Regularity and Transitivity in Graphs

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Abstract

Graphs with high regularity and transitivity conditions are studied. The first graphs considered are graphs where each vertex has an intersection array (possibly differing from that of other vertices). These graphs are called distance-regularised and are shown to be distance-regular or bipartite with each bipartition having the same intersection array. The latter graphs are called distance-biregular. This leads to the study of distance-biregular graphs. The derived graphs of a distance-biregular graph are shown to be distance-regular and the notion of feasibility for a distance-regular graph is extended to the biregular case. The study of the intersection arrays of distance-biregular graphs is concluded with a bound on the diameter in terms of the girth and valencies. Special classes of distance-biregular graphs are also studied. Distance-biregular graphs with 2-valent vertices are shown to be the subdivision graphs of cages. Distance-biregular graphs with one derived graph complete and the other strongly-regular are characterised according to the minimum eigenvalue of the strongly-regular graph. Distance-biregular graphs with prescribed derived graph are classified in cases where the derived graph is from some classes of classical distance-regular graphs. A graph theoretic proof of part of the Praeger, Saxl and Yokoyama theorem is given. Finally imprimitivity in distance-biregular graphs is studied and the Praeger, Saxl and Yokoyama theorem is used to show that primitive non-regular distance-bitransitive graphs have almost simple automorphism groups. Many examples of distance-biregular and distance-bitransitive graphs are given.

Contents

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1 Definitions and Examples

In this introductory section we first include the definitions which will be useful throughout the thesis. The second part of the section lists the main examples of distance-regular and distance-biregular graphs which we will meet at different points in the ensuing work.

1.1 Definitions

We have divided the definitions up into groups, each group having a title and number. It is hoped that this will not only make them more readable, but also make references from the text easier to trace.

Definition 1.1.1 : *Graphs*

A *graph* G is a pair of sets (VG, EG), where VG is the set of *vertices* and $EG \subset VG \times VG$ the set of *edges*. Unless explicitly referred to as a *digraph*, we will consider only non-directed graphs without loops, that is EG will be a symmetric areflexive relation and the *number* of edges will be $1/2|EG|$. A *finite* (*infinite*) graph is one in which the vertex set is finite (infinite). A *multigraph* is a digraph in which multiple edges are allowed (EG can be thought of as a multiset, or we can assign to each edge a number denoting its multiplicity). For $u, v \in VG$, we write $u \sim v$ if $(u, v) \in EG$, and say that v is a *neighbour* of u or that u and v are *adjacent*. Also the edge (u,v) is said to be *incident* with the vertices u and v. Two edges are *incident* if they have a common vertex. A *clique* in a graph G is a subset of the vertex set whose members are all pairwise adjacent.

A sequence $v_0, v_1, \ldots, v_k$ of vertices in a graph is a *path* (or *walk*) of *length* k from v_0 to v_k, if $v_i \sim v_{i-1}$, $i = 1, \ldots, k$. If $v_0 = v_k$ then the walk is called *closed*. By $\partial_G(u,v)$ (or $\partial(u,v)$, if G is clear from the context) we denote the length of the shortest path from u to v in G ($\partial(u,v) = \infty$, if no path exists). We also say that v is at *distance* $\partial(u,v)$ from u. A graph G is *connected* if $\partial(u,v) < \infty$ for all $u, v \in VG$. The *diameter* of a connected graph is the maximum value ∂ attains on G and is denoted $\mathrm{diam}(G)$.

A connected graph G is *bipartite* (*n-partite*), if the vertex set of G can be partitioned into 2 (n) non-empty subsets such that if $u \sim v$, then u and v are in different subsets.

The *complement* of a graph G is a graph G^c with $VG^c = VG$ and $u \sim v$ in G^c if $u \not\sim v$ in G.

We associate with a graph G on n vertices numbered $1, 2, \ldots, n$ the $n \times n$ *adjacency matrix* $\mathbf{A}(G)$ defined by

$$\mathbf{A}(G)_{ij} = \begin{cases} 1; & \text{if } i \sim j, \\ 0; & \text{otherwise.} \end{cases}$$

The eigenvalues $\lambda(G)$ of a graph G are the eigenvalues $\lambda(\mathbf{A}(G))$ of the adjacency matrix of G.

The *degree* (or *valency*) of a vertex v is the number of neighbours of v. A graph is *locally-finite* if each vertex has finite degree. A locally-finite graph G is *k-regular* if each vertex has valency k. A graph is *biregular* if it is bipartite and vertices in the same part of the bipartition have the same degree.

For a graph G, the i-th *derived* graph $G^{(i)}$ is the graph with vertex set $VG^{(i)} = VG$ and adjacent vertices those at distance i in G. For a bipartite graph G, the second derived graph is the disjoint union of two connected graphs also called the *derived* graphs of G.

A *cycle of length* k in a graph G is a path $v_0, \ldots, v_k$, for which $v_{i-1} \neq v_{i+1}$, $i = 1, \ldots, k-1$ and $v_0 = v_k$. The *girth* of a graph G is the length of the shortest cycle in G.

A *star* graph is one of the graphs $K_{1,n}$, with $n \geq 2$.

For a graph G and vertex $v \in VG$, the graph $G \setminus v$ is the graph obtained from G by deleting the vertex v and all edges incident it.

For graphs $G_1, \ldots, G_k$, their *cartesian product* is the graph $G_1 \times \ldots \times G_k$, with vertex set $VG_1 \times \ldots \times VG_k$ and $(v_1, \ldots, v_k) \sim (v_1', \ldots, v_k')$ if for some i, $v_j = v_j'$, for $j \neq i$ and $v_i \sim v_i'$ in G_i.

For a graph G, the *line graph* of G is the graph $L(G)$ with vertex set the edges of G, and adjacency between incident edges. The *subdivision graph* $S(G)$ of a graph G is the graph obtained from G by subdividing each edge with a new vertex. ■

Definition 1.1.2 : *Geometries and Designs*

An *incidence structure* (or *geometry*) $\mathcal{I}$ is a pair (P,L), where the set P are the *points* of $\mathcal{I}$ and the set L is a collection of at least two subsets of P, called the *lines* of $\mathcal{I}$. The points and lines are the *elements* of the geometry; a point being an element of point *type*, and a line an element of line *type*. If two points x and y of an incidence structure determine a unique line ℓ containing them both, then ℓ is referred to as the line xy.

The *incidence graph* $G = G(\mathcal{I})$ of an incidence structure $\mathcal{I} = (P,L)$ has vertex set $VG = P \cup L$ with pairs (p,ℓ), $p \in P$ and $\ell \in L$, adjacent if $p \in \ell$. The *distance* between elements of an incidence structure $\mathcal{I}$ is the distance in the graph $G(\mathcal{I})$. The incidence graph of an incidence structure is clearly a bipartite graph, so we will call an incidence structure *regular (biregular)* if its incidence graph is regular (biregular). The *line graph* of an incidence structure $\mathcal{I}$ is the derived graph of the structure's incidence graph with vertex set the lines of $\mathcal{I}$, while the *point graph* is the derived graph on the points of $\mathcal{I}$.

A *t-(v,k,λ) block design $\mathcal{D}$* is an incidence stucture $\mathcal{D} = (X, \mathcal{B})$ with v points and its lines (also called *blocks*) all of size k, such that each t-subset of points occurs in precisely λ blocks.

A *Steiner System* $S(d,m,n)$ is a d-$(n,m,1)$ block design, that is a block design for which $\lambda = 1$.

A *symmetric* block design is a 2-(v,k,λ) design for which each pair of blocks intersects in the same number of points.

A *quasisymmetric* block design with *intersection numbers* i_1, i_2 is a 2-(v,k,λ) design for which each pair of blocks intersect in either i_1 or i_2 points.

An incidence structure $\mathcal{D}$ is a 2-$(\alpha\beta, \alpha, 1)$-*transversal design* if the point set of $\mathcal{D}$ can be partitioned into α sets w_i, $i = 0, \ldots, \alpha - 1$, each containing β elements such that the β^2 blocks B_j, $j = 1, \ldots, \beta^2$, of size α satisfy

(1) each B_j and w_i have exactly one element in common,

(2) if $j \neq k$, then B_j and B_k have at most one element in common.

A 4-*point* in an incident structure $\mathcal{P}$ is set of 4 points no three of which lie on a single line. A *projective plane* is an incidence structure $\mathcal{P}$ with a 4-point, such that each pair of points determine a unique line containing them and each pair of lines intersect in a unique point.

A *generalised n-gon* is an incidence structure $\mathcal{P}$ satisfying

(i) it is biregular with line size $s + 1$ and point degree $t + 1$,

(ii) the distance between any two elements of the structure is at most n,

(iii) if the distance between two elements is less than n, there is a unique shortest path joining those two elements in $G(\mathcal{P})$,

(iv) for any element of the structure there is at least one element at distance n from it. ∎

Definition 1.1.3 : *Distance-regularity*

Let G be a connected graph. By $G_i(u)$ we denote the set of vertices of G at distance i from the vertex u, and by $k_i(u)$ the size of $G_i(u)$. An alternative notation for $G_1(u)$ is simply $G(u)$. Let $u, v \in VG$ with $i := \partial(u, v)$, then

$$c(u,v) = |G_{i-1}(u) \cap G(v)| \ ,$$
$$a(u,v) = |G_i(u) \cap G(v)| \quad \text{and}$$
$$b(u,v) = |G_{i+1}(u) \cap G(v)| \ .$$

If for fixed $u \in VG$ the numbers $c(u,v)$, $a(u,v)$ and $b(u,v)$ are independent of the choice of v in $G_i(u)$ for each $i = 1, \ldots, \mathrm{diam}(G)$, then u is *distance-regularised* and we denote by $c_i(u)$, $a_i(u)$ and $b_i(u)$ the numbers $c(u,v)$, $a(u,v)$ and $b(u,v)$, where v is any vertex in $G_i(u)$. If u is a distance-regularised vertex of a graph G, then the array

$$\iota(u) = \begin{bmatrix} * & c_1(u) & c_2(u) & \ldots & c_d(u) \\ 0 & a_1(u) & a_2(u) & \ldots & a_d(u) \\ b_0(u) & b_1(u) & b_2(u) & \ldots & * \end{bmatrix}$$

is the *intersection array* of u and the matrix

$$\mathbf{I}(u) = \begin{pmatrix} 0 & c_1(u) & 0 & 0 & \cdots & 0 \\ b_0(u) & a_1(u) & c_2(u) & 0 & \cdots & 0 \\ 0 & b_1(u) & a_2(u) & c_3(u) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{t-2}(u) & a_{t-1}(u) & c_t(u) \\ 0 & \cdots & 0 & 0 & b_{t-1}(u) & a_t(u) \end{pmatrix}$$

is the *intersection matrix* for u, where $d = \mathrm{diam}(G)$ and t is such that $G_t(u) \neq \emptyset$, but $G_{t+1}(u) = \emptyset$.

A graph G is *distance-regularised* if each vertex of G is distance-regularised. If every vertex of a distance-regularised graph G has the same intersection array then G is *distance-regular.* A bipartite distance-regularised graph is *distance-biregular* if vertices in the same part of the bipartition have the same intersection array. ∎

Definition 1.1.4 : *Distance-regular Graphs*

The *intersection array* $\iota(G)$ of a distance-regular graph G is the unique intersection array of its vertices. The standard notation for this array is

$$\iota(G) = \begin{bmatrix} * & c_1 & c_2 & \ldots & c_d \\ 0 & a_1 & a_2 & \ldots & a_d \\ k & b_1 & b_2 & \ldots & * \end{bmatrix}$$

where $d = \mathrm{diam}(G)$. Note that G is a k-regular graph.

Let G be a distance-regular graph with diameter d. The graph G is *antipodal* if the d-th derived graph of G is disconnected. If a distance-regular graph G is antipodal then the *antipodal derived* graph G' is obtained from G by taking VG' the components of its d-th derived graph with two components adjacent if there is an edge of G joining them. The antipodal derived graph is also distance-regular and the graph G is called an *antipodal covering* of its antipodal derived graph. A distance-regular graph is *primitive* if the i-th derived graph is connected for $i = 1, \ldots, d$, otherwise it is *imprimitive*. It is well known that an imprimitive distance-regular graph is either bipartite (the second derived graph disconnected) or antipodal (the d-th derived graph disconnected).

A (k,g)*-graph* or *cage* is a regular graph of valency k and girth g with diameter d satisfying $d = \lfloor g/2 \rfloor$ and which is bipartite if g is even. It is well known [1] that a

(k,g)-graph is distance-regular and has $n_0(k,g)$ vertices, where

$$n_0(k,g) = \begin{cases} 1 + \sum_{i=1}^{d} k(k-1)^{i-1}; & \text{g odd,} \\ 1 + \sum_{i=1}^{d-1} k(k-1)^{i-1} + (k-1)^{d-1}; & \text{g even.} \end{cases}$$

■

Definition 1.1.5 : *Strongly-regular Graphs*

A *strongly-regular* graph is a distance-regular graph of diameter 2. For a strongly-regular graph there are four standard parameters (v, k, λ, μ). They are $|VG|$, $\deg(G)$, the number of common neighbours of adjacent vertices and the number of common neighbours of non-adjacent vertices, respectively. We also use α for the absolute value of the smallest eigenvalue of a strongly-regular graph and β for the difference between the second and smallest eigenvalue.

A *conference* graph is a strongly-regular graph for which

$$v = 4\mu + 1,\ k = 2\mu,\ \lambda = \mu - 1,\ \beta = \sqrt{4\mu+1},\ \alpha = \frac{1}{2}(1 + \sqrt{4\mu+1})\ .$$

A *Steiner* graph $S_\alpha(\beta)$ is the line graph of an $S(2, \alpha, \alpha + \beta(\alpha - 1))$ Steiner system, with $\beta \geq \alpha + 1$. A Steiner graph is a strongly-regular graph. A *pseudo-Steiner* graph is a strongly-regular graph for which $\mu = \alpha^2$.

The line graph of a 2-$(\alpha\beta, \alpha, 1)$-transversal design with $\beta \geq \alpha + 1$ is called a *latin square* graph and denoted by $LS_\alpha(\beta)$. Latin square graphs are also strongly-regular. A *pseudo-latin square* graph is a strongly-regular graph satisfying $\mu = \alpha(\alpha - 1)$.

Definition 1.1.6 : *Distance-biregular Graphs*

Unless explicitly stated we will use the following standard notation for a distance-biregular graph G. The two parts of the bipartition of the vertex set VG are denoted by A and B. The diameter of G is d. A typical vertex in A is denoted by u and has intersection array

$$\iota(A) = \begin{bmatrix} * & c_1 & c_2 & \dots & c_d \\ 0 & 0 & 0 & \dots & 0 \\ r & b_1 & b_2 & \dots & * \end{bmatrix} \quad \text{or just} \quad \begin{bmatrix} * & c_1 & c_2 & \dots & c_d \\ r & b_1 & b_2 & \dots & * \end{bmatrix},$$

while v is a typical vertex of B and has intersection array

$$\iota(B) = \begin{bmatrix} * & f_1 & f_2 & \dots & f_d \\ s & e_1 & e_2 & \dots & * \end{bmatrix}.$$

The corresponding intersection matrices are denoted $\mathbf{I}(A)$ and $\mathbf{I}(B)$ respectively. Note that the valency of vertices in A is r, while that of vertices in B is s. We denote with k_i the numbers $k_i(u)$ for vertices $u \in A$ and with l_i the numbers $k_i(v)$ for vertices $v \in B$, $i = 0, 1, \dots, d$. Note that $l_{d-1} \neq 0$ and $k_{d-1} \neq 0$ though one of l_d and k_d may be zero.
A *CSR* graph is a distance-biregular graph for which one derived graph is complete and the other strongly-regular.
A non-regular distance-biregular graph G is *imprimitive* if the $2i$-th derived graph has more than 2 components for some i, $1 \leq i \leq \lfloor d/2 \rfloor$, otherwise it is *primitive*.

Definition 1.1.7 : *Permutation Groups and Automorphism Groups*

A pair (Γ, X) is a *permutation group* if Γ is a group with an implicit homomorphism to the group of all permutations of X. The *degree* of a permutation group is the size of the set X. The action is *transitive* if for all $x, y \in X$ there exists $\gamma \in \Gamma$ such that $\gamma(x) = y$. The action is *faithful* if the implicit homomorphism has trivial kernel. The action is *imprimitive* if there exists a non-trivial subset Y of X such that for all $\gamma \in \Gamma$, $\gamma(Y) = Y$ or $\gamma(Y) \cap Y = \emptyset$. If the permutation group is not imprimitive then it is *primitive*.
A permutation α of the vertex set of a graph G is an *automorphism* of G if it preserves adjacency (and non-adjacency). The set of all automorphisms of a graph G form a group $\mathrm{Aut}(G)$, the *automorphism group* of the graph.
A permutation of the points of a geometry $\mathcal{G}$ is an *automorphism* of the geometry if it maps lines to lines. The set of all automorphisms of $\mathcal{G}$ form a group $\mathrm{Aut}(\mathcal{G})$, the *automorphism group* of the geometry.

Definition 1.1.8 : *Distance-transitivity in Graphs and Geometries*

A pair (Γ, G), where G is a connected graph and Γ a subgroup of $\mathrm{Aut}(G)$ is *distance-transitive* if Γ acts transitively on ordered pairs of vertices at distance i apart,

$i = 1, 2, \ldots, \mathrm{diam}(G)$. A graph G is *distance-transitive* if $(\mathrm{Aut}(G), G)$ is a distance-transitive pair. It is well known that a distance-transitive graph is distance-regular and that it is primitive as a permutation group $(\mathrm{Aut}(G), VG)$ precisely when it is primitive as a distance-regular graph [16].

A pair $(\Gamma, \mathcal{G})$ where $\mathcal{G}$ is a geometry and Γ a subgroup of $\mathrm{Aut}(\mathcal{G})$ is *distance-transitive* if Γ acts transitively on pairs of elements at distance i apart provided their component types match, $i = 1, \ldots, \mathrm{diam}(G(\mathcal{G}))$. A geometry $\mathcal{G}$ is *distance-transitive* if the pair $(\mathrm{Aut}(\mathcal{G}), \mathcal{G})$ is distance-transitive.

A pair (Γ, G) is *distance-bitransitive* if G is the incidence graph of a geometry $\mathcal{G}$ such that $((\Gamma)_P, \mathcal{G})$ is distance-transitive, where $(\Gamma)_P$ is the subgroup of Γ stabilising the set P of points of the geometry. A graph G is *distance-bitransitive* if $(\mathrm{Aut}(G), G)$ is a distance-bitransitive pair.

A distance-transitive geometry $\mathcal{G}$ is *imprimitive* if $\mathrm{Aut}(\mathcal{G})$ acts imprimitively on either the points or the lines of $\mathcal{G}$, otherwise the geometry is *primitive*. A distance-bitransitive graph is *imprimitive (primitive)* if the corresponding geometry is imprimitive (primitive).

Definition 1.1.9 : *Fields and Finite Vector Spaces*

The real number field is denoted by $\mathcal{R}$, the complex field by $\mathcal{C}$, the integers by $\mathcal{Z}$ and the natural numbers by $\mathcal{N}$. The unique galois field of order q, where q is a prime power, is denoted by $GF(q)$. The number of j-dimensional subspaces of an n-dimensional vector space over $GF(q)$ is denoted by

$$\begin{bmatrix} n \\ j \end{bmatrix}_q = \prod_{i=0}^{j-1} \frac{q^n - q^i}{q^j - q^i} .$$

The subscript q can be omitted if it is apparent from the context.

A form on a vector space V is a bilinear mapping to the underlying field. A form f is non-degenerate if $f(u, v) = 0$ for all $v \in V$ implies $u = 0$. If f is such a form and $U \subset V$ then $U^{\perp}$ denotes the subspace

$$U^{\perp} = \{v \in V \mid f(u, v) = 0, \text{ for all } u \in U\} .$$

If U is a subspace of dimension j and $\dim V = n$ then $\dim U^{\perp} = n - j$. A subspace U is called *isotropic* if $U \subset U^{\perp}$.

Definition 1.1.10 : *Matrices*

A *Hadamard matrix* of order n is a real matrix $\mathbf{H}$ whose entries are 1 or -1, satisfying $\mathbf{H}^T\mathbf{H} = n\mathbf{I}$. Note that $|\det\mathbf{H}| = n^{n/2}$, being the maximum possible value for a real $n \times n$ matrix with entries having absolute value less than or equal to 1.

1.2 Examples of Distance-regular and Distance-biregular Graphs

The first examples will be of known families of distance-regular graphs. We then give some examples of distance-biregular and distance-bitransitive graphs.

Example 1.2.1 : *The Hamming Graph $H(d,n)$.*

Let X be a q-element set for some $q > 1$. Set $VG = X^d$, $d > 1$ and for $u, v \in VG$, let $u \sim v$ if u and v differ in exactly one coordinate. Clearly $H(d,n) \cong K_q \times \ldots \times K_q$ with n factors and $\partial(u,v) = i$ if u and v differ in precisely i coordinates. A subgroup of the automorphism group of G is $\mathrm{Sym}(q) \wr \mathrm{Sym}(d)$ acting in the obvious way. Thus fixing a vertex u, $(\mathrm{Aut}(G))_u$ acts transitively on vertices at distance i. As $\mathrm{Aut}(G)$ is transitive, G is distance-transitive. ∎

Example 1.2.2 : *The Johnson Scheme $J(d,n)$.*

Let X be the n-element set $\{1, 2, \ldots, n\}$ and set $VG = \{d\text{-subsets of } X\}$. For $u, v \in VG$, $u \sim v$ if $|u \cap v| = d - 1$. Vertices $u, v \in VG$ satisfy $|u \cap v| = d - i$ if $\partial(u,v) = i$. A subgroup of the automorphism group of G is $\mathrm{Sym}(n)$ acting in the obvious way. Fixing a vertex u, $(\mathrm{Aut}(G))_u$ clearly acts transitively on vertices at distance i from u. As $\mathrm{Aut}(G)$ acts transitively on VG, the graph G is distance-transitive. ∎

Example 1.2.3 : *The q-analogue of the Johnson Scheme $J_q(d,n)$.*

Let V be an n-dimensional vector space over $GF(q)$ and set $VG = \{d\text{-subspaces of } V\}$. For $u, v \in VG$, $u \sim v$ if $\dim(u \cap v) = d - 1$. Vertices $u, v \in VG$ satisfy $\dim(u \cap v) = d - i$

if $\partial(u,v) = i$. A subgroup of the automorphism group of $J_q(d,n)$ is $P\Gamma L(n,q)$. Fixing a vertex u, clearly $(\mathrm{Aut}(G))_u$ acts transitively on vertices at distance i from u. As $\mathrm{Aut}(G)$ acts transitively G is distance-transitive. ∎

Example 1.2.4 : *Dual Polar Space Graphs*

Let q and r be prime powers and V one of the following spaces equipped with the respective form:

$C_d(q) = GF(q)^{2d}$ with a non-degenerate symplectic form,
$(P\Gamma Sp(2d,q))$,

$B_d(q) = GF(q)^{2d+1}$ with a non-degenerate quadratic form,
$(P\Gamma O^+(2d+1,q))$,

$D_d(q) = GF(q)^{2d}$ with a non-degenerate quadratic form of Witt index d,
$(P\Gamma O^+(2d,q))$,

${}^2D_{d+1}(q) = GF(q)^{2d+2}$ with a non-degenerate quadratic form of Witt index d,
$(P\Gamma O^-(2d,q))$,

${}^2A_{2d}(r) = GF(q)^{2d+1}$ with a non-degenerate hermitian form $(q = r^2)$,
$(P\Gamma U(2d+1,r))$,

${}^2A_{2d-1}(r) = GF(q)^{2d}$ with a non-degenerate hermitian form $(q = r^2)$,
$(P\Gamma U(2d,r))$.

In each case the vertices of the dual polar space graph G are the maximal isotropic subspaces (of dimension d) with two subspaces adjacent if their intersection has dimension $d-1$. The exponent of the graphs are $e = 0, 0, -1, 1, 1/2, -1/2$ respectively. It is proved in [6] that these graphs are distance-transitive with the groups in brackets acting. ∎

Example 1.2.5 : *The complete bipartite graph $K_{r,s}$.*

The complete bipartite graph $K_{r,s}$ is an example of a distance-biregular graph. If $r \neq s$ it is not a distance-regular graph. Its two intersection arrays are:

$$\begin{bmatrix} * & 1 & r \\ 0 & 0 & 0 \\ r & s-1 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & s \\ 0 & 0 & 0 \\ s & r-1 & * \end{bmatrix}$$

∎

Example 1.2.6 : *The Johnson Biregular Graphs $JB(k,n)$*

Consider the set $\{1,\ldots,n\}$. Let $A = \{k\text{-subsets}\}$ and $B = \{k+1\text{-subsets}\}$ where k is a positive integer less than n. The vertex set of the graph G is $VG = A \cup B$ and adjacency is defined in the natural way: $u \sim v$, with $u \in A$ and $v \in B$ if $u \subset v$. It is not hard to show that this graph has automorphisms mapping pairs of vertices at the same distance to each other provided they lie in corresponding colour classes. This property clearly means that every vertex has an intersection array and so the graph is distance-biregular. The array for a vertex in A is:

$$\left[\begin{matrix} * & 1 & 1 & 2 & 2 & \ldots & i & i & \ldots \\ n-k & k & n-k-1 & k-1 & n-k-2 & \ldots & k-i+1 & n-k-i & \ldots \end{matrix}\right.$$

$$\left.\begin{matrix} \ldots & n-k-1 & n-k & n-k \\ \ldots & 1 & 2k-n+1 & * \end{matrix}\right] \text{ if } k \geq n-k \ (2k \geq n) \text{ or with the ending:}$$

$$\left.\begin{matrix} \ldots & k & k & k+1 \\ \ldots & 1 & n-2k & * \end{matrix}\right] \text{ if } k < n-k \ (2k < n).$$

∎

Example 1.2.7 : *The q-analogue Johnson biregular graphs $JB_q(k,n)$*

Consider an n-dimensional vector space over $GF(q)$, where q is the power of a prime and $GF(q)$ is the (unique) Galois field of order q. Let $A = \{k\text{-subspaces}\}$ and $B = \{k+1\text{-subspaces}\}$ and $VG = A \cup B$. Adjacency is defined for $u \in A$ and $v \in B$ with $u \sim v$ if $u \subset v$. Again we can find an automorphism of the vector space and so also of the graph which maps pairs of spaces at the same distance apart to each other provided they lie in corresponding colour classes. This means that all vertices are distance-regularised and that the graph G is distance-biregular.

The intersection array for a k-space is as follows:

$$\left[\begin{matrix} * & 1 & 1 & \ldots & \frac{q^i-1}{q-1} & \frac{q^i-1}{q-1} & \ldots \\ \frac{q^{n-k}-1}{q-1} & \frac{q^{k+1}-q}{q-1} & \frac{q^{n-k}-q}{q-1} & \ldots & \frac{q^{k+1}-q^i}{q-1} & \frac{q^{n-k}-q^i}{q-1} & \ldots \end{matrix}\right.$$

$$\left.\begin{matrix} \ldots & \frac{q^{n-k-1}-1}{q-1} & \frac{q^{n-k}-1}{q-1} & \frac{q^{n-k}-1}{q-1} \\ \ldots & q^{n-k-1} & \frac{q^{k+1}-q^{n-k}}{q-1} & * \end{matrix}\right] \text{ if } k \geq n-k,$$

$$\left.\begin{matrix} \cdots & \frac{q^k-1}{q-1} & \frac{q^k-1}{q-1} & \frac{q^{k+1}-1}{q-1} \\ \cdots & q^k & \frac{q^{n-k}-q^k}{q-1} & * \end{matrix}\right] \quad \text{if } k < n-k.$$

■

Example 1.2.8 : *Generalised n-gon*

A Generalised n-gon is a distance-biregular graph and not a distance-regular graph if the number $s+1$ of points on each line differs from the number $t+1$ of lines through each point. The intersection array for a 'point' as opposed to a 'line' vertex is:

$$\begin{bmatrix} * & 1 & 1 & 1 & \dots & 1 & t+1 \\ t+1 & s & t & s & \dots & s & * \end{bmatrix}$$

■

Example 1.2.9 : *Quasisymmetric 2-design*

Let D be a quasisymmetric 2-design with block intersection numbers i_1, i_2, with $i_2 = 0$. Then the incidence graph G of D is a distance-biregular graph of diameter 4 with intersection array for a point vertex:

$$\begin{bmatrix} * & 1 & \lambda_2 & b \\ \lambda_1 & b-1 & \lambda_1 - \lambda_2 & * \end{bmatrix}$$

and intersection array for a block vertex:

$$\begin{bmatrix} * & 1 & i_1 & c_3 = b\lambda_2/i_1 & b \\ b & \lambda_1 - 1 & b - i_1 & \lambda_1 - c_3 & * \end{bmatrix},$$

where b is the block size and λ_j is the number of blocks each j-element set of points is contained in, $j = 1, 2$. ■

Example 1.2.10 : *A Distance-bitransitive Graph in $PG(2,4)$.*

It is well known that the 2-$(21,5,1)$-design consisting of the points and lines of $PG(2,4)$ can be extended to a 3-$(22,6,1)$-design by adding an additional vertex to each line and a class of 56 ovals, determined by an equivalence relation on the set of all ovals in $PG(2,4)$ (an oval is a maximal set of points no three of which are collinear and the relation is given by $O \sim O'$ if $|O \cap O'| = 0, 2$ or 6). The graph G has vertex set the points of $PG(2,4)$ and the 56 ovals of a chosen class. The pair (x, O) is an edge of G if x is a point of the oval O. We will prove in section 8 that G is distance-bitransitive and so certainly distance-biregular. The intersection arrays of G are

$$\begin{bmatrix} * & 1 & 2 & 12 & 6 \\ 6 & 15 & 4 & 4 & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & 1 & 4 & 6 \\ 16 & 5 & 12 & * \end{bmatrix}.$$

■

2 Distance-Regularised Graphs

This section is concerned with proving that distance-regularised graphs (see Definition 1.1.3) are distance-regular (see Definition 1.1.4) or distance-biregular (see Definition 1.1.6). It is divided into two sections. The first deals with the non-bipartite case while the second is concerned with bipartite distance-regularised graphs.

2.1 Non-bipartite Distance-regularised Graphs

The following lemma will prove very helpful in the non-bipartite case.

Lemma 2.1.1 : *Let G be a non bipartite connected graph. Let f be a function from the vertices of G to the natural numbers such that for $u \in VG$, f is constant on $G_1(u)$. Then f is a constant function.*

Proof: Let u, w be vertices of G. Since G is not bipartite, G contains an odd cycle C: $x_1, \ldots, x_{2k} = x_1$. As G is connected we can find a path $u = y_1, \ldots, y_s = x_1$ from u to x_1 and a path $w = z_1, \ldots, z_t = x_1$ from w to x_1. If the path $y_1, \ldots, y_s = x_1 = z_t, \ldots, z_1$ from u to w is of odd length then by adding the odd cycle C to it we get an even path from u to w. In either case we can find a path $u = v_1, v_2, \ldots, v_{2k+1} = w$ from u to w of even length. Then $v_{2i-1}, v_{2i+1} \in G_1(v_{2i})$ and so $f(v_{2i-1}) = f(v_{2i+1})$, for $i = 1, \ldots, k$. Hence $f(u) = f(w)$. ∎

As an example of using Lemma 2.1.1 we give a corollary which will be useful later.

Corollary 2.1.2: *Let G be a non-bipartite distance-regularised graph. Then G is regular.*

Proof: Let $u \in VG$ and $v \sim u$. Then $deg(v) = b_1(u) + a_1(u) + c_1(u)$, which is independent of the choice of v in $G_1(u)$. ∎

We are now ready to tackle the main theorem of this section. Our method of proof will follow the spirit of Corollary 2.1.2.

Theorem 2.1.3: *Let G be a non-bipartite distance-regularised graph. Then G is distance-regular.*

Proof: We prove that all vertices have the same array by induction on the columns of the array. Let $u, v \in VG$ with $u \sim v$. First we calculate the number $|G_t(u) \cap G_t(v)|$. This is given by $k_t(u) - r_t - s_t$, where $r_t = |G_t(u) \cap G_{t+1}(v)|$ and $s_t = |G_t(u) \cap G_{t-1}(v)|$. Note that $s_1 = 1$ and $r_1 = b_1(v)$. By counting edges between $G_t(u) \cap G_{t-1}(v)$ and $G_{t-1}(u) \cap G_{t-2}(v)$ we obtain

$$c_{t-1}(v) s_t = s_{t-1} b_{t-1}(u)$$

as each vertex in $G_t(u)$ adjacent to a vertex in $G_{t-1}(u) \cap G_{t-2}(v)$ must be in $G_t(u) \cap G_{t-1}(v)$, while each of the $c_{t-1}(v)$ neighbours nearer to v of a vertex in $G_t(u) \cap G_{t-1}(v)$ must lie in $G_{t-1}(u) \cap G_{t-2}(v)$. Hence

$$s_t = \frac{b_{t-1}(u) \ldots b_1(u)}{c_{t-1}(v) \ldots c_1(v)} .$$

Similarly

$$r_t = \frac{b_t(v) \ldots b_1(v)}{c_t(u) \ldots c_1(u)} .$$

Note also that

$$k_t(u) = \frac{b_{t-1}(u) \ldots b_0(u)}{c_t(u) \ldots c_1(u)} .$$

We now start the induction on the columns of the intersection arrays. By Corollary 2.1.2 the first entry in each array is the same as G is regular. Now assume this is true for all entries up to and including the $(t-1)$-st column, for some $t \geq 1$. Since G is connected it will be sufficient to prove that the entries in the t-th column of the arrays for u and v agree, as u and v were chosen as any two adjacent vertices. The inductive assumption allows us to evaluate $|G_t(u) \cap G_t(v)|$ as

$$\frac{b_{t-1}(u) \dots b_1(u)}{c_t(u) \dots c_1(u)}(b_0(u) - c_t(u) - b_t(v)) \; .$$

We consider two cases.

Case 1 : $G_t(u) \cap G_t(v) = \emptyset$.

By the above formula $c_t(u) + b_t(v) = k$, the degree of G. Similarly $c_t(v) + b_t(u) = k$ and so

$$c_t(u) + b_t(u) + b_t(v) + c_t(v) = 2k$$

and we must have $c_t(u) + b_t(u) = k = b_t(v) + c_t(v)$. In this case $a_t(u) = a_t(v) = 0$. Note also that $b_t(v) = k - c_t(u) = b_t(u)$, so that the arrays of u and v agree in the t-th column.

Case 2 : $G_t(u) \cap G_t(v) \neq \emptyset$.

Let $w \in G_t(u) \cap G_t(v)$ and $q_i = |G_i(w) \cap G_{t-i}(u)|$. Clearly $q_1 = c_t(u)$ and we can readily evaluate

$$q_i = \frac{c_t(u) \dots c_{t-i+1}(u)}{c_1(w) \dots c_i(w)} \; .$$

Using the induction hypothesis $q_{t-1} = c_t(u)$. But $q_{t-1} = c_t(w)$ by definition and so $c_t(u) = c_t(w)$. Similarly $c_t(w) = c_t(v)$ and so $c_t(u) = c_t(v)$. Finally calculating $|G_t(u) \cap G_t(v)|$ in two ways we have $c_t(u) + b_t(v) = c_t(v) + b_t(u)$, so $b_t(v) = b_t(u)$ and the t-th column of the arrays of u and v agree. ∎

2.2 Bipartite Distance-regularised Graphs

The bipartite case will prove easier to treat. In fact the following lemma does all the essential work and the required result will be a simple corollary.

Lemma 2.2.1 : *Let G be a bipartite distance-regularised graph with $u, v \in VG$ and $u \sim v$. Then the intersection array for v can be determined from that of u.*

Proof : Assume the standard notation and let $v \sim u$. We will compute the intersection array for v from that of u. We have $G_i(v) \subset G_{i-1}(u) \cup G_{i+1}(u)$. Set $x_i = |G_i(u) \cap G_{i-1}(v)|$, for $i = 1, \ldots, d$. Then $x_1 = 1, x_2 = b_1$. Note also that $l_0 = 1$, $l_1 = b_1 + c_1$, $e_1 = b_0 - 1$, $f_1 = 1$, $e_0 = b_1 + c_1$. Assume now that we know e_j, f_j, l_j, x_j, $j < i$ and x_i, for some $i \geq 2$. Then

$$l_i = k_{i-1} - x_{i-1} + x_i b_i / f_i$$

since

$$|G_i(v) \cap G_{i-1}(u)| = |G_{i-1}(u)| - x_{i-1}$$

and each vertex $w \in G_{i+1}(u) \cap G_i(v)$ is adjacent to f_i vertices in $G_i(u)$. But we also have $l_i = l_{i-1} e_{i-1} / f_i$, so eliminating l_i we obtain

$$f_i = \frac{l_{i-1} e_{i-1} - x_i b_i}{k_{i-1} - x_{i-1}} .$$

If $k_{i-1} = x_{i-1}$ then $G_{i-1}(u) \subset G_{i-2}(v)$, forcing $G_i(v) = \emptyset$. In this case $e_{i-1} = 0$, and we have already determined $\iota(v)$. Hence we can evaluate f_i. Then of course $e_i = b_{i-1} + c_{i-1} - f_i$, and we can compute x_{i+1} by $x_{i+1} = x_i b_i / f_i$. This completes the calculations of another column of the array. The result follows by induction. ∎

Corollary 2.2.2 : *A bipartite distance-regularised graph is distance-biregular.*

Proof: Let u, w be vertices of a bipartite distance-regularised graph G which lie in the same colour class. Then there exists a path of even length from u to w. Alternate vertices along this path have the same intersection array by the lemma. Hence u and w have the same array. ∎

3 Intersection Arrays of Distance-biregular Graphs

This section is concerned with understanding the intersection arrays of a distance-biregular graph. We will begin by relating these arrays to those of its derived graphs, which are shown to be distance-regular. We then generalise the feasibility conditions of a distance-regular graph to the biregular case. After various other relations on the arrays have been proved we introduce the notion of a pair of feasible arrays for a distance-biregular graph. Finally we prove some results about the diameter of distance-biregular graphs including a bound in terms of the girth and the valencies of the graph.

3.1 The Derived Graphs of a Distance-biregular Graph

In this subsection we show that the derived graphs of a distance-biregular graph are distance-regular and also investigate the relations between the intersection arrays and eigenvalues of the graph and its derived graphs.

Proposition 3.1.1 : *Let G be a distance-biregular graph. Then the derived graphs of G are distance-regular and their intersection arrays can be calculated from the arrays of G.*

Proof: Assume that the arrays of G are in the standard notation. Let the derived graph on vertex set A be D. Let $u \in A$ and consider $D_j(u) = G_{2j}(u)$ (note that this set is contained in A). Pick $x \in D_j(u)$. In G, vertex x is adjacent to no vertices in $G_{2j}(u)$ but is adjacent to c_{2j} vertices in $G_{2j-1}(u)$ and b_{2j} vertices in $G_{2j+1}(u)$. Each of the vertices in $G_{2j-1}(u)$ is adjacent to $b_{2j-1} - 1$ vertices in $G_{2j}(u)$ other than x. Similarly the vertices in $G_{2j+1}(u)$ are adjacent to $c_{2j+1} - 1$ vertices in $G_{2j}(u)$ other than x. Let $a_j^* := |D_j(u) \cap D_1(x)|$. Then in G each of these a_j^* vertices is at distance 2 from x and so has c_2 common neighbours with x. Hence counting edges in G between $G_1(x)$ and $G_{2j}(u) \cap G_2(x)$ in two ways we have:

$$a_j^* c_2 = c_{2j}(b_{2j-1} - 1) + b_{2j}(c_{2j+1} - 1),$$

giving

$$a_j^* = \frac{1}{c_2}\left(c_{2j}(b_{2j-1} - 1) + b_{2j}(c_{2j+1} - 1)\right),$$

which is independent of the choice of x in $D_j(u)$.

Now by a similar argument we obtain

$$c_j^* = c_{2j}c_{2j-1}/c_2 \quad \text{and} \quad b_j^* = b_{2j}b_{2j+1}/c_2$$

for the number of vertices adjacent to x in $D_{j-1}(u)$ and $D_{j+1}(u)$, both independent of the vertex x. Hence the intersection array for u in the graph D exists and is given by

$$\iota(u) = \begin{bmatrix} * & 1 & c_2^* & \dots & c_t^* \\ 0 & a_1^* & a_2^* & \dots & a_t^* \\ b_0^* & b_1^* & b_2^* & \dots & * \end{bmatrix},$$

where $t = \lfloor d/2 \rfloor$. This array is independent of the choice of u in VD. Hence the graph D is distance-regular and its intersection array can be computed from those of G. A similar argument holds for the other derived graph on the vertex set B. ∎

Not only can we relate the intersection numbers of the derived graph with those of the original graph, we can also find relations between the eigenvalues of the two graphs.

Lemma 3.1.2 : *Let G be a distance-biregular graph with the standard notation such that $f_2 \neq 0$ and $c_2 \neq 0$, and let the derived graph on vertex set A be D and the derived graph on vertex set B be E. Then the squares of the eigenvalues of G, $\lambda(G)^2$, are related to the eigenvalues of the graphs D and E by the equation:*

$$\begin{aligned}\lambda(G)^2 \setminus \{0\} &= c_2(\lambda(D) \setminus \{-r/c_2\}) + r \\ &= f_2(\lambda(E) \setminus \{-s/f_2\}) + s\ .\end{aligned}$$

Also $-r/c_2 \leq \min\lambda(D)$ and $-s/f_2 \leq \min\lambda(E)$. If $0 \in \lambda(G)$ then at least one of $-r/c_2 \in \lambda(D), -s/f_2 \in \lambda(E)$ holds.

Proof: Consider the adjacency matrix $\mathbf{A}(G)$ of G. By indexing the vertices in A before those in B we give $\mathbf{A}(G)$ the following block structure:

$$\mathbf{A}(G) = \begin{pmatrix} \mathbf{0} & \mathbf{M}^T \\ \mathbf{M} & \mathbf{0} \end{pmatrix}.$$

$$\text{Then} \quad \mathbf{A}(G)^2 = \begin{pmatrix} \mathbf{M}^T\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}\mathbf{M}^T \end{pmatrix}.$$

$$\text{But} \quad \mathbf{A}(G)^2 = \begin{pmatrix} c_2\mathbf{A}(D) + r\mathbf{I} & \mathbf{0} \\ \mathbf{0} & f_2\mathbf{A}(E) + s\mathbf{I} \end{pmatrix}$$

as $\mathbf{A}(G)^2$ counts the paths of length 2 between vertices of G. Hence $\mathbf{M}^T\mathbf{M} = c_2\mathbf{A}(D) + r\mathbf{I}$ and $\mathbf{M}\mathbf{M}^T = f_2\mathbf{A}(E) + s\mathbf{I}$. As $\mathbf{M}^T\mathbf{M}$ and $\mathbf{M}\mathbf{M}^T$ are positive semi-definite they have non-negative eigenvalues. So $-r/c_2 \leq \min\lambda(D)$ and $-s/f_2 \leq \min\lambda(E)$. Further $\mathbf{M}^T\mathbf{M}$ and $\mathbf{M}\mathbf{M}^T$ have the same non-zero eigenvalues and hence

$$\begin{aligned}\lambda(G)^2 \setminus \{0\} &= \lambda(\mathbf{M}^T\mathbf{M}) \setminus \{0\} \\ &= c_2(\lambda(D) \setminus \{-r/c_2\}) + r \\ &= f_2(\lambda(E) \setminus \{-s/f_2\}) + s\ .\end{aligned}$$

Finally if $0 \in \lambda(G)$ then $0 \in \lambda(\mathbf{M}^T\mathbf{M}) \cup \lambda(\mathbf{M}\mathbf{M}^T)$ giving at least one of $-r/c_2 \in \lambda(D)$ and $-s/f_2 \in \lambda(E)$. ∎

3.2 Feasibility Conditions for a Distance-biregular Graph

We begin with elementary numerical conditions on the intersection arrays of a distance-biregular graph.

Proposition 3.2.1: *Using the standard notation for the parameters of a distance-biregular graph, we have the following relations:*

(i) $k_0 = 1$, $k_{i+1} = k_i b_i / c_{i+1}$, $l_0 = 1$, $l_{i+1} = l_i e_i / f_{i+1}$ *and the* k_i *and* l_i *are whole numbers.*

(ii) Alternate (non-zero) columns in the intersection arrays sum to r *and* s*:*

$$c_i + b_i = \begin{cases} r, & \text{if } i \text{ is even,} \\ s, & \text{otherwise.} \end{cases}$$

$$e_j + f_j = \begin{cases} r, & \text{if } j \text{ is odd,} \\ s, & \text{otherwise.} \end{cases}$$

(iii) $e_{i-1} \geq b_i \geq e_{i+1}, i = 1, \ldots, d-2.$

(iv) $f_{i+1} \geq c_i \geq f_{i-1}, i = 2, \ldots, d-1.$

(v) The following equations hold:

$$1 + k_2 + k_4 + \ldots + k_{d'} = l_1 + l_3 + \ldots + l_{d''} =: n$$

$$\text{and} \quad k_1 + k_3 + \ldots + k_{d''} = 1 + l_2 + l_4 + \ldots + l_{d'} =: m\ ,$$

where d' *is the largest even integer less than or equal to* d *and* d'' *is the largest such odd integer. Also* $nr = ms$.

Proof: (i) The first two relations follow from counting edges between $G_i(u)$ and $G_{i+1}(u)$, $u \in A$. The second two relations follow from counting edges between $G_i(v)$ and $G_{i+1}(v)$, $v \in B$.

(ii) a vertex in $G_i(u)$ has degree r if i is even and s if i is odd. The reverse holds for vertices in $G_i(v)$.

(iii) Let $u \in A$ and $v \in B$ with $u \sim v$. We can choose a vertex $x \in G_i(u) \cap G_{i-1}(v)$. The b_i neighbours of x in $G_{i+1}(u)$ lie in $G_i(v)$, and so $e_{i-1} \geq b_i$. By symmetry $b_i \geq e_{i+1}$.

(iv) With u and v as in (iii) we can choose a vertex $x \in G_i(u) \cap G_{i+1}(v)$. The c_i neighbours of x in $G_{i-1}(u)$ lie in $G_i(v)$, hence $f_{i+1} \geq c_i$. By symmetry $c_i \geq f_{i-1}$.

(v) This follows from considering $n = |A|$ and $m = |B|$ and the number of edges passing between A and B. ∎

We have already shown in section 2 (Lemma 2.2.1) that one array of a distance-biregular graph can be computed from the other. We present now a simpler expression of that relation.

Lemma 3.2.2: *Let G be a distance-biregular graph with the standard notation. Then $c_{2i+1}c_{2i} = f_{2i+1}f_{2i}$ and $b_{2i}b_{2i-1} = e_{2i}e_{2i-1}$, for $i = 1, \ldots, \lceil d/2 \rceil - 1$.*

Proof: Let $1 \leq i \leq \lceil d/2 \rceil - 1$ and consider two vertices u, v with $\partial(u,v) = 2i+1 (\leq d)$, with u in A and v in B. We wish to evaluate the size of the set $G_j(u) \cap G_{2i+1-j}(v)$. We claim that

$$m_j := |G_j(u) \cap G_{2i+1-j}(v)| = (f_{2i+1} \cdots f_{2i+2-j})/(c_1 \cdots c_j) \ .$$

We prove the claim by induction on j. For $j = 1, m_1 = f_{2i+1}$ by the definition of the intersection numbers. Suppose the equation holds for smaller numbers than j. Each vertex in $G_{j-1}(u) \cap G_{2i+2-j}(v)$ is adjacent to f_{2i+2-j} vertices in $G_{2i+1-j}(v)$ each of which lies in $G_j(u)$, while each vertex in $G_j(u) \cap G_{2i+1-j}(v)$ is adjacent to c_j vertices in $G_{j-1}(u)$, each of which lies in $G_{2i+2-j}(v)$. Hence $m_{j-1}f_{2i+2-j} = m_j c_j$. Using the induction hypothesis the claim follows. But then $m_{2i+1} = |G_{2i+1}(u) \cap G_0(v)| = 1$ and so

$$(f_{2i+1} \cdots f_1)/(c_1 \cdots c_{2i+1}) = 1$$

$$\text{and} \quad f_{2i+1} \cdots f_1 = c_{2i+1} \cdots c_1 \ .$$

$$\text{As} \quad f_{2i-1} \cdots f_1 = c_{2i-1} \cdots c_1 \neq 0$$

we have $f_{2i+1}f_{2i} = c_{2i+1}c_{2i}$.

To prove the second equation of the lemma we partition $G_{2i}(u)$, for u in A and $1 \leq i \leq \lceil d/2 \rceil - 1$, into two subsets, $G_{2i}(u) \cap G_{2i-1}(v)$ and $G_{2i}(u) \cap G_{2i+1}(v)$, where v is a vertex adjacent to u. We now estimate $k_{2i} = |G_{2i}(u)|$ in two ways. Firstly in the obvious fashion

$$k_{2i} = (b_0 b_1 \ldots b_{2i-1})/(c_1 c_2 \ldots c_{2i}) \neq 0, \quad \text{as } 2i < d \ .$$

To get the second estimate we first prove a claim that

$$n_j = |G_{j+1}(u) \cap G_j(v)| = (b_1 b_2 \ldots b_j)/(f_1 f_2 \ldots f_j) \; .$$

We again proceed by induction on j. For $j = 1$ it is true by the definition of b_1. Now assume it holds for integers less than j. Each vertex in $G_j(u) \cap G_{j-1}(v)$ is adjacent to b_j vertices in $G_{j+1}(u)$ all of which are distance j from v. Each vertex in $G_{j+1}(u) \cap G_j(v)$ is adjacent to f_j vertices in $G_{j-1}(v)$ all of which are distance j from u. Hence $n_{j-1} b_j = n_j f_j$. Using the induction hypothesis

$$n_j = (b_1 b_2 \ldots b_j)/(f_1 f_2 \ldots f_j) \; .$$

By the symmetry of the definition of a distance-biregular graph

$$|G_{j+1}(v) \cap G_j(u)| = (e_1 e_2 \ldots e_j)/(c_1 c_2 \ldots c_j) \; .$$

$$\text{Hence} \quad k_{2i} = |G_{2i}(u) \cap G_{2i-1}(v)| + |G_{2i}(u) \cap G_{2i+1}(v)|$$

$$= \frac{b_1 b_2 \ldots b_{2i-1}}{f_1 f_2 \ldots f_{2i-1}} + \frac{e_1 e_2 \ldots e_{2i}}{c_1 c_2 \ldots c_{2i}} \; .$$

$$\begin{aligned}
\text{By the first part} \quad & c_1 c_2 \ldots c_{2i-1} = f_1 f_2 \ldots f_{2i-1} \\
\text{and so} \quad & b_0 b_1 \ldots b_{2i-1} = b_1 b_2 \ldots b_{2i-1} c_{2i} + e_1 e_2 \ldots e_{2i} \\
\text{and} \quad & b_1 b_2 \ldots b_{2i-1}(b_0 - c_{2i}) = e_1 e_2 \ldots e_{2i} \\
\text{or} \quad & b_1 b_2 \ldots b_{2i} = e_1 e_2 \ldots e_{2i} \; . \\
\text{For } i > 1, \quad & b_1 b_2 \ldots b_{2i-2} = e_1 e_2 \ldots e_{2i-2} \neq 0 \; ,
\end{aligned}$$

and so we have $b_{2i} b_{2i-1} = e_{2i} e_{2i-1}$ as required. ∎

We now use techniques of quotient graphs to discover connections between the eigenvalues of a graph G and those of the intersection matrix of a distance-regularised vertex of G. The basic technique employed and some of the results presented here appear in [8], but we give the proofs in full for completeness.

Lemma 3.2.3: *Let $w \in VG$ be a distance-regularised vertex of a graph G. Let* $\mathbf{C}$ *be the incidence matrix between the vertices of G (columns) and the sets $G_i(w)$ (rows), $i = 0, \ldots, t$ (here t is the number such that $G_t(w) \neq \emptyset$ and $G_{t+1}(w) = \emptyset$). Let* $\mathbf{Q}$ *be the intersection matrix for w, and* $\mathbf{A}$ *the adjacency matrix of G. Then we have:*

(a) $\mathbf{CA} = \mathbf{QC}$

(b) If $\mathbf{Az} = \lambda\mathbf{z}$ *then* $\mathbf{QCz} = \lambda\mathbf{Cz}$

(c) $\mathbf{Q}$ *has a complete set of eigenvectors of the form* $\mathbf{Cz}$, *where* $\mathbf{z}$ *ranges over the eigenvectors of* $\mathbf{A}$

(d) all the eigenvalues of $\mathbf{Q}$ *are simple and are eigenvalues of G.*

Proof: (a) follows from the definition of $\mathbf{Q}$.

(b) Suppose $\mathbf{Az} = \lambda\mathbf{z}$, then $\lambda\mathbf{Cz} = \mathbf{CAz} = \mathbf{QCz}$.

(c) $\mathbf{C}$ has rank $t+1$ since its rows are clearly linearly independent. Thus if $\mathbf{z}_1, \ldots, \mathbf{z}_n$ is a complete set of eigenvectors for $\mathbf{A}$ then the non-zero vectors among $\mathbf{Cz}_1, \ldots, \mathbf{Cz}_n$ span $\mathcal{R}^{t+1}$ and form a complete set of eigenvectors for $\mathbf{Q}$.

(d) Clearly by (c) $\lambda(\mathbf{Q}) \subset \lambda(G)$. But since $\mathbf{Q}$ is the adjacency matrix of a multigraph of diameter t, $\mathbf{Q}$ has at least $t+1$ distinct eigenvalues [1], so they must all be simple. ∎

Definition 3.2.4: *The Walk Generating Function and Characteristic Polynomial*

For any square matrix $\mathbf{A}$ we define

$$\mathbf{W}(\mathbf{A}, x) = \sum_{k=0}^{\infty} x^k \mathbf{A}^k = (\mathbf{I} - x\mathbf{A})^{-1}$$

and

$$\phi(\mathbf{A}, x) = \det(x\mathbf{I} - \mathbf{A}) \ .$$

With a slight abuse of notation we write $\mathbf{W}(G, x)$ for $\mathbf{W}(\mathbf{A}, x)$ and $\phi(G, x)$ for $\phi(\mathbf{A}, x)$, where G is a digraph with adjacency matrix $\mathbf{A}$. The matrix function $\mathbf{W}(G, x)$ is often called the walk generating function for G, while $\phi(G, x)$ is the characteristic polynomial of G. For $u, v \in VG$ the uv entry of $\mathbf{W}(G, x)$ is denoted by $\mathbf{W}_{uv}(G, x)$. By $\mathbf{A} \setminus i$ we will denote the matrix obtained from $\mathbf{A}$ by deleting the i-th row and i-th column, where the columns and rows are numbered from 0 upwards.

Theorem 3.2.5 : *Let G be any digraph.*

(i) for $v \in VG$,

$$\mathbf{W}_{vv}(G,x) = \left(\frac{1}{x}\right) \frac{\phi(G \setminus v, 1/x)}{\phi(G, 1/x)}$$

(ii) The trace of the walk generating function of a graph G is given by

$$\operatorname{trace}(\mathbf{W}(G,x)) = \left(\frac{1}{x}\right) \frac{\phi'(G,1/x)}{\phi(G,1/x)},$$

where here and subsequently ϕ' is the derivative of ϕ with respect to its parameter (in this case $1/x$).

Proof: (i) We have

$$\begin{aligned} \mathbf{W}_{vv}(G,x) &= \sum_{k=0}^{\infty} x^k (\mathbf{A}^k)_{vv} \\ &= \left((\mathbf{I} - x\mathbf{A})^{-1}\right)_{vv} \qquad (*) \end{aligned}$$

$$\text{Now} \quad (x\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{\det(x\mathbf{I} - \mathbf{A})} \operatorname{adj}(x\mathbf{I} - \mathbf{A})$$

But the vv entry of $\operatorname{adj}(x\mathbf{I} - \mathbf{A})$ is the determinant of the vv cofactor of $x\mathbf{I} - \mathbf{A}$, that is $\phi(G \setminus v, x)$. Hence

$$(x\mathbf{I} - \mathbf{A})^{-1}_{vv} = \frac{\phi(G \setminus v, x)}{\phi(G,x)}.$$

Substituting this into $(*)$ we obtain the result.

(ii) The result follows from (i) and the equation:

$$\phi'(G,x) = \sum_{v \in VG} \phi(G \setminus v, x).$$

∎

Consider a distance-biregular graph with the standard notation. Let $\mathbf{P}$ be the intersection matrix $\mathbf{I}(A)$ of vertices in A. It can be readily verified by induction that for $u \in A$, the number of walks of length k in G which start at a specified vertex in $G_i(u)$ and finish anywhere in $G_j(u)$ is $(\mathbf{P}^k)_{ji}$. This in turn means that $\mathbf{W}_{uu}(G,x) = \mathbf{W}_{00}(\mathbf{P},x)$ for $u \in A$.

We are now ready to prove the main theorem of this subsection. This result is a stronger version of the one proved in [19].

Theorem 3.2.6: *Let G be a distance-biregular graph. Then the eigenvalues of G and their multiplicities can be determined from its two intersection arrays.*

Proof: Using the notation from above, we have:

$$\left(\frac{1}{x}\right)\frac{\phi'(G,1/x)}{\phi(G,1/x)} = \mathrm{trace}(\mathbf{W}(G,x))$$

$$= \sum_{u\in A} \mathbf{W}_{uu}(G,x) + \sum_{v\in B} \mathbf{W}_{vv}(G,x).$$

Consider the adjacency matrix $\mathbf{A}$ of the graph G with the vertices of A indexed before those of B. The matrix $\mathbf{A}$ then has a block structure:

$$\mathbf{A} = \begin{pmatrix} \mathbf{0} & \mathbf{M}^T \\ \mathbf{M} & \mathbf{0} \end{pmatrix},$$

$$\text{while} \quad \mathbf{A}(G)^2 = \begin{pmatrix} \mathbf{M}^T\mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}\mathbf{M}^T \end{pmatrix}.$$

$$\begin{aligned}
\text{Now} \quad \sum_{u\in A} \mathbf{W}_{uu}(G,x) &= \sum_{r=1}^{\infty} (\mathrm{trace}(\mathbf{M}^T\mathbf{M})^r) x^{2r} + n \\
&= \sum_{r=1}^{\infty} (\mathrm{trace}(\mathbf{M}\,\mathbf{M}^T)^r) x^{2r} + m + (n-m) \\
&= \sum_{v\in B} \mathbf{W}_{vv}(G,x) + n - m.
\end{aligned}$$

Hence

$$\begin{aligned}
\left(\frac{1}{x}\right)\frac{\phi'(G,1/x)}{\phi(G,1/x)} &= 2\sum_{u\in A} \mathbf{W}_{uu}(G,x) + m - n \\
&= 2n\mathbf{W}_{00}(\mathbf{P},x) + m - n \\
&= 2n\left(\frac{1}{x}\right)\frac{\phi(\mathbf{P}\setminus 0,1/x)}{\phi(\mathbf{P},1/x)} + m - n.
\end{aligned}$$

Applying a change of variable:

$$\frac{\phi'(G,x)}{\phi(G,x)} = 2n\frac{\phi(\mathbf{P}\setminus 0,x)}{\phi(\mathbf{P},x)} + \frac{m-n}{x}.$$

Since the eigenvalues of $\mathbf{P}$ are simple we can write

$$\frac{1}{\phi(\mathbf{P},x)} = \sum_{\theta\in\lambda(\mathbf{P})} \frac{1}{(x-\theta)\phi'(\mathbf{P},\theta)}.$$

For the LHS we have the following

$$\frac{\phi'(G,x)}{\phi(G,x)} = \sum_{\theta\in\lambda(G)} \frac{m(\theta)}{x-\theta},$$

where $m(\theta)$ is the multiplicity of θ in G. Hence

$$\sum_{\theta\in\lambda(G)} \frac{m(\theta)}{x-\theta} = 2n \sum_{\theta\in\lambda(\mathbf{P})} \frac{\phi(\mathbf{P}\setminus 0, x)}{\phi'(\mathbf{P},\theta)(x-\theta)} + \frac{m-n}{x}.$$

Equating residuals we obtain:

$$m(\theta) = 2n\frac{\phi(\mathbf{P}\setminus 0,\theta)}{\phi'(\mathbf{P},\theta)};\quad \theta \neq 0,$$

and

$$m(0) = 2n\chi_{\lambda(\mathbf{P})}(0)\frac{\phi(\mathbf{P}\setminus 0,0)}{\phi'(\mathbf{P},0)} + m - n.$$

This equation enables us to calculate the multiplicities and also tells us that we have all the eigenvalues, that is:

$$\lambda(G)\cup\{0\} = \lambda(\mathbf{P})\cup\{0\}\ .$$

By the symmetry of the arrays we could equally well have computed the eigenvalues and their multiplicities from the intersection matrix $\mathbf{I}(B)$. ∎

Though we can theoretically use Theorem 3.2.6 to calculate the eigenvalue multiplicities for G, it is in practice not easy to evaluate $\phi(\mathbf{P}\setminus 0,\theta)/\phi'(\mathbf{P},\theta)$ directly. However we can use a method analogous to that for distance-regular graphs [1] as we prove in the following proposition.

Proposition 3.2.7: *Let* $\mathbf{P}$ *be any tridiagonal matrix with all upper* and lower *diagonal elements non-zero. Let* λ *be an eigenvalue of* $\mathbf{P}$ *and* $\mathbf{y}$ $(\mathbf{x})$ *a left (right) eigenvector corresponding to* λ *normalised with* $\mathbf{x}_0 = \mathbf{y}_0 = 1$*. Then*

$$\frac{\phi'(\mathbf{P},\lambda)}{\phi(\mathbf{P}\setminus 0,\lambda)} = \mathbf{y}\ \mathbf{x}.$$

Proof: We first consider the polynomials in t defined by $p_0(t) = 1$, $p_1(t) = (t - a_0)/c_1$, and

$$c_{i+1}p_{i+1}(t) + (a_i - t)p_i(t) + b_{i-1}p_{i-1}(t) = 0,\ i = 1,\ldots,d,$$

where

$$\mathbf{P} = \begin{pmatrix} a_0 & c_1 & 0 & 0 & \cdots & 0 \\ b_0 & a_1 & c_2 & 0 & \cdots & 0 \\ 0 & b_1 & a_2 & c_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{d-2} & a_{d-1} & c_d \\ 0 & \cdots & 0 & 0 & b_{d-1} & a_d \end{pmatrix}$$

c_{d+1} is taken as 1 and $c_i \neq 0$ by assumption. Then if λ is an eigenvalue of $\mathbf{P}$ we have $p_{d+1}(\lambda) = 0$ and

$$\mathbf{x} = [p_0(\lambda), p_1(\lambda), \ldots, p_d(\lambda)]^T$$

is a right eigenvector with $\mathbf{x}_0 = 1$. Let B_i be the leading principal $i \times i$ minor of $t\mathbf{I} - \mathbf{P}$. We claim that $B_i = p_i(t)\prod_{j=1}^{i} c_j$. We prove this by induction:

$$\text{for}\quad i = 1,\ B_1 = t - a_0 = (t - a_0)c_1/c_1 = p_1(t)c_1,$$

and inductively

$$\begin{aligned} B_{i+1} &= (t - a_i)B_i - b_{i-1}c_iB_{i-1} \\ &= (t - a_i)p_i(t)\prod_{j=1}^{i} c_j - b_{i-1}p_{i-1}(t)\prod_{j=1}^{i} c_j \\ &= p_{i+1}(t)\prod_{j=1}^{i+1} c_j. \end{aligned}$$

Now consider forming the polynomials $q_i(t)$ by

$$q_d(t) = 1, \quad q_{d-1}(t) = (t - a_d)/c_d$$

$$\text{and} \quad c_i q_{i-1}(t) + (a_i - t)q_i(t) + b_i q_{i+1}(t) = 0, \; i = d-1, \ldots, 0,$$

setting $c_{-1} = 1$. Then if λ is an eigenvalue of $\mathbf{P}$, $q_{-1}(\lambda) = 0$ and the vector

$$[q_0(\lambda), q_1(\lambda), \ldots, q_d(\lambda)]$$

is a left eigenvector of $\mathbf{P}$. As above it can be shown that if C_i is the $(d-i) \times (d-i)$ minor of $t\mathbf{I} - \mathbf{P}$ on the rows and columns $i+1, \ldots, d$ then

$$C_i = q_i(t) \prod_{j=i+1}^{d} c_j.$$

But now we have

$$\phi(\mathbf{P} \setminus i, t) = B_i C_i = p_i(t) q_i(t) \prod_{j=1}^{d} c_j.$$

Hence for λ an eigenvalue of $\mathbf{P}$, $\phi(\mathbf{P} \setminus 0, \lambda) = q_0(\lambda) \prod_{j=1}^{d} c_j$, which is non zero, since the upper and lower diagonal elements are non zero. We can thus perform the following calculation:

$$\begin{aligned}
\phi'(\mathbf{P}, \lambda)/\phi(\mathbf{P} \setminus 0, \lambda) &= \sum_{i=0}^{d} \phi(\mathbf{P} \setminus i, \lambda)/\phi(\mathbf{P} \setminus 0, \lambda) \\
&= \frac{\sum_{i=0}^{d} p_i(\lambda) q_i(\lambda) \prod_{j=1}^{d} c_j}{q_0(\lambda) \prod_{j=1}^{d} c_j} \\
&= \sum_{i=0}^{d} p_i(\lambda)(q_i(\lambda)/q_0(\lambda)) = \mathbf{yx},
\end{aligned}$$

$$\text{where} \quad \mathbf{y} = [1, q_1(\lambda)/q_0(\lambda), \ldots, q_d(\lambda)/q_0(\lambda)]$$

is the left eigenvector with $\mathbf{y}_0 = 1$. ∎

We can in fact obtain even more precise information about the multiplicity calculation by comparing the graph with its derived graph, as the following proposition shows.

Proposition 3.2.8: *Let G be a distance-biregular graph and $\mathbf{x}$ the right eigenvector of $\mathbf{I}(A)$ corresponding to the non-zero eigenvalue λ and satisfying $\mathbf{x}_0 = 1$. Then the coordinates of $\mathbf{x}$ satisfy*

$$\sum_{i\ even} \mathbf{x}_i^2/k_i = \sum_{i\ odd} \mathbf{x}_i^2/k_i.$$

Proof: Let $m(\lambda)$ be the multiplicity of the eigenvalue λ in G. Then $\lambda' = (\lambda^2 - r)/c_2$ is an eigenvalue of the derived graph D on vertex set $VD = A$ (see Lemma 3.1.2). As G is bipartite $-\lambda$ is also an eigenvalue of G with multiplicity $m(\lambda)$. Hence λ^2 is an eigenvalue of $\mathbf{A}(G)^2$ with multiplicity $2m(\lambda)$. Assume we have indexed the rows and columns of $\mathbf{A}(G)$ so that it has the block structure of Lemma 3.1.2. As $\mathbf{M}^T\mathbf{M}$ and $\mathbf{M}\mathbf{M}^T$ have the same non zero eigenvalues with equal multiplicity each has λ^2 with multiplicity $m(\lambda)$.

We **claim** that the $(d'/2+1)$-vector (d' is the largest even coordinate of $\mathbf{x}$)

$$\mathbf{x}^* = [\mathbf{x}_0, \mathbf{x}_2, \ldots, \mathbf{x}_{d'}]^T$$

is a right eigenvector of the intersection matrix $\mathbf{I}(D)$ of D corresponding to λ'.

Let the intersection numbers of D be in the standard notation, but with a $*$ superscript. We prove the claim by induction. First note that

$$c_2\big((\mathbf{I}(D) - \lambda'\mathbf{I})\mathbf{x}^*\big)_0 = c_2(-\lambda'\mathbf{x}_0 + c_1^*\mathbf{x}_2) =$$

$$= c_2\left(-\lambda'\mathbf{x}_0 + \mathbf{x}_2 c_2 c_1/c_2\right) = -\lambda^2 + r + c_2 c_1 \mathbf{x}_2 = 0$$

as $\mathbf{x}_1 = \lambda$ and $r - \lambda\mathbf{x}_1 + c_2\mathbf{x}_2 = 0$.

Now inductively using the general formula $b_j\mathbf{x}_j = \lambda\mathbf{x}_{j+1} - c_{j+2}\mathbf{x}_{j+2}$,

$$\begin{aligned} b_{2i}\mathbf{x}_{2i} &= \lambda\mathbf{x}_{2i+1} - c_{2i+2}\mathbf{x}_{2i+2} \\ \Rightarrow b_{2i+1}b_{2i}\mathbf{x}_{2i} &= \lambda b_{2i+1}\mathbf{x}_{2i+1} - b_{2i+1}c_{2i+2}\mathbf{x}_{2i+2} \\ &= \lambda^2\mathbf{x}_{2i+2} - b_{2i+1}c_{2i+2}\mathbf{x}_{2i+2} - \lambda c_{2i+3}\mathbf{x}_{2i+3}. \end{aligned}$$

Similarly

$$c_{2i+4}c_{2i+3}\mathbf{x}_{2i+4} = \lambda^2\mathbf{x}_{2i+2} - b_{2i+2}c_{2i+3}\mathbf{x}_{2i+2} - \lambda b_{2i+1}\mathbf{x}_{2i+1}.$$

Using the above equations we can evaluate the $c_2 \times (i+1)$-st coordinate of $\mathbf{I}(D)\mathbf{x}^*$:

$$\begin{aligned}
&c_2(b_i^*\mathbf{x}_{2i} + a_{i+1}^*\mathbf{x}_{2i+2} + c_{i+2}^*\mathbf{x}_{2i+4}) = \\
&\quad = c_2\left(\frac{b_{2i}b_{2i+1}}{c_2}\mathbf{x}_{2i} + \left(k - \frac{b_{2i+2}b_{2i+3}}{c_2} - \frac{c_{2i+2}c_{2i+1}}{c_2}\right)\mathbf{x}_{2i+2} + \frac{c_{2i+4}c_{2i+3}}{c_2}\mathbf{x}_{2i+4}\right) \\
&\quad = \left(2\lambda^2 - b_{2i+1}c_{2i+2} - b_{2i+2}c_{2i+3} + r(s-1) - b_{2i+3}b_{2i+2} - c_{2i+2}c_{2i+1}\right)\mathbf{x}_{2i+2} \\
&\qquad - \lambda(c_{2i+3}\mathbf{x}_{2i+3} + b_{2i+1}\mathbf{x}_{2i+1}) \\
&\quad = \left(\lambda^2 - b_{2i+1}c_{2i+2} - b_{2i+2}c_{2i+3} + r(s-1) - b_{2i+3}b_{2i+2} - c_{2i+2}c_{2i+1}\right)\mathbf{x}_{2i+2} \\
&\quad = \left(\lambda^2 + r(s-1) - c_{2i+2}(b_{2i+1} + c_{2i+1}) - b_{2i+2}(c_{2i+3} + b_{2i+3})\right)\mathbf{x}_{2i+2} \\
&\quad = (\lambda^2 + r(s-1) - rs)\mathbf{x}_{2i+2} \\
&\quad = (\lambda^2 - r)\mathbf{x}_{2i+2} \\
&\quad = c_2\lambda'\mathbf{x}_{2i+2}
\end{aligned}$$

Finally if $\mathbf{x}$ is a $t+1$-vector we consider two cases when t is even and odd. If t is even

$$\begin{aligned}
c_2 b^*_{(t-2)/2}\mathbf{x}_{t-2} &= c_2\left(\frac{b_{t-2}b_{t-1}}{c_2}\mathbf{x}_{t-2}\right) \\
&= b_{t-1}(\lambda\mathbf{x}_{t-1} - c_t\mathbf{x}_t) \\
&= b_{t-1}\left(\lambda^2\mathbf{x}_{t-2}/b_{t-1} - c_t\mathbf{x}_t\right) \\
&= \mathbf{x}_t(\lambda^2 - rb_{t-1}) \\
&= \mathbf{x}_t\left((\lambda^2 - r) - c_t(b_{t-1} - 1)\right) \\
&= c_2(\lambda' - a^*_{t/2})\mathbf{x}_t.
\end{aligned}$$

while if t is odd

$$\begin{aligned}
c_2 b^*_{(t-3)/2}\mathbf{x}_{t-3} &= c_2\left(\frac{b_{t-3}b_{t-2}}{c_2}\mathbf{x}_{t-3}\right) \\
&= b_{t-2}(\lambda\mathbf{x}_{t-2} - c_{t-1}\mathbf{x}_{t-1}) \\
&= \lambda^2\mathbf{x}_{t-1} - \lambda c_t\mathbf{x}_t - b_{t-2}c_{t-1}\mathbf{x}_{t-1} \\
&= \lambda^2\mathbf{x}_{t-1} - c_t b_{t-1}\mathbf{x}_{t-1} - b_{t-2}c_{t-1}\mathbf{x}_{t-1} \\
&= \mathbf{x}_{t-1}\left(\lambda^2 - r - (c_t b_{t-1} + b_{t-2}c_{t-1} - r)\right) \\
&= c_2(\lambda' - a^*_{(t-1)/2})\mathbf{x}_{t-1}.
\end{aligned}$$

Hence $\mathbf{x}^*$ is an eigenvector of the intersection matrix $\mathbf{I}(D)$ of the graph D. The feasibility conditions for a distance-regular graph [1] state that the multiplicity of λ' in D

is given by

$$m_D(\lambda') = \frac{n}{\sum_{i \text{ even}} \mathbf{x}_i^2 / k_i} \, .$$

But

$$m_D(\lambda') = m(\lambda) = \frac{2n}{\sum_{i=0}^{t} \mathbf{x}_i^2 / k_i},$$

by Theorem 3.2.6, giving

$$\sum_{i=0}^{t} \mathbf{x}_i^2 / k_i = 2 \sum_{i \text{ even}} \mathbf{x}_i^2 / k_i \ ,$$

as required. ∎

Theorem 3.2.6 makes it reasonable to define a pair of feasible arrays for a distance-biregular graph in an analoguous way to feasible arrays for distance-regular graphs [1].

Definition 3.2.9 : *Two intersection arrays are said to be a pair of feasible arrays for a distance-biregular graph if*

(i) they satisfy the numerical conditions of Proposition 3.2.1;

(ii) Each can be calculated from the other using the equations of Lemma 3.2.2;

(iii) The values determined as multiplicities using the procedure of Theorem 3.2.6 and Proposition 3.2.7 are positive integers.

(iv) The equation of Proposition 3.2.8 involving the eigenvectors of the intersection matrix is satisfied.

As has been done in the case of distance-regular graphs, these requirements could be used in a computer programme to generate feasible arrays of moderate size. Once such arrays had been found the task would remain of deciding which feasible arrays can be "realised", in the sense that a distance-biregular graph exists with the given array.

3.3 The Diameter of Distance-biregular Graphs

This subsection is concerned with the diameter of distance-biregular graphs. First we prove that regular distance-biregular graphs are distance-regular and that non-regular distance-biregular graphs have even diameter. We then proceed to generalise a result of Terwilliger [17] bounding the diameter d of a bipartite distance-regular graph as a function of its girth g and valency k: $d \leq \frac{1}{2}(k-1)(g-2)+1$. If we consider a non-regular distance-biregular graph with girth g then its derived graphs have irreducible cycles of length $g/2$ and greater, hence we can apply a result of Ivanov [13] to bound the diameter in terms of the valency of the derived graph and g. However this bound is very weak, being exponential in the valency. The result we prove in this subsection parallels the work of Terwilliger and gives a bound $d \leq (r-1)(s-1)(g-2)/2+1$, which is linear in the valency of the derived graph, though even this bound we suspect is too large by a factor of $\max\{r-1, s-1\}$. To finish off the subsection we provide an even sharper bound in the special case when the graph has girth 4.

Lemma 3.3.1: *A regular distance-biregular graph is distance-regular.*

Proof: Assume the standard notation for the regular distance-biregular graph G. We prove by induction that $\iota(A) = \iota(B)$. As G is regular of degree $r = s$, the first two columns in each array are identical. Suppose now that the arrays are identical up to and including the $(2i-1)$-st column, for some $i \geq 1$. Then by Lemma 3.2.2 $b_{2i}b_{2i-1} = e_{2i}e_{2i-1}$, and so $b_{2i} = e_{2i}$. As $r = s$ this gives $c_{2i} = f_{2i}$. But again by Lemma 3.2.2, $c_{2i+1}c_{2i} = f_{2i+1}f_{2i}$ and so $c_{2i+1} = f_{2i+1}$, yielding $b_{2i+1} = e_{2i+1}$ and agreement of the next two columns of the two intersection arrays. The result follows. ∎

Lemma 3.3.2: *The diameter d of a non-regular distance-biregular graph is even.*

Proof: Suppose without loss of generality that $G_d(u) \neq \emptyset$ for u in A. By ordering the rows and columns of the adjacency matrix $\mathbf{A}(G)$ of G as in Lemma 3.1.2, it is clear that $\mathrm{rank}(\mathbf{A}(G)) \leq 2\min\{n, m\}$ as at most this many rows may be linearly independent. As $n \neq m$, $\mathbf{A}(G)$ is not full rank and so 0 is an eigenvalue of G. By Theorem 3.2.6 we know that $\lambda(G) = \lambda(\mathbf{I}(A)) \cup \lambda(\mathbf{I}(B))$. But then 0 is an eigenvalue of $\mathbf{I}(A)$ or $\mathbf{I}(B)$. As

both $\mathbf{I}(A)$ and $\mathbf{I}(B)$ are tridiagonal with off-diagonals non-zero and diagonal zero, they have 0 as an eigenvalue if and only if they have odd order.
Suppose d is odd. Let v in $G_d(u) \neq \emptyset$. But then v is in B and u is in $G_d(v)$, so both $\mathbf{I}(A)$ and $\mathbf{I}(B)$ have even order, a contradiction. ∎

We now begin to prove the bound on the diameter of a distance-biregular graph G. Throughout the next two lemmas and the main theorem we will assume the standard notation for G. It will be useful to consider the following subset of the vertex set of G:

$$C = C(u,w) = \{x \in VG | \partial(u,x) + \partial(x,w) = t\} ,$$

where $u, w \in VG$ and $\partial(u,w) = t$. The first two lemmas provide the groundwork for the theorem.

Lemma 3.3.3: *If x is a vertex in $C = C(u,w)$ and $\partial(x,u) = i$, with $1 \leq i \leq t-1$ and $u \in A$, then the valency of x in the induced subgraph on C is $c_i + c_{t-i}$, if t is even and $c_i + f_{t-i}$, if t is odd. The valency of w in the subgraph is c_t, while that of u is c_t or f_t, according as t is even or odd.*

Proof: The result follows directly from considering the intersection numbers and the fact that the vertices of $C(u,w)$ adjacent to x are precisely those closer than x to either u or w. ∎

Lemma 3.3.4: *Any pair of vertices a, b in $C(u,w)$ satisfy*

$$\partial(a,b) \leq \partial(u,w) = t .$$

Proof: By taking two paths from u to w one through a and the other through b, we obtain a circuit containing a, b, u and w of length $2t$. Any two vertices on such a circuit are clearly at most distance t apart. ∎

We now present the main theorem, with the required result as a corollary.

Theorem 3.3.5: *Let G be a distance-biregular graph with the standard notation. Let t be a positive integer less than or equal to d, such that $c_t > 1$. If t is even, there exists an i $(1 \le i \le t-1)$ such that*

$$c_i + c_{t-i} \le \begin{cases} c_t; & \text{if } i \text{ is even,} \\ f_t; & \text{if } i \text{ is odd,} \end{cases}$$

while if t is odd, there exists an i $(1 \le i \le t-1)$ such that

$$c_i + f_{t-i} \le \begin{cases} f_t; & \text{if } i \text{ is even,} \\ c_t; & \text{if } i \text{ is odd.} \end{cases}$$

Proof: For the corollaries we will need only the t odd case, so we omit the proof for the even case, though it is very similar. Assume that for some odd t the theorem does not hold, that is for all $i < t$, i even implies $c_i + f_{t-i} > f_t$, while i odd implies $c_i + f_{t-i} > c_t$. We will show that in this case $c_t = 1$. Choose u, w with u in A and $\partial(u,w) = t$ and let $C = C(u,w)$ as above. We define the map:

$$f : C \times C \longrightarrow Z$$

$$\text{by} \quad f(a,b) = \partial(a,b) - |\partial(a,u) - \partial(b,u)| \; .$$

Note that $f(u,w) = \partial(u,w) - \partial(u,w) = 0$. By Lemma 3.3.4 $\partial(a,b) \le t$ so $f(a,b) \le t$. Also we have

$$\partial(a,u) \le \partial(a,b) + \partial(b,u)$$

so

$$|\partial(a,u) - \partial(b,u)| \le \partial(a,b)$$

giving $f(a,b) \ge 0$. Let

$$f_0 = \max\{f(a,b) | a,b \in C\} \; .$$

Claim: We claim that $f_0 = 0$ and so $f(a,b) = 0$ for all $a, b \in C$. To show this consider the set

$$S = \{(a,b) \in C \times C | f(a,b) = f_0\}$$

and choose $(a,b) \in S$ such that $|\partial(a,u) - \partial(b,u)|$ is maximum. We will show that $(a,b) = (u,w)$ or (w,u) and so $f_0 = f(u,w) = 0$. Suppose $a \neq u$ or w. Let $\partial(a,u) = j$ so that $1 \leq j \leq t-1$, and let $\partial(a,b) = \ell \leq t$. Recall that the valency of a in C is $c_j + f_{t-j}$ which satisfies

$$c_j + f_{t-j} > \begin{cases} f_t; & \text{j even,} \\ c_t; & \text{j odd.} \end{cases}$$

We consider three cases as a and b lie in the same or different colour classes.

Case(i) : Vertices a and b are in A (ℓ is even and j is even as a and u are both in A). Exactly c_ℓ vertices in G adjacent to a are closer to b than a. While all the other vertices adjacent to a are further away from b. Since $\ell \leq t$, and t is odd while ℓ is even, $f_t \geq c_\ell$, so that more than c_ℓ vertices are adjacent to a in C. So at least one vertex a' in C is adjacent to a but further from b than a. But then as $|\partial(a',u) - \partial(b,u)|$ differs from $|\partial(a,u) - \partial(b,u)|$ by at most one, so

$$\begin{aligned} f(a',b) &= \partial(a',b) - |\partial(a',u) - \partial(b,u)| \\ &\geq \partial(a,b) + 1 - |\partial(a,u) - \partial(b,u)| - 1 \\ &= f(a,b) = f_0 \end{aligned}$$

and so $f(a',b) = f_0$ and

$$|\partial(a',u) - \partial(b,u)| > |\partial(a,u) - \partial(b,u)|$$

contradicting our choice of a and b.

Case (ii) Vertices a and b are in different parts of the bipartition. Without loss of generality assume a in A and b in B (ℓ is odd and j is even). Exactly f_ℓ vertices in G are adjacent to a and closer to b than a, while the other neighbours of a are further away from b. Since $t \geq \ell$, t and ℓ both odd, then $f_t \geq f_\ell$. Hence more than f_ℓ vertices are adjacent to a in C, and so there is at least one vertex a' in C adjacent to a but further away from b. As in case (i) we get a contradiction with the choice of a and b.

Case (iii) Vertices a and b are in B (ℓ is even but j is odd as a and u are not both in the same part). Exactly f_ℓ vertices in G are adjacent to a and closer to b. Since $c_t \geq f_\ell$ and the degree of a in C is greater than c_t, there is at least one vertex a' in C adjacent to a but further away from b. Again we get a contradiction as in case (i).

We conclude that a is either u or w. Similarly $b = u$ or w. Hence (a,b) is one of the pairs (u,w), (u,u), (w,w) or (w,u). To maximise $|\partial(a,u) - \partial(b,u)|$ we must have (u,w) or (w,u). Hence $f_0 = f(u,w) = 0$ as claimed.

Suppose now that $c_t \neq 1$. In this case we can find distinct vertices x and y both in $G(w) \cap G_{t-1}(u)$. But

$$\begin{aligned} f(x,y) &= \partial(x,y) - |\partial(x,u) - \partial(y,u)| \\ &= \partial(x,y) = 2 > 0 \end{aligned}$$

contradicting $f_0 = 0$. We conclude that $c_t = 1$, completing the proof of the theorem in the odd case. ∎

Corollary 3.3.6: *Let G be distance-biregular graph with girth g and diameter d. Then if t is an odd integer less than or equal to d,*

$$c_t c_{t-1} \geq \frac{2t}{g-2},$$

and

$$d \leq \frac{1}{2}(r-1)(s-1)(g-2) + 1 .$$

Proof: We prove the first inequality by induction. Consider first $t < g/2$. Then $c_t c_{t-1} = 1$, while $2t/(g-2) \leq (g-2)/(g-2) = 1$.

Now assume true for i odd, $i < t$, for some odd $t \geq g/2$. Since $c_{g/2} > 1$ and $f_{g/2} > 1$ certainly $c_t > 1$. Hence we can apply the theorem to find some $i < t$ such that

$$c_i + f_{t-i} \leq \begin{cases} c_t; & \text{i odd,} \\ f_t; & \text{i even.} \end{cases}$$

If i is odd then

$$\begin{aligned} c_t c_{t-1} &\geq c_i c_{t-1} + f_{t-i} c_{t-1} \\ &\geq c_i c_{i-1} + f_{t-i+1} f_{t-i} \\ &\geq c_i c_{i-1} + c_{t-i+1} c_{t-i} \end{aligned}$$

by Lemma 3.2.2, provided $i \neq 1$. But then by the induction hypothesis

$$c_t c_{t-1} \geq \frac{2i}{g-2} + \frac{2(t-i+1)}{g-2} > \frac{2t}{g-2} .$$

If $i = 1$ then $c_t \geq 1 + f_{t-1}$ so that

$$\begin{aligned} c_t c_{t-1} &\geq c_{t-1} + c_{t-1} f_{t-1} \\ &\geq c_{t-1} + f_{t-2} f_{t-3} \\ &\geq 1 + 2(t-2)/(g-2) = (2t+g-6)/(g-2) \\ &\geq 2t/(g-2) \ , \end{aligned}$$

provided $g \geq 6$. If $g = 4$ then, as $t \geq 2$ and t odd, $t - 1 \geq 2$. But then $c_{t-1} \geq 2$ and so

$$c_t c_{t-1} \geq 2 + \frac{2(t-2)}{g-2} = \frac{2(t+g-4)}{g-2} \geq \frac{2t}{g-2} \ .$$

Now consider the case when i is even. Then using Lemma 3.2.2 we obtain

$$\begin{aligned} c_t c_{t-1} = f_t f_{t-1} &\geq c_i f_{t-1} + f_{t-i} f_{t-1} \\ &\geq c_{i+1} c_i + f_{t-i} f_{t-i-1} \ , \end{aligned}$$

provided $i \neq t - 1$. Hence by the induction hypothesis

$$c_t c_{t-1} \geq \frac{2(i+1)}{g-2} + \frac{2(t-i)}{g-2} > \frac{2t}{g-2} \ .$$

If $i = t - 1$ then we have

$$\begin{aligned} f_t f_{t-1} &\geq c_{t-1} f_{t-1} + f_{t-1} \\ &\geq c_{t-2} c_{t-3} + 1 \\ &\geq 2(t-2)/(g-2) + 1 \\ &\geq (2t+g-6)/(g-2) \\ &\geq 2t/(g-2) \ , \end{aligned}$$

provided $g \geq 6$. If $g = 4$ then $t \geq 3$ and so $f_{t-1} > 1$ giving

$$f_t f_{t-1} \geq 2 + \frac{2(t-2)}{g-2} \geq \frac{2t}{g-2} \ .$$

The first result follows by induction.

Consider now a distance-biregular graph with girth g and diameter d. We know by Lemma 3.3.2 that if G is not regular then d is even. Applying the first part of the corollary with $t = d - 1$ we obtain

$$c_{d-1}c_{d-2} \geq \frac{2(d-1)}{g-2} .$$

Hence as $c_{d-1}c_{d-2} \leq (r-1)(s-1)$ we have

$$d \leq \frac{1}{2}(r-1)(s-1)(g-2) + 1 .$$

If however G is regular, then by Lemma 3.3.1 G is a bipartite distance-regular graph and Terwilliger's [17] bound on the diameter holds:

$$d \leq \frac{1}{2}(r-1)(g-2) + 1 .$$

Our inequality certainly follows from this. ∎

Corollary 3.3.7: *The only infinite locally-finite distance-biregular graphs are biregular trees.*

Proof: By Corollary 3.3.6, if a distance-biregular graph has finite girth it also has finite diameter. ∎

To show that the above results are by no means optimal, we will prove a much stronger result in the special case when $g = 4$.

Proposition 3.3.8: *Let G be a distance-biregular graph with diameter d, girth 4 and $r > s$ in the standard notation. Then $c_3 > f_2 + 1$, $f_3 > c_2 + 2$, $f_t \geq f_{t-2} + 2$ and $c_t \geq c_{t-2} + 2$, for t odd, with $t \leq d$.*

Proof: Let $u \in A$, $v \in B$ with $u \sim v$ and the other neighbours of v being $u_1, \ldots, u_{s-1}$. Choose $v' \in G(u_1) \cap G_3(u)$.

Assume $c_3 = f_2$. In this case the c_3 neighbours of v' in $G_2(u)$ are among $u_1, \ldots, u_{s-1}$ as they coincide with the f_2 common neighbours of v and v'. Consider some other vertex $v'' \in G(u) \cap G_2(v')$ (other than v – there are $f_3 - 1$ of them). The f_2 common neighbours of v' and v'' must be the common neighbours of v and v'. Hence v and v'' have f_2+1 common neighbours (count u as well), a contradiction. Hence $c_3 > f_2$.

Now suppose $c_3 = f_2 + 1$. Let $u_1, u_2, \ldots, u_{f_2}, u'$ be the neighbours of v' in $G_2(u)$ with $u' \not\sim v$. Then $\partial(v, u') = 3$ and so c_3 vertices are a distance 2 from u' and 1 from v. One of them is u. The other $c_3 - 1 = f_2$ of them must be precisely the common neighbours of v and v', $u_1, \ldots, u_{f_2}$.

Now consider $x \in G(u) \cap G(u')$. Vertex x must be adjacent to exactly $f_2 - 1$ of the vertices $u_1, \ldots, u_{f_2}$, in order to have f_2 common neighbours with v'. Each such vertex x must be adjacent to a different selection of $f_2 - 1$ vertices as otherwise two vertices x and x' adjacent to the same set would have $f_2 + 1$ common neighbours. Hence $|G(u) \cap G(u')| \leq f_2$, that is $c_2 \leq f_2$. But as $r > s$ and $(r-1)c_2 = (s-1)b_2$,

$$(r-1)(s-f_2) = (s-1)(r-c_2) ,$$

so

$$(r-1)(f_2-1) = (s-1)(c_2-1) ,$$

giving $f_2 < c_2$, a contradiction. We conclude that $c_3 > f_2+1$. But then by Lemma 3.2.2 $c_3 c_2 = f_3 f_2$, so

$$f_3 \geq \frac{(f_2+2)c_2}{f_2} = c_2 + 2\frac{c_2}{f_2} \geq c_2 + 3 ,$$

as required.

To prove the second two inequalities consider first a vertex $u \in A$ and $v \in G_{2i+1}(u)$, for some $i \leq \lfloor (d-1)/2 \rfloor$. Choose a vertex w in $G(v) \cap G_{2i}(u)$. We now count paths of length two between w and $X = (G(v) \cap G_{2i}(u)) \setminus \{w\}$. Each vertex v' adjacent to w in $G_{2i-1}(u)$ has f_2 neighbours in $G(v) \cap G_{2i}(u)$ (the common neighbours with v). Hence we obtain $c_{2i}(f_2-1)$ paths between w and X. There are a further $c_{2i+1} - 1$ paths through v. If however we consider a vertex $x \in X$, it is at distance 2 from w and so has c_2 common neighbours with w. As $|X| = c_{2i+1} - 1$ we must have

$$c_2(c_{2i+1} - 1) \geq c_{2i+1} - 1 + c_{2i}(f_2 - 1),$$

giving

$$c_2 - 1 \geq \frac{c_{2i}(f_2-1)}{c_{2i+1} - 1}.$$

Next consider $v \in G_{2i}(u)$ and $w \in G(v) \cap G_{2i-1}(u)$. Applying a similar argument to that above we get

$$f_2 - 1 \geq \frac{c_{2i-1}(c_2 - 1)}{c_{2i} - 1}.$$

Combining the two inequalities gives

$$f_2 - 1 \geq \frac{c_{2i-1}c_{2i}}{(c_{2i} - 1)(c_{2i+1} - 1)}(f_2 - 1),$$

$$\text{or} \quad c_{2i+1} \geq c_{2i-1}\frac{c_{2i}}{c_{2i} - 1} + 1$$
$$\geq c_{2i-1} + 2.$$

By the symmetry of the definition of a distance-biregular graph the second inequality $f_{2i+1} \geq f_{2i-1} + 2$ follows. ∎

Corollary 3.3.9 : *Let G be a distance-biregular graph with girth 4, diameter $d \geq 4$ and valencies r, s with $r > s$. Then*

$$d \leq s - f_2 + 2 \leq r - c_2 + 1.$$

Proof : By the proposition we have $c_3 \geq f_2 + 2$ and $c_{2i+1} \geq c_{2i-1} + 2$. Hence inductively $c_{2i+1} \geq 2i + f_2$ or $c_t \geq t - 1 + f_2$, for t odd. But if t is odd, $c_t \leq s$, giving $t - 1 + f_2 \leq s$ and so $d \leq s - f_2 + 2$ as we can choose $t \geq d - 1$.

Note that applying the same technique to the array $\iota(B)$ using the inequality $f_3 \geq c_2 + 3$ yields the bound $d \leq r - c_2 + 1$ for the diameter. We will now prove the second inequality of the corollary, which shows that this bound is in fact weaker. Since $r > s$ and $(r - c_2)(s - 1) = b_2 b_1 = e_2 e_1 = (s - f_2)(r - 1)$, we have $s - f_2 < r - c_2$, giving $s - f_2 + 2 \leq r - c_2 + 1$. ∎

4 Distance-biregular Graphs with 2-valent Vertices

In this section we study a special class of distance-biregular graphs, namely those with 2-valent vertices, though in the last subsection we extend this investigation to look at distance-biregular graphs whose derived graphs have minimum eigenvalue -2. The first subsection looks at distance-biregular graphs with 2-valent vertices and derives a characterisation in terms of subdivision graphs. This leads us in the next subsection to investigate distance-regular line graphs, which are shown to be the line graphs of cages (see Definition 1.1.4). In the penultimate subsection we use this result to classify distance-biregular graphs with 2-valent vertices. The final subsection extends the argument to look at distance-biregular graphs whose derived graphs have minimum eigenvalue -2.

4.1 Characterisation in terms of Subdivision Graphs

We begin with a theorem giving a fairly straightforward characterisation of distance-biregular graphs with 2-valent vertices. Later on, however, we will obtain a much more complete classification.

Theorem 4.1.1 : *Let H be the derived graph on the vertex set A of the bipartition of a distance-biregular graph G in the standard notation. Suppose H has standard intersection array with a *-superscript and that* $\mathrm{diam}(H)$ *is greater than 1. Then the following conditions are equivalent:*

(i) G is the subdivision graph of H,

(ii) $s = 2$,

(iii) $a_1^* = 0$.

Proof: (iii)$\Rightarrow$(ii). Let $v \in B$ and $s = \deg(v) > 2$. Pick u_1, u_2, u_3 distinct neighbours of v. The u_2 and u_3 belong to $G_2(u_1) = H_1(u_1)$. Since u_2 and u_3 are adjacent in H, we have $a_1^* > 0$, a contradiction.

(ii)$\Rightarrow$(i) We show that for any two adjacent vertices, u and u', of H there is precisely one vertex v which is adjacent to both of them in G (thus G is the subdivision graph of H). Suppose there is a second vertex $v' \in G(u) \cap G(u')$. Then $G(v) \cap G(v') = \{u, u'\}$ and so $f_2 = 2$. But then $e_2 = 0$, and $\mathrm{diam}(G) \leq 3$. This contradicts the assumption that $\mathrm{diam}(H) > 1$.

(i)$\Rightarrow$(iii) Suppose $a_1^* > 0$. Let $u \in A$, and let v_1 be a vertex in G which subdivides an edge of H joining two vertices in $H_1(u)$. Since $\mathrm{diam}(H) > 1$, there is a vertex v_2 in G which subdivides an edge of H joining a vertex in $H_1(u)$ to a vertex in $H_2(u)$. But v_1 and v_2 are in $G_3(u)$, while $|G_2(u) \cap G(v_1)| = 2$ and $|G_2(u) \cap G(v_2)| = 1$, contradicting that u is distance-regularised. ∎

Corollary 4.1.2 : *Let G and H be graphs as in the theorem. If* $\mathrm{diam}(H) > 1$ *and $s = 2$, then the derived graphs of G are H and its line graph $L(H)$.*

Proof: By the theorem G is the subdivision graph of H. The vertex sets of the derived graphs are then the vertices of H and the edges of H, with adjacency between adjacent vertices and incident edges. ■

To finish this subsection we cover for completeness the case when $\mathrm{diam}(H) = 1$.

Theorem 4.1.3: *Let G and H be as in Theorem 4.1.1. Suppose that $s = 2$ and $\mathrm{diam}(H) = 1$. Then either*
(a) $H = K_n$, $n \geq 3$, and G is the subdivision graph of H, or
(b) $H = K_2$, and $G = K_{2,r}$, for some $r \geq 1$.

Proof: If $H = K_2$ then clearly $G = K_{2,r}$. If $H = K_n$, $n \geq 3$, then for a vertex $v \in B = V(G) \setminus V(H)$ only two of the vertices of H are adjacent to v. Hence at least one vertex is at distance 3 from v giving $G_3(v) \neq \emptyset$. Applying the argument of (ii)⇒(i) from Theorem 4.1.1 shows that G is the subdivision graph of H. ■

Some examples of subdivision graphs which are distance-biregular are easily found, for example $S(K_n)$ and $S(K_{n,n})$. By Corollary 4.1.2, if $S(H)$ is to be distance-biregular its derived graphs and so the line graph of H must be distance-regular. Hence our next problem is to determine which graphs have distance-regular line graphs.

4.2 Distance-regular Line Graphs

In [2] Biggs proved that, if the line graph $L(G)$ of G is distance-transitive, then G is either $K_{1,n}$ or a cage. We now generalise this result to the case when $L(G)$ is distance-regular[20].

Theorem 4.2.1: *If the line graph $L(G)$ of a graph G is distance-regular, then either $G = K_{1,n}$ or G is a cage.*

Before giving the proof of Theorem 4.2.1 we need a simple lemma.

Lemma 4.2.2 : *If $L(G)$ is distance-regular, then either G is regular of degree greater than one, or it is a star $K_{1,n}$, $n \geq 1$.*

Proof: Since $L(G)$ is regular, G is either regular or biregular. Suppose that G is not regular. If it contains a vertex of degree 1, it must be $K_{1,n}$, for some n greater than or equal to 1. Assume now that G has no monovalent vertices. Let e be an edge of G joining vertices a and b where $2 \leq \deg(a) < \deg(b)$. We consider two cases.

(i) There are adjacent edges e' and e'' such that $e' \neq e$, $e'' \neq e$, e' is incident with a and e'' is incident with b (e, e' and e'' thus form a triangle).

(ii) There is an edge e' incident with a which is not adjacent to an edge e'' incident with b.

In both cases the numbers $|L(G)(e) \cap L(G)(e')|$ and $|L(G)(e) \cap L(G)(e'')|$ are different, contradicting the distance-regularity of $L(G)$. ∎

Proof of Theorem 4.2.1 : If G contains vertices of degree 1, then G is a star $K_{1,n}$ as guaranteed by Lemma 4.2.2. Otherwise G is regular of degree greater than 1. The case when G contains no cycles is also trivial. It must be an infinite regular tree which is a (k, ∞)-graph.

The rest of the proof will assume that G is k-regular, k greater than 1, and that the girth g of G is finite. The intersection array of $L(G)$ will be assumed in the standard notation. For each edge e of G, we denote by $L_i(e)$, $i = 0, 1, \ldots, d$, the set $L(G)_i(e)$. We must show that $\text{diam}(G) = \lfloor g/2 \rfloor$ and if g is even that G is bipartite. Denote $\lfloor g/2 \rfloor$ by ℓ. Let u be an arbitrary vertex of G. We show that no vertex of G is more than distance ℓ from u. Let $e \in EG$ be an edge which is incident with u. Since $L(G)$ is distance-regular, it is easy to see that e must lie on a cycle C in G which is of length g. Consider the case when g is odd. Choosing an edge e' which lies on C and is at distance ℓ from e, we see that $a_\ell \geq k - 2 + 1 = k - 1$, since e' is adjacent to an edge e'' on C which is also distance ℓ from e. Suppose that there is a vertex w in G which is at distance $\ell + 1$ from u. Let a sequence of edges $e_0, e_1, \ldots, e_\ell$ be a path of length $\ell + 1$ from u to w, and let v' be the common vertex of $e_{\ell-1}$ and e_ℓ and v be the common vertex of e_0 and e_1. The edge e_ℓ is in $L_\ell(e_0)$. Therefore e_ℓ has a_ℓ adjacent edges which also belong to $L_\ell(e_0)$. Since $a_\ell > k - 2$, there is at least one edge e' in $L_\ell(e_0) \cap L_1(e_\ell)$

which is not incident with v' but is adjacent to w. Let v'' be the other vertex of e'. To be at distance ℓ from e_0 there are two possibilities. Either there is a path of length $\ell - 1$ from u to v'' or there is a path of length $\ell - 1$ from v to v''. The former case is impossible since w is in $G_{\ell+1}(u)$. But in the latter case we obtain at v a cycle of length $\ell - 1 + 1 + \ell = 2\ell = g - 1$, which is also not possible. Hence $d = \ell$ as required. This completes the odd case.

Consider now g even. Let e' be the edge on C which is at distance ℓ from e (e' is opposite e). If $e' = (v_1, v_2)$, then no edge incident with either v_1 or v_2 other than the edges of C can be at distance less than ℓ from e, as we would otherwise have cycles of length less than g in G. Therefore all these edges lie in $L_\ell(e)$, giving $c_\ell = 2$, $a_\ell = 2k - 4$ and $b_\ell = 0$. Suppose there is a vertex $w \in G_{\ell+1}(u)$. Let $(u, v) = e_0, e_1, \ldots, e_\ell$ be a path of length $\ell + 1$ from u to w. As $e_\ell \in L_\ell(e_0)$ and $c_\ell = 2$, there is an edge e' adjacent to w at distance $\ell - 1$ from e_0. But the other end v' of e' is at distance $\ell - 1$ from either u or v. The former case contradicts $w \in G_{\ell+1}(u)$ while the latter gives a cycle of length

$$\ell - 1 + \ell - 1 + 1 = 2\ell - 1 < g.$$

The contradiction shows that $\mathrm{diam}(G) = \ell = g/2$.

Finally we most show that in the even case G is bipartite. It will be sufficient to show that for any vertex u, no two vertices in $G_d(u)$ are adjacent. Suppose $e = (v, w)$ is an edge with $v, w \in G_d(u)$. Let $(u, u_0) = e_0, e_1, \ldots, e_{d-1}$ be a path from u to v. Then $e \in L_d(e_0)$ and so as $c_d = 2$, there are precisely two edges closer to e_0 adjacent to e. One is e_{d-1}, let the other be e'_{d-1}, with $e_0 = e'_0, e'_1, \ldots, e'_{d-1}$ a path from e_0 to e'_{d-1}. As a vertex of e'_{d-1} is at distance d from u the edge e'_1 must be incident with the vertex u_0 of e_0 and not u. Hence if e'_{d-1} is incident with v we have a cycle in G of length $2g - 2$, while if it is incident with w we have a cycle of length $2g - 1$, in either case a contradiction. We conclude that when g is even G is indeed bipartite and so also a cage. ∎

4.3 Subdivision Graphs of Cages

So the graphs which have the required property that their line graphs are distance-regular are cages. The next lemma shows that they do actually behave as we would like.

Lemma 4.3.1: *If G is a (k,g)-graph, then the subdivision graph $S(G)$ of G is distance-biregular.*

Proof: Two cases must be considered.

(a) The girth g is odd. The intersection array for a vertex $u \in VG$ in $S(G)$ is

$$\begin{bmatrix} * & 1 & 1 & 1 & 1 & \dots & 1 & 1 & 2 \\ k & 1 & k-1 & 1 & k-1 & \dots & 1 & k-1 & * \end{bmatrix},$$

while the array for an edge of G in $S(G)$ is

$$\begin{bmatrix} * & 1 & 1 & 1 & \dots & 1 & 1 & 2 & 2 \\ 2 & k-1 & 1 & k-1 & \dots & k-1 & 1 & k-2 & * \end{bmatrix}.$$

The first array is immediate from the diameter and girth of the cage, while the second follows from the fact that a vertex v opposite an edge e on a cycle of length g in G has precisely two edges incident with it and closer to e. Note that the diameter d of $S(G)$ is $g+1$, though $S(G)_{g+1}(u) = \emptyset$ for $u \in VG$.

(b) The girth g is even. For $u \in VG$ we have intersection array in $S(G)$

$$\begin{bmatrix} * & 1 & 1 & 1 & 1 & \dots & 1 & 1 & k \\ k & 1 & k-1 & 1 & k-1 & \dots & k-1 & 1 & * \end{bmatrix},$$

while for an edge the array is

$$\begin{bmatrix} * & 1 & 1 & 1 & \dots & 1 & 1 & 2 \\ 2 & k-1 & 1 & k-1 & \dots & 1 & k-1 & * \end{bmatrix}.$$

In this case both arrays follow immediately from the girth and diameter of the cage and the fact that no edges are at distance $g+1$ in $S(G)$ from a given vertex u in G, as G being bipartite no edge can have both its ends at distance d from u in G. ∎

We can now present the more complete classification of distance-biregular graphs with 2-valent vertices as the corollary to the following theorem.

Theorem 4.3.2 : *For a graph G the following conditions are equivalent:*

(i) $L(G)$ is distance-regular and G is not a star graph,

(ii) G and $L(G)$ are both distance-regular,

(iii) $S(G)$ is a distance-biregular graph, and

(iv) G is a (k,g)-graph.

Proof: (i)⇒(iv) This is Theorem 4.2.1.

(iv)⇒(iii) This is Lemma 4.3.1.

(iii)⇒(ii) The derived graphs of $S(G)$ are G and $L(G)$. These are both distance-regular by Proposition 3.1.1.

(ii)⇒(i) Star graphs are not distance-regular. ∎

Corollary 4.3.3 : *A graph G with 2-valent vertices is distance-biregular if and only if either $G = K_{2,r}$ or G is the subdivision graph of a (k,g)-graph.*

Proof: By Theorem 4.1.1 and Theorem 4.1.3 a distance-biregular graph with 2-valent vertices is either $K_{2,r}$ or the subdivision graph of some graph H. By Theorem 4.3.2, H is a (k,g)-graph in the latter case. The converse follows as $K_{2,r}$ is clearly distance-biregular (see Example 1.2.5), while the subdivision graph of a (k,g)-graph is distance-biregular by Lemma 4.3.1. ∎

Theorem 4.3.2 has another interesting corollary, which excludes many distance-regular graphs from being the derived graphs of distance-biregular graphs.

Corollary 4.3.4 : *Let H be a distance-regular graph without triangles. Then H is the derived graph of a distance-biregular graph if and only if H is a (k,g)-graph with $g \geq 4$ or $H = K_2$.*

Proof: If H is a (k,g)-graph or K_2, it is the derived graph of $S(H)$. Conversely if H is the derived graph of a distance-biregular graph G and H has no triangles then in the notation of Theorem 4.1.1, as $a_1^* = 0$, either $\mathrm{diam}(H) = 1$ or G is the subdivision graph of H. In the latter case by the theorem, H is a (k,g)-graph and as H has no trangles, $g \geq 4$. If, however, $\mathrm{diam}(H) = 1$, H is a complete graph without triangles and hence K_2. ∎

4.4 Derived Graphs with Minimum Eigenvalue -2

It is well known that line graphs have minimal eigenvalue -2, which in our case we could also deduce from Lemma 3.1.2 and the fact that $r/c_2 = 2$. Graphs with minimal eigenvalue -2 have been classified as generalised line graphs or E_8 graphs [4]. It is therefore natural to ask if we can generalise the above result for distance-biregular graphs with 2-valent vertices to distance-biregular graphs with $r/c_2 = 2$, as this means the derived graph has minimum eigenvalue greater than or equal to -2. We begin by proving the following lemma.

Lemma 4.4.1: *Let G be a distance-biregular graph with girth 4 and $r/c_2 = 2$ in the standard notation. Then the derived graph on vertex set A is not a line graph.*

Proof: The intersection array $\iota(A)$ has initial entries

$$\begin{bmatrix} * & 1 & r/2 & \dots \\ r & s-1 & r/2 & \dots \end{bmatrix}.$$

Calculating e_2 of the array $\iota(B)$ we obtain

$$e_2 = \frac{r(s-1)}{2(r-1)}.$$

As $r > 2$ we must have $r-1 | s-1$. Let $s-1 = a(r-1)$ with $a \in \mathcal{N}$. Then $e_2 = ar/2$ and

$$f_2 = \frac{2s - ar}{2} = \frac{s+1-a}{2}.$$

Note that as $r \geq 3$ we have $a \leq (s-1)/2$. But then

$$c_3 \geq f_2 = \frac{s+1-a}{2} \geq \frac{s+3}{4}$$

and

$$b_3 \le s - \frac{s+3}{4} = \frac{3s-3}{4}.$$

$$\text{So} \quad b_1^* = \frac{b_2 b_3}{c_2} \le \frac{3(s-1)}{4}$$

and

$$\begin{aligned} a_1^* &= 2(s-1) - 1 - b_1^* \\ &\ge 2(s-1) - 3(s-1)/4 - 1 \\ &= 5(s-1)/4 - 1 \\ &> 2(s-1)/2 - 1 \\ &= k^*/2 - 1. \end{aligned}$$

Hence the derived graph is not a line graph as for line graphs $a_1^* = k^*/2 - 1$. ∎

The lemma tells us that all distance-biregular graphs with $r/c_2 = 2$ and derived graph D a line graph have 2-valent vertices ($r = 2$) and so are classified by Corollary 4.3.3. Next consider a distance-biregular graph with derived graph a generalised line graph. The only distance-regular generalised line graph which is not a line graph is the Cocktail Party graph, $K_{2k} \setminus kK_2$. The next theorem investigates this case.

Theorem 4.4.2: *Let G be a distance-biregular graph with derived graph the Cocktail Party graph, $K_{2r} \setminus rK_2$. Then either G is regular and a double cover of $K_{r,r}$ or G is a graph with intersection arrays*

$$\iota(B) = \begin{bmatrix} * & 1 & r-1 & r \\ 2r-1 & r-1 & r & * \end{bmatrix} \; \iota(A) = \begin{bmatrix} * & 1 & r/2 & 2(r-1) & r \\ r & 2(r-1) & r/2 & 1 & * \end{bmatrix} \; (*)$$

Proof: In the standard notation let the derived graph D on vertex set A of G be the Cocktail Party graph. The Cocktail Party graph has intersection array

$$\begin{bmatrix} * & 1 & 2(r-2) \\ 0 & 2(r-3) & 0 \\ 2(r-2) & 1 & * \end{bmatrix}.$$

But then $b_2 b_3 = c_2$ and so $b_3 = 1$ as $c_2 = b_2 = r/2$. Also $c_4 = r$ as

$$\frac{r(s-1)}{r/2} = k^* = 2(r-2) = \frac{c_4 c_3}{c_2}$$

and $c_3 = s - b_3 = s - 1$. Hence the array $\iota(A)$ is

$$\iota(A) = \begin{bmatrix} * & 1 & r/2 & s-1 & r \\ r & s-1 & r/2 & 1 & * \end{bmatrix}$$

with

$$\iota(B) = \begin{bmatrix} * & 1 & (s+1-a)/2 & f_3 & \dots \\ s & r-1 & (s-1+a)/2 & e_3 & \dots \end{bmatrix}$$

where $a \in \mathcal{N}$ is as in the proof of Lemma 4.4.1. The equation $f_3(s+1-a)/2 = r(s-1)/2$ and the inequality $f_3 \leq r$ together imply $s - 1 \leq s + 1 - a$ or $a \leq 2$.
If $a = 2$ then $s = 2r - 1$ and

$$\iota(A) = \begin{bmatrix} * & 1 & r/2 & 2(r-1) & r \\ r & 2(r-1) & r/2 & 1 & * \end{bmatrix},$$

while

$$\iota(B) = \begin{bmatrix} * & 1 & r-1 & r \\ 2r-1 & r-1 & r & * \end{bmatrix}.$$

If $a = 1$ we have $r = s$ and so a regular graph. Hence G is distance-regular by Lemma 3.3.1. Its intersection array is

$$\iota(G) = \begin{bmatrix} * & 1 & r/2 & r-1 & r \\ r & r-1 & r/2 & 1 & * \end{bmatrix}.$$

This is a double antipodal cover of the graph F with intersection array

$$\iota(F) = \begin{bmatrix} * & 1 & r \\ r & r-1 & * \end{bmatrix}.$$

Thus F is $K_{r,r}$ the regular complete bipartite graph. ∎

The following result is surprising as it shows the existence of the two kinds of distance-biregular graphs in Theorem 4.4.2 are both related to the existence of Hadamard matrices (see Definition 1.1.10).

Theorem 4.4.3 : *If $r \neq 1$ then the following are equivalent*

(i) the existence of a distance-biregular graph with intersection arrays $(*)$,

(ii) the existence of a double cover of $K_{2r,2r}$, and

(iii) the existence of a Hadamard matrix of dimension $2r$.

Proof: (i)$\Rightarrow$(ii) We start with a distance-biregular graph G in the standard notation with intersection arrays $(*)$. We will construct a new graph G' from G and then show that G' is a double cover of $K_{2r,2r}$. The vertices of G', $VG' = A' \cup B'$, will be the vertices of G except that each vertex v in B is duplicated to two vertices v and v' in B'. In addition 2 vertices x and y are added to A to give A'. The vertex x is adjacent to each vertex of B, while y is adjacent to each duplicate of a vertex in B. For v in B, the duplicate v' is adjacent to precisely the vertices in A' that v is not adjacent to. This completes the construction of G'.

We must now show that G' is distance-regular with intersection array

$$\begin{bmatrix} * & 1 & r & 2r-1 & 2r \\ 2r & 2r-1 & r & 1 & * \end{bmatrix}.$$

Each vertex v in B is adjacent to $2r-1$ vertices in A and to x, so that $\deg(v) = 2r$. As $|A'| = 4r - 2 + 2 = 4r$, the duplicate v' of v has $\deg(v') = 4r - 2r = 2r$. A vertex $u \in A$ had r neighbours in B and r non-neighbours as $|B| = 2r$. Hence in G', u is adjacent to r original vertices and r duplicate vertices, so $\deg(u) = 2r$. Clearly $\deg(x) = \deg(y) = 2r$. Hence G' is $2r$-regular. Consider now $v_1, v_2 \in B$. Clearly v_1 and v_2 have r common neighbours ($r-1$ in A and x). For v_1', v_2' duplicates of vertices v_1, v_2 in B, v_1' and v_2' have $r-1$ common non-neighbours in A (the common neighbours of v_1 and v_2) and each has $2r-1$ neighbours in A, so they have $4r-2+r-1-(4r-2) = r-1$ common neighbours in A, giving r common neighbours in all. Consider vertices v_1 in B and v_2' the duplicate of a vertex v_2 in B with $v_1 \neq v_2$. Vertices v_1 and v_2 have $r-1$ common neighbours in A, and so there are $2r - 1 - (r-1) = r$ neighbours of v_1 in A that v_2 is not adjacent to - these are the r common neighbours of v_1 and v_2'. Now consider two vertices u and w in A such that $\partial_G(u,w) = 2$. Then they have $r/2$ common neighbours in B, and $2r - 2r + r/2 = r/2$ common non-neighbours. But the duplicates of these non-neighbours will be new common neighbours. Hence u and w have r common neighbours. We must look next at common neighbours of vertices in A

and x or y. As x is adjacent to all of B, then any vertex in A has r common neighbours with x and as it has r non-neighbours in B it has r common neighbours with y. We have shown so far that columns 0, 1 and 2 of the intersection array exist and have the right entries. It will be sufficient to complete the proof if we show that each vertex determines a unique vertex at distance 4 from it, as this will force $b_3 = 1$ and the fact that the graph is bipartite proves the existence of the intersection array. For a vertex v in B, it is clear that the unique vertex distance 4 from v in G' is the duplicate v' of v. Vice versa for a duplicate vertex. For a vertex u in A, there was a unique vertex w in G at distance 4 from u. Suppose that w and u have a common neighbour in G'. It must be a duplicate vertex v', for some $v \in B$. But then v was adjacent to neither u or w, an impossibility if we consider the intersection array $\iota(A)$ and the fact that $G(u) = G_3(w)$ and vice versa. Hence w is still distance 4 from u in G'. As no edges have been deleted, distance can only have reduced from G to G', so no other vertices from A are distance 4 from u in G'. Finally x certainly has a common neighbour with u as does y. For x the unique vertex at distance 4 is y and vice versa.

(ii)⇒(iii) We start with a double cover of $K_{2r,2r}$, that is a distance-regular graph G with intersection array

$$\iota(G) = \begin{bmatrix} * & 1 & r & 2r-1 & 2r \\ 2r & 2r-1 & r & 1 & * \end{bmatrix}.$$

Note first that each vertex determines a unique vertex at distance 4 from it. We label each pair with a 1 and -1. These antipodal pairs fall into two classes determined by the bipartition each with $2r$ pairs in it. Let the pairs in one class be $p_1, \ldots, p_{2r}$ and those in the second class $q_1, \ldots, q_{2r}$. Note that if we choose a pair p_i and a pair q_j, each vertex in p_i is adjacent to exactly one vertex in q_j and vice versa. We will construct a matrix $\mathbf{H}$ with rows indexed by the pairs of the first class and columns indexed by the pairs of the second class. The i, j entry of $\mathbf{H}$ will be 1 if vertex 1 of p_i is adjacent to vertex 1 of q_j and -1 otherwise. It remains to show that $\mathbf{H}$ is a Hadamard matrix. Clearly the inner product of a column with itself is $2r$. What we must prove is that the inner product of different columns is 0. Consider columns j and j'. These correspond to pairs q_j and $q_{j'}$. The entries in row i of these two columns will agree if the pair p_i is connected the same way to q_j and $q_{j'}$. Each such connection will give vertices 1 of q_j

and $q_{j'}$ a common neighbour, while if the columns disagree they will have no common neighbours in p_i. Hence the number of rows in which the entries agree is r and the inner product of the two columns is $r - r = 0$.

(iii)$\Rightarrow$(i) In this proof we start with a Hadamard matrix $\mathbf{H}$ of order $2r$ and must construct a distance-biregular graph with intersection arrays $(*)$. First we adapt $\mathbf{H}$ by multiplying various rows by -1. This will not affect $\mathbf{H}^T\mathbf{H}$ and so leave $\mathbf{H}$ a Hadamard matrix. In this way we can take $\mathbf{H}$ to have its first column the all 1 vector. This in turn will mean that all subsequent columns will have half their entries 1 and half their entries -1. Delete from $\mathbf{H}$ the first column and call the resulting $2r \times (2r-1)$ matrix $\mathbf{H}'$. We now construct the graph G by taking the set A to be a pair of vertices u_1 and u_{-1} for each column u of $\mathbf{H}'$ and B to have a vertex for each row v of $\mathbf{H}'$. Vertex u_j in A $(j \in \{1, -1\})$ is adjacent to v in B if $\mathbf{H}'_{vu} = j$. We must now prove that G is distance-biregular with intersection arrays $(*)$. Each vertex u_j in A appears in r rows while each row has $2r-1$ entries so G is biregular with degrees r and $2r-1$. Consider first two vertices v and v' in B. These two rows had r agreements in $\mathbf{H}$ and so $r-1$ agreements in $\mathbf{H}'$ (they certainly agreed in the all ones column). Hence v and v' have $r-1$ common neighbours. This is sufficient to show that the vertices of B are distance-regularised with array $\iota(B)$ of $(*)$. We now turn our attention to vertices in A. First consider u_j and $w_{j'}$ with $u \neq w$. The two columns u and w each have an equal number of 1's and -1's but also agree in the same number of rows as they disagree. Hence exactly $r/2$ rows have a j in row u and j' in row w, as required. We complete the proof by determining the uniqueness of the vertex at distance 4 from a given vertex u_j in A. Clearly the only such vertex is u_{-j}. This shows that $b_3 = 1$ and so determines that the vertices of A are also distance-regularised with the array $\iota(A)$ of $(*)$. This completes the proof. ∎

The only other possible derived graphs with minimum eigenvalue -2 are E_8 graphs. It is known that regular E_8 graphs have at most 28 vertices, so an exhaustive search and check would certainly be feasible. We have not performed this task as we felt it was of little general interest. We finish this section by noting that one such E_8 graph is the Petersen graph P. It is also a $(3,5)$-graph and so is the derived graph of $S(P)$ the subdivision graph of P. In this case both derived graphs have minimum eigenvalue

-2. The graph $S(P)$ is probably the unique non-regular distance-biregular graph with this property.

5 CSR Graphs

This section looks at distance-biregular graphs of diameter 4 for which one derived graph is strongly-regular and the other is complete. These graphs are called CSR graphs (see Definition 1.1.6). To fix notation we will assume the standard notation for the CSR graph G and that the complete derived graph D is on the vertex set A, while the strongly-regular derived graph E is on the vertex set B.

5.1 Quasisymmetric Block Designs

Let G be a CSR graph as above. Consider the block design $\mathcal{D}$ we obtain by taking the set A as the set of points and the set of blocks as

$$\mathcal{B} = \{\{u \in A | u \sim v\} | v \in B\}.$$

Each pair of vertices in A occurs in exactly c_2 blocks – those indexed by their common neighbours. Hence $\mathcal{D}$ is a 2-$(|A|, s, c_2)$ block design. Further any pair of blocks intersect in exactly f_2 common points or no common points. Thus $\mathcal{D}$ is a quasisymmetric block design with intersection numbers $i_1 = f_2$ and $i_2 = 0$.

Conversely if we have a quasisymmetric 2-(v, k, λ) block design $\mathcal{D}$ with intersection numbers i_1, i_2, with $i_2 = 0$ the incidence graph G of $\mathcal{D}$ is biregular with degrees λ_1 (for point vertices) and k (for block vertices). Also each pair of point vertices have

$\lambda = \lambda_2$ common neighbours, while each pair of block vertices which are at distance 2 have i_1 common neighbours. Finally consider a block vertex v and a point vertex u, with $u \not\sim v$. We count in two ways the pairs (x, w) with x a point and w a block such that x is a common neighbour of w and v and $u \sim w$. Firstly for each point x adjacent to v, there are λ_2 choices for w and so $\lambda_2 k$ pairs. If however we consider choosing w first with $w \sim u$ and w and v having non-empty intersection, there are for each such w, $i_1 = |w \cap v|$ choices of x. Hence there are $k\lambda_2/i_1$ blocks w adjacent to u and distance 2 from v. This shows that G is a distance-biregular graph with the intersection arrays of Example 1.2.9. We have thus proved the following proposition.

Proposition 5.1.1 : *There is a one to one correspondence between CSR distance-biregular graphs and quasisymmetric block designs with one block intersection number zero.* ∎

5.2 Eigenvalues of the Derived Graphs of CSR Graphs

By Lemma 3.1.2 the eigenvalues of the derived graphs D (complete) and E (strongly-regular) are related to those of the CSR graph G by the equations

$$\begin{aligned}\lambda(G)^2 \setminus \{0\} &= c_2(\lambda(D) \setminus \{-r/c_2\}) + r \\ &= f_2(\lambda(E) \setminus \{-s/f_2\} + s.\end{aligned}$$

As

$$\lambda(D) = \left\{\frac{r(s-1)}{c_2}, -1\right\} = \lambda(K_{r(s-1)/c_2}),$$

we have

$$\lambda(E) = \left\{-\frac{s}{f_2}, \frac{-c_2 + r - s}{f_2}, \frac{s(r-1)}{f_2}\right\}.$$

So the least eigenvalue of E is $-\alpha$, where $\alpha = s/f_2$, and the second largest eigenvalue is $(b_2 - s)/f_2 = b_2/f_2 - \alpha$. So the parameter β, being the difference between the second and smallest eigenvalues is $\beta = b_2/f_2$ (see Definition 1.1.5).

Lemma 5.2.1 : *The strongly-regular derived graph E of a CSR graph G is not a conference graph (see Definition 1.1.5).*

Proof : For a conference graph we have

$$\begin{aligned} k = s(r-1)/f_2 = 2\mu \\ &= (\beta^2 - 1)/2 \\ &= (b_2^2 - f_2^2)/2f_2^2. \end{aligned}$$

Note also that $2\alpha - \beta = 1$ giving

$$2\frac{s}{f_2} - \frac{b_2}{f_2} = 1 \Rightarrow b_2 = 2s - f_2.$$

Hence

$$s(r-1) = \frac{4s^2 - 4sf_2}{2f_2}$$

and so

$$r = 2\frac{s}{f_2} - 1.$$

As $b_2 < r$ we have

$$2s - f_2 < 2\frac{s}{f_2} - 1$$

$$\begin{aligned} &\Rightarrow \quad 2sf_2 - f_2^2 - 2s + 1 < 0 \\ &\Rightarrow \quad (f_2 - 1)(f_2 - 2s + 1) > 0 \end{aligned}$$

giving $f_2 < 1$ or $f_2 > 2s - 1$, neither of which is possible. ∎

The following lemma is well known, see for instance [14].

Lemma 5.2.2 : *All the eigenvalues and hence also the parameters α and β of a strongly-regular graph which is not a conference graph are integers.* ∎

The two lemmas tell us that the parameters α and β for the strongly-regular derived graph of a CSR graph must be positive whole numbers. Thus the arrays for a CSR graph G are

$$\iota(A) = \begin{bmatrix} * & 1 & r-\beta f_2 & \alpha f_2 \\ r & \alpha f_2 - 1 & \beta f_2 & * \end{bmatrix}$$

$$(**)$$

$$\iota(B) = \begin{bmatrix} * & 1 & f_2 & \alpha(r-\beta f_2) & \alpha f_2 \\ \alpha f_2 & r-1 & (\alpha-1)f_2 & r-\alpha(r-\beta f_2) & * \end{bmatrix}$$

Using Lemma 3.2.2 we can calculate r as follows

$$(r-1)(\alpha-1)f_2 = e_1 e_2 = b_1 b_2 = \beta f_2(\alpha f_2 - 1)$$

$$\Rightarrow \quad r = \frac{\beta(\alpha f_2 - 1)}{\alpha - 1} + 1. \tag{$\dagger$}$$

So the arrays are determined by the positive integer parameters f_2, α and β. Finally we can find a simple expression for e_3 in terms of these parameters

$$\begin{aligned} e_3 = r - (r-\beta f_2)\alpha &= r - \alpha - \alpha\beta(f_2 - 1)/(\alpha - 1) \\ &= (\beta\alpha f_2 - \beta - \alpha\beta f_2 + \alpha\beta)/(\alpha-1) + \alpha + 1 \\ &= \beta - \alpha + 1. \end{aligned}$$

5.3 The Antipodal Case

In this subsection we continue out investigation of CSR graphs by turning our attention to the case when the strongly-regular derived graph is antipodal. As the parameter a_2^* of the strongly-regular graph is zero, we must have $f_4(e_3-1)=0$, by Proposition 3.1.1. Hence $e_3=1$. From the general arrays for a CSR graph computed in the last subsection we see that $\beta=\alpha$ and the intersection arrays are

$$\iota(A)=\begin{bmatrix} * & 1 & (\alpha f_2-1)/(\alpha-1) & \alpha f_2 \\ (\alpha^2 f_2-1)/(\alpha-1) & \alpha f_2-1 & \alpha f_2 & * \end{bmatrix}$$

$$\iota(B)=\begin{bmatrix} * & 1 & f_2 & \alpha(\alpha f_2-1)/(\alpha-1) & \alpha f_2 \\ \alpha f_2 & \alpha(\alpha f_2-1)/(\alpha-1) & (\alpha-1)f_2 & 1 & * \end{bmatrix}$$

The antipodal derived graph is a complete multipartite graph, the parts being the antipodal blocks. As $|G_4(v)|=\alpha-1$, these blocks have size α and so $E=K_{\ell(\alpha)}$ for some ℓ. We can compute ℓ since $|B|=\ell\alpha$ giving

$$\ell=\frac{\alpha^2 f_2-1}{\alpha-1}.$$

If $\alpha=2$ then E is the Cocktail Party graph with minimum eigenvalue $-\alpha=-2$. This case has already been treated in the penultimate subsection of the previous section, where the existence of such a CSR graph was shown to be equivalent to the existence of a Hadamard matrix of dimension $4f_2$.

We now consider the general case. First we use ℓ as a parameter to replace f_2. We can compute f_2 as $(\ell(\alpha-1)+1)/\alpha^2$. Hence the arrays for G are now

$$\begin{bmatrix} * & 1 & (\ell-1)/\alpha & (\ell(\alpha-1)+1)/\alpha \\ \ell & (\ell-1)(\alpha-1)/\alpha & (\ell(\alpha-1)+1)/\alpha & * \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & (\ell(\alpha-1)+1)/\alpha^2 & \ell-1 & (\ell(\alpha-1)+1)/\alpha \\ (\ell(\alpha-1)+1)/\alpha & \ell-1 & (\ell(\alpha-1)+1)(\alpha-1)/\alpha^2 & 1 & * \end{bmatrix}$$

Note that

$$\begin{aligned} & \alpha^2 \mid \ell(\alpha-1)+1-\alpha^2 \\ \Rightarrow\ & \alpha^2 \mid (\ell-\alpha-1)(\alpha-1) \\ \Rightarrow\ & \alpha^2 \mid \ell-\alpha-1 \end{aligned}$$

This condition is a generalistion of the condition that a non-trivial Hadamard matrix must have order divisible by 4.

To complete this section we will construct an infinite family of these graphs for α any prime power.

Example 5.3.1 : *Imprimitive CSR Graphs*

Let $\alpha = q$ be any prime power and k some positive integer greater than one (k is no longer the valency of E, nor is it the exponent of the prime in q). Consider the non-zero vectors of dimension k with entries from $GF(q)$. Choose representatives $\mathbf{p}_1, \ldots, \mathbf{p}_t$ of each line through the origin (projective point) in $GF(q)^k$. This gives us $t = (q^k - 1)/(q - 1)$ vectors, which we form as column vectors (in any order) into a matrix $\mathbf{M}$ of dimension $k \times t$. The rows of this matrix are k vectors $\mathbf{b}_1, \ldots, \mathbf{b}_k$ of dimension t, which are linearly independent as $\mathbf{M}$ is easily seen to have rank k. Hence they generate a k-dimensional subspace A of $GF(q)^t$. The graph G will have vertex set $VG = A \cup B$, where

$$B = GF(q) \times \{1, \ldots, t\}.$$

For $\mathbf{u} \in A$ and $(v, i) \in B$, we have $\mathbf{u} \sim (v,i)$ if $\mathbf{u}_i = v$. The definition is independent of the ordering and choice of the projective point representatives as a suitable permutation of the columns of the matrix $\mathbf{M}$, together with a multiplication of the vertices in each $B_i = GF(q) \times \{i\}$ will give an isomorphism of the first graph to the second.

It remains to show that G is distance-biregular with the parameters $\alpha = q$ and $\ell = t = (q^k - 1)/(q - 1)$, that is with arrays

$$\iota(A) = \begin{bmatrix} * & 1 & \left[{k-1 \atop 1}\right] & q^{k-1} \\ \left[{k \atop 1}\right] & q^{k-1} - 1 & \left[{k \atop 1}\right] - \left[{k-1 \atop 1}\right] & * \end{bmatrix}$$

$$\iota(B) = \begin{bmatrix} * & 1 & q^{k-2} & \left[{k \atop 1}\right] - 1 & q^{k-1} \\ q^{k-1} & \left[{k \atop 1}\right] - 1 & q^{k-2}(q-1) & 1 & * \end{bmatrix}$$

The degree of a vertex $\mathbf{u} \in A$ is clearly $t = \left[{k \atop 1}\right]$, while that of a vertex $(v, i) \in B$ is equal to the number of vectors having i-th coordinate v. Let A_a be the set of vectors having

i-th coordinate $a \in GF(q)$. Each set A_a as a ranges over $GF(q)$ is non-empty because there is a vector in A with non-zero i-th coordinate, together with all its multiples. Hence we can choose a representative $\mathbf{u}_a \in A_a$ for each $a \in GF(q)$. Let

$$t_a : A \longrightarrow A$$

be the translation of A by $\mathbf{u}_a$. Then $t_a(A_0) = A_a$ giving $|A_0| = |A_a|$ for all $a \in GF(q)$. Hence the degree of a vertex $(v,i) \in B$ is $|A_v| = q^k/q = q^{k-1}$.

Next consider two vectors $\mathbf{u}, \mathbf{u}' \in A$. The number of coordinates in which $\mathbf{u}$ and $\mathbf{u}'$ agree is the number of zero coordinates in $\mathbf{u} - \mathbf{u}'$.

Claim: The number of zero coordinates in any non-zero element of A is $\begin{bmatrix} k-1 \\ 1 \end{bmatrix}$.

Let $\mathbf{u} \in A$ and

$$\mathbf{u} = \sum_{j=1}^{k} x_j \mathbf{b}_j$$

be its expansion in the basis $\mathbf{b}_1, \ldots, \mathbf{b}_k$. Then

$$\mathbf{u}_i = \sum_{j=1}^{k} x_j (\mathbf{b}_j)_i = \sum_{j=1}^{k} x_j (\mathbf{p}_i)_j,$$

which is the inner product of the vector $\mathbf{x} = [x_1, \ldots, x_k]^T$ and $\mathbf{p}_i$. As all lines through the origin have exactly one representative vector, the vector $\mathbf{x}$ will be perpendicular to just $\begin{bmatrix} k-1 \\ 1 \end{bmatrix}$ of them, being those lying in a subspace of dimension $k-1$. Hence $c_2 = \begin{bmatrix} k-1 \\ 1 \end{bmatrix}$.

To prove the existence of the parameter f_2, consider two vertices $(v,i), (v',i') \in B$, with $i \neq i'$. Again we partition A into sets

$$A_{ab} = \{\mathbf{u} \in A | \mathbf{u}_i = a \text{ and } \mathbf{u}_{i'} = b\}, \quad a, b \in GF(q).$$

Consider the line representatives $\mathbf{p}_i$ and $\mathbf{p}_{i'}$. As these are different projective points there is a pair of coordinates j, l, with

$$(\mathbf{p}_i)_j (\mathbf{p}_{i'})_l \neq (\mathbf{p}_i)_l (\mathbf{p}_{i'})_j.$$

This means that for any a, b we can find a linear combination of the base vectors $\mathbf{b}_j$ and $\mathbf{b}_l$ with i-th coordinate a and i'-th coordinate b. Hence for all $a, b \in GF(q)$,

$A_{ab} \neq \emptyset$. Pick representatives $\mathbf{u}_{ab} \in A_{ab}$ and let t_{ab} be the translation of A by $\mathbf{u}_{ab}$. Then $t_{ab}(A_{00}) = A_{ab}$, and $|A_{ab}| = |A_{00}|$ for all $a, b \in GF(q)$. Hence

$$|A_{ab}| = q^k/q^2 = q^{k-2}$$

and the number of common neighbours of (v,i) and (v',i) is $|A_{vv'}| = q^{k-2}$, giving $f_2 = q^{k-2}$. Finally for $\mathbf{u} \in A$ and $(v,i) \in B_i$ with $\mathbf{u}_i \neq v$ there is only one $(w,j) \in B$ adjacent to $\mathbf{u}$ but not distance 2 from (v,i), this is $(\mathbf{u}_i, i)$. Hence G is distance-biregular with the required intersection arrays. ■

5.4 The Primitive Case

It is a result of Sims, see [14], that strongly-regular graphs with smallest eigenvalue $-\alpha$, $\alpha \geq 2$ integral can be classified as follows.

Theorem 5.4.1: *A strongly-regular graph with smallest eigenvalue $-\alpha$, $\alpha \geq 2$ integral is one of the following:*

(a) A complete multipartite graph with classes of size α,

(b) Latin square graphs or pseudo-latin square graphs, $\mu = \alpha(\alpha - 1)$,

(c) Steiner graphs or pseudo-Steiner graphs, $\mu = \alpha^2$,

(d) Finitely many other graphs, with $1 \leq \mu \leq \alpha^3(2\alpha - 3)$ and $\alpha < \beta < \alpha^5(\alpha - 1)$. ■

In the previous section we dealt with case (a). We now show that in our case μ can be bounded below by a larger value.

Proposition 5.4.2: *Let G be a CSR graph with E its derived strongly-regular graph. Then the parameters μ and α for E satisfy $\alpha^2|\mu$, in particular $\mu \geq \alpha^2$.*

Proof: From the arrays (**), we can calculate the parameter μ for E as

$$\mu = \alpha^2(r - \beta f_2) = \alpha^2 c_2.$$

So $\alpha^2|\mu$ and $\mu \geq \alpha^2$. ■

Proposition 5.4.2 show that case (b) of Theorem 5.4.1 is not possible for the strongly-regular derived graph of a CSR graph.

We consider next case (c). Recall (see Definition 1.1.5) that a Steiner graph is the line graph of a 2-$(\alpha+\beta(\alpha-1),\alpha,1)$-design with $\beta \geq \alpha+1$.

Proposition 5.4.3 : *Steiner graphs are precisely the strongly-regular derived graphs of primitive CSR graphs, such that $\mu=\alpha^2$.*

Proof : Consider a Steiner graph E and its corresponding 2-$(\alpha+\beta(\alpha-1),\alpha,1)$-design $\mathcal{D}$. Two blocks B_1 and B_2 of this design cannot intersect in more than one point as otherwise a pair of points contained in any intersection would be in two blocks, contradicting $\lambda=1$. Hence this is a quasisymmetric block design with intersection numbers $i_1=1$ and $i_2=0$. By Proposition 5.1.1 the incidence graph of $\mathcal{D}$ is a CSR graph with the given Steiner graph as its strongly-regular derived graph. From the array in Example 1.2.9, we have $f_2=i_1=1$, while $f_3=b\lambda/i_1=b=\alpha$, and so $\mu=\alpha^2$. Also as $\beta \geq \alpha+1$, the graph G is primitive.
Now consider a CSR graph G with derived graph E having parameter $\mu=\alpha^2$. From the general arrays (**) for such a graph we have

$$\mu=(r-\beta f_2)\alpha^2.$$

Hence in our case $r-\beta f_2=1$, and so

$$\beta\frac{\alpha f_2-1}{\alpha-1}=\beta f_2 \quad \text{by } (\dagger)$$

giving $f_2=1$. The arrays for G are thus

$$\iota(A)=\begin{bmatrix} * & 1 & 1 & \alpha \\ \beta+1 & \alpha-1 & \beta & * \end{bmatrix}$$

$$\iota(B)=\begin{bmatrix} * & 1 & 1 & \alpha & \alpha \\ \alpha & \beta & \alpha-1 & \beta-\alpha+1 & * \end{bmatrix}$$

Thus G is the incidence graph of a 2-$(\alpha+\beta(\alpha-1),\alpha,1)$-design. As G is primitive $\beta-\alpha+1\geq 2$, giving $\beta \geq \alpha+1$. ∎

It has been proved by Wilson [18] and Hanani [9] that 2-$(v, k, 1)$-designs exist for $v \geq v_0(k)$ with $k-1|v-1$ and $k(k-1)|v(v-1)$, for some $v_0(k)$. So CSR graphs with Steiner derived graphs are very numerous.

The remaining case of the Theorem 5.4.1 is case (d). In the CSR case we have by Theorem 5.4.1, Proposition 5.4.2 and Proposition 5.4.3, the following conditions on the parameters of the strongly-regular derived graph:

$$\alpha^2|\mu, \quad 2\alpha^2 \leq \mu \leq \alpha^3(2\alpha-3) \quad \text{and } \alpha < \beta < \alpha^5(\alpha-1).$$

Thus for given α there are finitely many possible triples (α, β, μ) of parameters of such graphs. For a given triple the arrays are fully determined as

$$\mu = (r - \beta f_2)\alpha^2 = \frac{\beta\alpha^2(f_2-1)}{\alpha-1} + \alpha^2,$$

and

$$f_2 = \frac{(\mu-\alpha^2)(\alpha-1)}{\beta\alpha^2} + 1,$$

while

$$r = \frac{\beta(\alpha f_2 - 1)}{\alpha - 1} + 1$$

by (†). For any given pair of arrays there are clearly only finitely many graphs. We have thus proved the following theorem.

Theorem 5.4.4: *Let G be a CSR graph with strongly-regular derived graph E having minimal eigenvalue $-\alpha$. Then α is an integer and one of the following holds:*

(i) E is multipartite with blocksize α,

(ii) E is a Steiner graph with G the incidence graph of the corresponding design,

(iii) G is one of finitely many exceptional graphs. ∎

We finish this section by listing the exceptional graph (case (iii) of Theorem 5.4.4) intersection arrays for $\alpha = 2, 3$.

(a) $\alpha = 2$.

In this case $8 \leq \mu \leq 8$ and $2 < \beta < 32$. As

$$f_2 = \frac{\mu - 4}{4\beta} + 1,$$

$\beta = 1$ contradicting $\beta > 2$. We conclude that no exceptional graphs exist when $\alpha = 2$.

(b) $\alpha = 3$.

Here μ can be one of the numbers

$$18, 27, 36, 45, 54, 63, 72, 81,$$

while $3 < \beta < 486$. Note also that

$$\begin{aligned} & r - \beta f_2 | r(\alpha f_2 - 1) \\ \Rightarrow\ & \beta(f_2 - 1)/(\alpha - 1) + 1 | r(\alpha f_2 - 1) \\ \text{or}\quad & \beta(f_2 - 1)/2 + 1 | \beta(3f_2 - 1)^2/2 + 3f_2 - 1. \end{aligned}$$

Checking through all the possible choices of β and μ and eliminating those not satisfying the above divisibility conditions leaves us with just three possible arrays.

(i) $\mu = 36, \beta = 6, f_2 = 2, r = 16$. The arrays are

$$\begin{bmatrix} * & 1 & 4 & 6 \\ 16 & 5 & 12 & * \end{bmatrix} \text{ and } \begin{bmatrix} * & 1 & 2 & 12 & 6 \\ 6 & 15 & 4 & 4 & * \end{bmatrix}.$$

These are the arrays of Example 1.2.10.

(ii) $\mu = 45, \beta = 8, f_2 = 2, r = 21$. The arrays are

$$\begin{bmatrix} * & 1 & 5 & 6 \\ 21 & 5 & 16 & * \end{bmatrix} \text{ and } \begin{bmatrix} * & 1 & 2 & 15 & 6 \\ 6 & 20 & 4 & 6 & * \end{bmatrix}.$$

These arrays are realised by the incidence graph of the Steiner system $S(3, 6, 22)$. In [12] this (unique) Steiner system is used to define a simple group of order 44,352,000. Higman and Sims list some of the properties of this system, among others that two distinct blocks intersect in 2 or 0 points. Hence the system is a quasisymmetric block design with second intersection number 0. By Proposition 5.1.1, its incidence graph is a CSR graph with $f_2 = 2$. This parameter together with the blocksize (6) and the number of blocks a single point is contained in ($\lambda_1 = 21$) are enough to determine the intersection arrays as those given above.

(iii) $\mu = 54, \beta = 5, f_2 = 3, r = 21$. The arrays are

$$\begin{bmatrix} * & 1 & 6 & 9 \\ 21 & 8 & 15 & * \end{bmatrix} \text{ and } \begin{bmatrix} * & 1 & 3 & 18 & 9 \\ 9 & 20 & 6 & 3 & * \end{bmatrix}.$$

This pair of arrays does not satisfy the simple numerical conditions of Proposition 3.2.1 (k_3 is not a whole number) and so is not feasible.

5.5 A Counter-example

Consider a bipartite graph G with vertex bipartition $A \cup B$ and set A the row vectors $\mathbf{u}_1, \ldots, \mathbf{u}_{12}$ of the following 12×11 matrix

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The set B consists of a pair of vertices $(j,0),(j,1)$ for each column j of the matrix. Adjacency is between $v = (j,k) \in B$ and $\mathbf{u}_i \in A$ if $\mathbf{M}_{ij} = k$. Each vertex in B has degree 6 and each two non-paired vertices in B are at distance 2. Hence the derived graph on B is $K_{22} \setminus 11K_2$, the cocktail party graph. Each vertex in A has degree 11 and each pair of vectors have common neighbours and so are at distance 2. Hence the derived graph on A is K_{12}. The graph G is not, however, distance-biregular as for instance $\mathbf{u}_1$ and $\mathbf{u}_2$ have 10 common neighbours while $\mathbf{u}_1$ and $\mathbf{u}_3$ have just 7. This is a counter-example to the conjecture that a biregular graph with distance-regular derived graphs is necessarily distance-biregular.

Note that in this case there is no distance-biregular graph with these derived graphs as by Theorem 4.4.3 it would imply the existence of a Hadamard matrix of order 22.

6 Distance-biregular Graphs with Specified Derived Graph

In this section we consider various classes of distance-regular graphs and ask whether they can occur as the derived graph of a distance-biregular graph. When the answer is affirmative, we also investigate all possible ways in which this can occur.

The following lemma relating the two derived graphs of a distance-biregular graph will prove central throughout this section.

Lemma 6.0.1: *Let D and E be the derived graphs of a distance-biregular graph G in the standard notation with $VD = A$ and suppose D is not a complete graph. Then a vertex of E corresponds to a maximal clique in D, being the set of neighbours of that vertex in G. Different vertices correspond to different cliques and all edges in D must be contained in a maximal clique that corresponds to a vertex of E.*

Proof: Consider a vertex $v \in B = VE$ as a vertex of G. Its neighbours $u_1, \ldots, u_s$ will form a clique in the derived graph D. We must show that this clique is maximal. Suppose a further vertex u is adjacent to each of $u_1, \ldots, u_s$ in D. We obtain a contradiction by showing that in this case D has diameter 1. The vertex v is distance 3 from u in G, but every neighbour of v is distance 2 from u. Hence $c_3 = s$ and $G_4(u) = \emptyset$, so $D_2(u) = \emptyset$. Now suppose two vertices v and v' determine the same maximal clique $u_1, \ldots, u_s$. Then all neighbours of v' are closer to v than v', giving $f_2 = s$. By Lemma 3.2.2 $c_2 = r$ and $G = K_{r,s}$, a contradiction. Finally for each edge (u, u') of D, there is a path u, v, u' of length 2 joining u to u'. Thus the edge (u, u') will be part of the maximal clique determined by v. ∎

The first proposition we will prove using Lemma 6.0.1 will tie up a loose end left over from section 4, namely precisely when a line graph may occur as the derived graph of a distance-biregular graph.

Proposition 6.0.2: *Let G be a distance-biregular graph with derived graph D on vertex set A a line graph $L(H)$. Then one of the following holds:*

(a) D is a complete graph $K_n (= L(K_{1,n}))$,

(b) $G = S(H)$, where H is a (k,g)-graph,

(c) $G = JB(2,n)$, the Johnson biregular graph.

Proof: Consider a maximal clique C of the line graph $L(H)$. If three edges $e_1, e_2, e_3 \in C$ they either have a vertex in common or they form a triangle v_1, v_2, v_3 in H. If they have a vertex in common then the only maximal clique they are contained in is that determined by that common vertex (all edges incident with it).

Consider first the case when H has no triangles. Then the vertices of B correspond to vertices of H. In order for two adjacent edges of H to be adjacent in D their common vertex must have a corresponding vertex of B. Hence provided H has no vertices of degree 1, $G = S(H)$. Also by Theorem 4.2.1 we know that in this case H is a (k,g)-graph. The only graph with its line graph distance-regular and vertices of degree 1 is $K_{1,n}$. In this case D is the complete graph K_n.

By Theorem 4.2.1 the only graphs with girth 3 whose line graphs are distance-regular are complete graphs. Hence in this case $D = L(K_n)$ for some n. But $L(K_n) = J(2,n)$, the Johnson scheme on 2-sets. We will prove later in this section that $J(2,n)$ can occur as the derived graph of a distance-biregular graph G only if G is $JB(1,n)$ or $JB(2,n)$. But $JB(1,n)$ is just $S(K_n)$, which is case (b) again. ∎

6.1 Hamming Graph Derived Graphs

In this subsection we investigate when the Hamming graphs can occur as derived graphs of distance-biregular graphs. We also investigate the graph $H(2,q)^c$, the complement of the Hamming graph of diameter 2. The results obtained here will prove useful in the section on distance-bitransitive graphs.

Proposition 6.1.1: *The only distance-biregular graph with Hamming derived graph is $S(K_{q,q})$, the subdivision graph of $K_{q,q}$. This graph is imprimitive and has derived graph $H(2,q)$.*

Proof: Suppose G is a distance-biregular graph with derived graph D on vertex set A isomorphic to $H(d,q)$. By Lemma 6.0.1 the vertices of the other derived graph E correspond to maximal cliques of $H(d,q)$ as D is not a complete graph. The maximal cliques of $H(d,q)$ are indexed by d-vectors over $X' = X \cup \{*\}$ in which precisely one coordinate is $*$, a symbol not in the set X used to define $H(d,q)$. The clique indexed by $c = (i_1,\ldots,i_d)$, with $i_k = *$, consists of all the vertices of $H(d,q)$ which agree with c in every component except the k-th. We claim that every such clique must correspond to a vertex of E. We prove this for the general clique c. The two vertices $(i_1,\ldots,i'_k,\ldots,i_d)$ and $(i_1,\ldots,i''_k,\ldots,i_d)$, where i'_k and i''_k are two distinct elements of X, are adjacent in $H(d,q)$, so there must be a vertex v of B adjacent to both of them in G. The only maximal clique containing both of them is c and so c must correspond to v. Hence the claim holds and G has vertex set the vertices of $H(d,q)$ together with its maximal cliques, with adjacency given by inclusion of a vertex in a clique.

Suppose now that $d > 2$. The clique $v = (*,i_2,\ldots,i_d)$ is distance 4 from $v' = (i_1,i'_2,*,i_4,\ldots,i_d)$ and $v'' = (*,i'_2,i_3,\ldots,i_d)$. But every neighbour of v'' is distance 3 from v, while just one neighbour $(i_1,i'_2,i_3,\ldots,i_d)$ of v' is distance 3 from v. This contradicts G being distance-biregular.

If $d=2$ the maximal cliques are indexed by $\{(i,*),(*,i) | i \in X\}$. Each vertex (i_1,i_2) of $H(2,q)$ can be viewed as the edge joining $(i_1,*)$ to $(*,i_2)$ in the complete bipartite graph with parts $X_1 = X \times \{*\}$ and $X_2 = \{*\} \times X$. Hence $G = S(K_{q,q})$. The derived graphs of G are $K_{q,q}$ and $L(K_{q,q}) = H(2,q)$. As $K_{q,q}$ is bipartite G is imprimitive. ∎

Proposition 6.1.2: *Let $q > 2$. The existence of a distance-biregular graph G with derived graph $H(2,q)^c$ is equivalent to the existence of a projective plane P of order q. The graph G is the incidence graph of the structure P' obtained from P by choosing two distinct points x and y and deleting all the lines through either of them and all the points on the line xy. The graph G is antipodal.*

Proof: ($\Rightarrow$) Suppose G is a distance-biregular graph with derived graph $D = H(2,q)^c$ on vertex set A in the standard notation. Let X denote the set used to define $H(2,q)$ (see Example 1.2.1) so that the set A can be regarded as $A = \{(i,j) | i,j \in X\}$, with $\partial_G((i,j),(i',j')) = 2$ iff $i \neq i'$ and $j \neq j'$. Then $G_4(u) \neq \emptyset$, for u in A, so by Lemma 6.0.1 the vertices of B correspond to maximal cliques of D.

We claim that any maximal clique of D has q elements, for suppose

$$c = \{(i_1,j_1),\ldots,(i_\ell,j_\ell)\}$$

is a maximal clique of $H(2,q)^c$. Then each pair differ in both coordinates and so $i_1,\ldots,i_\ell$ are all distinct and likewise $j_1,\ldots,j_\ell$. Hence $\ell \leq q = |X|$. If $\ell < q$ we can choose $i_{\ell+1} \in X \setminus \{i_1,\ldots,i_\ell\}$ and $j_{\ell+1} \in X \setminus \{j_1,\ldots,j_\ell\}$. Then $(i_{\ell+1},j_{\ell+1}) \sim (i_t,j_t)$, for $t = 1,\ldots,\ell$, contradicting the maximality of c.

We conclude that $s = q$ and as $H(2,q)^c$ has intersection array

$$\begin{bmatrix} * & 1 & (q-1)(q-2) \\ 0 & (q-2)^2 & q-1 \\ (q-1)^2 & 2(q-2) & * \end{bmatrix},$$

we can compute

$$\iota(A) = \begin{bmatrix} * & 1 & r/(q-1) & q-2 & r \\ r & q-1 & r(q-2)/(q-1) & 2 & * \end{bmatrix}.$$

By Lemma 3.2.2, $e_1e_2 = b_1b_2$ and so $e_2 = r(q-2)/(r-1)$. Hence $r-1 | q-2$ and $q-1 | r$. This forces $r = q-1$ and so

$$\iota(A) = \begin{bmatrix} * & 1 & 1 & q-2 & q-1 \\ q-1 & q-1 & q-2 & 2 & * \end{bmatrix}$$

and

$$\iota(B) = \begin{bmatrix} * & 1 & 1 & q-2 & q \\ q & q-2 & q-1 & 1 & * \end{bmatrix}.$$

The derived graph E on the vertex set B has intersection array

$$\begin{bmatrix} * & 1 & q(q-2) \\ 0 & q(q-3) & 0 \\ q(q-2) & q-1 & * \end{bmatrix}.$$

This is an antipodal graph of diameter 2 with $|\{u\} \cup G_2(u)| = q$. Hence $E = K_{(q-1)(q)}$, the complete $(q-1)$-partite graph with each part having q vertices. We label the parts of E from 1 to $q-1$. To complete the first half of the proof it remains to construct a projective plane P of order q from G. The points of the plane P will be the vertices of $A = VH$ together with $q+1$ points labelled $x, y, p_1, \ldots, p_{q-1}$. The lines of P will be labelled by the vertices of B together with $2q+1$ additional lines $l_i, m_i, i \in X$ and l_∞. Vertex v of B in block k of E labels a line composed of the points $\{u \in A | u \sim v\} \cup \{p_k\}$. The line l_i is the set of points $\{(i,j) | j \in X\} \cup \{x\}$ while m_i is the set $\{(j,i) | j \in X\} \cup \{y\}$. Finally l_∞ is the set of points $\{x, y, p_1, \ldots, p_{q-1}\}$. It is fairly straightforward to check that each pair of points lie on exactly one line and that each pair of lines intersect in exactly one point. Finally the four points $x, y, (i,i), (j,j)$ $(i,j \in X, i \neq j)$ form a four-point. So P is a projective plane of order q.

($\Leftarrow$) Suppose P is a projective plane of order q. Let x, y, P' and G be as in the proposition statement. Let u be any point of P'. The point u lies on $q+1$ lines in P, but the line through x and the line through y (distinct because u is not on xy) have been deleted, so u lies on $q-1$ lines in P'. Let v be a line of P'. The line v intersects xy in P in a point $p \neq x$ or y. Hence v is incident with q points in P' and G is a biregular graph. Two points lie on one line in P so the incidence graph of P has girth greater than 4. Hence girth$(G) \geq 6$. Now consider a point u of P' and a line v of P' not incident with u. Let u' be a point on v. The line uu' is in P' iff x and y are not on uu'. Now ux and uy intersect v in two distinct points of v as u is not on xy. Hence precisely $q-2$ points of v are collinear with u in P'. We thus see that a point vertex of G has the first seven intersection numbers well defined:

$$\begin{bmatrix} * & 1 & 1 & q-2 & \ldots \\ q-1 & q-1 & q-2 & 2 & \ldots \end{bmatrix}$$

But in the argument above we took any line not incident with u and found it was distance 3 from u. So $G_5(u) = \emptyset$ and the point vertices of G are distance-regularised with array

$$\begin{bmatrix} * & 1 & 1 & q-2 & q-1 \\ q-1 & q-1 & q-2 & 2 & * \end{bmatrix}.$$

Finally consider a line v of P' and a point u not incident with it. The only line through u which does not intersect v in P' is the line through the point $v \cap xy$. Again we choose any point u not incident with v, so $G_5(v) = \emptyset$ and the line vertices of G are distance-regularised with array

$$\begin{bmatrix} * & 1 & 1 & q-2 & q \\ q & q-2 & q-1 & 1 & * \end{bmatrix}.$$

So G is a distance-biregular graph. We now investigate its derived graph on the point vertices. A point vertex u of G can be labelled by an ordered pair of lines (ux, uy), which clearly determine u as their intersection. Conversely a pair of lines (l, m) with x on l and y on m, but neither the line xy, determine a point vertex of G. We now use this labelling, so that the point vertices of G are

$$\begin{aligned} A &= \{(l,m) | l \in G(P)(x) \setminus \{xy\} \text{ and } m \in G(P)(y) \setminus \{xy\}\} \\ &= X \times X, \quad \text{with } |X| = q. \end{aligned}$$

The distinct vertices (l, m) and (l', m') are adjacent in the derived graph of G iff they are collinear in P'. This will be true iff the line through them in P was not deleted, i.e. did not go through x or y. But the line $(l, m)(l', m')$ of P is incident with x iff it is $l = l'$, while it is incident with y iff it is $m = m'$. We conclude that (l, m) is adjacent to (l', m') in the derived graph iff $l \neq l'$ and $m \neq m'$, and so the derived graph is $H(2, q)^c$. ■

6.2 Johnson Scheme Derived Graphs

We now consider distance-biregular graphs with derived graph a Johnson scheme graph, or the q-analoque Johnson scheme graph. In order to use Lemma 6.0.1, we must identify the maximal cliques in each of these graphs. First consider the standard Johnson Scheme graphs.

Lemma 6.2.1: *The maximal cliques of $J(d,n)$ are of two types. Type 1 are the d-subsets contained in a fixed $(d+1)$-subset and type 2 are those subsets containing a fixed $(d-1)$-subset.*

Proof: Clearly maximal cliques of types 1 and 2 exist. Consider a maximal clique C of $J(d,n)$ and suppose C is not of type 1. Pick any $v_1, v_2 \in C$. Consider a $v_3 \in C$ such that $v_3 \not\subset v_1 \cup v_2$. But then $|v_3 \cap v_1 \cap v_2| = d-1$, as there is one element of v_3 which cannot be in v_1 or v_2. Set $u = v_1 \cap v_2 \cap v_3$ and let $\{x_i\} = v_i \setminus u$, $i = 1,2,3$. Pick any $v \in C$, and suppose $u \not\subset v$. But then $x_i \in v$, $i = 1,2,3$ and so $v \cap v_1 \subset v \setminus \{x_2, x_3\}$ giving $|v \cap v_1| \leq d-2$, a contradiction. ∎

Note that type 1 cliques are of size $d+1$ while type 2 are of size $n-d+1$. If G is a distance-biregular graph with derived graph $D = J(d,n)$, then the vertices of E correspond to maximal cliques of $J(d,n)$. If the clique corresponding to a vertex v is of type 1 or type 2, then $\deg(v) = d+1$ or $n-d+1$ respectively. Hence unless $n = 2d$ the cliques corresponding to vertices must all be of the same type. We treat the case $n = 2d$ separately.

Proposition 6.2.2: *The only distance-biregular graphs having $J(d,n)$, $n \neq 2d$, as a derived graph are $JB(d,n)$ and $JB(d-1,n)$.*

Proof: Let G be a distance-biregular graph with derived graph $D = J(d,n)$. We consider the two possible types of the cliques corresponding to vertices of E separately.
Case (i) The vertices of E correspond to cliques of type 1. Consider any $(d+1)$-subset X of $\{1,\ldots,n\}$, and choose two vertices u, u' of $J(d,n)$ such that $u \cup u' = X$. Then $|u \cap u'| = d-1$ and so $u \sim u'$ in $J(d,n)$. Hence they must be contained in a maximal

clique corresponding to a vertex of E. But the only type 1 clique containing u and u' is that determined by the set X. Thus there is a vertex v_X of E corresponding to the set X, adjacent to all the d-subsets contained in X. As X was chosen arbitrarily the graph G is $JB(d,n)$.

Case (ii) The vertices of E correspond to maximal cliques of type 2. Consider any $(d-1)$-subset X of $\{1,\ldots,n\}$ and choose vertices u,u' of $J(d,n)$ with $u \cap u' = X$. As $u \sim u'$ and there must be a vertex v_X of E corresponding to X, adjacent to all the d-subsets containing X. Hence G is the graph $JB(d-1,n)$. ∎

Proposition 6.2.3: *The only distance-biregular graph having $J(d,2d)$ as a derived graph is $JB(d,2d) \cong JB(d-1,2d)$.*

Proof: We first show that the vertices of E cannot correspond to maximal cliques of different types. Suppose this is the case. Then we can find adjacent vertices v,v' in E corresponding to cliques of type 1 and type 2 respectively. As $v \sim_E v'$ there is a vertex $u \in A$ adjacent to both of them. Note that v corresponds to a $(d+1)$-subset X, and v' to a $(d-1)$-subset Y such that $Y \subset u \subset X$. There is exactly one other d-subset u' satisfying $Y \subset u' \subset X$. Then v and v' have exactly two common neighbours, u and u', in G. Now suppose there are two adjacent vertices w,w' in E both of the same type. These can only have one common neighbour in G ($w \cup w'$ if of type 2, $w \cap w'$ if of type 1). Thus E must be bipartite with the vertices of one part corresponding to cliques of type 1 and the vertices of the other part to cliques of type 2. But as a bipartite graph is without triangles we can apply Corollary 4.3.4 to deduce that E is a (k,g)-graph with g even or K_2. In the former case G would be $S(E)$ and D the line graph $L(E)$, while in the latter case G would be $K_{2,r}$ for some $r \geq 1$ and D the complete graph K_r. By the above adjacent vertices v and v' in E have two common neighbours in G. This is not so if G is the subdivision graph of E. Hence the only possible case is $G \cong K_{2,2}$ when we must have $J(d,2d) \cong K_2$ and so $d=1$, a contradiction.

Hence all vertices of E correspond to maximal cliques of one type. A similar argument to that of Proposition 6.2.2 yields the result, together with the isomorphism

$$\phi : JB(d-1,2d) \longrightarrow JB(d,2d)$$

defined by mapping all subsets to their complements. ∎

Hemmeter [10] proves the following lemma for the q-analoque of the Johnson Schemes.

Lemma 6.2.4 : *There are two types of maximal cliques in $J_q(d,n)$. Type 1 consists of all d-subspaces contained in a fixed $(d+1)$-space. Type 2 consists of all d-subspaces containing a fixed $(d-1)$-space.* ■

The following two propositions are the q-analogues of Proposition 6.2.2 and Proposition 6.2.3. The proofs are omitted as they copy exactly the method employed in those two propositions.

Proposition 6.2.5 : *The only distance-biregular graphs having the graph $J_q(d,n)$, $n \neq 2d$, as a derived graph are $JB_q(d-1,n)$ and $JB_q(d,n)$.* ■

Proposition 6.2.6 : *The only distance-biregular graph having $J_q(d,2d)$ as a derived graph is $JB_q(d-1,2d) \cong JB_q(d,2d)$.* ■

6.3 Polar Space Graph Derived Graphs

Finally we turn our attention to the dual polar space distance-regular graphs. The results prove a little disappointing as they show that many very interesting distance-regular graphs cannot occur as the derived graphs of distance-biregular graphs. We begin with a lemma characterising the maximal cliques of the dual polar space graphs, which is again due to Hemmeter [11].

Lemma 6.3.1 : *The maximal cliques of a dual polar space graph consist of all the vertex (isotropic d-)subspaces containing a fixed $(d-1)$-dimensional istropic subspace.* ■

We are now ready to consider when the dual polar space graphs can occur as the derived graph of a distance-biregular graph. The next proposition covers all such graphs.

Proposition 6.3.2 : *Let G be a distance-biregular graph with derived graph D a non-trivial dual polar space graph of diameter d on vector space V. Then $d = 2$ and G is a generalised quadrangle.*

Proof : Let the exponent of the dual polar space graph D be e. Consider any isotropic $(d-1)$-subspace v of V. Then v contains $q^{e+1} + 1 > 1$ isotropic d-subspaces [5]. Picking two such d-subspaces u and u', then $u \sim u'$ in D and so there must be a vertex of E corresponding to a clique containing u and u'. But the only such clique is that determined by the $(d-1)$-subspace $u \cap u' = v$. Hence all the isotropic $(d-1)$-subspaces correspond to vertices of G. So now suppose that G is the graph with vertex set the isotropic d-spaces and $(d-1)$-spaces with adjacency given by inclusion. We must show that G is distance-biregular if and only if $d = 2$. We saw above that each isotropic $(d-1)$-subspace is contained in $q^{e+1} + 1$ isotropic d-subspaces. Also each isotropic d-subspace contains $\begin{bmatrix} d \\ 1 \end{bmatrix}$ $(d-1)$-subspaces all of which are isotropic. Hence G is biregular.

We will show that the d-subset vertices of G are distance-regularised. Consider two such vertices u and u' with $\partial(u, u') = 2j$. Then $\dim u \cap u' = d - j$. An isotropic $(d-1)$-subspace v adjacent to u' and closer to u must satisfy $u \cap u' \subset v \subset u'$. Such a v is closer to u as the space $u'' = v + v^{\perp} \cap u$ is an isotropic d-subspace with $\dim(u \cap u'') = d - j + 1$, since

$$\begin{aligned} u''^{\perp} &= (v + v^{\perp} \cap u)^{\perp} \\ &= v^{\perp} \cap (v^{\perp} \cap u)^{\perp} \\ &= v^{\perp} \cap (v + u^{\perp}) \\ &\supset v^{\perp} \cap (v + u) \quad \text{as } u^{\perp} \supset u \\ &= v + v^{\perp} \cap u \quad \text{as } v \subset v^{\perp} \\ &= u'', \end{aligned}$$

and $\dim u''^{\perp} - \dim u'' = \dim u^{\perp} - \dim u$. But u'' is also the only isotropic d-subspace containing v which is closer to u, since certainly any such subspace must be contained in $v + v^{\perp} \cap u = u''$. Hence we have determined the intersection numbers $c_{2j}(u) = \begin{bmatrix} j \\ 1 \end{bmatrix}$

and $c_{2j-1}(u) = 1$. Thus u is distance-regularised. If G is distance-biregular we can compute the opposite array by Lemma 3.2.2. Thus assuming $u \in A$ in the standard notation, we have

$$e_4 e_3 = b_4 b_3 = q^{e+1}\left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$$

$$\Rightarrow e_4 = q^{e+1}\left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) / \left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right).$$

Hence $e_4 < q^{e+1}$ and $f_4 > 1$. But by Proposition 3.2.1, $1 = c_5 \geq f_4 > 1$, a contradiction, if $\text{diam}(G) > 4$. But if $\text{diam}(G) = 4$, $d = 2$ and as G has girth 8 ($c_3(u) = c_2(u) = 1$), G is a generalised quadrangle. ∎

7 The Praeger, Saxl and Yokoyama Theorem

Praeger, Saxl and Yokoyama [15] recently proved a very strong result about the types of groups that can act primitively and distance-transitively on graphs. We begin this section by stating their result. We then give the proof of a well-known proposition covering the abelian socle case. Next we will present a result of O'Nan and Scott which forms the basis of Praeger, Saxl and Yokoyama's proof. Finally we will give an alternative proof of part of Praeger, Saxl and Yokoyama's theorem. The proof they use is group theoretic, while that presented here is perhaps more understandable to graph theorists.

Theorem [15] 7.0.1 : *Let G be a finite primitive distance-transitive graph of diameter d with Γ a group acting distance-transitively on G. Then one of the following holds:*

(a) G is the Hamming graph or $d = 2$ and G is the complement of the Hamming graph,

(b) Γ is almost simple,

(c) (Γ, VG) is affine. ∎

7.1 The Affine Case

Consider first the case when the permutation group (Γ, VG) has abelian socle.

Proposition 7.1.1 : *If (Γ, X) is an affine permutation group with socle $N \cong (Z_p)^m$, then N acts regularly on X and $\Gamma \leq AGL(m,p)$.*

Proof: First note that N acts transitively on X as the orbits of N would otherwise be non trivial blocks of imprimitivity:

Let g in Γ and O an orbit of N. Then Og is an orbit of N, as for $n \in N$, $x \in O$,

$$xgn = xn'g \in Og, \quad \text{as } n' = gng^{-1} \in N,$$

$$\text{and} \quad xng = xgn'', \quad \text{where } n'' = g^{-1}ng \in N.$$

Now suppose n in N fixes x in X. Let x' be any element of X and n' in N such that $xn' = x'$. Then

$$\begin{aligned} x'n &= x'n'^{-1}nn' \\ &= xnn' = xn' = x', \end{aligned}$$

so n is the identity and N acts regularly.

Finally an element g in Γ acts on $N \cong (Z_p)^m$ by conjugation. As $g^{-1}nn'g = g^{-1}ngg^{-1}n'g$, this gives us a map

$$\alpha : \Gamma \longrightarrow GL(m,p).$$

The kernel of α is $C_\Gamma(N)$. Let g in $C_\Gamma(N)$, so that g fixes an element x. Then as above for any x' in X, choose n' in N so that $xn' = x'$ and we have

$$x'g = xng = xgn = xn = x'.$$

So g is the identity, $C_\Gamma(N)$ acts regularly and as $C_\Gamma(N) \geq N$, we have $C_\Gamma(N) = N$. In conclusion we can write $\Gamma = \Gamma_x \ltimes N$ for some fixed x in X and α embeds Γ_x into $GL(m,p)$. Hence $G \leq AGL(m,p)$. ■

7.2 Proof of Praeger, Saxl and Yokoyama

We now present O'Nan and Scott's theorem. As there is no published reference due to O'Nan and Scott, we give a reference to a paper by Cameron [3] giving an exposition of the theorem. Unfortunately this exposition is deficient in the case when the wreath action is twisted. We merely note that case here.

Let Γ be a primitive permutation group on a set Ω, with degree n and non-abelian socle N. Then $N = T_1 \times T_2 \times \ldots \times T_m$, where $T_1, \ldots, T_m$ are all isomorphic to a fixed non abelian simple group T. Moreover either

(a) T is a normal subgroup of a primitive group Γ_0 of degree n_0 and $\Gamma \leq \Gamma_0 \wr Sym(m)$ (with the product action) where $n = n_0^m$, or

(b) For $u \in \Omega$, $N \cap \Gamma_u = D_1 \times \ldots \times D_l$ where $m = kl$ for some k and D_i is the diagonal subgroup of $T_{(i-1)k+1} \times \ldots \times T_{ik}$ and $n = |T|^{(k-1)l}$, or

(c) the action is analagous to that in (a) except that it is twisted.

We are now ready to present a graph theoretic proof of the following proposition, which constitutes a major part of the proof of the Praeger, Saxl and Yokoyama Theorem (see the remark that follows the exposition).

Proposition 7.2.2 : *If G is a graph on which a group Γ acts primitively and distance-transitively and further the permutation group (Γ, VG) satisfies case (a) of the O'Nan and Scott theorem with $m \geq 2$, then G is the Hamming graph $H(m,q)$, for some q, or possibly its complement $H(2,q)^c$, when $m = 2$.*

Remark : Before beginning our proof of this proposition we note that in the case when the permutation group (Γ, VG) satisfies case (a) of the O'Nan and Scott theorem with $m = 1$, the group Γ is almost simple and we thus have case (b) of the Praeger, Saxl and Yokoyama Theorem. If on the other hand the socle of Γ is abelian we have shown in Proposition 7.1.1 that (Γ, VG) satisfies case (c) of the Praeger, Saxl and Yokoyama result. Thus we fail to deal with the twisted wreath case and socles of diagonal type (case (b) of O'Nan and Scott). In the latter case the graph is a Cayley graph of the subgroup $M = E_1 \times \ldots \times E_l$, where $E_i = T_{(i-1)k+1} \times \ldots \times T_{ik-1}$, as $M \cap G_u = \{1\}$, for $u \in VG$ and $|M| = |T|^{(k-1)l} = n = |VG|$.

Proof of Proposition 7.2.2: The only case of the O'Nan and Scott Theorem where the socle is not a minimal normal subgroup is (b) with $l = 1$ and $k = 2 = m$. Hence in our case the subgroup N is minimal normal as well as the socle. We first fix notation for the proof. In our case N is non abelian and $VG = X^m$ for some set X on which a group Γ_0 acts primitively with normal subgroup T and $\Gamma \leq \Gamma_0 \wr Sym(m)$ (with the product action) and $N = T_1 \times \ldots \times T_m$, with $T_i \cong T$. Let $* \in X$ be some fixed element of X and $x = (*, \ldots, *) \in VG$. Also let $x(i, u) = (*, \ldots, *, u, *, \ldots, *) \in VG$ with u in the i-th coordinate. Then $N \cap \Gamma_x = S_1 \times \ldots \times S_m$, where $S_i = (T_i)_*$. We define the Hamming distance between two vertices x and y in VG to be the number of coordinates in which they differ and denote this number by $\partial_H(x, y)$. This function is clearly a metric on VG. Our proof will be divided into several lemmas. In the first three lemmas we incidentally do not require Γ to act distance-transitively.

Lemma 7.2.3: *Let $a \in \Gamma$, then there exists $\sigma \in \mathrm{Sym}(m)$ such that $a^{-1}T_i a = T_{\sigma(i)}$.*

Proof: Conjugation by a is an automorphism of N. Hence $N = a^{-1}T_1 a \times \ldots \times a^{-1}T_m a$. But by the Krull-Remak-Schmidt theorem, since $Z(N) = \{1\}$, there is only one way of decomposing N into a direct product up to permutation of the factors. ■

Lemma 7.2.4: *The subgroup Γ_x of Γ acts transitively on $\{T_1, \ldots, T_m\}$ and if $u \neq *$ then there exists $g \in \Gamma_x$ such that $x(i, u)g = x(j, u')$ for some $u' \in X \setminus \{*\}$.*

Proof: Let the subgroup generated by T_1 under the action of Γ_x be M. M is then the direct product of some of the T_i by Lemma 7.2.3. Now $N_\Gamma(M) \geq \Gamma_x$, but $N_\Gamma(M) \neq \Gamma_x$ as T_1 does not fix x. Hence, as Γ_x is a maximal subgroup, $N_\Gamma(M) = \Gamma$ and $M \triangleleft \Gamma$. But in this case as N is a minimal normal subgroup $M = N$, that is Γ_x acts transitively on the T_i. Now choose $a \in \Gamma_x$ so that $a^{-1}T_i a = T_j$. Let $t \in T_i$ such that $(x)t = x(i, u)$ and let $t' = a^{-1}ta \in T_j$. But then

$$x(i, u)a = (x)a^{-1}ta = (x)t'.$$

The vertex $(x)t'$ has all coordinates $*$ except the j-th as required. ■

Lemma 7.2.5 : *Γ preserves Hamming distance.*

Proof: Let $\partial_H(y,z) = \ell$ and $g \in \Gamma$. Let $\sigma \in \mathrm{Sym}(m)$ such that $g^{-1}T_i g = T_{\sigma(i)}$, as guaranteed by Lemma 7.2.3. Choose $t_i \in T_i$ so that $(y)n = z$, where $n = t_1 \ldots t_m$. If y and z agree in the i-th coordinate take $t_i = 1$. Then $(y)gg^{-1}ng = (z)g$. Now

$$g^{-1}ng = g^{-1}t_1 g \ldots g^{-1}t_m g$$

and precisely ℓ of the $g^{-1}t_i g \in T_{\sigma(i)}$ are non trivial. Hence $(y)g$ and $(z)g$ differ in at most ℓ coordinates. Applying g^{-1} to $(y)g$ and $(z)g$ we see that $(y)g$ and $(z)g$ differ in at least as many coordinates as y and z. We conclude that $\partial_H((y)g,(z)g) = \partial_H(y,z)$ and Γ preserves Hamming distance. ∎

This lemma shows that the group Γ is always a group of automorphisms of the Hamming graph. We wish to show that the distance-transitivity and primitivity of Γ forces the graph G to be the Hamming graph, or when $m = 2$ possibly its complement. The distance-transitivity of Γ gives us a simple corollary to the last lemma.

Corollary 7.2.6 : *There exists a function f from the distances of G to the set $\{1,\ldots,m\}$ of Hamming distances, such that*

$$\partial_H(y,z) = f(\partial(y,z)), \quad \textit{for all } y,z \in VG.$$

Proof: As Γ acts transitively on pairs of vertices a given distance k apart and at the same time Γ preserves Hamming distance by the lemma, all the pairs at distance k apart in G are at the same Hamming distance. ∎

In the following the function f will be that referred to in this Corollary. We will also refer to the image of a distance in G under the function f as its *type*.

Lemma 7.2.7: *If adjacency is of type 1, that is $f(1) = 1$, then the graph is the Hamming graph.*

Proof: Consider the induced subgraph $G(i, v_1, \ldots, v_m)$ of G on the set of vertices

$$X(i, v_1, \ldots, v_m) = \{(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_m) | u \in X\}.$$

We will often refer to the vertex $(v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_m)$ as u if it is clear from the context that it lies in the set $X(i, v_1, \ldots, v_m)$. By choosing $t_j \in T_j$ such that $(v_j)t_j = v'_j$, $(t_i = 1)$, we get an automorphism $n = t_1 \ldots t_m$ of G mapping the vertices of $X(i, v_1, \ldots, v_m)$ to those of $X(i, v'_1, \ldots, v'_m)$. Hence

$$G(i, v_1, \ldots, v_m) \cong G(i, v'_1, \ldots, v'_m) \cong G(i)$$

for some graph $G(i)$. Next consider $b \in \Gamma_x$ such that $b^{-1}T_i b = T_j$ (such a b exists by Lemma 7.2.4) and consider the subgraphs $G(i, *, \ldots, *)$ and $G(j, *, \ldots, *)$. Clearly $X(i, *, \ldots, *)b = X(j, *, \ldots, *)$ so that b is an automorphism of G taking a copy of $G(i)$ to a copy of $G(j)$. Hence $G(i) \cong D$, for all i and some graph D. Let E be the graph

$$E = G(1, *, \ldots, *) \times \ldots \times G(m, *, \ldots, *)$$

and define a map

$$\varphi : G \longrightarrow E$$

by

$$\varphi(v_1, \ldots, v_m) = (v_1, \ldots, v_m).$$

We will show that φ is an isomorphism.

Suppose $(v_1, \ldots, v_m) \sim (v'_1, \ldots, v'_m)$ in G. As $f(1) = 1$ there exists some i such that for all $j \neq i$, $v_j = v'_j$ and so $v_i \sim v'_i$ in $G(i, v_1, \ldots, v_m)$. Then $v_i \sim v'_i$ in $G(i, *, \ldots, *)$ and so $(v_1, \ldots, v_m) \sim (v'_1, \ldots, v'_m)$ in E.

Now suppose $(v_1, \ldots, v_m) \sim (v'_1, \ldots, v'_m)$ in E. Then there exists some i such that for $j \neq i$, $v_j = v'_j$ and $v_i \sim v'_i$ in $G(i, *, \ldots, *)$, by the definition of the cartesian product.

But then $v_i \sim v_i'$ in $G(i, v_1, \ldots, v_m)$ and so $(v_1, \ldots, v_m) \sim (v_1', \ldots, v_m')$ in G. We conclude that φ is an isomorphism and that

$$G \cong D \times \ldots \times D.$$

Clearly then $f(i) = i$, $i = 1, \ldots, m$. The graph D must be connected as G is connected. If D has diameter greater than one, then G has vertices at distance two but Hamming distance one, contradicting $f(2) = 2$. We deduce that D is the complete graph and so G is the Hamming graph. ∎

Lemma 7.2.8: *$S_i = (T_i)_*$ has no fixed points in $X \setminus \{*\}$, for $i = 1, \ldots, m$.*

Proof: Suppose S_i has a fixed point $u \in X \setminus \{*\}$. Then $S_i \leq (T_i)_u$. But $|S_i||X| = |T_i| = |(T_i)_u||X|$, and so $(T_i)_u = S_i$. As T_i acts transitively on X we can find $t \in T_i$ such that $(*)t = u$. Then $t^{-1}S_i t = (T_i)_u = S_i$, so $t \in N_{T_i}(S_i) \setminus S_i$. Hence $S_i \lhd N_{T_i}(S_i) \neq S_i$ and as $S_i \ntrianglelefteq T_i$, $N_{T_i}(S_i) \neq T_i$. Let $W_i = N_{T_i}(S_i)$. The subgroup $W = W_1 \times \ldots \times W_m$ is invariant under the action of Γ_x as for $g \in \Gamma_x$, if $g^{-1}T_i g = T_j$, then $g^{-1}S_i g = S_j$ and so $g^{-1}N_{T_i}(S_i)g = N_{T_j}(S_j)$. But then

$$\Gamma_x \leq N_\Gamma(W) \neq \Gamma, \quad \text{as } W \ntrianglelefteq \Gamma.$$

Also $\Gamma \neq N_\Gamma(W)$ as $W \nleq \Gamma_x$. Hence Γ_x is not a maximal subgroup, contradicting the primitivity of Γ's action on G. ∎

Lemma 7.2.9: *The function f satisfies $f(1) < 3$. Also if $f(1) = 2$ then $f(2) = 1$.*

Proof: Let adjacency be of type $k > 1$. Without loss of generality

$$x = (*, \ldots, *) \sim (u_1, \ldots, u_k, *, \ldots, *) = x'.$$

Choose $a_i \in S_i$ such that $(u_i)a_i = u_i' \neq u_i$, $i = 1, \ldots, k$. This is possible by Lemma 7.2.8. Then

$$(x')a_1 = x'' = (u_1', u_2, \ldots, u_k, *, \ldots, *) \sim (*, \ldots, *) = (x)a_1.$$

Hence $\partial(x', x'') = 2$ as $\partial_H(x', x'') = 1 \neq k$. So $f(2) = 1$.

Now assume $k > 2$ and consider

$$(x')a_1a_2 = x^\dagger = (u_1', u_2', u_3, \ldots, u_k, *, \ldots, *) \sim (*, \ldots, *) = (x)a_1a_2.$$

This gives $\partial(x', x^\dagger) = 2$ and $f(2) = 2$, a contradiction. ∎

Lemma 7.2.10: *If $m > 2$ then $f(1) = 1$.*

Proof: Suppose $f(1) \neq 1$. Then by Lemma 7.2.9, $f(1) = 2$. Let

$$x = (*, \ldots, *) \sim (u_1, u_2, *, \ldots, *) = x'.$$

We can choose $b \in \Gamma_x$ so that $b^{-1}T_1b = T_3$. Then

$$(x')b = (v_1, v_2, u_1', v_4, \ldots, v_m) = x'', \quad \text{with } u_1' \neq *.$$

So $x'' \sim x$, with one v_j not equal to $*$ corresponding to u_2, say v_k. Choose $a \in S_k$ so that $(v_k)a = v_k' \neq u_k$. But then $x^\dagger = (x'')a \sim x$ and $x^\dagger$ and x' differ in 3 or 4 coordinates, while $\partial(x^\dagger, x') = 1$ or 2. As $f(1) = 2$ and $f(2) = 1$ by Lemma 7.2.9 we have a contradiction. ∎

Lemma 7.2.11: *If $f(2) = 1$ then $G^{(2)} \cong D \times D$ for some connected graph D.*

Proof: Adapting the notation of Lemma 7.2.7, let $G(i, v_{3-i})$ be the induced subgraph of $G^{(2)}$ on the set of vertices

$$X(i, v_{3-i}) = \begin{cases} \{(u, v_2) | u \in X\}; & \text{if } i = 1, \\ \{(v_1, u) | u \in X\}; & \text{if } i = 2. \end{cases}$$

Again

$$G(i, v_{3-i}) \cong G(i, v_{3-i}') \cong G(i),$$

for some graph $G(i)$, and by choosing $b \in \Gamma_x$ so that $b^{-1}T_1b = T_2$, we see that $G(1) \cong G(2) = D$. Now let E be the graph $G(1, *) \times G(2, *)$ and define a map

$$\varphi : G^{(2)} \longrightarrow E$$

by $\varphi(v_1, v_2) = (v_1, v_2)$. We again show that φ is an isomorphism. Suppose $(v_1, v_2) \sim (v_1', v_2')$ in $G^{(2)}$. As $f(2) = 1$ then either $v_1 = v_1'$ or $v_2 = v_2'$. If $v_1 = v_1'$, $v_2 \sim v_2'$ in $G(2, v_1')$ and so also in $G(2, *)$. Hence $(v_1, v_2) \sim (v_1', v_2')$ in E. Similarly if $v_2 = v_2'$. Now suppose $(v_1, v_2) \sim (v_1', v_2')$ in E. Again $v_1 = v_1'$ or $v_2 = v_2'$. If $v_1 = v_1'$ then $v_2 \sim v_2'$ in $G(2, *)$ and so $v_2 \sim v_2'$ in $G(2, v_1)$. Hence $(v_1, v_2) \sim (v_1', v_2')$ in $G^{(2)}$. Similarly if $v_2 = v_2'$. We conclude that φ is an isomorphism and that $G^{(2)} \cong D \times D$. The graph D is connected as $G^{(2)}$ is connected by the primitivity of the action of Γ. ∎

For the rest of this section D will be the graph introduced in Lemma 7.2.11. We will also need some further notation.

Definition 7.2.12: *For two vertices $x = (x_1, x_2), y = (y_1, y_2) \in VG$ we define*

$$\partial'(x, y) = \{\partial_D(x_1, y_1), \partial_D(x_2, y_2)\}.$$

This is well defined as $\partial_{G(1,*)}(x_1, y_2) = \partial_{G(1,v)}(x_1, y_2)$ for all $v \in X$.

Lemma 7.2.13: *Γ preserves the function ∂'.*

Proof: Let $g \in \Gamma$ and $x = (x_1, x_2), y = (y_1, y_2)$ in VG. Choose $t_1 \in T_1$ and $t_2 \in T_2$ so that $(x_1)t_1 = y_1$ and $(x_2)t_2 = y_2$. Then $(x)gg^{-1}t_1t_2g = (y)g$. But $g^{-1}t_1t_2g = t_1^g t_2^g$. Now let $y' = (y_1, x_2)$. Then $(x)gt_1^g = (y')g$. If $g^{-1}T_1g = T_2$ then $(x)g$ and $(y')g$ differ in the second coordinate, otherwise they differ in just the first. Let j be the coordinate in which they differ. Then $((y')g)_j = ((y)g)_j$. But $(x)g$ and $(y')g$ lie in $G(j, ((x)g)_{3-j})$ as x and y' lie in $G(1, x_2)$, while g is an isomorphism between these two copies of D. Hence

$$\begin{aligned}\partial_D(((x)g)_j, ((y)g)_j) &= \partial_D(((x)g)_j, ((y')g)_j)\\ &= \partial_D(x_1, y_1')\\ &= \partial_D(x_1, y_1).\end{aligned}$$

Applying a similar argument for $y'' = (x_1, y_2)$ we see that $\partial'(x, y) = \partial'((x)g, (y)g)$. Hence Γ preserves the function ∂'. ∎

Corollary 7.2.14: *There is a function g from the distances in G to the set of pairs of distances in D such that $g(\partial(x,y)) = \partial'(x,y)$, for all $x, y \in VG$.*

Proof: The corollary follows from the lemma and the fact that Γ acts transitively on pairs of vertices at a given distance in G. ∎

In the following lemmas the function g will be that defined in Corollary 7.2.14.

Lemma 7.2.15: *Suppose $g(i) = \{n, m\}$ with $n > 0$. Then there exists $j \neq i$ with $|j - i| \leq 2$ and $g(j) = \{n-1, m\}$.*

Proof: Consider a vertex $(u, v) \in G_i(*,*)$ with $\partial_D(*, u) = n$ and $\partial_D(*, v) = m$. Choose $u' \in VD$ so that $u' \sim u$ in D and $\partial_D(*, u') = n - 1$. Let $j = \partial((*,*),(u',v))$. Then $|j - i| \leq 2$ as $\partial((u,v),(u',v)) = 2$ and $j \neq i$ as $g(j) = \{n-1, m\} \neq g(i)$. ∎

We can consider the situation diagrammatically as a half grid with nodes indexed by pairs of distances in D. The distances of G are then assigned by the function g to the nodes of the grid. If distance i is assigned to some given node then an adjacent node nearer the origin must be assigned a distance differing from i by at most 2. Note that the node $\{0,1\}$ is assigned only the distance 2. Further note that vertices in $G_{2j+2}(x) = G^{(2)}_{j+1}(x)$ are adjacent in $G^{(2)}$ to vertices in $G_{2j}(x) = G^{(2)}_j(x)$. Hence $2j$ and $2j+2$ are assigned either to the same or adjacent nodes.

Lemma 7.2.16: *If $\mathrm{diam}(G) > 5$ then $g(2i) = \{0, i\}$.*

Proof: We use an induction argument on i including in the induction hypothesis the claim that if $g(j) = \{n, m\}$ with $n + m = i$, then $j \leq 2i$. We know that $g(2) = \{0,1\}$. Consider first $g(4)$. This cannot be $\{0,1\}$ as 2 is the only distance assigned to this node. Hence by the above $g(4) = \{0,2\}$ or $\{1,1\}$. If $g(4) = \{1,1\}$ then $g(6) = \{2,1\}$ as $g(6) \neq \{1,1\}$ by Lemma 7.2.15. Hence there exists by Lemma 7.2.15 some distance j with $|j - 6| < 3$ and $g(j) = \{0,2\}$, but then again by Lemma 7.2.15 $|j - 2| < 3$ as 2 is the only distance assigned to the node $\{0,1\}$. Hence $j = 4$ which is impossible as $g(4) = \{1,1\}$. Hence $g(4) = \{0,2\}$ and the induction hypothesis holds for $i \leq 2$. We now proceed with induction on i. Suppose the result holds for $i - 1 > 1$. Then by the above $g(2i) = \{0, i\}$ or $\{1, i-1\}$. The possibility $g(2i) = \{0, i-1\}$ is excluded as no distance assigned to $\{0, i-2\}$ can be larger than $2i - 4$ by the hypothesis. Suppose $g(2i) = \{1, i-1\}$. By Lemma 7.2.15 there exists a distance j with $|2i - j| \leq 2$ and $g(j) = \{1, i-2\}$. By the hypothesis $j \leq 2i-2$ and so $j = 2i-2$, contradicting $g(2i-2) = \{0, i-1\}$. We conclude that $g(2i) = \{0, i\}$. Finally suppose $g(j) = \{n, m\}$ with $n + m = i$ and assume without loss of generality that $n > 0$. By Lemma 7.2.15 there exists j' with $|j - j'| \leq 2$ and $g(j') = \{n-1, m\}$. As $n - 1 + m = i - 1$ we can apply the induction hypothesis giving $j' \leq 2i - 2$. But then $j \leq 2i$ as required. ∎

Corollary 7.2.17: $\mathrm{diam}(D) < 3$.

Proof: Suppose $\mathrm{diam}(D) = d \geq 3$. Then $\mathrm{diam}(G^{(2)}) = 2d$, so Γ_u has at least t orbits, where

$$t = (d+2)(d+1)/2$$

and so

$$\mathrm{diam}(G) \geq t - 1 = (d+1)(d+2)/2 - 1 > 2d + 2$$

as $d \geq 3$. But then by the lemma $g(2d+2) = \{0, d+1\}$, a contradiction. ∎

Lemma 7.2.18: $\mathrm{diam}(D) \neq 2$.

Proof: Suppose $\mathrm{diam}(D) = 2$. Then $\mathrm{diam}(G^{(2)}) = 4$, so Γ_u has at least 6 orbits and $\mathrm{diam}(G) \geq 5$. But if $\mathrm{diam}(G) \geq 6$ then $g(6) = \{0, 3\}$ by Lemma 7.2.16, contradicting $\mathrm{diam}(D) = 2$. Hence $\mathrm{diam}(G) = 5$ and each node of the half grid corresponds to exactly one distance in G. Applying Lemma 7.2.15 we can eliminate all but two possible functions g. In both cases $g(3) = \{1, 2\}$ and $g(5) = \{2, 2\}$ while $g(1)$ and $g(4)$ must be assigned between $\{1, 1\}$ and $\{0, 2\}$. If $g(1) = \{0, 2\}$, consider two vertices y, y' of D which are not adjacent. Then $(y, y') \sim (y, y) \sim (y', y)$ in G. But $\partial'((y', y), (y, y')) = \{2, 2\}$ and so $\partial((y', y), (y, y')) = g^{-1}(\{2, 2\}) = 5$ clearly contradicting their having a common neighbour (y, y). Now suppose $g(1) = \{1, 1\}$ and again choose y and y' non adjacent in D with $y \sim y'' \sim y'$. Now $(y, y) \sim (y'', y'') \sim (y, y')$ in G. But $\partial'((y, y), (y, y')) = \{0, 2\}$ and so

$$\partial((y, y), (y, y')) = g^{-1}(\{0, 2\}) = 4,$$

again contradicting their having a common neighbour (y'', y'') in G. ∎

It remains to consider the case when $\mathrm{diam}(D) = 1$ so that $G^{(2)} = K_q \times K_q$, for some $q > 4$ and $\mathrm{diam}(G) \leq 4$. By simply comparing the intersection arrays for G and $G^{(2)}$, we can show that in this case G is the complement of the Hamming graph $H(2, q)$. This completes the proof of Proposition 7.2.2. ∎

8 Distance-bitransitive Graphs

This section is concerned with distance-bitransitive graphs. We start by showing an analogous result for distance-bitransitive graphs to those in section 3 about the derived graphs of a distance-biregular graph. Next there is a subsection on imprimitive distance-biregular graphs. The central subsection is concerned with applying Praeger and Saxl's result to distance-bitransitive graphs. Finally we present some examples of distance-bitransitive graphs, both primitive and imprimitive.

It is shown in section 3 (Proposition 3.1.1) that if G is a distance-biregular graph then $G^{(2)}$ is the disjoint union of two distance-regular graphs called the derived graphs of G. The following lemma presents an analogous result for distance-bitransitive graphs.

Proposition 8.0.1 : *Let (Γ, G) be a distance-bitransitive pair. Then $G^{(2)}$ is the disjoint union of two connected graphs D and E on each of which Γ acts faithfully and distance-transitively.*

Proof : Let $A \cup B = VG$ be the bipartition of G. In $G^{(2)}$ no vertex of A is adjacent to a vertex of B. Hence $G^{(2)}$ is the disjoint union of two graphs D and E with $VD = A$ and $VE = B$. For u, u' vertices in A, $\partial_{G^{(2)}}(u, u') = \partial_G(u, u')/2$. Similarly for v, v' vertices in B. So D and E are connected graphs and Γ acts transitively on pairs at a given distance apart in both D and E. It remains to show that the action of Γ is faithful. Suppose g in Γ is the identity on D. Let v in B and $u_1, \ldots, u_s$ be the neighbours of v in G. Since g fixes $u_1, \ldots, u_s$, $(v)g$ is also adjacent to precisely $u_1, \ldots, u_s$. Suppose

$(v)g \neq v$. Considering the intersection array for v we must have:

$$\iota(B) = \begin{bmatrix} * & 1 & s \\ s & r-1 & * \end{bmatrix}.$$

So $G = K_{r,s}$ the complete bipartite graph. We cannot have $r > 1$ as this would mean that we have two lines being the same subset of the points (see Definition 1.1.8). But $r = 1$ means we have just one line, which was excluded in our definition of a distance-bitransitive pair. We conclude that g fixes every vertex of G. Hence g is the identity and Γ acts faithfully on D. Similarly Γ acts faithfully on E. ∎

This result shows that our definition of imprimitivity for a non-regular distance-bitransitive graph coincides with the definition of imprimitivity when it is viewed as a distance-biregular graph (This follows from the result of [16] mentioned in Definition 1.1.8). It is therefore natural to turn our attention to the study of imprimitivity in distance-biregular graphs.

8.1 Imprimitivity in Distance-biregular Graphs

It is known that the intersection array of an antipodal distance-regular graph is 'palindromic'. To be precise if a distance-regular graph G has intersection array

$$\begin{bmatrix} * & c_1 & \dots & c_{d-1} & c_d \\ 0 & a_1 & \dots & a_{d-1} & a_d \\ b_0 & b_1 & \dots & b_{d-1} & * \end{bmatrix}$$

then G is antipodal if and only if $b_i = c_{d-i}$, $i = 0, 1, \dots, d$, $i \neq \lfloor d/2 \rfloor$. The proof of this is in [7] though it is not explicitly stated there. This result means that one of the intersection arrays of an antipodal distance-biregular graph must be 'palindromic', as the next proposition makes explicit.

Proposition 8.1.1 : *Let G be a non regular distance-biregular graph with derived graph D on vertex set $VD = A$. Then D is antipodal if and only if $G_d(u) \neq \emptyset$ for u in A, and $\iota(A)$ satisfies $b_i = c_{d-i}$, $i = 0, 1, \dots, d$, $i \neq d/2$.*

Proof: ($\Rightarrow$) Suppose $G_d(u) = \emptyset$. Then $c_{d-1} = s$ and the derived graph D has diameter $d' = d/2 - 1$, as d is even by Lemma 3.3.2. Let D have intersection array:

$$\iota(D) = \begin{bmatrix} * & c'_1 & \dots & c'_{d'} \\ 0 & a'_1 & \dots & a'_{d'} \\ b'_0 & b'_1 & \dots & * \end{bmatrix}.$$

Then

$$a'_{d'} = \frac{(c_{d-2}(b_{d-3} - 1) + b_{d-2}(c_{d-1} - 1))}{c_2}$$

(see Proposition 3.1.1).

But $c_{d-1} = s > 1$. Hence $a'_{d'} > 0$ and so $b'_0 \neq c'_{d'}$, and D is not antipodal. We conclude that $G_d(u) \neq \emptyset$. Suppose now that $b_j = c_{d-j}$ for $j < i$, for some i, $1 \leq i < d/2$. This is true for $i = 1$, as $b_0 = c_d = r$. We consider the possible parities of i separately.

case (i) : i odd. Here

$$\begin{aligned} b_{i-1}b_i/c_2 = b'_{(i-1)/2} &= c'_{d/2-(i-1)/2} \\ &= c'_{(d-i+1)/2} = c_{d-i+1}c_{d-i}/c_2, \\ \text{as} \quad (i-1)/2 \neq \lfloor d'/2 \rfloor. \end{aligned}$$

But $b_{i-1} = c_{d-i+1}$ and so $b_i = c_{d-i}$ as required.

case (ii) : i even. Here

$$\begin{aligned} c_i c_{i-1}/c_2 = c'_{i/2} &= b'_{d/2-i/2} \\ &= b'_{(d-i)/2} = b_{d-i}b_{d-i+1}/c_2, \\ \text{as} \quad d/2 - i/2 \neq \lfloor d'/2 \rfloor. \end{aligned}$$

But $b_{i-1} = c_{d-i+1}$ so $c_{i-1} = b_{d-i+1}$ as

$$b_{i-1} + c_{i-1} = s = c_{d-i+1} + b_{d-i+1}.$$

We conclude that $c_i = b_{d-i}$ and so $b_i = c_{d-i}$. The result follows by induction.

($\Leftarrow$) Let $d' = d/2$, the diameter of D as $G_d(u) \neq \emptyset$. Then

$$\begin{aligned} b'_j = b_{2j}b_{2j+1}/c_2 &= c_{d-2j}c_{d-2j+1}/c_2 \\ &= c_{d'-j}, \quad j = 0, 1, \dots, d', \ j \neq \lfloor d'/2 \rfloor. \end{aligned}$$

■

We conclude this subsection by showing that both derived graphs of a non-regular distance-biregular graph cannot be imprimitive.

Proposition 8.1.2 : *Let G be a non regular distance-biregular graph. Then at least one of the derived graphs is primitive. Suppose the derived graph E is imprimitive. Then one of the following holds.*

(a) G is the subdivision graph of E, which is a bipartite (k,g)-graph,

(b) E is an antipodal, non bipartite graph with $diam(E) \geq diam(D)$.

Proof: We consider first the case when G has vertices of valency two.

Case (i) : G has vertices of valency 2. By Corollary 4.3.3 we know that in this case G is the subdivision graph of one of its derived graphs, which is a (k,g)-graph or $G = K_{2,r}$. The latter graph is primitive if $r \neq 2$, so we must consider only the first possibility. Let E be the derived graph satisfying $G = S(E)$. Then in the standard notation $r = 2$ and $s = k$ the degree of E. The intersection array of the second derived graph D may be computed as:

$$\begin{bmatrix} * & 1 & \dots & 1 & 1 & 4 \\ 0 & k-2 & \dots & k-2 & k-1 & 2(k-3) \\ 2(k-1) & k-1 & \dots & k-1 & k-2 & * \end{bmatrix}$$

if g is odd, while if g is even it is

$$\begin{bmatrix} * & 1 & \dots & 1 & 2 \\ 0 & k-2 & \dots & k-2 & 2(k-2) \\ 2(k-1) & k-1 & \dots & k-1 & * \end{bmatrix}.$$

In no case is D bipartite, as we must have $k = s > 2 = r$ for the non regularity of G. The only case when the array is antipodal is when $k = 3, g = 3$. This means that E is K_4 and $G = S(K_4)$. Here E is primitive while D is antipodal and non bipartite with $diam(D) > diam(E)$. This is case (b) of the proposition, with D and E interchanged. For all other values of k and g, D is primitive, while the (k,g)-graph E is imprimitive only if bipartite (g even). This is case (a) of the proposition.

Case (ii) : G has no vertices of valency 2. It is immediate that both derived graphs contain triangles and so neither is bipartite. If derived graph E is antipodal, then $diam(E) = d/2$ by Proposition 8.1.1. But for the second derived graph D, $diam(D) \leq$

$d/2$ and so $diam(E) \geq diam(D)$. Hence it remains to prove that both derived graphs cannot be antipodal. Suppose this to be the case. By Proposition 8.1.1 both intersection arrays for G are 'palindromic' with $G_d(u)$ and $G_d(v)$ non empty for u in A, v in B. Let

$$\iota(A) = \begin{bmatrix} * & c_1 & \dots & c_{\ell-1} & c_\ell & b_{\ell-1} & \dots & b_1 & r \\ r & b_1 & \dots & b_{\ell-1} & b_\ell & c_{\ell-1} & \dots & c_1 & * \end{bmatrix}$$

and

$$\iota(B) = \begin{bmatrix} * & f_1 & \dots & f_{\ell-1} & f_\ell & e_{\ell-1} & \dots & e_1 & s \\ s & e_1 & \dots & e_{\ell-1} & e_\ell & f_{\ell-1} & \dots & f_1 & * \end{bmatrix},$$

where $\ell = d/2$. Consider first ℓ odd. Here by Proposition 8.1.1 and Lemma 3.2.2 $b_\ell c_{\ell-1} = e_\ell f_{\ell-1}$ and $c_\ell c_{\ell-1} = f_\ell f_{\ell-1}$. Adding we obtain

$$c_{\ell-1}(b_\ell + c_\ell) = f_{\ell-1}(e_\ell + f_\ell)$$

and so $f_{\ell-1}/c_{\ell-1} = s/r$. But then

$$b_\ell/e_\ell = f_{\ell-1}/c_{\ell-1} = s/r.$$

For ℓ even $b_{\ell-1}b_\ell = e_{\ell-1}e_\ell$ and $b_{\ell-1}c_\ell = e_{\ell-1}f_\ell$, by Lemma 3.2.2. Adding we have

$$b_{\ell-1}(c_\ell + b_\ell) = e_{\ell-1}(e_\ell + f_\ell)$$

and so $b_{\ell-1}/e_{\ell-1} = s/r$. Now suppose that for some $2i+1 \leq \ell$, $b_{2i+1}/e_{2i+1} = s/r$. As $c_{2i+1} + b_{2i+1} = s$ and $e_{2i+1} + f_{2i+1} = r$, we have

$$c_{2i+1}/f_{2i+1} = (s - b_{2i+1})/(r - e_{2i+1}) = s/r.$$

Then as

$$c_{2i+1}c_{2i} = f_{2i+1}f_{2i}$$

$$f_{2i}/c_{2i} = c_{2i+1}/f_{2i+1} = s/r$$

and as $e_{2i} + f_{2i} = s$, while $b_{2i} + c_{2i} = r$, we have

$$e_{2i}/b_{2i} = (s - f_{2i})/(r - e_{2i}) = s/r.$$

Further as

$$b_{2i-1}b_{2i} = e_{2i-1}e_{2i}$$

$$b_{2i-1}/e_{2i-1} = e_{2i}/b_{2i} = s/r.$$

Hence by induction $b_1/e_1 = (s-1)/(r-1) = s/r$ and so $r = s$, a contradiction. ∎

8.2 Automorphism Groups of Distance-bitransitive Graphs

We now state and prove the main result of this section.

Theorem 8.2.1: *If (Γ, G) is a primitive distance-bitransitive pair and G is not regular, then Γ is almost simple.*

Proof: By Proposition 8.0.1, Γ acts distance-transitively (and faithfully) on each of the derived graphs D and E of G. As G is primitive, so are both derived graphs and we can apply Theorem [15] 7.0.1 to each of the graphs D and E. We consider the three possible cases for the graph D:

(a) $D \cong H(d,q)$ or $D \cong H(2,q)^c$. By Proposition 6.1.1 and Proposition 6.1.2 this cannot occur if G is primitive.

(b) Γ is almost simple.

(c) (Γ, VD) is affine. In this case the socle N of Γ acts regularly on VD (see Proposition 7.1.1) and so $|VD| = |N|$. But consider the action of Γ on E. As Γ is not almost simple and we can exclude the case when E is of Hamming type, (Γ, VE) is also affine and so $|VE| = |N|$. But then $|VD| = |VE|$ and so G is regular. ∎

8.3 Examples of Distance-bitransitive Graphs

We conclude with some examples of distance-bitransitive graphs some of which are primitive.

Example 8.3.1 : *The q-analoque Johnson biregular Graphs*

Consider a vector space V of dimension m over the Galois field $GF(q)$, where q is a prime power. The vertices of the graph $G = JB_q(k,m)$ are the k-dimensional and $(k+1)$-dimensional subspaces of V with (X,Y) an edge in G if $X \subset Y$. The group $PGL(m,q)$ acts as a group of automorphisms on G and it is not hard to check that $(PGL(m,q),G)$ is a distance-bitransitive pair. The group $PGL(m,q)$ has simple socle $PSL(m,q)$. ∎

Example 8.3.2 : *A Distance-bitransitive Graph in $PG(2,4)$*

For the definition of this graph see Example 1.2.10. The group $PSL(3,4)$ is a group of automorphisms of G as it is the vertex stabiliser of M_{22}, the automorphism group of the 3-(22,6,1) design. Using the fact that $PSL(3,4)$ acts transitively on quadruples of points, exactly three of which are collinear and that three non collinear points uniquely determine an oval vertex adjacent to them in G, we can check that the simple group $PSL(3,4)$ acts distance-bitransitively. ∎

We finish by showing that Example 5.3.1 is actually distance-bitransitive.

Proposition 8.3.3 : *Example 5.3.1 is an antipodal distance-bitransitive graph.*

Proof : We first consider what automorphisms the graph G of Example 5.3.1 has. We consider them initially as automorphisms of the vector space A. Consider any non-zero vector $\mathbf{u} \in A$, and the translation

$$\tau_{\mathbf{u}} : A \longrightarrow A,$$

given by $\tau_{\mathbf{u}}(\mathbf{a}) = \mathbf{u} + \mathbf{a}$. We extend this map to B by defining $\tau_{\mathbf{u}}(x,i) = (x + \mathbf{u}_i, i)$. This preserves adjacency in G and so defines an automorphism. There is a one to one correspondence between the elements of A and the translations. Hence we can view A as a set of automorphisms of G. This set is a subgroup acting transitively on the vector space A. Now consider the zero vector $\mathbf{0} \in A$, and the subgroup H of $\mathrm{Aut}(G)$ which fixes this vector. We have $\mathrm{Aut}(G) = H \rtimes A$.

Let $\alpha \in GL(k,q)$ map the basis $\mathbf{b}_1,\ldots,\mathbf{b}_k$ to the basis $\mathbf{b}'_1,\ldots,\mathbf{b}'_k$. No two columns of the matrix $\mathbf{M}'$ obtained by taking $\mathbf{b}'_1,\ldots,\mathbf{b}'_k$ as rows are linearly dependent as otherwise those two columns would be linearly dependent in $\mathbf{M}$. Hence the columns of $\mathbf{M}'$ are a set of t pairwise independent k-vectors and so must be a set of representatives of the projective points of $PG(q,k-1)$. Hence there exists a permutation matrix $\mathbf{S}$ of order t and scalars $\tau_1,\ldots,\tau_t$ such that

$$\mathbf{M}\mathbf{diag}(\tau_1,\ldots,\tau_t)\mathbf{S} = \mathbf{M}'.$$

So by applying $\mathbf{diag}(\tau_1,\ldots,\tau_t)\mathbf{S}$ to B in the obvious fashion (if $\mathbf{S}$ corresponds to $\sigma \in \mathrm{Sym}(t)$ then $(x,i) \mapsto (\tau_i x, \sigma(i))$), we can extend α to an automorphism $\hat{\alpha}$ of G. Hence $GL(k,q) \leq H$ and $\Gamma = A \rtimes GL(k,q)$ is a subgroup of $\mathrm{Aut}(G)$.

Consider our original choice of projective point representatives. It is pointed out in the definition that the choice does not affect the graph obtained. Hence we can without loss of generality assume that $\mathbf{p}_1$ was chosen as $[0,\ldots,0,1]^T$ and $\mathbf{p}_2$ as $[0,\ldots,1,0]^T$. Assume that $\mathbf{p}_i = [x_1,\ldots,x_k]^T$, for some $i > 1$. As $\mathbf{p}_1$ and $\mathbf{p}_i$ are not parallel there exists some $\ell < k$ with $x_\ell > 0$. We will choose a specific $\alpha_i \in GL(k,q)$ and show that then $\hat{\alpha}_i$ maps B_1 to B_i. This will prove useful later. The transformation α will map $\mathbf{b}_1,\ldots,\mathbf{b}_k$ to $\mathbf{b}'_1,\ldots,\mathbf{b}'_k$, where

$$\begin{aligned}
\mathbf{b}'_j &= \mathbf{b}_j - x_j x_\ell^{-1}\mathbf{b}_\ell, \quad j \neq \ell, k \\
\mathbf{b}'_k &= a x_\ell^{-1}\mathbf{b}_\ell \\
\mathbf{b}'_\ell &= \mathbf{b}_k - x_k x_\ell^{-1}\mathbf{b}_\ell.
\end{aligned}$$

This defines a non-singular transformation and the corresponding matrix $\mathbf{M}'$ has i-th column equal to $a\mathbf{p}_1$. Hence the corresponding permutation matrix $\mathbf{S}$ must map 1 to i and the corresponding $\tau_1 = a$.

We will choose another set of special transformations from $GL(k,q)$. Again consider $\mathbf{p}_i = [x_1,\ldots,x_k]^T$, this time for some $i > 2$. Consider first the case when there exists some $\ell < k-1$ with $x_\ell > 0$. In this case we choose $\beta_i \in GL(k,q)$ to map $\mathbf{b}_1, \ldots \mathbf{b}_k$ to $\mathbf{b}'_1, \ldots, \mathbf{b}'_k$, where

$$\begin{aligned} \mathbf{b}'_j &= \mathbf{b}_j x_j x_\ell^{-1} \mathbf{b}_\ell, \quad j \neq \ell, k-1 \\ \mathbf{b}'_{k-1} &= a x_\ell^{-1} \mathbf{b}_\ell \\ \mathbf{b}'_\ell &= \mathbf{b}_{k-1} - x_{k-1} x_\ell^{-1} \mathbf{b}_\ell. \end{aligned}$$

This defines a non-singular transformation and the i-th column of $\mathbf{M}'$ is $a\mathbf{p}_2$ while the first column is $\mathbf{p}_1$. Hence the corresponding permutation matrix $\mathbf{S}$ maps 2 to i, while fixing 1 and $\tau_2 = a$. If $x_j = 0$ for all $j < k-1$ then both x_{k-1} and x_k must be non-zero. In this case choose $\mathbf{b}'_j = \mathbf{b}_j$ for $j < k-1$, $\mathbf{b}'_{k-1} = a x_{k-1}^{-1} \mathbf{b}_{k-1}$ and $\mathbf{b}'_k = \mathbf{b}_k - x_k x_{k-1}^{-1} \mathbf{b}_{k-1}$. Then in the matrix $\mathbf{M}'$ again the i-th column is $a\mathbf{p}_2$ while the first column is $\mathbf{p}_1$. Hence the corresponding permutation matrix again maps 2 to i while fixing 1, with $\tau_2 = a$.

We now show that (Γ, G) is a distance-bitransitive pair. As Γ acts transitively on A we need consider first Γ_0 the vertex stabiliser of the $\mathbf{0}$ vector in A and show that it acts transitively on vertices at distance 1,2 and 3 from $\mathbf{0}$. The vertices at distance 1 from $\mathbf{0}$ are $(0,i) \in B$, $i = 1, \ldots, t$. As the automorphism $\hat{\alpha}_i$ takes $(0,1)$ to $(0,i)$, while fixing $\mathbf{0}$, it follows that Γ_0 acts transitively on $G_1(\mathbf{0})$.

Next consider vertices at distance 2 from $\mathbf{0}$. These are all the non zero vectors of A. As $GL(k,q)$ acts transitively on them, so does Γ_0.

For the vertex $\mathbf{0}$ we must lastly consider vertices at distance 3. These are of the form (x,i), with $x \neq 0$. The automorphism $\hat{\alpha}_i$ maps the vertex $(1,1)$ to (a,i), while fixing $\mathbf{0}$. By choosing all non-zero values for a we obtain the required transitivity. We conclude that $GL(k,q)$ and so also Γ_0 acts transitively on $G_3(\mathbf{0})$.

We have seen that Γ_0 acts transitively on vertices in B with 1-st coordinate 0 and on those with first coordinate non-zero. To map a vertex with zero first coordinate to one with non-zero first coordinate we can use a translation by any non-zero $\mathbf{u} \in A$. Hence Γ acts transitively on B. Consider $v = (0,1) \in B$ and the subgroup Γ_v of Γ fixing v. To complete our proof we must show that Γ_v acts transitively on vertices of $G_i(v)$, $i = 1,2,3$ and 4. The vertices in $G(v)$ are those whose first coordinate is zero. Clearly $\mathbf{0} \in A$ is one such vertex. Let $\mathbf{u} \in A$ be any other such vertex. Then translation by $\mathbf{u}$ takes $\mathbf{0}$ to $\mathbf{u}$ while fixing v. Hence Γ_v acts transitively on $G(v)$.

Vertices in $G_2(v)$ have the form (x,i) with $i \neq 1$. The automorphism $\hat{\beta}_i$ fixes column 1 (and so $v = (0,1)$) while mapping $(1,2)$ to (a,i) and $(0,2)$ to $(0,i)$. This shows that Γ_v acts transitively on vertices of the form (x,i), $x \neq 0$, $i \neq 1$ and also on vertices of the form $(0,i)$, $i \neq 1$. By choosing $\mathbf{u} \in A$ non-zero with $\mathbf{u}_1 = 0$ we can also map a member of the first set to a member of the second while fixing v. Hence Γ_v acts transitively on $G_3(v)$. Finally $G_4(v)$ are vertices of the form $(x,1)$, $x \neq 0$. By choosing $\gamma \in GL(k,q)$ to be scalar multiplication by a the automorphism $\hat{\gamma}$ maps $(1,1)$ to $(a,1)$ while fixing $(0,1)$. Hence Γ_v acts transitively on $G_4(v)$ completing the proof that the pair (Γ, G) is distance-bitransitive. Hence G is a distance-bitransitive graph. ■

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