb{N}$, we have

$$\mathrm{tr}(e^{-\beta H})\pi(a^{\#}(e_r))\Omega_\pi)\eta$$

$$= \sum_{n \geq 0} \sum_{J_n} e^{-\frac{1}{2}\beta\sum_k w(k)n_k} <\eta, \Psi_e^n>\pi_0(a^{\#}(e_r))\Psi_e^n$$

$$= \sum_{n \geq 0} \sum_{J_n} e^{-\frac{1}{2}\beta\sum_k w(k)n_k} (n+1)^{\frac{1}{2}} <\eta, \Psi_e^n>\Psi_e^{n+1}$$

$$= \sum_{n \geq 0} \sum_{J_n} e^{-\frac{1}{2}\beta\sum_k w(k)n_k - w(r)} <\eta, \pi_0(a(\overline{e}_r)\Psi_e^n>\Psi_e^n$$

$$= e^{\frac{1}{2}\beta w(r)} \sum_{n \geq 0} \sum_{J_n} e^{-\frac{1}{2}\beta\sum_k w(k)n_k} <\pi_0(a^{\#}(e_r))\eta, \Psi_e^n> \Psi_e^n .$$

Thus $\quad (\pi(a^{\#}(e_r))\Omega_\pi)\eta = e^{\frac{1}{2}\beta w(r)}\Omega_\pi\pi_0(a^{\#}(e_r))\eta = \Omega_\pi\pi_0(\Delta(-\frac{1}{2})a^{\#}(e_r))\eta$

and similarly, we have

$$(\pi(a(\overline{e}_r))\Omega_\pi)\eta = e^{-\frac{1}{2}\beta w(r)}\Omega_\pi\pi_0(a(\overline{e}_r))\eta = \Omega_\pi\pi_0(\Delta(-\tfrac{1}{2})a(\overline{e}_r))\eta \quad .$$

For general $x \in A$, we thus get $(\pi(x)\Omega_\pi)\eta = \Omega_\pi\pi_0(\Delta(-\frac{1}{2})x)\eta$.

Hence for $\xi, \eta \in \mathcal{D}_\pi$ we have

$$<\pi_0(x)\Omega_\pi\eta, \xi> = <\Omega_\pi(\pi_0(\Delta(-\tfrac{1}{2})x)\eta, \xi> = <(\pi_0(\Delta(-\tfrac{1}{2})x)\eta, \Omega_\pi\xi>$$

$$= <\eta, \pi_0(\Delta(-\tfrac{1}{2})x)^{\#}\Omega_\pi\xi> = <\eta, \pi_0(x^+)\Omega_\pi\xi>$$

since $(\Delta(-\frac{1}{2})x)^{\#} = \Delta(\frac{1}{2})x^{\#} = x^+$. The claim now follows since $\pi_0(x)\Omega_\pi$ is bounded . ///.

Now we have all the ingredients to prove our main result for $\tilde{A}$.

THEOREM 4.3. *The commutants* $\pi_\beta(\tilde{A})'$ *and* $\rho_\beta(\tilde{A})'$ *are von Neumann algebras in* $B(H_+)$ *, commutants of each other. Moreover, the operator* J *induces a spatial anti-isomorphism of either one of them onto the other, that is,*

$$J\pi_\beta(\tilde{A})'J = \rho_\beta(\tilde{A})' \qquad (4.5)$$
$$J\rho_\beta(\tilde{A})'J = \pi_\beta(\tilde{A})'$$

Proof. 1. Since $\bar{\pi}$ and $\bar{\pi}_\beta$ (respectively ρ and ρ_β) are unitarily equivalent via U, and $J_{\boxtimes}$ maps the the domain $\mathcal{D}_{\bar{\pi}}$ onto $\mathcal{D}_\rho$ by lemmas 4.1 and 4.2, we find that $\rho(x) = J_{\boxtimes}\pi(x^+)J_{\boxtimes}$ and $J_{\boxtimes}\pi(\tilde{A})'J_{\boxtimes} = \rho(\tilde{A})'$.

2. Next we show that $\rho(\tilde{A})' = \pi(\tilde{A})'' = L(H_{\boxtimes})$. Recall from section 3.1 (1.3) that $\pi(\tilde{A})' = R(H_{\boxtimes})$ where R denotes the right von Neumann algebra of the Hilbert algebra $H_{\boxtimes}$. Since $\pi(\tilde{A})' = \pi(A)'$, every $C \in \pi(\tilde{A})'$ is of the form $C(T) = TC_1$ for some $C_1 \in B(H_+)$ and for $T \in H_{\boxtimes}$ by thm. 2.6. Hence for $T \in H_{\boxtimes}$ we get

$$J_{\boxtimes}CJ_{\boxtimes}(T) = J_{\boxtimes}C(T^*) = J_{\boxtimes}(T^*C_1) = C_1^*T$$

by lemma 4.2. This equality, together with $J_{\boxtimes}\pi(\tilde{A})'J_{\boxtimes} = \rho(\tilde{A})'$, then implies that $\rho(\tilde{A})' = L(H_{\boxtimes}) = \pi(\tilde{A})''$, where $L(H_{\boxtimes})$ is the left von Neumann algebra of $H_{\boxtimes}$.

3. To show (4.5), one makes use of the following relationships:

$$U\pi_\beta(\tilde{A})'U^{-1} = \pi(\tilde{A})' \qquad \text{and} \qquad U\rho_\beta(\tilde{A})'U^{-1} = \rho(\tilde{A})' \quad .$$

Finally, by using the equations obtained in parts 1 and 2 of this proof, we have

$$J\pi_\beta(\tilde{A})'J = \rho_\beta(\tilde{A})' \qquad \text{and hence} \qquad \pi_\beta(\tilde{A})'' = \rho_\beta(\tilde{A})' \quad . \qquad ///.$$

The extension from $\tilde{A}$ to A has become almost trivial now. Recall that $\bar{\pi}_\beta$ is a right closed regular representation of A, and as in the case of $\tilde{A}$, $\bar{\pi}_\beta$ and $\bar{\pi}$ are unitarily equivalent.

THEOREM 4.4. *The commutant* $\pi_\beta(A)'$ *is a von Neumann algebra which satisfies the following:*

1. $J\pi_\beta(A)'J = \pi_\beta(A)''$.
2. $\Delta^{it}\pi_\beta(A)'\Delta^{-it} = \pi_\beta(A)'$ *for* $t \in \mathbb{R}$.

Proof. $\pi_\beta(A)'$ is spatially isomorphic to $\pi(A)' = \pi(\tilde{A})'$ via U. Since the latter is a von Neumann algebra so is $\pi_\beta(A)'$ (see thm. 4.3). Thus $\pi_\beta(A)' = \pi_\beta(\tilde{A})'$ and

$$J\pi_\beta(A)'J = J\pi_\beta(\tilde{A})'J = \rho_\beta(\tilde{A})' = \pi_\beta(\tilde{A})'' = \pi_\beta(A)'' \quad .$$

This follows again from the previous theorem and 2. now follows immediately by applying Ch.2 prop. 4.4 ///.

In conclusion, we have seen that it is possible to extend Tomita-Takesaki theory to a certain physically important class of unbounded *-algebras.

PART B

PERTURBATIONS OF THE DYNAMICS

Chapter 4

PERTURBATIONS OF DERIVATIONS

4.1 INTRODUCTION

In this chapter we consider W*-dynamical systems (M,α) , $M \quad \mathcal{B}(H)$. The generator of α is denoted by δ, and δ is a *-derivation on M (see Ch.1 section 1.3) with domain $\mathcal{D}(\delta)$. Let Δ denote another *-derivaton on M and define the perturbed derivation δ_p by

$$\delta_p = \delta + \Delta \tag{1.1}$$

We are interested in those derivations δ_p which generate automorphism groups of M. Two kinds of problems arise.

1. <u>The convergence problem</u> : What properties of Δ or δ_p guarantee that δ_p is the generator of an automorphism group β of $\mathcal{B}(H)$ such that (M,β) is a W*-dynamical system?

2. <u>The approximation problem</u> : Let (M,α) and (M,β) be two W*-dynamical systems with generators δ_α and δ_β respectively. How can the proximity of α and β be described in terms of their generators?

Both kinds of problem are addressed in the subsequent sections. The two problems are of course closely related and similar methods of proof can be applied (at least in the case of bounded perturbing derivations). The approximation problem was first considered by Buchholz and Roberts who characterised α and β in terms of their generating selfadjoint operators. Their result is briefly reviewed in section 4.4. Independently of them, I was interested in the convergence problem which is dealt with in sections 4.2 and 4.3.

We now turn to the classes of derivations which are considered as the perturbing derivations Δ in the following sections. Let Δ be a bounded *-derivation on a C*-algebra (by Ch.1 section 1.3, such a *-derivation is everywhere defined and continuous). We say Δ is *inner* if there exists $h = h^* \in A$ such that

$$\Delta(x) = i[h\ ,\ x] \qquad \text{for every } x \in A\ . \tag{1.2}$$

Note that on von Neumann algebras, every bounded *-derivation is inner (see [B&R1] cor. 3.2.47). Inner derivations (respectively the implementing element $h \in M$) are the objects of interest in section 4.2 and in Buchholz and Roberts' work.

For linear operators on Banach spaces one is also interested in perturbations which are no longer bounded. Let T and S be linear operators on a Banach space $\mathcal{B}$ with domains $\mathcal{D}(T)$ and $\mathcal{D}(S)$ respectively. The operator S is called T-*bounded* or *relatively bounded with respect to* T if $\mathcal{D}(S) \supseteq \mathcal{D}(T)$ and if there exist positive numbers a, b such that

$$\|S(x)\| \leq a\|x\| + b\|T(x)\| \qquad \text{for every } x \in \mathcal{D}(T). \tag{1.3}$$

If S is T-bounded, the greatest lower bound b_o of all possible constants b in (1.3) is called the T-*bound* of S. If S is a bounded linear operator, then S is T-bounded with T-bound $b_o=0$. Of particular importance are those T-bounded linear operators S whose T-bound is less than one, as the next two theorems demonstrate.

THEOREM 1.1 ([Kat] Ch. 4.1 thm. 1.1) *Let* S , T *be operators on a Banach space* X, *and* S *is* T-*bounded with* T-*bound less than 1. Then*

1. S + T *is closable if and only if* T *is closable.*
2. *If* T *is closable then* $\mathcal{D}(\overline{S+T}) = \mathcal{D}(\overline{T})$.
3. S + T *is closed if and only if* T *is closed.*

Relatively bounded derivations are clearly special cases of the above, and we shall meet them in sections 4.3 and 4.4. Another special case of the above are selfadjoint operators on a Hilbert space, and for perturbation theory on Hilbert spaces it is important to know when the sum of a selfadjoint and a symmetric operator is selfadjoint. The answer to this question is due to Kato and Rellich.

THEOREM 1.2 (Kato-Rellich) ([Wei] Satz 5.28)
Let T *be a selfadjoint (essentially selfadjoint) operator on the Hilbert space* H. *Let* S *be a symmetric and* T*-bounded operator on* H *with* T*-bound less than 1 . Then* $T + S$ *is selfadjoint (essentially selfadjoint) on* $\mathcal{D}(T)$ *(and* $\mathcal{D}(\overline{T+S}) = \mathcal{D}(\overline{T})$*).*

4.2 INNER PERTURBATIONS AND THE COCYCLE CONDITION

In this section we treat perturbations of derivations which arise from elements in the algebra. For such perturbing derivations we show:

THEOREM 2.1 *Let (A,α) be a C^*-dynamical system with generator δ and $A \subseteq B(H)$. Let Δ be an inner *-derivation on A and put $\delta_p = \delta + \Delta$. Then δ_p generates a one-parameter group of *-automorphisms β such that (A,β) is a C^*-dynamical system and $\|(\alpha_t - \beta_t)\| = O(t)$.*

We prove the theorem by means of 1-cocycles over $\mathbb{R}$. These objects are standard in cohomology but they also occur naturally in differential equations (see [Ara]). We start our proof with a proposition on these cocycles. The idea of the proof of thm. 2.1 is to show that Δ gives rise to a norm continuous cocycle γ_t, and the automorphism group β will then be $\beta_t = \gamma_t\alpha_t$ $(t \in \mathbb{R})$. We first define the candidate for γ:

$$\gamma_t = T\exp\left(\int_0^t ds\ \alpha_s\Delta\alpha_{-s}\right) \qquad \text{for } t \in \mathbb{R}, \tag{2.1}$$

where T denotes the time-ordered product.

PROPOSITION 2.2 *Let (A,α), Δ be as in thm. 2.1, γ as in (2.1). Then*

1. $t \to \gamma_t$ is norm continuous,

2. γ_t satisfies the cocycle equation with respect to α :

$$\gamma_{t+s} = \gamma_t\alpha_t\gamma_s\alpha_{-t} \qquad \text{for } t,s \in \mathbb{R} \tag{2.2}$$

3. for $t \in \mathbb{R}$, $\gamma_t \in aut(A)$.

Proof. 1. Let $h = h^*$ denote the selfadjoint element in A which implements Δ. For $s \in \mathbb{R}$, put $\Delta(s) = \alpha_s\Delta\alpha_{-s}$. Then $\Delta(s)$ is an inner *-derivation which is implemented by $h_s = \alpha_s(h) \in A$.

Let $t_i \leq t$ $(i=1,2,\ldots,n-1)$ and put

$$\tau_t = \sum_{n\geq 0} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n\, \Delta(t_n)\ldots\Delta(t_1)\ . \tag{2.3}$$

We show that $t \mapsto \tau_t$ is a norm continuous map on A. Once this is done it is easy to see that γ_t, given by (2.1), equals τ_t for $t \in \mathbb{R}$.

Let $S_{n,t}$ denote the n-th term in (2.3). For $x \in A$ we have:

$$\|S_{n,t}(x)\| = \| \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n\, \Delta(t_n)\ldots\Delta(t_1)(x)\|$$

$$\leq \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n\, \|\Delta(t_n)\ldots\Delta(t_1)(x)\|$$

$$\leq \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n\, 2^n\|h\|^n\|x\| \leq \frac{t^n}{n!}(2\|h\|)^n\|x\|\ . \tag{2.4}$$

Hence $\|S_{n,t}\| \leq \frac{t^n}{n!}(\|2h\|)^n$, and thus the n-th term is bounded in norm. Similarly it follows that the partial sums $\sum S_{n,t}$ are bounded in norm; and they converge to τ_t uniformly in x , since h is bounded. Hence for $\varepsilon > 0$ there exists an N_0 such that

$$\|\tau_t(x) - \sum_{n=0}^{N} \frac{1}{n!}\int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n\, (\Delta(t_n)\ldots\Delta(t_1))(x)\|$$

$$\leq \sum_{n\geq N} \frac{t^n}{n!}(2\|h\|)^n\, \|x\| < \varepsilon \qquad \text{for } N > N_0\ .$$

Hence τ_t is norm continuous. But the last inequality also implies that $\tau_t(A) \in A$, since A is norm closed.

We now have all the ingredients required in Araki's set-up of exponentials (see [Ara]). Note that Araki's proofs hold for the element $h_s \in A$, but it is not difficult to generalise the relevant proofs to $\Delta(s) = i[h_s,.]$.

Define the time-ordered product T on elements $\Delta(t_1)\ldots\Delta(t_n)$ by

$$T(\Delta(t_1)\ldots\Delta(t_n)) = \Delta(t_{\pi_n})\ldots\Delta(t_{\pi_1})$$

where $\{\pi_1,\ldots,\pi_n\}$ is a permutation of the integers $\{1,\ldots,n\}$ such that

$t_{\pi_n} < t_{\pi_{n-1}} < \ldots < t_{\pi_1}$. Following [Ara] section 2 (2.2) and (2.6), by using his "right" notation E_r, and T instead of $\overline{T}$, it is clear that τ_t of (2.3) satisfies

$$\tau_t = \sum_{n\geq 0} \int_0^t dt_1 \ldots \int_0^t dt_n \frac{1}{n!} T(\Delta(t_1)\ldots\Delta(t_n)) , \tag{2.5}$$

and hence $\tau_t = T \exp(\int_0^t ds\, \Delta(s)) = \gamma_t$ for $t \in \mathbb{R}$.

2. For convenience of notation we shall sometimes omit the T in front of the exponential, since it is clear how γ_t is defined. Note that

$$\begin{aligned}\gamma_{t+s} = \exp(\int_0^{t+s} dr\Delta(r)) &= \exp(\int_0^t dr\Delta(r)) \exp(\int_0^s dr\Delta(r+t)) \\ &= \gamma_t \exp(\int_0^s dr\Delta(r+t)) \quad ,\end{aligned}$$

by [Ara], prop.5. It thus remains to show that

$$\exp(\int_0^s dr\Delta(r+t)) = \alpha_t \exp(\int_0^s dr\Delta(r))\, \alpha_{-t} \quad .$$

For this, we first consider finite sums and put

$$T_{n,s+t} = \int_0^s ds_1 \ldots \int_0^{s_{n-1}} ds_n \Delta(s_n+t)\ldots\Delta(s_1+t). \tag{2.6}$$

From this definition it follows that

$$\exp(\int_0^s dr\Delta(r+t)) = \lim_{N\to\infty} \sum_{n=0}^{N} T_{n,s+t} \quad ,$$

where the limit refers to the norm limit in A. For fixed $n \in \mathbb{N}$, we have

$$\begin{aligned}T_{n,s+t} &= \int_0^s ds_1 \ldots \int_0^{s_{n-1}} ds_n\, \alpha_t\Delta(s_n)\alpha_{-t}\alpha_t\Delta(s_{n-1})\alpha_{-t}\ldots\alpha_t\Delta(s_1)\alpha_{-t} \\ &= \int_0^s ds_1 \ldots \int_0^{s_{n-1}} ds_n\, \alpha_t\Delta(s_n)\Delta(s_{n-1})\ldots\Delta(s_1)\alpha_{-t} \\ &= \alpha_t \left(\int_0^s ds_1 \ldots \int_0^{s_{n-1}} ds_n\, \Delta(s_n)\ldots\Delta(s_1) \right)\alpha_{-t} = \alpha_t S_{n,s}\alpha_{-t} \quad ,\end{aligned}$$

since α_t is independent of the integration and $S_{n,s}$ is as in (2.4).
A similar relationship holds for finite sums of the T's and S's, namely

$$\sum_{n=0}^{N} T_{n,s+t} = \alpha_t \left(\sum_{n=0}^{N} S_{n,s} \right) \alpha_{-t} \quad .$$

For $N \to \infty$, we now have for $x \in A$

$$\left\|\left(\exp\left(\int_0^s dr\Delta(r+t)\right) - \alpha_t\left(\exp\left(\int_0^s dr\Delta(r)\right)\alpha_{-t}\right)\right)(x)\right\|$$

$$\leq \left\|\left(\exp\left(\int_0^s dr\,\Delta(r+t)\right) - \sum_{n=0}^{N} T_{n,s+t}\right)(x)\right\| + \left\|\left(\sum_{n=0}^{N} T_{n,s+t} - \alpha_t \sum_{n=0}^{N} S_{n,s}\alpha_{-t}\right)(x)\right\|$$

$$+ \left\|\alpha_t\left(\sum_{n=0}^{N} S_{n,s} - \exp\left(\int_0^s dr\Delta(r)\right)\right)y_t\right\| < 2\varepsilon \quad ,$$

where $y_t = \alpha_{-t}(x)$. The last inequality holds, since $\sum T_{n,s+t}$ and $\sum S_{n,s}$ converge uniformly, and the second summand is zero. Hence γ satisfies the cocycle condition with respect to α.

3. Note that $\gamma_t(A) \subseteq A$ follows from the proof of part 1. For t=0, γ_0 = id follows from the definition. Also, $\gamma_t(x^*) = \gamma_t(x)^*$, since $\Delta(s)$ is a *-derivation. It remains to show that for x, $y \in A$, $\gamma_t(xy) = \gamma_t(x)\gamma_t(y)$. Let $x, y \in A$, and put $z_{-t} = \gamma_{-t}(x)\gamma_{-t}(y)$. Then

$$\frac{d}{dt}\left(\gamma_t(\gamma_{-t}(x)\gamma_{-t}(y))\right) = \gamma_t\Delta(t)(z_{-t}) - \gamma_t(\Delta(t)\gamma_{-t}(x))\gamma_{-t}(y))$$

$$- \gamma_t(\gamma_{-t}(x)\Delta(t)\gamma_{-t}(y)) \quad ,$$

by the derivation property of $\Delta(t)$ and by [Ara] props. 2 and 3 which deal with derivatives and inverses of exponentials respectively. The above implies that

$$\frac{d}{dt}\ \gamma_t(\gamma_{-t}(x)\gamma_{-t}(y)) = 0 \qquad \text{for } t \in \mathbb{R} \ ,$$

and hence $\gamma_t(xy) = \gamma_t(x)\gamma_t(y)$ follows immediately.
Putting $y = 1$, one observes that $\gamma_t(1) = 1$. And this now implies that γ_t is a *-automorphism of A. ///.

Put $\beta_t = \gamma_t\alpha_t$. The proof of thm. 2.1 follows now easily from the following lemma.

LEMMA 2.3 *(A,β) is a C*-dynamical system; β is generated by δ_p and $\|\beta_t - \alpha_t\| = O(t)$.*

Proof. Let $x \in A$, then $\beta_t(x) = \gamma_t\alpha_t(x) \in A$, since both α and γ leave A invariant. From the group property of α and the cocycle property of γ it is easily verified that β is a group. To check the continuity of β, let $t \in \mathbb{R}$, $x \in A$, then $\|\beta_t(x) - x\| = \|\gamma_t\alpha_t(x) - x\| \leq$ $= \|\gamma_t(x) - x\| + \|\gamma_t(\alpha_t - \mathrm{id})(x)\| \to 0$ as $t\to 0$, since α is strongly continuous and γ is norm preserving and norm continuous.

Next consider $x \in \mathcal{D}(\delta)$, then

$$(\beta_t - \alpha_t)(x) = \int_0^t ds\beta_s(\delta_p - \delta)\alpha_{t-s}(x).$$

The right hand side is well-defined, since $\mathcal{D}(\delta_p) = \mathcal{D}(\delta)$ and since α_t leaves $\mathcal{D}(\delta)$ invariant (because α_t and δ commute on $\mathcal{D}(\delta)$). Thus

$$\|(\beta_t - \alpha_t)(x)\| = \|\int_0^t ds\beta_s\Delta\alpha_{t-s}(x)\| \leq |t|\sup_{0\leq s\leq t}\|\Delta\alpha_{t-s}(x)\| \leq 2|t|\|h\|\|x\|, \quad (2.7)$$

since $\Delta = i[h,.]$ is a bounded inner derivation (see the beginning of proof of part 1 of prop. 2.2). But $\mathcal{D}(\delta)$ is norm-dense in A, and $\|(\gamma_t - \mathrm{id})(x)\| = \|(\beta_t - \alpha_t)(x)\|$ with $t \mapsto \gamma_t$ a norm continuous map. It follows that (2.7) can therefore be extended to the norm-closure of $\mathcal{D}(\delta)$, that is, to A. ///.

4.3 PERTURBATIONS AND THE TROTTER PRODUCT

In this section we shall employ a different technique for proving that perturbations of *-derivations are generators. We first assume that Δ is a bounded *-derivation (see (1.1) of section 4.1) and show:

THEOREM 3.1 *Let (A,α) be a C^*-dynamical system with generator δ. Let ω be a faithful α-invariant state on A with GNS-representation (π,H,Ω). Let Δ be a bounded *-derivation on $B(H)$ such that $\Delta\pi(A) \subseteq \pi(A)$. Then $\delta_p = \pi\circ\delta + \Delta$ is a *-derivation on $\pi(A)$ which generates a σ-weakly continuous group of *-automorphisms of $\pi(A)''$.*

The proof of thm. 3.1 follows from the next three lemmas.

LEMMA 3.2 *Let $A \subseteq B(H)$ be a C^*-algebra, and let Δ be a bounded *-derivation on $B(H)$. The following are equivalent.*

1. $\Delta(x) \in A$ for $x \in A$.

2. $e^{t\Delta} A \subseteq A$ for $t \in \mathbb{R}$. (3.1)

Proof. Recall from Ch. 1 thm. 3.2 that Δ is bounded if and only if $\gamma_t = e^{t\Delta}$ is norm continuous.

We first show 1. $\Rightarrow$ 2.

Since $\Delta(x) \in A$ for every $x \in A$, it follows that $\Delta^n(x) \in A$ for $x \in A$. Polynomials in Δ converge to γ in norm for every $x \in A$, since Δ is bounded. Hence γ is the limit in norm of a sequence of elements in A, and thus $\gamma_t(x) \in A$, for $x \in A$ and $t \in \mathbb{R}$.

2. $\Rightarrow$ 1.

From $\gamma_t(x) \in A$ for $x \in A$, it follows that $\frac{1}{t}(\gamma_t(x) - x) \in A$ for $x \in A$ and $t > 0$. Recall that $\Delta(x) = \lim_{t\to 0} \frac{1}{t}(\gamma_t(x) - x)$. Since γ is norm continuous, this limit exists in norm for every $x \in A$, whence it follows that $\Delta(x) \in A$. ///.

As in the previous section, we now have to show that $\beta_t = e^{t\delta_p}$ exists and is connected to $e^{t\delta}$ and $e^{t\Delta}$. Here we proceed via the implementing selfadjoint operators.

LEMMA 3.3 *The *-derivation δ_p is implemented by a selfadjoint operator H_p on $\mathcal{H}$.*

Proof. Let H denote the selfadjoint operator which implements δ on $\mathcal{H}$ (such an H exists by Ch.1 (3.3) and it is selfadjoint by Ch.1 thms. 1.3 and 3.1). Next observe that Δ is bounded, and thus by [Sak] thm. 4.1.6, there exists an $h = h^* \in \mathcal{B}(\mathcal{H})$ which implements Δ. Put $H_p = H + h$, then H_p is selfadjoint with $\mathcal{D}(H_p) = \mathcal{D}(H)$ by thm. 1.2. Furthermore, let $\delta_\pi = \pi\circ\delta$, then δ_π is a *-derivation on $\pi(A)$ satisfying $\delta_\pi(\pi(x)) = \pi(\delta(x))$ for $x \in \mathcal{D}(\delta)$. Let $x \in \mathcal{D}(\delta)$, $\xi \in \mathcal{D}(H)$. It follows that $\pi(x)\xi \in \mathcal{D}(H)$, since $\mathcal{D}(\delta) = \mathcal{D}(\delta_p)$, $\mathcal{D}(H) = \mathcal{D}(H_p)$ and since $\mathcal{D}(\delta_\pi)\mathcal{D}(H) \subseteq \mathcal{D}(H)$. Hence δ_p is spatial as the following calculation shows:

$$\delta_p(\pi(x))\xi = (\delta_\pi + \Delta)(\pi(x))\xi = i[H + h, \pi(x)]\xi = i[H_p, \pi(x)]\xi \;. \qquad ///.$$

LEMMA 3.4 *There exists a σ-weakly continuous one-parameter group of *-automorphisms $\beta \in aut(\pi(A))''$ such that $\beta_t = e^{t\delta_p}$ $(t \in \mathbb{R})$.*

Proof. Since H_p of lemma 3.3 is selfadjoint, $\nu_t = e^{itH_p}$ is a strongly continuous group of unitaries on $\mathcal{H}$ by Stone's theorem for $t \in \mathbb{R}$. We want to show that $\nu_t\pi(A)''\nu_t^* = \pi(A)''$.
For this purpose we apply a theorem by Trotter - a proof can be found in [Wei] thm. 7.40 - which states that for selfadjoint operators S,T, S + T the following relationship holds:

$$e^{it(S+T)} = \text{s-}\lim_{n\to\infty} \left(e^{i(t/n)S}e^{i(t/n)T}\right)^n \qquad (t \in \mathbb{R}) \qquad (3.2)$$

where the limit denotes the strong limit. Applying Trotter's "product formula" to H_p, we get

$$e^{itH_p} = \text{s-}\lim_{n\to\infty} (e^{i(t/n)H} e^{i(t/n)h})^n \quad .$$

Consider the term $\upsilon_{n,t} = e^{i(t/n)H} e^{i(t/n)h}$. Let $t \in \mathbb{R}$, then

$$\begin{aligned}(\upsilon_{n,t})^n \pi(A)(\upsilon^*_{n,t})^n &= (\upsilon_{n,t})^{n-1} \upsilon_{n,t} \pi(A) \upsilon^*_{n,t} (\upsilon^*_{n,t})^{n-1} \\ &= (\upsilon_{n,t})^{n-1} \pi(A)(\upsilon^*_{n,t})^{n-1} = \ \ldots \ = \pi(A),\end{aligned} \tag{3.3}$$

since H and h generate automorphism groups of $\pi(A)$ by assumption and by lemma 3.2 respectively. Let $\beta_{n,t} = (\upsilon_{n,t})^n \cdot (\upsilon^*_{n,t})^n$. Then β_n extends to a σ-weakly continuous *-automorphism of $\pi(A)''$. For $t \in \mathbb{R}$, let β_t denote the σ-weak limit of the sequence $\beta_{n,t}$. This limit exists σ-weakly, since the $(\upsilon_{n,t})^n$ form a strongly convergent group of unitaries on H, and hence

$$\beta_t = \sigma\text{-}\lim_{n\to\infty} (\upsilon_{n,t})^n \cdot (\upsilon^*_{n,t})^n = e^{itH_p} \cdot e^{-itH_p} \ .$$

The group property of β follows from that of the unitary group e^{itH_p} . Furthermore, for $y \in \pi(A)''$ and $t \in \mathbb{R}$, $\beta_t(y) \in \pi(A)''$, since $\pi(A)''$ is σ-weakly closed and β is σ-weakly continuous. Therefore β is a σ-weakly continuous group of *-automorphisms of $\pi(A)''$ with $\frac{d}{dt}\beta_t\big|_{t=0} = \delta_p$. ///.

This completes the proof of thm 3.1. We now turn to relatively bounded perturbations and apply the above-mentioned techniques there.

THEOREM 3.5 *Let (A,α) be a C*-dynamical system with generator δ. Let ω be a faithful α-invariant state on A with GNS-representation (π,H,Ω). Let Δ be a norm-densely defined norm-closed *-derivation on A such that Δ has a dense set of analytic elements and such that $\mathcal{D}(\Delta) \supseteq \mathcal{D}(\delta)$. Let S and H denote the implementing operators of Δ and δ on H*

respectively with $\mathcal{D}(S) \supseteq \mathcal{D}(H)$. Assume further that there exist constants $a,b > 0$ and $0 < c < 1$ such that for every $\xi \in \mathcal{D}(H)$ the following are satisfied:

$$\begin{aligned} &1.\ \ \| S\xi\| = c\| H\xi\| \\ &2.\ \ \| (\mathrm{ad}S)^n H\xi\| = ab^n n!\,\| H\xi\| \qquad n=1,2\ldots \end{aligned} \tag{3.4}$$

Then $\delta_p = \delta + \Delta$ generates a σ-weakly continuous one-parameter group of $*$-automorphisms of $\pi(A)''$.

We first concentrate on Δ and S, which are no longer bounded.

PROPOSITION 3.6 *S is essentially selfadjoint and $e^{it\overline{S}}\pi(A)''e^{-it\overline{S}} = \pi(A)''$ for $t \in \mathbb{R}$.*

Proof. We first show that the symmetric operator S is essentially selfadjoint, thus has a selfadjoint closure (see section 1.1). Recall that for a linear operator T on H and for $\eta \in H$, $(\mathrm{ad}S)T\eta = [S,T]\eta$. Using the bounds given in (3.4), it follows that the analytic elements of H, denoted by $\mathcal{D}^a(H)$, are contained in $\mathcal{D}^a(S)$, the analytic elements of S, by [Far] thm. 16.4. But H is selfadjoint, and hence by [Wei] thm. 8.31 Folgerung 1, the set $\mathcal{D}^a(H)$ is dense in H. It follows that the symmetric operator S has a dense set of analytic vectors, and, applying Nelson's theorem to S(see [Wei] thm. 8.31 or [R&SII] thm. X.39), we find that S is essentially selfadjoint (note that we do not claim here that $S\restriction\mathcal{D}^a(S)$ is essentially selfadjoint). Let $\overline{S}$ denote the selfadjoint closure of S. Then $\gamma_t = e^{it\overline{S}}\,.\,e^{-it\overline{S}}$ defines a σ-weakly continuous group of $*$-automorphisms on $B(H)$. Let $X = \frac{d}{dt}\gamma_t\big|_{t=0}$ denote its generator. We want to show that $\gamma_t(\pi(A)'') \subseteq \pi(A)''$. First note that X restricted to the set $\pi(\mathcal{D}(\Delta))$ equals $\pi o\Delta$. The representation π is faithful and thus all properties of the derivation Δ with respect to A apply to $\pi o\Delta$ and $\pi(A)$. Put $\Delta_\pi = \pi o\Delta$, then $\Delta_\pi(\pi(x)) = \pi(\Delta(x))$ for every $x \in$

$\mathcal{D}(\Delta)$. Since Δ is a *-derivation on A, $X\pi(\mathcal{D}(\Delta)) \subseteq \pi(A)$. Next let $\mathcal{D}^a(\Delta)$ denote the set of analytic elements of Δ. Fix $x \in \mathcal{D}^a(\Delta)$. Then there exists $t_x > 0$ such that

$$\|\gamma_t(\pi(x)) - \sum_{n=0}^{N} \frac{t^n}{n!} \Delta_\pi^n(\pi(x))\| \to 0 \text{ as } N \to \infty \quad \text{for } |t| < t_x. \tag{3.5}$$

Since Δ is a derivation on A, $\Delta(x) \in A$ for every $x \in \mathcal{D}(\Delta)$; and hence the norm limit $\gamma_t(\pi(x))$ in (3.5) belongs to $\pi(A)$ for $|t| < t_x$. Furthermore

$$\Delta_\pi \sum_{n=0}^{N} \frac{t^n}{n!} \Delta_\pi^n(\pi(x)) = \sum_{n=0}^{N} \frac{t^n}{n!} \Delta_\pi^{n+1}(\pi(x)) \quad , \tag{3.6}$$

and since Δ is a norm-closed operator on A, $\gamma_t(\pi(x)) \in \pi(\mathcal{D}(\Delta))$ for $|t| < t_x$. Because Δ_π and γ commute on $\pi(\mathcal{D}(\Delta))$ and because γ is norm preserving, a repeated application of Δ_π as in (3.6) shows that $\gamma_t(\pi(x))$ is analytic for Δ_π for $|t| < \frac{1}{2}t_x$ and radius of convergence t_x. Hence for $|s| < \frac{1}{2}t_x$, $\gamma_t(\gamma_s(\pi(x))) \in \pi(A)$ for $|t| < t_x$ by (3.5). But γ_t is a group and it thus follows that

$$\gamma_t(\gamma_s(\pi(x))) = \gamma_{t+s}(\pi(x)) = \gamma_{t'}(\pi(x)) \in \pi(A) \quad \text{for } |t'| < \frac{3}{2}t_x .$$

For general $t \in \mathbb{R}$, $\gamma_t(\pi(x)) \in \pi(A)$, since one defines

$$\gamma_t(\pi(x)) = (\gamma_{t/n})^n(\pi(x)) \qquad \text{for } n > 2\frac{|t|}{t_x} .$$

This shows that for $x \in \mathcal{D}^a(\Delta)$, $\gamma_t(\pi(x)) \in \pi(A)$ for every $t \in \mathbb{R}$. Since Δ has a dense set of analytic elements, $\gamma_t\pi(x) \in \pi(A)$ for a norm-dense set of elements of A. To show that γ leaves $\pi(A)$ invariant, it is sufficient to show that $\|\lambda\Delta(x) + x\| \geq \|x\|$ for $x \in \mathcal{D}(\Delta)$ and $\lambda \in \mathbb{R}$ by Ch.1 thm. 3.3. But this last property of Δ can be derived from the corresponding property of X, since X is the generator of γ and since $\mathcal{D}(X) \supseteq \pi\mathcal{D}(\Delta)$. Hence Δ is a generator and $\gamma_t = e^{t\Delta}$ leaves $\pi(A)$ invariant. The conclusion of the proposition follows now easily. ///.

LEMMA 3.7 *The *-derivation δ_p is implemented by a selfadjoint operator H_p and $e^{itH_p}\pi(A)''e^{-itH_p} = \pi(A)''$ for $t \in \mathbb{R}$.*

Proof. The operator S is symmetric and H-bounded with H-bound < 1 (by (3.4)). This guarantees, by thm. 1.2, that $H_p = H + \overline{S}$ is selfadjoint on $\mathcal{D}(H)$. Since $\mathcal{D}(\Delta) \supseteq \mathcal{D}(\delta)$, $\delta_p = \delta + \Delta$ is defined on $\mathcal{D}(\delta_p) = \mathcal{D}(\delta)$, $\pi(X)$ leaves $\mathcal{D}(H_p)$ invariant for every $X \in \mathcal{D}(\delta_p)$ and

$$\pi(\delta_p(X))\xi = \pi(\delta(X) + \Delta(X))\xi = i[H + S, \pi(X)]\xi \tag{3.7}$$

for $X \in \mathcal{D}(\delta)$, $\xi \in \mathcal{D}(H)$, whence it follows that δ_p is implemented by a selfadjoint operator. Note that δ_p is closable by thm. 1.1. Let $\overline{\delta}_p$ denote its closure, then (3.7) holds for $\overline{\delta}_p$, too.

Put $\beta_t = e^{itH_p} \cdot e^{-itH_p}$ for $t \in \mathbb{R}$. It remains to show that $\beta \in \mathrm{aut}(A)''$. As in the proof of thm. 3.1, this follows by applying Trotter's product formula (3.2) to H_p, and thus $e^{itH_p} = \text{s-}\lim_{n\to\infty} (e^{i(t/n)H} . e^{i(t/n)\overline{S}})^n$ for $t \in \mathbb{R}$. Recall that $e^{itH}\, \pi(A)\, e^{-itH} = \pi(A)$ by assumption, and $e^{it\overline{S}}\, \pi(A)''\, e^{-it\overline{S}} = \pi(A)''$ by lemma 3.6. Hence the desired result may be derived in a fashion similar to that given in the proof of lemma 3.4. ///.

This completes the proof of thm. 3.5.

4.4 PROXIMITY OF AUTOMORPHISMS

In this final section, we consider perturbation of a dynamical system as an approximation problem. Let (M,α) and (M,β) denote two W*-dynamical systems. We are interested in the relationship between α and β in terms of their closeness for small t. This study was originated by Buchholz and Roberts whose results are quoted in prop. 4.1.

PROPOSITION 4.1 [BuR] *Let (M,α) and (M,β) be W*-dynamical systems; $\alpha_t = e^{itH} \cdot e^{-itH}$, $\beta_t = e^{itK} \cdot e^{-itK}$ for $t \in \mathbb{R}$. Then the following are equivalent.*

1. $\|\alpha_t - \beta_t\| \to 0$ *as* $t \to 0$. (4.1)
2. *There exists a unitary u and a selfadjoint h , both in M , such that* $K\psi = uHu^{-1}\psi + h\psi$ *for* $\psi \in \mathcal{D}(K)$.

The proof uses the cocycle $\gamma_t = \beta_t \alpha_{-t}$ which is norm continuous, and hence inner, since M is a von Neumann algebra, and condition 1 is shown to be equivalent to the existence of a norm continuous cocycle u_t of unitaries in M such that $\beta_t = \mathrm{ad}u_t\ \alpha_t$.

A partial extension of this result was obtained by Olesen and Pedersen for the case of a simple C*-algebra, which we present here for the case of a simple C*-algebra with unit (see [O&P] cor.8.2)

PROPOSITION 4.2 [O&P] *Let (A,α) and (A,β) be C*-dynamical systems where A is simple and contains a unit. If*

$$||\phi\circ\alpha_t - \phi\circ\beta_t|| \to 0 \quad \text{as} \quad t \to 0 \tag{4.2}$$

for every pure state ϕ on A, then there exists a norm continuous α-1-cocycle of unitaries u_t of A such that $\beta_t = \mathrm{ad}u_t\ \alpha_t$ *for* $t \in \mathbb{R}$.

We shall be concerned with groups of *-automorphisms which do not converge to zero everywhere on M, as in prop. 4.1. As a consequence, our generators will no longer differ by a bounded operator on $\mathcal{D}(\delta_\alpha)$ and the "twist", which led to u in the case of Buchholz and Roberts, is no longer of this nice nature. As in Ch.1 section 1.3, we shall assume that the unit element in M belongs to the domain of the derivations unless otherwise specified. We show the following.

THEOREM 4.3 *Let (M,α) and (M,β) be separable W^*-dynamical sustems with generators δ_α and δ_β respectively. Put $M^\gamma = \mathcal{D}(\delta_\alpha) \cap \mathcal{D}(\delta_\beta)$. Assume that*

there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that for $0 \leq |t|, |s| \leq \delta_0$

and for $x \in M^\gamma$ $\quad \|(\alpha_s\beta_t - \beta_t\alpha_s)(x)\| \leq \varepsilon_0|ts|\;\|x\|$. (4.3)

Then the following are equivalent.

1. For $\varepsilon_1 > 0$ there exists $\delta_1 > 0$ such that for $0 \leq |t| \leq \delta_1$ and for $x \in M^\gamma$

$$\|(\alpha_t - \beta_t)(x)\| \leq \varepsilon_1\|x\| \quad . \tag{4.4}$$

2. There exist a bounded operator Γ on M mapping $\mathcal{D}(\delta_\alpha)$ into $\mathcal{D}(\delta_\beta)$ and a linear operator Λ which is bounded on M^γ such that

a. $B = \Gamma\delta_\alpha\Lambda - \delta_\beta$ is bounded on M^γ

b. $\Gamma\Lambda = \Lambda\Gamma \restriction M^\gamma = 1 \restriction M^\gamma$

c. for $\varepsilon_2 > 0$ there exists $\delta_2 > 0$ such that for $0 \leq |t| \leq \delta_2$ and for $x \in M^\gamma$

$$\|(\alpha_t\Gamma - \Gamma\alpha_t)(x)\| \leq \varepsilon_2\|x\| . \tag{4.5}$$

If these conditions are satisfied, then the operator

$$P = \Gamma\delta_\alpha - \delta_\beta\Gamma \tag{4.6}$$

is bounded on M^γ and extends to a $\Gamma\delta_\alpha$-bounded operator on $\mathcal{D}(\delta_\alpha)$.

Before proving the theorem, we make some general observations about automorphism groups on M, which are used in the proof of the theorem. Let (M,τ) denote a W*-dynamical system with generator Δ. Put

$$M^{\tau} = \{ x \in M : \|(\tau_t - \mathrm{id})(x)\| \to 0 \text{ as } t \to 0 \} \quad . \tag{4.7}$$

Then M^{τ} is σ-weakly dense in M by [Ped] p.250. Furthermore

$$\mathcal{D}(\Delta) = \{ x \in M : \sup_{0<t\leq 1} \frac{1}{t} \|(\tau_t - \mathrm{id})(x)\| < \infty \} \tag{4.8}$$

by [B&R I] p.182, since τ is a C_0^*-group in their terminology. Clearly, $\mathcal{D}(\Delta) \subseteq M^{\tau}$, and both sets are invariant under τ .

LEMMA 4.4 *Let (M,α) and (M,β) be W*-dynamical systems which satisfy (4.3). Then for $s \in \mathbb{R}$, $\alpha_s(M^{\gamma}) \subseteq M^{\gamma}$, where M^{γ} is as in the theorem.*

Proof. Clearly $\alpha_s(x) \in \mathcal{D}(\delta_\alpha)$ for every $s \in \mathbb{R}$ and $x \in \mathcal{D}(\delta_\alpha)$. It thus suffices to show that α_s leaves $\mathcal{D}(\delta_\beta)$ invariant. Let ε_0, δ_0 be as in (4.3). Let $|s|,|t| \leq \delta_0$ and let $x \in \mathcal{D}(\delta_\beta)$ with $\|x\| \leq 1$. Then

$$\begin{aligned}\|(\beta_t - \mathrm{id})\alpha_s(x)\| &\leq \|(\beta_t\alpha_s - \alpha_s\beta_t)(x)\| + \|\alpha_s(\beta_t - \mathrm{id})(x)\| \\ &\leq |st|\varepsilon_0 + \|(\beta_t - \mathrm{id})(x)\|\end{aligned}$$

holds by (4.3), since α is an isomorphism. And therefore one has

$$\sup_{0<t\leq 1} \frac{1}{t} \|(\beta_t - \mathrm{id})\alpha_s(x)\| \leq \sup_{0<t\leq 1} \left[|s|\varepsilon_0 + \frac{1}{t} \|(\beta_t - \mathrm{id})(x)\| \right] < \infty$$

by (4.8), since $x \in \mathcal{D}(\delta_\beta)$. It follows that $\alpha_s(x) \in \mathcal{D}(\delta_\beta)$ for $|s| \leq \delta_0$. For arbitrary $s \in \mathbb{R}$, put $\alpha_s = (\alpha_{s/n})^n$ where $n > (s/\delta_0)$ and then note that $(\alpha_{s/n})^n M^{\gamma} \subseteq (\alpha_{s/n})^{n-1} M^{\gamma} \subseteq \ldots \subseteq M^{\gamma}$. ///.

Next let α and β be as in the theorem and define

$$\gamma_t = \beta_t\alpha_{-t} \qquad \text{for } t \in \mathbb{R}. \tag{4.9}$$

LEMMA 4.5 *For $t \in \mathbb{R}$, γ_t is a *-automorphism of M ; γ is a σ-weakly continuous α-1-cocycle and it is a group if and only if α and β commute. Furthermore, if α and β satisfy (4.4), then for $x \in M^{\gamma}$ with $\|x\| \leq 1$*

$$\|(\gamma_t - id)(x)\| \leq \varepsilon_1 \quad \text{if } |t| \leq \delta_1 .$$

Proof. The automorphism property of γ follows immediately from the definition (4.9), since α and β are automorphisms. The cocycle identity (see prop. 2.2) $\gamma_{t+s} = \gamma_t \alpha_t \gamma_s \alpha_{-t}$ also follows from (4.9).

Now assume that γ is a group; then for $x \in M$, $\gamma_{t+s}(x) = \gamma_t \gamma_s(x)$. Take $x = \alpha_t(y)$ for $t \in \mathbb{R}$ and $y \in M$. Then

$$\beta_t \alpha_{-s}(y) = \beta_{-s} \gamma_{s+t}(x) = \beta_{-s} \gamma_s \gamma_t(x) = \alpha_{-s} \beta_t(y) .$$

This implies that α and β commute. The reverse direction follows along the same line.

Next observe that for $x \in M$

$$\|(\gamma_t - \mathrm{id})(x)\| = \|(\alpha_t - \beta_t)(x)\| \tag{4.10}$$

follows from the definition of γ, too, since α and β are norm preserving. Fix $\varepsilon_1 > 0$. Let $x \in M^{\gamma}$ with $\|x\| \leq 1$. Then for $|t| \leq \delta_1$, $\|(\gamma_t - \mathrm{id})(x)\| \leq \varepsilon_1$ by (4.10) and (4.4).

To show that γ is σ-weakly continuous, let $x \in M$, $t \in \mathbb{R}$ and $\phi \in M_*$.

$$|\phi(\gamma_t - \mathrm{id})(x)| = |\phi(\alpha_t - \beta_t)(x)| \leq |\phi(\alpha_t - \mathrm{id})(x)| + |\phi(\beta_t - \mathrm{id})(x)| \to 0$$

as $t \to 0$, since α and β are σ-weakly continuous. ///.

COROLLARY 4.6 *In the notation of the theorem, let $x \in M^{\gamma}$ and $s, t \in \mathbb{R}$. Then the following are equivalent.*

1. $\|(\gamma_{s+t} - \gamma_s \gamma_t)(x)\| = O(st)$
2. $\|(\alpha_s \beta_t - \beta_t \alpha_s)(x)\| = O(st)$.

We now show that 1 implies 2 of theorem 4.3. Fix $0 < \varepsilon_1 < 1$. By (4.4) there exists $\delta_1 > 0$ such that for $x \in M^\gamma$ with $\|x\| \leq 1$,

$$\|(\alpha_t - \beta_t)(x)\| \leq \varepsilon_1 \qquad \text{for } |t| \leq \delta_1 .$$

By (4.3) there exists $\delta_0 > 0$ corresponding to the ε_0. Put

$$\tau = \min\{\delta_1 , \delta_0\} ,$$

and define

$$\Gamma = \Gamma_\tau = \frac{1}{\tau}\int_0^\tau ds\, \gamma_s = \frac{1}{\tau}\int_0^\tau ds\, \beta_s \alpha_{-s} . \tag{4.11}$$

LEMMA 4.7

1. *Γ is a linear contraction on M, thus uniformly continuous.*
2. *Γ maps $D(\delta_\alpha)$ into $D(\delta_\beta)$ and leaves M^γ invariant.*

Proof. 1. Let $x \in M$, then $\|\Gamma x\| = \|\frac{1}{\tau}\int_0^\tau ds \gamma_s(x)\| \leq \sup_{0\leq s\leq\tau} \|\gamma_s(x)\|$ $\leq \|x\|$, since γ is isometric. Thus Γ is uniformly bounded, and hence uniformly continuous.

2. We first show that $\Gamma M^\alpha \subseteq M^\beta$ (see (4.7)). Let $x \in M^\alpha$, then using

$$\beta_t \int_0^\tau ds \beta_s(x) = \int_t^{\tau+t} ds \beta_s(x) ,$$

we have

$$\|(\beta_t - id)\Gamma(x)\| = \|\frac{1}{\tau}\int_0^\tau (\beta_t - id)\beta_s\alpha_{-s}(x) ds\|$$

$$= \|\frac{1}{\tau}\left(\int_t^{\tau+t} \beta_s\alpha_{-s}\alpha_t(x) ds - \int_0^\tau \beta_s\alpha_{-s}(x) ds\right)\|$$

$$= \|\frac{1}{\tau}\left(\int_\tau^{\tau+t} \gamma_s\alpha_t(x) ds - \int_0^t \gamma_s(x)\, ds + \int_t^\tau \gamma_s(\alpha_t - id)(x) ds\right)\|$$

$$\leq \frac{2t}{\tau}\|x\| + \frac{\tau - t}{\tau}\|(\alpha_t - id)(x)\|$$

$$= \frac{t}{\tau}(2\|x\| - \|(\alpha_t - id)(x)\|) + \|(\alpha_t - id)(x)\| \to 0 \text{ as } t \to 0$$

since $\chi \in M^{\alpha}$.

Next let $\chi \in \mathcal{D}(\delta_\alpha)$. By the above calculation it follows that

$$\sup_{0<t\leq 1} \frac{1}{t} \| (\beta_t - \mathrm{id})\Gamma(\chi) \| < \infty \quad ,$$

since the first term above becomes independent of t and less than $4\|\chi\|/\tau$ while the second term is finite by (4.8). This implies that $\Gamma(\chi) \in \mathcal{D}(\delta_\beta)$.

It remains to show that M^{γ} is invariant under Γ. Because of the argument above, it suffices to show that $\Gamma(\chi) \in \mathcal{D}(\delta_\alpha)$ for $\chi \in M^{\gamma}$. Let $\in M^{\gamma}$ and note that

$$\| \gamma_t \Gamma(\chi) \| = \| \frac{1}{\tau} \int_0^\tau ds\, \gamma_t \gamma_s(\chi) \|$$

$$\leq \| \frac{1}{\tau} \int_0^\tau ds\, (\gamma_t\gamma_s - \gamma_{t+s})(\chi) \| + \| \frac{1}{\tau} \int_0^\tau ds\, \gamma_{t+s}(\chi) \|$$

$$\leq \sup_{0\leq s\leq \tau} \| (\gamma_t\gamma_s - \gamma_{t+s})(\chi) \| + \| \frac{1}{\tau} \int_t^{\tau+t} ds\, \gamma_s(\chi) \|$$

Put $S = \sup_{0\leq s\leq \tau} \| (\gamma_t\gamma_s - \gamma_{t+s})(\chi) \|$. Then $S \leq \varepsilon_0 |t|\tau \|\chi\|$ by (4.3) and cor. 4.6, if $|t|, |s| \leq \delta_0$ (with ε_0 and δ_0 as in (4.3.)). Hence

$$\| (\gamma_t - \mathrm{id})\Gamma(\chi) \| \leq \| \frac{1}{\tau} (\int_t^{\tau+t} - \int_0^\tau) ds\, \gamma_s(\chi) \| + S$$

$$= \| \frac{1}{\tau} (\int_\tau^{\tau+t} - \int_0^t) ds\, \gamma_s(\chi) \| + S$$

$$\leq \frac{2t}{\tau} \|\chi\| + \varepsilon_0 |t|\tau \|\chi\| \; .$$

By (4.10) it now follows that

$$\| \frac{1}{t} (\alpha_t - \mathrm{id})\Gamma(\chi) \| \leq \| \frac{1}{t} (\alpha_t - \beta_t)\Gamma(\chi) \| + \| \frac{1}{t} (\beta_t - \mathrm{id})\Gamma(\chi) \|$$

$$\leq \frac{2}{\tau} \|\chi\| + \varepsilon_0 \tau \|\chi\| + \| \frac{1}{t} (\beta_t - \mathrm{id})\Gamma(\chi) \| \; ,$$

and therefore $\sup_{0<t\leq 1} \| \frac{1}{t} (\alpha_t - \mathrm{id})\Gamma(\chi) \| < \infty$ follows since $\Gamma(\chi) \in \mathcal{D}(\delta_\beta)$.

///

We now turn to the linear operator Λ and show:

PROPOSITION 4.8 *There exists a linear operator* Λ *on* M *which maps its domain* M^γ *into* M^γ *and satisfies* $\Lambda\Gamma\restriction_{M^\gamma} = \Gamma\Lambda = 1\restriction_{M^\gamma}$.

Proof. We construct Λ as the limit of $\sum_{n\geq 0} (1 - \Gamma)^n$. Put $\Lambda_1 = 1 - \Gamma$. The operator Λ_1 is uniformly bounded on M, hence continuous and leaves M^γ invariant, since Γ enjoys all these properties by lemma 4.7. Thus the same is true for the operator Λ_1 . Furthermore, if $x \in M^\gamma$, then $\| \Lambda_1^2(x) \| \leq \varepsilon_1^2 \| x \| \leq \| x \|$ since $\| \Gamma(x) - x \| = \leq \frac{1}{\tau}\int_0^\tau ds \, \| (\gamma_s - \mathrm{id})(x) \| \leq \varepsilon_1 \| x \|$ by (4.11). Now put $\Lambda_N = \sum_{n=0}^{N} \Lambda_1^n = \sum_{n=0}^{N} (1 - \Gamma)^n$. One shows similarly that the partial sum operator Λ_N is continuous on M, leaves M^γ invariant and is bounded on M^γ by

$$\| \Lambda_N(x) \| \leq \sum_{n=0}^{N} \| \Lambda_1^n(x) \| \leq \frac{1 - \varepsilon_1^{N+1}}{1 - \varepsilon_1} \| x \| < \frac{1}{1 - \varepsilon_1} \| x \|$$

for $x \in M^\gamma$. Note that for $x \in M^\gamma$ $\{\Lambda_N(x)\}$ forms a Cauchy sequence in M, as the following argument shows: Let $x \in M^\gamma$ with $\| x \| \leq 1$, and let $M > N$. Then

$$\| \Lambda_M(x) - \Lambda_N(x) \| = \sum_{n=N+1}^{M} \varepsilon_1^n \to 0 \quad \text{as } N , M \to \infty \quad \text{since } \varepsilon_1 < 1 .$$

But M is complete, and therefore the sequence $\{\Lambda_N(x)\}$ converges to a limit, say y, in M. Put

$$y = \Lambda(x) = \lim_{N\to\infty} \Lambda_N(x) = \lim_{N\to\infty} \sum_{n=0}^{N} \Lambda_1^n(x) . \tag{4.12}$$

This process defines a linear operator Λ on M^γ.

To show that Λ is the inverse of Γ on M^γ, note that

$$\Lambda_N\Gamma = \sum_{n=0}^{N} (1 - \Gamma)^n\Gamma = \sum_{n=0}^{N} \Gamma(1 - \Gamma)^n = \Gamma\Lambda_N = 1 - (1 - \Gamma)^{N+1} \quad ,$$

and hence for $x \in M^\gamma$, we have

$$\Gamma\Lambda(x) = \Gamma\ (\lim_{N\to\infty} \Lambda_N(x)) = \lim_{N\to\infty} \Gamma\Lambda_N(x) = \lim_{N\to\infty} \Lambda_N\Gamma(x) = \Lambda\Gamma(x)\ ,$$

since Γ is continuous and leaves M^γ invariant and since the sequence $\{\Lambda_N(x)\}$ converges to $\Lambda(x)$ uniformly for $x \in M^\gamma$. The above equality also implies that Λ and Γ commute on M^γ. And since $\lim_{N\to\infty} (1 - \Gamma)^{N+1} \to 0$ uniformly on M^γ, we have for $x \in M^\gamma$

$$\lim_{N\to\infty} \Lambda_N\Gamma(x) = \lim_{N\to\infty} (1 - (1 - \Gamma)^{N+1})(x) = 1(x),$$

and hence $\Gamma\Lambda = \Lambda\Gamma\restriction_{M^\gamma} = 1\restriction_{M^\gamma}$. ///.

So far we have shown that 2b of thm 4.3 holds. We now turn to 2c where assumption (4.3) is required.

LEMMA 4.9 *Let (M,α) and (M,β) satisfy (4.3). Then 2c of theorem 4.3 follows.*

Proof. Let $\varepsilon_2 > 0$. Put $\delta_2 = \min\{\delta_0 , \frac{\varepsilon_2}{\varepsilon_0 \tau}\}$ where δ_0 corresponds to ε_0 via (4.3). We show that for $x \in M^\gamma$ and for $|t| \leq \delta_2$ $\|(\Gamma\alpha_t - \alpha_t\Gamma)(x)\| \leq \varepsilon_2\|x\|$.

Let $x \in M^\gamma$ and let $|t| \leq \delta_2$. Then

$$\|(\Gamma\alpha_t - \alpha_t\Gamma)(x)\| = \|\frac{1}{\tau}\int_0^\tau ds(\gamma_s\alpha_t - \alpha_t\gamma_s)(x)\|$$

$$= \|\frac{1}{\tau}\int_0^\tau ds(\beta_s\alpha_t\alpha_{-s} - \alpha_t\beta_s\alpha_{-s})(x)\|$$

$$\leq \sup_{0\leq s\leq\tau} \|(\beta_s\alpha_t - \alpha_t\beta_s)(\alpha_{-s}(x))\| \quad .$$

For $0 \leq s \leq \tau$, put $y_s = \alpha_{-s}(x)$. By lemma 4.4, $y_s \in M^\gamma$ and hence $\|(\beta_s\alpha_t - \alpha_t\beta_s)(y_s)\| \leq \varepsilon_0|ts|\|y_s\| = \varepsilon_0|ts|\|x\|$, by (4.3). Since $\tau \leq \delta_0$, $\sup_{0\leq s\leq\tau} \|(\beta_s\alpha_t - \alpha_t\beta_s)(x)\| \leq \varepsilon_0|t|\tau\|x\|$. And hence for $|t| \leq \delta_2$, $x \in M^\gamma$ we get

$$\|(\Gamma\alpha_t - \alpha_t\Gamma)(x)\| \leq \varepsilon_0 . \delta_2 .\tau . \|x\| = \varepsilon_2\|x\| \quad . \quad ///.$$

It only remains to show that the difference between the operators $\Gamma\delta_\alpha\Lambda$ and δ_β is bounded on M^γ.

PROPOSITION 4.10 *For* $x \in M^\gamma$, $\| \Gamma\delta_\alpha\Lambda(x) - \delta_\beta(x) \| \leq k\|x\|$ $(k < \infty)$.

Proof. For $t \in \mathbb{R}$, put

$$\eta_t = -\beta_t\Gamma\alpha_{-t}\Lambda \quad . \tag{4.13}$$

We will show that on M^γ the difference between $\Gamma\delta_\alpha\Lambda$ and δ_β equals $\frac{d}{dt}\eta_t\big|_{t=0}$ and that for $x \in M^\gamma$, this operator is bounded. First recall from prop. 4.8 that on M^γ Λ is the uniform limit of $\Lambda_N = \sum_{n=0}^{N} (1 - \Gamma)^n$, by (4.12). Since the operators Λ_N are bounded, we have for $x \in M^\gamma$

$$\| \Lambda(x) \| = \frac{1}{1-\varepsilon_1} \|x\|$$

by prop. 4.8 , with $0 < \varepsilon_1 < 1$. Since Λ is defined and bounded on M^γ and α, β and Γ are uniformly bounded on M, the last calculation shows that η_t is defined and bounded on M^γ for $t \in \mathbb{R}$. Next observe that for $t, r \in \mathbb{R}$

$$\eta_{t+r} = -\frac{1}{\tau}\int_0^\tau ds\ \beta_{t+r}\beta_s\alpha_{-s}\alpha_{-t-r}\Lambda = -\frac{1}{\tau}\int_r^{\tau+r} ds\ \gamma_{t+s}\Lambda$$

and therefore

$$-\frac{1}{r}(\eta_{t+r} - \eta_t) = -\frac{1}{\tau r}\{\int_r^{\tau+r} ds\ \gamma_{t+s} - \int_0^\tau ds\ \gamma_{t+s}\}\Lambda$$

$$= -\frac{1}{\tau r}\{\int_\tau^{\tau+r} ds\ \gamma_{t+s} - \int_0^r ds\ \gamma_{t+s}\}\Lambda \quad .$$

The last equality implies that η_t is σ-weakly differentiable, since α and β (and hence also γ) are. Furthermore, its derivative

$$\frac{d}{dt}\eta_t = -\frac{1}{\tau}\beta_t(\beta_\tau\alpha_{-\tau} - id)\alpha_{-t}\Lambda$$

exists σ-weakly, being the σ-weak limit of the above. Let B denote this

σ-weak limit for $t\to 0$, then

$$B = \sigma\text{-}\lim_{t\to 0} \frac{1}{t}(\eta_t - \mathrm{id}) = \frac{d}{dt}\eta_t\Big|_{t=0} = -\frac{1}{\tau}(\beta_\tau\alpha_{-\tau} - \mathrm{id})\Lambda\,, \tag{4.14}$$

since $\eta_0 = \mathrm{id}\restriction_{M^\gamma}$. Clearly, B is defined on M^γ. Now let $x \in M^\gamma$. Then the following is true for $0 < t \in \mathbb{R}$

$$\frac{1}{t}\Gamma(\alpha_t - \mathrm{id})\Lambda(x) = \frac{1}{t}(\beta_t - \mathrm{id})(x) - \frac{1}{t}(\eta_t - \mathrm{id})\Gamma\alpha_t\Lambda(x) \tag{4.15}$$

since $\alpha_t(M^\gamma) \subseteq M^\gamma$ and $\Lambda\Gamma\restriction_{M^\gamma} = 1\restriction_{M^\gamma}$ by lemma 4.4 and prop. 4.8 respectively. The σ-weak limit for (4.15) exists as $t \to 0$, since η_t is σ-weakly differentiable and α and β are σ-weakly continuous. Hence for $x \in M^\gamma$ and for $\phi \in M_*$, this limit becomes (by (4.14))

$$\phi(\Gamma\delta_\alpha\Lambda(x)) = \phi(\delta_\beta(x)) + \phi(B(x)). \tag{4.16}$$

To conclude that $\Gamma\delta_\alpha\Lambda(x) = \delta_\beta(x) + B(x)$ for $x \in M^\gamma$, note that M is separable. This implies that every $\phi \in M_*$ is of the form $\phi(.) = \langle .\xi, \zeta\rangle$ for some vectors $\xi, \zeta \in H$. But these vectors are clearly separating, and it thus follows that for $x \in M^\gamma$

$$\Gamma\delta_\alpha\Lambda(x) = \delta_\beta(x) + B(x) \quad .$$

It remains to estimate B. Let $x \in M^\gamma$ with $\|x\| \leq 1$. Then

$$\|\delta_\beta(x) - \Gamma\delta_\alpha\Lambda(x)\| = \left\|\frac{1}{\tau}(\beta_\tau\alpha_{-\tau} - \mathrm{id})\Lambda(x)\right\|$$

$$= \frac{1}{\tau}\|(\beta_\tau - \alpha_\tau)\Lambda(x)\| \leq \frac{\varepsilon_1}{(1-\varepsilon_1)\tau}$$

by the above estimate for Λ and by (4.4). ///

This completes the direction $1 \Rightarrow 2$ of the proof of thm. 4.3. The converse is shown in the next proposition.

PROPOSITION 4.11 *Let (M,α) and (M,β) be separable W^*-dynamical systems as in theorem 4.3 and assume that 2 of theorem 4.3 holds. Then 1 follows.*

Proof. We first show that for $x \in M^{\gamma}$, $\|(\beta_t\Gamma - \Gamma\alpha_t)(x)\| \to 0$ as $t \to 0$. The conclusion of the proposition will then follow easily.

We start by showing that for $x \in M^{\gamma}$ i. and ii. below are equivalent.

$$\begin{aligned} &\text{i.} \quad \|(\Gamma\delta_\alpha\Lambda - \delta_\beta)(x)\| \leq k_1\|x\| \qquad (\text{some } k_1 < \infty), \\ &\text{ii.} \quad \|(\Gamma\delta_\alpha - \delta_\beta\Gamma)(x)\| \leq k_2\|x\| \qquad (\text{some } k_2 < \infty). \end{aligned} \tag{4.17}$$

To see that these two statements are equivalent, let $x \in M^{\gamma}$. Put $y = \Gamma(x)$. Then $\Lambda(y) = x$, since Γ and Λ are inverses of each other on M^{γ} by assumption. Assume i. Then

$$\|(\Gamma\delta_\alpha - \delta_\beta\Gamma)(x)\| = \|(\Gamma\delta_\alpha - \delta_\beta\Gamma)\Lambda(y)\| = \|(\Gamma\delta_\alpha\Lambda - \delta_\beta)(y)\|$$

$$\leq k_1\|y\| \leq k_2\ \|x\| \ ,$$

since Γ is a bounded operator. The other direction can be shown similarly.

Next observe that for groups of *-automorphisms ρ and σ with generators R and S respectively, the following relationship holds for every $x \in M$ for which the right hand side is defined (R and S are unbounded and hence not everywhere defined).

$$\rho_t - \sigma_t = \int_0^t ds\ \rho_s(R - S)\sigma_{t-s} \quad .$$

Applying this formula to $\rho_t = \beta_t\Gamma$ and to $\sigma_t = \alpha_t$, one obtains

$$\beta_t\Gamma - \Gamma\alpha_t = \int_0^t ds\ \beta_s(\delta_\beta\Gamma - \Gamma\delta_\alpha)\alpha_{t-s} \quad . \tag{4.18}$$

We now want to estimate (4.18) for elements in M^{γ}. Note that $\mathcal{D}(\Gamma\delta_\alpha) = \mathcal{D}(\delta_\alpha)$ since Γ is bounded, and $\mathcal{D}(\delta_\beta\Gamma) = \{ x \in M : \Gamma(x) \in \mathcal{D}(\delta_\beta) \} \supseteq \mathcal{D}(\delta_\alpha)$, since Γ maps $\mathcal{D}(\delta_\alpha)$ into $\mathcal{D}(\delta_\beta)$ by assumption. Furthermore, for $x \in M^{\gamma}$ we have

$$\|(\beta_t\Gamma - \Gamma\alpha_t)(x)\| = \|\int_0^t ds\ \beta_s(\delta_\beta\Gamma - \Gamma\delta_\alpha)\alpha_{t-s}(x)\|$$

$$\leq |t| \sup_{0\leq s\leq t} \|\beta_s(\delta_\beta\Gamma - \Gamma\delta_\alpha)\alpha_{t-s}(x)\| \quad .$$

The last expression can be further simplified using (4.17). For fixed $s \in \mathbb{R}$ one gets

$$\| \beta_s(\delta_\beta \Gamma - \Gamma\delta_\alpha)\alpha_{t-s}(\chi) \| = \| (\delta_\beta \Gamma - \Gamma\delta_\alpha)\alpha_{t-s}(\chi) \| \leq k_1 \| \chi \|$$

for some constant $k_1 < \infty$, since α and β are isometries. This last inequality does not depend on s.

Thus for $\chi \in M^\gamma$ we get

$$\| (\beta_t \Gamma - \Gamma\alpha_t)(\chi) \| \leq |t| k_1 \| \chi \| . \tag{4.19}$$

To show that $\|\alpha_t - \beta_t\| \to 0$ on M^γ, let $\chi \in M^\gamma$. Put $y = \Lambda(\chi)$. Then $\chi = \Gamma\Lambda(\chi)$ and

$$\| (\alpha_t - \beta_t)(\chi) \| = \| (\alpha_t - \beta_t)\Gamma\Lambda(\chi) \|$$

$$\leq \| (\alpha_t \Gamma - \Gamma\alpha_t)\Lambda(\chi) \| + \| (\Gamma\alpha_t - \beta_t\Gamma)\Lambda(\chi) \| \quad .$$

Since Λ is bounded on M^γ and leaves M^γ invariant, the first term goes to zero as $t \to 0$ by 2c of thm. 4.3. The second term goes to zero by (4.18), and therefore $\|\alpha_t - \beta_t\|$ on M^γ becomes small for small t. ///.

This concludes the proof of the equivalence of 1 and 2 in thm. 4.3. If Γ and Λ are as in (4.11) and as in the proof of prop. 4.8 respectively, the operator $\Gamma\delta_\alpha\Lambda$ is in general not a *-derivation since Γ is strictly positive, that is, $\Gamma(\chi^*\chi) > \Gamma(\chi)\Gamma(\chi^*)$. This is true since for $\chi \in M$

$$2(\Gamma(\chi^*\chi) - \Gamma(\chi^*)\Gamma(\chi)) = \frac{1}{\tau^2}\int_0^\tau ds \int_0^\tau dr\, (\gamma_s(\chi^*\chi) + \gamma_r(\chi^*\chi) - 2\gamma_r(\chi^*)\gamma_s(\chi))$$

$$= \frac{1}{\tau^2}\int_0^\tau ds \int_0^\tau dr |\gamma_r(\chi) - \gamma_s(\chi)|^2 \geq 0 \quad .$$

Now note that $|\gamma_r(\chi) - \gamma_s(\chi)| = 0$ if and only if $\gamma_r(\chi) = \gamma_s(\chi)$ for s, $r \in [0,\tau]$ if and only if $\alpha_t = \beta_t$ for $0 \leq |t| \leq \tau$.

Because of the positivity of Γ, there is no loss in generality by considering the operator $P = \Gamma\delta_\alpha - \delta_\beta\Gamma$ instead of the operator $B = \Gamma\delta_\alpha\Lambda - \delta_\beta$. By (4.17) P is bounded on M^γ if and only if B is bounded on M^γ. Furthermore P can be extended to $\mathcal{D}(\delta_\alpha)$. Unless $M^\gamma = \mathcal{D}(\delta_\alpha)$, P will no longer be bounded on $\mathcal{D}(\delta_\alpha)$, in general, however, it is still $\Gamma\delta_\alpha$-bounded, as the next lemma and proposition show.

PROPOSITION 4.12 *$G(\delta_\alpha)$ and $G(\delta_\beta)$ are norm-closed subspaces in $M \times M$ and $\Gamma\delta_\alpha$ and $\delta_\beta\Gamma$ are norm-closed operators.*

Proof. We first show that $G(\delta_\alpha)$ is norm-closed in $M \times M$. Let $(x_n, \delta_\alpha(x_n))$ be a norm-convergent sequence in $G(\delta_\alpha)$ whose limit (x, y) belongs to $M \times M$. Clearly (x_n) is a σ-weakly convergent sequence. But δ_α is σ-weakly closed (being the generator of α), and hence $x \in \mathcal{D}(\delta_\alpha)$, by the uniqueness of the limit. Furthermore $\phi(\delta_\alpha(x)) = \phi(y)$ for every $\phi \in M_*$. It remains to show that $\delta_\alpha(x) = y$. This follows since M is separable. Recall that each $\phi \in M_*$ is of the form $\phi = \langle .\xi, \eta\rangle$ for $\xi, \eta \in H$. Hence $\phi(\delta_\alpha(x) - y) = \langle(\delta_\alpha(x) - y)\xi, \eta\rangle$. Since these vectors are clearly separating, it follows that $\delta_\alpha(x) = y$, and therefore $G(\delta_\alpha)$ is norm-closed in $M \times M$. By Ch. 1 section 1.1, this also implies that δ_α is a norm-closed operator. Next we want to show that $\Gamma\delta_\alpha$ is a norm-closed operator. $\mathcal{D}(\Gamma\delta_\alpha) = \mathcal{D}(\delta_\alpha)$, since Γ is a bounded operator on M. It therefore follows immediately that $\Gamma\delta_\alpha$ is norm-closed on $\mathcal{D}(\delta_\alpha)$, since δ_α is norm-closed and Γ is uniformly continuous.

In a similar fashion one shows that $G(\delta_\beta)$ is norm-closed in $M \times M$. To show that $\delta_\beta\Gamma$ is a norm-closed operator, note that $\mathcal{D}(\delta_\beta\Gamma) = \{ x \in M : \Gamma(x) \in \mathcal{D}(\delta_\beta) \}$. In genral, $\mathcal{D}(\delta_\beta\Gamma) \neq \mathcal{D}(\delta_\beta)$. But $\delta_\beta\Gamma$ is norm-closed on its domain as the following argument shows. Let $(x_n, \delta_\beta\Gamma(x_n))$ be a norm convergent sequence in $M \times M$ with limit (x, y). Then $(x_n, \Gamma(x_n))$ converges to $(x, z) \in M \times M$ in norm, $x \in \mathcal{D}(\Gamma) = M$ and $\Gamma(x) = z$ by the uniform continuity of Γ . Hence $(\Gamma(x_n), \delta_\beta\Gamma(x_n))$ is a norm convergent sequence in $M \times M$ with limit (z, y). But δ_β is norm-closed, and therefore $z \in \mathcal{D}(\delta_\beta)$ and $\delta_\beta(z) = y$. However, since $z = \Gamma(x)$, it follows that $x \in \mathcal{D}(\delta_\beta\Gamma)$ and $\delta_\beta(z) = \delta_\beta\Gamma(x) = \lim_{n\to\infty} \delta_\beta\Gamma(x_n)$. ///.

LEMMA 4.13 *$\delta_\beta\Gamma$ is $\Gamma\delta_\alpha$-bounded on $\mathcal{D}(\delta_\alpha)$.*

Proof. By lemma 4.7, $\Gamma\mathcal{D}(\delta_\alpha) \subseteq \mathcal{D}(\delta_\beta)$, whence $\mathcal{D}(\delta_\beta\Gamma) \supseteq \mathcal{D}(\delta_\alpha)$ follows. By the previous proposition, $\Gamma\delta_\alpha$ and $\delta_\beta\Gamma$ are norm-closed operators, and thus $G(\Gamma\delta_\alpha)$ is complete in the graph norm defined by $\|\cdot\|_G = \|\cdot\| + \|\Gamma\delta_\alpha\cdot\|$. We now define a new operator C from $G(\Gamma\delta_\alpha)$ into M by

$$C: (x, \Gamma\delta_\alpha(x)) \longmapsto \delta_\beta\Gamma(x) .$$

C is clearly linear, and we now show that it is norm-closed. Let $(x_n, \Gamma\delta_\alpha(x_n), \delta_\beta\Gamma(x_n))$ converge to (x, y, z) in norm. Since $G(\Gamma\delta_\alpha)$ is norm-closed, the limit (x, y) of the convergent sequence $(x_n, \Gamma\delta_\alpha(x_n))$ belongs to $G(\Gamma\delta_\alpha)$. In particular, this implies that $x \in \mathcal{D}(\Gamma\delta_\alpha) \subseteq \mathcal{D}(\delta_\beta\Gamma)$, and since $\delta_\beta\Gamma$ is a norm-closed operator, it follows that $\delta_\beta\Gamma(x) = z$. But this implies that C is closed. We can now apply the Closed Graph theorem which provides us with a constant $c > 0$ such that

$$\|\delta_\beta\Gamma(x)\| \leq c\|x\|_G = c\|x\| + c\|\Gamma\delta_\alpha(x)\| \qquad \text{for } x \in \mathcal{D}(\delta_\alpha).$$

It now follows easily that

$$\|\Gamma\delta_\alpha(x) - \delta_\beta\Gamma(x)\| \leq a\|x\| + b\|\Gamma\delta_\alpha(x)\| \qquad \text{for } x \in \mathcal{D}(\delta_\alpha) .$$

The last calculation with $a = c$ and $b = c + 1$ gives the required result. ///.

We have thus shown that $\delta_\beta\Gamma$ is $\Gamma\delta_\alpha$-bounded on $\mathcal{D}(\delta_\alpha)$, and hence the proof of thm. 4.3 is complete.

So far we have not made any assumptions about the 'size' of M^γ. It certainly is not required that M^γ be σ-weakly dense in M. Clearly M^γ contains the identity, since the domains of δ_α and δ_β do. If M^γ consists just of multiples of the identity, theorem 4.3 is trivially true since $\alpha_t(1) = \beta_t(1) = 1$ for $t \in \mathbb{R}$, and $\delta_\alpha(1) = \delta_\beta(1) = 0$. However, if $M^\gamma = \mathcal{D}(\delta_\alpha)$, we get the following.

COROLLARY 4.14 *If $\mathcal{D}(\delta_\alpha) \subseteq \mathcal{D}(\delta_\beta)$ and the assumptions of theorem 4.3 hold, then B and P are bounded on $\mathcal{D}(\delta_\alpha)$.*

On the other hand, if one replaces the 'almost commutativity' (4.3) by the assumption that α and β commute, theorem 4.3 can be strengthened in the following way:

COROLLARY 4.15 *Let (M,α) and (M,β) be separable W^*-dynamical systems which satisfy (4.4) or (4.5) and assume that α and β commute on M. Then $B = \Gamma\delta_\alpha\Lambda - \delta_\beta$ is a *-derivation which is defined on M^γ and is bounded there.*

Proof. If Γ is given by (4.11), then $\alpha_t\Gamma = \Gamma\alpha_t$ for $t \in \mathbb{R}$ follows since α and β commute. Furthermore, $\alpha_t\mathcal{D}(\delta_\beta) \subseteq \mathcal{D}(\delta_\beta)$ for $t \in \mathbb{R}$. From the commutativity of Γ and α (and hence also of Λ) one deduces that δ_α commutes with Γ. It thus follows that $\delta_\alpha\restriction_{M^\gamma} = \Gamma\delta_\alpha\Lambda\restriction_{M^\gamma}$. ///.

We conclude this section by giving an example of two generators H and K whose "twisted" difference is bounded on $\mathcal{D}(H) \cap \mathcal{D}(K)$, but for which $\mathcal{D}(H) \cap \mathcal{D}(K)$ is not dense. This kind of situation is clearly not covered by Buchholz and Roberts' theorem (prop. 4.1), while it nevertheless fits into the framework of theorem 4.3. The example was inspired by one given in a paper by van Daele [Dae].

Let H denote a separable Hilbert space and let $\{\xi_n : n=1,2...\}$ be a fixed basis in H. Define an operator S on H by $S\xi_n = n\xi_n$ for $n=1,2...$ Then S is a selfadjoint operator with domain $\mathcal{D}(S) = \{\psi \in H: \psi = \sum a_n\xi_n\ ,\ \sum n^2|a_n|^2 < \infty\}$.

Next define a partition of the set of indices $\mathbb{N}_o$ into disjoint subsets $\Gamma_1, \Gamma_2, ...$ as follows:

$$\Gamma_1 = \{1\}\quad ,\quad \Gamma_2 = \{(2k-1) : k=2,3...\}\quad ,\quad \Gamma_n = \{(2k-1)2^{n-2} : k=1,2...\}\ .$$

Define sequences of vectors $\{\zeta_n\}$, $\{\eta_n\}$ $(n=1,2...)$ by

$$\zeta_1 = \xi_1\quad ,\quad \zeta_2 = \sum_{k=2}^{\infty} \frac{1}{k}\, \xi_{2k-1}\quad ,\quad \zeta_n = \sum_{k=1}^{\infty} \frac{1}{k}\, \xi_{(2k-1)2^{n-2}}\ ,$$

$$\eta_n = \zeta_n / \| \zeta_n \|\ .$$

Clearly, the sequence $\{\eta_n : n=1,2...\}$ forms an orthonormal basis for H, since the Γ_n's are disjoint, and for $n \geq 2$, $\eta_n \notin \mathcal{D}(S)$.

On these two orthonormal bases $\{\xi_n\}$ and $\{\eta_n\}$ there exists an isometry V which is defined by $V\xi_n = \eta_n$ $(n=1,2...)$. Put $R = 1 + \frac{1}{2}V$. Then R has an everywhere defined and bounded inverse R^{-1}. Next define the operator T by $T = SR^{-1}$. The operator T is a densely defined closed operator with $\mathcal{D}(T) = R\mathcal{D}(S)$. Furthermore, $\mathcal{D}(S) \cap \mathcal{D}(T) = \{\lambda\xi_1 : \lambda \in \mathbb{C}\}$.

The operators H and K are now defined in the following way:
$H = S^*S$ and $K = T^*T$. H and K are positive selfadjoint and selfadjoint respectively, and $\mathcal{D}(H) \cap \mathcal{D}(K) = \{\lambda\xi_1 : \lambda \in \mathbb{C}\}$. Hence H and K intersect on a one-dimensional subspace of H. Put $I = \mathcal{D}(H) \cap \mathcal{D}(K)$.
For any pair of linear operators u and v on H which satisfy condition 2b of theorem 4.3 the following can easily be verified. Let $\xi \in I$. Then

$$\| (uHv - K)(\xi) \| = \| (H - K)(\xi) \| = \| (1 - (R^*R)^{-1})(\xi) \|\quad ,$$

and $1 - (R^*R)^{-1}$ is clearly a bounded operator.

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<u>Errata</u>

p8 17 Delete sentence starting with 'Then x'.

p8 $\overline{15}$ Replace line by 'if it can be extended to a closed operator; every closable operator has a unique smallest closed extension, called its closure and denoted by $\overline{\mathbf{x}}$. If $\mathbf{x}$ is continuous (relative to the norm topology that $\mathbf{D(x)}$ inherits from $\mathbf{H}$), then $\mathbf{x}$ has an extension to $\mathbf{B(H)}$.'.

p9 $\overline{14}$ Add after 'space.': 'Note that trace class and Hilbert-Schmidt operators are <u>compact</u> operators (i.e. operators which map bounded sets into relatively compact sets).'

p9 $\overline{12}$ Replace 'thm.3' by 'thms. 3 & 4'.

p9 $\overline{11}$ Replace '$\sum_{n\geqslant 1} \lambda_n^2 < \infty$' by '$\mathbf{x}$ is compact and $\sum_{n\geqslant 1} |\lambda_n|^2 < \infty$'.

p10 12 Replace '$\sum_{n\geqslant 1} \lambda_n < \infty$' by '$\mathbf{x}$ is compact and $\sum_{n\geqslant 1} |\lambda_n| < \infty$'.

p10 15 Replace 'λ_n' by '$\{\lambda_n\}$'.

p11 11 Change 'thm. 13.30' to 'thms. 13.30 & 13.24'.

p11 17 Replace line by: 'there exists a densely defined closed operator $\mathbf{y}$ on $\mathbf{H}$, which is characterised by

$$\langle \mathbf{y}\xi,\eta\rangle = \int f dE_{\xi,\eta} \qquad (\xi \in \mathbf{D(y)}, \eta \in \mathbf{H}) \qquad (1.6)\text{'.}$$

p11 19 Append: 'Note if $\mathbf{E}$ is the projection-valued measure of a selfadjoint operator $\mathbf{x}$ and f is an $\mathbf{E}$-measurable function on sp($\mathbf{x}$), then it is customary to denote the operator $\mathbf{y}$ of (1.6) by $\int f d\mathbf{E}$ or by f($\mathbf{x}$) (for details see [Rud] 12.24, 13.24).'.

p11 $\overline{14}$ Replace 'sup norm' by 'operator norm'.

p14 $\overline{110}$ Replace '*-subalgebra $\mathbf{M}$' by '*-subalgebra $\mathbf{M}$ with unit'.

p18 16 Change 'Let S be a finite set' to 'For each finite set S' and delete 'and'.

p18 110-11 Replace 'these seminorms' by 'the family $\{\|\ \|_S : S \subset \mathbf{A}$, S finite$\}$'.

p19 110 Replace 'this follows ...' by 'this follows by [B&RI]

Cor.3.2.30, if δ is norm-closed and norm-densely defined.'.

p20 $\overline{19-8}$ Replace '$t \longrightarrow \alpha_t(x)$' to end of sentence by: '$\alpha : \mathbf{R} \rightarrow \mathrm{aut}(\mathbf{A})$ satisfies $\|(\alpha_t - \mathrm{id})(x)\| \rightarrow 0$ as $t \rightarrow 0$ uniformly for $x \in \mathbf{A}$.'.

p21 14 Replace 'Covariant representations' by: 'In fact, these two topologies are the same for unitary representations, since u_t belongs to the unit ball of **B(H)**. In what follows we consider those covariant representations which '.

p21 $\overline{110-9}$ Replace sentence starting 'However' by: 'However, if $(\mathbf{B(H)},\alpha)$ is a W*-dynamical system, then α is implemented. Furthermore, the C*-dynamical system $(\mathbf{A},\alpha)$ is implemented, if **A** is a simple C*-algebra with unit and α is norm continuous (see [Sak] thm. 4.1.19).'.

p25 $\overline{11}$ Replace '$\beta=-1$' by '$\beta=+1$'.

p29 $\overline{17-6}$ Delete from 'and' to end of sentence.

p29 $\overline{14}$ Change '$>$' to '$\gg$'.

p32 16 Replace the next 11 lines by the following:

'Let $\{e_k\}_{k\geq 0}$ denote the basis of $\mathbf{L}^2(\mathbf{R})$ consisting of the Hermite functions (see [R&S] p.142). Then $e_k \in \mathbf{S(R)}$ for k=0,1... . Let

$$e^n(n_0 n_1 \ldots) = (n!(\prod_k n_k!)^{-1})^{1/2}\ \Pi_n(e_0^{n_0} \otimes e_1^{n_1} \otimes \ldots) \qquad (2.9)$$

where the vector e_j appears n_j-times on the right (before symmetrisation), and $n=\sum n_j$. Elements of this kind belong to $\mathbf{S}_+^n$ and form a basis for $\mathbf{H}_+^n$ (see [Gui] Ch.2.1). We write Ψ_e^n or $\Psi^n(e^n(n_0 n_1 \ldots))$ for the elements in **D** which are obtained from $e^n(n_0 n_1 \ldots)$ by the natural embedding $\Psi^n : \mathbf{S}_+^n \longrightarrow \mathbf{D}$.

We now define a *-representation $(\pi_0, \mathbf{H}, \mathbf{D})$ of **A**. Put

$$\pi_0(a(\overline{e_r}))\Psi^n(e^n(n_0 n_1 \ldots)) = \begin{cases} \sqrt{n_r}\ \Psi^{n-1}(e^{n-1}(n_0 \ldots n_r - 1 \ldots)) \\ 0 \quad \text{if } n=0 \end{cases} \qquad (2.10)$$

$$\pi_0(a^{\#}(e_r))\Psi^n(e^n(n_0 n_1 \ldots)) = \sqrt{n_r+1}\ \Psi^{n+1}(e^{n+1}(n_0 \ldots n_r + 1 \ldots))$$

It is clear from this definition that $\pi_o(a^{\#}(f))$ and $\pi_o(a(\bar{g}))$ (f,g $\in$ **S(R)**) map symmetrised spaces into symmetrised spaces.´.

p32 $\overline{17ff}$ Replace ´H_+´ by ´**D**´ in statement of lemma 2.3. Replace the proof of lemma 2.3 by the following:

´**Proof.** It is enough to show that $\pi_o(a^{\#}(e_r))$ is the adjoint of $\pi_o(a(\bar{e}_r))$. The result follows since every element in **S(R)** can be expressed as a linear combination of e_r´s. Consider the elements $\Psi^n_e = \Psi^n(e^n(n_o\ldots n_r\ldots))$ and $\Psi^{n+1}_e = \Psi^{n+1}(e^{n+1}(n\ \ldots n_r+1\ldots))$ in **D**. Take $e_r \in$ **S(R)**, then

$$\langle \pi_o(a^{\#}(e_r)\Psi^n_e, \Psi^{n+1}_e\rangle = \langle\sqrt{n_r+1}\,\Psi^{n+1}_e, \Psi^{n+1}_e\rangle = \sqrt{n_r+1} =$$

$$= \sqrt{n_r+1}\langle\Psi^n_e,\Psi^n_e\rangle = \langle\Psi^n_e, \pi_o(a(\bar{e}_r))\Psi^{n+1}_e\rangle.$$

The first and last equalities follow by (2.10), the others follow since the elements Ψ^{n+1}_e and Ψ^n_e are orthogonal in **D**. Note that it suffices to show the above relationship on elements Ψ^n_e and Ψ^{n+1}_e, since $\langle\Psi^n_e,\Psi^m_{e'}\rangle=0$ unless n=m and e=e´.´.

p33 13 Change ´S^n´ to ´S^n_+´ and ´H^n´ to ´H^n_+´.

p33 14 Replace ´(2.9)´ by ´(2.10)´.

p33 15ff Replace to end of page by: ´$e_{k_i} \in$ **S(R)**, i=1,...,n (not necessarily all distinct). Elements of the form

$$\pi_o(a^{\#}(e_{k_1})\ldots a^{\#}(e_{k_n}))\Psi^o \qquad (2.11)$$

belong to **D**. (2.11) is the image of the element $e^n(\ldots n_{k_1}\ldots n_{k_n}\ldots) \in S^n_+$ (with $n_j=0$ unless $j=k_i$) under the natural embedding $\Psi^n: S^n_+ \to$ **D** (see also (2.9)). Clearly, if $e^n(n_o n_1\ldots) \in S^n_+$ and $e'^{\,n+1}(n_o n_1\ldots) \in S^{n+1}_+ \setminus S^n_+$, then Ψ^n_e and $\Psi^{n+1}_{e'}$ are orthogonal in **D**. Let $\phi \in$ **D**, then ϕ is of the form $\{\varphi^n:\ \varphi^n \in S^n_+,\ n=0,1\ldots\}$. Using also that **D** is an algebraic sum, we can get a representation for elements of **D**. This is a combination of the results given in [KMT] thm. 3.14 and (3.19), which we state as:

LEMMA 2.4 Every ϕ e **D** has a unique representation as a finite orthogonal sum

$$\phi = \Sigma' \Psi^n(\varphi^n), \tag{2.12}$$

where Σ' denotes that the sum is finite, and φ^n e $\mathbf{S}_+^n$.´.

p34 l12 Replace ´#-homomorphism´ by ´*-homomorphism´.

p34 l13 Replace ´$(\pi_o(\mathbf{x}))^{\#}$´ by ´$(\pi_o(\mathbf{x}))^*$´.

p34 l14-17 Replace from ´and then´ by: ´and note that these operators leave **D** invariant by (2.10) and (2.11). The result for general elements $\pi_o(\mathbf{x})$ then follows by linearity. ///.

p36 l8 Change ´$\mathbf{h}_n$´ to ´$\mathbf{h}_n \Pi_n$´ on left, and first ´f_i´ to ´f_1´ on right.

p36 l9 Append at end of sentence: ´and then extend by continuity.´.

p36 $\overline{l3}$ to p37 l4 Delete paragraph.

p37 $\overline{l1}$ and p44 l6 Change ´(3.3)´ to ´(2.9)´.

p42 $\overline{l3-1}$ Replace sentence starting ´This is´ by ´Suppose $\omega_\beta(\mathbf{xx}^*)=0$. Then $\|\pi_o(\mathbf{x})\Psi_e^n\|=0$ for every Ψ_e^n e **D**. Note that $\mathbf{x}$ can be decomposed uniquely into the following sum: $\mathbf{x} = \sum_{k\geq 1} \alpha_k A_k + \alpha_o$, such that α_k e $\mathbb{C}$ and A_n differs from A_m by at least one factor $\underline{\mathbf{a}}(e_r)$ ($\underline{\mathbf{a}}$ denotes $\mathbf{a}$ or $\mathbf{a}^{\#}$), where $A_k = \prod_{j=1}^{\ell_k} \mathbf{a}^{n_{kj}}(e_{k_j}) \prod_{j=\ell_k+1}^{m_k} \mathbf{a}^{\# n_{kj}}(e_{k_j})$, $e_{k_i} \neq e_{k_{i'}}$ and $n_{k_j}>0$. Take $\Psi^n(e^n(n_o n_1 \ldots n_r \ldots))$ e **D** such that n_r is greater than the largest power of $\mathbf{a}(e_r)$ occurring in the sum. By (2.10) we have

$$\|\pi_o(\mathbf{x})\Psi_e^n\| = \|\alpha_o\Psi_e^n + \sum_k \alpha_k \pi_o(A_k)\Psi_e^n\| = \|\alpha_o\Psi_e^n + \sum_k \alpha_k c_k \phi^k\|,$$

where ϕ^k is of the form $\Psi^m(e^m(m_o m_1 \ldots))$. By the choice of Ψ_e^n, $c_k>0$. Furthermore, since the A_k are distinct, Ψ_e^n and the ϕ^k are all distinct and, in fact, orthogonal. Therefore the last expression above can only be zero if all α_k are zero.´.

p91 l4&5 Replace ´=´ by ´≤´ in both places.

p95 thm 4.3: Delete the last 3 lines starting with 'If these ...' and change '2.' in the following way:

2. There exist a bounded operator Γ on $\mathbf{M}$ mapping $\mathbf{D}(\delta_\alpha)$ into $\mathbf{D}(\delta_\beta)$ with $\Gamma(\mathbf{M}^\gamma) \subseteq \mathbf{M}^\gamma$, and a linear operator Λ which is bounded on $\mathbf{M}^\gamma$ such that

a. $\mathbf{P} = \Gamma\delta_\alpha - \delta_\beta\Gamma$ is bounded on $\mathbf{M}^\gamma$ (4.5)

b. on $\mathbf{M}^\gamma$ the following holds: $\Lambda\Gamma = \Gamma\Lambda = \mathbf{1} \restriction \mathbf{M}^\gamma$

c. for $\varepsilon_2 > 0$ there exists $\delta_2 > 0$ such that for $0 \leq |t| \leq \delta_2$ and for $\mathbf{x} \in \mathbf{M}^\gamma$

$$\|(\beta_t\Gamma - \Gamma\beta_t)(\mathbf{x})\| \leq \varepsilon_2 \|\mathbf{x}\| . \tag{4.6}$$

p97 $\overline{17-5}$ Replace 3 lines by: 'To see that γ is σ-weakly continuous, observe that α and β are σ-weakly continuous automorphism groups. ///.'

p100 12-3 Delete: 'maps its domain $\mathbf{M}^\gamma$ into $\mathbf{M}^\gamma$ and'.

p100 17 Change 'Λ_1' to 'Λ_1^2'.

p100 $\overline{18}$ Change '=' to '$\leq$'.

p101 $\overline{110-1}$ Replace by: '$\|(\Gamma\beta_t - \beta_t\Gamma)(\mathbf{x})\| \leq \varepsilon_2\|\mathbf{x}\|$. Let $\mathbf{x} \in \mathbf{M}^\gamma$ and let $|t| \leq \delta_2$. Then

$$\begin{aligned}\|(\Gamma\beta_t - \beta_t\Gamma)(\mathbf{x})\| &= \Big\|\frac{1}{\tau}\int_0^\tau ds(\gamma_s\beta_t - \beta_t\gamma_s)(\mathbf{x})\Big\| = \\ &= \Big\|\frac{1}{\tau}\int_0^\tau ds(\beta_s\alpha_{-s}\beta_t - \beta_s\beta_t\alpha_{-s})(\mathbf{x})\Big\| \leq \\ &\leq \sup_{0\leq s\leq\tau} \|(\alpha_{-s}\beta_t - \beta_t\alpha_{-s})(\mathbf{x})\| \leq \varepsilon_0|t\tau|\|\mathbf{x}\|,\end{aligned}$$

by (4.3), since $\tau \leq \delta_0$. We get for $|t| \leq \delta_2$, $\mathbf{x} \in \mathbf{M}^\gamma$:

$$\|(\Gamma\beta_t - \beta_t\Gamma)(\mathbf{x})\| \leq \varepsilon_0\delta_2\tau\|\mathbf{x}\| \leq \varepsilon_2\|\mathbf{x}\| . \qquad ///.'$$

p102 12,3,6 Change '$\Gamma\delta_\alpha\Lambda$' to '$\Gamma\delta_\alpha$' and '$\delta_\beta$' to '$\delta_\beta\Gamma$' in all 3 lines.

p102 17 Change '$\frac{d}{dt}\eta_t|_{t=0}$' to '$(\frac{d}{dt}\eta_t|_{t=0})\Gamma$'.

p103 12 Change first '-id' to '+id'.

p103 13 Change 'η_0 = id' to 'η_0 = -id'.

p103 15 Change eq (4.15) to:

$$\frac{1}{t}\Gamma(\alpha_t - \mathrm{id})\Lambda\Gamma(\mathbf{x}) = \frac{1}{t}(\beta_t - \mathrm{id})\Gamma(\mathbf{x}) - \frac{1}{t}(\eta_t + \mathrm{id})\Gamma\alpha_t\Lambda\Gamma(\mathbf{x}) \qquad (4.15).$$

p103 l6 Insert after 'since': '$\Gamma(\mathbf{M}^\gamma) \subseteq \mathbf{M}^\gamma$', and change 'lemma 4.4' to 'lemmas 4.7 & 4.4'.

p103 l10 Change eq (4.16) to:

$$\phi(\Gamma\delta_\alpha(\mathbf{x})) = \phi(\delta_\beta\Gamma(\mathbf{x})) + \phi(\mathrm{B}\Gamma(\mathbf{x})) \qquad (4.16).$$

p103 l11&15 Replace '$\Gamma\delta_\alpha\Lambda(\mathbf{x}) = \delta_\beta(\mathbf{x}) + \mathrm{B}(\mathbf{x})$' by '$\Gamma\delta_\alpha(\mathbf{x}) = \delta_\beta\Gamma(\mathbf{x}) + \mathrm{B}\Gamma(\mathbf{x})$' in both places.

p103 l16 Replace '$\mathbf{B}$' by '$\mathbf{P} = \mathbf{B}\Gamma$'.

p103 l17-19 Replace by: '$\|\delta_\beta\Gamma(\mathbf{x}) - \Gamma\delta_\alpha(\mathbf{x})\| = \|\frac{1}{\tau}(\beta_\tau\alpha_{-\tau} - \mathrm{id})\Lambda\Gamma(\mathbf{x})\| = = \|\frac{1}{\tau}(\beta_\tau\alpha_{-\tau} - \mathrm{id})(\mathbf{x})\| \leq \frac{\varepsilon_1}{\tau}\|\mathbf{x}\|$'.

p104 l3-18 Replace by: 'Integration by parts shows that the following relationship holds for every $\mathbf{x} \in \mathbf{M}$ for which the rhs is defined.'.

p104 l7&2 Insert 'i' before 'β_s' in both places.

p105 l8-12 Replace from sentence starting 'Put' to end of l12 by: 'Note that α and β leave $\mathbf{M}^\gamma$ invariant (by lemma 4.4) and therefore

$$\|(\beta_t - \alpha_t)(\mathbf{x})\| = \|\Lambda\Gamma(\beta_t - \alpha_t)(\mathbf{x})\| \leq$$
$$\leq \|\Lambda(\beta_t\Gamma - \Gamma\beta_t)(\mathbf{x})\| + \|\Lambda(\Gamma\alpha_t - \beta_t\Gamma)(\mathbf{x})\| \leq$$
$$\leq k\ \{\ \|(\beta_t\Gamma - \Gamma\beta_t)(\mathbf{x})\| + \|(\Gamma\alpha_t - \beta_t\Gamma)(\mathbf{x})\|\ \}$$

since Λ is bounded on $\mathbf{M}^\gamma$.' Then start next sentence with 'The first term'.

p105 $\overline{\text{l10}}$ Replace '(4.18)' by '(4.19)'.

p105 $\overline{\text{l8}}$ - p106 l4 Delete up to 'extended to $\mathbf{D}(\delta_\alpha)$.'.

p106 l6 Replace line by: 'if α and β commute, for example.'.

p106 l7-8 Replace 2 lines by:

COROLLARY 4.12 If α and β commute and the assumptions of theorem 4.3 hold, then $\delta_\beta\Gamma$ is $\Gamma\delta_\alpha$-bounded on $\mathbf{D}(\delta_\alpha)$.

p106 $\overline{\text{l10-7}}$ Delete sentences starting from 'Next we want...'.

p107 14 Change '///.' to 'To show that $\Gamma\delta_\alpha$ is norm continuous, we use that α and β commute, as this fact implies that $\alpha_t\Gamma = \Gamma\alpha_t$ for $t \in \mathbf{R}$. From this it follows that $\Gamma(D(\delta_\alpha)) \subseteq D(\delta_\alpha)$, $\Gamma\delta_\alpha = \delta_\alpha\Gamma$ and $D(\Gamma\delta_\alpha) = D(\delta_\alpha)$. The proof of the norm-closedness of the operator $\Gamma\delta_\alpha$ follows closely the corresponding proof for $\delta_\beta\Gamma$.'.

p107 15 Replace 'Lemma 4.13' by 'It remains to show that'

p107 16 Delete 'Proof.'.

p107 $\overline{12\text{-}1}$ Delete.

p108 17 Change '4.14' to '4.13'.

p108 19-19 Delete.

p109 $\overline{14\text{-}1}$ Replace by: 'Let **u** be a bounded operator on $\mathcal{H}$ which acts on I by: $\mathbf{u}: \xi \longrightarrow c\xi$ (c=constant), and put $\mathbf{v}\xi = \frac{1}{c}\xi$ for $\xi \in$ I. It is easy to see that a.-c. below are satisfied.

a. $\mathbf{u}H - K\mathbf{u}$ is bounded on $D(H) \cap D(K)$

b. $\mathbf{uv} = \mathbf{vu} = 1$ on $D(H) \cap D(K)$

c. $\|(\exp(itK)\mathbf{u} - \mathbf{u}\exp(itK))(\xi)\| \leq \varepsilon\|\xi\|$ for $|t|$ small and $\xi \in D(H) \cap D(K)$.

This is the situation described in 2. of thm. 4.3. Note that the operators **H** and **K** commute on **I**, and therefore the general assumptions of thm. 4.3 are also satisfied.'.

Background to Errata

p8 $l7$ An appropriate statement about continuous operators has been included on p8 $\overline{15}$ff.

p10 $l4$-5 $\mathbf{x}$ compact not needed, since $\mathbf{x} \in \mathbf{HS(H)}$ implies that $\mathbf{x}$ is compact, while, on the other hand, a representation of the kind given for $\mathbf{x}$ shows that $\mathbf{x}$ is compact, since it is the limit of finite dimensional operators (see e.g. [G&V] I.2.2 thm. 3 or [Wei] Satz 6.5).

p29 $l6$ The isomorphism between $\mathbf{H}^n$ and $\mathbf{L}^2(\mathbf{R}^n)$ can be constructed in the following way: Let $\{\varphi_k\}$ denote a basis for $\mathbf{H}$; $\{\varphi_{k_1} \otimes \varphi_{k_2} \otimes \ldots \otimes \varphi_{k_n}\}$ is then a basis for $\mathbf{H}^n$. The map $\mathbf{U}$ which is given by

$(\mathbf{U}(\varphi_{k_1} \otimes \ldots \otimes \varphi_{k_n}))(t_1, \ldots t_n) = \varphi_{k_1}(t_1)\varphi_{k_2}(t_2) \ldots \varphi_{k_n}(t_n)$

maps the above basis of $\mathbf{H}^n$ 1-1 onto a basis of $\mathbf{L}^2(\mathbf{R}^n)$ and thus extends uniquely to an isomorphism between $\mathbf{H}^n$ and $\mathbf{L}^2(\mathbf{R}^n)$.

p36 $\overline{13}$ - p37 $l5$ This paragraph has been moved forward to section 2.3, and eq (3.3) has become eq (2.9).

p95 thm 4.3: Changes related to thm 4.3.

In prop 4.8 it was claimed that $\Lambda(\mathbf{M}^\gamma) \subseteq \mathbf{M}^\gamma$. This is not true in general, and some changes are therefore required. The claim $\Lambda(\mathbf{M}^\gamma) \subseteq \mathbf{M}^\gamma$ was used in defining the operator $\mathbf{B} = \Gamma\delta_\alpha\Lambda - \delta_\beta$ (old thm 4.3). However, instead of first introducing this operator $\mathbf{B}$, we now work directly with the operator $\mathbf{P}$ defined by $\mathbf{P} = \Gamma\delta_\alpha - \delta_\beta\Gamma$ (old thm 4.3). As a consequence, thm 4.3 is slightly modified; the general assumptions and 1. remain unchanged, in 2a. we change $\mathbf{B}$ to $\mathbf{P}$, and in 2c. a corresponding estimate for $(\Gamma\beta_t - \beta_t\Gamma)$ is used instead of the previous one given in terms of $(\Gamma\alpha_t - \alpha_t\Gamma)$. Lastly, the last 3 lines of the thm are deleted, since they are not true without further assumptions (see new cor 4.12).

The changes in the thm have the following consequences on its proof:

p100 prop 4.8: The statement is changed, the proof remains unchanged, since we had not shown that $\Lambda(\mathbf{M}^{\gamma}) \subseteq \mathbf{M}^{\gamma}$.

p101 lemma 4.9: The roles of α and β are interchanged in the proof in order to show 2c, but the proof remains essentially the same.

p102 prop 4.10: We now have to find a bound for **P** and not for **B**, in order to show 2a.

p103 prop 4.11: In the proof the new conditions 2a. and 2c. are used to show that 2. $\Rightarrow$ 1., but the essence of the proof is the same. The proof of thm 4.3 is then complete at the end of prop 4.11.

p106 prop 4.12 and lemma 4.13: These do not follow from the general asumptions of thm 4.3. However the results still hold under the additional assumption that α and β commute (this assumption was made in the old cor 4.15). The content of the prop and lemma now become cor 4.12, and the old cor 4.15 becomes redundant.

p97 16 This line was wrong, however γ is σ-weakly continuous; this can be seen as follows: Since **M** is a von Neumann algebra, $\mathbf{M} \subseteq \mathbf{B(H)}$ for some **H** and α and β can be extended to **B(H)** where they are implemented by strongly continuous groups of unitaries say **u** and **v** (see e.g. [B&RI] p.243). Put $\mathbf{w}_t = \mathbf{v}_t \mathbf{u}_{-t}$, then **w** is strongly continuous, since $\|(\mathbf{w}_t - \mathbf{1})\xi\| = \|(\mathbf{v}_t \mathbf{u}_{-t} - \mathbf{1})\xi\| \leq \|(\mathbf{v}_t \mathbf{u}_{-t} - \mathbf{v}_t)\xi\| + \|(\mathbf{v}_t - \mathbf{1})\xi\| = \|(\mathbf{u}_{-t} - \mathbf{1})\xi\| + \|(\mathbf{v}_t - \mathbf{1})\xi\| \to 0$ as $t \to 0$. Furthermore, if $\gamma_t = \mathbf{w}_t . \mathbf{w}_t^*$, then γ is σ-weakly continuous and $\gamma_t \restriction \mathbf{M} = \beta_t \alpha_{-t} \restriction \mathbf{M}$.