THE SHEAF CONSTRUCTION AND ITS APPLICATIONS

BY

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To my parents and to Esther.

ABSTRACT

This thesis deals with the theory of sheaves and its applications.

Definitions and constructions of sheaves are given and used to represent algebraic structures, reduced products and limit reduced products of $\mathcal{L}$ -structures. The notion of forcing in sheaves is discussed and used with the representation theorems to derive Łoś' theorem.

The notion of h-limit theories is given and used to prove that a theory is an h-limit theory iff it is invariant under global sections. Some other preservation theorems for first order sentences are proved. These results are applied to derive some model theoretical properties of some classes of rings.

CONTENTS

INTRODUCTION

In recent years the theory of sheaves has played an increasingly important role in mathematics.

It has been used in the study of functions with complex variables (cf. Cartan [53]), in the study of algebraic geometry, Serre [55] and algebraic topology, Godement [58].

These methods proved to be very useful in studying rings, since many interesting rings can be constructed as sheaves of irreducible rings; this generalises the construction of rings as products of irreducibles. R.S.Pierce [67], Dauns and Hoffman [66] and Keimel [71] used sheaves in this setting to study regular, biregular and lattice ordered rings.

The representation of regular rings given by Pierce was used by Lipshitz and Saracino [73] and Carson [73] to prove that the theory of commutative rings with no non-zero nilpotent elements has a model-companion.

This application of sheaves to model theory led to the study of the model theoretical properties of sheaves. (cf. Macintyre [73], Ellerman [74], Mansfield [76], Volger [79], Weispfenning [78])

In this thesis we shall discuss the notion of sheaves, their construction and some of their applications in Model theory.

In Chapter 1 we discuss the notions of s-sheaf (stalk-structures) p-sheaf (sheaves constructed as presheaves) and translate Ellerman's constructions of prime and ultrasheaves into s-sheaves.

In Chapter 2, Section 1. collects some known results on the structure of the lattice of congruences on algebras. Section 2. defines a rather general class of algebras which can be decomposed as sheaves of irreducibles. We present Wolf's construction of s-sheaves of algebras with permutable distributive congruence lattices and deduce (a corrected version of) Swamy's result *[theorem 2.5]*. In Section 3. we give some algebraic applications of the previous construction. Section 4 presents a representation theorem for limit reduced products of $\mathcal{L}$-structures by S-sheaves (theorem 4.10).

In addition to proving a preservation theorem on sheaves due to Volger *[79]* and using some of these results to extend some of Keisler's, Hodges and Shelah's results to sheaves, we translate in Chapter 3 Ellerman's notion of forcing to S-sheaves, and present a proof to Łoś's type theorems (1.D.3 and 1.E.1).

Chapter 4 deals with examples of s-sheaves. In the first part we deduce Łoś's theorem using theorem 1.E.1. Among other results in part 2, we discuss Carson's proof of theorem 2.22 and give a counter example to Macintyre's proof of theorem 2.11.

Some topological notions and facts as well as some model-theoretical notions are assumed to be known. The reader is referred to Kelly *[57]*, Rasiowa and Sikorski *[63]*, Bell and Slomson *[69]* and Chang and Keisler *[73]*.

The notation used is the standard notation. Unless otherwise stated, a language $\mathcal{L}$ means a first order language.

We sometimes used $\bar{v}$ and $\bar{a}$ to indicate sequences of variables and elements of the same length. We use the same letters to indicate $\mathcal{L}$-structures and their domains.

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CHAPTER 1

SHEAVES DEFINITIONS AND CONSTRUCTION

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In recent years the theory of sheaves provided mathematicians with a powerful tool to study mathematical objects. It has been used in studying algebraic geometry Godement (58). This has led to a category theoretical notion of sheaves. Later on J.P.Serre (55) discovered that sheaves can be used to study algebraic structures; which in turn led to a series of representation theorems of algebraic structures as global section structures. This study needed a different approach to sheaves. In 1974 Ellerman used sheaves to generalize the construction of ultra products. This construction provides us with a method of constructing generic structures as we shall see in later chapters.

This chapter will be concerned in providing the basic tool of our discussion. In 1. we study the stalk-spaces. In 2. we depart from our original discussion to define the directed limits of   -structures. In 3. we discuss the presheaves and the category theoretical definition of sheaves. In 4. we show that the stalk-space and presheaf approaches to sheaves are equivalent in some sense. In 5. and 6. we shall be concerned with generalized constructions of ultraproducts mainly the construction of prime and ultra-sheaves.

1. Stalk-spaces

In what follows X will denote a topological space and $\mathcal{O}(X)$ is the set of open subsets of X.

Def 1.1: By a stalk-space of sets over X we mean a pair (S, π) of a topological space S and a map π from S onto X such that:

(SS 1) - π is a local homeomorphism

Remarks: 1 - For every $x \in X$; $\pi^{-1}(x) \neq \phi$. let $S_x = \pi^{-1}(x)$

S_x is called the stalk of S at x.

2. $S = \bigsqcup_{x \in X} S_x$

3. Condition 1 means that: For every $s \in S$ there exist open subsets V_s and N_s of S and X so that:

i - $s \in V_s$

ii - $\pi(s) \in N_s$

iii - $\pi_{/V_s}$ is a continuous and open bijection from V_s onto N_s.

Def.1.2: Let (S, π) be a stalk-space of sets over X and N be a subset of X. Endow N with the restricted topology, then we say that the map f from N into S is a section over N iff f is continuous and $\pi \circ f = 1_N$. We write $\Gamma(N,S)$ for the set of sections over N. In particular if $N = X$ then f is called a global section and $\Gamma(X,S)$ is the set of all global sections over X. The relation between the sections over open subsets of X and the topology on S is given in the following proposition:

prop 1.3: the set $B = \{ f(N);\, N \in \mathcal{O}(X);\, f \in \Gamma(N,S) \}$ forms a basis for the topology on S.

pf: i - $\emptyset \in B$ trivially.

ii - Let N be in $\mathcal{O}(X)$, $f \in \Gamma(N,S)$ consider $x \in N$ and $s \in S$ so that $s = f(x)$) then there is V_s and N_s open subsets of S and X so that $\pi : V_s \to N_s$ is a homeomorphism and $s \in V_s$ but since $\pi \circ f(x) = x = \pi(s)$ then $x \in N_s$. Let $M = N_s \cap N$ this is an open subset of X and $x \in M$ moreover $\pi \circ f/_M = 1_M$ and $\pi^{-1}_{/M}(x) \in \pi^{-1}(M)$ i.e. $s \in \pi^{-1}(M)$ but by the choice of N_s, $\pi^{-1}(M)$ must be open in V_s hence open in $\mathcal{S}$. Now remark that: $f(M) = \pi^{-1}(M) \subseteq f(N)$. And hence $f(N)$ is a neighbourhood of all its elements so it must be open.

Now let V be any open subset of S we ~~need to show~~ claim that $V = \bigcup_i f_i(N_i)$

where $f_i(N_i)$'s are in B. For; for every $s \in V$ let x be $\pi(s)$ and choose open sets V_s and N_s so that $s \in V_s$; $\pi_{/V_s} : V_s \to N_s$ is a homeomorphism.

Consider $V_s \cap V = U_s$ $\pi_{/V_s}(U_s) = N_s'$ and let f_s be π^{-1}/N_s' is a section over N_s' and $f_s(N_s') = U_s$

But $V = \bigcup_{s\in V} (V_s \cap V) = \bigcup_{s\in V} U_s = \bigcup_{s\in V} f_s(N_s')$.

<u>prop 1.4</u>: Let N_1, N_2 be elements of $\mathcal{O}(X)$, $f_1 \in \Gamma(N_1, S)$ and $f_2 \in \Gamma(N_2, S)$ so that for some $x \in N_1 \cap N_2$; $f_1(x) = f_2(x)$; then there is an open subset N of X so that $N \subsetneq N_1 \cap N_2$; $x \in N$ and $f_{1/N} = f_{2/N}$

<u>pf:</u> Let $s = f_1(x) = f_2(x)$ so by definition of (S, π) there is V_s and N_s so that $\pi_{/V_s} : V_s \to N_s$ is a homomeomorphism; but $\pi_{/V_s}(s) = \pi \circ f_1(x) = \pi \circ f_2(x) = x$ so $x \in N_s$. Let $N = N_1 \cap N_2 \cap N_s$ by the choice of N_s we have $f_{1/N} = \pi^{-1}{}_{/N} = f_{2/N}$.

Given (S, π) and (S', π') as stalk-spaces over X we wish to define morphisms between them so that we can talk about the category of stalk-spaces.

<u>Def.1.5</u>: We say that a map $\alpha : S \longrightarrow S'$ is ~~said to be~~ a homomorphism from (S, π) into (S', π') iff α is continuous and $\pi' \circ \alpha = \pi$

It is easy to see that the collection of stalk-spaces over X together with the homomorphism defined above form a category. We shall write <u>SS(X)</u> for this category.

Our next aim is to show that this category admits finite "products". We snall do that in 2 steps; first define what we shall call finite sum of stalk-spaces and then prove that the following hold:

(C i) - If $\sum_1^n S_i$ denotes the sum of $(S_i, \pi_i)_{i \leq n}$ then for each i there is a morphism $P_i : \sum_1^n S_i \longrightarrow S_i$

(C ii) - If (S, π) is a stalk-space over X and $f_i : S \longrightarrow S_i$ $i = 1,\cdots,n$ are morphisms then there is a unique morphism $f : S \to \Sigma S_i$ so that $P_i \circ f = f_i$ for all $i = 1,\cdots,n$. So let $(S_i, \pi_i)_{i\in I}$ be a finite family of stalk-spaces. Let $T = \{ (s_i)_{i\in I} ; \pi_i(s_i) = \pi_j(s_j) \ \forall i, j \in I \}$.

We note that $T \subset \prod S_i$ (the cartesian product of S_i's). So we may endow T with the induced product topology. We can define $\theta : T \to X$ so as $\theta((s_i)) = \pi_i(s_i)$ the next proposition shows that (T, θ) is a stalk space over X.

<u>prop 1.6</u>: (T, θ) is a stalk-space over X

<u>pf</u>: We have to verify condition (SS 1). We start by showing that θ is onto. For; since π_i is onto for every $i = 1,\cdots,n$ we may consider for $x \in X$ an element $s_i \in S_i$ so that $\pi_i(s_i) = x$ take $t = (s_i)$

thus: $\theta(t) = \pi_i(s_i) = x$.

Now we show that θ is a local homeomorphism. We first make the following remark: let $t = (s_i)$ be an element of T; since each π_i is a local homeomorphism then for each $i = 1,\cdots,n$ there are V_{si} and N_{si} open subsets of S_i and X so that π_i/V_{s_i} is a homeomorphism from V_{si} onto N_{si}. We consider: $V_t = T \cap (\prod_i V_{s_i})$ and $N_t = \bigcap_i N_{si}$ those are open subsets of T and X respectively. Let t' be an element of V_t so $t' = (s'_i)$ so that $s'_i \in V_{s_i}$

for each i, and $\pi_i(s'_i) = \pi_j(s'_j)$ for all $i, j = 1,\cdots,n$.

Thus $\theta(t') = \pi_i(s'_i) = \pi_j(s'_j)$.

for all $i, j = 1,\cdots,n$ i.e. $\theta(t') \in N_{s_i}$ for all $i = 1,\cdots,n$ so $\theta(t') \in N_t$

We claim that $\theta/V_t : V_t \longrightarrow N_t$ is a homeomorphism.

<u>i - $\theta/_{V_t}$ is a surjection</u>: Let $x \in N_t$ thus there are $s'_i \in V_{s_i}$ so that

$\pi_i/V_{s_i}(s'_i) = x$ for all $i = 1,\cdots,n$. Let $t = (s'_i)$

By choice of t' we have $t' \in V_t$ and $\theta(t') = x$.

<u>ii - $\theta_{/V_t}$ is injective:</u> Let $t' = (s_i')$ and $t'' = (s_i'')$ be elements of V_t so that $\theta/V_t(t') = \theta/V_t(t'')$, that is to say $\pi_i(s_i') = \pi_i(s_i'')$ for all $i = 1, \cdots, n$ but π_i/V_{s_i} 's are injective so $s_i' = s_i''$ for all $i = 1, \cdots, n$ thus $t' = t''$.

<u>iii - $\theta_{/V_t}$ is continuous:</u> Let $t' = (s_i')$ be an element of V_t and $x = \theta(t')$. Let $U \subseteq N_t$ be open subset so that $x \in U$. Since every π_i/V_{s_i} is continuous for every $i = 1, \cdots, n$ then there are M_i open subsets of V_{si} so that $s_i' \in M_i$ and $\pi_i/_{V_{s_i}}(M_i) \subseteq U$. Let $M' = (\prod_i M_i) \cap T$

M' is an open subset of V_t and $\theta/V_t(M') \subseteq U$

<u>iv - $\theta_{/V_t}$ is an open map:</u> Let V' be an open subset of T so

$V' = T \cap \prod_i V_i'$ Now $\theta_{/V_t}(V') = \bigcap_i \pi_i/_{V_{s_i}}(V_i')$

because: if $t' \in V'$ then $\theta/V_t(t') = \pi_i(s_i')$

for all i so $\theta/V_t(t') \in \pi_i/V_{s_i}(V_i')$ for all $i = 1, \cdots, n$.

Conversely if $x \in \bigcap_i \pi_i/V_{s_i}(V_i')$; then as before there is $t' \in V'$ so that $\theta_{/V_t}(t') = x$. But each π_i/V_{s_i} is an open map.

So $\theta_{/V_t}(V')$ is a finite intersection of open subsets so it must be open. Q.E.D.

The next proposition shows that the above defined (T, θ) is the finite product of the (S_i, π_i) in the category <u>SS(X)</u>.

<u>prop 1.7</u>: Let $(S_i, \pi_i)_{i \leq n}$ and (T, θ) be as above then (T, θ) is the product of the $(S_i, \pi_i)_{i \leq n}$ in <u>SS(X)</u>.

<u>pf:</u> We need to show that the conditions (C-i) and (C-ii) hold for (T, θ)

Let $P_i : T \to S_i$ be the usual projection i.e. $P_i((s_i)) = s_i$

It is easily verified that for every i P_i is continuous and so condition (C - i) holds.

Now let (S, π) be a stalk space over X and $f_i : S \to S_i$ be homomorphisms from (S, π) into (S_i, π_i)

Define $f : S \longrightarrow T$ as $f(s) = (f_i(s))$ We claim that f is homomorphism of stalk-spaces.

pf. of the claim:

i - f is well defined, in fact we have $\pi_i \circ f_i = \pi$ for all $i = 1, \ldots, n$ thus $\pi_i \circ f_i(s) = \pi_j \circ f_j(s)$ and so $f(s) \in T$ since the f_i's are well defined so f must be well defined as well.

ii - f is continuous. This follows from the fact that the f_i's are continuous and we may repeat an argument similar to the one used in prop. 1.6.

iii - $\theta \circ f = \pi$ in fact $\theta \circ f(s) = \pi_i \circ f_i(s) = \pi(s)$ for every $s \in S$ and every $i = 1, \ldots, n$; so f is a homomorphism.

Now we claim that $P_i \circ f = f_i$ Let $s \in S$, $P_i \circ f(s) = P_i((f_i(s))) = f_i(s)$

From this we deduce that f is unique for say f' has the above property so $f'(s) = (s'_i)$ but $P_i \circ f'(s) = s'_i = f_i(s)$ for every i so $(s'_i) = (f_i(s))$ and $f'(s) = f(s)$ so $f' = f$ and thus condition (C-ii) holds Q.E.D.

Remarks 1.8: i - If X contains more than 1 element then the finite sums of stalk-spaces are proper subset of the cartesian products.

ii - The construction given in prop.1.6 need not work for infinite family of stalk-spaces. For we give an example to show that:

Let $X = \{x\}$ Let S be a set with more than 1-element. Topologize S with the discrete topology, let $\pi : S \to X$ be the constant map. Thus (S,π) is a stalk-structure. Now consider an infinite number of copies of $(S,\pi)_{i\in I}$ Now the sum of $(S,\pi)_{i\in I}$ is the cartesian power of Card.I-copies of S. Let $T = \prod_{i\in I} S$ the topology on T is not the discrete topology since its basis $\mathcal{B}$ is of the form $\prod_{i\in I} A_i$ where all of the A_i's but a finite number of them are equal to S. Thus clearly (T, θ) is not a stalk space.

iii - Let κ be any cardinal. We say that a topological space X is κ -complete (resp. $<\kappa$ -complete) iff whenever $(\mathcal{O}_i)_{i\in I}$ is a family of $\leq\kappa$ (resp. $<\kappa$) elements of $\mathcal{O}(X)$ then $\bigcap \mathcal{O}_i \in \mathcal{O}(X)$ Let X be a κ -complete topological space. And assume that we endow the cartesian product of S_i's (where (S_i, π_i) is a stalk-space over X) with the Box topology (i.e. the topology generated by sets of the form $\prod_{i\in I} A_i$ where $|I| \leq\kappa$ and all the A_i open subsets of S_i). Then in <u>SS(X)</u> products of κ -elements exist and so does their sums.

Given (S, π) a stalk-space over X we wish to discuss the case when every $\pi^{-1}(x)$ $(=S_x)$ is an $\mathcal{L}$ -structure. To do that we need to introduce some notation.

$\mathcal{L}$ will be a fixed first order Language with relation, function, and constant symbols.

$\underset{\sim}{2}$ will denote the 2 elements Boolean algebra endowed with Sierpinski's topology (i.e. the topology whose open sets are $\underset{\sim}{2}$, ϕ, and $\{1\}$) unless otherwise stated.

We also denote by R_x (resp. F_x , C_x) the interpretation of R (resp. F, C) in S_x .

Let F be a function symbol of $\mathcal{L}$. (S,π) a stalk-space over X with S_x is an $\mathcal{L}$-structure for every $x \in X$. If F is an n-ary function symbol we may define a function F_S from $\sum_1^n S$ into S as follows:

$$F_S((s_1,\dots,s_n)) = F_{\pi(s_i)}((s_1,\dots,s_n)).$$

Similarly if R is a n-ary relation symbol we may define :

$$\chi_R : \sum_1^n S \longrightarrow \underset{\sim}{2} \quad \text{as } \chi_R((s_1,\dots,s_n)) = 1 \quad \text{iff } S_{\pi(s_i)} \models R_{\pi(s_i)}(s_1,\dots,s_n)$$

Using this notation we are now in position of defining the S-sheaf of $\mathcal{L}$ -structures over X.

<u>Def 1.9</u>: Let (S,π) be a stalk-space over X. Then (S,π) is said to be an S -sheaf of $\mathcal{L}$ -structures over X iff:

i - Every $S_x = \pi^{-1}(x)$ is the domain of an $\mathcal{L}$ -structure S_x

ii - For every function symbol F the map F_S is continuous

iii - For every relation symbol R the map χ_R is continuous.

We again consider $\Gamma(N,S)$ the set of sections over N - (N is a subset of X) - let us first note:

1 - If C is a constant symbol of $\mathcal{L}$ then the map $\overline{C} : X \to S$ so that $C(x) = c_x$ is a global section and $\overline{C}_{/N}$ is a section over N.

2 - If $f_1,\dots,f_n \in \Gamma(N,S)$ and F is an n-ary function symbol of then $F(f_1,\dots,f_n)(x) = F_x(f_1(x),\dots,f_n(x))$ is a section over N

3 - If R is an n-ary relation symbol of $\mathcal{L}$. Define for $f_1,\dots,f_n \in \Gamma(N,S)$

$R(f_1,\dots,f_n)$ iff $\forall x \in N \quad R_x(f_1(x),\dots,f_n(x))$

4 - It follows from remark 1 above that if $\mathcal{L}$ has a constant symbol then no $\Gamma(N,S)$ is empty.

5 - From 1 - 2 - and 3 we deduce that for $N \subseteq X$; $\Gamma(N,S)$ can be made into an $\mathcal{L}$ -structure. Such structures will be called local section structures

unless N = X then we shall call Γ (X,S) the global section structure providing Γ (X,S) is not empty.

We shall pause here to study the connection between sections over open subsets of X and the topology on S and on X.

For we first recall that the set $B = \{ f(N) \,/\, N \in \mathcal{O}(X)$ and $f \in \Gamma(N,S) \}$ form a basis for the topology on S. We now make the following notation:

Def 1.10: i - Let $N \subseteq X$ and f,g be elements of $\Gamma(N,S)$ we write $|(f,g)|$ for the set $\{ x \in N \,/\, f(x) = g(x) \}$

ii - If $\varphi(v_0,\ldots,v_n)$ is an $\mathcal{L}$ -formula with free variables among $v_0,\ldots,v_n$ and $f_1,\cdots,f_n$ are elements of $\Gamma(N,S)$ then we write $S(\varphi(f_1, \quad ,f_n))$ for the set: $\{ x \in N \,/\, S_x \models \varphi(f_1(x),\ldots,f_n(x)) \}$

prop 1.11: 1 - Let (S,π) be a stalk-space (or an S-sheaf of $\mathcal{L}$ -structures) and N be open subset of X and $f,g \in \Gamma(N,S)$ then $|(f,g)|$ is a closed subset of N; furthermore if S is Hausdorff then $|(f,g)|$ is clopen.

2 - Let (S,π) be an S-sheaf of $\mathcal{L}$-structures over X. And $\varphi(v_0,\ldots,v_n)$ be a positive $\mathcal{L}$ -formula with no quantifier, assume N is an open subset of X and $f_0,\ldots,f_n$ are elements of $\Gamma(N,S)$ then $S(\varphi(f_0,\ldots,f_n))$ is open.

3 - If $\mathcal{L}$ has no relation symbol or if $\underset{\sim}{2}$ is endowed with the discrete topology and $\varphi(v_0,\ldots,v_n)$ is a positive formula with no quantifier, $f_0,\ldots,f_n \in \Gamma(N,S)$ and N is an open subset of X. Then if S is Hausdorff then $S(\varphi(f_0,\ldots,f_n))$ is clopen.

pf: 1 - We claim that if $x \notin |(f,g)|$ then there is an open subset of N; N_x say; so that $x \in N_x$ and $N_x \subseteq |(f,g)| = \phi$.

So let $x \notin |(f,g)|$ this implies that $f(x) = g(x) = s$ say, so there is V_s and N_s open subsets of S and X so that $\pi/V_s : V_s \longrightarrow N_s$ is a homeomorphism. We note that $x \in N_s$ so consider $N_x = N_s \cap N$

We claim that $N_x \cap |(f, g)| = \phi$ For let $y \in N_x$ we have:

$y = \pi(f(y)) = \pi(g(y))$ but π/V_s is 1-1.

And $\pi(f(y))$ and $\pi(g(y))$ are in N_s so $f(y)$ and $g(y)$ are in V_s hence $f(y) = g(y)$ and so $y \in |(f, g)|$ .proving the first assertion of 1.

Now if S is Hausdorff then clearly $|(f,g)|$ is open so $|(f,g)|$ is clopen.

2 - The proof is by induction on the degree of complexity of φ

We first note that If $\varphi(\bar{v}) = \psi(\bar{v}) \wedge \chi(\bar{v})$

then $S(\varphi(\bar{f})) = S(\psi(\bar{f})) \cap S(\chi(\bar{f}))$.

Similarly $S(\psi(\bar{f}) \vee \chi(\bar{f}) = S(\psi(\bar{f})) \cup S(\chi(\bar{f}))$.

thus it is enough to prove 2 - for atomic formulae.

The case when $\varphi(\bar{v}) =$ " $\tau_1(\bar{v}) = \tau_2(\bar{v})$ " follows from 1

If $\varphi(\bar{v}) = R(\bar{v})$ Then consider the map $h : N \longrightarrow \underset{\sim}{2}$ defined as $h(x) = \chi_R(\bar{f}(x))$ thus $S(R(\bar{f})) = h^{-1}(\{1\})$.

but h is the composition of continuous maps so it is continuous and $\{1\}$ is open subset of $\underset{\sim}{2}$ so $h^{-1}(\{1\})$ is open subset of N. Thus proving 2.

3 - If $\mathcal{L}$ has no relation symbol and S Hausdorff then the case $\varphi(\bar{v}) = R(\bar{v})$ does not occur but in the 1st case the problem is as in 1. So $S(\varphi(\bar{f}))$ must be clopen.

Now if $\mathcal{L}$ has relation symbol and $\underset{\sim}{2}$ is endowed with the discrete topology all we have to note is that: $\{1\}$ is clopen so is $h^{-1}(\{1\})$.

And hence the result.

In our discussion of stalk-spaces we induced the concept of homomorphism between stalk-space in what follows we generalize this concept and we discuss how such homomorphisms induce homomorphisms between the section structures.

Def 1.12: Let (S,π) and (S',π') be 2 S-sheaves of $\mathcal{L}$-structures over X a map $\alpha: S \to S'$ is said to be a homomorphism of S-sheaves iff it satisfies:

i - α is a homomorphism of stalk-spaces

ii - for every $x \in X$; $\alpha/S_x = \alpha_x$ is an $\mathcal{L}$-homomorphism between S_x and S'_x.

With this definition in mind we are able to talk about the category of S-sheaves of $\mathcal{L}$- structures over X, whose objects are the S-sheaves and morphisms are the homomorphisms defined as in Def 1.12. We write $SS_{\mathcal{L}}(X)$ for this category.

Let $\alpha: S \to S'$ be a homomorphism of S-sheaves over X. Let $N \subset X$ be an open subset of X. Define $\alpha^*_N : \Gamma(N,S) \to \Gamma(N,S')$ as:

$\alpha^*_N(f)(x) = \alpha(f(x))$ For $x \in N$

Claim 1: $\alpha^*_N(f) \in \Gamma(N,S')$ whenever $f \in \Gamma(N,S)$.

For $\alpha^*_N(f) = \alpha \circ f : N \to S'$ moreover α and f are continuous so $\alpha \circ f$ is continuous. Let $x \in N$, $\pi' \circ \alpha^*_N(f)(x) = \pi' \circ \alpha(f(x)) = \pi(f(x)) = x$, so $\alpha^*_N(f) \in \Gamma(N,S')$

Claim 2: α^*_N is an $\mathcal{L}$-homomorphism from $\Gamma(N,S)$ into $\Gamma(N,S')$

pf: We show that if R is a Relation symbol of $\mathcal{L}$, of arity n $f_1,\ldots,f_n \in \Gamma(N,S)$ so that $\Gamma(N,S) \models R(f_1,\ldots,f_n)$ then $\Gamma(N,S') \models R(\alpha^*_N(f_1),\ldots,\alpha^*_N(f_n))$.

$\Gamma(N,S) \models R(f_1,\ldots,f_n)$ iff $\forall x \in N$, $S_x \models R(f_1(x),\ldots,f_n(x))$.

since α_x is an $\mathcal{L}$-homomorphism we have : $\forall x \in N$

$S'_x \models R(\alpha_x f_1(x),\ldots,\alpha_x f_n(x))$

i.e. $\forall x \in N$, $S'_x \models R(\alpha^*_N(f_1(x)),\ldots,\alpha^*_N(f_n(x)))$.

thus $\Gamma(N,S') \models R(\alpha^*_N(f_1),\ldots,\alpha^*_N(f_n))$. Q.E.D.

Hence we have proved:

prop 1.13: Let α be a homomorphism of S-sheaves of $\mathcal{L}$-structures over a space X then α induces $\mathcal{L}$-homomorphisms between the section structures of the S-sheaves. We end this section with the following remark:

Remark 1.14: Let N be an open subset of X, $(N_i)_{i\in I}$ be an open cover of N and $(f_i)_{i\in I}$ a family of sections so that for each $i \in I$, $f_i \in \Gamma(N_i, S)$ and for any $i, j \in I$ $f_i/N_i \cap N_j = f_j/N_i \cap N_j$ then the map $f : N \longrightarrow S$ defined as $f(x) = f_i(x)$ iff $x \in N_i$ is well defined and it is a section over N. Moreover if $g \in \Gamma(N, S)$ is such that $g/N_i = f_i$ then g = f, so f is unique.

2. Direct Limits (Construction):

In what follows we shall give a construction of direct limits to which we shall refer later on.

Let I be any set a relation $\leqslant$ on I is called preorder ($(I, \leqslant)$ is a pre-ordered set) iff $\leqslant$ is reflexive and transitive.

We say that $(I, \leqslant)$ is $\uparrow$-directed (resp. $\downarrow$-directed) set iff $\forall i, j \in I \quad \exists k \in I \quad [\, i \leq k$ and $j \leq k \,]$ (resp. $k \leq i$ and $k \leq j$)

A $\uparrow$-directed system of sets (resp. $\mathcal{L}$-structures) is a family of sets (resp. $\mathcal{L}$-structures) $(F_i)_{i\in I}$ together with a family of morphisms

$(\rho^{\alpha}_{\beta})_{\alpha \leq \beta \in I}$ such that: i - $(I, \leqslant)$ is a -directed set

ii. $\forall \alpha \in I \quad \rho^{\alpha}_{\alpha} = 1_{F_\alpha}$

iii. $\forall \alpha, \beta, \gamma \in I \quad \alpha \leq \beta \leq \gamma \Rightarrow \rho^{\gamma}_{\alpha} = \rho^{\beta}_{\alpha} \circ \rho^{\gamma}_{\beta}$

Similarly one can define $\downarrow$-directed system of sets or ($\mathcal{L}$-structures).

Given a directed set one can consider it as a category. And the conditions ii. and iii. tell us that a directed system of sets ($\mathcal{L}$-structures) is a

Functor from a directed set into the category of sets (resp. $\mathcal{L}$ -structures).

Def 2.1: Let $F : (I, \leq) \to$ set (resp. $Str\mathcal{L}$ the category of $\mathcal{L}$ -structures) be a functor from a $\uparrow$- directed set into set (resp. $Str\mathcal{L}$). We say that F has a limit $(F_0, (\sigma_\alpha)_{\alpha\in I})$ iff: $(F_0, (\sigma_\alpha)_{\alpha\in I})$ verifies the following conditions:

i - F_0 is an object of set (resp. $Str\mathcal{L}$) and $\sigma_\alpha : F_\alpha \longrightarrow F_0$ are morphisms.

ii - for any $(\alpha, \beta) \in I^2$ and $\alpha \leq \beta$
$$\begin{tikzcd} F_\alpha \arrow[r, "\sigma_\alpha"] \arrow[d, "\rho_\alpha^\beta"'] & F_0 \\ F_\beta \arrow[ur, "\sigma_\beta"'] & \end{tikzcd}$$
commutes i.e. $\sigma_\beta \circ \rho_\alpha^\beta = \sigma_\alpha$

iii - If $(F', (\sigma'_\alpha)_{\alpha\in I})$ are an object and a family of morphisms verifying i and ii then there is a unique morphism $f : F_0 \longrightarrow F'$ so that for every $f \circ \sigma_\alpha = \sigma'_\alpha$

We note here:

i - A similar definition can be given to $\downarrow$ -directed limits.

ii - From condition i ii in the previous definition we deduce:

prop 2.2: If $(F_0, (\sigma_\alpha)_{\alpha\in I})$, $(F', (\sigma'_\alpha)_{\alpha\in I})$ are limits of the same directed system then $F \simeq F'$ and σ_α's are equivalent to σ'_α's i.e. they differ by an isomorphism.

Construction of a direct limit 2.3: Let $\{(F_\alpha)_{\alpha\in I}, (\rho_\alpha^\beta)_{\alpha\leq\beta\in I}\}$ be a $\uparrow$ -directed system of sets. We construct the direct limit of this system as follows:

i - Let $F_1 = \bigcup_{\alpha\in I} F_\alpha$ and define on F_1 the following relation If $a, b \in F_1$, $a \in F_\alpha$, $b \in F_\beta$ then $a \sim b$ iff there is $\gamma \in I$; $\gamma \geq \alpha$ and $\gamma \geq \beta$ so that $\rho_\alpha^\gamma(a) = \rho_\beta^\gamma(b)$.

It is easy to check that $\sim$ is an equivalence relation on F_1 so consider $F_0 = F_1 / \sim$

ii - For every $\alpha \in$ I let $\sigma_\alpha : F_\alpha \longrightarrow F_0$ be defined as $\sigma_\alpha(a) = |a|$

(the equivalence class of a with respect to $\sim$).

prop 2.4: (F_0 , $(\sigma_\alpha)_{\alpha \in I}$) is the direct limit of $\{(F_\alpha)_{\alpha \in I}, (f_\alpha^\beta)_{\alpha \leq \beta}\}$

pf: Clearly F_0 is an object of set and σ_α are morphisms.

Moreover if $\alpha \leq \beta \in I$ then $\sigma_\beta \circ f_\alpha^\beta(a) = |f_\alpha^\beta(a)|$

but $a \sim f_\alpha^\beta(a)$ since if we consider β we have $f_\beta^\beta = 1_{F_\beta}$

$f_\alpha^\beta(a) = f_\beta^\beta \circ f_\alpha^\beta(a)$ so $\sigma_\beta \circ f_\alpha^\beta = \sigma_\alpha$

Finally say (F', $(\sigma'_\alpha)_{\alpha \in I}$) verifies cond i and ii of Def 2.1.

consider $f : F_0 \longrightarrow F'$ so that:

if $|a| \in F_0$ so that $a \in F_\alpha$ let $f(|a|) = \sigma'_\alpha(a)$

This is a well defined morphism. Which verifies condition iii.

We now indicate how one can impose an $\mathcal{L}$ -structure on F_0 when F_α's are $\mathcal{L}$ -structures.

i - If C is a constant put $C^{F_0} = |C^{F_\alpha}|$

ii - If R is an n-ary relation symbol then $\langle |a_1|, \ldots, |a_n| \rangle \in R$ iff $\exists \alpha, b_1, \ldots, b_n \in F_\alpha$ with $b_i \sim a_i$ for all $i = 1, \quad ,n$ so that $\langle b_1, \ldots, b_n \rangle \in R_{F_\alpha}$.

iii - If G is an n-ary function symbol then $G((|a_1|, \ldots, |a_n|)) = G_{F_\alpha}(b_1, \ldots, b_n)$ where $\alpha \in I$, $b_1, \ldots, b_n \in F_\alpha$ and $a_i \sim b_i$ for all $i = 1, \ldots, n$.

Clearly this interpretation makes F_0 an $\mathcal{L}$ -structure it is as well easy to verify that (F_0 , $(\sigma_\alpha)_{\alpha \in I}$) form a $\uparrow$-direct limit of $\{(F_\alpha)_{\alpha \in I}, (f_\alpha^\beta)_{\alpha \leq \beta}\}$.

We end this section by the following proposition which will be an application of direct limit and will be useful later.

prop 2.5: Let (S, Π) be a stalk space (or an s-sheaf of $\mathcal{L}$-structures) over X. Let x be an element of X then:

S_x is the direct limit of the system:

$\{ \Gamma(U, S) \ / \text{ the } U\text{'s}$ are open subsets of X containing $x \}$

pf: In fact $\{ \Gamma(U,S) \ / \ x \in U \in \mathcal{O}(X) \}$ and $\rho_{U}^{U_1}(f) = f/U_1$ whenever $U_1 \subset U$ and $f \in \Gamma(U, S)$ form a direct system.

Let $\sigma_U : \Gamma(U,S) \to S_x$ so that: $\sigma_U(f) = f(x)$.

It is easy to see that $(S_x, (\sigma_U)_{x \in U \in \mathcal{O}(X)})$ is the direct limit of $\{ (\Gamma(U, S))_{x \in U}, \ (\rho_{U}^{U_1})_{U_1 \subseteq U} \}$. Q.E.D.

3. Presheaves and P-sheaves

If X is a topological space and $\mathcal{O}(X)$ is the set of open subsets of X then the inclusion relation on $\mathcal{O}(X)$ is an order relation which makes $\mathcal{O}(X)$ a category. The following notation will be used throughout. If C is a category we write C^{op} for the category whose objects are those of C and arrows are those of C such that $a \to b$ in C^{op} iff $b \to a$ in C. By Set we mean the category of sets. And if $\mathcal{L}$ is any 1st order Language then $Str_h(\mathcal{L})$ (resp. $Str_e(\mathcal{L})$, $Str_{ee}(\mathcal{L})$) denotes the category whose objects are the $\mathcal{L}$-structures and morphisms are homomorphisms (resp. embeddings, elementary embeddings) between the $\mathcal{L}$-structures. Given 2 categories C and D we recall that $F : C \to D$ is a functor.

Iff F associates to every object a of C an object F (a) of D and for every arrow $f : a \to b$ in C an arrow $F(f) : F(a) \to F(b)$ in D such that:

1 - for all objects a of C, $F(1_a) = 1_{F(a)}$

2 - If $f : a \to b$ and $g : b \to c$ then $F(g \circ f) = F(g) \circ F(f)$

for any a,b,c,f and g in C.

Def 3.1: By a presheaf of sets (resp. $\mathcal{L}$ -structures) we mean a pair (X,p) where X is a topological space and p is a functor from $\mathcal{O}(X)^{op}$ into set (resp. $Str_h(\mathcal{L})$).

Thus if $\mathcal{U} \subseteq \mathcal{U}_1$; $\mathcal{U}, \mathcal{U}_1 \in \mathcal{O}(X)$ then $P^{\mathcal{U}}_{\mathcal{U}_1} : P(\mathcal{U}_1) \to P(\mathcal{U})$ exists and is called the restriction map.

We note that every presheaf of $\mathcal{L}$ -structures is a presheaf of sets.

For every $x \in X$ let $\mathcal{T}(x) = \{\mathcal{U} \in \mathcal{O}(X); x \in \mathcal{U}\}$; $\mathcal{T}(x)$ is a $\downarrow$ -directed family and so is $\{P(\mathcal{U}) / \mathcal{U} \in \mathcal{O}(X)\}$ so we can define the direct limit of the latter. We shall denote this limit by P_x

Thus from 2. we conclude that for every $\mathcal{U} \in \mathcal{O}(X)$ there is a morphism $P^{\mathcal{U}} : P(\mathcal{U}) \to P_x$ so that whenever $x \in \mathcal{U} \subseteq \mathcal{U}_1$;

$$\begin{tikzcd} P(\mathcal{U}_1) \arrow[r, "P^{\mathcal{U}_1}"] \arrow[d, "P^{\mathcal{U}_1}_{\mathcal{U}}"'] & P_x \\ P(\mathcal{U}) \arrow[ur, "P^{\mathcal{U}}"'] & \end{tikzcd}$$

commutes i.e. $P^{\mathcal{U}} \circ P^{\mathcal{U}}_{\mathcal{U}_1} = P^{\mathcal{U}_1}$

Def 3.2: Let (X, P), (X,Q) be presheaves of sets (resp. $\mathcal{L}$ -structures). We say that $\alpha : (X,P) \to (X,Q)$ is a presheaves homomorphisms iff α is a family $(\alpha_{\mathcal{U}})_{\mathcal{U} \in \mathcal{O}(X)}$ of morphisms of the category set (resp. $Str_h(\mathcal{L})$) such that: (1) For every $\mathcal{U} \in \mathcal{O}(X)$ $\alpha_{\mathcal{U}} : P(\mathcal{U}) \to Q(\mathcal{U})$.

(2) Given $\mathcal{U} \subseteq \mathcal{U}_1$, elements of $\mathcal{O}(X)$ then

$$\begin{tikzcd} P(\mathcal{U}_1) \arrow[r, "\alpha_{\mathcal{U}_1}"] \arrow[d, "P^{\mathcal{U}}_{\mathcal{U}_1}"'] & Q(\mathcal{U}_1) \arrow[d, "Q^{\mathcal{U}}_{\mathcal{U}_1}"] \\ P(\mathcal{U}) \arrow[r, "\alpha_{\mathcal{U}}"'] & Q(\mathcal{U}) \end{tikzcd}$$

commutes.

It is easy to verify that the presheaves over X together with presheaves homomorphisms form a category. Furthermore If $(X,P_i)_{i \in I}$ is a family of presheaves then $(X, \prod_{i \in I} P_i)$ is a presheaf and it is the product of the (X,P_i)'s. In particular If (X,P) is a presheaf so is (X,P^n) where $P^n(\mathcal{U}) = [P(\mathcal{U})]^n$ and $P^{n\,\mathcal{U}}_{\mathcal{U}_1} : P^n(\mathcal{U}_1) \to P^n(\mathcal{U})$ is defined as

$P^{n\,\mathcal{U}}_{\mathcal{U}_1}(\langle a_1,\dots,a_n\rangle) = (P^{\mathcal{U}}_{\mathcal{U}_1}(a_1),\dots, P^{\mathcal{U}}_{\mathcal{U}_1}(a_n))$

Let (X,P) be a presheaf of $\mathcal{L}$ -structures; and R be an n-ary relation symbol of $\mathcal{L}$. Let $\mathcal{U} \in \mathcal{O}(X)$ and note by R ($\mathcal{U}$) the set $\{\langle a_1,\dots,a_n\rangle \in P^n(\mathcal{U});\ R(a_1,\dots,a_n)\}$ Thus we have $R(\mathcal{U}) \subseteq P^n(\mathcal{U})$. for every $\mathcal{U} \in \mathcal{O}(X)$. Furthermore if $\mathcal{U} \subseteq \mathcal{U}_1$ then

$$P^{n\,\mathcal{U}}_{\mathcal{U}_1}/_{R(\mathcal{U}_1)} : R(\mathcal{U}_1) \longrightarrow R(\mathcal{U}). \text{ Put } P^{n\,\mathcal{U}}_{\mathcal{U}_1}/_{R(\mathcal{U})} = R^{\mathcal{U}}_{\mathcal{U}_1}$$

Thus (X,R) is a presheaf of sets. This suggests the following definition:

Def 3.3: Let (X,P) be a presheaf of sets ($\mathcal{L}$ -structures resp.). We say that (X,Q) is a subpresheaf of (X,P) iff (X,Q) is a presheaf and verifies the following conditions:

i - For every $\mathcal{U} \in \mathcal{O}(X)$; $Q(\mathcal{U}) \subseteq P(\mathcal{U})$

ii - For every $\mathcal{U}$ and $\mathcal{U}_1$ in $\mathcal{O}(X)$ such that $\mathcal{U} \subseteq \mathcal{U}_1$; $Q^{\mathcal{U}}_{\mathcal{U}_1} = P^{\mathcal{U}}_{\mathcal{U}_1}/_{Q(\mathcal{U}_1)}$.

Thus the above presheaf of sets (X,R) is a subpresheaf of the presheaf of sets (X,P^n).

Now we are in position to define $\mathcal{P}$ -sheaves.

Def 3.4: We say that the presheaf of sets (X, P) is a $\mathcal{P}$ -sheaf of sets iff whenever $\mathcal{U}$ is an open subset of X; $(\mathcal{U}_i)_{i\in I}$ is an open cover of $\mathcal{U}$ and $(a_i)_{i\in I}$ a family of elements so that $a_i \in P(\mathcal{U}_i)$ for every $i \in I$. such that $P^{\mathcal{U}_i \cap \mathcal{U}_j}_{\mathcal{U}_i}(a_i) = P^{\mathcal{U}_i \cap \mathcal{U}_j}_{\mathcal{U}_j}(a_j)$ for every pair $(i,j) \in I^2$ there exists a unique element $a \in P(\mathcal{U})$ so that $P^{\mathcal{U}_i}_{\mathcal{U}}(a) = a_i$

We observe that the above condition is equivalent to the following:

1 - If $(\mathcal{U}_i)_{i\in I}$ is an open cover of $\mathcal{U}$ and a,b are elements of $P(\mathcal{U})$ so that for all $i \in I$ $P^{\mathcal{U}_i}_{\mathcal{U}}(a) = P^{\mathcal{U}_i}_{\mathcal{U}}(b)$ then a = b.

2. If $(\mathcal{U}_i)_{i\in I}$ is an open cover of $\mathcal{U}$ and $(a_i \in P(\mathcal{U}_i))_{i\in I}$ is a

family of elements so that for all $i, j \in I$ $P^{\mathcal{U}_i \cap \mathcal{U}_j}_{\mathcal{U}_i}(a_i) = P^{\mathcal{U}_i \cap \mathcal{U}_j}_{\mathcal{U}_j}(a_j)$ then there is $a \in P(\mathcal{U})$ so that $P^{\mathcal{U}_i}_{\mathcal{U}}(a) = a_i$ for all $i \in I$.

The P-sheaves of sets over X with their homomorphisms (defined as presheaves homomorphisms) form a subcategory of the category of presheaves of sets over X. We denote this category by P.S (X)

Prop.3.5: P.S (X) admits products

pf: Let $(X, P_i)_{i \in I}$ be a family of P-sheaves and let $(X, \prod P_i)$ be the presheaf product of (X, P_i)'s. It is enough to show that $(X, \prod P_i)$ is a P-sheaf. So let $\mathcal{U} \in \mathcal{O}(X)$, $(\mathcal{U}_j)_{j \in J}$ be an open cover of $\mathcal{U}$ and $(a_j)_{j \in J}$ a family of elements $a_j \in \prod P_i(\mathcal{U}_j)$ so that: $(\prod P_i)^{j,k}_j (a_j) = (\prod P_i)^{j,k}_k (a_k)$

for all $j, k \in J$; where

$(\prod P_i)^{j,k}_j = (\prod (P_i))^{\mathcal{U}_j \cap \mathcal{U}_k}_{\mathcal{U}_j}$.

Thus $(P_i)^{j,k}_j (a^i_j) = (P_i)^{j,k}_k (a^i_k)$ for all $i \in I$ and $j, k \in J$.

Since (X, P_i) is a P-sheaf it follows that for all $i \in I$ there is a unique $a^i \in P_i(\mathcal{U})$ so that $(P_i)^j_{\mathcal{U}} (a^i) = a^i_j$ for all $i \in I$ and $j \in J$. Let $a = (a^i)$. It is now clear that a has the property of the definition 3.4.

Def 3.6: Let $\mathcal{L}$ be a 1st order language. We say that a presheaves of $\mathcal{L}$-structures (X,P) is a P-sheaf of $\mathcal{L}$-structures iff:

1 - (X,P) is a P-sheaf of sets.

2 - If R is any n-ary relation symbol of $\mathcal{L}$; $\mathcal{U}$ an element of $\mathcal{O}(X)$; then for any open cover $(\mathcal{U}_i)_{i \in I}$ of $\mathcal{U}$; and $a_1, \cdots, a_n$ elements of $P(\mathcal{U})$ then $P(\mathcal{U}) \models R(a_1, \ldots, a_n)$ iff $\forall i \in I$ $P(\mathcal{U}_i) \models R(P^{\mathcal{U}_i}_{\mathcal{U}}(a_1), \ldots, P^{\mathcal{U}_i}_{\mathcal{U}}(a_n))$

prop 3.7: For a presheaf of $\mathcal{L}$ -structures (X,P) to be a P -sheaf it is necessarily and sufficient that:

i - (X,P) is a P -sheaf of sets

ii - (X,R) is a P -sheaf of sets for all relation symbols R of $\mathcal{L}$.

pf: Let (X,P) be a P -sheaf of $\mathcal{L}$ -structures then (X,P) is a P -sheaf of sets (Def 3.6). Now since (X,P) is a P -sheaf of sets, then for any $n \in \omega$,(X,P^n) is a P -sheaf of sets (prop 3.5). Let R be an n-ary relation of $\mathcal{L}$ then (X,R) is a subpresheaf of (X,P^n).

Let $\mathcal{U} \in \mathcal{O}(X)$; $(\mathcal{U}_i)_{i \in I}$ be any cover of $\mathcal{U}$ and $\bar{a},\bar{b}$ elements of R ($\mathcal{U}$)

so that $R^{\mathcal{U}_j}_{\mathcal{U}}(\bar{a}) = R^{\mathcal{U}_j}_{\mathcal{U}}(\bar{b})$ for all $j \in I$. Thus

$(P^{\mathcal{U}_j}_{\mathcal{U}}(\bar{a})) = (P^{\mathcal{U}_j}_{\mathcal{U}}(\bar{b}))$ i.e. $(\overline{P^{\mathcal{U}_j}_{\mathcal{U}}(a)}) = (\overline{P^{\mathcal{U}_j}_{\mathcal{U}}(b)})$

So for every $k = 1,\cdots,n$ we have $P^{\mathcal{U}_j}_{\mathcal{U}}(a_k) = P^{\mathcal{U}_j}_{\mathcal{U}}(b_k)$ for all $j \in J$

so $a_k = b_k$ because (X , P) is a p-sheaf and hence $\bar{a} = \bar{b}$.

Now assume that $(\bar{a}_i)_{i \in I}$ is a family of elements $\bar{a}_i \in R(\mathcal{U}_i)$ so that

$R^{i,j}_i(\bar{a}_i) = R^{i,j}_j(\bar{a}_j)$ i.e. $(\overline{P^{i,j}_i(a_i)}) = (\overline{P^{i,j}_j(a_j)})$

but (X,P^n) is a P -sheaf. Hence there is $\bar{a} \in P^n(\mathcal{U})$ so that

$P^{n\,\mathcal{U}_i}_{\mathcal{U}}(\bar{a}) = \bar{a}_i$ So to end the proof it is enough to show that

$\bar{a} \in R(\mathcal{U})$; but since $P(\mathcal{U}_i) \models R(\bar{a}_i)$ for all $i \in I$

and $\bar{a}$ defined above is unique thus we have $P(\mathcal{U}_i) \models R(P^{\mathcal{U}_i}_{\mathcal{U}}(\bar{a}))$ for

all $i \in I$ but by def 3.6.2 $P(\mathcal{U}) \models R(\bar{a})$. thus $\bar{a} \in R(\mathcal{U})$.

Conversely: All that we have to show is that if given an n-ary relation R of $\mathcal{L}$; $\mathcal{U} \in \mathcal{O}(X)$; $(\mathcal{U}_i)_{i \in I}$ an open cover of $\mathcal{U}$ and $\bar{a} \in P^n(\mathcal{U})$

so that $P(\mathcal{U}_i) \models R(P^{\mathcal{U}_i}_{\mathcal{U}}(\bar{a}))$ for all $i \in I$; then $P(\mathcal{U}) \models R(\bar{a})$.

Thus by hypothesis we have $\overline{P^{\mathcal{U}_i}_{\mathcal{U}}(a)} \in R(\mathcal{U}_i)$ for all $i \in I$

and (X,R) is a P -sheaf of sets; hence it is enough to show that $\bar{a} \in R(\mathcal{U})$ We note that $R_i^{i,j}(\bar{a}) = (\overline{P_i^{i,j}(a)}) = (\overline{P_j^{i,j}(a)}) = R_j^{i,j}(\bar{a})$ so there must be a unique $b \in R(\mathcal{U})$ $R_{\mathcal{U}}^{\mathcal{U}_i} = ((P_{\mathcal{U}}^{\mathcal{U}_i})(\bar{a}))$; so $b \in (X,P^n)$ and (X,P^n) is a P -sheaf and $P^{n}{}_{\mathcal{U}}^{\mathcal{U}_i}(b) = P^{n}{}_{\mathcal{U}}^{\mathcal{U}_i}(\bar{a})$ so $b = \bar{a}$ and $\bar{a} \in R(\mathcal{U})$.

4. Equivalence of Notions of s-sheaves and p-sheaves

Theorem 4.1: Let (X,P) be a p-sheaf of $\mathcal{L}$ -structures then there is an s-sheaf of $\mathcal{L}$ -structures (S,π) over X such that:

i - $S = P(x)$

ii - For every open subset $\mathcal{U}$ of X $P(\mathcal{U}) \simeq \Gamma(\mathcal{U}, S)$ as $\mathcal{L}$ -structures

pf: Let $S = \dot{\bigcup\limits_{x \in X}} P(x)$ and denote by S_x the $\mathcal{L}$ -structure $P(x)$.

We define $\pi : S \to X$ as follows: $\pi(s) = x$ iff $s \in P(x) = S_x$

Let $\mathcal{U}$ be an open subset of X. For $y \in \mathcal{U}$ and $a \in P(\mathcal{U})$ we write $|a|_y$ for the equivalence class of a with respect to the equivalence relation defining $P(y)$; and we denote by $\mathcal{V}_a$ the set $\{|a|_y \ ; y \in \mathcal{U}\}$

Claim 1: the set $\mathcal{B} = \{\mathcal{V}_a \ ; a \in \bigcup\limits_{\mathcal{U} \in \mathcal{O}(x)} P(\mathcal{U})\}$ form a "sub"basis for a topology on S.

pf of the claim: we remark first that $\phi = \mathcal{V}_a$ whenever $a \in P(\phi)$ so $\phi \in \mathcal{B}$ Now assume $\mathcal{V}_a, \mathcal{V}_{a'} \in \mathcal{B}$, $\mathcal{V}_a \cap \mathcal{V}_{a'} \neq \phi$ To prove the claim we shall show that there is $\mathcal{V}_c \in \mathcal{B}$ so that $\mathcal{V}_c \subseteq \mathcal{V}_a \cap \mathcal{V}_{a'}$.

For let $\mathcal{V}_a = \{|a|_y \ ; y \in \mathcal{U}\}$ and $\mathcal{V}_{a'} = \{|a'|_y \ ; y \in \mathcal{U}'\}$

Say $|b|_x \in \mathcal{V}_a \cap \mathcal{V}_{a'}$. so $|b|_x = |a|_y = |a'|_z$

But then we must have $x = y = z$ and hence $\mathcal{U} \cap \mathcal{U}' \neq \phi$ so let $\mathcal{U}_0 = \mathcal{U} \cap \mathcal{U}'$

Furthermore if $|a'|_x = |a|_x$ there must be an open subset $\mathcal{U}_1$ of X such that $x \in \mathcal{U}_1 \subset \mathcal{U}_0$ such that $P_{\mathcal{U}}^{\mathcal{U}_1}(a) = P_{\mathcal{U}'}^{\mathcal{U}_1}(a')$

Let $c = P_{\mathcal{U}}^{\mathcal{U}_1}(a) = P_{\mathcal{U}'}^{\mathcal{U}_1}(a')$ and let $\mathcal{V}_c$ be $\{ |c|_z \; ; \; z \in \mathcal{U}_1 \}$

$\mathcal{V}_c \in \mathcal{B}$ Now if $z \in \mathcal{U}_1$ we have that $|a|_z = |a'|_z$ because

$P_{\mathcal{U}}^{\mathcal{U}_1}(a) = P_{\mathcal{U}'}^{\mathcal{U}_1}(a')$ and so $|c|_z = |a|_z = |a'|_z$ Hence

$\mathcal{V}_c \subseteq \mathcal{V}_a \cap \mathcal{V}_{a'}$.

Endow S with the topology generated by $\mathcal{B}$.

Claim 2: $\pi : S \to X$ is a local homeomorphism from S onto X.

pf of the claim:

That π is onto follows from the definition of π ; to show that π is a local homeomorphism consider $s = |a|_x \in S$ where $a \in P(\mathcal{U})$ and $x \in \mathcal{U}$. Let $\mathcal{V}_s = \{ |a|_z \; ; \; z \in \mathcal{U} \}$ We remark:

If $s_1 \in \mathcal{V}_s$ then $s_1 = |a|_z$ and $\pi(s_1) = z$ so $\pi/\mathcal{V}_s$ is a map from $\mathcal{V}_s$ into $\mathcal{U}$ moreover for every $z \in \mathcal{U}$; $\pi(|a|_z) = z$ so $\pi/\mathcal{V}_s$ is onto.

Now to show that $\pi/\mathcal{V}_s$ is injective note that if $x, y \in \mathcal{U}$ and $x = y$ then $|a|_x = |a|_y$ so if $\pi(|a|_x) = \pi(|a|_y)$ then $|a|_x = |a|_y$ and $\pi/\mathcal{V}_s$ is injective.

$\pi/\mathcal{V}_s$ is an open map for if $\mathcal{V}_0 \subset \mathcal{V}_s$ and $\mathcal{V}_0$ is open then $\mathcal{V}_0 = \{ |a|_x \; ; \; x \in \pi(\mathcal{V}_0) \}$ so by definition of the elements of $\mathcal{B}$, $\pi(\mathcal{V}_0)$ must be open.

Finally we need to show that $\pi/\mathcal{V}_s$ is continuous. For let $\mathcal{U}_1 \subseteq \mathcal{U}$ be an open subset of $\mathcal{U}$ consider $\pi^{-1}_{/\mathcal{V}_s}(\mathcal{U}_1) = \{ |a|_y \; / \; y \in \mathcal{U}_1 \} \in \mathcal{B}$ so $\pi^{-1}/\mathcal{V}_s(\mathcal{U}_1)$ is open.

If F is an n-ary function symbol of $\mathcal{L}$ and $F : \sum_1^n S \to s$ is defined as $F(|a_1|_x, \ldots, |a_n|_x) = |F_{\mathcal{U}_0}(\alpha_1, \ldots, \alpha_n)|$ where $\alpha_1, \ldots, \alpha_n \in P(\mathcal{U}_0)$ and $x \in \mathcal{U}_0 \in \mathcal{O}(x)$ such that $|\alpha_i|_x = |a_i|_x$ Then F is well defined.

To see that note given $|a_1|_x, \ldots, |a_n|_x$ then there are $\mathcal{U}_1, \ldots, \mathcal{U}_n \in \mathcal{O}(x)$

with $x \in \bigcap_{i=1}^{n} \mathcal{U}_i$ and $b_1, \cdots, b_n$ such that $b_i \in P(\mathcal{U}_i)$ and $|b_i|_x = |a_i|_x$ for all $i = 1, \cdots, n$ put $\mathcal{U}_0 = \bigcap_{i=1}^{n} \mathcal{U}_i$ and $\alpha_i = P_{\mathcal{U}_i}^{\mathcal{U}_0}(b_i)$ for all $i = 1, \cdots, n$.

Since $P(\mathcal{U}_0)$ is an $\mathcal{L}$-structure and by definition of $P(x)$ we have $|\alpha_i|_x = |b_i|_x$ for all $i = 1, \cdots, n$ and $\alpha_i \in P(\mathcal{U}_0)$ for all $i = 1, \cdots, n$ then we can define $|F_{\mathcal{U}_0}(\alpha_1, \cdots, \alpha_n)|_x$. Now if $(\beta_i)_{i \leq n}$ $\beta_i \in P(\mathcal{U}')$ and $x \in \mathcal{U}'$ and $|\beta_i|_x = |a_i|_x$ for all $i = 1, \cdots, n$ then we have $|\beta_i|_x = |\alpha_i|_x$ for all $i = 1, \cdots, n$ so there must be $\mathcal{U}_1 \subset \mathcal{U}_0 \cap \mathcal{U}'$ so that

$P_{\mathcal{U}_0}^{\mathcal{U}_1}(\alpha_i) = P_{\mathcal{U}'}^{\mathcal{U}_1}(\beta_i)$ for all $i = 1, \cdots, n$. Furthermore

$$|F_{\mathcal{U}_1}(P_{\mathcal{U}'}^{\mathcal{U}_1}(\beta_1), \cdots, P_{\mathcal{U}'}^{\mathcal{U}_1}(\beta_n))|_x = |F_{\mathcal{U}_1}(P_{\mathcal{U}_0}^{\mathcal{U}_1}(\alpha_1), \cdots, P_{\mathcal{U}_0}^{\mathcal{U}_1}(\alpha_n))|_x$$

so $|F_{\mathcal{U}'}(\beta_1, \cdots, \beta_n)|_x = |F_{\mathcal{U}_0}(\alpha_1, \cdots, \alpha_n)|_x$ and F is well defined.

<u>Claim 3</u>: Let F be an n-ary function symbol of $\mathcal{L}$; then the F defined above from $\sum_1^n S$ into S is continuous.

<u>pf</u> of the claim: Let $|a_{n+1}|_x = F(|a_1|_x, \text{—}, |a_n|_x)$

Let $\mathcal{V}_{a_{n+1}} = \{|a_{n+1}|_y \, ; \, y \in \mathcal{U}\}$ be an open subset of S containing $|a_{n+1}|_x$ so $x \in \mathcal{U}$. Consider $N = \mathcal{U} \cap \mathcal{U}_0$ where $\mathcal{U}_0$ is as above and let $\mathcal{V}_{\alpha_i} = \{|\alpha_i|_y \, ; \, y \in N\}$ and $\mathcal{V} = (\prod_{i=1}^{n} \mathcal{V}_{\alpha_i}) \cap \sum_1^n S$ this is an open subset of $\sum_1^n S$. If $b_1, \cdots, b_n \in \mathcal{V}$ then $F(b_1, \cdots, b_n) =$

$= F(|\alpha_1|_z, \text{—}, |\alpha_n|_z)$) for some $z \in N$. But this is equal to

$|F_{\mathcal{U}_0}(\alpha_1, \text{—}, \alpha_n)|_z = |a_{n+1}|_z \in \mathcal{V}_{a_{n+1}}$ so $F(\mathcal{V}) \subseteq \mathcal{V}_{a_{n+1}}$ and F is continuous.

Now if R is an n-ary relation symbol we define $\chi_R : \sum_1^n S \longrightarrow \underset{\sim}{2}$ as $\chi_R(|a_1|_x, \cdots, |a_n|_x) = 1$ iff $P(x) \models R(|a_1|_x, \cdots |a_n|_x)$

Claim 4: Endow 2 with Sierpinski's topology then χ_R is continuous

pf of the claim: Let $(|a_1|_x, -, |a_n|_x) \in \sum_1^n S$ so that $\chi_R(|a_1|_x, -, |a_n|_x) = 1$

We need to show that there is $\mathcal{V}$ an open subset of Σ S containing $(|a_1|_x, -, |a_n|_x)$ so that $\chi_R(\mathcal{V}) \subseteq \{1\}$.

$\chi_R(|a_1|_x, -, |a_n|_x) = 1$ implies that $P(x) \models R(|a_1|_x, ---, |a_n|_x)$.

thus there exist $\mathcal{U} \in \mathcal{O}(X)$; $x \in \mathcal{U}$ and $b_1, \cdots, b_n \in P(\mathcal{U})$ with $P(\mathcal{U}) \models R(b_1, \cdots, b_n)$ and $|b_i|_x = |a_i|_x$ for all $i = 1, \dots n$

Let $\mathcal{V}_{b_i} = \{|b_i|_y ; y \in \mathcal{U}\}$, $\mathcal{V} = \Pi \mathcal{V}_{b_i} \cap \sum_1^n S$ is open in $\sum_1^n S$.

Let $(|b_1|_z, \cdots, |b_n|_z)$ be an element of $\mathcal{V}$, then $P_z \models R(|b_1|_z, -, |b_n|_z)$ because $P(\mathcal{U}) \models R(b_1, -, b_n)$; so $\chi_R(\mathcal{V}) \subset \{1\}$

Furthermore since $|b_i|_x = |a_i|_x \; \forall i = 1, \dots, n$ and $x \in \mathcal{U}$ then $(|a_1|_x, ---, |a_n|_x) \in \mathcal{V}$ so χ_R is continuous.

The claims 1,2,3 and 4 and the definition of S -sheaves of $\mathcal{L}$ -structures prove that the above constructed (S, Π) is an S-sheaf of $\mathcal{L}$ -structures over X. Moreover by construction we have that $S_x = P(x)$ for all $x \in X$.

To end the proof of the theorem we claim:

Claim 5: For every $\mathcal{U} \in \mathcal{O}(X)$ $P(\mathcal{U}) \simeq \Gamma(\mathcal{U}, S)$.

pf of the claim: For every $a \in P(\mathcal{U})$ let $\hat{a} : \mathcal{U} \longrightarrow S$ be defined as $\hat{a}(x) = |a|_x$ for all $x \in \mathcal{U}$ We first show that $\hat{a} \in \Gamma(\mathcal{U}, S)$ for all $a \in P(\mathcal{U})$ To see that we note that $\Pi \circ \hat{a}(x) = \Pi(|a|_x) = x$ for every $x \in \mathcal{U}$ so $\Pi \circ \hat{a} = 1_{\mathcal{U}}$ Now we need to show that $\hat{a}$ is continuous. So let $\mathcal{V}_a$ be an open subset of S so that $|a|_x \in \mathcal{V}_a$ By definition of $\mathcal{V}_a$ we have $\mathcal{V}_a = \{|a|_z, z \in \mathcal{U}_1\}$ so $x \in \mathcal{U}_1$ Consider $\mathcal{U}_0 = \mathcal{U}_1 \cap \mathcal{U}$. this is an open subset of $\mathcal{U}$ moreover $\hat{a}(\mathcal{U}_0) \subseteq \mathcal{V}_a$ and so $\hat{a}$ is continuous. Let $\alpha : P(\mathcal{U}) \longrightarrow \Gamma(\mathcal{U}, S)$ so that $\alpha(a) = \hat{a}$.

This map is well defined. For if a = b then $|a|_x = |b|_x$ for all $x \in \mathcal{U}$

and so $\hat{a} = \hat{b}$.

To show that α is surjective consider $f \in \Gamma(\mathcal{U}, S)$ so f is continuous and $\pi \circ f = 1_{\mathcal{U}}$, hence $f(x) \in (P(x))$. So there is an open subset $\mathcal{U}_x$ of $\mathcal{U}$ and b^x in $P(\mathcal{U}_x)$ so that $|b^x|_x = f(x)$ Consider the family $(\mathcal{U}_x)\ x \in \mathcal{U}$ this forms an open cover of $\mathcal{U}$; moreover If we consider b^x and b^y as before we have $P^{\mathcal{U}_x \cap \mathcal{U}_y}_{\mathcal{U}_x}(b^x) = P^{\mathcal{U}_x \cap \mathcal{U}_y}_{\mathcal{U}_y}(b^y)$ so by definition of P-sheaves there is a unique $a \in P(\mathcal{U})$ so that $P^{\mathcal{U}_x}_{\mathcal{U}}(a) = b^x$ thus $|a|_x = f(x)$ for all $x \in \mathcal{U}$ and so $\hat{a} = f$ the uniqueness of a proves that α is injective. Now consider an n-ary relation symbol of $\mathcal{L}$ and $a_1, \cdots, a_n \in P(\mathcal{U})$ We claim that $P(\mathcal{U}) \models R(a_1, \cdots, a_n)$ iff $\Gamma(\mathcal{U}, S) \models R(\hat{a}_1, \cdots, \hat{a}_n)$

If $P(\mathcal{U}) \models R(a_1, \cdots, a_n)$ then for all $x \in \mathcal{U}$ $P_x \models R(|a_1|_x, \cdots, |a_n|_x)$ thus for all $x \in \mathcal{U}$ $P_x \models R(\hat{a}_1(x), \cdots, \hat{a}_n(x))$ and so $\Gamma(\mathcal{U}, S) \models R(\hat{a}_1, \cdots, \hat{a}_n)$.

Conversely say that $\Gamma(\mathcal{U}, S) \models R(\hat{a}_1, \cdots, \hat{a}_n)$ then for every $x \in \mathcal{U}$ $P_x \models R(|a_1|_x, \cdots, |a_n|_x)$ i.e. there are open subsets $\mathcal{U}_x$'s of $\mathcal{U}$, $x \in \mathcal{U}_x$ and $b^x_1, \cdots, b^x_n \in P(\mathcal{U}_x)$ such that $P(\mathcal{U}_x) \models R(b^x_1, \cdots, b^x_n)$ and $|b^x_i|_x = |a_i|_x$ Those $\mathcal{U}_x$'s form an open cover of $\mathcal{U}$ and since (X,P) is a P-sheaf we have $P^{\mathcal{U}_x}_{\mathcal{U}}(a_i) = b^x_i$ and $P(\mathcal{U}_x) \models R(b^x_1, \cdots, b^x_n)$ so by definition of P-sheaf $P(\mathcal{U}) \models R(a_1, \cdots, a_n)$; thus we proved that α is an $\mathcal{L}$-homomorphism. And so $P(\mathcal{U}) \simeq \Gamma(\mathcal{U}, S)$.

Now we recall that Remark 1.14 and the definition of P-sheaves together with prop 2.5 give us the following theorem:

Theorem 4.2: Let (S, π) be an S-sheaf of $\mathcal{L}$-structures over X then there is a P-sheaf of $\mathcal{L}$-structures (X,P) so that i - $\Gamma(\mathcal{U}, S) = P(\mathcal{U})$ for all $\mathcal{U} \in \mathcal{O}(X)$ and $S_X = P_X$ for all $x \in X$

With these 2 theorems in mind one can consider the 2 categories $SS_{\mathcal{L}}(X)$ and $P_{\mathcal{L}}(X)$ of S-sheaves and P-sheaves of $\mathcal{L}$-structures over X. Theorem 4.1 induces a map from the objects of $P_{\mathcal{L}}(X)$ into those of $SS_{\mathcal{L}}(X)$ Consider (X,P) and (X,Q) to be 2 P-sheaves of $\mathcal{L}$-structures over X and α a sheaf homomorphism from (X,P) into (X,Q). Thus α is a family $(\alpha_{\mathcal{U}})_{\mathcal{U}\in\mathcal{O}(X)}$ of $\mathcal{L}$-homomorphisms $\alpha_{\mathcal{U}} : P(\mathcal{U}) \longrightarrow Q(\mathcal{U})$

By the construction of P_x and Q_x one can find a map: $\alpha_x : P_x \longrightarrow Q(x)$ so that whenever $\mathcal{U} \in \mathcal{O}(X)$ and $x \in \mathcal{U}$ the following diagram commutes:

$$\begin{array}{ccc} P(\mathcal{U}) & \xrightarrow{\alpha_{\mathcal{U}}} & Q(\mathcal{U}) \\ {\scriptstyle P^{\mathcal{U}}}\downarrow & & \downarrow{\scriptstyle Q^{\mathcal{U}}} \\ P_x & \xrightarrow[\alpha_x]{} & Q_x \end{array}$$

where $P^{\mathcal{U}}$ and $Q^{\mathcal{U}}$ are the canoninical homomorphisms.

Consider $\bar{\alpha} = \bigcup_{x\in X} \alpha_x$ thus $\bar{\alpha} : \dot{\bigcup}_{x\in X} P_x \longrightarrow \bigsqcup_{x\in X} Q_x$ and If we denote By (S, π) (resp. (S', π')) the S-sheaf of $\mathcal{L}$-structure corresponding to (X,P) (resp. (X,Q)) we get $\bar{\alpha} : S \longrightarrow S'$; and it is trivial to see that i - $\pi' \circ \bar{\alpha} = \pi$ and ii - $\bar{\alpha}/S_x : S_x \longrightarrow S'_x$ is an $\mathcal{L}$-homomorphism. To show that $\bar{\alpha}$ is a homomorphism of S-sheaves of $\mathcal{L}$-structures it is enough to show that $\bar{\alpha}$ is continuous. To see that let $a \in P(x)$ thus $a = |b|_x$ for some b in some $P(\mathcal{U})$ where $\mathcal{U} \in \mathcal{O}(x)$ and $x \in \mathcal{U}$.
Thus $\alpha_x(a) = |\alpha_{\mathcal{U}}(b)|_x = C$ so we may take an open subset of S containing C. We may as well consider such open to be of the form:
$\mathcal{V}' = \{ |\alpha_{\mathcal{U}}(b)|_y \; ; \; y \in \mathcal{U}_1 \subset \mathcal{U}, \mathcal{U}_1 \in \mathcal{O}(X)\}$ Let $\mathcal{V} = \{ |b|_y \; ; \; y \in \mathcal{U}_1\}$.
and we calculate $\bar{\alpha}(\mathcal{V}) = \{ \bar{\alpha}(|b|_y) \; ; \; y \in \mathcal{U}_1\} = \{ |\alpha_{\mathcal{U}_1}(b)|_y \, / \, y \in \mathcal{U}_1\}$

$= \{ |\alpha_{\mathcal{U}}(b)|_y ; \; y \in \mathcal{U}_1\} \subseteq \mathcal{V}'$. thus $\bar{\alpha}$ is continuous

Thus we have proved the following Lemma:

<u>Lemma 4.3</u>: Let $\alpha : (X,P) \longrightarrow (X,Q)$ be a homomorphism of P-sheaves of $\mathcal{L}$-structures and let (S, π) and (S, π') be the corresponding S-sheaves

of $\mathcal{L}$ -structures over X to (X,P) and (X,Q) then α induces an S-sheaf homomorphism $\bar{\alpha}$ from S into S'.

Theorem 4.1 and Lemma 4.3 put together give us the following result:

Theorem 4.4: Let $F : P_{\mathcal{L}}(X) \longrightarrow SS_{\mathcal{L}}(X)$ be defined as:

i - If $(X,P) \in ob\ P_{\mathcal{L}}(X)$ then $F((X,P)) = (S, \pi)$ where (S, π) is the S-sheaf of theorem 4.1

ii - If $\alpha : (X,P) \longrightarrow (X,Q)$ is a P-sheaf homomorphism then $F(\alpha) = \bar{\alpha}$ where $\bar{\alpha}$ is the S-sheaf homomorphism of Lemma 4.3

Then F is a functor from $P_{\mathcal{L}}(X)$ into $SS_{\mathcal{L}}(X)$

pf: It is easy to show that if $\alpha : (X,P) \longrightarrow (X,P)$ is the identity then $F(\alpha)$ is the identity from (S, π) onto (S', π')

Now if $\alpha : (X,P) \longrightarrow (X,Q)$ and $\beta : (X,Q) \longrightarrow (X,T)$ are P-sheaves homomorphisms so that $\beta \circ \alpha : (X,P) \longrightarrow (X,T)$ is defined and it is a P-sheaves homomorphism. We claim that: $F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$

Let $s \in S$, $\overline{\beta \circ \alpha}(s) = (\beta \circ \alpha)_x(s)$ for some $x \in X$ but $(\beta \circ \alpha)_x = \beta_x \circ \alpha_x$ so $(\beta \circ \alpha)_x(s) = \beta_x(\alpha_x(s)) = \beta_x(\bar{\alpha}(s)) = \bar{\beta}(\bar{\alpha}(s))$. and hence $\overline{\beta \circ \alpha}(s) = \bar{\beta} \circ \bar{\alpha}(s)$.

Similarly theorem 4.2 and prop 1.13 give us:

Theorem 4.5: Let $F' : SS_{\mathcal{L}}(X) \longrightarrow P_{\mathcal{L}}(X)$ be defined as:

i - If $(S, \pi) \in ob\ SS_{\mathcal{L}}(X)$ then $F'(S, \pi) = (X, \Gamma(-, S))$

ii - If $\alpha : (S, \pi) \longrightarrow (S', \pi')$ is an S-sheaves homomorphism then $F'(\alpha)$ is the P-sheaf homomorphism of 1.13

Then F' is a functor from $SS_{\mathcal{L}}(X)$ into $P_{\mathcal{L}}(X)$. Furthermore

$F' \circ F = 1_{P_{\mathcal{L}}(X)}$ and $F \circ F' = 1_{SS_{\mathcal{L}}(X)}$.

pf: That F' is a functor is trivial to see. Now we show that

$F' \circ F = 1_{P_{\mathcal{L}}(X)}$ For let $(X,P) \in Ob(P_{\mathcal{L}}(X))$ and consider

$F((X,P)) = (S,\Pi)$ where $\Gamma(\mathcal{U},S) \simeq P(\mathcal{U})$ for all $\mathcal{U} \in \mathcal{O}(X)$

Now take $F'((S,\Pi)) = (X, \Gamma(-,S))$ but by above $(X, \Gamma(-,S)) \simeq (X,P)$

thus proving $F' \circ F = 1_{P_{\mathcal{L}}(X)}$. Similarly let $(S,\Pi) \in ob\ SS_{\mathcal{L}}(X)$

and consider: $F'(S,\Pi) = (X, \Gamma(-,X))$ but $F(X, \Gamma(-,X)) = (\dot{\bigcup}_{x \in X} \Gamma(x,S), \Pi)$

but $\Gamma(x,S) = S_x$ so $F(X, \Gamma(-,S)) = (S,\Pi)$.

5. Prime Sheaves:

In all that follows X is a fixed topological space and (S,Π) is an S-sheaf of $\mathcal{L}$-structures over X.

Def 5.1: We say that a subset F of $\mathcal{O}(X)$ is a filter on X iff F satisfies:

i - $X \in F$ and $\phi \notin F$.

ii - If $\mathcal{U}_1, \mathcal{U}_2 \in F$ then $\mathcal{U}_1 \cap \mathcal{U}_2 \in F$

iii - If $\mathcal{U} \in F$ and $V \in \mathcal{O}(X)$ such that $\mathcal{U} \subseteq V$ then $V \in F$.

We say that a filter F on X is prime iff whenever $\mathcal{U}_1 \cup \mathcal{U}_2 \in F$ then either $\mathcal{U}_1 \in F$ or $\mathcal{U}_2 \in F$ We write $P_r(X)$ for the set of Prime filters on X.

We say that a filter F on X is an ultrafilter iff it is a maximal filter. We write $M(X)$ for the set of maximal filters.

Here we shall recall some of the elementary properties of ultrafilters and prime filters.

prop 5.2: i - Every ultrafilter is a prime filter and hence $M(X) \subseteq P_r(X)$

ii - Every filter on X can be extended to an ultrafilter.

iii - For every $x \in X$ the set $\mathcal{F}(x) = \{\mathcal{U} \in \mathcal{O}(X) \; ; \; x \in \mathcal{U}\}$ is a prime filter on X.

iv - The set $\{V_{\mathcal{U}} = \{F \; ; \; F \in P_r(X), \mathcal{U} \in F\} \; ; \; \mathcal{U} \in \mathcal{O}(X)\}$ form a basis for a topology on $P_r(X)$ making the map $x \longrightarrow \mathcal{F}(x)$ from X into $P_r(X)$ continuous.

The proof of this proposition is very easy and can be found in any book on topology (see Kelly [57]).

We aim to construct an s-sheaf over $P_r(X)$ using the s-sheaf (S,Π) over X. To do that let us denote by η the map defined in prop 5.2 iv. and by using the same proposition we get that whenever V is an open subset of $P_r(X)$; $\eta^{-1}(V)$ is an open subset of X. So we can define a functor: $P^o : (\mathcal{O}(P_r(X), \subseteq)^{op} \longrightarrow Str_r \mathcal{L}$ as follows:

$P^o(V) = \Gamma(\eta^{-1}(V), S)$ and if $V_1 \subseteq V$ then:

$P^o{}^V_{V_1} : \Gamma(\eta^{-1}(V), S) \longrightarrow \Gamma(\eta^{-1}(V_1), S)$

is the map defined as : $P^o{}^V_{V_1}(f) = f/\eta^{-1}(V_1)$

An argument similar to that given in the proof of theorem 4.2 gives the following result:

<u>Theorem 5.3</u>: $(P_r(X), P^o)$ is a P-sheaf of $\mathcal{L}$-structures. Such a sheaf will be called a Prime P-sheaf.

Now we use theorem 4.1 to build an S-sheaf over $P_r(X)$. The stalks of this S-sheaf are $T^o_F = \lim\limits_{F \in V} P^o(V) = \lim\limits_{F \in V} \Gamma(\eta^{-1}(V), S) = \lim\limits_{U \in F} \Gamma(U, S)$

The final limit is $\uparrow$-directed limit and it is built as in 2.

Now if $V \in \mathcal{O}(P_r(X))$ then $\Gamma(V, T^o) \simeq \Gamma(\eta^{-1}(V), S)$ by theorem 4.1.. The S-sheaf (T^o, σ^o) over $P_r(X)$ will be called Prime S-sheaf.

Similarly one may consider a P-sheaf (X,P) and then construct a Prime P-sheaf $(P_r(X), P^o)$ by putting $P^o(V) = P(\eta^{-1}(V))$ and deduce the construction of the Prime s-sheaf (T^o, σ^o) with $\Gamma(V, T^o) \simeq P^o(V)$

6. <u>Ultrasheaves</u>:

We again use the same data and notation as in 5. We need some more information from topology which we summarize here:

Def 6.1: We denote by $\overline{\mathcal{U}}$ ($Int\,\mathcal{U}$, $-\mathcal{U}$ resp.) the topological closure (resp. interior; and the interior of the closure) of the subset $\mathcal{U}$ of X. And we say that $\mathcal{U}$ is a regular open (Reg open for short) iff $\mathcal{U} = -\mathcal{U}$. We write Reg (X) for the set of reg open subsets of X. And Ult (X) will denote the set of ultrafilter in Reg (X).

Prop 6.2: i - If $F \in$ Ult (X) then $F^{o} = \{\mathcal{U} \in \mathcal{O}(X) ; -\mathcal{U} \notin F\} \in M(X)$

ii - If $F \in M$ (X) then $F^* = \{-\mathcal{U} ; \mathcal{U} \in F\} \in Ult(X)$

Furthermore if $F \in$ Ult (X) then $(F^o)^* = F$ and if $F \in M(X)$ then $(F^*)^{o} = F$

iii - The set $\{ V_{\mathcal{U}} = \{ F \in Ult(X) ; \mathcal{U} \in F \} ; \mathcal{U} \in Reg(X)\}$ form a basis for a topology on Ult (X) so that every $V_{\mathcal{U}}$ is compact. And if $V_{\mathcal{U}} \subset V_{\mathcal{U}_o}$ then $\mathcal{U} \subseteq \mathcal{U}_o$.

We do not give a proof for this easy proposition.

Finally we note that $\mathcal{U} \subseteq_d \mathcal{U}_o$ means that $\mathcal{U}$ is a dense subset of $\mathcal{U}_o$ and we remark that if $\mathcal{U} \subset_d \mathcal{U}_o$ and $\mathcal{U}'_o \subseteq \mathcal{U}_o$ then $\mathcal{U} \cap \mathcal{U}'_o \subset_d \mathcal{U}'_o$

Now we are in position to define the ultrasheaves.

Def 6.3: Let $P^* : (\mathcal{O}(Ult(X), \leq)^{op} \longrightarrow Str\mathcal{L}$ be defined as follows:

$P^*(V_{\mathcal{U}_o}) = \varinjlim_{\mathcal{U} \subset_d \mathcal{U}_o} \Gamma(\mathcal{U}, S)$; and if $V_{\mathcal{U}_o} \subset V_{\mathcal{U}'_o}$ then $P^{*\,V_{\mathcal{U}'_o}}_{\;V_{\mathcal{U}_o}} : P^*(V_{\mathcal{U}'_o}) \longrightarrow P^*(V_{\mathcal{U}_o})$ is defined as $P^{*\,V_{\mathcal{U}'_o}}_{\;V_{\mathcal{U}_o}}(|f|) = |f/_{\mathcal{U} \cap \mathcal{U}_o}|$ where $\mathcal{U} \subseteq_d \mathcal{U}'_o$ and $f \in \Gamma(\mathcal{U}, S)$. Clearly P* is a functor from $\mathcal{O}(Ult(X), \leq)^{OP}$ into $Str\mathcal{L}$. And we call (Ult (X),P*) Ultra P-sheaf.

The next theorem will justify the definition: "Ultra-P-sheaf".

Theorem 6.4: (Ult (X),P*) is a P-sheaf of $\mathcal{L}$ -structures.

pf: First we note that (Ult (X),P*) is a presheaf of $\mathcal{L}$ -structures.

We claim that this is a P-sheaf of sets. (1)

Let $V_{\mathcal{U}_0}$ be a basic open subset of Ult (X) and $(V_{\mathcal{U}_j})_{j\in J}$ be an open cover of $V_{\mathcal{U}_0}$ Thus -- $\bigcup_{j\in J} \mathcal{U}_j = \mathcal{U}_0$ so $\bigcup_{j\in J} \mathcal{U}_j \subset_d \mathcal{U}_0$ Let $a,b \in P^*(V_{\mathcal{U}_0})$ and denote by $P^{*\,j}_{\kappa,j}$ the map $P^{*\,V_{\mathcal{U}_j}\cap V_{\mathcal{U}_\kappa}}_{V_{\mathcal{U}_\kappa}}$ Assume that $P^{*\,j}_0(a) = P^{*\,j}_0(b)$ for every $j \in J$ We claim that a = b. For since $a,b \in P^*(V_{\mathcal{U}_0})$ then there is $\mathcal{U} \subset_d \mathcal{U}_0$ so that there are $f, g \in \Gamma(\mathcal{U},S)$ such that $|f| = a$ and $|g| = b$ Now $P^{*\,j}_0(a) = P^{*\,j}_0(b)$ implies $|f/\mathcal{U}\cap\mathcal{U}_j| = |g/\mathcal{U}\cap\mathcal{U}_j|$ for all $j \in J$. Thus for every $j\in J$ there exist $\mathcal{U}'_j \subset_d \mathcal{U}_j \cap \mathcal{U}$ so that $f/\mathcal{U}'_j = g/\mathcal{U}'_j$ Let $V = \bigcup_{j\in J} \mathcal{U}'_j$ From above we have $f_{/V} = g_{/V}$ Since $\mathcal{U}'_j \subset_d \mathcal{U}\cap\mathcal{U}_j \subset_d \mathcal{U}_j$ we get that $V \subset_d \bigcup_{j\in J} \mathcal{U}_j \subset_d \mathcal{U}_0$ so $V \subset_d \mathcal{U}_0$ but $|f| = |f/V|$ and $|g| = |g/V|$ so it follows that $a = |f| = |f_{/V}| = |g_{/V}| = |g| = b$

Thus condition 1 of discussion following definition 3.4 is proved. To prove the claim(1) all we need to show is that condition 2. holds.

So again let $V_{\mathcal{U}_0}$, $(V_{\mathcal{U}_j})_{j\in J}$ be as above and consider a family $(a_j)_{j\in J}$ of elements $a_j \in P^*(V_{\mathcal{U}_j})$ so that $P^{*\,j}_{j,\kappa}(a_j) = P^{*\,\kappa}_{j,\kappa}(a_\kappa)$ for all $j,\kappa \in J$. We claim that there is $a \in P^*(V\mathcal{U}_0)$ such that $P^{*\,j}_0(a) = a_j$

Since $V\mathcal{U}_0$ is compact then there is a finite subcover $V_{\mathcal{U}_1}, \ldots, V_{\mathcal{U}_n}$ of the cover $(V_{\mathcal{U}_j})_{j\in J}$ of $V\mathcal{U}_0$ It is now enough to prove the assertion for $a_1, \ldots, a_n$. Because if such a exist, then $P^{*\,m}_0(a) = a_m$ for all $m = 1, \ldots, n$.

Now let $j \in J$ and consider $V_{\mathcal{U}_j} = \bigcup_{m=1}^{n} (V_{\mathcal{U}_j} \cap V_{\mathcal{U}_m})$ Furthermore

$P^{*\,0}_{j,m}(a) = P^{*\,j}_{j,m}(P^{*\,j}_0(a)) = P^{*\,m}_{j,m}(a_m) = P^{*\,m}_{j,m}(P^{*\,m}_0(a))$.

and that is for all $m = 1, \cdots, n$.

Now consider $P^{*\,j}_{j,m}(a_j) = P^{*\,m}_{j,m}(a_m)$ for all $m = 1, \ldots, n$ so

$P^{*\,j}_{j,m}(P^{*\,j}_0(a)) = P^{*\,j}_{j,m}(a_j)$ for all $m = 1, \ldots, n$ by the above argument we have that: $P^{*j}_{o}(a) = a_j$.

So all that we need to prove is that condition 2 holds for finite open covers. We prove that by induction on the number n of the elements of the cover:

If n = 1 then condition 2 holds trivially.

If n > 2 and condition 2 holds for any cover of cardinality m less than n then consider $\bigcup_{k=1}^{n-1} V_{\mathcal{U}_k}$ so this is a cover of a basic open subset $V_{\mathcal{U}'}$ of Ult (X). By hypothesis there must be $a' \in V_{\mathcal{U}'}$ so that $P^{*\,V_{\mathcal{U}'}}_{k}(a') = a_k$

so again it is enough to show that there is a $\in P^*(V_{\mathcal{U}_o})$ so that

$P^{*\,V_{\mathcal{U}'}}_{V_{\mathcal{U}_o}}(a) = a'$ and $P^{*\,n}_{o}(a) = a_n$ Thus this case is reduced to the case of n = 2.

So let $V_{\mathcal{U}_1} \cup V_{\mathcal{U}_2} = V_{\mathcal{U}_o}$, $a_1 \in P^*(V_{\mathcal{U}_1})$; $a_2 \in P^*(V_{\mathcal{U}_2})$ so that $P^{*1}_{1,2}(a_1) = P^{*2}_{1,2}(a_2)$.

Take $V_{\mathcal{U}_1} \cap V_{-\mathcal{U}_2}$; $V_{-\mathcal{U}_1} \cap V_{\mathcal{U}_2}$ and $V_{\mathcal{U}_1} \cap V_{\mathcal{U}_2}$ This forms a dijoint cover of $V_{\mathcal{U}_o}$ Let $b_1 = P^{*1}_{1,-2}(a_1)$, $b_2 = P^{*2}_{-1,2}(a_2)$ and $c_1 = P^{*1}_{1,2}(a_1) = P^{*2}_{1,2}(a_2)$

so $b_1 \in P^*(V_{\mathcal{U}_1} \cap V_{-\mathcal{U}_2})$ and $b_2 \in P^*(V_{-\mathcal{U}_1} \cap V_{\mathcal{U}_2})$ and $c_1 = P^*(V_{\mathcal{U}_1} \cap V_{\mathcal{U}_2})$

By definition of $P^*(V_{\mathcal{U}})$ we get that there exist

$\mathcal{U}_{b_1} \subset_d \mathcal{U}_1 \cap -\mathcal{U}_2$ and $\mathcal{U}_{b_2} \subset_d -\mathcal{U}_1 \cap \mathcal{U}_2$ and $\mathcal{U}_{c_1} \subset_d \mathcal{U}_1 \cap \mathcal{U}_2$ with $f \in \Gamma(\mathcal{U}_{b_1}, S)$; $g \in \Gamma(\mathcal{U}_{b_2}, S)$ and $h \in \Gamma(\mathcal{U}_{c_1}, S)$ such that:

$|f| = b_1$ $|g| = b_2$ and $|h| = c_2$ Let $\mathcal{U} = \mathcal{U}_{b_1} \cup \mathcal{U}_{b_2} \cup \mathcal{U}_{c_1}$ clearly

$\mathcal{U} \subset_d \mathcal{U}_1$ and $t = f \cup g \cup h \in \Gamma(\mathcal{U}, S)$ let $a = |t|$

We claim that $P^{*1}_{o}(a) = a_1$ and $P^{*2}_{o}(a) = a_2$ for note that

$P^{*o}_{1,-2}(a) = P^{*}_{1,-2}(a_1)$ and $P^{*o}_{-1,2}(a) = P^{*1}_{-1,2}(a_2)$

and $P^{*o}_{1,2}(a) = P^{*1}_{1,2}(a_1) = P^{*2}_{1,2}(a_2)$ but that gives us

$P^{*1}_{1,2}(P^{*1}_{o}(a)) = P^{*1}_{1,2}(a_1)$

and $P^{*1}_{1,2}(a_1) = P^{*1}_{1,2}(P^{*1}_{o}(a))$ and $(-\mathcal{U}_2 \cap \mathcal{U}_1) \cup (\mathcal{U}_1 \cap \mathcal{U}_2) = \mathcal{U}_1$
so by condition 1. $P^{*1}_{o}(a) = a_1$, similarly for $P^{*2}_{o}(a) = a_2$.
A similar argument shows that given an n-ary relation symbol R_j
Then the Presheaf (Ult (X), R) is a p-sheaf of sets. Now using prop 3.7 we get the needed result .
Again we note that the argument works if we replace (S, π) by a P-sheaf (X,P).
We can again use theorem 4.1 to get an S-sheaf (T, σ) over Ult (X) which corresponds to (Ult (X), P*) Such an S-sheaf will have as stalks

$T_F = \varinjlim_{\mathcal{U}_o \in F} \varinjlim_{\mathcal{U} \subset_d \mathcal{U}_o} \Gamma(\mathcal{U}, S)$ and for every open subset V of Ult (X)

$$\Gamma(V, T) = P^*(V) = \varinjlim_{V_{\mathcal{U}_o} \in V} \varinjlim_{\mathcal{U} \subset_d \mathcal{U}_o} \Gamma(\mathcal{U}, S).$$

In later chapters we shall see that this construction is a generalization of the ultraproduct construction and that ultrasheaves are generic structures in some sense.

CHAPTER 2

REPRESENTATION OF $\mathcal{L}$-STRUCTURES BY SHEAVES

When studying an algebraic structure, mathematicians try to reduce this study to some simpler structures of the same type; by means of representing the initial structure as some kind of product of simpler ones. For example, Birkhoff's theorem represents algebras as subdirect product of irreducible ones. The theory of ring provides us with the following representation theorem: Every semi-simple ring is the subdirect product of primitive rings (see Herstein 68). The usefulness of this theorem can be seen in the following strategy of tackling ring theoretic problems: one proves a result for division rings then passes to primitive rings. Now using another theorem of ring theory (Herstein p.43) induces the result into Matrix rings. Now the result holds for semi simple rings so does for every ring R/J(R) where J/R) is the Jacobson Radical. All that remains to do is to show that the result holds in J (R); thus one gets the result for R. For examples of this program see Herstein (68).

In recent years sheaves provided mathematicians with a new tool in representing algebras as global sections structures and proved to be a powerful tool in analyzing and studying some algebraic structures like regular rings and modules over regular rings etc. These representation theorems were first worked out for rings by Pierce (67) then by Dauns and Hofmann (Haufmann (72)) and Keimel (71). These constructions proved to work in a much wider setting mainly a Universal-algebraic setting; under some conditions as in Wolf's (74).

In representing algebras as subdirect products or as global sections structures the congruence relations play an important role. Thus in this chapter we start by a study of the congruence relations.

In 2. we give Wolf's construction and in 3. we deduce some of the consequences of this construction. In 4. we show that every reduced product and every limit reduced product can be represented as a global section structure.

To fix our terminology let us call a 1st order Language $\mathcal{L}$ algebraic iff $\mathcal{L}$ has no relation symbols. We call A an algebra iff A is an $\mathcal{L}$ -structure where $\mathcal{L}$ is an algebraic Language.

1. Congruence Relations;

In what follows let $\mathcal{L}$ be a fixed algebraic Language; and A an algebra (i.e. $\mathcal{L}$-structure).

Def 1.1: A relation θ on A is said to be a congruence relation iff

1 - θ is an equivalence relation on A

2 - If F is any n-ary function symbol of $\mathcal{L}$; $a_1, \cdots, a_n$, $b_1, \cdots, b_n$ are elements of A so that $(a_i, b_i) \in \theta$ for all $i = 1, \cdots, n$ then $(F(a_1, \cdots, a_n), F(b_1, \cdots, b_n)) \in \theta$

Remarks: 1-This definition is equivalent to saying that θ is an equivalence relation on the set A and θ is a subalgebra of the product algebra A X A.

2-Given θ a congruence relation on A then $A/_\theta$ is an algebra of the same type as A.

3-We denote by $L(A)$ the set of all congruences on A. And if θ and ψ are elements of $L(A)$ we write $\theta \leqslant \psi$ for $\theta \subseteq \psi$

4- $L(A)$ with $\leqslant$ form a lattice with largest element $\omega = A \times A$ and smallest element $\Delta = \{(x, x) ; x \in A\}$ This assertion will follow from the next few lemmas.

Lemma 1.2: If θ and ψ are conguences on A then so is $\theta \cap \psi$ furthermore

$\theta \cap \psi$ is the largest congruence relation contained in θ and ψ.

pf: Is a simple verification of the definition we shall not write it.

Notation: Let θ and ψ be 2 congruences on A. We write $\theta \vee \psi$ for the smallest congruence relation containing both θ and ψ. The following Lemma whose proof will be sketched characterize $\theta \vee \psi$.

Lemma 1.3: For any pair $(a,b) \in A \times A$ the following are equivalent

i - $(a,b) \in \theta \vee \psi$

ii - there exists a finite sequence of elements $a_0, \cdots, a_n$ of A such that $a_0 = a$, $a_n = b$ and for all $i = 0, \cdots, n$ either $(a_i, a_{i+1}) \in \theta$ or $(a_i, a_{i+1}) \in \psi$.

pf: It is easy to show that the relation $\sim$ defined as follows: $a \sim b$ iff there exist $(a_0, \ldots, a_n) \in A^n$ with $a_0 = a$, $a_n = b$ and for all $i = 0, \ldots, n-1$ $(a_i, a_{i+1}) \in \theta \cup \psi$.

is an equivalence relation. One checks that $\sim$ is a congruence by using the notion of uniform sequences. This notion is defined as follows: Let $a_1, \cdots, a_m$, $b_1, \cdots, b_m$ be elements of A so that $a_i \sim b_i$ for all $i = 1, \cdots, m$. Hence there are sequences $a_i^0, \cdots, a_i^{n_i}$ for all $i = 1, \cdots, m$ verifying ii. These sequences are called uniform iff $n_i = n_j$ for all $i,j = 1, \cdots, m$ and If $(a_0^j, a_0^{j+1}) \in \theta$ then $(a_i^j, a_i^{j+1}) \in \theta$ for all $i = 1, \quad ,m$.

Finally; it is not difficult to see that $\sim$ is the smallest congruence containing $\theta \cup \psi$.

Corollary 1.4: $\theta \vee \psi$ is the smallest congruence relation containing θ and ψ.

Theorem 1.5: $L(A)$ is a complete lattice with 0 and 1.

pf: By Lemma 1.2 and cor.1.4 we have that $L(A)$ is a lattice moreover Δ and ω are the 0 and 1 elements of $L(A)$. Now to prove that $L(A)$ is complete it is enough to show that whenever $(\theta_i)_{i \in I}$ is a family of elements of L(A) then the neet of this family exists in L(A) but this is trivial since $\bigcap_{i\in I} \theta_i$ is clearly the meet of the θ_i 's.

Notation: Let θ and ψ be congruences on A; we write $\theta\cdot\psi$. for the set $\{(a,b) \in A \times A \;/\; \exists c \in A \;;\; (a,c) \in \theta$ and $(c,b) \in \psi\}$

As we shall see later the representation theorem which will be proved will depend on the structure of L(A); thus the need of developing this structure. It will be of great use to have L(A) distributive and permutable in what follows we discuss these notions and the conditions which if imposed on A gives a distributive and permutable L(A). We start by :

Def 1.6: i - We say that the congruences θ and ψ on A permute iff

$$\theta \vee \psi = \theta\cdot\psi$$

ii - We say that L(A) is permutable iff every 2 elements of it permute.

The following proposition characterizes the permutable elements of L(A).

prop 1.7: Let θ and ψ be elements of L(A) then θ and ψ permute iff $\theta\cdot\psi = \psi\cdot\theta$

pf: If θ and ψ permute then $\theta \vee \psi = \theta\cdot\psi$. but $\theta \vee \psi = \psi \vee \theta = \psi\cdot\theta$ so $\theta\cdot\psi = \psi\cdot\theta$

Conversely, assume $\theta\cdot\psi = \psi\cdot\theta$ It is enough to show that $\theta\cdot\psi$ is the join of θ and ψ. First we claim that $\theta\cdot\psi$ is a congruence relation to see that $\theta\cdot\psi$ is an equivalence relation. We remark the following:

i - Reflexivity: $\Delta \subseteq \theta$ and $\Delta \subseteq \psi$ but $\Delta = \Delta^2 \subseteq \theta\cdot\psi$ so $\theta\cdot\psi$ is reflexive.

ii - Symmetry: say $(x,y) \in \theta\cdot\psi$ this implies that $(x,z) \in \theta$ and $(z,y) \in \psi$ for some $z \in A$ which implies $(y, z) \in \psi$ and $(z,x) \in \theta$

implying $(y,x) \in \psi\cdot\theta = \theta\cdot\psi$ thus $(y,x) \in \theta\cdot\psi$

iii - Transitivity: $(\theta\cdot\psi)^2 = \theta\cdot\psi\cdot\theta\psi = \theta\cdot\theta\cdot\psi\cdot\psi = \theta\cdot\psi$.

iv - That $\theta\cdot\psi$ is a congruence relation follows from the cor.1.4 and the facts that $\theta \cup \psi \subseteq \theta\cdot\psi \subseteq \theta \vee \psi$.

As we mentioned before, it will turn out later on that certain conditions on L(A) will allow us to prove a strong representation theorem. Those conditions will be L(A) permutable and distributive. Here we give a characterization of the varieties (i.e. classes of algebras which are models of sets of universally quantified equations) in which every element has permutable and distributive lattice of congruences. The following Lemma is due to Mal'cev:

<u>Lemma 1.8</u>: Let Σ be a set of universally quantified equation and $K(\Sigma)$ be the class of models of Σ assume that there exist a term q (x,y,z) in the Language of Σ so that $\Sigma \vdash \forall x \forall y \forall z\, [q(x,x,z)=x \wedge q(x,y,x)=x \wedge q(x,z,z)=z]$ Then for every A $\in K(\Sigma)$, L(A) verifies the infinite distributive Law i.e. $(\theta \wedge (\bigvee_{i\in I} \theta_i) = \bigvee_{i\in I} (\theta \wedge \theta_i)$ for all $\theta, \theta_i \in L(A)$.

<u>pf</u>: Let $(\theta_i)_{i\in I}$ be a family of elements of L(A); then $\bigvee_{i\in I} \theta_i$ exists (Theorem 1.5). Furthermore it is easy to see that $(a,b) \in \bigvee_{i\in I} \theta_i$ iff there exist a finite sequence $a_0,\ldots,a_n$ of elements of A and a finite sequence $\theta_{i_0},\ldots,\theta_{i_{n-1}}$ of the family $(\theta_i)_{i\in I}$ so that $a_0 = a$ and $a_n = b$ and $(a_j, a_{j+1}) \in \theta_{i_j}$ for all $j = 0,\ldots, n-1$. Now consider $\theta \wedge (\bigvee_i \theta_i)$.

The following holds in L(A):

(i) $\theta \wedge (\bigvee_i \theta_i) \geq \bigvee_i (\theta \wedge \theta_i)$ because L(A) is a lattice. To end the proof of the lemma we need to show that

(ii) $\bigvee_{i\in I} (\theta \wedge \theta_i) \geq (\bigvee_i \theta_i) \wedge \theta$.

To do that assume $(a,b) \in \theta \wedge (\bigvee_i \theta_i)$

By the above we have $(a,b) \in \theta$ and there exist $x_o, x_1, \cdots, x_n$ and $\theta_{i_o}, \cdots, \theta_{i_{n-1}}$ so that $a = x_o$ and $b = x_n$ and:

$(x_j, x_{j-1}) \in \theta_{ij}$ for all $j = 1, \cdots, n - 1$

Now to show that $(a,b) \in \bigvee_i (\theta \wedge \theta_i)$ we need to exhibit a sequence $y_o, \cdots, y_m$ and a sequence $\theta_{i_o}, \cdots, \theta_{i_{m-1}}$ so that $a = y_o$; $b = y_m$ and $(y_j, y_{j+1}) \in \theta_{ij}$ for all $j = 1, \ldots, m-1$. We do that as follows: set $y_j = q(a,b,x_j)$ for all $j = 1, \cdots, n$. Thus

$y_o = q(a,b,x_o) = q(a,b,a) = a$ and

$y_n = q(a,b,x_n) = q(a,b,b) = b$. Furthermore

$y_j = q(a,b,x_j)$ and $y_{j+1} = q(a,b,x_{j+1})$ but since $a\ \theta_{ij}\ a$ and $b\ \theta_{ij}\ b$ and $x_j\ \theta_{ij}\ x_{j+1}$ thus $(q(a,b,x_j), q(a,b,x_{j+1})) \in \theta_{ij}$

and $y_j = q(a,b,x_j)\ \theta\ (q(a,a,x_j) = a\ \theta\ b = q(b,b,x_{j+1})\ \theta\ q(a,b,x_{j+1}) = y_{j+1}$

so $(y_j, y_{j+1}) \in \theta$ Hence $(y_j, y_{j+1}) \in \theta \wedge \theta_{ij}$ for all $j = 0, \ldots, n-1$ and the lemma is proved.

<u>Theorem 1.9</u>: Let $K(\Sigma)$ be the class of models of a set Σ of universally quantified equations then the following are equivalent:

i) For every $A \in K(\Sigma)$; $L(A)$ is a permutable lattice.

ii) There exists a term $P(x,y,z)$ so that $\Sigma \vdash \forall x \forall z [P(x,x,z) = z \wedge P(x,z,z) = x]$

<u>pf</u>: i) $\Rightarrow$ ii). Let $\mathcal{F}_3(\Sigma)$ be the free algebra in $K(\Sigma)$ generated by 3-elements a,b, and c. The identification of a and b defines a congruence relation θ_1 on $\mathcal{F}_3(\Sigma)$ similarly the identification of b and c,

so $\mathcal{F}_3(\Sigma)/\theta_1 \simeq \mathcal{F}_3(\Sigma)/\theta_2 \simeq \mathcal{F}_2(\Sigma)$ so it will be enough to show that there exist a term $p(x,y,z)$ so that $\mathcal{F}_2(\Sigma) \models \forall x \forall z [P(x,x,z) = z] \wedge$ $\mathcal{F}_2(\Sigma) \models \forall x \forall z [P(x,z,z) = x]$.

We show that as follows:

We have first that a θ_1 b and b θ_2 c thus a $\theta_1 \cdot \theta_2$ c but $\theta_1 \cdot \theta_2 = \theta_2 \cdot \theta_1$ so a $\theta_2 \cdot \theta_1$ c and there must be an element of $\mathcal{F}_3(\Sigma)$, $P(a,b,c)$ say such that $a\ \theta_2\ P(a,b,c)\ \theta_1\ C$. Consider p(x,y,z) and take the following identities p(x,x,z) = z and p(x,z,z) = x. By definition of θ_1 and θ_2 we have $\mathcal{F}_3(\Sigma)/\theta_1 \models \forall x\ \forall z\ [P(x,x,z)=z]$ and : $\mathcal{F}_3(\Sigma)/\theta_2 \models \forall x\ \forall z\,[P(x,z,z)=x]$ Hence $\mathcal{F}_L(\Sigma) \models \forall x\,\forall z\,[P(x,x,z)=z \wedge P(x,z,z)=x]$ Thus proving that ii) holds.

ii) $\Rightarrow$i) Let $A \in K(\Sigma)$ and $\theta_1, \theta_2 \in L(A)$. Assume a $\theta_1 \cdot \theta_2$ b then there exist $C \in A$ so that $a\ \theta_1 C\ \theta_2\ b$. But by(ii) we have that there exist p(x,y,z) so that p(a,b,b) = a and p(a,a,b) = b so a = p(a,b,b) θ_2 p(a,c,b) and b = p(a,a,b) θ_1 p(a,c,b) so a $\theta_2 \cdot \theta_1$ b thus we proved that $\theta_1 \cdot \theta_2 \leq \theta_2 \cdot \theta_1$ similarly we can show that $\theta_1 \cdot \theta_2 \geq \theta_2 \cdot \theta_1$ Hence $\theta_1 \cdot \theta_2 = \theta_2 \cdot \theta_1$ for any $\theta_1, \theta_2 \in L(A)$.

By prop 1.7 we have that L(A) is permutable.

The following theorem characterizes all the varieties $K(\Sigma)$; whose elements have distributive and permutable lattices of congruence relations.

<u>Theorem 1.10</u>: Let $K(\Sigma)$ be a variety then the following are equivalent:

i - For every $A \in K(\Sigma)$, L(A) is distributive and permutable.

ii - There exist terms (p(x,y,z) and q(x,y,z) such that:

$$\Sigma \vdash \forall x,y,z\,[P(x,x,z)=z \wedge P(x,z,z)=x \wedge q(x,x,z)=x \wedge q(x,y,x)=x \wedge q(x,z,z)=z]$$

<u>pf</u>: ii $\Rightarrow$ i It is just theorem 1.9 and lemma 1.8.

i $\Rightarrow$ ii If for every $A \in K(\Sigma)$ L(A) is permutable then p(x,y,z) exists by theorem 1.9. If in addition L(A) is distributive then again consider $\mathcal{F}_3(\Sigma)$ and assume $\mathcal{F}_3(\Sigma)$ is generated by a,b and c.

Consider θ_1, θ_2 as in theorem 1.9 and θ_3 to be the congruence relation identifying a and c thus $\mathcal{F}_3(\Sigma)/\theta_1 \simeq \mathcal{F}_3(\Sigma)/\theta_2 \simeq \mathcal{F}_3(\Sigma)/\theta_3 \simeq \mathcal{F}_2(\Sigma)$

Thus $(a,b) \in \theta_3 \cdot \theta_2 = \theta_2 \cdot \theta_3$ so $(a,b) \in \theta_2 \vee \theta_3$ and $(a,b) \in \theta_1$

thus $(a,b) \in \theta_1 \wedge (\theta_2 \vee \theta_3) = (\theta_1 \wedge \theta_2) \vee (\theta_1 \wedge \theta_3) = (\theta_1 \wedge \theta_2) \cdot (\theta_1 \wedge \theta_3)$

Thus there exists an element $q(a,b,c) \in \mathcal{F}_3(\Sigma)$ so that $(a, q(a,b,c)) \in (\theta_1 \wedge \theta_2)$ and $(q(a,b,c), b) \in (\theta_1 \wedge \theta_3)$. Hence:

$a\ \theta_1\ q(a,a,c)$ and $a\ \theta_2\ q(a,b,a)$ and $b\ \theta_3 (q,a,b,b)$ thus by definition of θ_1, θ_2 and θ_3 we have:

$\mathcal{F}_3(\Sigma)/\theta_1 \models \forall x \forall y (q(x,x,y) = x)$ and $\mathcal{F}_3(\Sigma)/\theta_2 \models \forall x \forall y [q(x,y,x) = x]$ and $\mathcal{F}_3(\Sigma)/\theta_3 \models \forall x \forall z [q(x,z,z) = z]$ but $\mathcal{F}_3(\Sigma)/\theta_i \simeq \mathcal{F}_2(\Sigma)$ for all i=1,2,3 so $\mathcal{F}_2(\Sigma) \models \forall x \forall y \forall z [q(x,x,z) = x \wedge q(x,y,x) = x \wedge q(x,z,z) = z]$

Thus proving that ii holds.

<u>Examples</u>: 1. Let Σ be the axioms of group theory then:

$$\Sigma \vdash \forall x \forall z\ [p(x,x,z) = z \wedge p(x,z,z) = x] \quad \text{Where} \quad p(x,y,z) = x \cdot y^{-1} z$$

Thus if G is a group then L(G) is permutable. Similarly if G is a ring.

2. Let Σ be the axioms of "distributive" lattices.

Consider $q(x,y,z) = [(x \wedge y) \vee z] \wedge (x \vee y)$ then:

$$\forall x\ \forall z \quad q(x,x,z) = [(x \wedge x) \vee z] \wedge (x \vee x) = (x \vee z) \wedge x = x$$
$$\forall x\ \forall y \quad q(x,y,x) = [(x \wedge y) \vee x] \wedge (x \vee y) = x \wedge (x \vee y) = x$$
$$\forall x\ \forall z \quad q(x,z,z) = [(x \wedge z) \vee z] \wedge (x \vee z) = z \wedge (x \vee z) = z$$

Hence for every lattice M; L(M) is distributive.

3. Let Σ be the axioms of lattice ordered ring.

Consider p(x,y,z) and q(x,y,z) of 1 and 2. Theorem 1.10 implies that for every lattice ordered ring A; L(A) is distributive and permutable. The following theorem will be very useful in proving the representation theorem.

<u>Theorem 1.11</u> (The Chinese remainder theorem): Let A be an algebra and be a finite family of congruences such that

i - $\theta_i \vee \theta_j = \theta_i \cdot \theta_j$ for all $i,j = 1,\cdots,n$

ii - the sublattice C of L(A) generated by $\theta_1,\cdots,\theta_n$ is distributive

assume that $a_1,\cdots,a_n \in A$ so that $(a_i, a_j) \in \theta_i \cdot \theta_j$ For all $i,j=1,\dots,n$;

then there exists an element $a \in A$ so that $(a,a_i) \in \theta_i$ for all $i \leq n$.

In particular if L(A) is distributive and permutable then for any $\theta_1,\cdots,\theta_n \in L(A)$ and $a_1,\cdots,a_n \in A$ such that $(a_i, a_j) \in \theta_i \cdot \theta_j$ there exists $a \in A$ such that $(a, a_i) \in \theta_i$ for all $i \leq n$.

pf: The proof is by induction on n.

If $n = 1$ there is nothing to prove.

If $n = 2$ and $(a_1,a_2) \in \theta_1 \cdot \theta_2$ then by definition of $\theta_1 \cdot \theta_2$ there is $c \in A$ so that $(a_1,c) \in \theta_1$ and $(c,a_2) \in \theta_2$.

Now assume the assertion has been proved for all $k \leq n$ where $n > 2$ and assume $\theta_1,\cdots,\theta_n$ and $a_1,\cdots,a_n$ are as in the hypothesis of the theorem. We consider first $a_1,\cdots,a_{n-1}$ and $\theta_1,\cdots,\theta_{n-1}$ by induction there exists $b \in A$ so that $(a_i,b) \in \theta_i$ for all $i = 1,\cdots,n-1$

Now we know that $(a_i,a_n) \in \theta_i \cdot \theta_n$ for all $i = 1,\cdots,n$ so there are $b_1,\cdots,b_n$ so that $(a_i,b_i) \in \theta_i$ and $(b_i,a_n) \in \theta_n$ thus $(b,b_i) \in \theta_i$ and $(b_i,a_n) \in \theta_n$ for all $i = 1,\cdots,n-1$ so $(b,a_n) \in \theta_i \vee \theta_n$ for all $i = 1,\cdots,n-1$ thus $(b,a_n) \in \bigwedge_{i=1}^{n-1} (\theta_i \vee \theta_n)$ but since C is distributive then $\bigwedge_{i=1}^{n-1} (\theta_i \vee \theta_n) = \theta_n \vee (\bigwedge_{i=1}^{n-1} \theta_i)$ thus $(b,a_n) \in (\bigwedge_i \theta_i) \cdot \theta_n$

Therefore: there exists $a \in A$ so that $(b,a) \in \bigwedge_i \theta_i$ and $(a,a_n) \in \theta_n$ so $(b,a) \in \theta_i$ for all $i = 1,\cdots,n-1$ and thus $(a_i,a) \in \theta_i$ for all $i = 1,\cdots,n$

In what follows we shall discuss the topological structure of L(A).

We need the following definition:

Def 1.12: A congruence relation θ is said to be irreducible iff $\theta \neq \Delta$ whenever $\theta_1 \wedge \theta_2 \leq \theta$ then either $\theta_1 \leq \theta$ or $\theta_2 \leq \theta$ We write X for the set of irreducible congruences on A.

Let ψ be an element of L(A), we write X for the set of irreducible congruences θ such that $\psi \nleq \theta$. And write $\mathcal{O}$(X) for the set $\{X_\psi / \psi \in L(A)\}$

prop 1.13: i - $\mathcal{O}$(X) is a topology on X

ii - The map f : L(A) $\longrightarrow \mathcal{O}$(X) defined by f $(\psi) = X_\psi$ is a lattice isomorphism from L(A) onto $\mathcal{O}$(X)

iii - For any $Y \subseteq X$, Y is closed iff whenever $\theta \in X$ such that $\bigwedge_{\psi \in Y} \psi \leq \theta$ then $\theta \in Y$

iv - For any $Y \subseteq X$, Y is dense iff $\bigwedge_{\psi \in Y} \psi = \Delta$

pf: i - First we note that $\phi = X_\Delta$ and $X = X_\omega$ and hence ϕ and X are in $\mathcal{O}$(X). Now we need to show:

(1) If X_{ψ_1} and X_{ψ_2} are in $\mathcal{O}$(X) then $X_{\psi_1} \cap X_{\psi_2} \in \mathcal{O}$(X) and

(2) If $(X_{\psi_i})_{i \in I}$ is a family of elements of $\mathcal{O}$(X) then $\bigcup_I X_{\psi_i} \in \mathcal{O}(X)$

For (1) say $\theta \in X_{\psi_1} \cap X_{\psi_2}$ that is iff $\psi_1 \nleq \theta$ and $\psi_2 \nleq \theta$ and $\theta \in X$ iff

$$\psi_1 \wedge \psi_2 \nleq \theta \quad \text{thus } X_{\psi_1} \cap X_{\psi_2} = X_{\psi_1 \wedge \psi_2} \in \mathcal{O}(X)$$

For (2) say $\theta \in \bigcup_i X_{\psi_i}$ that is iff $\exists j \in I$ so that $\theta \in X_{\psi_j}$ iff $\exists j \in I$ $\psi_j \nleq \theta$ iff $\theta \ngeq \bigvee_{j \in I} \psi_j$ iff $\theta \in X_{\bigvee_{i \in I} \psi_i}$ thus $\bigcup_{i \in I} X_{\psi_i} = X_{\bigvee_i \psi_i} \in \mathcal{O}(X)$

ii - Clearly f is a homomorphism from L(A) onto $\mathcal{O}$(X) so all that is left to show is that f is injective, for:

Let $\psi_1 \neq \psi_2$ then there are $a \neq b \in A$ so that $a \psi_1 b$ and $(a,b) \notin \psi_2$

Write $\psi(a,b)$ for the largest congruence relation θ such that $(a,b) \notin \theta$

Clearly $\psi(a,b)$ is irreducible and $\psi(a,b) \in X_{\psi_1}$ and $\psi(a,b) \notin X_{\psi_2}$

so $X_{\psi_1} \neq X_{\psi_2}$ proving that f is injective.

iii - Let $Y \subseteq X$ then Y is closed iff $Y = X - X_\psi$ for some $\psi \in L(A)$ iff there is $\psi \in L(A)$ such that $Y = \{\theta \in X \,/\, \psi \leq \theta\}$

Now assume that Y is closed so there is $\psi \in L(A)$ such that $Y = \{\theta \in X;\ \psi \leq \theta\}$

Let $\varphi \in X$ be such that $\bigwedge_{\theta \in Y} \theta \leq \varphi$ Hence $\psi \leq \varphi$ and so $\varphi \in Y$

Conversely let $\psi = \bigwedge_{\theta \in Y} \theta$ clearly $Y = \{\theta \in X;\ \theta \geq \psi\}$ and so Y is closed.

iv - Let $Y \subseteq X$ be such that $\bigwedge_{\theta \in Y} \theta = \Delta$ We denote by $\overline{Y}$ the closure of Y in X. Now $Y \subseteq \overline{Y}$ so $\bigwedge_{\theta \in \overline{Y}} \theta \leq \bigwedge_{\theta \in Y} \theta = \Delta$ and therefore $\overline{Y} = \{\theta \in X;\ \Delta \in \theta\} = X$ so Y is dense.

Conversely: assume Y is dense since $Z = \{\theta \in X;\ \bigwedge_{\psi \in Y} \psi \leq \theta\}$ is closed and $Y \subseteq Z$ then Z must be equal to X and therefore $\bigwedge_{\psi \in Y} \psi = \Delta$.

2. Sheaf Representation of Algebraic System

In what follows L(A), X and X_ψ where $\psi \in L(A)$ will have the same meaning as before. Let Z be a topological space and $f : Z \to L(A)$ such that for any pair $(a,b) \in A \times A$ the set $\{z \in Z;\ (a,b) \in f(z)\}$ is an open subset of Z. In what follows we shall construct an S-sheaf of algebras of the same type as A so that there is a homomorphism from A into the global section structure of this s-sheaf.

Let $S_z = A/f(z)$ for every $z \in Z$ and consider $S = \dot{\bigcup_{z \in Z}} S_z$ we denote by $\pi : S \to Z$ the map defined by $\pi(s) = z$ iff $s \in S_z$

If $a \in A$ we shall write a_z for the equivalence class of a w.r.t $f(z)$ and $\hat{a}$ for the map which corresponds to every element z of Z the element a_z of S. Finally for every $\mathcal{U}$ open in Z, we shall denote by $V_{\mathcal{U}}^{a}$ the set $\hat{a}(\mathcal{U}) = \{\hat{a}(z);\ z \in \mathcal{U}\}$

Let $\mathcal{B}$ be the set of all those $V_{\mathcal{U}}^{a}$'s i.e. the set $\{ V_{\mathcal{U}}^{a} ; a \in A ; \mathcal{U} \in \theta(Z) \}$
We shall show that (S, π) is an $\mathcal{S}$-sheaf. The proof of this fact follows immediately from the following series of lemmas:

Lemma 2.1: $\mathcal{B}$ is a basis of a topology on S which makes all the maps $\hat{a}$ continuous and open.

pf: We note first that $\phi = V_{\phi}^{a}$ for all $a \in A$ so $\phi \in \mathcal{B}$ Now to show that $\mathcal{B}$ is a basis it is enough to show that whenever $V_{\mathcal{U}}^{a}$ and $V_{\mathcal{U}_1}^{b}$ are in $\mathcal{B}$ then $V_{\mathcal{U}}^{a} \cap V_{\mathcal{U}_1}^{b}$ is in $\mathcal{B}$. We may assume that $V_{\mathcal{U}}^{a} \cap V_{\mathcal{U}_1}^{b} \neq \phi$
Let $\mathcal{U}_o = \{ z \in Z ; \hat{a}(z) = \hat{b}(z) \} = \{ z \in Z ; (a, b) \in f(z) \}$ by assumption $\mathcal{U}_o$ is an open subset of Z. We take $\mathcal{U}_o \cap \mathcal{U}_1 \cap \mathcal{U} = \mathcal{U}_2$ this is an open subset of Z clearly $\hat{a}(\mathcal{U}_2) \subseteq V_{\mathcal{U}}^{a} \cap V_{\mathcal{U}_1}^{b}$ We show that $V_{\mathcal{U}}^{a} \cap V_{\mathcal{U}_1}^{b} \subseteq \hat{a}(\mathcal{U}_2)$
For say $a_z = b_y \in V_{\mathcal{U}}^{a} \cap V_{\mathcal{U}_1}^{b}$ so $z = \pi(a_z) = \pi(a_y) = y$ and hence $a_z = b_y$ and $z \in \mathcal{U}_o \cap \mathcal{U}_1 \cap \mathcal{U} = \mathcal{U}_2$ So $\hat{a}(z) = a_z \in \hat{a}(\mathcal{U}_2)$
hence $\hat{a}(\mathcal{U}_2) = V_{\mathcal{U}}^{a} \cap V_{\mathcal{U}_1}^{b}$ and $\mathcal{B}$ is a basis for a topology on S.
To see that this topology makes all the a's continuous and open consider a basic open $V_{\mathcal{U}_1}^{b}$ and take $\hat{a}^{-1}(V_{\mathcal{U}_1}^{b}) = \{ x \in \mathcal{U}_1 ; \hat{a}(x) = \hat{b}(x) \}$? which is equal to the set $\{ x \in Z , (a, b) \in f(x) \} \cap \mathcal{U}_1$ which is an open subset of Z. So $\hat{a}$ is continuous.
Now if $\mathcal{U}$ is open subset of Z, then $\hat{a}(\mathcal{U})$ is an open subset of S by definition so $\hat{a}$ is open.

Lemma 2.2: π is a local homeomorphism from S onto Z.

pf: first we note that for every $z \in Z$, $A/f(z) \neq \phi$ thus π is an onto map.
Now we show that π is a local homeomorphism.
Let $t = a_z$ be an element of S. Hence there is a basic open $V_{\mathcal{U}}^{a}$ such that $t \in V_{\mathcal{U}}^{a}$ i.e. $z \in \mathcal{U}$ Consider $\pi / V_{\mathcal{U}}^{a} : V_{\mathcal{U}}^{a} \longrightarrow \mathcal{U}$

It is easy to see that $\pi / V_{\mathcal{U}}^{a}$ is onto; to show that $\pi / V_{\mathcal{U}}^{a}$ is 1-1 say $\hat{a}(x), \hat{a}(y) \in V_{\mathcal{U}}^{a}$ so that $\pi(\hat{a}(x)) = \pi(\hat{a}(y))$ i.e. $x = y$ so clearly $a(x) = a(y)$.

To see that $\pi / V_{\mathcal{U}}^{a}$ is continuous an open remark that $\pi / V_{\mathcal{U}}^{a} \circ \hat{a} = 1_{V_{\mathcal{U}}^{a}}$ and $\hat{a} \circ \pi / V_{\mathcal{U}}^{a} = 1_{\mathcal{U}}$ so; since $\hat{a}$ is continuous and open $\pi / V_{\mathcal{U}}^{a}$ must be continuous and open. Thus Lemma 2.2 is proved.

Lemma 2.3: Let F be any function symbol of the Language $\mathcal{L}$; say F is n-ary; then $F : \sum_{1}^{n} S \to S$ as defined in Chapter 1 is continuous.

pf: Let $\langle s_1, \quad, s_n \rangle$ be an element of $\sum_{1}^{n} S$. Then $\forall i, j, \pi(s_i) = \pi(s_j) = x$ say, so $s_i \in A/f(x)$ for all $i = 1, \cdots, n$. We may write $s_i = a_x^i = \hat{a}^i(x)$

Now consider $F(s_1, \cdots, s_n) = F(\bar{s}) = F(\overline{\hat{a}(x)}) = F(\bar{a}_x) = (F(\bar{a}))_x$

Let $V_{\mathcal{U}}^{b}$ be a basic open containing $(F(\bar{a}))_x$ Hence $(F(\bar{a}))_x = b_x$ and consider $\mathcal{U}_0 = \{x \in \mathcal{U} ;\ (F(\bar{a}))_x = b_x \}$ this is an open subset of Z

Now since every $\hat{a}^i$ is open then $V_i = \hat{a}^i(\mathcal{U}_0)$ is an open subset of S.

Let $V_0 = (\prod_{i=1}^{n} V_i) \cap \sum_{1}^{n} S$ thus V_0 is an open subset of $\sum_{1}^{n} S$

Clearly $F(V_0) \subseteq V_{\mathcal{U}}^{b}$ Thus F is continuous at $\langle s_1, \cdots, s_n \rangle$ but $\langle s_1, \cdots, s_n \rangle$ is an arbitrary element of $\sum_{1}^{n} S$; so F is continuous.

We summarize:

Theorem 2.4: The above constructed (S, π) is an S-sheaf of algebras of the same type as A over Z; furthermore the map $a \to \hat{a}$ defines a homomorphism of algebras from A into $\Gamma(Z, S)$. This homomorphism is injective iff $\bigwedge_{z \in Z} f(z) = \Delta$

pf: The first assertion is just the Lemmas 2.1; 2.2 and 2.3.

Now let $a \in A$ As we have seen in Lemma 1. $\hat{a}$ is continuous moreover for every $z \in Z$, $\hat{a}(z) = a_z \in S_z$ so $\pi(\hat{a}(z)) = z$ and so $\pi \circ \hat{a} = 1_Z$

Hence $\hat{a} \in \Gamma(Z, S)$ Let $\varphi : A \to \Gamma(Z, S)$ defined as $\varphi(a) = \hat{a}$

To show that φ is a homomorphism take an atomic sentence $\tau_1(a_1,\dots,a_n) =$ $= \tau_2(a_1,\dots,a_n)$ and assume $A \models \tau_1(a_1,\dots,a_n) = \tau_2(a_1,\dots,a_n)$ Consider $\tau_1(\hat{a}_1,\dots,\hat{a}_n)$ and $\tau_2(\hat{a}_1,\dots,\hat{a}_n)$ By Lemma 1.3 we have $\tau_1(\hat{a}_1,\dots,\hat{a}_n) = \widehat{\tau_1(a_1,\dots,a_n)}$ but $\tau_1(a_1,\dots,a_n) = \tau_2(a_1,\dots,a_n)$ so $\widehat{\tau_1(a_1,\dots,a_n)} = \widehat{\tau_2(a_1,\dots,a_n)}$ and $\Gamma(Z,S) \models \tau_1(\hat{a}_1,\dots,\hat{a}_n) = \tau_2(\hat{a}_1,\dots,\hat{a}_n)$

Now assume $\bigwedge_{z\in Z} f(z) = \Delta$ we show that φ is injective.

Let $\hat{a} = \hat{b}$ (i.e. $\hat{a}(x) = \hat{b}(x)$) for all $x \in Z$ thus $(a,b) \in f(x)$ for all $x \in Z$ so $(a,b) \in \bigwedge_{x\in Z} f(x) = \Delta$ thus a = b.

Conversely assume φ injective, we claim that $\bigwedge_{z\in Z} f(z) = \Delta$

For we need to show that if $(a,b) \in \bigwedge_{z\in Z} f(z)$ then a = b.

If $(a,b) \in \bigwedge_{z\in Z} f(z)$ then $(a,b) \in f(z)$ for all $z \in Z$ i.e. $a_z = b_z$ for all $z \in Z$ that is to say $\hat{a}(z) = \hat{b}(z)$ for all $z \in Z$ so $\hat{a} = \hat{b}$ or equivalently $\varphi(a) = \varphi(b)$ but φ is injective so a = b.

An immediate application of this theorem is the following:

<u>Theorem 2.5</u>: Let Y be a dense subspace of the space X and let Z be any topological space such that there exist a continuous map g from Y into Z Then there is an S-sheaf (S, π) of algebras of the same type as A over Z such that A is a subalgebra of $\Gamma(Z, S)$.

<u>pf</u>: For any $\mathcal{U} \in \mathcal{O}(Z)$ let us denote by $\theta_{\mathcal{U}}$ the congruence relation $\left(\bigwedge_{\psi\in g^{-1}(\mathcal{U})} \psi\right)$ and let θ_z be $\bigvee_{z\in\mathcal{U}\in\mathcal{O}(Z)} \theta_{\mathcal{U}} = \bigvee_{z\in\mathcal{U}} \theta_{\mathcal{U}}$

Let $f : Z \longrightarrow L(A)$ so that $f(z) = \theta_z$ Let us first remark that $\bigwedge_{z\in\mathcal{U}} f(z) = \theta_{\mathcal{U}}$ For in fact $\theta_{\mathcal{U}} \leq f(z)$ for all $z \in Z$ so $\theta_{\mathcal{U}} \leq \bigwedge_{z\in\mathcal{U}} f(z)$ Now assume that $(a,b) \notin \theta_{\mathcal{U}}$ thus there exist $\psi \in g^{-1}(\mathcal{U})$ so that $(a,b) \notin \psi$ so $g(\psi) = x \in \mathcal{U}$ therefore $(a,b) \notin \theta_V$ for any open subset V of Z such that $x \in V$ For the sake of contradiction assume $(a,b) \in \bigwedge_{z \in \mathcal{U}} f(z)$; hence $(a,b) \in f(x)$ and so there must be $V_1, \cdots, V_n$ and $a_1, \ldots, a_n$ sequences of open subsets of Z and elements of A so that: $(a_i, a_{i+1}) \in \theta_{V_i}$ and $a_1 = a$ and $a_n = b$ this is because $f(x) = \bigvee_{x \in V} \theta_V$ and $x \in \bigcap_{i=1}^{n} V_i$ By definition of θ_{V_i} we shall have that $(a_i, a_{i+1}) \in \psi$ for all $i = 1, \cdots, n-1$ but is transitive so $(a,b) \in$ contradicting the choice of and so $\theta_{\mathcal{U}} = \bigwedge_{z \in Z} f(z)$

It follows from that, that $\bigwedge_{z \in Z} f(z) = \bigwedge_{\psi \in g^{-1}(z)} \psi = \bigwedge_{\psi \in Y} \psi$ but Y is dense in X so $\bigwedge_{\psi \in Y} \psi = \Delta$ by prop (1.13.iv).

Now to be able to apply our construction and theorem 2.4 it is enough to show that for any $(a,b) \in A \times A$ the set $Z_{(a,b)} = \{z \in Z ; (a,b) \in f(z)\}$ is an open subset of Z. We do that as follows:

Let $x \in Z_{(a,b)}$ We shall show that there is an open set V of Z containing x and such that $V \subseteq Z_{(a,b)}$ For we consider θ_x by definition of $Z_{(a,b)}$ We have that $(a,b) \in \theta_x$ so there are $\mathcal{U}_1, \ldots, \mathcal{U}_{n-1} \in \mathcal{O}(z)$ with $x \in \bigcap_{i=1}^{n} \mathcal{U}_i$ and $a_1, \cdots, a_n \in A$ with $a = a_1$; $b = a_n$ and $(a_i, a_{i+1}) \in \theta_{\mathcal{U}_i}$

Let $V = \bigcap_{i=1}^{n} \mathcal{U}_i$; $x \in V$ We claim that $V \subseteq Z_{(a,b)}$

<u>pf</u> of the claim: Let $y \in V$ hence $y \in \mathcal{U}_i$ for all $i = 1, \cdots, n-1$ but we have proved that $\theta_{\mathcal{U}_i} = \bigwedge_{z \in \mathcal{U}_i} f(z)$ so $(a_i, a_{i+1}) \in f(y)$ for all $i = 1, \cdots, n-1$. Thus $(a,b) \in f(z)$ and hence $y \in Z_{(a,b)}$

Now the hypothesis of theorem 2.4 holds so an application of this theorem gives us the result needed.

In what follows we shall assume that either L(A) or a proper sublattice of L(A) is distributive and permutable. This assumption will allow us to strengthen the 2 previous theorems.

Let us consider an algebra A with distributive and permutable lattice of congruences L(A); let M be any distributive lattice with 0 and 1. We denote by $Pr(M)$ the set of ideals P of M which have the following property if $a \wedge b \in P$ then either $a \in P$ or $b \in P$ (i.e. the set of Prime ideals of M). Let $Z \subseteq Pr(M)$ For any element $c \in M$ we write Z_c for the set $\{P \in Z ; c \notin P\}$ Clearly the set $\{Z_c ; c \in M\}$ forms a basis of a topology on Z. Finally let f be a map from M into L(A) and let us denote by: $\tilde{c}$

$\tilde{c}$ the element $\bigvee\{f(d) ; d \wedge c = 0\}$ these elements exist because L(A) is complete; for every $c \in M$; and for every Prime ideal P of M Let $r(P)$ be the element $\bigvee_{c \notin P} \tilde{c}$ Further we assume that f verifies the following conditions:

(2.I) $f(0) = \Delta$

(2.II) $f(c \wedge d) = f(c) \wedge f(d)$

(2.III) $f(c \vee d) = f(c) \vee f(d) = f(c) \cdot f(d)$

(2.IV) For every $c \in M$ $\tilde{c} = \bigwedge_{P \in Z_c} r(P)$

Remark: Condition (2.IV) implies the following: $f(1) = \omega = A \times A$

For $f(1) = \bigvee_{d \in M} f(d) = \bigvee\{f(d) ; d \wedge 0 = 0\} = \tilde{0} = \bigwedge_{0 \notin P} r(P) = \wedge \phi$

but $\wedge \phi = A \times A = \omega$ Thus f is a lattice homomorphism from M into L(A).

Now we are in position to construct an S-sheaf of algebras over Z.

For every $P \in Z$ Let $S_P = A/r(P)$ and for every element $a \in A$ let a_P be the equivalence class of a w. r. t. $r(P)$. Let S be $\bigcup_{P \in Z} S_P$ and define $\pi : S \longrightarrow Z$ as $\pi(s) = P$ iff $s \in S_P$; further denote by $\hat{a}$ the map which corresponds to every element P of Z the element a_P of S. We endow S with the finest topology making all the maps $\hat{a}$ continuous.

The following proposition is similar to theorem 2.2. We omit its proof.

<u>prop 2.6</u>: 1 - The set $\{\hat{d}(Z_c) ; d \in A ; c \in M\}$ forms a basis of a topology on S.

2 - (S, π) is an S-sheaf of algebra; of the same similarity type as A; over Z.

3 - $\Gamma(Z, S)$ is an algebra of the same similarity type as A.

4 - the map $a \to \hat{a}$ is a homomorphism from A into $\Gamma(Z, S)$

Now we prove the main theorem:

<u>Theorem 2.7</u>: With the same notation as above we assume further that Z contains all the maximal ideals of M and that f verifies conditions I, II, III and IV then $A \simeq \Gamma(Z, S)$ where (S, π) is the S-sheaf of prop 2.6.

<u>pf</u>: All that we have to show is that the map $a \to \hat{a}$ is a bijection.

We start by injectivity:

We first remark that $\tilde{1} = \vee\{f(d) / d \wedge 1 = 0\} = f(0) = \Delta$ since $1 \notin P$ for all $P \in Z$ then by condition IV we have that $\tilde{1} = \bigwedge_{P \in Z} r(P) = \Delta$

Now let a_1, a_2 be elements of A such that $\hat{a}_1 = \hat{a}_2$ therefore for every $P \in Z$ $a_{1P} = a_{2P}$ i.e. $(a_1, a_2) \in r(P)$ thus $(a_1, a_2) \in \bigwedge_{P \in Z} r(P) = \Delta$ and hence $a_1 = a_2$ proving that $a \to \hat{a}$ is injective.

Now we prove the surjectivity of the map $a \to \hat{a}$.

Let $g \in \Gamma(Z, S)$ thus for every $P \in Z$ there is an element a^P of A. such that $g(P) = \hat{a}^P(P)$ Now consider $\hat{a}^P(Z)$. This is an open subset of S by 1 of prop 2.6 Thus $g^{-1}(\hat{a}^P(Z))$ is an open subset of Z because g is continuous, and $P \in g^{-1}(\hat{a}^P(Z))$ therefore there is a basic open Z_{c_P} say such that $P \in Z_{c_P} \subseteq g^{-1}(\hat{a}^P(Z))$ [i.e. $c_P \notin Q$ for all $Q \in Z_{c_P}$] And $g / Z_{c_P} = \hat{a}^P / Z_{c_P}$

Now let I be the ideal generated by c_P. We claim that I is not a proper ideal. For if it were, then there would be a maximal ideal P_0

containing it. But $P_o \in Z$ by hypothesis so $c_P \notin P_o$ which contradicts the fact that $I \subseteq P_o$; so $I = M$ Thus there must be elements $c_1, \cdots, c_n \in M$ and $a_1, \cdots, a_n \in A$ such that $c_1 \vee \cdots \vee c_n = 1$ (since $I = M$ and $1 \in I$) and $\hat{a}_i / z_{c_i} = g / z_{c_i}$ $(i = 1, \ldots, n)$

We claim that for any $i, j = 1, \cdots, n$ $\quad (a_i, a_j) \in (c_i \wedge c_j)^{\sim}$

To prove the claim we remark that $Z_{c_i \wedge c_j} = Z_{c_i} \cap Z_{c_j}$ Thus for any $P \in Z_{c_i \wedge c_j}$ we have $g(P) = \hat{a}_i(P) = \hat{a}_j(P)$ and hence $(a_i, a_j) \in r(P)$ for all $P \in Z_{c_i \wedge c_j}$ and so $(a_i, a_j) \in \bigwedge_{c_i \wedge c_j \notin P} r(P) = (c_i \wedge c_j)^{\sim}$ (by condition IV) proving the claim.

Now since $(c_i \wedge c_j)^{\sim} = \vee \{ f(d) ; d \wedge c_i \wedge c_j = 0 \}$ there must be an element $d_{ij} \in M$ such that $d_{ij} \wedge c_i \wedge c_j = 0$ and $(a_i, a_j) \in f(d_{ij})$

We may assume that $d_{ij} = d_{ji}$

Now for any $k = 1, \ldots, n$ We have:

$(a_i, a_k) \in f(d_{ik})$ and $(a_k, a_j) \in f(d_{kj})$ but since L(A) is permutable : $(a_i, a_j) \in f(d_{ik}) \cdot f(d_{kj}) = f(d_{ik} \vee d_{kj})$

$\therefore \quad (a_i, a_j) \in \bigwedge_{k=1}^{n} f(d_{ik} \vee d_{kj}) = f\left(\bigwedge_{k=1}^{n} (d_{ik} \vee d_{kj}) \right)$

But $\bigvee_{\ell=1}^{n} c_\ell = 1$ $\quad$ thus $(a_i, a_j) \in f\left[\bigwedge_{k=1}^{n} (d_{ik} \vee d_{kj}) \wedge \left(\bigvee_{\ell=1}^{n} c_\ell \right) \right]$

But M is distributive so:

$$f\left[\bigwedge_{k=1}^{n} (d_{ik} \vee d_{kj}) \wedge \left(\bigvee_{\ell=1}^{n} c_\ell \right) \right] = f\left(\bigvee_{\ell=1}^{n} \left[\bigwedge_{k=1}^{n} (d_{ik} \vee d_{kj}) \wedge c_\ell \right] \right) \leq$$

$$f\left(\bigvee_{\ell=1}^{n} (d_{i\ell} \vee d_{\ell j}) \wedge c_\ell \right) = f\left(\bigvee_{\ell=1}^{n} [(d_{i\ell} \wedge c_\ell) \vee (d_{\ell j} \wedge c_\ell)] \right) =$$

$$f\left(\bigvee_{\ell=1}^{n} (d_{i\ell} \wedge c_\ell) \vee \left(\bigvee_{\ell=1}^{n} (d_{i\ell} \wedge c_\ell) \right) \right] = f\left(\bigvee_{\ell=1}^{n} (d_{\ell i} \wedge c_\ell) \right) \cdot f\left(\bigvee_{\ell=1}^{n} (d_{\ell j} \wedge c_\ell) \right)$$

put $d_i = \bigvee_{\ell=1}^{n} (d_{\ell i} \wedge c_\ell)$ thus we have :

$(a_i, a_j) \in f(d_i) \cdot f(d_j)$

But L(A) is distributive and permutable therefore the Chinese Remainder Theorem (Theorem 1.11) holds well for A. Thus there is $a \in A$ such that

$\forall i = 1, \ldots, n \quad (a, a_i) \in f(d_i)$

Now remark that $d_i \wedge c_i = \left[\bigvee_{\ell=1}^{n} (d_{i\ell} \wedge c_\ell) \right] \wedge c_i = \bigvee_{\ell=1}^{n} (d_{i\ell} \wedge c_\ell \wedge c_i) = 0$

thus $(a, a_i) \in \tilde{c}_i$ $\qquad$ and for every $P \in z_{c_i}$; $(a, a_i) \in r(P) \therefore g/z_{c_i} = \hat{a}/z_{c_i}$

for all $i = 1, \cdots, n$ But $\bigcup_{i=1}^{n} Z_{c_i} = Z_{\bigvee_{i=1}^{n} c_i} = Z_1 = Z$ so it follows that for every $P \in Z$; $g(P) = \hat{a}(P)$ Thus $g = \hat{a}$ proving that $a \to \hat{a}$ is surjective.

3. Applications

First let us make some remarks and observations.

Theorem 2 becomes interesting when M is a sublattice of $L(A)$. And the map f is the injection map. Thus we are led to study the conditions I, II, III and IV. The first 3 conditions are natural ones in the sense that f becomes a lattice homomorphism taking 0 to the smallest congruence relation on A. The fourth condition, though, seems a bit unnatural; still holds in quite general cases as the following Lemma shows:

Lemma 3.1: Consider the following 4 assertions:

1 - For every $c \in M$, $\tilde{c} = \bigwedge_{P \in Z_c} \left(\bigvee_{d \in P} f(d) \right)$.

2 - Whenever $c \in M$ and $a, b \in A$ such that the set $K = \{ c \wedge d \,/\, (a,b) \in f(d) \}$ generates a proper filter F of M then there exist a prime ideal P in Z such that $P \cap F = \phi$.

3 - $\bigcap_{P \in Z} P = \phi$ and for every pair $(a,b) \in A \times A$ there is a smallest element c of M such that $(a,b) \in f(c)$.

4 - Z contains all the minimal prime ideals of M.

Then: i - 1) $\Leftrightarrow$ 2) $\Rightarrow$ condition IV;

ii - 3) $\Rightarrow$ 2) $\Rightarrow$ condition IV;

iii - 4) $\Rightarrow$ 2) $\Rightarrow$ condition IV.

<u>pf</u>: i - 1) $\Rightarrow$ 2). Let c, (a,b), and K be as in 2) and assume that K generates a filter F (proper). Thus if $c \wedge d \in K$ then $c \wedge d \neq 0$ and $(a,b) \in f(d)$. So by definition of $\tilde{c}$ we have $(a,b) \notin \tilde{c}$ and by 1) there is $P_0 \in Z_c$ such that $(a, b) \notin \bigvee_{d \in P_0} f(d)$; we claim that $P_0 \cap F = \phi$.

In fact, if it were not then there would be $e \in M$ and $d \in M$ such that $c \wedge d \in K$ and $c \wedge d \leq e \in P_0$ so $c \wedge d \in P_0$ but P_0 is Prime and $c \notin P_0$ thus $d \in P_0$. But now $c \wedge d \in K$ implies that $(a,b) \in f(d)$ and $d \in P_0 \Rightarrow (a,b) \notin f(d)$, a contradiction. Thus proving the fact that 2) holds.

2) $\Rightarrow$ 1). Clearly $\tilde{c} \leq \bigwedge_{p \in Z_c} (\bigvee_{d \in P} f(d))$.

Now assume $(a,b) \notin \tilde{c} = \bigvee_{d \wedge c = 0} f(d)$ and let $K = \{c \wedge d / (a,b) \in f(d)\}$ thus $0 \notin K$, moreover if $c \wedge d_1$, $c \wedge d_2$ are elements of K, $(c \wedge d_1) \wedge (c \wedge d_2) = c \wedge (d_1 \wedge d_2) \in K$, and so it is not equal to 0 hence K generates a proper filter F by 2), there is $P_0 \in Z$ s.t. $P_0 \cap F = \phi$ so since $c \in F$, $c \notin P_0$; and $P_0 \in Z_c$ but if $(a,b) \in \bigvee_{d \in P_0} f(d)$ there would be $d \in P_0$ s.t. $(a,b) \in f(d)$ and thus $c \wedge d \in P_0 \cap F$ contradicting $P_0 \cap F = \phi$. Hence $(a,b) \notin (\bigvee_{d \in P_0} f(d))$ and so $(a,b) \notin \bigwedge_{P \in Z_c} (\bigvee_{d \in P_0} f(d))$ proving 1).

1) $\Rightarrow$ condition IV. First we remark that for any $p \in Z_c$ $(c \in M)$, $r(p) = \bigvee_{d \notin P} \tilde{d}$ so $\tilde{c} \leq r(p)$ for all $p \in Z_c$ and thus $\tilde{c} \leq \bigwedge_{p \in Z_c} r(p)$.

To prove that $\bigwedge_{p \in Z_c} r(p) \leq \tilde{c}$: remark that $\tilde{c} = \bigvee_{d \wedge c = 0} f(d) \leq \bigvee_{d \in P} f(d)$ for any $p \in Z_c$; that, because if $d \wedge c = 0$ and $p \in Z_c$ then $d \in P$ (because P is a Prime ideal). So it follows that:

$r(p) = \bigwedge_{c \notin P} \tilde{c} \leq \bigvee_{d \in P} f(d)$, $\bigwedge_{p \in Z_c} r(p) \leq \bigwedge_{p \in Z_c} (\bigvee_{d \in P} f(d)) = \tilde{c}$, by (1).

ii - 3) $\Rightarrow$ 2). Let c, a, b, K, F be as in 2). Let c_1 be the smallest element of M such that $(a,b) \in f(c_1)$. Hence $c \wedge c_1 \leq c \wedge d$ for all $d \in M$ such that $(a,b) \in f(d)$ So F is generated by $c \wedge c_1$. Now since $\bigcap_{P \in Z} P = 0$ there must be $P_0 \in Z$ such that $c \wedge c_1 \notin P_0$. If $P_0 \cap F \neq \phi$ there would be an element $d \in P_0$ such that $d \geq c \wedge c_1$ but P_0 is an ideal so $c \wedge c_1 \in P_0$, contradicting the choice of P_0.

iii - 4) $\Rightarrow$ 2). Again let C,a,b,k and F be as in 2). We denote by $I = \{ c \wedge d \ / \ (a,b) \notin f(d) \}$ and by P_o the minimal Prime ideal generated by I. We claim that $P_o \cap F = \phi$.

Otherwise there would be an $e \in P_0 \cap F$, thus there are $d_1, d_2 \in M$ with $(a,b) \notin f(d_1)$ and $(a,b) \in f(d_2)$ such that $c \wedge d_1 \geqslant e \geqslant c \wedge d_2$. So $c \wedge d_1 \in P_0 \cap F$. If $c \wedge d_1 \in I$ then let $I_1 = I - \{ c \wedge d_1 \}$. I_1 generates an ideal. Let P_1 be the smallest prime ideal containing I_1, $c \wedge d_1 \notin P_1$ so $P_1 \subsetneqq P_0$, contradicting the minimality of P_o so $c \wedge d_1 \notin I$ and $(a,b) \in f(d_1)$, again contradicting the choice of d_1; and thus $P_0 \cap F = \phi$. Now since Z contains all minimal prime ideals, proving 2).

Now we are ready to apply the theorem 2.7 to the cases where M is a sublattice of L(A).

<u>Cor 3.2</u>: Let M be a sublattice of L(A) with W, Δ in M. Let $Z = Pr(M)$, then there is an s-sheaf (S, π) over Z such that $A \simeq \Gamma(Z, S)$

<u>pf:</u> Since $Z = Pr(M)$ then all the minimal prime ideal of M are in Z. By 4 of Lemma 3.1 the injection map verifies conditions I,II,III and IV, moreover every maximal ideal of M is prime thus it is in Z so theorem 2.7 can be applied.

<u>Def 3.3</u>: A congruence relation θ on A is called a factor iff there is a congruence ψ on A such that $\theta \wedge \psi = \Delta$ and $\theta \vee \psi = W$.

<u>Remark 3.4</u>: The set F(A) of factors is a Boolean algebra providing L(A) is distributive and permutable.

<u>Cor 3.5</u>: Let Z be the stone space of F(A), then there is an S-sheaf (S, π) over Z such that $A \simeq \Gamma(Z,S)$.

<u>Remark 3.6</u>: If $Z \subset Pr(M)$ where M is a distributive lattice as before

then Z is dense iff $\bigcap_{P \in Z} P = \{0\}$.

We say that Z is full it contains all the maximal ideals.

Cor 3.7: Let M be a sublattice of L(A) with $\nabla, \Delta \in M$. Assume that Z is either $P_r(M)$ or a full and dense subset of it. And that for every $(a,b) \in A \times A$ there is a smallest element $\theta \in M$ such that $(a,b) \in \theta$, then there is an s-sheaf over Z such that $A \simeq \Gamma(Z, S)$.

pf: Clearly 3) of Lemma 3.1 holds so conditions I,II,III and IV hold for the injection map from M to L(A) and Z contains all the maximal ideals of M. Thus theorem 2.7 gives the result.

A particular case is when $M = L(A)$ and Z is either $P_r(L(A))$ or a dense and full subspace of it.

Cor 3.8: Let K be an equational class such that there are 3-ary terms $p(x,y,z)$ and $q(x,y,z)$ such that the following identities hold in

1. $p(x,x,z) = z$ and $p(x,z,z) = x$,
2. $q(x,x,z) = x$ and $q(x,y,x) = x$ and $q(x,z,z) = z$.

Then for every $A \in K$, there is a topological space Z and an s-sheaf (S,π) over Z such that $A \simeq \Gamma(Z,S)$.

pf: By theorem 1.10 L(A) is distributive and permutable. Thus cor.3.2 3.5 or 3.7 can be applied to get the result.

Remark: In fact theorem 1.11. allows us to just consider the case where the sublattice of L(A) generated by $f(M)$ is distributive and permutable since we do not use the congruences which are outside this sublattice. This remark allows us to apply theorem 2.7 to a wider range of algebraic structures. In particular to those algebras whose L(A) generated by those elements which either permute with every element of L(A) or which are in the centre of L(A) i.e. which are distributive with every pair of elements of L(A). Thus we may apply this theorem (Theorem 2.7) to the classes of groups; rings, lattices and Boolean algebras.

The next paragraph will be very useful in discussing the sentences preserved by the global sections, it will allow us to use a theorem of Keisler in that discussion.

4. Representation of Reduced Products and Limit Reduced Products

Let J be a non-empty set of index; by $E(J)$ we denote the set of equivalence relation on J.

It is easy to check that $E(J)$ together with the inclusion relation form a complete lattice. Let F be a proper filter in this lattice.

For every $j \in J$ let A_j be an $\mathcal{L}$ -structure and let $A = \prod_{j \in J} A_j$ be the product of these structures.

For every element f of A we define the relation $[f]$ on J as follows: $(i,j) \in [f]$ iff $A_i = A_j$ and $f(i) = f(j)$.

$[f]$ is trivially an equivalence relation.

We shall say that f is an F-element iff $[f] \in F$.

Finally let D be a filter on J and we consider the Boolean algebra (B.A. for short) $2^J/D$.

Here we note that for every ultrafilter in $2^J/D$ there exists an ultrafilter on J corresponding to it and containing D.

Let X be the stone space of $2^J/D$ and denote by $\{F_x \mid x \in X\}$ the set of all ultrafilters on J containing D and corresponding to the elements of X.

Def 4.1: i - The structure $\prod A_j/D = A/D$ is called the reduced product of the family $(A_j)_{j \in J}$ with respect to D.

ii - The substructure of $\prod A_j/D = A/D$ whose domain is formed by the equivalence classes of F-elements of A is called the limit reduced product of A_j's w.r.t. to D and F we write A_D/F for this structure.

Def 4.2: Let (S, π) be an s-sheaf over X.

i - We say that (S, π) is quasi-Boolean iff X is a Boolean space.

ii - We say that (S, π) is Boolean iff X is Boolean and $\underset{\sim}{2}$ is endowed by the discrete topology and for every n-ary relation R of $\mathcal{L}$; the map $\chi_R : \sum_1^n S \longrightarrow \underset{\sim}{2}$ is continuous.

Remark: If $\mathcal{L}$ is algebraic (i.e. $\mathcal{L}$ has no relation symbols) then i and ii in the above definition are equivalent.

Now let $\prod_{j\in J} A_j/D$ (resp. $\prod A_j/D \,/\, F = A_D/F$) be a reduced product (resp. a limit reduced product). We use the same notation of above to construct s-sheaves (S, π) and (S', π') over X (X the stone space of $2^J/D$) so that $\prod_{j\in J} A_j/D \simeq \Gamma(X, S)$ and $A_D/F \simeq \Gamma(X, S)$.

Construction 4.3: For every $x \in X$, let $S_x = \prod A_j/F_x$ and $S_x' = A_{F_x}/F$

Let $S = \bigcup_{x\in X} S_x$ and $S' = \bigcup_{x\in X} S'_x$ and $\pi : S \to X$; $\pi' : S' \to X$ be defined as $\pi(s) = x$ iff $s \in X$ and $\pi'(s') = x$ iff $s' \in S'_x$.

We note that S'_x is a substructure of S_x and so : $\pi' = \pi/S'$.

Let $A = \prod_{j\in J} A_j$ and for every element $a \in A$ denote by $\hat{a}$ the map $\hat{a} : X \to S$ so that $\hat{a}(x) = |a|_{F_x}$.

Similarly for every F-element a of A; $\hat{a} : X \to S'$ is defined as $\hat{a}(x) = |a|_{F_x}$.

Endow S and S' with the finest topology making all the $\hat{a}$'s continuous (In the case of S' we restrict ourselves to the $\hat{a}$'s where a is an F-element.)

Lemma 4.4: (S, π) is a Boolean S-sheaf over X.

pf: We prove that π is a local homeomorphism from X onto S. Clearly is surjective. To show that π is a local homeomorphism:

Let $s \in S$ and put $\pi(s) = x$. we may assume $s = |a|_{F_x}$. Let N be any basic open of X so that $x \in N$ and put $M = \hat{a}(N)$. M is an open subset of S containing s.

Claim I: π is surjective from M onto N.

pf: For every $x \in N$ take $|a|_{F_x}$.

Claim II: π is injective from M onto N.

pf: Let s,t be in M so that $\pi(s)=\pi(t)=x$; since s,t are in M we have that $s = |a|_{F_x}$ and $t = |a|_{F_y}$ by the above y = x so $F_x = F_y$ and thus s = t.

Claim III: π is continuous from M onto N.

pf: Let $s = |a|_{F_x}$ be in M. Let $V = N \cap U$ where U is a basic open in X containing x, put H = a ($\hat{V}$) clearly H is open subset of M, $s \in H$ and $\pi(H) \subseteq V$, so π is continuous at every point in M.

Claim IV: π is open from M onto N.

pf: Say H is an open subset of M and put $V = \pi\ (H)$; clearly $V \subseteq N$. Let $x \in V$, we show that there is an open subset of N containing x and contained in V.

We may consider H as a basic open i.e. $H = M \cap \hat{b}\ (U)$ where U is basic open of X. We note that if $y \in N \cap U$ then $|a|_{F_y} = |b|_{F_y}$.

Thus $\hat{a}$ (H) = V but $\hat{a}$ (H) is open. Thus to prove our claim it is enough to choose U in the above as to contain x.

Thus π is a local homeomorphism from S onto X.

To prove that (S, π) is a Boolean s-sheaf:remark that X is Boolean and endow $\underset{\sim}{2}$ with the discrete topology and consider R an n-ary relation symbol of $\mathcal{L}$ and let $\chi_R : \sum_1^n S \to \underset{\sim}{2}$ be the map $\chi_R(s_1, \cdots, s_n) = 1$ iff $S_x \models R(s_1, \cdots, s_n)$.

Claim V: χ_R is continuous.

pf: Let $\langle s_1, \cdots, s_n\rangle$ be in $\sum_1^n S$ then $\chi_R(s_1,\cdots,s_n)=1$ iff $A/F_x \models R(s_1,\cdots,s_n)$ iff $\{j\in J / A_j \models R(a_1(j),\cdots,a_n(j))\} \in F_x$ (Łos 's theorem) where the a_i's are elements of A so that $|a_i|_{F_x} = s_i$ for all $i = 1,\cdots,n$.
Let $U=\{j\in J / A_j \models R(a_1(j),\cdots,a_n(j))\}$ and let N_U be the set $\{y\in X / U\in F_y\}$ clearly N_U is a basic open in X. We consider $V_i = \hat{a}_i(N_U)$ for all $i = 1,\cdots,n$ and $V = \prod_{i=1}^n V_i \cap \sum_1^n S$, this is an open subset of $\sum_1^n S$ and it is easy to see that $\chi_R(V) \subseteq \{1\}$. Thus if $\langle s_1,\cdots,s_n\rangle \in \sum_1^n S$ is such that $\chi_R(\langle s_1,\cdots,s_n\rangle)=1$ one can find an open subset V of $\sum_1^n S$ containing $\langle s_1,\cdots,s_n\rangle$ with $\chi_R(V) \subseteq \{1\}$. Similarly if $\chi_R(\langle s_1,\cdots,s_n\rangle)=0$. Thus the claim is proved and so is the lemma.
A similar argument shows that:

Lemma 4.5: (S', π') is a Boolean s-sheaf over X.

Lemma 4.6: The congruence relations on $A/_D$ determined by the F_x's; where $x\in X$; are permutable and distributive.

pf: Say that $\sim_{F_x}$'s are those congruence relations and let $a \sim_{F_x}\cdot\sim_{F_y} b$ where $a, b\in A/D$; this is to say that there exist C in $A/_D$ so that $a\sim_{F_x} c\sim_{F_y} b$ or equivalently the sets $V=\{j / a(j)=c(j)\}$ and $U=\{j / b(j)=c(j)\}$ are respectively in F_x and F_y . Let d be the element of A defined as

$d_j = b_j$ if $j\in V$
$d_j = a_j$ if $j\in U$
$d_j = c_j$ if $j\in J-(V\cup U)$ and consider $|d|_D$. This exists because d is well defined (if $j\in U\cap V$ then $d(j)=a(j)=c(j)=b(j)$).
Now we note that the set $V'=\{j / d(j)=b(j)\}=V$ and $U'=\{j / d(j)=a(j)\} = U$, so $V'\in F_x$ and $U'\in F_y$ and hence

$a \sim_{F_y} d \sim_{F_x} b$ so $\sim_{F_x} \cdot \sim_{F_y} \subseteq \sim_{F_y} \cdot \sim_{F_x}$.

A similar argument shows that $\sim_{F_y} \cdot \sim_{F_x} \subseteq \sim_{F_x} \cdot \sim_{F_y}$.

Thus $\sim_{F_x}$ permutes with $\sim_{F_y}$.

A similar construction shows the distributivity of those congruences.

Lemma 4.7: Again let $\sim_{F_x}$'s be the congruences on A_D/F determined by the F_x's then they are permutable and distributive.

~~pf~~: with the same notation as in Lemma 4.6. We assume that a,b,c are all F-elements. We show that the element d is an F-element.

We remark that $[a] \wedge [b] \wedge [c] \leq [d]$. For say $(i,j) \in [a] \wedge [b] \wedge [c]$ then

$A_i = A_j$ and $a_i = a_j$ and $b_i = b_j$ and $c_i = c_j$ thus $A_i = A_j$ and $d_i = d_j$ so $(i,j) \in [d]$ Hence $[d] \in F$ (F is a filter and $[a], [b], [c] \in F$)

So $|d|_D \in A_D/F$ and the proof of the Lemma is now similar to the Lemma 4.6.

Lemma 4.8: A/D is isomorphic to $\Gamma(X, S)$.

pf: The map $\varphi : A/D \longrightarrow \Gamma(X,S)$ defined as $\varphi(|a|_D) = \hat{a}$ will be proved to be the isomorphism needed.

It is easy to check that every $\hat{a}$ is a global section over X. And φ is well fined because $D \subseteq F_x$ for all $x \in X$. Furthermore we have that $D = \bigcap_{x \in X} F_x$. So we claim:

1 - φ is injective: say $\hat{a} = \hat{b}$ thus $|a|_{F_x} = |b|_{F_x}$ for all $x \in X$ and hence $\{ j \in J \,/\, a(j) = b(j)\} \in \bigcap_{x \in X} F_x = D$, so $|a|_D = |b|_D$.

2 - for the surjectivity of φ one uses Lemma 4.6 and an argument similar to the one used in the representation theorems of algebraic structures Theorem 2.7.

3 - to prove that φ is an isomorphism let $\varphi(v_1, \cdots, v_n)$ be an atomic

formula. We may restrict ourselves to the case where

$\psi(v_1, \cdots, v_n) = R(v_1, \cdots, v_n)$ and assume $A/D \models R(|a_1|_D, \cdots, |a_n|_D)$.

This is equivalent to say $\{ j \in J / A_j \models R(a_1(j), \cdots, a_n(j))\} \in D = \bigcap_{x \in X} F_x$.

i.e. $\{ j \in J / A_j \models R(a_1(j), \cdots, a_n(j))\} \in F_x$

for all $x \in X$, i.e. $A/F_x \models R(\hat{a}_1(x), \cdots, \hat{a}_n(x))$ for all $x \in X$

or equivalently $\Gamma(X, S) \models R(\hat{a}_1, \cdots, \hat{a}_n)$

One proves in a similar manner:

Lemma 4.9: A_D/F is isomorphic to $\Gamma(X, S')$.

We summarize all these Lemmas:

Theorem 4.10: Every reduced product and every limit reduced product is isomorphic to a global section structure of a Boolean S-sheaf (S, π) over a Boolean space X.

CHAPTER 3

FORCING; AND PRESERVATION THEOREMS IN SHEAVES

In the first chapter we defined sheaves and gave a slight generalization to the ultraproduct construction which we called ultrasheaf, and we defined the prime sheaf. In the first part of this chapter we aim to relate these notions. To do that we define forcing in sheaves which, apart from its own inteiest, provides us with Łos' type theorems for prime sheaves and ultrasheaves.

In the second part we shall be concerned with the following question: Given an $\mathcal{S}$-sheaf (S, π) off structures over a topological space X. What sentences are preserved by $\Gamma(X, S)$?
Feferman-Vaught theorem and Keisler theorem provide us with a powerful tool in answering this question for Boolean S-sheaves. Thus we give the needed version of this theorem, then we study the theories preserved by limits and we prove that those theories are exactly those preserved by global sections.

1. Forcing

The idea of "Forcing" was first introduced by Cohen in 1963 in his proofs of the independence of the axiom of choice and the continuum hypothesis. Later Robinson adopted this idea to prove some results in Model theory. In 1963 too Kripke made use of the forcing in analyzing the semantic of intuitionistic logic. In 1974 Ellerman introduced this idea to the P-sheaves and proved that forcing can provide the Łoś' type theorems for prime and ultra-P-sheaves. In what follows we shall consider the $\mathcal{S}$-sheaves and define "the forcing-value" in a similar way to Ellerman. We also give the definition of Robinson's infinite forcing and Kripke's forcing and study the relation between those notions. Then we proceed to prove the Łoś's type theorems.

Our notation will be the one used in Chang and Keisler (1973)

A - Robinson's infinite forcing

Let $\mathcal{L}$ be a first-order Language; $\sum$ a class of $\mathcal{L}$ -structures and $M \in \sum$

Let φ be a sentence defined in M .

Def 1.A.1: The relation $M \Vdash_R \varphi$ (read M infinitely forces φ) is defined inductively as follows:

1 - If φ is atomic then $M \Vdash_R \varphi$ iff $M \models \varphi$

2 - If $\varphi = \psi \vee \chi$ then $M \Vdash_R \varphi$ iff either $M \Vdash_R \psi$ or $M \Vdash_R \chi$

3 - If $\varphi = \psi \wedge \chi$ then $M \Vdash_R \varphi$ iff $M \Vdash_R \psi$ and $M \Vdash_R \chi$

4 - If $\varphi = \exists v\, \psi(v)$ then $M \Vdash_R \varphi$ iff there exists $a \in M$ such that $M \Vdash_R \psi(a)$

5 - If $\varphi = \neg \psi$ then $M \Vdash_R \varphi$ iff there is no $M' \in \sum$ such that $M \subseteq M'$ and $M' \Vdash_R \psi$.

And we say that M weakly forces φ iff $M \Vdash_R \neg\neg\varphi$. We write $M \Vdash_R^* \varphi$ for M weakly forces φ .

Def 1.A.2: We say that M is generic in Σ iff for every sentence φ defined in M either $M \Vdash_R \varphi$ or $M \Vdash_R \neg\varphi$

The following proposition summarizes the elementary properties of forcing and generic structures given by the above definitions. All the structures considered are in .

prop 1.A.3: i - If φ is a sentence defined in M then M cannot infinitely force both φ and $\neg\varphi$

ii - If $M \Vdash_R \varphi$ then $M \Vdash_R^* \varphi$ and if $M \Vdash_R^* \neg\varphi$ then $M \Vdash_R \neg\varphi$

iii - If $M \subseteq M'$ and $M \Vdash_R \varphi$ then $M' \Vdash_R \varphi$

iv - If $\sum$ is inductive (i.e. closed under the union of chain) then every $M \in \Sigma$ is a substructure of a generic structure

v - The following are equivalent:

1 - M is generic

2 - For every φ defined in M; $M \models \varphi$ iff $M \Vdash_R \varphi$

3 - For every sentence of the form $\neg\varphi$ defined in M

$M \models \neg\varphi$ iff $M \Vdash_R \neg\varphi$

vi - If M, M' are generic and $M \subseteq M'$ then $M \preccurlyeq M'$

vii - Assume Σ inductive then every generic structure M is existentially closed (i.e. if φ is an existential sentence defined in M and true in an extension M' of M such that $M' \in \Sigma$ then φ is true in M)

viii - Every elementary substructure of a generic structure is generic.

pf: i - Follows from part 5 of def 1.A.1.

ii - We claim first that if $M \subseteq M' \in \Sigma$ and $M \Vdash_R \varphi$ then $M' \Vdash_R \varphi$ We show this by an inductive argument. As an example we prove the case of $\neg\varphi$. Assume $M \Vdash_R \neg\varphi$ since $M \subseteq M'$ and $M' \in \Sigma$ then $M' \nVdash_R \varphi$ Now if $M'' \in \Sigma$ and $M' \subseteq M''$ then $M'' \supseteq M$ and so $M'' \nVdash \varphi$ So by def 1.A.1( ) $M' \Vdash_R \neg\varphi$

Now say $M \Vdash_R \varphi$ we want to show that $M \Vdash_R \neg\neg\varphi$ If not then there is $M' \in \Sigma$; such that $M \subseteq M'$ and $M' \Vdash_R \neg\varphi$ but $M' \Vdash_R \varphi$ so $M' \Vdash \varphi \wedge \neg\varphi$ contradicting i.

Now we show that if $M \Vdash_R^* \neg\varphi$ then $M \Vdash_R \neg\varphi$. Say $M \Vdash_R^* \neg\varphi$ then $M \Vdash_R \neg\neg\neg\varphi$ that is iff for no $M' \in \Sigma$, $M \subseteq M'$, $M' \Vdash_R \neg\neg\varphi$ but by definition 1.A.1 this is to say that $M \Vdash_R \neg\varphi$ h

iii - proved in ii

iv - First let $(\varphi_\alpha)_{\alpha < \beta}$ be a list of all the sentences defined on M then construct the following chain:

$M_0 = M$ and suppose M_α has been constructed let :

$M_{\alpha+1} = M_\alpha$ if either $M_\alpha \Vdash_R \varphi_\alpha$ or $M_\alpha \Vdash_R \neg\varphi_\alpha$ otherwise

Let $M_{\alpha+1}$ be any extension M' of M such that $M' \Vdash_R \varphi_\alpha$ (such extension must exist because otherwise $M_\alpha \Vdash_R \neg\varphi_\alpha$)

If δ is a limit ordinal let $M_\delta = \bigcup_{\alpha<\delta} M_\alpha$ ($M_\delta \in \Sigma$ because Σ is inductive)

Consider $M' = \bigcup_{\alpha<\beta} M_\alpha$ and build the following chain :

$M^0 = M$

$M^{n+1} = (M^n)'$ and put $M^\omega = \bigcup_{n<\omega} M^n$ clearly $M^\omega \in \Sigma$,

and it is easy to see from the construction that M^ω is generic.

v - Assume M is generic. Then by an easy induction one can prove that if φ is a sentence defined in M ; $M \models \varphi$ iff $M \Vdash_R \varphi$ hence $1 \Rightarrow 2$ and $1 \Rightarrow 3$. To prove that $3 \Rightarrow 2$ one uses an induction argument the only difficulty is when $\varphi = \neg\psi$ but this is assertion 3 itself. To prove that $2 \Rightarrow 1$ let φ be a sentence defined in M thus either $M \models \varphi$ or $M \models \neg\varphi$ so by 2 either $M \Vdash_R \varphi$ or $M \Vdash_R \neg\varphi$ and hence M is generic.

vi - Let φ be defined in M then :

$M \models \varphi \Rightarrow M \Vdash_R \varphi \Rightarrow M' \Vdash \varphi \Longrightarrow$ (by genericity of M') $M' \models \varphi$

thus $M \preceq M'$

vii - If M is generic let $\exists v\, \varphi(v)$ be an existential sentence defined in M and true in $M' \supseteq M$. Since Σ is inductive there is a generic structure M'' with $M'' \geq M' \geq M$ so $M'' \models \exists v\, \varphi(v)$ by vi - $M \leq M''$ so $M \models \exists v\, \varphi(v)$

viii - Let $M \leq M'$ and M' be generic let φ be a sentence defined in M then $M \Vdash_R \varphi$ implies that $M' \Vdash_R \varphi$ which implies that $M' \models \varphi$ which implies that $M \models \varphi$

Now an induction argument shows that for any structure $M \in \Sigma$ and any sentence φ defined in M: if $M \models \varphi$ then $M \Vdash_R \varphi$ therefore by(v -) M is generic.

B - Kripke's Structure and Forcing: In his paper "A Semantic Analysis of Intuitionistic Logic I", Kripke defined what he called intuitionistic models and the truth value of formulae in these models. These ideas amounted to an independent discovery of forcing. In what follows we shall give the definition of these models and truth values. In this paragraph we shall denote by $P_{\mu(i)}$'s the propositional and predicate letters of arity $\mu(i)$ of $\mathcal{L}$; by $For\mathcal{L}$ we mean the set of all formulae of $\mathcal{L}$

Def 1.B.1: A Kripke structure is a triple $< \mathcal{G} , R, dom>$ where $\mathcal{G}$ is a non-empty set; R a transitive and reflexive relation on $\mathcal{G}$ and dom is a map from $\mathcal{G}$ to a set of non-empty sets such that if $H, H' \in \mathcal{G}$ and $H R H'$ then $dom(H) \subseteq dom(H')$

Def 1.B.2: A Kripke truth value function is a map ϑ from $For\mathcal{L} \times \mathcal{G}$ in to the set $\{T, F\}$ such that:

K_1 - If $P_{\mu(i)}$ is a propositional letter then $\vartheta(P_{\mu(i)}, H)$ is either T or F.

K_2 - If $P_{\mu(j)}$ is a predicate let $V(P_{\mu(j)}, H)$ be a subset of $[dom(H)]^{\mu(j)}$ and define for $a_1, \dots, a_{\mu(j)} \in dom(H)$ the value $\vartheta(P_{\mu(j)}(a_1, \dots, a_{\mu(j)}), H) = T$ iff $< a_1, \dots, a_{\mu(j)} > \in V(P_{\mu(j)}, H)$ and $\vartheta(P_{\mu(j)}(a_1, \dots, a_{\mu(j)}), H) = F$ otherwise

K_3 - If $H R H'$ for $H, H' \in \mathcal{G}$ and for $a_1, \dots, a_{\mu(j)} \in dom(H)$ and $\vartheta(P_{\mu(j)}(a_1, \dots, a_{\mu(j)}), H) = T$ then $\vartheta(P_{\mu(j)}(a_1, \dots, a_{\mu(j)}), H') = T$.

Now we continue the definition of Kripke's truth value by induction on the degree of complexity of the formula φ. So we assume that $\varphi(\overline{v})$

is a formula and $\bar{a}$ is a sequence of elements of dom (H) so that the length of $\bar{a}$ equals the length of $\bar{v}$.

K4 - If $\varphi(\bar{v}) = \psi(\bar{v}) \wedge \chi(\bar{v})$ then
$\vartheta(\varphi(\bar{a}), H) = T$ iff $\vartheta(\psi(\bar{a}), H) = \vartheta(\chi(\bar{a}), H) = T$
otherwise $\vartheta(\varphi(\bar{a}), H) = F$

K5 - If $\varphi(\bar{v}) = \psi(\bar{v}) \vee \chi(\bar{v})$ then $\vartheta(\varphi(\bar{a}), H = T$ iff either $\vartheta(\psi(\bar{a}), H) = T$ or $\vartheta(\chi(\bar{a}), H) = T$ otherwise $\vartheta(\varphi(\bar{a}), H) = F$.

K6 - If $\varphi(\bar{v}) = \exists w\, \psi(\bar{v}, w)$ then $\vartheta(\varphi(\bar{a}), H) = T$ iff there is $b \in dom(H)$ so that $\vartheta(\psi(\bar{a}, b), H) = T$ otherwise $\vartheta(\varphi(\bar{a}), H) = F$.

K7 - If $\varphi(\bar{v}) = \neg\psi(\bar{v})$ then $\vartheta(\varphi(\bar{a}), H) = T$ iff for all $H' \in \mathcal{G}$ such that $H R H'$, $\vartheta(\psi(\bar{a}), H') = F$ otherwise $\vartheta(\varphi(\bar{a}), H) = F$.

<u>Remarks 1.B.3</u>:

(1) In definition 1.B.1 one could consider an indexed set $\mathcal{G}$; say $\{H_i; i \in I\}$ and a relation R' on I so that $(i, j) \in R'$ iff $H_i R H_j$ Hence one could take a functor from a preordered set into a preordered set (by considering both sets as categories).

(2) $\mathcal{G}$ is intended to be a set of $\mathcal{L}$ -structures and dom is the domain map. Thus if $H, H' \in \mathcal{G}$ and $H R H'$ then $\mathrm{dom}(H) \subseteq dom(H')$.

(3) Condition K3 is equivalent to the following condition:
K3' - If $H, H' \in \mathcal{G}$, $H R H'$ and $\varphi(\bar{v})$ is a formula, $\bar{a} \in dom(H)$ then $\vartheta(\varphi(\bar{a}), H) = T$ implies $\vartheta(\varphi(\bar{a}), H') = T$

To show that $K3 \Rightarrow K3'$ one argues by induction on the degree of complexity of $\varphi(\bar{v})$.

Def 1.B.4: Let $\langle \mathcal{G}, R, \mathrm{dom} \rangle$ be a Kripke structure and ϑ a Kripke truth value function. Let $\varphi(\bar{v})$ be a formula in $\mathcal{L}$ and $\bar{a}$ a sequence of elements of dom (H). Then we say that H forces $\varphi(\bar{a})$ (write $H \Vdash_K \varphi(\bar{a})$) iff $\vartheta(\varphi(\bar{a}), H) = T$

And we say that H is generic iff for every formula $\varphi(\bar{v})$ of $\mathcal{L}$ and every $\bar{a} \in dom H$ either $H \Vdash_K \varphi(\bar{a})$ or $H \Vdash_K \neg\varphi(\bar{a})$

The following proposition relates Kripke forcing to Robinson forcing:

prop 1.B.5: Let $\mathcal{G}$ be a set of $\mathcal{L}$-structures and consider $\langle \mathcal{G}, /, dom \rangle$ then $\langle \mathcal{G}, \subseteq, dom \rangle$ is a Kripke structure. Furthermore if $H \in \mathcal{G}$ and $\varphi(\bar{a})$ is a sentence defined in H then $H \Vdash_K \varphi(\bar{a})$ implies $H \Vdash_R \varphi(\bar{a})$

pf: That $\langle \mathcal{G}, \subseteq, dom \rangle$ is a Kripke structure is easy to verify. Now the truth value defined in the usual manner for atomic formula, i.e. if P is an n-ary predicate symbol of $\mathcal{L}$ then

$\vartheta(P(a_1, \ldots, a_n), H) = T$ iff $\langle a_1, \ldots, a_n \rangle \in \underset{\sim}{P}$ provides the truth map which can be extended to a Kripke truth value function. Now definition 1.B.4 is equivalent to the following inductive definition:

1. if $\varphi(\bar{v})$ is atomic then $H \Vdash_K \varphi(\bar{a})$ iff $H \models \varphi(\bar{a})$
2. if $\varphi(\bar{v}) = \psi(\bar{v}) \wedge \chi(\bar{v})$ then $H \Vdash_K \varphi(\bar{a})$ iff $H \Vdash_K \psi(\bar{a})$ and $H \Vdash_K \chi(\bar{a})$
3. if $\varphi(\bar{v}) = \psi(\bar{v}) \vee \chi(\bar{v})$ then $H \Vdash_K \varphi(\bar{a})$ iff either $H \Vdash_K \psi(\bar{a})$ or $H \Vdash_K \chi(\bar{a})$
4. if $\varphi(\bar{v}) = \exists w\, \psi(\bar{v}, w)$ then $H \Vdash_K \varphi(\bar{a})$ iff $H \Vdash_K \psi(\bar{a}, b)$ for some $b \in dom(H)$
5. if $\varphi(\bar{v}) = \neg\psi(\bar{v})$ then $H \Vdash_K \varphi(\bar{a})$ iff there is no H' such that $H R H'$ and $H' \Vdash_K \varphi(\bar{a})$

It is clear that if we replace R in close 5 by $\subseteq$ we have Robinson's forcing for the sentences defined in H.

We note as well that Def 1.B.4 differs from definition 1.A.1 by the following:

1 - $\mathcal{G}$ is a set while $\sum$ is a class.

2 - Robinson's morphisms are embeddings while Kripke allows arbitrary homomorphisms.

3 - Robinson's forcing conditions are considered to be the diagrams of the models under consideration, i.e. he allows negated atomics; while Kripke's definition considers the positive diagrams only.

C. Forcing Value in Sheaves: here we adopt Ellerman's definition to / -sheaves. So let (S, π) be an s-sheaf of $\mathcal{L}$- structures over X. For every open set of U of X let Γ (U,S) be the set of sections over U. We define inductively the forcing value of a formula $\varphi(\bar{v})$ at $\bar{f} \in \Gamma(U,S)$ as the value of the map $\Vdash_E \varphi_U : [\Gamma(U,S)]^n \longrightarrow \mathcal{O}(U)$ where n = length of $\bar{v}$; such value is defined as follows:

Def 1.C.1: 1-If $\varphi(\bar{v})$ is atomic then $\Vdash_E \varphi_U(\bar{f}) = \{x \in U \; ; \; S_x \models \varphi(\bar{f}(x))\}$

2-If $\varphi(\bar{v}) = \psi(\bar{v}) \wedge \chi(\bar{v})$ then $\Vdash_E \varphi_U(\bar{f}) = \Vdash_E \psi_U(\bar{f}) \cap \Vdash_E \chi_U(\bar{f})$

3-If $\varphi(\bar{v}) = \psi(\bar{v}) \vee \chi(\bar{v})$ then $\Vdash_E \varphi_U(\bar{f}) = \Vdash_E \psi_U(\bar{f}) \cup \Vdash_E \chi_U(\bar{f})$

4-If $\varphi(\bar{v}) = \exists w \, \psi(\bar{v}, w)$ then $\Vdash_E \varphi_U(\bar{f}) = \bigcup_{u_0 \subset U} \{ \bigcup_{g \in \Gamma(u_0,S)} [\Vdash_E \psi (\bar{f}_{/u_0}, g)]\}$

5-If $\varphi(\bar{v}) = \neg \psi(\bar{v})$ then $\Vdash_E \varphi_U(\bar{f}) = Int(U - \Vdash_E \psi_U(\bar{f}))$

where $Int(A)$ is in the interior of A.

Def 1.C.2: We say that $\Gamma(U,S) \Vdash_E \varphi(\bar{f})$ iff $\Vdash_E \varphi_U(\bar{f}) = U$ and that $S_x \Vdash_E \varphi(\bar{f}(x))$ iff $x \in \Vdash_E \varphi_U(\bar{f})$ where $U \in \mathcal{O}(X)$ and $x \in U$.

To see the connection between Kripke's forcing and Ellerman forcing we need to represent Kripke structures as sheaves; we hsall do that here. We first note that given a topological space X with base $\mathcal{B}$ any functor P defined on $\mathcal{B}$ can be extended to a functor on $\mathcal{O}(X)$ by taking the

$\varprojlim_{\mathcal{U} \subset \mathcal{U}_1} P(\mathcal{U}) = P_1(\mathcal{U}_1)$. Thus any presheaf defined on $\mathcal{B}$ can be extended to a presheaf defined on $\mathcal{O}(X)$. Moreover it is easy to check that this presheaf is a P-sheaf iff it verifies the following condition:

($*$) For any $\mathcal{U} \in \mathcal{B}$ and any open cover $(V_r)_{r \in T}$ of U and any family $(a_r)_{r \in T}$ of elements $a_r \in P_1(V_r)$ if $P_{1\,V_r}^{\;\mathcal{U}'}(a_r) = P_{1\,V_t}^{\;\mathcal{U}'}(a_t)$ where $\mathcal{U}' \in \mathcal{B}$ and $\mathcal{U}' \subseteq V_r \cap V_{r'}$ then there exists $a \in P_1(U)$ such that $P_{1\,\mathcal{U}}^{\;V_r}(a) = a_r$ and a is unique.

With this in mind consider $\langle \mathcal{G}, R, dom \rangle$ as a Kripke structure where $\mathcal{G} = \{ H_i \,;\, i \in I \}$ and R is a transitive reflexive relation on I such that if $i R j$ then $dom(H_i) \subseteq dom(H_j)$. Endow I with the topology whose subbasis is the set of sets $\mathcal{U}_i$ where $\mathcal{U}_i = \{ j \in I \,;\, i R j \}$ And define $P(\mathcal{U}_j) = H_j$ Clearly P is a functor from $\{\mathcal{U}_i \,;\, i \in I\}$ into $\mathcal{G}$ So as before we can extend P to a functor $P_1 : (\mathcal{O}(I), \subseteq)^{op} \longrightarrow Str_R \mathcal{L}$ to make (I, P_1) a presheaf of $\mathcal{L}$-structures (here we consider all the H_i to be $\mathcal{L}$-structures).

Now we note that if given an open cover $(V_\alpha)_{\alpha \in K}$ of $\mathcal{U}_i$ then there must be an element V_{α_0} of this cover so that $\mathcal{U}_i = V_{\alpha_0}$ Thus condition ($*$) must hold for (I, P_1) therefore (I, P_1) is a P-sheaf of $\mathcal{L}$-structures. Now we use theorem 1.4.1 to get an s-sheaf (S, π) over I so that:

$$S_i \simeq P_1(i) \simeq H_i \simeq \Gamma(\mathcal{U}_i, S)$$

Thus we have proved:

<u>Theorem 1.C.3</u>: Let $\langle \{H_i \,;\, i \in I\}, R, dom \rangle$ be a Kripke structure then there is an S-sheaf (S, π) of sets ($\mathcal{L}$-structures if the H_i's are $\mathcal{L}$-structures) so that $S_i = H_i$ and such that for every element U_i of the subbasis of the topology on I $\Gamma(\mathcal{U}_i, S) \simeq H_i$

The connection between Ellerman's forcing and Kripke's forcing is given in the following theorem:

<u>Theorem 1.C.4</u>: Let $\langle \{H_i ; i \in I\} = \mathcal{G}, R, dom \rangle$ be a Kripke structure and (S, π) be the S-Sheaf of theorem 1.C.3. Assume that the truth value function on $\mathcal{G}$ is defined for atomic formula as follows:

$\vartheta(P_{\mu(i)}(\bar{a}), H_i) = T$ iff $H_i \models P_{\mu(i)}(\bar{a})$ i.e. $\bar{a} \in P_{\mu(i)}$

Then for any formula $\varphi(\bar{v})$ and elements $\bar{a} \in H_i$

$H_i \Vdash_E \varphi(\bar{a})$ implies $H_i \Vdash_K \varphi(\bar{a})$

<u>pf</u>: by induction on the degree of complexity of $\varphi(\bar{v})$

For $\varphi(\bar{v})$ atomic; $H_i \Vdash_E \varphi(\bar{a})$ iff $\{j \in \mathcal{U}_i ; H_j \models \varphi(\bar{a}(j))\} = \mathcal{U}_i$

iff $H_i \models \varphi(\bar{a}) \Rightarrow H_i \Vdash_K \varphi(\bar{a})$

For $\varphi(\bar{v}) = \psi(\bar{v}) \wedge \chi(\bar{v})$ and $\varphi(\bar{v}) = \psi(\bar{v}) \vee \chi(\bar{v})$ it is immediate

For $\varphi(\bar{v}) = \exists w \, \psi(\bar{v}, w)$

$H_i \Vdash_E \exists w \, \psi(\bar{a}, w)$ then $\mathcal{U}_i = \bigsqcup_{\mathcal{U}_j \subset \mathcal{U}_i} \{ \bigsqcup_{g \in H_j} \Vdash_E \psi_{\mathcal{U}_j}(\bar{a}, g) \}$ thus

$(\Vdash_E \psi_{\mathcal{U}_j}(\bar{a}, g))_{\substack{\mathcal{U}_j \subset \mathcal{U}_i \\ g \in H_j}}$ form an open cover of the open subset $\mathcal{U}_i$

so $\Vdash_E \psi_{\mathcal{U}_j}(\bar{a}, g) = \mathcal{U}_i$ for some $\mathcal{U}_j$ and $g \in H_j$ thus

$\mathcal{U}_j = \mathcal{U}_i$ and $H_j = H_i$ and $g \in H_i$ and $H_i \Vdash_E \psi(\bar{a}, g)$ by induction

$H_i \Vdash_K \psi(\bar{a}, g)$ so $H_i \Vdash_K \exists w \, \psi(\bar{a}, w)$ and $H_i \Vdash_K \varphi(\bar{a})$

If $\varphi(\bar{v}) = \neg\psi(\bar{v})$ and $H_i \Vdash_E \varphi(\bar{a})$ then $\mathcal{U}_i = \mathrm{Int}(\mathcal{U}_i - \Vdash_E \psi_{\mathcal{U}_i}(\bar{a}))$

so $\Vdash_E \psi_{\mathcal{U}_i}(\bar{a}) = \emptyset$ Thus for any $j \in I$ such that iRj $H_j \nVdash_E \psi(\bar{a})$

So by induction hypothesis $H_j \nVdash_K \psi(\bar{a})$ thus $H_i \Vdash_K \neg\psi(\bar{a})$ i.e.

$H_i \Vdash_K \varphi(\bar{a})$.

Thus theorem 1.C.4 proves that Ellerman's forcing implies Kripke's forcing which in turn generalizes Robinson's forcing(prop 1.B.5).

D - The Prime Sheaf Theorem: In what follows we shall establish the relation between prime-s-sheaves and s-sheaves. To do that we need to develop the notion of weak forcing in s-sheaves.

Def 1.D.1: Let (S,π) be an s-sheaf of $\mathcal{L}$ -structures over X. Let U_o be an open subset of X and $\varphi(v_1,--,v_n)$ be a formula of $\mathcal{L}$. Consider $f_1,\ldots,f_n$ in $\Gamma(\mathcal{U}_o,S)$. We say that $\Gamma(\mathcal{U}_o,S)$ weakly forces $\varphi(f_1,\ldots,f_n)$ iff $\Gamma(\mathcal{U}_o,S)\Vdash_E \neg\neg\varphi(f_1,\ldots,f_n)$, we write $\Gamma(\mathcal{U}_o,S)\Vdash_E^* \varphi(f_1,\ldots,f_n)$ for $\Gamma(\mathcal{U}_o,S)$ weakly forces $\varphi(f_1,\ldots,f_n)$

The following proposition gives an inductive definition of weak forcing:

prop 1.D.2: The weak forcing can be defined inductively as the value of the map $\Vdash_E^*\varphi_{\mathcal{U}_o}:[\Gamma(\mathcal{U}_o,S)]^n\longrightarrow \mathcal{O}(\mathcal{U}_o)$ as follows:

1. If $\varphi(\bar v)$ is atomic then $\Vdash_E^*\varphi_{\mathcal{U}_o}(\bar f)=--[\{x\in\mathcal{U}_o\ ;\ S_x\models\varphi(\bar f(x))\}]\cap\mathcal{U}_o$ where $-A=\mathrm{Int}\,(X-A)$ for any $A\subseteq X$
2. If $\varphi(\bar v)=\psi(\bar v)\wedge\chi(\bar v)$ then $\Vdash_E^*\varphi_{\mathcal{U}_o}(\bar f)=\Vdash_E^*\psi_{\mathcal{U}_o}(\bar f)\cap\Vdash_E^*\chi_{\mathcal{U}_o}(\bar f)$
3. If $\varphi(\bar v)=\psi(\bar v)\vee\chi(\bar v)$ then $\Vdash_E^*\varphi_{\mathcal{U}_o}(\bar f)=--[\Vdash_E^*\psi_{\mathcal{U}_o}(\bar f)\cup\Vdash_E^*\chi_{\mathcal{U}_o}(\bar f)]\cap\mathcal{U}_o$
4. If $\varphi(\bar v)=\exists w\,\psi(\bar v,w)$ then $\Vdash_E^*\varphi_{\mathcal{U}_o}(\bar f)=--[\bigcup_{\mathcal{U}\subset\mathcal{U}_o}\{\bigcup_{g\in\Gamma(\mathcal{U},S)}\Vdash_E^*\varphi_{\mathcal{U}}((\bar f/_{\mathcal{U}}),g)\}]\cap\mathcal{U}_o$
5. if $\varphi(\bar v)=\neg\psi(\bar v)$ then $\Vdash_E^*\varphi_{\mathcal{U}_o}(\bar f)=\mathrm{Int}[\mathcal{U}_o-\Vdash_E^*\psi_{\mathcal{U}_o}(\bar f)]$ and $\Gamma(\mathcal{U}_o,S)\Vdash_E^*\varphi(\bar f)$ iff $\mathcal{U}_o=\Vdash_E^*\varphi_{\mathcal{U}_o}(\bar f)$

pf: To show that it is enough to show that:

$\Vdash_E\neg\neg\varphi_{\mathcal{U}_o}(\bar f)=\Vdash_E^*\varphi_{\mathcal{U}_o}(\bar f)$ for every formula $\varphi(\bar v)$ of $\mathcal{L}$ and every sequence $\bar f$ of elements of $\Gamma(\mathcal{U}_o,S)$. We do that by induction on the degree of complexity of $\varphi(\bar v)$

1 - If $\varphi(\bar v)$ is atomic then $\Vdash_E\neg\neg\varphi_{\mathcal{U}_o}(\bar f)=\mathrm{Int}[\mathcal{U}_o-\Vdash_E\neg\varphi_{\mathcal{U}_o}(\bar f)]=$

$=\mathrm{Int}[\mathcal{U}_o-\mathrm{Int}[\mathcal{U}_o-\Vdash_E\varphi_{\mathcal{U}_o}(\bar f)]=--\{x\in\mathcal{U}_o\,;\,S_x\models\varphi(\bar f(x))\}]\cap\mathcal{U}_o=\Vdash_E^*\varphi_{\mathcal{U}_o}(\bar f)$.

2 - If $\varphi(\bar{v}) = \psi(\bar{v}) \wedge \chi(\bar{v})$

$$\Vdash_E^* \varphi_{\mathcal{U}_0}(\bar{f}) = \Vdash_E^* \psi_{\mathcal{U}_0}(\bar{f}) \cap \Vdash_E^* \chi_{\mathcal{U}_0}(\bar{f}) = \Vdash_E \neg\neg \psi_{\mathcal{U}_0}(\bar{f}) \cap \Vdash_E \neg\neg \chi_{\mathcal{U}_0}(\bar{f}) \text{(by induction hyp.)}$$
$$= \Vdash_E \neg\neg(\psi_{\mathcal{U}_0}(\bar{f}) \wedge \chi_{\mathcal{U}_0}(\bar{f}))] = \Vdash_E \neg\neg \varphi_{\mathcal{U}_0}(\bar{f})$$

3 - If $\varphi(\bar{v}) = \psi(\bar{v}) \vee \chi(\bar{v})$.

$$\Vdash_E^* \varphi_{\mathcal{U}_0}(\bar{f}) = --[\Vdash_E^* \psi_{\mathcal{U}_0}(\bar{f}) \cup \Vdash_E^* \chi_{\mathcal{U}_0}(\bar{f})] \cap \mathcal{U}_0$$
$$= --[\Vdash_E^* \psi_{\mathcal{U}_0}(\bar{f}) \cap \mathcal{U}_0] \cup --[\Vdash_E^* \chi_{\mathcal{U}_0}(\bar{f}) \cap \mathcal{U}_0]$$
$$= --[\Vdash_E \neg\neg\psi_{\mathcal{U}_0}(\bar{f}) \cap \mathcal{U}_0] \cup --[\Vdash_E \neg\neg\chi_{\mathcal{U}_0}(\bar{f}) \cap \mathcal{U}_0] \text{ (by induction)}$$
$$= --[\Vdash_E \neg\neg\psi_{\mathcal{U}_0}(\bar{f}) \cup \Vdash_E \neg\neg\chi_{\mathcal{U}_0}(\bar{f})] \cap \mathcal{U}_0$$
$$= \Vdash_E \neg\neg \varphi_{\mathcal{U}_0}(\bar{f})$$

4 - If $\varphi(\bar{v}) = \exists w\, \psi(\bar{v}, w)$ the proof is the same as in 3.

5 - If $\varphi(\bar{v}) = \neg\psi(\bar{v})$

$$\Vdash^* \varphi_{\mathcal{U}_0}(\bar{f}) = \mathrm{Int}(\mathcal{U}_0 - \Vdash_E^* \psi_{\mathcal{U}_0}(\bar{f})) = \mathrm{Int}\,[\mathcal{U}_0 - \Vdash_E \neg\neg\psi_{\mathcal{U}_0}(\bar{f})] \text{(by induction)}$$
$$= \Vdash_E (\neg\neg(\neg\psi_{\mathcal{U}_0}(\bar{f})) = \Vdash_E \neg\neg\varphi_{\mathcal{U}_0}(\bar{f}) .$$

For the construction of prime s-sheaf we refer the reader to Chapter 1 part 5. We shall use here the same notation as in 1.5.

The following theorem establishes the relation between prime sheaves and sheaves.

Theorem 1.D.3: Let (S, π) be an s-sheaf of $\mathcal{L}$-structures over X; (T°, σ°) be the prime s-sheaf of $\mathcal{L}$-structures over $Pr(X)$ and let F be a prime filter filter and $\varphi(\bar{v})$ an $\mathcal{L}$-formula and $f_0, \ldots, f_n$ elements of $\Gamma(\mathcal{U}_0, S)$ such that $\mathcal{U}_0 \in F$ write $|f_0|, \ldots, |f_n|$ for the elements in T°_F which correspond to $f_0, \ldots, f_n$ (i.e. the equivalence classes of $f_0, \ldots, f_n$ w.r.t. the equivalence relation defined by F). Then

$T^\circ_F \Vdash_E^* \varphi(|f_0|, \ldots, |f_n|)$ iff $\{x \in \mathcal{U}_0 ; S_x \Vdash_E^* \varphi(f_0(x), \ldots, f_n(x))\} \in F$

i.e. $T^\circ_F \Vdash_E^* \varphi(|f_0|, \ldots, |f_n|)$ iff $\Vdash_E^* \varphi_{\mathcal{U}_0}(f_0, \ldots, f_n) \in F$.

pf: We shall prove this theorem by induction on the degree of complexity of $\varphi(\bar{v})$; but before doing that we need some notation and topological facts which we shall state here; the proof of these facts is too easy and can be found in Rasiowa and Sikorski [63].

As usual we note by $-\mathcal{U}$ the interior of $X-\mathcal{U}$ and for $V \in P_r(X)$
$-V = Int(P_r(X) - V)$

1 - $--V_{\mathcal{U}} = V_{--\mathcal{U}}$

2 - $V_{\mathcal{U}_1} \cap V_{\mathcal{U}_2} = V_{\mathcal{U}_1 \cap \mathcal{U}_2}$

3 - $V_{\mathcal{U}_1} \sqcup V_{\mathcal{U}_2} = V_{\mathcal{U}_1 \cup \mathcal{U}_2}$

4 - $-V_{\mathcal{U}} = V_{-\mathcal{U}}$

5 - $--\bigsqcup_{j \in J} V_{\mathcal{U}_j} = V_{--\bigcup_j \mathcal{U}_j}$

Now by definition of forcing we have $T_F^o \Vdash_E^* \varphi(|f_o|, \dots, |f_n|)$ iff $F \in \Vdash_E^* \varphi_{V_{\mathcal{U}_o}}(|f_o|, \dots, |f_n|)$. Hence it is enough to show that
$\Vdash_E^* \varphi_{V_{\mathcal{U}_o}}(f_o, \dots, f_n) = V_{\Vdash_E^* \varphi_{\mathcal{U}_o}(f_o, \dots, f_n)}$.

1. If $\varphi(\bar{v})$ is atomic then :

$\Vdash_E^* \varphi_{\mathcal{U}_o}(\bar{f}) = --\{x \in \mathcal{U}_o \,;\, S_x \models \varphi(\bar{f}(x))\} \cap \mathcal{U}_o$.

so: $\Vdash_E^* \varphi_{V_{\mathcal{U}_o}}(\bar{f}) = --\{F \in V_{\mathcal{U}_o} \,;\, T_F^o \models \varphi(|\bar{f}|)\} \cap V_{\mathcal{U}_o} =$

$= --V_{\{x \in \mathcal{U}_o \,;\, S_x \models \varphi(\bar{f}(x))\}} \cap V_{\mathcal{U}_o} =$ (by definition of limit)

$= V_{--\{x \in \mathcal{U}_o \,;\, S_x \models \varphi(\bar{f}(x))\}} \cap V_{\mathcal{U}_o} =$ (by 1 above)

$= V_{--\{x \in \mathcal{U}_o \,;\, S_x \models \varphi(f(x))\} \cap \mathcal{U}_o} =$ (by 2 above)

$= V_{\Vdash_E^* \varphi(\bar{f})}$

2. If $\varphi(\bar{v}) = \psi(\bar{v}) \wedge \chi(\bar{v})$

$\Vdash_E^* \varphi_{V_{\mathcal{U}_o}}(\bar{f}) = \Vdash_E^* \psi_{V_{\mathcal{U}_o}}(\bar{f}) \cap \Vdash_E^* \chi_{V_{\mathcal{U}_o}}(\bar{f}) =$

$= V_{\Vdash_E^* \psi_{\mathcal{U}_o}(\bar{f})} \cap V_{\Vdash_E^* \chi_{\mathcal{U}_o}(\bar{f})}$ (by induction hypothesis)

$= V_{\Vdash_E^* \psi_{\mathcal{U}_o}(\bar{f}) \wedge \Vdash_E^* \chi_{\mathcal{U}_o}(\bar{f})}$ (by 2 above)

$= V_{\Vdash_E^* \varphi_{\mathcal{U}_o}(f)}$

3. If $\varphi(\bar{v}) = \psi(\bar{v}) \vee \chi(\bar{v})$.

$$\Vdash^*_E \varphi_{V\mathcal{U}_0}(\bar f) = --[\Vdash^*_E \psi_{V\mathcal{U}_0}(\bar f) \cup \Vdash^*_E \chi_{V\mathcal{U}_0}(\bar f)] \cap V\mathcal{U}_0 =$$

$$= --[V\Vdash^*_E \psi_{\mathcal{U}_0}(\bar f) \cup V\Vdash^*_E \chi_{\mathcal{U}_0}(\bar f)] \cap V\mathcal{U}_0 \qquad \text{by induction}$$

$$= V--[\Vdash^*_E \psi_{\mathcal{U}_0}(\bar f) \cup \Vdash^*_E \chi_{\mathcal{U}_0}(\bar f)] \cap \mathcal{U}_0 \qquad \text{by 3,1 and 2 above}$$

$$= V\Vdash^*_E \varphi_{\mathcal{U}_0}(\bar f)$$

4. If $\varphi(\bar v) = \neg\psi(\bar v)$

$$\Vdash^*_E \varphi_{V\mathcal{U}_0}(\bar f) = \mathrm{Int}\,(V\mathcal{U}_0 - \Vdash^*_E \psi_{V\mathcal{U}_0}(\bar f))$$

$$= \mathrm{Int}(V\mathcal{U}_0 - V\Vdash^*_E \psi_{\mathcal{U}_0}(\bar f)) \qquad \text{by induction hypothesis}$$

$$= -V\Vdash^*_E \psi_{\mathcal{U}_0}(\bar f) \cap V\mathcal{U}_0$$

$$= V - \Vdash^* \psi_{\mathcal{U}_0}(\bar f) \cap \mathcal{U}_0$$

$$= V\Vdash^*_E \varphi_{\mathcal{U}_0}(\bar f)$$

5. If $\varphi(\bar v) = \exists w\, \psi(\bar v, w)$

$$\Vdash^*_E \varphi_{V\mathcal{U}_0}(\bar f) = --[\bigcup_{\mathcal{U}\subset\mathcal{U}_0}\{\bigcup_{g\in\Gamma(\mathcal{U},S)}\{\Vdash^*_E \psi_{\mathcal{U}}(\bar f, g)\}\}] \cap V\mathcal{U}_0$$

$$= --[\bigcup_{\mathcal{U}}\{\bigcup_{g}\{V\Vdash^*_E \psi_{\mathcal{U}}(\bar f, g)\}\}] \cap V\mathcal{U}_0 \qquad \text{by induction hypothesis}$$

$$= V--[\bigcup_{\mathcal{U}}\{\bigcup_{g} \Vdash^*_E \psi_{\mathcal{U}}(\bar f, g)\}\}] \cap \mathcal{U}_0 = V\Vdash^*_E \varphi(\bar f)\,.$$

<u>E - The Ultrastalk Theorem:</u> We refer the reader to part 6 of Chapter 1 for the construction of ultra-s-sheaves and the notation which we shall use in here.

<u>Theorem 1.E.1:</u> Let $\mathcal{U}_0 \in \mathcal{O}(X)$ and $F \in \mathcal{U}lt(X)$ so that $\mathcal{U}_0 \in F^0$ (i.e. $--\mathcal{U}_0 \in F$) then for any $\mathcal{L}$-formula $\varphi(\bar v$ and sequence $\bar f$ of elements of $\Gamma(\mathcal{U}_0, S)$ where (S, π) is an s-sheaf over X. Let (T, σ) be the ultra-s-sheaf over ult(X) then

$T_{F^0} \vDash \varphi(|f_1|, \ldots, |f_n|)$ iff $\{x \in \mathcal{U}_0 \,;\, S_x \Vdash^*_E (f_1(x), \ldots, f_n(x))\} \in F^0$

i.e. $T_{F^0} \vDash \varphi(|\bar f|)$ iff $\Vdash^*_E \varphi_{\mathcal{U}_0}(\bar f) \in F^0$

<u>pf:</u> By the prime stalk theorem (Th 1.D.3) it is enough to show that $T_{F^0} \vDash \varphi(|\bar f|)$ iff $T_{F^0} \Vdash^*_E \varphi(|\bar f|)$ We do that by induction on the degree of complexity of $\varphi(\bar v)$.

1 - If $\varphi(\bar{v})$ is atomic then

$T_{F^o} \Vdash^*_E \varphi(|\bar{f}|)$ iff $\{x \in \mathcal{U}_o \ ; \ S_x \Vdash^*_E \varphi(\bar{f}(x))\} \in F^o$

iff $-- \{x \in \mathcal{U}_o \ ; \ S_x \models \varphi(\bar{f}(x))\} \cap \mathcal{U}_o \in F^o$

iff $\{x \in \mathcal{U}_o \ ; \ S_x \models \varphi(\bar{f}(x))\} \in F^o$

iff $T_{F^o} \models \varphi(|\bar{f}|)$

2 - If $\varphi(\bar{v}) = \psi(\bar{v}) \vee \chi(\bar{v})$ then

$T_{F^o} \Vdash^*_E \varphi(|\bar{f}|)$ iff $\Vdash^*_E \psi_{\mathcal{U}_o}(\bar{f}) \vee \chi_{\mathcal{U}_o}(\bar{f}) \in F^o$

iff either $\Vdash^*_E \psi_{\mathcal{U}_o}(\bar{f}) \in F^o$

or $\Vdash^*_E \chi_{\mathcal{U}_o}(\bar{f}) \in F^o$

iff $T_{F^o} \models \psi(|\bar{f}|)$ or $T_{F^o} \models \chi(|\bar{f}|$

(by induction hypothesis)

iff $T_{F^o} \models \varphi(|\bar{f}|)$

3 - If $\varphi(\bar{v}) = \psi(\bar{v}) \vee \chi(\bar{v})$ then it is similar to 2.

4 - If $\varphi(\bar{v}) = \neg\psi(\bar{v})$ then :

$T_{F^o} \Vdash^*_E \neg\psi(|\bar{f}|)$ iff $Int(\mathcal{U}_o - \Vdash^*_E \psi_{\mathcal{U}_o}(|\bar{f}) \in F^o$

iff $\Vdash^*_E \psi_{\mathcal{U}_o}(\bar{f}) \notin F^o$

iff $T_{F^o} \not\Vdash^*_E \psi_{\mathcal{U}_o}(|\bar{f}|)$

iff $T_{F^o} \not\models \psi(|\bar{f}|)$ (by induction hypothesis)

iff $T_{F^o} \models \varphi(|\bar{f}|)$

5 - If $\varphi(\bar{v}) = \exists w\, \psi(\bar{v}, w)$ then

$T_{F^o} \Vdash^*_E \varphi(|\bar{f}|)$ iff $\mathcal{U}_1 = \bigsqcup_{\mathcal{U} \subset \mathcal{U}_o} [\bigsqcup_{g \in \Gamma(\mathcal{U}, S)} \Vdash^*_E \psi_{\mathcal{U}}(\bar{f}, g)] \in F^o$

We need to continue the proof for the following topological fact:

$(*)$ Let $\mathcal{U} \in \mathcal{O}(X)$ and $(\mathcal{U}_i)_{i \in I}$ be an open cover of $\mathcal{U}$ then there exists an open cover $(V_j)_{j \in J}$ of a dense subset of $\mathcal{U}$ such that $V_j \cap V_K = \phi$ iff $j \neq K$ and for every $j \in J$ there is an $i \in I$ so that $V_j \subseteq \mathcal{U}_i$

Let $\mathcal{M}$ be the set $\{G$, family of open subsets of $\mathcal{U}$ such that whenever $\mathcal{U}'$

and $\mathcal{U}''$ are in C ; $\mathcal{U}' \neq \mathcal{U}''$ then $\mathcal{U}' \cap \mathcal{U}'' = \phi$ and for every $\mathcal{U}' \in C$; $\exists \mathcal{U}_i$ with $\mathcal{U}' \subseteq \mathcal{U}_i$ }

1 - $\mathcal{M}$ is not empty choose $i \in I$ and take $C = \{\mathcal{U}_i\}$

2 - If $C, C_1 \in \mathcal{M}$ we say that $C \leqslant C_1$ iff $\mathcal{U}' \in C \Rightarrow \mathcal{U}' \in C_1$

3 - For any chain $\mathcal{C}$ of elements of $\mathcal{M}$; $\cup\{C ; C \in \mathcal{C}\} \in \mathcal{M}$

By Zorn's lemma $\mathcal{M}$ has a maximal element, C_0 say. Now $\bigcup_{V \in C_0} V \subset_d \mathcal{U}$

This is due to the maximality of C_0.

Now we use ($*$) for $\mathcal{U}_1$ and its open cover formed by $\Vdash_E^* \psi_{\mathcal{U}}(\bar{f}, g)$ and we get a family $(V_j)_{j \in J}$ of open subsets of $\mathcal{U}_0$ such that $j \neq k \Rightarrow V_j \cap V_k = \phi$; $\bigcup_j V_j \subset_d \mathcal{U}_0$ and for every $j \in J$ $V_j \subseteq \Vdash_E^* \psi_{\mathcal{U}_j}(\bar{f}, g_j)$ for some $\mathcal{U}_j \subset \mathcal{U}_0$ and $g_j \in \Gamma(\mathcal{U}_j, S)$ Let $\mathcal{U}_2 = \bigcup_j V_j$

Since $V_j \cap V_k \neq \phi$ for $j \neq k$ then there is $g \in \Gamma(\mathcal{U}_2, S)$ such that $g/V_j = g_j$ since $T_{F^0} = \varinjlim_{\mathcal{U}_0 \in F^0} \varinjlim_{\mathcal{U} \subset_d \mathcal{U}_0} \Gamma(\mathcal{U}, S)$ Let $|g|$ be the equivalence class of g w.r.t. the equivalence relation defined by F^0

Now we continue the proof:

$$
\begin{aligned}
T_{F^0} \Vdash_E^* \exists W\, \psi(\bar{f}, W) \quad & \text{iff } \mathcal{U}_1 \in F^0 \\
& \text{iff } \exists\, \mathcal{U}_2 \subset_d \mathcal{U}_1 \text{ and } g \in \Gamma(\mathcal{U}_2, S) \text{ such} \\
& \qquad \text{that } T_{F^0} \Vdash_E^* \psi(|\bar{f}|, |g|) \\
& \text{iff } T_{F^0} \models \psi(|\bar{f}|, |g| \quad (\text{ by induction hypothesis }) \\
& \text{iff } T_{F^0} \models \exists W\, \psi(|\bar{f}|, W) \\
& \text{iff } T_{F^0} \models \varphi(|\bar{f}|)
\end{aligned}
$$

<u>Corollary 1.E.2</u>: i -Let $F \in Ult(X)$ and $\mathcal{U}_0 \in F$ then $T_F \models \varphi(|\bar{f}|)$

iff $\neg\neg \Vdash_E^* \varphi_{\mathcal{U}_0}(\bar{f}) \in F$

ii- $T_{F^0} \models \varphi(|\bar{f}|)$ iff $\Vdash_E \varphi_{\mathcal{U}_0}(\bar{f}) \in F^0$

<u>pf</u>: i- is just theorem 1.E.1

ii- In fact $\Vdash_E^* \varphi_{\mathcal{U}_0}(\bar{f}) = \neg\neg \Vdash_E \varphi_{\mathcal{U}_0}(\bar{f}) \cap \mathcal{U}_0$ by prop 1.D.2

This together with theorem 1.E.1 gives the result.
In Chapter 4 we shall apply this theorem to get Łoś' theorem for ultraproducts.

2. Theories and Sentences Preserved by Global Sections of S-Sheaves

In this section we shall be interested in first order theories and sentences preserved by global sections of S-sheaves of $\mathcal{L}$-structures. Such theories will be called Global section theories and sentences (G.S. theories and G.S. sentences for short). This section will be divided into 3 parts:

Part A: will contain some notation and definitions of limits, colimits and special equalizers as well as some results concerning theories and sentences preserved by limits; products and reduced products.

Part B: will be concerned with first order theories, which are preserved by global sections. We make use of the result stated in A, and prove that those are the theories preserved by limits.

Part C: will be concerned with G.S. sentences when the S-sheaves are Boolean. We shall also study some of the topological properties of the truth value and forcing value of sentences of $\mathcal{L}$.

A - Notation; limits, equalizers; and limit theories.

In what follows we denote by Str $\mathcal{L}$ the class of all $\mathcal{L}$-structures; all the structures are supposed to be non-empty. Sent $\mathcal{L}$ denotes the set of sentences of $\mathcal{L}$. At.(resp. Bas, pos) denotes the set of all atomic (resp. basic, positive) elements of Sent $\mathcal{L}$.

If $\Delta, \Delta' \subseteq Sent\mathcal{L}$ then $\wedge[\Delta]$ (resp. $\vee[\Delta]$, $\exists[\Delta]$, $\forall[\Delta]$) denotes the closure of Δ under conjunction (resp. disjunction, existential quantification, universal quantification) and $[\Delta, \Delta']$ will be the set of sentences φ of the form: i - $\delta_1 \to \delta_2$ with $\delta_1 \in \Delta$ and $\delta_2 \in \Delta'$

ii - $\neg\delta_1$ with $\delta_1 \in \Delta$

iii - δ_2 with $\delta_2 \in \Delta'$

Similar notation will be used if $\Delta, \Delta' \subseteq$ For $\mathcal{L}$ the set of all formulae of $\mathcal{L}$.

Considering Str $\mathcal{L}$ one realizes that homomorphisms (resp. embeddings, elementary embeddings) are morphisms which together with Str $\mathcal{L}$ form a category. We shall denote this category by $\mathrm{Str}_h \mathcal{L}$ (resp. $\mathrm{Str}_e \mathcal{L}$; $\mathrm{Str}_{ee} \mathcal{L}$). We shall use the letters h (resp. e, ee) to note in which category we are working. Thus, for example, h-limit means a limit in $\mathrm{Str}_h \mathcal{L}$.

We recall here the definitions of limits, colimits and equalizers.

In general let $\mathcal{C}$ and I be 2 categories and $F : I \to \mathcal{C}$ be a functor

Def 2.A.1: Colimit of F (resp. lim F) is defined as an object of $\mathcal{C}$ together with a set of morphisms $(f_i)_{i \in ob I}$ such that:

i - For every pair $(i, i_1) \in (ob\ I)^2$ and $g : i \to i_1$, the following diagram commutes

$$\begin{tikzcd} F(i) \arrow[r, "f_i"] \arrow[d, "F(g)"'] & \mathrm{Colim}\, F \\ F(i_1) \arrow[ur, "f_{i_1}"'] & \end{tikzcd} \quad \left(\text{resp. } \begin{tikzcd} \mathrm{Lim}\, F \arrow[r, "f_i"] \arrow[d, "f_{i_1}"'] & F(i) \arrow[dl, "F(g)"] \\ F(i_1) & \end{tikzcd} \right)$$

ii - If for any object A of $\mathcal{C}$ and family $(f'_i)_{i \in ob I}$ of morphisms the following diagram commutes

$$\begin{tikzcd} F(i) \arrow[r, "f'_i"] \arrow[d, "F(g)"'] & A \\ F(i_1) \arrow[ur, "f'_{i_1}"'] & \end{tikzcd} \quad \left(\text{resp. } \begin{tikzcd} A \arrow[r, "f'_i"] \arrow[d, "f'_{i_1}"'] & F(i) \arrow[dl, "F(g)"] \\ F(i_1) & \end{tikzcd} \right) \quad \text{for all } g : i \to i_1$$

Then there is a unique morphism $h : \mathrm{Colim}\, F \to A$ (resp. $A \xrightarrow{h} \mathrm{Lim}\, F$) such that the following diagram commutes

$$\begin{tikzcd} F(i) \arrow[dr, "f_i"] \arrow[rr, dashed, "f'_i"] \arrow[dd, "F(g)"'] & & A \\ & \mathrm{Colim}\,F \arrow[ur] & \\ F(i_1) \arrow[ur, "f_{i_1}"] \arrow[uurr, dashed, "f'_{i_1}"'] & & \end{tikzcd} \left[\text{resp.} \quad \begin{tikzcd} A \arrow[rr, dashed, "f'_i"] \arrow[dd, dashed, "f'_{i_1}"'] & & F(i) \arrow[ddll, "F(g)"] \\ & \mathrm{Lim}\,F \arrow[ur, "f_i"] \arrow[dl, "f_{i_1}"] & \\ F(i_1) & & \end{tikzcd} \right]$$

Thus one can talk about limits in $Str_h \mathcal{L}$ (resp. $Str_e \mathcal{L}$, $Str_{ee} \mathcal{L}$) if they do exist. We shall call them h-limits, h-colimits (resp. e-limits, e-colimits and ee-limits, ee-colimits).

A special case is when I is a $\uparrow$-directed set (resp. $\downarrow$-directed set) then we have $\uparrow$-directed h-colimit (resp. $\downarrow$-directed h-limits) we write $\uparrow$-h-colimits and $\downarrow$-h-limits for those notions.

Def 2.A.2: Let A,B be $\mathcal{L}$-structures f_1, f_2 be morphisms from A into B. We denote by $E(f_1,f_2) = \{ a \in A \,;\, f_1(a) = f_2(a) \}$ $E(f_1,f_2)$ is called the equalizer of f_1 and f_2. We say that $E(f_1,f_2)$ is special equalizer iff f_2 is an elementary embedding and $f_2(E(f_1,f_2)) = f_1(A) \cap f_2(A)$

Again the letters h, e, ee in front of equalizer and special equalizer denote the category in which the equalizer and the special equalizer is considered.

We note that every ee-special is an e-special and every e-special is an h-special.

Lemma 2.A.3:

1 - Every elementary substructure of an $\mathcal{L}$-structure is an ee-special

2 - Every $\uparrow$-h-colimit is an h-special

3 - Every $\downarrow$-h-limit is an h-special.

pf: 1. Assume $E \preccurlyeq A$ and let $g : A \longrightarrow E^I/D$ be an elementary embedding such that $g/_E = d_E : E \longrightarrow E^I/D$ where d_E is the natural embedding i.e. $d_E(e) = |e|_D$ Let $j : E \longrightarrow A$ be the inclusion map and $j^I/D : E^I/D \longrightarrow A^I/D$ be the elementary embedding induced by j;

finally let $d_A : A \to A^I/D$ be the natural embedding. We claim that $E(j^I/D \circ g, d_A)$ is an ee-special. For it is enough to show that $d_A(A) \cap j^I/D \circ g(A) \subseteq d_A(E(j^I/D \circ g, d_A))$.

So let $x \in d_A(A) \cap j^I/D \circ g(A)$. Hence $x \in |d|_D$ for some $d \in A$ and $x = |\langle e(i)\rangle|_D$ for $\langle e(i)\rangle \in E^I$ But $\{i \in I, e(i) = d\} \in D$ Hence $\{i ; d \in E\} \in D$ so $\{i ; d \in E\} \neq \phi$ and $a \in E$ Furthermore $j^I/D \circ g(d) = |\langle e(i)\rangle|_D = x = d_A(d)$ thus $d_A(d) \in d_A(E(j^I/D \circ g, d_A))$ Moreover if $d \in E(j^I/D \circ g, d_A)$ then $d \in E$ it is easy to check that if $e \in E$ then $e \in E(j^I/D \circ g, d_A)$; hence $E = E(j^I/D \circ g, d_A)$ and E is ee-special.

The proof of 3 is similar to the proof of 2 so we shall give the proof of 2 and indicate how to prove 3.

2 - Let D be an ultrafilter on I ; where I is $\uparrow$-directed set and C is the colimit of a functor $F : I \longrightarrow Str_h \mathcal{L}$. Assume that every subset J of I which has the following property: [If $i \in J$ and $j \geqslant i$ then $j \in J$]; is in D . Let $A = \prod_{i \in I} F(i)/D$ and $B = A^I/D$ remark that there is a homomorphism $f : A \to B$ induced by the functor F and let d_A be the natural elementary embedding from A into B.

For every $a_i \in F(i)$ and $\alpha \in \prod_I F(i)$

Let $S(a_i, \alpha) = \{j \geqslant i ; F(i,j)(a_i) = \alpha(j)\}$ Thus the homomorphism f is defined by $f(|\alpha|_D) = |\gamma|_D$ iff $\{i ; \{j \geqslant i ; F(i,j)(\alpha(i)) = \gamma_i(j)\} \in D\} \in D$ where $\alpha(i) = |\alpha_i|_D$ and $\gamma_i(j) \in F(j)$

We note as well that $d_A(|\alpha|_D) = |\gamma|_D$ iff $\{i ; |\alpha|_D = |\gamma_i|_D\} \in D$

Therefore $f(|\alpha_1|_D) = d_A(|\alpha_2|_D)$ iff $\{i ; S(\alpha_1(i), \alpha_2) \in D\} \in D$ Hence $|\alpha|_D \in E(f, d_A)$ iff $\{i ; S(\alpha(i), \alpha) \in D\} \in D$.

We claim $E(f, d_A)$ is h-special. For it is enough to show that if $\{i ; S(\alpha_1(i), \alpha_2) \in D\} \in D$ then $\{i ; S(\alpha_2(i), \alpha_2) \in D\} \in D$

Assume $\{i ; S(\alpha_1(i), \alpha_2) \in D\} \in D$ then there is $i_0 \in I$ so that

$S(\alpha_1(i_0),\alpha_2) \in D$ but $S(\alpha_1(i_0),\alpha_2) \subseteq \{i ; S(\alpha_2(i),\alpha_2) \in D\}$ so the latter is in D.

Now we claim that $C \simeq E(f, d_A)$.

In fact $C = \coprod_I F(i)/Q$ where Q is the following equivalence relation $|a_i|_Q = |a_j|_Q$ iff $\exists k \geq i,j$ $F(i,k)(a_i) = F(j,k)(a_j)$

So let $e : C \longrightarrow E(f, d_A)$ $e(|a_i|_Q) = |\alpha|_D$ iff $S(a_i, \alpha) \in D$

e is easily seen to be a surjective embedding so $C \simeq E(f, d_A)$

3 - The set I considered in 2 must now be taken as $\downarrow$ -directed the ultrafilter must contain all $J \subseteq I$ such that if $j \in J$ and $i \leq j$ then $i \in J$; and $A = \prod_I F(i)/D$ $B = \prod_{i \in I} (F(i)^I/D)/D$ and $d = \prod_I d_{F(i)}/D$ and f the homomorphism induced by F.

Let us call a theory T an h-limit theory iff the class M = Mod (T) is closed under h-limit (i.e. if $F : I \longrightarrow Str_h \mathcal{L}$ is a functor such that lim F exists and every $F(i) \models T$ then $\lim F \models T$).

It is a very well known fact from category theory that limits exist iff products and equalizers exist. Hence a theory is an h-limit theory iff it is invariant under products and equalizers. Furthermore, if a theory T is invariant under equalizers then it is invariant under $\uparrow$ -h-colimits. (This follows from the above lemma.)

Our main aim is to characterize the h-limit theories to do that we need some more definition and notation.

Let M be a class of $\mathcal{L}$ -structures. We write $SW_h(M)$ for the class of $\mathcal{L}$ -structures A which verifies the following property: there exist $\mathcal{L}$ -structures B and C and $f : A \longrightarrow B$ and $g : B \longrightarrow C$ such that: i - $B \in M$

ii - f is an embedding and g is a homomorphism

iii - $g \circ f$ is an elementary embedding from A into C.

Let Δ be the set $\forall [V \wedge [At], \exists V \wedge [At]]$ The following relates those notions to $\uparrow$ -h-colimits:

Lemma 2.A.4: Let T be a 1st order theory in $\mathcal{L}$. Then

1 - T is invariant under SW_h

2 - T is invariant under $\uparrow$ -h-colimits

3 - T is axiomatised by $\top \cap \Delta$

pf: $1 \Rightarrow 2$. Let M be Mod T , M is ultraclosed; furthermore we have that T is invariant under SW_h so M is closed under SW_h. Let $F : I \longrightarrow Str_h \mathcal{L}$. Such that $F(i) \in M$ and I is $\uparrow$ -directed Let C = colm F. By Lemma 2.A.3 we have that $d : C \longrightarrow C^I/_D$ is equal to he where $h : C \longrightarrow \prod F(i)/_D$ and $e : \prod F(i)/_D \longrightarrow C^I/_D$ but e is an embedding and $\prod F(i)/_D \in M$ and d is an elementary embedding so $C \in SW_h(M)$ and hence $C \in M$.

$2 \Rightarrow 1$. Since M = Mod T then M is elementary and assume M closed under $\uparrow$ -h-colimits. We need to show that $SW_h(M) \subseteq M$ so let $A \in SW_h(M)$ Then there is $B_1 \in M$ s.t. $A \underset{g}{\subsetneq} B_1$ and C so that $f : B_1 \to C$ is a homomorphism and $f \circ g$ is an elementary embedding so $A \prec C$ thus since $SW_h(M)$ is elementary then $C \in SW_h(M)$ so let $C = C_1$ and we repeat the same argument, thus we get B_2 and C_2 for C_1 similarly we continue to build a sequence $B_1, \cdots, B_n$ ... of elements of M. Now we note that $A \prec \underset{n \in \omega}{\mathrm{colim}} (B_n)$ so $A \in M$.

$3 \Rightarrow 1$. Assume that T is axiomatized by $\top \cap \Delta$ we want to show that $SW_h(M) \subseteq M$ So let A be an element of $SW_h(M)$.

We shall use a Lemma from Chang and Keisler which says that If Δ is a set of sentences closed under finite disjunction then the following are equivalent:

i - A theory T is axiomatized by a subset of Δ

ii - If $B \models T$ and every sentence of Δ which is true in B is true in A then $A \models T$. Since $A \in SW_h(M)$ then there exist $B \models T$ such that:

1. A is embedded in B ($f : A \longrightarrow B$ the embedding)

2. there exist an $\mathcal{L}$-structure C and a homomorphism from B into C, g say.

3. $g \circ f$ is an elementary embedding

Let $\delta = \forall \bar{v} [\bigvee_{i \le n} \bigwedge_{j \le m} \delta ij (\bar{v}) \longrightarrow \exists \bar{w} \bigvee_{k \le \ell} \bigwedge_{\ell \le q} \psi_{k\ell} (\bar{v}, \bar{w})]$ be an element of Δ true in B.

We show that δ is true in A for:

Assume $A \models \bigvee_{i \le n} \bigwedge_{j \le m} \delta ij (\bar{a})$ then $B \models \bigvee_i \bigwedge_j \delta ij (f(\bar{a}))$

but then $B \models \exists \bar{w} \bigvee_k \bigwedge_\ell \psi_{k\ell} (f(\bar{a}), \bar{w})$ i.e. B

$B \models \bigvee_k \bigwedge_\ell \psi_{k\ell} (f(\bar{a}), \bar{b})$

so $C \models \bigvee_{k \le \ell} \bigwedge_{\ell \le q} \psi_{k\ell} (gf(\bar{a}), g(\bar{b}))$ we can identify $gf(\bar{a})$ with $\bar{a}$ because $A < C$ so $A \models \exists \bar{w} \bigvee_k \bigwedge_\ell \psi_{k\ell} (\bar{a}, \bar{w})$ and hence $A \models \delta$.

$1 \Rightarrow 3$. We again use the Lemma used in $3 \Rightarrow 1$. and let $A \models T$, B an $\mathcal{L}$-structure so that if $A \models \delta$ then $B \models \delta$ for all $\delta \in \Delta$. We need to show that $B \in SW_h(M)$ or $B \models T$. Consider A' an elementary extension of A such that A' is $|B|$-saturated. By above we have that whenever $\delta \in \Delta$ such that $B \models \neg \delta$ then $A \models \neg \delta$ so $A' \models \neg \delta$. Let b, be an enumeration of elements of B, then, and because $\neg \Delta$ is closed under existential quantification and conjunction; there is a map $f : B \rightarrow A'$ so that f (b) = a say. This map is a homomorphism because $[At] \subseteq \neg \Delta$

Now let (B', b) be an elementary extension of (B, b) so that (B', b) is $|A'|$-saturated because $\exists \vee \wedge [At] \subset \Delta$ and if $A' \models \delta$ for $\delta \in \Delta$ then $B' \models \delta$ then there is $g : A' \rightarrow B'$ a homomorphism so that $g(a) = b$. Hence $g \circ f(b) = b$ and gf is an elementary embedding from B into B'.

Let T be a fixed theory in $\mathcal{L}$ and let A be an $\mathcal{L}$-structure and D a substructure of A

We define $Def_T(D, A)$ as follows:

$a \in Def_T(D,A)$ iff there exists $\mu(x,\bar{y}) \in \exists\wedge[At]$ so that $A \models \mu(a,\bar{d})$ for some $\bar{d} \in D$ and $T \vdash \forall\bar{y}\,\exists^{\leq 1}_x\, \mu(x,\bar{y})$

Lemma 2.A.5: i - Every h-equalizer is a $Def_T(D,A)$

ii - T is invariant under h-equalizer iff T is invariant under $Def_T(-,-)$.

pf: i - Let $E = E(f_1,f_2)$ so that $f_1,f_2 : A \to B$ and $B \models T$. We claim that $E = Def_T(E,A)$ if $a \in Def_T(E,A)$ then there is $\mu(x,\bar{y})$ as above so $A \models \mu(a,\bar{e})$ for some $\bar{e} \in E$ so $B \models \mu(f_1(a), f_1(\bar{e})) \wedge \mu(f_2(a), f_1(\bar{e}))$ but $B \models T$ so $f_2(a) = f_1(a)$ and $a \in E$.

Now let $T_1 = T \cup D^+(f_1(A)) \cup D^+(f_2(A)) \cup \{a^1 = a^2 ; a \in E\}$

$(B,e)_{e\in E} \models T_1$ What we need to show is that $e \in Def_T(E,A)$ for $T_1 \vdash e^1 = e^2$ (because $e \in E$ so $e^1 = e^2$ and $T_1 \vdash e^1 = e^2$) thus there is $\mu(x,\bar{y}) \in \exists\wedge[At]$ so that $A \models \mu(e,\bar{e})$ for some $\bar{e} \in E$ and $T \vdash \forall\bar{y}\,\exists^{\leq 1}_x\, \mu(x,\bar{y})$ so $e \in Def_T(E,A)$.

ii - By i - we have that if T is invariant under Def_T then T is invariant under h-equalizer.

Conversely we show that every $Def_T(D,A)$ is a $\downarrow$-h-limit.

By the proof of i - we have that $Def_T(D,A) = \bigcap E(f,g)$ where $(*)$ $f,g : A \longrightarrow B$; $B \models T$, $|B| \leq Max\{|A|,|\mathcal{L}|\}$ and $f/_D = g/_D$

Let $\{(f_\varepsilon, g_\varepsilon) ; \varepsilon \leq \alpha\}$ be such that $f_\varepsilon, g_\varepsilon$ verify $(*)$ $\forall \varepsilon \leq \alpha$

It is easy to see that $Def_T(D,A) = \bigcap_{\varepsilon\leq\alpha} A_\varepsilon$ where A_ε is defined as follows: $A_0 = A$ and $A_{\varepsilon+1} = E(f_{\varepsilon/A_\varepsilon}, g_{\varepsilon/A_\varepsilon})$ and $A_\lambda = \bigcap_{\varepsilon<\lambda} A_\varepsilon$ for limits λ. Now assume $A \models T$. We show that $A_\varepsilon \models T$ for all ε. We do that by induction, say $A_\varepsilon \models T$ And T is invariant under h-equalizer so $A_{\varepsilon+1} \models T$. Now $Def_T(D,A)$ is $\downarrow$-h-limit of models of T so by Lemma 2.A.3 we have the result.

Let C be a fixed set of constants and consider $\mathcal{L}(C)$ the Language augmented by the elements of C. Let Γ be a set of atomic sentences of $\mathcal{L}(C)$. Then $\langle C, \Gamma \rangle$ is called a representation. We say that A is represented by $\langle C, \Gamma \rangle$ over T iff there exists a map $f : C \to A$ such that:

i - $A \models \gamma(f(\bar{c}))$ for all $\gamma(\bar{c}) \in \Gamma$ and $A \models T$.

ii - If $g : C \to B$ and $B \models T$ such that $B \models \gamma(g(\bar{c}))$ for all $\gamma(\bar{c}) \in \Gamma$ then there exists a unique homomorphism h from A into B so that $h \circ f = g$

If C and Γ are finite then $\langle C, \Gamma \rangle$ is said to be a finite presentation and if A has a finite presentation over T we say that A is finitely presented (f.p. for short).

By Lemma 2.A.5.i - and the uniqueness of h above one has if A is represented by $\langle C, \Gamma \rangle$ then $\mathrm{Def}_T(f(C), A) = A$ and so A is generated by $f(C)$

Lemma 2.A.6: Let $C = \{c_1, \ldots, c_m\}$ and $\Gamma = \{\gamma_1(\bar{c}), \ldots, \gamma_n(\bar{c})\}$ form a finite presentation. Let $A \models T$ and $f : C \to A$ so that $A \models \bigwedge_{i \le n} \gamma_i(f(\bar{c}))$ and $\mathrm{Def}_T(f(\bar{c}), A) = A$ then the following are equivalent.

i - A is f.p. by $\langle C, \Gamma \rangle$ over T.

ii - For every $\delta(\bar{v}) \in \exists\wedge[At]$ If $A \models \delta(f(\bar{c}))$ then

$$T \vdash \forall \bar{v} [\bigwedge_{i \le n} \gamma_i(\bar{v}) \to \delta(\bar{v})]$$

pf: i $\Rightarrow$ ii. It is clear that $A \models \forall \bar{v} [\bigwedge_i \gamma_i(\bar{v}) \to \delta(\bar{v})]$. Now let $B \models T$ and assume $B \models \bigwedge_i \gamma_i(\bar{b})$ consider $g : C \to B$ s.t. $g(c_i) = b_i$

By i - there is homomorphism $h : A \to B$ so that $h \circ f = g$ thus we have

$A \models \delta(f(\bar{c}))$ so $B \models \delta(h \circ f(\bar{c}))$ i.e. $B \models \delta(g(\bar{c}))$ i.e. $B \models \delta(\bar{b})$

ii $\Rightarrow$ i. Let $B \models T$ and $g : C \to B$ be so that $B \models \bigwedge_i \gamma_i(g(\bar{c}))$

For every $a \in A$, $a \in \mathrm{Def}_T(f(C), A)$ so there exists a formula in $\exists\wedge[At]$ $\mu(v, \bar{w})$, say, so that $A \models \mu(a, f(\bar{c}))$ and $T \vdash \forall \bar{w} \exists^{\le 1}_v \mu(v, \bar{w})$ thus $B \models \forall \bar{w} \exists^{\le 1}_v \mu(v, \bar{w})$ Let $b \in B$ so that $B \models \mu(b, g(\bar{c}))$ and define

$h: A \longrightarrow B$ by $h(a) = b$ Remark that b exists because of ii - and the fact $B \models \bigwedge_i \gamma_i(g(\bar{c}))$ Now it is easy to see that h is a homomorphism and that $h \circ f = g$.

prop 2.A.7: Let $\langle C, \Gamma \rangle$ be a finite presentation so that $T \cup \Gamma$ is consistent in $\mathcal{L}(C)$, where $\mathcal{L}(C)$ contains at least one constant. Let $C = \{c_1, \ldots, c_m\}$ and $\Gamma = \{\gamma_1(\bar{c}), \ldots, \gamma_n(\bar{c})\}$ and assume $A \models T$ and A is f.p. by $\langle C, \Gamma \rangle$ then T verifies the following:

($*$) For every finite sequences of formulae $\varphi_1(\bar{v}), \ldots, \varphi_k(\bar{v})$ in $\exists \wedge [At]$ If $T \vdash \forall \bar{v} [\bigwedge_{i \leq n} \gamma_i(\bar{v}) \longrightarrow \bigvee_{j \leq k} \varphi_j(\bar{v})]$ then $T \vdash \forall \bar{v} [\bigwedge_{i \leq n} \gamma_i(\bar{v}) \rightarrow \varphi_j(\bar{v})]$ for some $j \leq k$. Furthermore if T is invariant under h-equalizer and T verifies ($*$) then there is $A \models T$ s.t. A is f.p. by $\langle C, \Gamma \rangle$

pf: Let $A \models T$ so that A is f.p. by $\langle C, \Gamma \rangle$ and assume that:

$$T \vdash \forall \bar{v} [\bigwedge_i \gamma_i(\bar{v}) \rightarrow \bigvee_j \varphi_j(\bar{v})]$$

Assume that $T \nvdash \forall \bar{v} [\bigwedge_i \gamma_i(\bar{v}) \rightarrow \varphi_j(\bar{v})]$ for all $j \leq k$ Then for every $j \leq k$ there exist A_j and $\bar{a}_j$ sequence of elements of A_j so that $A_j \models T$, $A_j \models \wedge \gamma_i(\bar{a}_j)$ and $A_j \models \neg \varphi_j(\bar{a}_j)$

Let $g_j: C \longrightarrow A_j$ be defined by $g_j(c_i) = a_{ji}$ since A is f.p. by $\langle C, \Gamma \rangle$ there exist h_j's, s.t. $h_j \circ f$ extends g_j but this gives us that $A \models \bigwedge_i \gamma_i(f(\bar{c})) \wedge \bigwedge_j \neg \varphi_j(f(\bar{c}))$ but which contradicts the fact that $T \vdash \forall \bar{v} [\wedge \gamma_i(\bar{v}) \longrightarrow \bigvee_j \varphi_j(\bar{v})]$.

Conversely: assume T is invariant under h-equalizers and T verifies ($*$). Let Δ^+ be the set of sentences $\delta(\bar{c})$ so that $\delta(\bar{c}) \in \exists \wedge [At(\mathcal{L}(C))]$ and $T \vdash \forall \bar{v} [\bigwedge_i \gamma_i(\bar{v}) \rightarrow \delta(\bar{v})]$ and Δ^- be the set of $\delta(\bar{c})$ so that $\delta(\bar{c}) \in \exists \wedge [At(\mathcal{L}(C))]$ and $T \nvdash \forall \bar{v} [\bigwedge_i \gamma_i(\bar{v}) \rightarrow \delta(\bar{c})]$

Let $T^* = T \cup \Delta^+ \cup \Delta^-$ by ($*$) and the compactness theorem one deduces that T^* is consistent. Let $(B, \bar{b}) \models T^*$, and consider $Def_T(\bar{b}, B) \models T$ because T is invariant under h-equalizers. And it is easily seen that $Def_T(\bar{b}, B)$ is f.p. by $\langle C, \Gamma \rangle$.

In what follows we shall give 2 theorems concerning the sentences preserved by products which will be useful in characterizing the h-limit theories and which we shall use later on in studying the sentences preserved by global sections of Boolean sheaves. The first is due to Keisler; a full proof of it can be found in Chang and Keisler [73]. The second is due to Feferman and Vaught [59]. For definition of reduced products we refer the reader to Chapter 2, part IV. We say that a sentence is a reduced product sentence iff it is invariant under reduced products.

We say that a sentence φ is a basic Horn formula (sentence resp.) iff φ is of the form $\bigwedge_{i=1}^{n} \psi_i \longrightarrow \psi_{n+1}$

Where all the ψ_i's are atomic. Thus a basic Horn formula (sentence resp.) is a formula (sentence resp.) of the set $[\wedge[At], [At]]$

A Horn formula (sentence resp.) is a formula (sentence resp.) built from the basic Horn formulae (sentences resp.) using the connectives $\wedge, \exists, \forall$.

<u>Theorem 2.A.8</u>: φ is a reduced product sentence iff it is a Horn sentence.

<u>pf</u>: Assume φ is a Horn sentence. To show that φ is a reduced product sentence, one takes a reduced product $\prod_I A_i/D$, and assumes that $A_i \models \varphi$ for all $i \in I$. Then one argues by induction on the degree of complexity of φ . We do it here for the basic Horn sentences.

Case 1: Say $\varphi = \neg\psi_1 \vee \cdots \vee \neg\psi_n$ where all the ψ_i's are atomic

The set $X = \{i \in I \,;\, A_i \models \varphi\} = I \in D$

So let $X_j = \{i \in I \,;\, A_i \models \psi_j\}$

If for all $j \leq n$ $X_j \in D$ then $X \cap X_1 \cap \cdots \cap X_n = \phi \in D$ but D is a proper filter so $\phi \notin D$ and hence there must be an $X_j \notin D$.

But ψ_j is atomic so if $X_j \notin D$ then $\prod_I A_i/D \nvDash \psi_j$ so $\prod A_i/D \models \neg\psi_j$ and hence $\prod A_i/D \models \varphi$

Case 2: Say $\varphi = \psi_1 \wedge \cdots \wedge \psi_n \longrightarrow \psi_{n+1}$ where all the ψ_j's are atomic. Let $X_j = \{ i \in I ; A_i \models \psi_j \}$ and assume $A_i \models \varphi$ for all $i \in I$ and $\prod A_i/D \models \psi_1 \wedge \cdots \wedge \psi_n$ then $\prod A_i/D \models \psi_j$ for all $j \leq n$ and hence $X_j \in D$ thus $\bigcap_{j \leq n} X_j \in D$

Now let $i \in X \cap (\bigcap_j X_j)$ then $A_i \models \varphi$ and $A_i \models \bigwedge_{j \leq n} \psi_j$ thus $A_i \models \psi_{n+1}$ so $X \cap (\bigcap_j X_j) \subseteq X_{n+1}$ and hence $\prod A_i/D \models \psi_{n+1}$ so $\prod A_i/D \models \varphi$.

Conversely: We shall prove the converse when (C. H.) holds, then indicate briefly how to eliminate the C. H.. The elaborate proof of that can be found in Chang and Keisler [73]; a more direct proof can be found in Sheiloh [72]. So assume A_i $2^\omega = \omega^+$ Let φ be a reduced product sentence. If φ is inconsistent then $\varphi \leftrightarrow \forall x [x \neq x]$ which is a Horn sentence. So we may assume that φ is consistent and $\mathcal{L}$ is countable.

Let $\Sigma = \{ \psi ; \psi$ Horn sentence and $\vdash \varphi \to \psi \}$; $\Sigma \neq \emptyset$ and is closed under conjunction so to show our theorem it is enough to show that there exists an element $\psi \in \Sigma$ so that $\vdash \psi \longrightarrow \varphi$. Using a compactness argument one can show the theorem by proving that $\Sigma \vdash \varphi$ So we do that and we consider $B \models \Sigma$. One may assume that B is either finite or saturated of power ω_1.

Let $K = \{ \psi ; \psi$ is a Horn sentence and $\varphi \wedge \neg\psi$ is consistent $\}$ $K \neq \emptyset$ otherwise Σ will be the set of all Horn sentences and again φ will be inconsistent. So for each element $\psi \in K$ let $A_\psi \models \varphi \wedge \neg\psi$ so that A_ψ is countable. Let $I = \omega \times K$ and for each $i = \langle n, \psi \rangle$ Let $A_i = A_\psi$ Let ψ be an arbitrary Horn sentence which holds in almost all A_i. Then

$\psi \notin K$ and so $\psi \in \Sigma$ and $B \models \psi$. Now it is enough to show that $B \simeq \prod A_i/_D$ for some filter D on I . We shall sketch the proof of that since all the A_i's are countable then $|\prod_I A_i| \leq 2^{\omega} = \omega^+$ Hence one defines a surjective map $h : A = \prod A_i \to B$ so that:

1 - For all Horn formulae $\varphi(\bar{v})$ and all $\bar{y} \in A$ iffor almost all $i \in I$ $A_i \models \varphi(\bar{y}(i))$ then $B \models \varphi(\bar{h}(y))$

So let $a \in {}^{\omega^+}A$ and $b \in {}^{\omega^+}B$ enumerate A and B respectively. We construct by induction 2 sequences $\bar{a} \in {}^{\omega^+}A$ and $\bar{b} \in {}^{\omega^+}B$ so that for each $\nu < \omega^+$

2 - Every Horn sentence in $\mathcal{L} \cup \{c_\mu, \mu < \nu\}$ holding in almost every $(A_i, \bar{a}_\mu(i))$ holds in $(B, \bar{b}_\mu)$ and

3 - range (a) $\subseteq$ range ($\bar{a}$), range (b) $\subseteq$ range ($\bar{b}$).

Assume *2* holds for μ . We consider the 2 cases $\mu = \eta + 2k$ and $\mu = \eta + 2k + 1$

Case 1. $\mu = \eta + 2k$ and assume $\bar{a}_\nu = a_{\eta+k}$ Let

$\Sigma = \{\varphi(x) \; ; \; \varphi(x)$ a Horn formula in $\mathcal{L} \cup \{c_\mu \; ; \mu < \nu\}$ and for almost all $i \in I$ $(A_i, \bar{a}_\mu(i))_{\mu<\nu} \models \varphi(\bar{a}_\nu(i))\}$

It is easily seen that Σ is closed under $\wedge$ by induction and because $\exists x \varphi$ is a Horn sentence of $\mathcal{L} \cup \{c_\mu ; \mu < \nu\}$ we have: for every $\varphi \in \Sigma$ $(B, \bar{b}_\mu)_{\mu<\nu} \models \exists x \, \varphi(x)$ So one extends Σ to a type of $(B, \bar{b}_\mu)_{\mu<\nu}$ But B is ω^+- saturated so there is $\bar{b}_\nu \in B$ realizing this type. So 2 - holds for $\mu+1$

Case 2. $\mu = 2k + 1 + \eta$ and $\bar{b}_\mu = b_{\eta+k}$ Let

$\Sigma = \{\varphi(x) \; ; \; \varphi$ a Horn formula of $\mathcal{L} \cup \{c_\mu \; ; \mu < \nu\}$ and $(B, b_\delta)_{\delta<\mu} \models \neg\varphi(\bar{b}_\delta) \; \}$

Let $\varphi \in \Sigma$ so $(B, b_\delta)_\delta \models \exists x \neg \varphi(x)$ so $(B, \bar{b}_\delta) \not\models \forall x \varphi$ so the set $I_\varphi = \{i \; ; \; (A_i, \bar{a}_\mu(i)) \not\models \forall x \varphi\}$ is a subset of I of power ω and since $\mathcal{L}$ is countable then for each $\varphi \in \Sigma$ there is a subset $J_\varphi \subseteq I_\varphi$

of power α such that $J_{\varphi} \cap J_{\psi} = \phi$ whenever $\varphi, \psi \in \Sigma$ and $\varphi \neq \psi$
Define now $a_{\mu} \in \prod_{I} A_i$ as follows:
if $i \in J_{\varphi}$ let $a_{\mu}(i)$ be so that $(A_i, (\overline{a_{\xi}(i)}) \models \neg \varphi(a_{\mu}(i))$
otherwise choose any element of A_i. The definition of a_{μ} proves that (2) holds for $\mu + 1$. And hence $\bar{a}$ and $\bar{b}$ are so defined clearly (3) holds. Let $h : A \to B$ so that $h(\bar{a}_{\nu}) = \bar{b}_{\nu}$ for $\nu < \omega^{+}$. It is easy to see that h is a well defined surjection which verifies (1).
For each atomic formula $\varphi(\bar{v})$ and each sequence $\bar{y}$ of elements of A let
$K_{\varphi,y} = \{ i \in I ; A_i \models \varphi(\bar{y}(i)) \}$
If $B \models \varphi(h(\bar{y}))$ then $|K_{\varphi,y}| = \alpha$ Otherwise $\neg\varphi$ is a Horn formula $B \models \neg\varphi(h(\bar{y}))$ Let E be the collection of $K_{\varphi,\bar{y}}$ such that $B \models \varphi(h(\bar{y}))$
(4) Every finite intersection of elements of E has cardinality α .
Otherwise we have $B \models \bigvee_{i \leq n} \neg\varphi_i(h(\bar{y}))$ and $\bigvee \neg\varphi_i(h(\bar{y}))$ is a Horn sentence.
Let D be the filter generated by E . Now one shows that $B \simeq \prod A_i / D$
So $B \models \varphi$ And the theorem is proved with $C \cdot H$. holding.
Now one uses Gödel numbering for elements of $\mathcal{B}$. And the fact that If γ is an arithmetical statement then γ is provable in ZF iff it is provable in ZFL i.e. $ZF + V = L$ Since $ZF + V = L \vdash ZF + C.H.$ then $ZF + V = L \vdash \varphi$ is reduced product sentence iff φ is a Horn sentence.
Let us say that P(X) is an arithmetical predicate iff it is obtained from some recursive predicate $R(x, \bar{y})$ and such that

$P(x)$ iff $Q_1 y_1, \ldots, Q_n \bar{y}_n \; R(x, y_1, \ldots, y_n)$ where Q_i is either $\exists$ or $\forall$

One now proves that: 1 - "φ is a Horn sentence" is an arithmetical predicate

2 - "φ is a reduced product sentence" is an arithmetical predicate

(We refer the reader to Chang and Keisler for the proof of 2.)
Thus the statement: φ is reduced product sentence iff φ is a Horn sentence is an arithmetical statement. Thus finishing the proof of the theorem.

Now we are in position to characterize the h-limit theories:

Theorem 2.A.9: Let T be a theory then the following are equivalent:

1 - T is invariant under h-equalizers and products.

2 - T is invariant under h;equalizers and T has finite presentations.

3 - T can be axiomatized by $T \cap \forall[\wedge[At], \exists\wedge[At]]$ and it satisfies the following conditions: (**) If φ and ψ are in $\wedge[At]$ and $T \vdash \forall\bar{v}\, \varphi(\bar{v}) \Rightarrow \exists\bar{w}\, \psi(\bar{v},\bar{w})$ then there exists $\mu(\bar{v},\bar{w}) \in \exists\wedge[At]$ such that $T \vdash \forall\, \bar{v}\, [\varphi(\bar{v} \longrightarrow \exists\bar{w}\; [\psi(\bar{v},\bar{w}) \wedge \mu(\bar{v},\bar{w})]]$ and
$T \vdash \forall\, \bar{v}\; \exists^{\leq 1}\bar{w}\;\; \mu(\bar{v},\bar{w})$

pf: $1 \Rightarrow 2$.

Since T is invariant under products then T satisfies condition * of prop 2.A.7. Moreover T is invariant under h-equalizers so T must have finite presentation (by prop 2.A.7).

$2) \Rightarrow 3)$. Since T is invariant under h-equalizers so it is invariant under $\uparrow$-h-colimits. So by prop 2.A.7 condition (*) holds in T and by Lemma 2.A.4 T is axiomatized by $T \cap \forall[\wedge[At], \exists\forall\wedge[At]]$ Now one uses (*) to deduce that T is axiomatized by $T \cap \forall[\wedge[At], \exists\wedge[At]]$.
To show that the condition in (3) holds. Let $\varphi(\bar{v}), \psi(\bar{v},\bar{w})$ be as in condition 3. We use $\mu(\bar{v},\bar{w})$ of the proof of Lemma 2.A.5. Clearly $\mu(\bar{v},\bar{w})$ verifies the condition of 3.

$3) \Rightarrow 2)$. can be done by an easy verification since the elements of $\forall[\wedge[At], \exists\wedge[At]]$ are Horn sentences, so they are preserved by products. Furthermore the condition in (3) can be used to verify that

T is invariant under h-equalizers.

Cor 2.A.10: T is an h-limit theory iff T is axiomatized by

$T \cap \forall[\wedge[At], \exists\wedge[At]]$ and T verifies the condition (**) of Theorem 2.A.9 (3).

Another theorem due to Feferman and Vaught will be useful later, we shall state it here and sketch its proof.

Theorem 2.A.11: A sentence φ is preserved by arbitrary products iff it is preserved by finite products.

The proof needs some definitions and notations: let us denote by T_{AB} the theory of Boolean algebras.

We say that an $\mathcal{L}$-formula $\varphi(\bar v)$ is determined iff there exists a sequence of formulae $(\sigma, \psi_1, \dots, \psi_m)$ such that

i - Every ψ_j is an $\mathcal{L}$-formula with free variables among $\bar v$.

ii - $\sigma(\bar w)$ is a formula in the language of B.A. with

$$T_{BA} \vdash \forall \bar z\, \forall \bar w\, [\sigma(\bar w) \wedge \bigwedge_{j \le m} (w_j \le z_j) \to \sigma(\bar z)].$$

iii - Let I be an index set D a filter on I, $(A_i)_{i \in I}$ be a family of $\mathcal{L}$-structures and $\bar a$ a sequence of elements of $\prod A_i$ and let

$X_j = \{i \in I \,;\, A_i \models \psi_j(a(j))\}$ then :

$$\prod A_i/D \models \varphi(|\bar a|_D) \quad \text{iff} \quad \mathcal{P}(I)/D \models \sigma(|X_1|_D, \dots, |X_n|_D)$$

One proves by induction that: (1) Every formula is determined.

Now let Δ be a finite set of formula We denote by $\neg\Delta = \{\neg\delta \,/\, \delta \in \Delta\}$ and by $\wedge[\Delta]$ and $\exists v\,\Delta$ the closure under $\wedge$ and $\exists v$ of Δ. And by $B(\Delta)$ the set of Boolean combination of elements of Δ. $B(\Delta)$ is a finite $B.A.$

We say that Δ is self-determining iff Δ is finite and each $\psi \in \Delta$ is determined by $(\sigma, \psi_1, \dots, \psi_m)$ where all the ψ_i 's are in Δ. And we say that Δ can be found effectively as a self-determining set iff Δ is explicitly given and for each $\varphi \in \Delta$ the determining sequence $(\sigma, \psi_1, \dots, \psi_m)$

is also explicitly given. The proof of (1) gives rise to

(2) If Δ , Δ' are self-determining then so are $\Delta \cup \Delta', \neg\Delta, \exists v(\Delta)$ and $B(\Delta)$

we say that Δ is autonomous iff Δ is self-determining and $\Delta = B(\Delta)$

Using (2) one proves by induction that

(3) For every formula φ we can effectively find an autonomous set Δ_φ such that $\varphi \in \Delta_\varphi$

If Δ is finite then $B(\Delta)$ is finite so it has atoms. Let $At_0(\Delta) = \{\varphi_1, -, \varphi_n\}$ be the set of atoms in $B(\Delta)$. Let I be an index set so that $\varphi_i \in At_0(\Delta)$

We claim that there is $\sigma \in At_0(\Delta)$ such that if $A_i \models \varphi_i$ then $\prod A_i \models \sigma$

To do that one shows that if $B_i \models \varphi_i$ for all $i \in I$ then:

for all $\varphi \in \Delta$ $\prod A_i \models \varphi$ iff $\prod B_i \models \varphi$

Take $\varphi \in \Delta$ and $(\sigma, \psi_1, ---, \psi_m)$ its determining sequence.

$X^j = \{i \in I ; A_i \models \psi_j\} = \{i ; B_i \models \psi_j\}$ so $\prod A_i \models \varphi$ iff $\mathcal{P}(I) \models \sigma[X^1, --, X^m]$ iff $B \models \varphi$. We write $\sigma = \prod_{i \in I} \varphi_i$

Let Δ be an autonomous set, we say that $\prod_{i \in I}(-)$ is essentially finite with index m on Δ iff for all $\varphi \in \Delta$ and all index sets with at least m elements. There exists a subset J of I ; $|J| = m$ and such that

$\prod_J \varphi = \prod_K \varphi = \prod_I \varphi$ for all K such that $J \subseteq K \subseteq I$

The above definition of $\sigma = \prod_I \varphi_i$ gives us:

(4) (i) φ is preserved under direct powers iff φ is equivalent to a disjunction of sentences preserved under direct products.

(ii) For any sentence φ in $\mathcal{L}$ one can find effectively an autonomous set of sentences Δ such that $\varphi \in \Delta$ and $\prod_{i \in I}(-)$ is essentially finite with index m and m can be effectively determined by Δ .

<u>pf</u> of the theorem: We shall prove a stronger statement, that is to say,

given a sentence φ we can effectively find a number n such that for all index sets I and all $\mathcal{L}$-structures $(A_i)_{i \in I}$ there is a subset J of I with at most n elements such that for all K, $J \subset K \subset I$; $\prod_K A_i \models \varphi$ iff $\prod_I A_i \models \varphi$, By (4) (ii) we can choose Δ to be an autonomous set for φ with $\Pi(-)$ is essentially finite on Δ with index m

Let n = m $|\Delta|$ Let $\{ \psi_i ; i \le n \}$ be the set of atoms of Δ suppose $A_i \models \psi_i$; if an atom ψ occurs in the list $\psi_i, i \in I$ less than m times then put into J all indices i such that $\psi_i = \psi$ if it occurs more than select any m indices i so that $\psi = \psi_i$ and put them into J so J will have at most n elements. Now remark that it is enough to prove the statement for all the atoms ψ in Δ such that $\psi \le \varphi$ so let K be such that $J \subset K \subset I$ and $\psi \in At_0(\Delta)$ with $\psi \le \varphi$

$\Pi(-)$ has index m so one gets $\prod_J \psi_i = \prod_K \psi_i = \prod_I \psi_i$ Thus $\prod_K \psi_i = \psi$ iff $\prod_I \psi_i = \psi$ and $\prod_K A_i \models \psi$ iff $\prod_I A_i \models \psi$

B - G.S. Theories:

The following theorem classifies the G.S. Theories.

Theorem 2.B.1: Let T be a first order theory then the following are equivalent:

1 - T is a G.S. Theory

2 - T is invariant under the global section structures of sheaves over compact spaces

3 - T is invariant under the global section structures of sheaves over P.O. sets endowed with the topology whose basis is the set of sets of the form $[x, \rightarrow] = \{ y ; y \ge x \}$.

4 - T is an h-limit theory.

pf: $1 \Rightarrow 2$ and $1 \Rightarrow 3$ trivially. We only need to show that $2 \Rightarrow 4$ and

$3 \Rightarrow 4$ and $4 \Rightarrow 1$.

$4 \Rightarrow 1$. Let (S, π) be an s-sheaf over X so that $S_x \models T$ for all $x \in X$
Since T is an h-limit then by Cor 2.A.10 we have: T is axiomatized by
$T \cap \forall[\wedge[At], \exists\wedge[At]] = \Delta$ and for all $\varphi(\bar{v}), \psi(\bar{v},\bar{w}) \in \wedge[At]$
if $T \vdash \forall\bar{v}[\varphi(\bar{v}) \rightarrow \exists\bar{w}\,\psi(\bar{v},\bar{w})]$ then there is $\mu(\bar{v},\bar{w}) \in \exists\wedge[At]$
so that $T \vdash \forall\bar{v}[\varphi(\bar{v}) \rightarrow \exists\bar{w}[\psi(\bar{v},\bar{w}) \wedge \mu(\bar{v},\bar{w})]]$ and $T \vdash \forall\bar{v}\,\exists^{\leq 1}\bar{w}\,\mu(\bar{v},\bar{w})$
so let $\chi(\bar{v},\bar{w}) = \forall\bar{v}[\varphi(\bar{v}) \rightarrow \exists\bar{w}\,\psi(\bar{v},\bar{w})] \in \Delta$
We show that $\Gamma(X,S) \models \chi(\bar{v},\bar{w})$ so let $\bar{f} \in \Gamma(X,S)$ so that
$\Gamma(X,S) \models \varphi(\bar{f})$ thus for all $x \in X$ $S_x \models \varphi(\bar{f}(x))$ Consider $\mu(\bar{v},\bar{w})$
the above described formula, hence $S_x \models \exists\bar{w}\,\psi(\bar{f}(x),\bar{w}) \wedge \mu(\bar{f}(x),\bar{w})$
So $S_x \models \psi(\bar{f}(x), \bar{s}_x) \wedge \mu(\bar{f}(x), \bar{s}_x)$ where s^j_x is a
unique element of S_x. This holds for all $x \in X$ and all $j = 1, \ldots, \ell(\bar{w}) = n$
Note that $\psi(\bar{v},\bar{w})$ and $\mu(\bar{v},\bar{w})$ are conjunction of atomic formulae
and (S, π) is an s-sheaf over X. Thus for every $j = 1, \cdots, n$ and every
s^j_x there exists an open subset $\mathcal{U}^j_x$ of X with $x \in \mathcal{U}^j_x$ and a local
section g^j_x over $\mathcal{U}^j_x$ such that $g^j_x(x) = s^j_x$ Thus we have:
1 - $\forall j = 1, \ldots, n$ $\quad \bigcup_{x \in X} \mathcal{U}^j_x = X$
2 - $\forall j = 1, \ldots, n$ $\quad \forall x_1, x_2 \in X$ $\quad g^j_{x_1} / \mathcal{U}_{x_1} \cap \mathcal{U}_{x_2} = g^j_{x_2} / \mathcal{U}_{x_1} \cap \mathcal{U}_{x_2}$ So let
$g^j = \bigcup_{x \in X} g^j_x$ this is a global section Moreover $g^j(x) = s^j_x$ for all $j = 1, \ldots, n$
Thus $S_x \models \psi(\bar{f}(x), \bar{g}(x)) \wedge \mu(\bar{f}(x), \bar{g}(x))$ but ψ is a conjunction of
atomic so $\Gamma(X,S) \models \psi(\bar{f},\bar{g})$ i.e. $\Gamma(X,S) \models \exists\bar{w}\,\psi(\bar{f},\bar{w})$

$3 \Rightarrow 4$. Let T be invariant under the global sections of s-sheaves over sets endowed with the topology described in 3.

<u>Claim 1</u>: T is invariant under products.

Let X be any set order X by $\leq$ defined as follows $x \leq y$ iff $x = y$
Thus the topology defined in 3 is the discrete topology on X. Now an
s-sheaf of $\mathcal{L}$ -structures over X verifies that $\Gamma(X,S) \simeq \prod_{x \in X} S_x$ so T is

invariant under products.

Claim 2: T is invariant under intersection of substructures.

For let $X = \{x_0, x_1, x_2\}$ with $x_0 \leq x_1$ and $x_2 \leq x_1$ Assume $S_{x_0} \subseteq S_{x_1}$ and $S_{x_2} \subseteq S_{x_1}$ and $S_{x_1} \models T$ We claim that $S_{x_0} \cap S_{x_2} \models T$

Topologize X as in 3. Thus the S_x's form a Kripke structure. By theorem 1.C.3. there is an s-sheaf (S, π) over X so that:

$\Gamma(\{x_0, x_1\}, S) \simeq S_{x_0}$ and $\Gamma(\{x_2, x_1\}, S) \simeq S_{x_2}$ consider

$\Gamma(X, S) \simeq \Gamma(\{x_0, x_1\}, S) \cap \Gamma(\{x_2, x_1\}, S) = S_{x_0} \cap S_{x_2}$ and by hypothesis $\Gamma(X, S) \models T$

Now remark that every h-limit can be represented as products and intersection of $\mathcal{L}$-structures so T is invariant under h-limit.

$2 \Rightarrow 4$. Claim 2 of the above holds for this case because X is finite so it is compact.

Claim 1 holds for finite products because again X is finite.

By Theorem 2.A.11 (Feferman-Vaught theorem) we have that T is invariant under products - so again T is an h-limit theory.

The following example gives a theory which is a G.S. theory which is not axiomatized by G.S. sentences. It is due to Volger but the proof given in Volger's [79] does not work. Here we give a variation of his proof.

Example 2.B.2: Let $\mathcal{L}$ be a language containing predicates $(P_n)_{n\in\omega}$ so that each P_n has arity n ; let T be the theory whose axioms are:

i - $\varphi_n = \forall x \forall \bar{y} \forall y [P_{n+1}(x, \bar{y}, y) \rightarrow P_n(x, \bar{y})]$ for all $n \in \omega$

ii - $\psi_n = \forall x \forall \bar{y} \forall y \forall \bar{z} \forall z [P_{n+1}(x, \bar{y}, y) \wedge P_{n+1}(x, \bar{z}, z) \rightarrow \bigwedge_i y_i = z_i]$ for all $n \in \omega$

iii - $\chi_n = \forall x \exists \bar{y}\ P_n(x, \bar{y})$ for all $n \in \omega$

Clearly T is an h-limit theory. Similarly $\Sigma_0 = \{\varphi_n \wedge \psi_n ; n \in \omega\}$ is invariant under h-limits.

Let $A \models \Sigma_0 \cup \{\chi_n\}$ and let $D = \{a \in A ; A \models \forall \bar{y}\ P_{n+1}(a, \bar{y})\}$

Put $A_j = A \dot{\cup} D^{(\omega - j)}$ for all $j = 1, \ldots, n, \ldots$ Thus $A = \bigcap_{j \in \omega} A_j$

Every $A_j \models \Sigma_0 \cup \{\chi_{n+1}\}$ For it is enough to show that

$A_j \models \chi_{n+1}$ i.e. $A_j \models \forall x \; \exists \bar{y} \; P_{n+1}(x, \bar{y})$ We do that as follows

if $x \in A$ and $x \in D$ then $A \models \exists \bar{y} \, P_{n+1}(x, \bar{y})$ so $A_j \models \exists \bar{y} \; P_{n+1}(x, \bar{y})$

Furthermore $A \models \exists \bar{y} \, P_n(x, \bar{y})$ so assume $x \in D$

For A_1 choose $y_{n+1} = x$ where x is in the first copy of D added to A

For A_2 choose $y_{n+1} = x$ where x is in the second copy of D added to A

etc. ... Clearly $A_j \models P_{n+1}(x, \bar{y}, y_{n+1})$ for all j. We claim that:

Claim 1: T is not axiomatized by a set of sentences which are invariant under h-limits.

pf of claim 1: If not then there exists a set of h-limit sentences Δ say such that $T \cap \Delta$ axiomatizes T. Thus there is $\delta \in \Delta$ such that $T \vdash \delta$ and $\delta \vdash \chi_2$. By compactness there must be an $n_0 \in \omega$ such that

$\Sigma_0 \cup \{\chi_{n_0+1}\} \vdash \delta$ Let κ be the smallest such n_0 so

$\Sigma_0 \cup \{\chi_\kappa\} \nvdash \delta$ By the above there is a family of $\mathcal{L}$ -structures

$(A_j)_{j \in \omega}$ so that for all $j \in \omega$ $A_j \models \Sigma_0 \cup \{\chi_{\kappa+1}\}$ and $A = \bigcap_j A_j$

so A is a limit of the A_j 's and hence $A \models \delta$ and that is true for any model of $\Sigma_0 \cup \{\chi_\kappa\}$ contradicting the choice of κ , thus proving the claim.

This theory T is a G.S. theory. Now note:

1 - If T is a G.S. theory then T is invariant under h-limits

2 - Since every $\downarrow$ -e-limit is an h-limit then any theory which is invariant under global section is invariant under $\downarrow$ -e-limits.

Claim 2: T cannot be axiomatized by G.S. sentences.

pf: follows from Claim 1 since every G.S. sentence is an h-limit sentence.

C.G.S. Sentences

We refer the reader to Chapter 2, part 4 for the definition of Boolean s-sheaves. We say that a sentence is a Boolean Global section sentence (B.G.S. sentence for short) iff it is invariant under the global section structures of Boolean s-sheaves.

Theorem 2.C.1: i - Every B.G.S. sentence is equivalent to a Horn sentence.

ii - If φ is a B.G.S. sentence then φ is invariant under limit, reduced product and direct products.

iii - Every sentence in $\forall\wedge[Pos, \forall\exists[\wedge[At];[At]]]$ is a B.G.S. sentence; in particular every universal Horn sentence is a B.G.S. sentence.

iv - More generally, every universal Horn sentence is a G.S. sentence.

v - There exists a non-universal Horn sentence which is a G.S. sentence.

pf: i - By part 4 of Chapter 2 we have that every reduced product is isomorphic to a global section structure of a Boolean s-sheaf; hence every B.G.S. sentence is a reduced product sentence by theorem 2.A.8. it is equivalent to a Horn sentence.

ii - We have that every limit reduced product (and hence every direct product is isomorphic to a global section structure of a Boolean s-sheaf. Thus ii follows

iii - The proof of this part follows from the following Lemmas:

Lemma I: Let (S, π) be a Boolean s-sheaf over X. Then $\Gamma(X, S)$ is isomorphic to a subdirect product of $(S_x)_{x\in X}$.

pf: Let $C \subset X$ be closed and $g: C \to S$ be a continuous map so that $\pi\circ g = 1_C$ We claim that:

there exists $f \in \Gamma(X,S)$ such that $f/_C = g$. To show that let $x \in X - C$ there is N_x an open subset of X such that $\pi^{-1}/_{N_x}$ is a local section over N_x. Similarly for every $x \in C$ such N_x does exist. Now note that $\bigcup_{x \in X} N_x = X$ so we may consider a finite disjoint cover of them say $N_{x_1}, \dots, N_{x_n}$ and take $f = \bigcup_{i \in I} \pi^{-1}/N_{x_i}$. Thus $f \in \Gamma(X,S)$ and clearly $f/_C = g$. Now since every $\{x\} \subseteq X$ is closed the lemma follows.

Lemma II: i - If $\varphi(\bar{v})$ is a positive formula then $\Gamma(X,S) \models \varphi(\bar{f})$ iff

$$\{x \in X;\ S_x \models \varphi(f(x))\} = X.$$

ii - Let $\forall \bar{v}\, \varphi(\bar{v})$ be a B.G.S. sentence and $\psi(\bar{v})$ be a positive sentence, then $\forall \bar{v}\, [\psi(\bar{v}) \rightarrow \varphi(\bar{v})]$ is a B.G.S. sentence.

pf: i - follows from Lemma I.

ii - Assume $\Gamma(X,S) \models \psi(\bar{f})$ and (S, π) is a Boolean s-sheaf over X. Thus by i - we have $S_x \models \psi(\bar{f}(x))$ for all $x \in X$. Thus $S_x \models \varphi(\bar{f}(x))$ for all $x \in X$ but $\forall \bar{v}\, \varphi(\bar{v})$ is a B.G.S. sentence so $\Gamma(X,S) \models \varphi(\bar{f})$.

Lemma III: Every sentence of the form $\forall \bar{v}\, \exists \bar{w}\, [\bigwedge_{i \leq n} \varphi_i(\bar{v},\bar{w}) \rightarrow \varphi_{n+1}(\bar{v},\bar{w})]$ where the φ_i's are atomic, is a B.G.S. sentence.

pf: Assume $\varphi(\bar{v},\bar{w})$ is a q.f. formula assume that for $\bar{f} \in \Gamma(X,S)$, where (S,π) is a Boolean s-sheaf over X; the set $\{x \in X$

$$\{x \in X;\ S_x \models \exists \bar{w}\, \varphi(\bar{f}(x), \bar{w})\} = X.$$

Let $\bar{h} \in \Gamma(X,S)$ then $\{x \in X;\ S_x \models \varphi(\bar{f}(x), \bar{h}(x))\}$ is a clopen. An argument similar to the one in Lemma I shows that there exists $\bar{g} \in \Gamma(X,S)$ so that $\{x \in X;\ S_x \models \varphi(\bar{f}(x), \bar{g}(x))\} = X$. Now the Lemma follows from this fact.

A combination of Lemma II and III gives the proof of iii.

iv - Follows from theorem 2.B.1.

v - Let $\varphi(x,y) = \forall x\, \forall y\, [R(x) \wedge R(y) \rightarrow x=y] \wedge \exists x\, R(x)$ where R is a uniray predicate of $\mathcal{L}$. $\varphi(x,y)$ is a G.S. sentence by theorem 2.B.1. but clearly it is not a universal Horn. Since $\exists x\, R(x)$ is not preserved by submodels.

Before we derive some of the consequences of the above theorem let us recall the definition of a complete structure.

Let A be an $\mathcal{L}$ -structure say A = $\langle A, (f_i), (R_j), (C_k) \rangle$ then A is said to be complete iff whenever a function F of $\mathcal{L}$ is defined on the domain A of A then it is equal to one of the f_i's.

We combine the above result with a result of Keisler to get:

<u>Theorem 2.C.2</u>: Let A,B be 2 $\mathcal{L}$ -structures and assume A complete then the following are equivalent:

i - Every universal Horn sentence of $\mathcal{L}$ which is true in A is true in B.

ii - B is isomorphic to a limit reduced power of A

iii - B is isomorphic to a global section structure of a Boolean sheaf whose stalks are limit ultrapowers of A.

<u>pf</u>: ii $\Rightarrow$ iii. This is just the representation theorem of limit reduced products, Chapter 2, theorem 4.10.

iii $\Rightarrow$ i. Assume $\forall \bar{v}\, \varphi(\bar{v})$ is a universal Horn sentence of $\mathcal{L}$ which holds in A; using Los' theorem we get that $\forall \bar{v}\, \varphi(\bar{v})$ holds in every ultrapower of A. Since limit ultrapowers are substructures of ultra-powers then $\forall \bar{v}\, \varphi(\bar{v})$ holds in all the stalks of the sheaf described in iii. Thus by the above theorem (2.C.1) it holds in $\Gamma(X,S) \simeq B$.

i $\Rightarrow$ ii. We follow Hodges and Shelah's argument.

Let us call a family $(f_i)_{i<2^\mu}$ of functions from μ into X where

$\mu \geq |X|$ (μ, ω, X) -independent iff for every increasing sequence $(\alpha_i ; i < n < \omega)$ $\alpha_i < 2^\mu$ and every family $(x_i, i < n)$ of elements of X there is $j < \mu$ such that $f_{\alpha_i}(j) = x_i$ for all $i < n$.

Now let $\mu > |A|$ and $(f_i)_{i<2^\mu}$ be a (μ, ω, A) -independent family. Let F be a filter on $E(\mu)$. Denote by $[f_i]$ the equivalence relation defined in Chapter 2, section 4. We call $A^\mu/_F$ the (μ, ω) independent limit power of A generated by $\{f_i / i < 2^\mu\}$; and if $g \in A^\mu/_F$ then there is a finite set of $\{f_i \mid i \in Z\}$ so that $[g] \geq \bigwedge_{i \in Z} [f_i]$. This Z is called the support of g.

Let the above μ be chosen so that $2^\mu \geq |B|$. And let $h : \{f_i / i < 2^\mu\} \longrightarrow \mathrm{dom}\, B$ be any surjective map. Define D to be the filter on μ generated by the finite intersections of sets of the form: $X(\varphi) = \{\alpha < \mu ; A \models \varphi[f_i(\alpha)]_{i<2^\mu}\}$ such that φ is atomic and $B \models \varphi[h(f_i)]_{i<2^\mu}$.

<u>Claim 1</u>: For every atomic φ If $X(\varphi) \in D$ then $B \models \varphi[h(f_i)]_{i<2^\mu}$.

<u>pf</u>: Suppose $X(\varphi) \in D$ then there are atomic formulae $(\psi_k)_{k<n}$ so that

$B \models \psi_k[h(f_i)]_{i<2^\mu}$ and $\bigcap_i^n X(\psi_k) \subseteq X(\varphi)$.

If $A \models \forall \bar{v}[\bigwedge_{k<n} \psi_k \rightarrow \varphi]$ then by i - $B \models \varphi[h(f_i)]_{i<2^\mu}$.

Thus it is enough to show that $A \models \forall \bar{v}[\bigwedge_{k<n} \psi_k \rightarrow \varphi]$. Assume not, then there must be $\bar{a}$ in A so that $A \models \bigwedge_{k<n} \psi_k(\bar{a})$ and $A \models \neg\varphi(\bar{a})$. Since $(f_i)_{i<2^\mu}$ is (μ, ω, A) -independent and a finite number of variables occur in φ and the ψ_k so we can find $\alpha < \mu$ so that $(f_i(\alpha))_{i<2^\mu}$ agrees with $\bar{a}$ but this contradicts the fact that $\bigcap_{k<n} X(\psi_k) \subseteq X(\varphi)$. Thus claim 1 is proved since $\perp$ (falsehood) is atomic then D must be proper. The claim proves that h is an isomorphism from B into $C/_D$ where C is the substructure of A^μ generated by the $(f_i)_{i<2^\mu}$. Now if $C \in A^\mu/_F$ then for some

$Y \subseteq 2^{\mu}$ with $|Y| < \omega$; $[c] \geq \bigwedge_{i \in Y} [f_i]$ since A is complete

It follows that $c \in C$. Hence $A^{\mu}/_F = C$ and $A^{\mu}_D/F \simeq C/_D \simeq B$.

Cor 2.C.1: Let A be a complete $\mathcal{L}$ -structure and let B be any $\mathcal{L}$ -structure so that every universal formula which is true in A is true in B; then B is isomorphic to a global section structure of a Boolean sheaf whose stalks are limit ultrapower of A.

Remark: The above results can be extended to any regular cardinal κ providing we have A κ -complete (i.e. every function of arity less than κ and defined on the domain of A is one of the function of the Language of A). Filters are to be taken κ -complete. The representation theorem of κ -limit reduced products as global section structures of Boolean s-sheaves still holds and $\mathcal{L}$ is replaced by $\mathcal{L}_{\kappa\kappa}$.

2.D. Relation between Truth Value and Forcing Value

Let (S, π) be an s-sheaf of $\mathcal{L}$ -structures over X; $\varphi(v_1, \ldots, v_n)$ be any first order $\mathcal{L}$ -formula, and $f_1, \ldots, f_n$ be any elements of $\Gamma(X,S)$. Let us denote by $S_\varphi(f_1, \ldots, f_n)$ the set $\{x \in X / S_x \models \varphi(f_1(x), \ldots, f_n(x)\}$. We call $S_\varphi(f_1, \ldots, f_n)$ the truth value of the formula $\varphi(v_1, \ldots, v_n)$ at $(f_1, \ldots, f_n)$.

We note that S_φ defines a map from $[\Gamma(X, S)]^n$ into $\mathcal{P}$ (X) (the power set of X).

Here we pause to recall some topological definitions and results:

Def 2.D.1: Let T be a topological space.

i - T is said to be 2nd countable iff it has a countable basis.

ii - T is said to be separable iff it has a countable dense subset (i.e. there exists a countable subset D of T so that $\overline{D}$ = T).

iii - A subset D of T is said to be nowhere dense iff $Int(\overline{D}) = \emptyset$.

iv - A subset D of T is said to be meagre or (of the 1st Baire Category) iff D is the union of a countable family of nowhere dense subsets of X. Otherwise we say that D is of the 2nd Baire category.

The proof of the following facts can be found in Kelly [57].

Theorem 2.D.2: i - If T is a 2nd countable topological space then T is separable.

ii - If U is any open subset of T then U differs from $\mathrm{Int}(T - \mathrm{Int}(T-U))$ by a set of the 1st Baire category.

iii - If U_1 and U_2 are clopen subsets of T; which differs by a set of the 1st Baire category then $U_1 = U_2$

Theorem 2.D.3: Let (S,Π) be an s-sheaf of $\mathcal{L}$-structures over X, assume S is 2nd countable, and let $\varphi(v_1, \cdots, v_n)$ be any 1st order $\mathcal{L}$-formula then the weak forcing value of $\varphi(v_1, \cdots, v_n)$ at $(f_1, \cdots, f_n)$ differs by a set of the 1st Baire category from the truth value of $\varphi(v_1, \cdots, v_n)$ at $(f_1, \cdots, f_n)$.

pf: Let us denote by $--U$ the set $\mathrm{Int}(X-\mathrm{Int}(X-U))$ whenever U is a subset of X. Now we shall prove the theorem by induction on the degree of complexity of $\varphi(v_1, \cdots, v_n)$.

1 - If $\varphi(v_1, \cdots, v_n)$ is atomic, then the result follows from the definition of weak forcing value; which is in this case $--S_\varphi(\bar{f})$; and theorem 2.D.1 (ii).

2 - The case of $\wedge, \vee$ and $\neg$ are easy to prove. In fact in the first two cases the difference of the values will be a union of intersection of meager sets so it must be meager. In the latter case the difference will be included in a meager set so again it is meager.

3 - For the case of $\varphi(\bar{v}) = \exists w\, \gamma(\bar{v}, w)$ one considers $f_0 \in \Gamma(X, S)$ so that the value $S_\gamma(\bar{f}, f_0)$ is dense in $S_\varphi(\bar{f})$. Now since $--[\Vdash_E^* \gamma(\bar{f}, f_0) = --[\Vdash_E^* \varphi(\bar{f})]$ Then we use the induction hypothesis to get: $S_\gamma(\bar{f}, f_0) \supseteq \Vdash_E^* \varphi(\bar{f})$ and the difference between these values is a meager set.

Now assume for the sake of contradiction that the difference between $S_\varphi(\bar{f})$ and $\Vdash_E^* \varphi(\bar{f})$ is of the 2nd Baire category. And choose a countable basis of S; $(V_n)_{n \in \omega}$ say. Then $S = \bigcup_{n \in \omega} V_n$ and $f_0^{-1}(V_n) = U_n$ is open in X furthermore $\bigcup_{n \in \omega} U_n = X$.

Thus $S_\varphi(\bar{f}) = \bigcup_{n \in \omega} [\bigcup_{g_n \in \Gamma(U_n, S)} S_\gamma(\bar{f}/U_n, g_n)]$.

Using the above assumption there must be a $g_n \in \Gamma(U_n, S)$ so that $S_\gamma(\bar{f}/U_n, g_n)$ differs from $\Vdash_E^* \gamma(\bar{f}/_{U_n}, g_n)$ by a set of the 2nd Baire category. Hence $S_\gamma(\bar{f}, f_0)$ and $\Vdash_E^*(\bar{f}, f_0)$ must differ by a set of the 2nd Baire category which contradicts the choice of f_0.

Remark: Looking back one realizes that the assumption "S is 2nd countable" was not used with its full strength. In fact it was enough to assume that: Given $f_1, \dots, f_n$ a finite family of local sections, then there exists a countable family F of local sections so that the set: $\{x \in X / x \in \bigcap_{k=1}^{n} \operatorname{dom} f_k$ $\{$ and $\{g(x) / g \in F\}$ is not an elementary submodel of S_x or it does not contain $\{f_1(x), \dots, f_n(x)\}$ $\}$ is meager.

We shall apply this remark and the above theorem to two particular cases for that we need the following definitions:

Def 2.D.4: Let X be any topological space and A be an $\mathcal{L}$ -structure then we endow A with the discrete topology and consider $S = X \times A$ and π the first projection then (S, π) is called the constant s-sheaf over X. An s-sheaf (S, π) over X is said to be locally constant iff there is a Basis B for the topology on X so that for every $U \in B$, S/U is a constant s-sheaf.

Def 2.D.5: A theory T of $\mathcal{L}$ is called positively model complete iff T is model complete and relative to T every existential formula is equivalent to a positive existential formula.

Cor 2.D.6: i - Assume (S, π) is a locally constant s-sheaf over X. Then for every formula $\varphi(v_1, \ldots, v_n)$ of $\mathcal{L}$ and $f_1, \ldots, f_n \in \Gamma(X, S)$ $\Vdash_E^* \varphi(\bar{f})$ and $S_\varphi(f_1, \ldots, f_n)$ differs by a meager set.

ii - Assume the theory $T = Th(\{S_x / x \in X\})$ is positively model complete, X is Boolean and S is 2nd countable. Then for every formula $\varphi(v_1, \ldots, v_n)$ and every $f_1, \ldots, f_n \in \Gamma(X, S)$ $S_\varphi(f_1, \ldots, f_n) = \Vdash_E^* \varphi(f_1, \ldots, f_n)$.

pf: i - We use the above remark and we show that if (S, π) is locally constant over X then the condition cited in the remark is fulfilled.

Let $\mathcal{B}$ be a basis for the topology on X so that $S/_{\mathcal{U}}$ is constant whenever $\mathcal{U} \in \mathcal{B}$. Let $S/_{\mathcal{U}} = \mathcal{U} \times A_{\mathcal{U}}$.

Let $f_1, \ldots, f_n$ be local section, for every $\mathcal{U} \in \mathcal{B}$ $f_i/_{\mathcal{U}}$ is a continuous map. from $\mathcal{U}$ into $A_{\mathcal{U}}$; for each $\langle a_1, \ldots, a_n \rangle \in A_{\mathcal{U}}^n$ let:

$$\mathcal{U}_{a_1, \ldots, a_n} = \left(\bigcap_{k=1}^{n} f_k^{-1}(a_k)\right) \cap \mathcal{U},$$

and $C_{\mathcal{U}_{a_1, \ldots, a_n}}$ be an elementary substructure of $A_{\mathcal{U}}$ so that $C_{\mathcal{U}_{a_1, \ldots, a_n}}$ is countable; enumerate the elements of C and for every $n \in \omega$ let g_k be the section defined as $g_k(x)$ = the k th element of $C_{\mathcal{U}_{a_1, \ldots, a_n}}$ whenever x is in $\mathcal{U} \cap \mathcal{U}_{a_1, \ldots, a_n}$. It is easy to check that the family of g_k 's verifies the condition of the remark.

ii - Follows from prop 1.11 (ii) and theorem 2.D.3 and part (iii) of theorem 2.D.2.

CHAPTER 4

APPLICATION OF SHEAVES TO MODEL THEORY

This chapter will be concerned with applications of the previous notions studied in the first three chapters. Our aim is twofold:

1 - To give some examples of s-sheaves which help to clarify the definitions

2 - To prove the usefulness of these notions in solving some problems in model theory.

The first part will be concerned with discrete s-sheaves. This will allow us to deduce Los's theorem from the ultrastalk theorem.

The second part will be concerned with s-sheaves of rings. This will provide us with some typical examples of s-sheaves over Boolean spaces. These sheaves will be used later in studying the model companion of some classes of rings.

1 - Discrete s-sheaves and Los's Theorem:

Let X be any set and $(A_x)_{x \in X}$ be any indexed family of $\mathcal{L}$-structures Endow X with the discrete topology and let A be $\bigcup_{x \in X} A_x$ and endow A with the discrete topology; finally let $\pi : A \longrightarrow X$ be defined as $\pi(a) = x$ iff $a \in A_x$. The following is very easy to check:

Lemma 1.1: With the above notation (A, π) is an s-sheaf of $\mathcal{L}$-structures over X; and $\Gamma(X, A) \simeq \prod_{x \in X} A_x$

Such s-sheaf is called the discrete s-sheaf of $(A_x)_{x \in X}$ over X.

Discrete s-sheaves are in reality the most trivial examples of s-sheaves.

In what follows (A, π) will denote the discrete s-sheaf of $(A_x)_{x \in X}$ over X and F will be an ultrafilter on X. The following are trivial remarks on discrete s-sheaves.

(1) - Since X is a discrete space then $\mathcal{O}(X) = \mathrm{Reg}(X) = P(X)$ and so we have that every prime filter in $\mathcal{O}(X)$ is an ultrafilter on X.

(2) - If $\varphi(v_1, \ldots, v_n)$ is an $\mathcal{L}$-formula and $f_1, \ldots, f_n \in \Gamma(X, S)$ then:

$$\Vdash_E^* \varphi(f_1, \text{—} f_n) = \Vdash_E \varphi(f_1, \text{—}, f_n) = S_\varphi(f_1, \text{—}, f_n).$$

Thus using the same notation as in Chapter 1 we deduce from the construction of prime s-sheaves and (1) above that $T_F = \prod_{x \in X} A_x / F$ for every ultrafilter F on X, and the prime stalks of this s-sheaf are exactly the ultrastalks. Now using (2) and the ultrastalk theorem we have:

$T_F \models \varphi(|f_1|_F, \text{—}, |f_n|_F)$ iff $\Vdash_E^* \varphi(f_1, \text{—}, f_n) \in F$

iff $S_\varphi(f_1, \text{—}, f_n) \in F$ iff $\{x \in X / A_x \models \varphi(f_1(x), \text{—}, f_n(x))\} \in F$.

Thus:

Theorem 1.2 (Los): Let $\prod_{x \in X} A_x / F$ be the ultraproduct of the family $(A_x)_{x \in X}$ of $\mathcal{L}$ -structures with respect to F. Then for any formula $\varphi(v_1, \text{—}, v_n)$ of $\mathcal{L}$ and elements $|f_1|_F, \text{——}, |f_n|_F$ of $\prod_{x \in X} A_x / F$

$\prod_{x \in X} A_x / F \models \varphi(|f_1|_F, \text{—}, |f_n|_F)$ iff $\{x \in X / A_x \models \varphi(f_1(x), \text{—}, f_n(x))\} \in F$

The second remark on the ultrastalk theorem is that:

Let us recall that given an s-sheaf over X and $\varphi(v_1, \text{—}, v_n)$ an $\mathcal{L}$ -formula and $f_1, \text{—}, f_n$ elements of $\Gamma(X,S)$; we say that $S_x \Vdash_E \varphi(f_1(x), \text{—}, f_n(x))$ iff $x \in \Vdash_E \varphi(f_1, \text{—}, f_n)$.

We shall say that S_x is generic iff for any formula $\varphi(v_1, \text{—}, v_n)$ and any elements $f_1, \text{—}, f_n \in \Gamma(X,S)$:

$$S_x \models \varphi(f_1(x), \text{——}, f_n(x)) \quad \text{iff} \quad S_x \Vdash_E \varphi(f_1(x), \text{—}, f_n(x))$$

With this definition in mind we deduce (using the ultrastalk theorem) that every ultrastalk is generic.

Thus the ultrastalk construction provides us with a method to construct generic structures. One may remark that if F is a prime filter in $\mathcal{O}(X)$ and $F_1 \supset F$ (where F_1 is a filter in $\mathcal{O}(X)$) then

$T_F \Vdash_E \varphi(|f_1|_F, \text{——}, |f_n|_F)$ implies that $T_{F_1} \Vdash_E \varphi(|f_1|_{F_1}, \text{—}, |f_n|_{F_1})$

Thus one can view the prime stalks and ultrastalks as "extensions" of the stalks of the s-sheaf under consideration which leads to generic stalks.

2 - s-sheaves of Rings

In what follows Rings are assumed to be commutative and to have an identity element 1.

Let R be a ring. We denote by B(R) the set $\{e \in R / e^2 = e\}$.

For e,f $\in$ B(R) define $e \wedge f = e.f$ and $e \vee f = e + f - ef$ and $e^* = 1 - e$ then $\langle B(R), 0, 1, \wedge, \vee, * \rangle$ is a Bolean algebra.

R is said to be indecomposable iff B (R) = $\{0,1\}$

Let J be an ideal in B(R) define $\overline{J} = \{r.e / e \in J \text{ and } r \in R\}$ then $\overline{J}$ is an ideal in R.

Remark: When we talk about ideals in B(R) we mean subsets J of B(R) satisfying: i - If e,f $\in J$ then $e \wedge f \in J$,

ii - If $e \in J$ and e.f = f then f $\in J$,

iii - $0 \in J$.

On the other hand, an ideal in R means a subset I of R so that

i - I is a subgroup of R and

ii - If $r \in R$ and $a \in I$ then $r \cdot a \in I$.

Now let X(R) be the set of maximal ideals of B(R). Endow X(R) with the Hull-Kernel topology (i.e. If $A \subset X(R)$ then $\overline{A} = \{M \in X(R) / M \supset \bigcap_{J \in A} J\}$))

This makes X(R) a Boolean space.

For every M $\in$ X(R) let K_M be $R/\overline{M}$ and for every $r \in R$ let $\hat{r} : X(R) \to \dot{\bigcup}_{M \in X(R)} K_M$ be defined as $\hat{r}(M) = |r|_{\overline{M}}$ the equivalence class of r w.r.t. $\overline{M}$. Endow $K = \dot{\bigcup}_{M \in X(R)} K_M$ with the finest topology making all the $\hat{r}$'s continuous.

Theorem 2.1: Let $\pi : K \to X(R)$ be defined as $\pi(t) = M$ iff $t \in K_M$; where K, K_M, and X(R) are as above, then:

i - (K, π) is an s-sheaf over X(R),

ii - $\Gamma(X(R), K) \simeq R$,

iii - $B(\Gamma(X(R), K) = \{\hat{e} / e \in B(R)\}$.

pf: By the representation theorem of algebraic structure we see that i,ii, and iii will be proved as soon as we establish the following: The set $\{\overline{M} / M \in X(R)\}$ generates a permutable and distributive sublattice of the lattice L(R) (the lattice of congruences on R).
So let M, N be elements of X(R) consider $\overline{M}$ and $\overline{N}$ and let $(a,b) \in \overline{M} . \overline{N}$
That is to say that there exists $c \in R$ so that
$a - c \in \overline{M}$ and $c - b \in \overline{N}$ or equivalently
$a - c = r_1 e_1$ and $c - b = r_2 e_2$ with $r_1, r_2 \in R$ and $e_1 \in M$ and $e_2 \in N$
Let $d = c + r_1 e_1 + r_2 e_2$ Thus
$a - d = a - c - r_1 e_1 - r_2 e_2 = -r_2 e_2 \in \overline{N}$ and
$d - b = c + r_1 e_1 + r_2 e_2 - c - r_2 e_2 = r_1 e_1 \in \overline{M}$ Thus
$(a,b) \in \overline{N} . \overline{M}$ and hence $\overline{N} . \overline{M} \supseteq \overline{M} . \overline{N}$, a similar argument shows that $\overline{M} . \overline{N} \supseteq \overline{N} . \overline{M}$ thus $\overline{M} . \overline{N} = \overline{N} . \overline{M}$ This shows that $\{\overline{M} / M \in X(R)\}$ generates a permutable sublattice of L(R).
For the distributivity it is enough to show that:
for $M, N, P \in X(R)$: $\overline{M \cup (N \cap P)} = \overline{(M \cup N) \cap (M \cup P)}$
So it is enough to show that $M \cup (N \cap P) = (M \cup N) \cap (M \cup P)$
but that follows from the fact that B(R) is a Boolean algebra.
We recall that a ring R is said to be (Von Neuman) regular iff for every element x of R there is an element x' in R such that $x x' x = x$
Thus if R is regular and $x x' x = x$ then $e = x' x$ is an idempotent.
For $e^2 = x' x x x' = x' x = e$ thus $x' x \in B(R)$.

We also remark that If R is regular and I is an ideal in R then $I \cap B(R) = J$ is an ideal in B(R) and $\overline{J} = I$. So it follows that If R is regular and M is a maximal ideal in B(R) then $\overline{M}$ is a maximal ideal in R and hence $K_M = R/\overline{M}$ is a field for every $M \in X(R)$.

Thus we have proved:

<u>prop 2.2</u>: Let R be a regular ring then R is isomorphic to a global section structure of an s-sheaf of fields over a Boolean space.

<u>Conversely</u>: Let (K, π) be an s-sheaf of fields over a Boolean space X and we consider $\Gamma(X, K)$. Let φ be the sentence $\forall v \; \exists w \; [v.w.v = v]$ Clearly φ holds in every K_x by theorem 2.C.1 of Chapter 3. φ holds in $\Gamma(X, K)$. Thus $\Gamma(X, K)$ is a regular ring. Thus we have:

<u>Theorem 2.3</u>: A ring R is regular iff R is isomorphic to a global section structure of an s-sheaf of fields over a Boolean space.

As a particular case of this theorem: we may consider the constants s-sheaf of fields over a Boolean space X; we recall the construction of this s-sheaf. Let K be a field. Endow K with the discrete topology and let S be $X \times K$. Thus $S = \bigcup_{x \in X} \{x\} \times K$ and π be the first projection from S onto X. Thus (S, π) is an s-sheaf over X and $\Gamma(X, S) = \mathcal{C}(X, K)$ the set of continuous maps from X into K. Clearly K is naturally embedded in $\mathcal{C}(X, K)$. Furthermore $\mathcal{C}(X, K)$ is a regular ring.

In what follows we shall discuss the case of existentially closed regular rings:

For the moment we shall pause to recall some model theoretic notions which we shall use later on in our discussion. These notions and results are due to the BERS group and can be found in Hershfield and Wheeler [75]: Let $\mathcal{L}$ be a first order Language and T be a fixed theory in $\mathcal{L}$.

Def 2.4: i - T is said to be model complete iff whenever A and B are models of T and $A \subset B$ then $A \prec B$

ii - A model A of T is said to be existentially closed (e.c.) iff whenever $B \models T$ and $A \subset B$ then every existential sentence defined in A and true in B is true in A.

iii - A formula $\varphi(\bar{v})$ is called primitive iff $\varphi(\bar{v})$ is of the form $\exists \bar{w} \bigwedge_{i \leq n} \psi_i(\bar{v},\bar{w}) \wedge \bigwedge_{j \leq n} \neg \chi_j(\bar{v},\bar{w})$ where the ψ's and χ_j 's are atomic.

Facts 2.5: i - T is model complete iff every model of T is e.c.

ii - T is model complete iff whenever $A \subset B$ and $A, B \models T$ and $\varphi(\bar{v})$ is a primitive formula and $\bar{a}$ elements of A. Then $A \models \varphi(\bar{a})$ iff $B \models \varphi(\bar{a})$

iii - T is model complete iff every existential formula is equivalent to a universal formula relative to T.

iv - If the negation of every primitive formula is equivalent to an existential formula relative to T then T is model complete.

We recall from Chapter 3 that a theory T is said to be positively model complete iff it is model complete and relative to T every existential formula is equivalent to a positive existential formula.

As an example to that we consider the theory of algebraically closed fields which is model-complete. Now every existential formula in the Language of field is of the form $\exists \bar{w}\ p(\bar{v},\bar{w}) = q(\bar{v},\bar{w}) \wedge p_1(\bar{v},\bar{w}) \neq q_1(\bar{v},\bar{w})$.

We just consider the negative part which we can transform to $\exists \bar{w}\ p_2(\bar{v},\bar{w}) \neq 0$ which is equivalent to say $\exists u \exists \bar{w}\ p_2(\bar{v},\bar{w}) \cdot u = 1$.

In what follows let $\mathcal{L}$ be the language of ring theory i.e. $(\mathcal{L} = \langle 0,1,+,\cdot,= \rangle)$ and let T be the theory of commutative regular rings.

Let R be an e.c. model of T. By theorem 2.3 R $\simeq \Gamma(X(R), K)$.
Since R is an e.c. model of T it is easy to see that every sentence of the form $\forall a_1, \ldots, a_n \exists x [a_0 + a_1 x + \cdots + a_n x^n = 0]$ holds in R whenever $n \in \omega$ and $n > 0$. Thus $\Gamma(X(R), K)$ verifies all those sentences by considering the equivalence classes of $a_1, \cdots, a_n$ with respect to $\overline{M}$ for any $M \in X(R)$. When the $a_1, \cdots, a_n$ ranges over R their equivalence classes will range over K_M.
Hence $K_M \models \forall a_1, \ldots, a_n \exists x [a_0 + a_1 x + \cdots + a_n x^n = 0]$ This means that every K_M is an algebraically closed field. Thus we have proved:

prop 2.6: If R is an e.c. model of T then R is isomorphic to the global section structure of an s-sheaf of a.c.fields over a Boolean space. Now we prove the following proposition:

prop 2.7: Let R be an e.c. regular ring then B(R) is an atomless Boolean algebra.

pf: We first note the following well known fact:

(*) Every Boolean algebra can be embedded in an atomless Boolean algebra so let R be as above and let B be an atomless Boolean algebra in which B(R) is embedded. Let X' be the Stone space of B' and $h : X' \longrightarrow X(R)$ to be the continuous surjection induced by the embedding.
Let (K, π) be an s-sheaf of fields over X(R) such that $\Gamma(X(R), K) \simeq R$
For every $x' \in X'$ choose $F_{x'}$ to be a copy of $K_{h(x')}$. Let $F = \bigcup_{x' \in X'} F_{x'}$ with $\pi' : F \to X'$ so that $\pi'(f) = x'$ iff $f \in F_{x'}$, put $S = \Gamma(X', F)$.
S is a regular ring and by its construction one can easily show that R is embeddable in S.
Consider the formula $\varphi(v) = \exists w [v^2 = v \longrightarrow (w^2 = w \wedge wv = w \wedge w \neq 0 \wedge w \neq 0)]$
For any $a \in R$ consider the sentence $\varphi(a)$.

This is an existential sentence defined in R, since B' is atomless $S \models \varphi(a)$ but R is e.c. so $R \models \varphi(a)$ Thus B(R) is atomless.

As a corollary of that we note:

Cor.2.8: If R is an e.c. regular ring then X(R) has no isolated points.

pf: In fact X(R) has isolated points iff B(R) has atoms.

In what follows we aim to prove that if R is a regular ring such that

i - X(R) has no isolated points and

ii - Every K_M where $M \in X(R)$ is an a.c. field,

then R is an e.c. regular ring.

In the process of doing so we shall give a counter-example to a claim of Macintyre [73].

So from now on let (K, π) be an s-sheaf of a.c. fields over a Boolean space X with no isolated points.

To show that $\Gamma(X, K)$ is e.c. it is enough to show that every primitive sentence defined in $\Gamma(X, K)$ and true in an extension of it then it holds in $\Gamma(X, K)$. We shall write R for $\Gamma(X, K)$.

Let $\varphi(\bar{v}) = \exists \bar{w}\, \psi(\bar{v}, \bar{w}) \wedge \bigwedge_{j \leq m} \neg \delta_j(\bar{v}, \bar{w})$ where $\psi(\bar{v}, \bar{w})$ is a conjunction of atomic formula and each δ_j is an atomic. Let $f_1, \ldots, f_n$ be elements of $\Gamma(X, K)$.

Lemma 2.9: Let $x \in X$ so that $K_x \models \varphi_j(\bar{f}(x))$ where $\varphi_j(\bar{v}) = \exists \bar{w}\, \psi(\bar{v}, \bar{w}) \wedge \neg \delta_j(\bar{v}, \bar{w})$. Then there exists a clopen subset N of X such that: $x \in N$ and for every $y \in N$ $K_y \models \varphi_j(\bar{f}(y))$.

pf: Since atomic formulae in the language of rings are of the form $u + v = w$, $u \cdot v = w$ or $u = v$, then the problem is reduced to the case $u \neq 0$. Thus it is enough to show that if $f \in \Gamma(X, K)$ then the set $\{x \in X;\ f(x) \neq 0\}$ is clopen since R is regular then there is $g \in R$ so that $fgf = f$.

Let $T_f = \{x \in X;\ f(x) \neq 0\}$, then

$T_f \supseteq T_{fg} \supseteq T_{fgf} = T_f$ Hence $T_f = T_{fg}$ but $fg \in B(R)$

as we noted earlier.

Now we note that if $h \in B(R)$ then either $h(x) = 0$ or $h(x) = 1$

for all $x \in X$. So T_{fg} must be a clopen.

Lemma 2.10: With the same notation as above:

$R \models \varphi(\bar{f})$ iff $\forall j = 1, \dots, n$ there is an x_j so that $K_{x_j} \models \varphi_j(\overline{f(x_j)})$

and for all $x \in X$ $K_x \models \exists \bar{w}\, \psi(\bar{f}(x), \bar{w})$.

pf: Assume $R \models \varphi(\bar{f})$ then there are $\bar{g} \in R$ so that:

$R \models \psi(\bar{f}, \bar{g}) \wedge \bigwedge_{j \le m} \neg \delta_j(\bar{f}, \bar{g})$. Hence for each $j \le m$

$R \models \psi(\bar{f}, \bar{g}) \wedge \neg \delta_j(\bar{f}, \bar{g})$ i.e. (1) $R \models \psi(\bar{f}, \bar{g})$ and for all

$j \le m$ (2) $R \models \neg \delta_j(\bar{f}, \bar{g})$.

Now (1) implies that $K_x \models \psi(\bar{f}(x), \bar{g}(x))$ (3) and that is for every

$x \in X$ so $K_x \models \exists \bar{w}\, \psi(\bar{f}(x), \bar{w})$ for all $x \in X$.

(2) implies that there exist $x_j \in X$ so that $K_{x_j} \not\models \delta_j(\bar{f}(x_j), \bar{g}(x_j))$

i.e. $K_{x_j} \models \neg \delta_j(\bar{f}(x_j), \bar{g}(x_j))$ (4) that holds for every $j \le m$.

using (3) and (4) together we get that for every $j \le m$ there is $x_j \in X$

so that $K_{x_j} \models \varphi_j(\bar{f}(x_j))$.

Conversely Assume $\forall j = 1, \dots, m$. there is an x_j so that $K_{x_j} \models \varphi_j(\bar{f}(x_j))$

and $K_x \models \exists \bar{w}\, \psi(\bar{f}(x), \bar{w})$ for all $x \in X$.

First we show that the above x_j 's can be taken to verify:

$x_j \neq x_i$ if $j \neq i$.

Let $H = \{x_\ell\ ;\ \ell \le m\}$ so that $K_{x_\ell} \models \varphi_\ell(\bar{f}(x_\ell))$ and H is of maximal

cardinality.

If $x_\ell = x_r$ for $\ell \neq r$ then using Lemma 2.9 we choose a clopen subset

N of X so that $x_\ell \in N$ and for all y in N $K_y \models \varphi_\ell(\bar{f}(y_\ell))$

Since X has no isolated points then N contains an element $y \neq x_\ell$ and

$K_y \models \varphi_\ell(\bar{f}(y))$. But by the maximality of H we must have $N \cap H = \{x_\ell\}$ which is a contradiction to the existence of the above y.

Now we prove the converse with the assumption that $x_i \neq x_j$ whenever $i \neq j$ and $i, j \leq m$.

Since $K_x \models \exists \bar{w}\, \psi(\bar{f}(x), \bar{w})$ one chooses elements $\bar{a}_x$ in K_x so that $K_x \models \psi(\bar{f}(x), \bar{a}_x)$. We note that the elements $\bar{a}_x$ may be chosen so that for all $j \leq m$ $K_{x_j} \models \psi(\bar{f}(x_j), \bar{a}_{x_j}) \wedge \neg \delta_j(\bar{f}(x_j), \bar{a}_{x_j})$.

Now for every $x \in X$ there exist a neighbourhood N_x of x and elements $\bar{g}_x \in \Gamma(N_x, K)$ so that $\bar{g}_x(y) = \bar{a}_y$ for all $y \in N_x$.

Thus for all $y \in N_x$ we have:

$K_y \models \psi(\bar{f}(y), \bar{g}_x(y))$ and furthermore if $y = x_j$ for some $j \leq m$ then $K_y \models \neg \delta_j(\bar{f}(x_j), \bar{g}_x(y))$.

Clearly $X = \bigcup_{x \in X} N_x$ so there are a finite number of clopen subsets $N_1, \ldots, N_t$ so that $N_i \subseteq N_x$ for some x and $N_i \cap N_j = \emptyset$ iff $i \neq j$ and $X = \bigcup_{i=1}^{t} N_i$.

Thus we have for every $i = 1, \cdots, t$ there are $\bar{g}_{x_i}$ so that for $y \in N_i$ $\bar{g}_{x_i}(y) = \bar{a}_y$ and $\bar{g}_{x_i} \in \Gamma(N_i, K)$.

Let $\bar{g}$ be the sequence of elements $g^s = \bigcup_{i=1}^{t} g^s_{x_i}$ The g^s's are elements of $\Gamma(X, K)$ and they verify:

(1) $K_x \models \psi(\bar{f}(x), \bar{g}(x))$ for all $x \in X$,

(2) $K_{x_j} \models \psi(\bar{f}(x_j), \bar{g}(x_j)) \wedge \neg \delta_j(\bar{f}(x_j), \bar{g}(x_j))$ for all $j \leq m$.

(1) implies that $\Gamma(X, K) \models \psi(\bar{f}, \bar{g})$ because ψ is a conjunction of atomics.

(2) implies that $\Gamma(X, K) \models \bigwedge_{j \leq m} \neg \delta_j(\bar{f}, \bar{g})$ because the δ_j's are atomics so $\Gamma(X, K) \models \psi(\bar{f}, \bar{g}) \wedge \bigwedge_{j \leq m} \neg \delta_j(\bar{f}, \bar{g})$ i.e.

$$\Gamma(X, K) \models \exists \bar{w}\, \psi(\bar{f}, \bar{w}) \wedge \bigwedge_{j \leq m} \neg \delta_j(\bar{f}, \bar{g}).$$

We note here that the above Lemma says that:

$\Gamma(X,K) \models \varphi(\bar{f})$ iff $S_{\varphi_j}(\bar{f}) \neq \emptyset$ for all $j \leq m$ and

$$S_{\exists \bar{w} \psi(\bar{f},\bar{w})} = X.$$

In his paper Model-completeness for sheaves of structures [73] Macintyre proves the following:

Theorem 2.11 (Macintyre): Let (S,π) be an s-sheaf of $\mathcal{L}$-structures over X. Assume that:

(i) X is Boolean with no isolated points,

(ii) $Th(\{S_x / x \in X\})$ is complete and positively model complete,

(iii) $\mathcal{L}$ includes the Language of rings $\langle +, \cdot, 0, 1 \rangle$,

(iv) Every S_x is a ring with 1 and has 0 and 1 as the only idempotents.

Then $T = Th(\Gamma(X,S))$ is model complete.

In his proof Macintyre claims that if $\varphi(\bar{v})$ is primitive of the form used in the previous Lemmas then

$\Gamma(X,S) \models \varphi(\bar{f})$ iff $S_{\varphi_j}(\bar{f}) \neq \emptyset$ for all $j \leq m$ and

$$X = \bigcup_{j \leq m} S_{\varphi_j}(\bar{f}).$$

Here we give a counter-example to this claim:

Let X be a Boolean space with no isolated points. Let V be a clopen subset of X. And consider the constant sheaf over X whose stalks are isomorphic to $\mathbb{C}$ (the field of complex number). As we mentioned before $\Gamma(X,S) \simeq \mathcal{C}(X,\mathbb{C})$ Clearly this s-sheaf verifies all the conditions i, ii,..., iv. above:

Let $\varphi(v) = \exists w \; v \cdot w \neq 0$, this is a primitive formula.

Let $f_1 : X \to \mathbb{C}$ be defined as $f_1(x) = 1$ if $x \in V$ and $f_1(x) = 0$ if $x \notin V$,

and $f_2 : X \to \mathbb{C}$ be defined as $f_2(x) = 1$ if $x \in V$ and $f_2(x) = -1$ if $x \notin V$.

$f = f_1 - f_2 \in \mathcal{C}(X, \mathbb{C})$ and $f \neq 0$ since $\forall x \notin V$ $f(x) = 1$,
and $f(x) = 0$ $\forall x \in V$.
$\Gamma(X, S) \models \exists w \; f \cdot w \neq 0$ for choose w to be the constant map $g : X \to S$ so that $g(x) = 1$,
but for all $x \in V$ $S_x \not\models \exists w \, f(x) \cdot w \neq 0$. Thus if Macintyre's claim holds we must have $X = S_{\varphi}(\bar{f})$ but as we have shown above, this is not true.

Lemma 2.12: Let (K, π) be an s-sheaf over X which has the following properties: i - X has no isolated points and ii - every K_x is a.c. field. Let $\varphi(\bar{v})$ be $\exists \bar{w} \, \psi(\bar{v}, \bar{w}) \wedge \bigwedge_{j \leq m} \neg \delta_j(\bar{v}, \bar{w})$ and $\varphi_j(\bar{v})$ be $\exists \bar{w} \, \psi(\bar{v}, \bar{w}) \wedge \neg \delta_j(\bar{v}, \bar{w})$ then $\Gamma(X, K) \models \varphi(\bar{f})$ iff $\Gamma(X, K) \models \varphi_j(\bar{f})$ for all $j \leq m$ and where $\bar{f}$ are elements of $\Gamma(X, K)$.

pf: It is immediate from Lemma 2.9 and Lemma 2.10.

Let $R \simeq \Gamma(X, K)$ and $S \simeq \Gamma(Y, F)$ be two regular rings with $R \subset S$ and X is Boolean and has no isolated points and K_x is a.c.field for all $x \in X$. For every $r \in R$ let $\bar{r}$ denote the image of r by the isomorphism from R into $\Gamma(X, K)$, similarly let $\hat{s}$ be the image of the element s of S in $\Gamma(Y, F)$.

Lemma 2.12: Assume $\varphi(\bar{v})$ be as in Lemma 2.11 and let $r_1, \ldots, r_n$ be elements of R so that $S \models \varphi(r_1, \ldots, r_n)$, then $R \models \varphi(r_1, \ldots, r_n)$.

pf: By Lemma 2.12 it is enough to show this for $\varphi_j(\bar{v})$.
To say $S \models \varphi_j(r_1, \ldots, r_n)$ is equivalent to say $\Gamma(Y, F) \models \varphi_j(\hat{r}_1, \ldots, \hat{r}_n)$.
Now we have that $S \supset R$ so there is a continuous surjection $f : Y \longrightarrow X$ and a map $g : Y \times K \longrightarrow F$ so that

a) $g(y, -) : K_{f(y)} \longrightarrow F_y$ which is injective because $K_{f(y)}$ is a field.

b) $g(y, \bar{r}(f(y))) = \hat{r}(y)$ for each $y \in Y$ and $r \in R$.

By a) and b) we get an isomorphism h_y from $\hat{K}_y$ and $K_{f(y)}$ where $\hat{K}_y$ is a subfield of F_y so that $h_y(\hat{r}(y)) = \bar{r}(f(y))$ for every $r \in R$.

Thus K_y is embedded in F_y.

To say that $\Gamma(X,F) \models \varphi_j(\hat{r}_1,\dots,\hat{r}_n)$ this means there is an element $y \in Y$ so that $F_y \models \varphi_j(\hat{r}_1(y),\dots,\hat{r}_n(y))$. Since $K_{f(y)}$ is a.c. then $K_{f(y)} \models \varphi_j(\bar{r}_1(f(y)),\dots,\bar{r}_n(f(y)))$ thus $\Gamma(X,K) \models \varphi_j(\bar{r}_1,\dots,\bar{r}_n)$ so $S \models \varphi_j(r_1,\dots,r_n)$.

<u>Theorem 2.14</u>: Let R be a regular ring so that $R \simeq \Gamma(X,K)$ where X is Boolean space with no isolated points and every K_x is an a.c.field. Then R is an e.c.regular ring.

<u>pf</u>: Let $\varphi(\bar{v}) = \exists \bar{w}\ \psi(\bar{v},\bar{w}) \wedge \bigwedge_{j \le m} \neg \delta_j(\bar{v},\bar{w})$ be a primitive formula $\bar{f}$ elements of R and assume S an extension of R (S regular ring) so that $S \models \varphi(\bar{f})$. Then $S \models \varphi_j(\bar{f})$ for all $j \le m$ (Lemma 2.11). By Lemma 2.12 $R \models \varphi_j(\bar{f})$ for all $j \le m$ and again by Lemma 2.11 we get: $R \models \varphi(\bar{f})$.

Theorem 2.14, cor 2.8 and prop 2.6 allow us to axiomatize the theory of e.c. regular rings.

<u>prop 2.15</u>: Let T_n be the set of the following axioms in $\mathcal{L} = \langle 0, 1, +, \cdot, = \rangle$:

(1) $\forall x\, \forall y\, \forall z\, [x+(y+z) = (x+y)+z \wedge x\cdot(y\cdot z) = (x\cdot y)\cdot z \wedge x\cdot y = y\cdot x \wedge x+y = y+x]$,

(2) $\forall x\, \exists y\, [x+0 = 0+x \wedge x+y = 0]$,

(3) $\forall x\, \forall y\, \forall z\, [x\cdot(y+z) = xy + xz \wedge x\cdot 1 = 1\cdot x = x]$,

(4) $\forall x\, [\, nx = \underbrace{x+\cdots+x}_{n \text{ times}} = 0]$,

(5) $\forall x\, \exists y\, [xyx = x]$,

(6) $\{\forall a_0, \dots, \forall a_n\, \exists x\, [a_0 + a_1 x + \cdots + a_n x^n = 0] \,/\, n > 0\}$,

(7) $\forall x\, \exists y\, [x^2 = x \longrightarrow y^2 = y \wedge yx = y \wedge x \ne y \wedge y \ne 0]$.

Then $R \models T_n$ iff R is an e.c. regular ring of characteristic n.

pf: If R is an e.c. regular ring of characteristic n, then clearly (1), (2), (3), (4) and (5) are satisfied in R.

(6) is satisfied in R by prop 2.7 and (7) is satisfied because R is e.c.

Now assume $R \models T_n$ then by (1), (2), (3) (4) and (5) we have that R is a regular ring of characteristic n. Since R verifies (6) then B(R) is atomless so X(R) has no isolated points. Now $R \models (7)$, this implies that K_M for every $M \in X(R)$ is a.c.field. By theorem 2.14 R must be e.c.

Cor 2.16: For every $n \in w$ T_n is model complete.

pf: Every model of T_n is e.c. so by facts 2.5 T_n is model complete.

Def 2.17: Let T by any theory we say that a model A of T is prime iff A is embeddable in every model of T.

As an example we note that for every prime number P and 0 $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Q}$ are the prime models for the theory of fields.

The following is a well known theorem due to Robinson.

Theorem 2.18: Let T be a model complete theory, then if T has a prime model then T is complete.

pf: It is enough to show that if B,C are models of T then $B \equiv C$. For say A is a prime model of T then $A \subset B$ and $A \subset C$ but T is model complete so $A \prec B$ and $A \prec C$. Thus $A \equiv B$ and $A \equiv C$ so $B \equiv C$.

Theorem 2.19: Let T_n be the theory of prop 2.15. Assume that n is either prime or n = 0, then T_n is complete.

pf: By the above if n is prime or n = 0 then there is a prime field L. Let F be the algebraic closure of L and X be the Cantor space. We claim that:

(*) $\mathcal{C}(X,F)$ is a prime model of T_n.

Clearly $\mathcal{C}(X,F) \models T_n$. Let $R \models T_n$ say $R \models \Gamma(X(R), K)$.

Now X(R) has no isolated points so B(R) must be atomless. Let B' be an atomless countable subalgebra of B(R) then $X = S(B')$ the Stone space of B' So there is a continuous surjection from X(R) onto X. This induces an embedding from $\mathcal{C}(X,F)$ into $\mathcal{C}(X(R),F)$.

So it is enough to embed $\mathcal{C}(X(R),F)$ into R.

Let $G = \{ r(x) / r \in R \;\; f(r) = 0$ for some polynomial f over L and $x \in X(R) \}$.

G defines a sub-s-sheaf of K over X(R). Furthermore by its definition there exists an isomorphism between $\Gamma(X(R),G)$ and $\mathcal{C}(X(R),F)$.

Since if $g : X(R) \to G$ then $g(x) = r(x)$ for some $r \in R$ so that r is a root of some polynomial $P(v)$ over L. Since F is the algebraic closure of L then $P(v)$ must have a root in F ; α say, take the map $\sigma_g : X(R) \to F$ $\sigma_g(x) = \alpha$ we have $\sigma_g \in \mathcal{C}(X(R),F)$ and the map $g \to \sigma_g$ is an isomorphism since $G_x \simeq F \; \forall x$. But now $\Gamma(X(R),G)$ is embedded in $\Gamma(X(R),K)$ by construction so $\mathcal{C}(X,F)$ is embedded in R. Thus proving (*). Now this together with theorem 2.17 and cor. 2.15 gives the theorem.

We pause now to recall some more model theoretical notions.

Let T, T_1 be two first order theories in $\mathcal{L}$ so that $T \subset T_1$

Def 2.20: i - T_1 is said to be model consistent relative to T iff every model of T is embeddable in a model of T_1.

ii - T_1 is said to be model complete relative to T iff whenever $A \models T$,

Then $T_1 \cup D(A)$ is complete in $\mathcal{L}(A)$ where the D(A) is the set of all atomic and negated atomic sentences of the Language $\mathcal{L}$ augmented by a set of constants naming the elements of A true in A.

iii - T_1 is said to be model companion of T iff T_1 is model consistent relative to T and T_1 is model complete.

iv - T_1 is said to be model completion of T iff T_1 is model consistent and model complete relative to T.

v - T is said to have the amalgamation property(A.P.) iff whenever $A, B_1, B_2 \models T$, $f_i : A \to B_i$ are embeddings then there is $C \models T$ g_1, g_2 embeddings $g_i : B_i \to C$ so that

$$\begin{array}{ccc} A & \xrightarrow{f_1} & B_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ B_2 & \xrightarrow[g_2]{} & C \end{array}$$

is commutative

Facts 2.21: Let $T \subset T_1$ be two first order theories then:

i - If T_1 is model completion of T then (a) T_1 is model companion of T

(b) T_1 is unique.

ii - A theory T has at most one model companion.

iii - If T_1 is model companion of T then T_1 is model completion of T iff T has the A.P.

iv - If T_1 is model companion of T then every model of T_1 is an e.c. model of T

v - If T is axiomatized by $\forall\exists$ -sentences then every model of T is embeddable in an e.c. model of T.

Now we return to our discussion.

Let R be a commutative ring with no nilpotent elements except 0.

i.e. $R \models \forall x[x^2 = 0 \to x = 0]$. Then R is a semi-simple and so it is embedded in $\prod_M R/M$ where M ranges over the set of Maximal ideals of R.

It is a well known fact that every field is embedded in an a.c.field so for every M let F_M be an a.c. field so that $R/M \subset F_M$.

Let C be the Cantor space. Thus we have $F_M \subset \mathcal{C}(C, F_M)$ and

$R \subset \prod_M \mathcal{C}(C, FM) = R_1$ but $\mathcal{C}(C, F_M)$ is regular thus R_1 is regular but the theory of regular rings is inductive so R_1 is embeddable in an e.c. regular ring R_2 say, so $R \subset R_2$. Thus we have proved

Theorem 2.22: Let T'_n (resp. T''_n) be the theory of commutative rings with no non-zero nilpotent elements (resp. theory of regular rings) of characteristic n.

Then: i - T_n (The theory of prop 2.14)is model consistent with both T'_n and T''_n.

ii - T_n is the model companion of T'_n and T''_n

Lemma 2.23: T''_n has the A.P.

pf: Let $R, S_1, S_2 \models T''_n$ and $f_i : R \to S_i$, $i=1,2$ be embeddings, consider $S_1 \otimes_R S_2$ the tensor product of S_1 and S_2 relative to R. This is a module over R. Moreover it is flat (since every module over a regular ring is flat i.e. it satisfies the following:whenever $a_1, \cdots, a_n$ are linearly independent in R $f_1, \cdots, f_n$ are elements of the module so that $\sum f_i a_i = 0$ then $f_i = 0 \; \forall i=1,\dots,n$)

So the canonical maps $h_i : S_i \to S_1 \otimes_R S_2$ are embeddings. Let $N = \{ t \in S_1 \otimes_R S_2 \,/\, t^n = 0$ for some $n > 0 \}$ then $S = S_1 \otimes_R S_2 / N$ is a ring with no nilpotent elements so it can be embedded in a regular ring S' say.

Thus the h_i's induce embeddings g_i from S_i into S which in turn induce embeddings t_i from S_i into S' . Now one verifies easily that $t_1 \circ f_1 = t_2 \circ f_2$.

prop 2.24: T_n is the model completion of T''_n.

The second application is similar to the one above. In what follows we shall discuss the case of f-rings with no non-zero nilpotent elements. We shall work in the Language of rings augmented by a binary relation $\leq$ which will be interpreted as an order relation. We denote by $a \vee b$ (resp. $a \wedge b$) the join (resp. the meet) of a and b. If they exist. Again all rings are assumed to be commutative and have an identity element 1.

Def 2.25: 1 - An $\mathcal{L}$-structure A is said to be an ℓ-ring iff

i - $\langle A, \cdot, +, 0, 1\rangle$ is a ring,

ii - $\langle A, \leq \rangle$ is a lattice,

iii - the sentence $\forall x \forall y \forall a\, [[x \geq y \rightarrow a+x \geq a+y] \wedge [x \geq 0 \wedge y \geq 0 \rightarrow xy \geq 0]]$ holds in A.

2 - An $\mathcal{L}$-structure A is called f-ring iff

i - A is an ℓ-ring,

ii - the sentence $\forall a \forall b \forall c\, [a \wedge b = 0 \wedge c \geq 0 \rightarrow ac \wedge b = 0]$.

The following proposition will be useful to apply the representation theorem of Chapter 2.

prop 2.26: i - Every ℓ-ring is a distributive lattice

ii - An ℓ-ring with no non-zero nilpotent elements is an f-ring iff it satisfies: $\forall a \forall b\, [a \wedge b = 0 \rightarrow ab = 0]$.

pf: i - It is enough to show that

$*\ A \models \forall x \forall y \forall a\, [a \wedge x = a \wedge y \wedge a \vee x = a \vee y \rightarrow x = y]$.

To do that we first claim

(1) $A \models \forall a \forall b \forall c\, [a + (b \vee c) = (a+b) \vee (a+c)]$.

For let a,b,c be elements of A since $b \vee c \geq b$ then $(b \vee c) + a \geq b + a$, and similarly $(b \vee c) + a \geq a + c$ so $a + (b \vee c) \geq (b+a) \vee (c+a)$.

Let $x \geq a + b$ and $x \geq a + c$ then $x - a \geq c$ and $x - a \geq b$ so $x - a \geq b \vee c$ and hence $x \geq a + (b \vee c)$. Thus $(b \vee c) + a = (b+a) \vee (c+a)$.

Similarly one proves

(2) $A \models \forall a \forall b \forall c\, [a + (b \wedge c) = (a+b) \wedge (a+c)]$.

Now if $a \in A$ $a \geq 0$ then $0 = a - a \geq 0 - a = -a$ Thus

(3) $A \models \forall a\, [a \geq 0 \rightarrow -a \leq 0]$ so using (3), (2) and (1) it follows that

(4) $A \models \forall a \forall b\, [a \wedge b = -(-a \vee -b)]$ But this implies that

(5) $A \models \forall a \forall x \forall y\, [a - (x \vee y) + b = a + (-x \wedge -y) + b = [(a-x) \wedge (a-y)] + b = (a - x + b) \wedge (a - y + b)]$.

So (6) $A \models \forall a \, \forall b \, [a - (a \vee b) + b = a \wedge b]$.

Now say $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ then by (6):

$x = (a \wedge x) - a + (a \vee x) = (a \wedge y) - a + (a \vee y) = y$. Thus proving *.

ii - Say A is an f-ring we have if $a \wedge b = 0$ then $ab \wedge b = 0$ and so

$ab \wedge ab = 0 \Rightarrow ab = 0$.

Conversely: if $a \wedge b = 0 \Rightarrow ab = 0$. Let $c \geq 0$ and $a \wedge b = 0$, thus $ab = 0$, and so $0 \leq (ac \wedge b)^2 \leq acb = 0$ but A has no non-zero nilpotent elements so $ac \wedge b = 0$.

<u>Cor 2.27</u>: Let A be an ℓ-ring then L(A) (the lattice of congruences on A) is permutable and distributive.

<u>pf</u>: Note that A is an abelian group thus L(A) is permutable. And A is a distributive lattice so L(A) is distributive.

As usual a congruence relation on an ℓ-ring can be represented as an ideal but since A is a lattice, those ideals must have some more properties. The following definition is needed:

<u>Def 2.28</u>: Let A be an ℓ-ring by an ℓ-ideal J of A we mean a non-empty subset of A such that i - $\forall a \, \forall b \in J \quad a - b \in J$,

ii - $\forall a \in J \ \forall b \in A \quad ab \in J$,

iii - If $a \in J$ and $b \in A$ so that

$a \vee -a \geq b \vee -b$ then $b \in J$.

Let us denote by $|a|$ the element $a \vee -a$ then iii is equivalent to

(iii$'$) If $a \in J$ $b \in A$ so that $|a| \geq |b|$ then $b \in J$.

<u>Remark</u>: It is easy to see that if θ is congruence on A then $J = \{a \in A / (a, 0) \in \theta\}$ is an ℓ-ideal and that if J is an ℓ-ideal then the relation θ defined on A as follows: $(a, b) \in \theta$ iff $a - b \in J$ is a congruence relation.

Using Cor 2.26 and the above remark together with the representation theorem of Chapter 2 one proves that:

Every ℓ -ring A is isomorphic to a global section structure of an s-sheaf of ℓ -rings. We note that this does not need the commutativity assumption. Since Cor 2.26 holds even when A is not commutative.

We now prove the following embedding theorem for f-rings.

Theorem 2.29 (Pierce): Let A be an f-ring, then the following are equivalent: i - A has no nilpotent elements except 0.

ii - A is a subdirect sum of totally ordered ring with no divisors of 0.

pf: Let us first call an ℓ -ideal J prime iff whenever $a.b \in J$ then either $a \in J$ or $b \in J$. The proof of this theorem needs the following Lemma:

Lemma 2.30: 1) If A is a subdirectly irreducible f-ring then A is totally ordered.

2) If A is an f-ring and P is an ℓ -ideal then A/P is totally ordered and has no divisors of 0 iff P is prime.

pf of the Lemma:

1) - Let a, b $\in$ A so that $a \wedge b = 0$ Let X be the set $\{c \in A / c \wedge b = 0\}$ and Y be the set $\{d \in A / d \wedge x = 0 \ \forall x \in X\}$. Since A is an f-ring it is easy to see that X and Y are ℓ -ideals; moreover $X \cap Y = 0$. Since A is irreducible then either X = 0 or Y = 0 i.e. either a = 0 or b = 0 and so A is totally ordered.

2) Say A/P is totally ordered and with no divisors of 0. Let $a \cdot b \in P$ so $0 = \overline{ab} = \bar{a} . \bar{b}$ thus either $\bar{a} = 0$ or $\bar{b} = 0$ so ither $a \in P$ or $b \in P$.

Conversely: If A/P is not totally ordered by 1) A/P is reducible so there are ℓ -ideals $I, J \supsetneqq P$ so that $I . J \subseteq P$ but then P is not prime so A/P must be totally ordered.

If $\bar{a}$ and $\bar{b} \in A/_P$ so that $\bar{a}, \bar{b} \neq 0$ and $\bar{a} \cdot \bar{b} = 0$ then $a \cdot b \in P$ and $a \notin P$ and $b \notin P$ so P is not prime, contradiction. So $A/_P$ must have no divisors of 0.

pf of the theorem:

ii $\Rightarrow$ i is trivial.

i $\Rightarrow$ ii, Let $M = \{x^n / n \geq 1$ and $x \neq 0\}$, $0 \notin M$ by Zorn's lemma we can find a prime ℓ-ideal P_x so that $P_x \cap M = 0$.

Take $\prod_{x \in A, x \neq 0} A/P_x$ Every A/P_x is totally ordered ring with no divisors of 0.

Remark: If A is commutative then A/P_x is an integral domain which is totally ordered so A/P_x can be embedded in its field of fraction F_x say so A is embeddable in a product of totally ordered fields $\prod_{x \in A} F_x$.

Now let T be the theory of real closed fields. The following are well known:

Theorem 2.31: i - T is the model companion of totally ordered fields,

ii - T has the J.E.P. (joint embedding property) i.e. for any A, B $\models$ T there is $C \models T$ so that $A \subset C$ and $B \not\models C$,

iii - T is complete and every existential formula is equivalent relative to T to a positive existential formula.

Thus it follows from the above that every f-ring A with no non-zero nilpotent elements is embeddable in a product of real closed fields. As in the case of commutative rings with no nilpotent elements we have that A is embedded in $\prod_{x \in X} \mathcal{C}(X, F_x)$ where X is the Cantor space. Using ii of Theorem 2.30 we may reduce $\prod_{x \in X} \mathcal{C}(X, F_x)$ to $\mathcal{C}(X, F)$.

Hence we have proved:

Theorem 2.32: Every f-ring with no nilpotent elements except zero can be embedded in the global section structure of an s-sheaf of real closed fields over a Boolean space with no isolated points.

Now since F is a field then $\mathcal{C}(X, F)$ is a regular ring.

Let $\mathcal{M}$ be the class of all global sections of s-sheaf of real closed fields over Boolean spaces with no isolated points and let $T = \mathrm{Th}(\mathcal{M})$.

The above theorem shows that T is model consistent relative to the theory of f-rings.

Theorem 2.33: T is model complete.

pf: Let R and $S \models T$ so that $R \subset S$ and let $\varphi(\bar{v})$ be a primitive formula $\varphi(\bar{v}) = \exists \bar{w}\, \psi(\bar{v}, \bar{w}) \wedge \bigwedge_{j \leq m} \neg \delta_j(\bar{v}, \bar{w})$ where $\psi(\bar{v})$ is a conjunction of atomics and each $\delta_j(\bar{v}, \bar{w})$ is an atomic. For every $j \leq m$ put

$$\varphi_j(\bar{v}) = \exists \bar{w}\, \psi(\bar{v}, \bar{w}) \wedge \neg \delta_j(\bar{v}, \bar{w}).$$

Let $R = \Gamma(X, K)$ and $S = \Gamma(Y, F)$. We prove the theorem by a series of Lemmas:

Lemma I: If for some $\bar{f} \in R$ $K_x \models \varphi_j(\bar{f}(x))$ then there is a clopen neighbourhood N of x so that $K_y \models \varphi_j(\bar{f}(y))$ for every y in N.

pf: is similar to Lemma 2.9.

Lemma II: $R \models \varphi(\bar{f})$ iff for all $j \leq m$ there is $x_j \in X$ so that $K_{x_j} \models \varphi_j(\bar{f}(x_j))$ and for all $x \in X$ $K_x \models \exists \bar{w}\, \psi(\bar{f}(x), \bar{w})$.

pf: the same as the proof of Lemma 2.10.

Lemma III: $R \models \varphi(\bar{f})$ iff $R \models \varphi_j(\bar{f})$ for all $j \leq m$.

pf: by Lemma II and I.

Lemma IV: Let $\bar{f}$ be elements of R and $h : R \to S$ be the embedding then if $S \models \varphi_j(\overline{h(f)})$ then $R \models \varphi_j(\bar{f})$.

pf: The argument is the same as in Lemma 2.13 but with one change at the end of it one uses the fact that all the fields used in R and S are elementary equivalent and that the theory of real closed fields is model complete.

Remark: Macintyre's theorem (Theorem 2.11) can be proved in a similar way to theorem 2.33.

As a consequence of this theorem we have:

Cor 2.34: The theory T of theorem 2.33 is the model companion of the theory of ~~f-~~rings.

prop 2.35: T is axiomatized by the following set of sentences:

(1) A sentence which says that a model of T is a regular f-rings with no atoms.

(2) A set of sentences which says that every polynomial of odd degree whose parameters are in a model of T has a root in this model

(3) The sentence $\forall v \exists w [v \geq 0 \rightarrow w^2 = v]$.

pf: Clearly a model A in M is a model of (1), (2) and (3).

To prove the converse, let us first remark the following:

If $R \models$ (1), (2) and (3). And $a \in R$ so that $0 \leq x \leq ba$ for some x and b in R. Then and since R is regular there must be $e \in B(R)$ so that $ba = ce$.

$e \in B(R) \Rightarrow (1-e)^2 = 1-e$ so $1-e \geq 0$ (because R is an f-ring).

So $x = xe + x(1-e)$ and $0 \leq xe + x(1-e) \leq ce$.

Thus $0 \leq [xe + x(1-e)](1-e) \leq ce(1-e)$ but this implies

$0 \leq x(1-e) \leq 0$ so $x = xe$.

Thus every principal ℓ-ideal is exactly the principal ring ideal.

Now let J be an ℓ-ideal so that $x^2 \in J$ then $x \in J$. So by regularity of R we have that R/J has no non-zero nilpotent elements.

If J is prime then R/J is a totally ordered ring with no divisors of 0.

Again we use the regularity of R to establish that if J is prime then J is maximal and hence R/J is an ordered field.

Moreover $X = \{ J / J$ prime ℓ -ideals$\}$ form a Boolean space with no isolated points because R has no atoms. And by Chapter 2 we get that $R \simeq \Gamma(X, K)$ where $K_J = R/J$ is an ordered field.

Since R saitsfies (2) so does K_J .

Finally we establish that K_J satisfies (3).

So let $a \in K_J$ so that $a \geq 0$. So there is $f : X \to K$ so that $f(J) = a$. Consider the section $f \wedge 0$ and 0, since $a \wedge 0 = 0$ then $f \wedge 0 (J) = 0(J)$ and so there is a clopen N of X so that $J \in N$ and $f \wedge 0/_N = 0/_N$. Hence there must be an idempotent e of R $e \neq 0$ so that $ef \wedge 0 = 0$ and $e(J) = 1$. So by (3) and the fact that $R \simeq \Gamma(X, K)$ there must be $g \in \Gamma(X, K)$ so $g^2 = ef$ so $g^2 e = fe$ thus $g^2(J) = f(J) = a$. So let $b = g(J)$, thus $b^2 = a$ and hence $R \in M$.

<u>prop 2.36</u>: T has a prime model and hence T is complete.

<u>pf</u>: Let F be a real closed field and consider $\varphi(X, F)$ where X is the Cantor space as in the case of e.c. regular rings. $\varphi(X, F)$ is the prime model of T. That T is complete follows from the fact that T is model complete and has a prime model.

<u>Cor 2.37</u>: T is decidable.

<u>pf</u>: prop 2.35 shows that T is recursively axiomatizable and by 2.36 T is complete so it is decidable.

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