MORLEY (H.)

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ABSOLUTE SUMMABILITY AND ITS APPLICATION TO FOURIER SERIES

A dissertation presented by Hilda Morley for the degree of M.A. of the University of London.

1. Introduction.

The object of this dissertation is to present results on absolute summability by the methods of Hausdorff, Cesàro, Hölder and Abel and to show as far as possible their application in the theory of Fourier Series. Cesàro and Hölder absolute summability are treated as special cases of Hausdorff absolute summability but the more elementary original definitions are also given.

The idea of absolute summability (C,α) of a series was first introduced in 1911 by Fekete[1] in the case where α is a positive integer. The next development was in 1925 when Kogbetlianz[2] proposed a definition for absolute summability of order α where α is any real number other than a negative integer, and developed some of the properties of absolutely summable series giving results analagous to those already found for summable series and some new results on the multiplication of absolutely summable series. Within the last fifteen years the subject has been developed by Bosanquet[3], Chow[4], Hyslop[5], Wang[6] and other writers, particularly with regard to its use in the study of Fourier series and power series.

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(1) Fekete, 4.
(2) Kogbetlianz, 9.
(3) Bosanquet, 1, 2.
(4) Chow, 3.
(5) Hyslop, 8.
(6) Wang, 12.

Absolute Abel summability was first introduced by Whittaker(1) in a paper published in 1930 and was immediately applied to Fourier series. Prasad(2) added theorems of a similar nature in the same year and Fekete(3) and Bosanquet(4) proceeded to investigate the subject, producing further results in 1933 and 1934. The relationship between absolute Cesàro and absolute Abel summability was given by Fekete(3) in 1933 for Cesàro summability of positive integral order and was extended by Bosanquet(5) in 1936 to include Cesàro summability of order α where α is any real number greater than -1.

Hausdorff ordinary summability was introduced by Hausdorff(6) in 1921. Garabedian(7) in 1939 gave an English version of Hausdorff's paper with some slight modifications, and in 1942 Rogosinski(8) produced further developments. The present writer has been unable to find any printed definition of Hausdorff absolute summability but has made use of a paper published by Winn(9) in 1932 in which is used by implication a definition of absolute summability of the transform of a sequence by a T-matrix.

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(1) Whittaker, 13.
(2) Prasad, 10.
(3) Fekete, 5.
(4) Bosanquet.
(5) Bosanquet, 1, 2.
(6) Hausdorff, 7.
(7) Garabedian, 6.
(8) Rogosinski, 11.
(9) Winn, 14.

Hölder absolute summability was defined by Chow[1] in 1939 and the equivalence of Hölder and Cesàro absolute summability was established in the same paper.

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(1) Chow, 13.

2. Preliminary definitions.

The definition of linear transformations given at the beginning of this section holds for matrices of real or complex terms, but for the purpose of this dissertation only real terms will be considered. Suppose that $\{s_n\}$ is the sequence of partial sums of a series $\sum_0^\infty u_n$ and $A = (a_{mn})$ is a row-finite triangular matrix; then the sequence $\{t_n\}$, where $t_n = \sum_{\nu=0}^{n} a_{n\nu} s_\nu$, is called the linear transform of $\{s_n\}$ by the matrix A. If $\lim_{n\to\infty} t_n = s$, where s is finite, the sequence $\{s_n\}$ is said to be summable to s by the matrix A. If $s_n \to s$ implies $t_n \to s'$ where the limits s and s' are finite but not necessarily equal, A is called a K-matrix; if, in addition, $s_n \to 0$ implies $t_n \to 0$, A is said to be essentially regular, and if $s_n \to s$ implies $t_n \to s$, A is said to be regular. Necessary and sufficient conditions for a K-matrix have been shown[1] to be:-

(i) $\sum_{\nu=0}^{n} |a_{n\nu}| \leq M$ where M is independent of n and $n = 0, 1, 2, \ldots\ldots$,

(2.1) (ii) $\lim_{n\to\infty} a_{n\nu} = l_\nu$, $(\nu = 0, 1, 2, \ldots\ldots)$, where l_ν is finite,

(iii) $\lim_{n\to\infty} \sum_{\nu=0}^{n} a_{n\nu} = l$, where l is finite.

For an essentially regular matrix we have the additional condition $l_\nu = 0$ and for a regular matrix $l_\nu = 0$ and $l = 1$.

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(1) see Hausdorff 7, 75 , where references are given.

Hausdorff methods of summability are defined(1) by the linear transformation $t_n = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu}(\mu_\nu) s_\nu, (n = 0, 1, 2, \ldots\ldots\ldots)$, of a sequence $\{s_n\}$, where $\Delta^0(\mu_\nu) = \mu_\nu$, $\Delta'(\mu_\nu)$ denotes the difference $\mu_\nu - \mu_{\nu+1}$, and $\Delta^p(\mu_\nu) = \Delta^{p-1}(\mu_\nu) - \Delta^{p-1}(\mu_{\nu+1})$ is defined by induction. The matrix $(a_{n\nu})$ where $a_{n\nu} = \binom{n}{\nu}\Delta^{n-\nu}(\mu_\nu)$ is called an H-matrix. If μ_n is defined by $\mu_n = \int_0^1 u^n \, d\chi(u), (n = 0, 1, 2, \ldots\ldots)$, where $\chi(u)$ is a function of bounded variation in $[0,1]$, $\{\mu_n\}$ is called a moment sequence. Hausdorff has shown(2) that every moment sequence gives rise to a K-matrix and that every H-matrix which satisfies the conditions (2.1) for a K-matrix can be obtained from a moment sequence. A Hausdorff matrix of this form is called a C-matrix. A Hausdorff transform by a C-matrix is essentially regular if $l_0 = \lim_{n\to\infty} \binom{n}{0} \Delta^{n-0}(\mu_0) = 0$, and regular if in addition $\mu_0 = 1$. The Cesàro and Hölder methods of summability are special cases of Hausdorff summability; they will be introduced later from a more elementary stand-point.

Any sequence $\{\mu_n\}$ is said to be totally monotonic if $\Delta^m(\mu_n) \geq 0$, $(m, n = 0, 1, 2, \ldots\ldots\ldots)$. In the Hausdorff transform by a C-matrix, where $\chi(u)$ is a function of bounded variation in $[0,1]$, we can write $\chi(u) = \chi_1(u) - \chi_2(u)$

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(1) Rogosinski , 11 , introduction to the paper.
(2) Hausdorff , 7 , theorems I and II.

where $\chi_1(u)$ and $\chi_2(u)$ are positive and increasing. Thus $\mu_n = \int_0^1 u^n d\chi_1(u) - \int_0^1 u^n d\chi_2(u) = \alpha_n - \beta_n$ where $\{\alpha_n\}$ and $\{\beta_n\}$ are totally monotonic moment sequences.

In the case of a Hausdorff transform, the necessary and sufficient conditions for a K-matrix have been shown by Hausdorff(1) to reduce to the condition (2.1)(i):-

$$\sum_{v=0}^{n} \binom{n}{v} \left| \Delta^{n-v}(\mu_v) \right| \leq M \quad \text{where } M \text{ is independent of } n$$

and $n = 0, 1, 2, \ldots\ldots\ldots$.

If the sequence $\{\mu_n\}$ is totally monotonic, this condition is automatically satisfied, since

$$\sum_{v=0}^{n} \binom{n}{v} \left| \Delta^{n-v}(\mu_v) \right| = \sum_{v=0}^{n} \binom{n}{v} \Delta^{n-v}(\mu_v) = \mu_0 .$$

Thus, every totally monotonic sequence gives rise to a C-matrix, and hence must be a moment sequence.

If $t_n' = \sum_{v=0}^{n} \binom{n}{v} \Delta^{n-v}(\mu_v) s_v$ and $t_n'' = \sum_{v=0}^{n} \binom{n}{v} \Delta^{n-v}(\mu_v') t_v'$

are two Hausdorff transforms, we can form the product transform $t_n'' = \sum_{v=0}^{n} \binom{n}{v} \Delta^{n-v}(\mu_v \mu_v') s_v$ (2). If the two given transforms are regular, by the fundamental definition of regularity the product transform will also be regular. Provided that $\mu_v \neq 0, (v = 0, 1, 2, \ldots\ldots)$, we may put $\mu_v' = \frac{1}{\mu_v}$ in the second transform, obtaining $t_n'' = \sum_{v=0}^{n} \binom{n}{v} \Delta^{n-v}(1) s_v = s_n$ or $s_n = \sum_{v=0}^{n} \binom{n}{v} \Delta^{n-v}\left(\frac{1}{\mu_v}\right) t_v'$. This is called the inverse transform. By induction the product transform of any

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(1) Hausdorff 7, 79 and 81.
(2) Hausdorff 7, 77

finite number of transforms can be found; if all the original transforms are regular the product must also be regular.

3. Absolute summability: definitions and some theorems.

Definition 1. If we denote by v_n the difference $t_n - t_{n-1}$, $(n=1,2,\ldots\ldots)$, $v_0 = t_0$, and if the series $\sum_0^\infty |v_n|$ is convergent, the sequence $\{s_n\}$, or the series $\sum u_n$, is said to be absolutely summable by the matrix A.

THEOREM 1.(1) *If $\sum_0^\infty u_n$ is absolutely convergent and if $A=(a_{mn})$ is a K-matrix and $\sum_{v=i}^{n}(a_{n,v}-a_{n-1,v}) \geqslant 0$, $(0 \leq i \leq n,\ n=0,1,2,\ldots\ldots\ldots)$, then $\sum_0^\infty v_n$ is also absolutely convergent. Thus if $\{s_n\}$ is of bounded variation, so also is $\{t_n\}$ when the stated condition holds.*

We have

$$v_n = \sum_{v=0}^{n}(a_{n,v}-a_{n-1,v})(u_0+u_1+\ldots\ldots\ldots+u_v)$$

$$= \sum_{i=0}^{n} u_i \sum_{v=i}^{n}(a_{n,v}-a_{n-1,v})$$

so that

$$|v_n| \leq \sum_{i=0}^{n}|u_i|\sum_{v=i}^{n}(a_{n,v}-a_{n-1,v})$$

and

$$\sum_{n=0}^{N}|v_n| \leq \sum_{n=0}^{N}\sum_{i=0}^{n}|u_i|\sum_{v=i}^{n}(a_{n,v}-a_{n-1,v})$$

$$= \sum_{i=0}^{N}|u_i|\sum_{i\leq v\leq n\leq N}(a_{n,v}-a_{n-1,v})$$

$$= \sum_{i=0}^{N}|u_i|\sum_{v=i}^{N}a_{N,v}$$

$$= \sum_{v=0}^{N}a_{N,v}\sum_{i=0}^{v}|u_i|\,.$$

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(1) Winn, 14. This proof is an adaptation of the proof given by Winn.

Now by the definition of a K-matrix, since $\sum_{i=0}^{\nu} |u_i| \to$ a finite limit as $\nu \to \infty$, $\sum_{\nu=0}^{N} a_{N,\nu} \sum_{i=0}^{\nu} |u_i| \to$ a finite limit as $N \to \infty$. The result is now immediate.

The next theorem is really a special case of Theorem 1.

THEOREM 2. <u>If $\sum_0^\infty u_n$ is absolutely convergent and if $t_n = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu}(\mu_\nu) s_\nu$ is a Hausdorff transform and the sequence $\{\mu_n\}$ is totally monotonic, then $\sum_0^\infty v_n$ is absolutely convergent.</u>

We have

$$\binom{n}{\nu}\Delta^{n-\nu}(\mu_\nu) - \binom{n-1}{\nu}\Delta^{n-\nu-1}(\mu_\nu)$$
$$= \binom{n}{\nu}\Delta^{n-\nu}(\mu_\nu) - \binom{n-1}{\nu}\left\{\Delta^{n-\nu}(\mu_\nu) + \Delta^{n-\nu-1}(\mu_{\nu+1})\right\}$$
$$= \binom{n-1}{\nu-1}\Delta^{n-\nu}(\mu_\nu) - \binom{n-1}{\nu}\Delta^{n-\nu-1}(\mu_{\nu+1}), \quad (n \geq \nu \geq 1).$$

If $n \geq i \geq 0$,

$$\sum_{\nu=i}^{n}\left\{\binom{n}{\nu}\Delta^{n-\nu}(\mu_\nu) - \binom{n-1}{\nu}\Delta^{n-\nu-1}(\mu_\nu)\right\}$$
$$= \binom{n-1}{i-1}\Delta^{n-i}(\mu_i) - \binom{n-1}{i}\Delta^{n-i-1}(\mu_{i+1})$$
$$+ \binom{n-1}{i}\Delta^{n-i-1}(\mu_{i+1}) - \cdots\cdots + \binom{n-1}{n-1}(\mu_n) - 0$$
$$= \binom{n-1}{i-1}\Delta^{n-i}(\mu_i)$$
$$\geq 0.$$

From the previous theorem and the result that $\left(\binom{n}{\nu}\Delta^{n-\nu}(\mu_\nu)\right)$ ia a C-matrix we can now deduce that if $\sum_0^\infty u_n$ is absolutely convergent so also is $\sum_0^\infty v_n$.

In view of the fact that any Hausdorff transform by a C-matrix can be expressed as the difference of two transforms $t_n' - t_n''$ where $t_n' = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu}(\alpha_\nu) s_\nu$ and $t_n'' = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu}(\beta_\nu) s_\nu$ and $\{\alpha_n\}, \{\beta_n\}$ are totally monotonic moment sequences, the condition that $\{\mu_n\}$ should be totally monotonic in the previous theorem can be discarded, for a C-matrix transform, and the theorem restated:-

THEOREM 2'. <u>If $\sum_0^\infty u_n$ is absolutely convergent and if $t_n = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu}(\mu_\nu) s_\nu$ is a Hausdorff transform by a C-matrix, then $\sum_0^\infty v_n$ is absolutely convergent.</u>

4. Cesàro Summability and Absolute Summability.

The definitions for Cesàro summability of any order are well-known; they will be quoted briefly(1). We denote by A_n^α the coefficient of x^n in the expansion of $(1-x)^{-\alpha-1}$ for $|x|<1$ and for any real α, so that $A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\ldots\ldots(\alpha+n)}{n!}$ $(n=1,2,\ldots\ldots)$, $A_0^\alpha = 1$ for all α, and if α is a negative integer $A_n^\alpha = 0$ for $n \geqslant -\alpha$. Let $S_n^\alpha = \sum_{\nu=0}^{n} A_{n-\nu}^\alpha u_\nu = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} s_\nu$ where s_ν is the ν'th partial sum of the infinite series $\sum_0^\infty u_n$. Let $\sigma_n^\alpha = \frac{S_n^\alpha}{A_n^\alpha}$, $\alpha \neq -1,-2,-3,\ldots\ldots\ldots\ldots$. S_n^α and σ_n^α are called the n'th Cesàro sum and the n'th Cesàro mean of order α of the series $\sum u_n$. If $\sigma_n^\alpha \to s$ as $n \to \infty$, the series is said to be summable (C,α) with sum s. If we now put $u_n^\alpha = \sigma_n^\alpha - \sigma_{n-1}^\alpha$, we see that summability (C,α) is equivalent to convergence of the series $\sum_0^\infty u_n^\alpha$. This series is called the transform of order α of the original series. Summability $(C,0)$ is thus equivalent to convergence of the original series.

Alternative and equivalent definitions for Cesàro summability can be given by referring to the preceding section. If we take $\mu_\nu = \frac{1}{A_\nu^\alpha}$, $(\alpha > -1)$, it can be shown by induction that $\binom{n}{\nu} \Delta^{n-\nu}(\mu_\nu) = \frac{A_{n-\nu}^{\alpha-1}}{A_n^\alpha}$. Thus the

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(1) see Zygmund 15, 42. Useful relations between the A_n^α and S_n^α are given here.

Hausdorff transform $t_n = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu}(\mu_\nu)\, s_\nu$ is identical with σ_n^α. For $\alpha \geqslant 0$, $\Delta^{n-\nu}(\mu_\nu) \geqslant 0$ so that $\{\mu_n\}$ is a totally monotonic sequence. Also the transform satisfies the conditions for regularity for $\alpha \geqslant 0$.

<u>Definition 2</u>. The series $\sum_0^\infty u_n$ is said to be summable $|C,\alpha|$, that is absolutely summable (C,α) where $\alpha > -1$ if the series $\sum_0^\infty |u_n^\alpha|$ is convergent. It follows directly from this definition that summability $|C,0|$ is equivalent to absolute convergence. Also, a series which is summable $|C,\alpha|$ is necessarily summable (C,α) since convergence of the series $\sum |u_n^\alpha|$ implies convergence of the series $\sum u_n^\alpha$. The converse is not true.

If T_n^α, τ_n^α denote respectively the n'th Cesàro sum and the n'th Cesàro mean of order α of the sequence $\{n u_n\}$, it is well-known that

$$(3.1) \qquad \tau_n^\alpha = n(\sigma_n^\alpha - \sigma_{n-1}^\alpha).$$

Summability $|C,\alpha|$ is therefore equivalent to convergence of the series $\sum_1^\infty \left|\frac{\tau_n^\alpha}{n}\right|$.

THEOREM 3. <u>If $\sum u_n$ is summable $|C,\alpha|$ where $\alpha > -1$, then it is also summable $|C,\alpha+\delta|$ for all $\delta \geqslant 0$ and $\sum |u_n^{\alpha+\delta}| \leq \sum |u_n^\alpha|$.</u>

We may write

$$t_n = \sigma_n^{\alpha+\delta} = \sum_{\nu=0}^{n} a_{n\nu}\, \sigma_\nu^\alpha$$

where

$$a_{n\nu} = \frac{A_{n-\nu}^{\delta-1} A_\nu^\alpha}{A_n^{\alpha+\delta}}.$$

Now

$$\sigma_n^\alpha = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu}(\mu_\nu') s_\nu, \text{ where } \mu_\nu' = \frac{1}{A_\nu^\alpha}.$$

Forming the inverse transform,

$$s_n = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu}\left(\frac{1}{\mu_\nu'}\right) \sigma_\nu^\alpha,$$

Also

$$\sigma_n^{\alpha+\delta} = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu}(\mu_\nu'') s_\nu, \text{ where } \mu_\nu'' = \frac{1}{A_\nu^{\alpha+\delta}}.$$

If we now form the product of the last two transforms, we have

$$\sigma_n^{\alpha+\delta} = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu}(\mu_\nu) \sigma_\nu^\alpha, \text{where } \mu_\nu = \frac{A_\nu^\alpha}{A_\nu^{\alpha+\delta}}.$$

Since $\Delta^{n-\nu}(\mu_\nu) \geqslant 0$, $\{\mu_n\}$ is a totally monotonic sequence, and hence it follows from Theorem 2 that if $\sum u_n^\alpha$ is absolutely convergent so also is $\sum(t_n - t_{n-1}) = \sum u_n^{\alpha+\delta}$: that is, if $\sum u_n$ is summable $|C,\alpha|$ it is summable $|C,\alpha+\delta|$ for $\delta \geqslant 0$. Moreover, since $\sum_{\nu=0}^{n} a_{n\nu} = 1$, it follows from the proof of Theorem 1 that

$$\sum_{n=0}^{N} |u_n^{\alpha+\delta}| \leqslant \sum_{i=0}^{N} |u_i^\alpha| \sum_{\nu=i}^{N} a_{N,\nu}$$

$$\leqslant \sum_{i=0}^{N} |u_i^\alpha|$$

Hence

$$\sum_0^\infty |u_n^{\alpha+\delta}| \leqslant \sum_0^\infty |u_n^\alpha|.$$

If we take the value $\alpha = 0$ the result takes the form:- a series which is absolutely convergent is summable $|C, \delta|$ for every $\delta \geq 0$.

The following two complementary theorems were first proved by Kogbetlianz in his publication of 1925[1]. The proofs given here depend on the preceding results on transformation of sequences.

THEOREM 4. <u>If $-1 < \alpha \leq \beta$ and $\sum u_n$ is summable $|C, \beta|$, then $\sum u_n^\alpha$ is summable $|C, \beta - \alpha|$</u> .

THEOREM 5. <u>If $-1 < \alpha \leq \beta$ and $\sum u_n^\alpha$ is summable $|C, \beta - \alpha|$, then $\sum u_n$ is summable $|C, \beta|$</u> .

<u>Proof of Theorem 4.</u> We can write

(4.1) $$\sigma_n^\beta = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu} (\mu_\nu') s_\nu, \quad \text{where} \quad \mu_\nu' = \frac{1}{A_\nu^\beta}.$$

Now, forming the inverse transform,

(4.2) $$s_n = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu} \left(\frac{1}{\mu_\nu'}\right) \sigma_\nu^\beta.$$

Also

(4.3) $$\sigma_n^\alpha = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu} (\mu_\nu'') s_\nu, \quad \text{where} \quad \mu_\nu'' = \frac{1}{A_\nu^\alpha},$$

and

(4.4) $$t_n = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu} (\mu_\nu''') \sigma_\nu^\alpha, \quad \text{where} \quad \mu_\nu''' = \frac{1}{A_\nu^{\beta-\alpha}}.$$

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(1) Kogbetlianz 9 .

If we now form the product of the last three transforms, we have

(4.5) $$t_n = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu}(\mu_\nu)\, \sigma_\nu^\beta$$

where

$$\mu_\nu = \frac{\mu_\nu''' \mu_\nu''}{\mu_\nu'} = \frac{A_\nu^\beta}{A_\nu^{\beta-\alpha} A_\nu^\alpha}.$$

We will write

$$\mu_\nu = \frac{A_\nu^\beta}{(\nu+1)^\beta} \cdot \frac{(\nu+1)^{\beta-\alpha}}{A_\nu^{\beta-\alpha}} \cdot \frac{(\nu+1)^\alpha}{A_\nu^\alpha}.$$

We may assume the regularity of the transforms obtained from the sequences $\left\{\frac{A_\nu^\beta}{(\nu+1)^\beta}\right\}$, $\left\{\frac{(\nu+1)^{\beta-\alpha}}{A_\nu^{\beta-\alpha}}\right\}$, $\left\{\frac{(\nu+1)^\alpha}{A_\nu^\alpha}\right\}$ (1); these are transforms from Cesàro means to Hölder means and vice-versa, and will be considered more fully in the next section. Since the product of three regular transforms is itself regular, the transform (4.5) is regular. Since $\sum |u_n^\beta|$ is convergent by hypothesis, the result stated in the theorem can now be inferred from Theorem 2'.

<u>Proof of Theorem 5</u>. As in the previous theorem we write

$$t_n = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu}(\mu_\nu''')\, \sigma_\nu^\alpha, \text{ where } \mu_\nu''' = \frac{1}{A_\nu^{\beta-\alpha}},$$

$$\sigma_n^\alpha = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu}(\mu_\nu'')\, s_\nu, \text{ where } \mu_\nu'' = \frac{1}{A_\nu^\alpha},$$

(4.1) $$\sigma_n^\beta = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu}(\mu_\nu')\, s_\nu, \text{ where } \mu_\nu' = \frac{1}{A_\nu^\beta}.$$

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(1) Garabedian 6, 408.

Now, forming the inverse transforms,

(4.6)
$$s_n = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu} \left(\frac{1}{\mu_\nu''}\right) \sigma_\nu^\alpha$$

and

(4.7)
$$\sigma_n^\alpha = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu} \left(\frac{1}{\mu_\nu'''}\right) t_\nu .$$

Hence, taking the product of (4.1), (4.6) and (4.7),

(4.8)
$$\sigma_n^\beta = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu} \left(\frac{\mu_\nu'}{\mu_\nu''\mu_\nu'''}\right) t_\nu .$$

By writing
$$\frac{\mu_\nu'}{\mu_\nu''\mu_\nu'''} = \frac{(\nu+1)^\beta}{A_\nu^\beta} \cdot \frac{A_\nu^\alpha}{(\nu+1)^\alpha} \cdot \frac{A_\nu^{\beta-\alpha}}{(\nu+1)^{\beta-\alpha}} ,$$

we see that $\left\{\frac{\mu_\nu'}{\mu_\nu''\mu_\nu'''}\right\}$ is a regular sequence and (4.8) a regular transform. Since $\sum |t_n - t_{n-1}|$ is convergent it follows from Theorem 2' that $\sum |u_n^\beta|$ is convergent: that is, the series $\sum u_n$ is summable $|C, \beta|$.

5. Hölder Summability and Absolute Summability.

Hölder means of positive integral order can be defined by induction from the simple formulae $H_n^0 = s_n$, $H_n^r = \frac{H_0^{r-1} + H_1^{r-1} + \dots\dots + H_n^{r-1}}{n+1}, (r = 1,2,3,\dots\dots)$, where s_n is the n'th partial sum of the series $\sum_0^\infty u_n$. For means of any order, integral or non-integral, the neatest definition is by a Hausdorff transform. We define $H_n^\alpha = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu} \left(\frac{1}{\nu+1^\alpha}\right) s_\nu$ for all α. The two definitions are equivalent where α is a positive integer; for, since $\binom{n}{\nu} \Delta^{n-\nu}\left(\frac{1}{\nu+1}\right) = \frac{1}{n+1}$, $H_n' = \sum_{\nu=0}^{n} \frac{1}{n+1} s_\nu = \sum_{\nu=0}^{n} \binom{n}{\nu}\Delta^{n-\nu}\left(\frac{1}{\nu+1}\right) s_\nu$, so by using the results on products of sequences and induction we obtain

$$H_n^r = \sum_{\nu=0}^{n} \binom{n}{\nu}\Delta^{n-\nu}\left(\frac{1}{\nu+1}\right) H_\nu^{r-1} = \sum_{\nu=0}^{n} \binom{n}{\nu}\Delta^{n-\nu}\left(\frac{1}{\nu+1^2}\right) H_\nu^{r-2} = \dots\dots\dots\dots$$

$= \sum_{\nu=0}^{n} \binom{n}{\nu}\Delta^{n-\nu}\left(\frac{1}{\nu+1^r}\right) s_\nu$. The transform is known to be regular when $\alpha > 0$. If $H_n^\alpha \to s$ as $n \to \infty$ the series $\sum u_n$ is said to be summable (H,α) with sum s. It is evident from the first definition that summability $(H,1)$ is equivalent to summability $(C,1)$ and it is, in fact, a well-known result that summability (H,α) is equivalent to summability (C,α) for all $\alpha > -1$. If we write $h_n^\alpha = H_n^\alpha - H_{n-1}^\alpha$, we see that summability (H,α) is equivalent to convergence of the series $\sum_0^\infty h_n^\alpha$.

Definition 3. If the series $\sum |h_n^\alpha|$ is convergent, the series $\sum u_n$ is said to be summable $|H,\alpha|$ that is

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(1) Hausdorff 7, 83.
(2) Garabedian 6, Theorem 18.

absolutely summable (H,α). A series which is summable $|H,\alpha|$ is necessarily summable (H,α) since convergence of the series $\sum|h_n^\alpha|$ implies convergence of the series $\sum h_n^\alpha$.

If we combine with the Hölder transform

$H_n^\alpha = \sum_{\nu=0}^{n}\binom{n}{\nu}\Delta^{n-\nu}\left(\frac{1}{\overline{\nu+1}\,\alpha}\right)s_\nu$ and its inverse

$s_n = \sum_{\nu=0}^{n}\binom{n}{\nu}\Delta^{n-\nu}\left(\overline{\nu+1}\,\alpha\right)H_\nu^\alpha$ the Cesàro transform

$\sigma_n^\alpha = \sum_{\nu=0}^{n}\binom{n}{\nu}\Delta^{n-\nu}\left(\frac{1}{A_\nu^\alpha}\right)s_\nu$ and its inverse

$s_n = \sum_{\nu=0}^{n}\binom{n}{\nu}\Delta^{n-\nu}\left(A_\nu^\alpha\right)\sigma_\nu^\alpha$, we obtain

(5.1) $$\sigma_n^\alpha = \sum_{\nu=0}^{n}\binom{n}{\nu}\Delta^{n-\nu}\left(\frac{\overline{\nu+1}\,\alpha}{A_\nu^\alpha}\right)H_\nu^\alpha$$

and

(5.2) $$H_n^\alpha = \sum_{\nu=0}^{n}\binom{n}{\nu}\Delta^{n-\nu}\left(\frac{A_\nu^\alpha}{\overline{\nu+1}\,\alpha}\right)\sigma_\nu^\alpha .$$

These transforms are known to be regular(1) for $\alpha > -1$.

<u>THEOREM 6. If the series $\sum u_n$ is summable $|C,\alpha|$ where $\alpha > -1$ it is also summable $|H,\alpha|$, and vice-versa.</u>(2)

From the regular transform (5.2) and Theorem 2' we can deduce that if $\sum|u_n^\alpha|$ is convergent so also is $\sum|h_n^\alpha|$.

From the regular transform (5.1) and the same theorem we can deduce that if $\sum|h_n^\alpha|$ is convergent so also is $\sum|u_n^\alpha|$.

The results stated follow immediately.

Theorem 6 is of fundamental importance in the

(1) Garabedian 6, Theorem 18.
(2) A proof of this theorem has been given by Chow, 3, for the case when α is a positive integer.

study of Hölder summability. In consequence of this theorem, some results on absolute Hölder summability can be deduced from results on absolute Cesàro summability, and vice-versa, by restating the appropriate theorems and without further proof. This does apply to Theorem 3 of the preceding section but not to Theorem 4 and 5. The results analogous to Theorem 4 and 5 are true, but independent proof is necessary and will be given in Theorem 7 and 8.

THEOREM 7. *If $\sum u_n$ is summable $|H,\beta|$, then $\sum h_n^\alpha$ is summable $|H,\beta-\alpha|$ for all α and β.*

THEOREM 8. *If $\sum h_n^\alpha$ is summable $|H,\beta-\alpha|$, then $\sum u_n$ is summable $|H,\beta|$ for all α and β.*

Proof of Theorem 7. We can write

(5.1) $$H_n^\beta = \sum_{\nu=0}^{n}\binom{n}{\nu}\Delta^{n-\nu}(\mu_\nu')\, s_\nu \quad \text{where} \quad \mu_\nu' = \frac{1}{\overline{\nu+1}^{\,\beta}}.$$

Now, forming the inverse transform,

(5.2) $$s_n = \sum_{\nu=0}^{n}\binom{n}{\nu}\Delta^{n-\nu}\left(\frac{1}{\mu_\nu'}\right) H_\nu^\beta .$$

Also

(5.3) $$H_n^\alpha = \sum_{\nu=0}^{n}\binom{n}{\nu}\Delta^{n-\nu}(\mu_\nu'')\, s_\nu \quad \text{where} \quad \mu_\nu'' = \frac{1}{\overline{\nu+1}^{\,\alpha}},$$

and

(5.4) $$t_n = \sum_{\nu=0}^{n}\binom{n}{\nu}\Delta^{n-\nu}(\mu_\nu''')\, H_\nu^\alpha \quad \text{where} \quad \mu_\nu''' = \frac{1}{\overline{\nu+1}^{\,\beta-\alpha}},$$

so that t_n is the Hölder mean of order $(\beta-\alpha)$ of the series $\sum h_n^\alpha$. If we now form the product of the last three transforms, we have

$$t_n = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu} (\mu_\nu)\, H_\nu^\beta$$

where

$$\mu_\nu = \frac{\mu_\nu''' \mu_\nu''}{\mu_\nu'} = 1 .$$

Thus t_n is identical with the Hölder mean H_n^β . It follows that if $\sum h_n^\beta$ is absolutely convergent, $\sum |t_n - t_{n-1}|$ is convergent: that is $\sum h_n^\alpha$ is summable $|H, \beta-\alpha|$.

<u>Proof of Theorem 8.</u> As in the previous theorem, we write

$$t_n = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu} (\mu_\nu''')\, H_\nu^\alpha, \text{ where } \mu_\nu''' = \frac{1}{(\nu+1)^{\beta-\alpha}} ,$$

$$H_n^\alpha = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu} (\mu_\nu'')\, s_\nu , \text{ where } \mu_\nu'' = \frac{1}{(\nu+1)^{\alpha}} ,$$

and

(5.1) $$H_n^\beta = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu} (\mu_\nu')\, s_\nu , \text{ where } \mu_\nu' = \frac{1}{(\nu+1)^{\beta}} .$$

Now, forming the inverse transforms,

(5.5) $$s_n = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu} \left(\frac{1}{\mu_\nu''}\right) H_\nu^\alpha ,$$

(5.6) $$H_n^\alpha = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu} \left(\frac{1}{\mu_\nu'''}\right) t_\nu .$$

If we now form the product transform of (5.1), (5.5) and (5.6) we obtain

(5.7) $$H_n^\beta = \sum_{\nu=0}^{n} \binom{n}{\nu} \Delta^{n-\nu} (\mu_\nu)\, t_\nu ,$$

where $\mu_\nu = \dfrac{\mu_\nu'}{\mu_\nu'' \mu_\nu'''} = 1$.

Since, by hypothesis, $\sum |t_n - t_{n-1}|$ is convergent, it follows that $\sum |h_n^\beta|$ is convergent: that is $\sum u_n$ is summable $|H, \beta|$.

6. Abel Summability and Absolute Summability.

For the definition of Abel summability of a series $\sum u_n$ (1), we associate with this series the function $f(x) = \sum_{n=0}^{\infty} u_n x^n$, the series being assumed convergent for $|x| < 1$. If $\lim_{x \to 1-0} f(x)$ exists and is finite the series is said to be summable (A). In view of Abel's theorem it is evident that every convergent series is summable (A).

Definition (1) 4. A series $\sum u_n$ is said to be summable $|A|$ or absolutely summable (A) if $\lim_{x \to 1-0} f(x)$ exists and is finite and if, in addition, $f(x)$ is a function of bounded variation in the interval $(0, 1)$.

Since, by hypothesis, the series $\sum u_n x^n$ has radius of convergence $\geqslant 1$, $f(x)$ is continuous and has derivatives of all orders for $0 \leq x < 1$; thus $f(x)$ is absolutely continuous in $[0, 1-\varepsilon]$ where $0 < \varepsilon < 1$, and
$\int_0^{1-\varepsilon} |d f(x)| = \int_0^{1-\varepsilon} |f'(x)|\, dx$.
Hence $\sum u_n$ is summable $|A|$ if $\lim_{\varepsilon \to 0} \int_0^{1-\varepsilon} |f'(x)|\, dx$ exists and is finite. It can easily be shown that every absolutely convergent series is summable $|A|$. For if $\sum u_n$ is absolutely convergent,

$$\begin{aligned} \int_0^{1-\varepsilon} |f'(x)|\, dx &\leq \int_0^{1-\varepsilon} \left(\sum_1^{\infty} n |u_n| x^{n-1} \right) dx \\ &= \sum_1^{\infty} |u_n| (1-\varepsilon)^n \\ &\leq \sum_1^{\infty} |u_n| \end{aligned}$$

(1) Whittaker, 13.

and the right hand side of this inequality is finite. Letting $\varepsilon \to 0$ we see that the condition for summability $|A|$ is satisfied.

The connection between summability $|A|$ and absolute Cesàro summability (or absolute Hölder summability) is an intimate one. It will be proved in the next theorem that if a series is summable $|C,\alpha|$ for any positive order α, then it is also summable $|A|$, the sum being the same in each case. It can be shown by means of an example that the converse is not true. Thus summability $|A|$ includes summability $|C,\alpha|$ for all $\alpha \geqslant 0$, but summability $|C,\alpha|$ does not include summability $|A|$. In other words, Abel absolute summability is applicable to a wider range of series than Cesàro absolute summability.

THEOREM 9 (1). <u>If $\sum u_n$ is summable $|C,\alpha|$ where $\alpha \geqslant 0$ then it is also summable $|A|$.</u>

We have to prove that if $\sum_0^\infty |u_n^\alpha|$ is convergent then $\int_0^1 |f'(x)|\,dx$ is finite. Now convergence of $\sum_0^\infty |u_n^\alpha|$ is equivalent to convergence of the series $\sum_1^\infty \left|\frac{\tau_n^\alpha}{n}\right|$ where τ_n^α is the Cesàro mean defined in (4.1). We have, for $|x| < 1$,

$$f'(x) = \sum_{n=1}^\infty n u_n x^{n-1}$$

---

(1) Bosanquet 2, footnote on p.518. Fekete previously proved the result for α a positive integer.

$$= (1-x)^{\alpha} . (1-x)^{-\alpha} \sum_{n=1}^{\infty} n u_n x^{n-1}$$

$$= (1-x)^{\alpha} \sum_{n=1}^{\infty} T_n^{\alpha} x^{n-1} .$$

Thus $$\int_0^1 |f'(x)|\, dx \leq \int_0^1 (1-x)^{\alpha} \sum_1^{\infty} |T_n^{\alpha}|\, x^{n-1}\, dx$$

$$= \sum_1^{\infty} |T_n^{\alpha}| \int_0^1 (1-x)^{\alpha} x^{n-1}\, dx$$

$$= \sum_1^{\infty} |T_n^{\alpha}| . \frac{1}{n A_n^{\alpha}}$$

$$= \sum_1^{\infty} \frac{|T_n^{\alpha}|}{n} .$$

It follows that if $\sum_1^{\infty} \frac{|T_n^{\alpha}|}{n}$ is convergent, $\int_0^1 |f'(x)|\, dx$ is finite, $f(x)$ is of bounded variation in $(0, 1)$ and the series is summable $|A|$.

The following example of a series summable $|A|$ but not summable $|C, r|$ for any r is quoted in a paper by Fekete(1); it is originally due to Bohr. Suppose that $f(x) = e^{\frac{1}{1+x}}$. Then $f(x)$ can be expanded in a power series $\sum_0^{\infty} u_n x^n$ which is convergent for $|x| < 1$. Since $\lim_{x \to 1-0} f(x) = e^{\frac{1}{2}}$ and $\int_0^1 |f'(x)|\, dx = \int_0^1 - f'(x)\, dx = e - e^{\frac{1}{2}}$, it follows that $\sum_0^{\infty} u_n$ is summable $|A|$. Now suppose, if possible, that $\sum u_n$ is summable (C, r) for some positive integral value of r ; there is no real restriction here in assuming r to be an integer. We must have $u_n = o(n^r)$ as $n \to \infty$, and hence, for all values of n ,

$|u_n| < K' n^r$ where K' is a positive constant.

Thus

$$|u_n| < K' . A_n^r . \frac{r!}{(1+\frac{1}{n})(1+\frac{2}{n}) \ldots\ldots\ldots (1+\frac{r}{n})} ,$$

and hence

(1) Fekete , 5

$$|u_n| < K A_n^r \text{ where } K = K' r!.$$

Now, for $0 \leq x < 1$,

$$\begin{aligned} e^{\frac{1}{1-x}} &= f(-x) \\ &\leq \sum_0^\infty |u_n| x^n \\ &< K \sum_0^\infty A_n^r x^n \\ &= K(1-x)^{-r-1}. \end{aligned}$$

Since $\frac{e^y}{y^r} \to \infty$ as $y \to \infty$, this inequality is false if x is sufficiently close to 1. Thus our supposition was not correct and $\sum u_n$ is not summable (C, r) and hence not summable $|C, r|$ for any r.

7. The Absolute Summability of Fourier Series.

Some of the preceding results on Abel and Cesàro summability have in recent years been found useful in the study of Fourier series. A comprehensive account of the research which has been done and is still being done in this field would be a considerable undertaking, but the theorems given in this section serve to illustrate the power of summability processes.

Here we will give the terminology used in this section. Suppose

(7.1) $$\tfrac{1}{2}a_0 + \sum_{k=1}^{\infty}\left(a_k \cos kx + b_k \sin kx\right)$$

is the Fourier series of a function $f(t)$ which has a finite Lebesgue integral in $(-\pi, \pi)$ and is of period 2π, the coefficients a_k, b_k being found by the formulae

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\, dt, \quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, dt,$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\, dt.$$ Let

(7.2) $$\phi(t) = \phi_0(t) = \tfrac{1}{2}\left\{f(x+t) + f(x-t) - 2s\right\},$$

(7.3) $$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\phi(u)\, du, \quad (t>0,\ \alpha>0),$$

the integral existing almost everywhere,

$$\Phi_0(t) = \phi(t),$$

$$\phi_\alpha(t) = \frac{\Gamma(\alpha+1)}{t^\alpha}\Phi_\alpha(t)$$

(7.4) $$= \frac{\alpha}{t^\alpha}\int_0^t (t-u)^{\alpha-1}\phi(u)\, du.$$

$\Phi_\alpha(t)$ and $\phi_\alpha(t)$ are thus analogous to Cesàro sums and Cesàro means of order α of $\phi(t)$. Assuming the existence of $\Phi_{\alpha+\beta}(t)$, it can be shown that

(7.5) $$\Phi_{\alpha+\beta}(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-u)^{\beta-1} \Phi_\alpha(u)\, du.$$

The Poisson series corresponding to the given Fourier series is defined by the equation

$$P(r) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} r^k (a_k \cos kx + b_k \sin kx), \quad (0 \leq r < 1)$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \left\{ \frac{1}{2} + \sum_{k=1}^{\infty} r^k \cos k(x-t) \right\} dt$$

(7.6) $$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) \cdot \frac{1-r^2}{1-2r\cos(x-t)+r^2}\, dt,$$

the infinite series being uniformily convergent in $(-\pi, \pi)$ for $0 \leq r < 1$. Taking $f(t) \equiv 1$ in this expression, all the Fourier coefficients except a_0 vanish and we have

(7.7) $$2\pi = \int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(x-t)+r^2}\, dt = \int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos t + r^2}\, dt,$$

replacing $(x-t)$ by t and using the periodicity of $f(t)$.

The following theorem, published by Whittaker in 1933, is the earliest of a series of three theorems giving tests for summability $|A|$ of Fourier series.

THEOREM 10.[1] <u>The Fourier series (7.1) is summable $|A|$ to s at the point x if $\int_0^\delta \left|\frac{\phi(t)}{t}\right| dt$ exists and is finite, where $0 < \delta \leq \pi$. In other words, every Fourier</u>

(1) Whittaker, 13.

<u>series which converges in virtue of Dini's condition is summable $|A|$.</u>

Let $Q(r) = P(r) - s, \ (0 \le r < 1)$,

where $P(r)$ is the Poisson series (7.6). Replacing $(x-t)$ by t,

$$Q(r) = \frac{1}{2\pi}\int_{x-\pi}^{x+\pi} f(x-t) \cdot \frac{1-r^2}{1-2r\cos t + r^2}\,dt - s$$

$$= \frac{1}{2\pi}\int_0^{\pi} + \int_{-\pi}^{0} f(x-t) \cdot \frac{1-r^2}{1-2r\cos t + r^2}\,dt - s$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left\{f(x-t) + f(x+t)\right\}\frac{1-r^2}{1-2r\cos t + r^2}\,dt - s,$$

replacing t by $-t$ in the second integral. Hence, using (7.7),

$$Q(r) = \frac{1}{\pi}\int_0^{\pi} \phi(t) \cdot \frac{1-r^2}{1-2r\cos t + r^2}\,dt$$

If $0 \le r_1 < 1$, then

$$\int_0^{r_1} |Q'(r)|\,dr = \frac{1}{\pi}\int_0^{r_1} \left|\int_0^{\pi} \phi(t)\,\frac{\partial}{\partial r}\left(\frac{1-r^2}{1-2r\cos t + r^2}\right)dt\right| dr$$

$$\le \frac{2}{\pi}\int_0^{r_1} dr \int_0^{\pi} |\phi(t)| \left|\frac{(1+r^2)\cos t - 2r}{(1-2r\cos t + r^2)^2}\right| dt$$

$$\le \frac{1}{\pi}\int_0^{\pi} |\phi(t)|\, V(r_1, t)\,dt \tag{7.8}$$

where

$$V(r_1, t) = 2\int_0^{r_1} \left|\frac{(1+r^2)\cos t - 2r}{(1-2r\cos t + r^2)^2}\right| dr$$

The inversion of the order of integration is justified here by Fubini's theorem since the integrand is non-negative.

We proceed to show that, for $0 \le r_1 < 1$, $V(r_1, t) < \frac{\pi}{t}$ when $0 < t \le \frac{\pi}{2}$ and $V(r_1, t) < 1$ when $\frac{\pi}{2} \le t \le \pi$. We write

$$t_1 = \cos^{-1}\left(\frac{2r_1}{1+r_1^2}\right), \text{ so that } 0 < t_1 \le \frac{\pi}{2},$$

and

$$X(r) = \frac{1-r^2}{1-2r\cos t + r^2}.$$

Then, for $0 \leq t \leq t_1$, and $0 \leq r \leq r_1 < 1$, the expression

$$X'(r) = \frac{(1+r^2)\cos t - 2r}{(1-2r\cos t + r^2)^2}$$

is essentially positive, since

$$\begin{aligned}(1+r^2)\cos t - 2r &\geqslant (1+r^2).\frac{2r_1}{1+r_1^2} - 2r \\ &= \frac{2(r_1-r)(1-rr_1)}{(1+r_1^2)} \\ &\geqslant 0.\end{aligned}$$

It follows that, for $0 < t \leq t_1$,

$$\begin{aligned}V(r_1, t) &= \int_0^{r_1} X'(r)\,dr \\ &= X(r_1) - 1 \\ &< \frac{1-r_1^2}{1-2r_1\cos t + r_1^2} \\ &\leq \frac{\sin t_1}{1-\cos t_1} \\ &= \frac{\cos\frac{t_1}{2}}{\sin\frac{t_1}{2}} \\ &< \frac{\pi}{t_1} \quad \text{(by Jordan's Inequality)} \\ &\leq \frac{\pi}{t}.\end{aligned}$$

If $t_1 \leq t \leq \frac{\pi}{2}$, $X'(r)$ is negative when $\cot\left(\frac{t}{2}+\frac{\pi}{4}\right) < r \leq r_1$, and

$$\begin{aligned}V(r_1, t) &= \int_0^{\cot\left(\frac{t}{2}+\frac{\pi}{4}\right)} X'(r)\,dr - \int_{\cot\left(\frac{t}{2}+\frac{\pi}{4}\right)}^{r_1} X'(r)\,dr \\ &= \frac{2}{\sin t} - 1 - X(r_1) \\ &< \frac{\pi}{t} \quad \text{, (by Jordan's Inequality).}\end{aligned}$$

Finally, if $\frac{\pi}{2} \leq t \leq \pi$,

$$\begin{aligned}V(r_1, t) &= -\int_0^{r_1} X'(r)\,dr \\ &= 1 - X(r_1) \\ &< 1.\end{aligned}$$

Substituting now in (7.8),

$$\int_0^{r_1} |Q'(r)|\,dr \le \frac{1}{\pi}\left(\int_0^{t_1} + \int_{t_1}^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi}\right) |\phi(t)|\, V(r_1,t)\,dt$$
$$< \int_0^{\frac{\pi}{2}} \left|\frac{\phi(t)}{t}\right| dt + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} |\phi(t)|\,dt .$$

Each of the integrals on the right hand side is finite, the first one by hypothesis and the second from the Lebesgue-integrability of $f(t)$ and hence of $\phi(t)$; also the right hand side is independent of r_1 . Now, letting $r_1 \to 1-0$, we see that $\int_0^1 |Q'(r)|\,dr$ exists and is finite. Hence $Q(r)$ is of bounded variation in $(0,1)$ and the condition for absolute Abel summability is satisfied. Thus the Fourier series (7.1) is summable $|A|$ with sum s .

Dini's condition is not, however, necessary for summability $|A|$, nor even for absolute convergence. An example will be given later of a function whose Fourier series does not satisfy Dini's condition at the origin but is summable $|A|$.

Here it may be observed that summability $|A|$ is a local property: it depends only on the behaviour of the function in the immediate neighbourhood of the point under consideration.

The next two theorems give further tests for summability $|A|$. They are due to Prasad.

THEOREM 11.[1] <u>The Fourier series (7.1) is summable $|A|$ to s at the point x if $\phi(t)$ is absolutely continuous in $(0,\delta)$ where $\delta > 0$</u>.

As in the previous theorem, if we define $Q(r) = P(r) - s$, then

$$\pi . Q(r) = \int_0^\pi \phi(t) . \frac{1-r^2}{(1-2r\cos t + r^2)} dt$$
$$= \int_0^\delta + \int_\delta^\pi \phi(t) . \frac{1-r^2}{(1-2r\cos t + r^2)} dt, \quad (0 < \delta \leq \pi)$$
$$= Q_1(r) + Q_2(r), \quad \text{(say)}.$$

By reasoning similar to that in the previous theorem it can be proved that $Q_2(r)$ is of bounded variation in $(0,1)$. Now suppose that $\phi(t)$ is absolutely continuous in $(0,\delta)$; then $\phi(t)$ is of bounded variation in $(0,\delta)$ and the total variation of $\phi(t)$ in $(0,\delta)$ is $\int_0^\delta |d\phi(t)| = \int_0^\delta |\phi'(t)| dt$. Now

$$(7.9) \qquad Q_1(r) = \int_0^\delta \phi(t) . \frac{1-r^2}{(1-2r\cos t + r^2)} dt$$
$$= \left[\phi(t) . 2\tan^{-1}\left(\frac{1+r}{1-r}\tan\frac{t}{2}\right)\right]_0^\delta - \int_0^\delta 2\tan^{-1}\left(\frac{1+r}{1-r}\tan\frac{t}{2}\right)\phi'(t) dt$$
$$= J(r) - K(r), \text{ (say), so that}$$
$$J(r) = 2\phi(\delta) . \tan^{-1}\left(\frac{1+r}{1-r}\tan\frac{\delta}{2}\right).$$

We can say immediately that $J(r)$ is a function of bounded variation in $(0,1)$ since $\tan^{-1}\left(\frac{1+r}{1-r}\tan\frac{\delta}{2}\right)$ is of bounded variation in $(0,1)$. Choose r_1 so that $0 < r_1 < 1$. Then

$$\int_0^{r_1} |K'(r)| dr = \int_0^{r_1} \left| \int_0^\delta 2\phi'(t) . \frac{\partial}{\partial r}\left\{\tan^{-1}\left(\frac{1+r}{1-r}\tan\frac{t}{2}\right)\right\} dt \right| dr$$

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(1) Prasad, 10

$$\leq \int_0^{r_1} dr \int_0^{\delta} 2\,|\phi'(t)| \left| \frac{\sin t}{1 + r^2 - 2r\cos t} \right| dt$$

$$= \int_0^{\delta} |\phi'(t)|\, V_1(r_1, t)\, dt,$$

where

$V_1(r_1, t) = \int_0^{r_1} 2 \left| \frac{\sin t}{1 + r^2 - 2r\cos t} \right| dr$ and t lies in $(0, \delta)$

The inversion of the order of integration is legitimate here, by Fubini's theorem. Since $0 \leq t \leq \delta < \pi$, $\frac{\sin t}{(1 + r^2 - 2r\cos t)}$ is essentially positive.

Thus

$$V_1(r_1, t) = 2\tan^{-1}\left\{\frac{1 + r_1}{1 - r_1}\tan\frac{t}{2}\right\} - t$$

$$\leq \pi$$

Hence

$$\int_0^{r_1} |K'(r)|\, dr \leq \pi \int_0^{\delta} |\phi'(t)|\, dt.$$

The integral on the right is independent of r_1 and by hypothesis it is finite. By letting $r_1 \to 1-0$ we see that $\int_0^1 |K'(r)|\, dr < \infty$ and $K(r)$ is a function of bounded variation in $(0, 1)$. It follows that $Q_1(r)$ and hence $Q(r)$ is of bounded variation in $(0, 1)$ and the condition for summability $|A|$ of the Fourier series is satisfied.

THEOREM 12.[1] The Fourier series (7.1) is summable $|A|$ at the point x with sum S if the integral $\int_0^{\delta} |\Phi_1(t)/t^2|\, dt$ exists where $\delta > 0$ and $\Phi_1(t)$ is given by the equation $\Phi_1(t) = \int_0^t \phi(u)\, du$.

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(1) Prasad, 10

We will write

$$W(r,t) = \frac{1-r^2}{(1-2r\cos t+r^2)} .$$

Then

$$\frac{\partial W}{\partial r} = \frac{2(1+r^2)\cos t - 4r}{(1-2r\cos t+r^2)^2},$$

$$\frac{\partial W}{\partial t} = \frac{-2r(1-r^2)\sin t}{(1-2r\cos t+r^2)^2},$$

and

$$\frac{\partial^2 W}{\partial r\,\partial t} = \frac{\partial^2 W}{\partial t\,\partial r} = \frac{2\sin t\left[-r^4-1+6r^2-2r(r^2+1)\cos t\right]}{(1-2r\cos t+r^2)^3}.$$

Using the notation of the previous theorem, and replacing δ by $\frac{\pi}{2}$ in line (7.9),

$$\int_0^{r_1} |\varphi_1'(r)|\,dr = \int_0^{r_1}\left|\int_0^{\frac{\pi}{2}} \frac{\partial W}{\partial r}\,\phi(t)\,dt\right| dr,$$

differentiation under the integral sign being legitimate since $\frac{\partial W}{\partial r}$ is a continuous function of r and t for $0 \le t \le \frac{\pi}{2}$ and $0 \le r \le r_1 < 1$. Integrating by parts,

$$\int_0^{r_1} |\varphi_1'(r)|\,dr = \int_0^{r_1}\left| \left[\Phi_1(t)\frac{\partial W}{\partial r}\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \Phi_1(t)\cdot \frac{\partial^2 W}{\partial t\,\partial r}\,dt\right| dr$$

$$\le A + \int_0^{r_1} dr \int_0^{\frac{\pi}{2}} |\Phi_1(t)| \left|\frac{\partial^2 W}{\partial t\,\partial r}\right| dt,$$

where A is a constant. By Fubini's theorem it is permissible to invert the order of integration in the repeated integral.

Hence

$$\int_0^{r_1} |\varphi_1'(r)|\,dr \le A + \int_0^{\frac{\pi}{2}} |\Phi_1(t)|\,dt \int_0^{r_1} \left|\frac{\partial^2 W}{\partial t\,\partial r}\right| dr.$$

Our object now is to show that $\int_0^{r_1}\left|\frac{\partial^2 W}{\partial t\,\partial r}\right| dr < \frac{K}{t^2}$ where K is some positive constant and $0 \le t \le \frac{\pi}{2}$. If we write $t_1 = 2\sin^{-1}\frac{1-r_1}{\sqrt{2(1+r_1^2)}}$, then $0 \le t_1 \le \frac{\pi}{2}$; thus when $0 \le t \le t_1$, and $0 \le r \le r_1$, $-\frac{\partial^2 W}{\partial t\,\partial r}$ is essentially positive. For

$$-\frac{\partial^2 W}{\partial t\,\partial r} \ge \frac{2\sin t\left[r^4-6r^2+1+2r(r^2+1)\left\{1-\frac{2(1-r_1)^2}{2(1+r_1^2)}\right\}\right]}{(1-2r\cos t+r^2)^3}$$

$$\geq \frac{2\sin t}{(1-r)^6}\left[(1-r^2)^2 + \frac{4r}{(1+r_1^2)}\left\{(r^2+1)r_1 - (1+r_1^2)r\right\}\right]$$

$$\geq \frac{2\sin t}{(1-r)^6}\left[(1-r^2)^2 + \frac{4r(r_1-r)(1-rr_1)}{(1+r_1^2)}\right]$$

$$\geq 0.$$

Thus, for $0 \leq t \leq t_1 \leq \frac{\pi}{2}$,

$$\int_0^{r_1}\left|\frac{\partial^2 W}{\partial t\,\partial r}\right| dr = \int_0^{r_1} -\frac{\partial^2 W}{\partial r\,\partial t}\,dr$$

$$= \left[-\frac{\partial W}{\partial t}\right]_0^{r_1}$$

$$= \frac{2r_1(1-r_1^2)\sin t}{(1-2r_1\cos t + r_1^2)^2}.$$

Now, since

$$(1-2r_1\cos t + r_1^2)^2 = \left\{(1-r_1)^2 + 4r_1\sin^2\frac{t}{2}\right\}^2$$

$$= (1-r_1)^4 + 8r_1(1-r_1)^2\sin^2\frac{t}{2} + 16r_1^2\sin^4\frac{t}{2}$$

$$\geq 8r_1(1-r_1)^2\sin^2\frac{t}{2},$$

it follows that, for $0 \leq t \leq t_1$,

$$\int_0^{r_1}\left|\frac{\partial^2 W}{\partial t\,\partial r}\right| dr \leq \frac{2r_1(1-r_1)(1+r_1)\sin t}{8r_1(1-r_1)^2\sin^2\frac{t}{2}}$$

$$\leq \frac{(1+r_1)}{(1-r_1)}\cdot\frac{\sin t_1}{4\sin^2\frac{t}{2}}$$

$$\leq \frac{(1+r_1)}{(1-r_1)}\cdot\frac{(1-r_1^2)}{(1+r_1^2)}\cdot\frac{\pi^2}{4t^2}$$

$$\leq \frac{\pi^2}{2t^2}.$$

To obtain a similar inequality for $t_1 \leq t \leq \frac{\pi}{2}$, let $\cos t = \dfrac{6y^2-y^4-1}{2y(1+y^2)}$ where $\sqrt{2}-1 \leq y < 1$. Then

$$-\frac{\partial^2 W}{\partial t\,\partial r} = \frac{2\sin t\left[r^4-6r^2+1+2r(r^2+1)\frac{(6y^2-y^4-1)}{2y(1+y^2)}\right]}{(1-2r\cos t + r^2)^3}$$

$$= \frac{2\sin t\,(y-r)(1-yr)\left[(1+y^2)(1+r^2)+8ry\right]}{y\,(1+y^2)(1-2r\cos t+r^2)^3}.$$

Thus $-\frac{\partial^2 W}{\partial t\,\partial r} \geq 0$ for $0 \leq r \leq y$ and $\frac{\partial^2 W}{\partial t\,\partial r} \geq 0$ for $y \leq r < 1$.

In this case,

$$\int_0^{r_1}\left|\frac{\partial^2 W}{\partial t\,\partial r}\right| dr \leq \int_0^1 \left|\frac{\partial^2 W}{\partial t\,\partial r}\right| dr$$

$$= \left(\int_0^{\sqrt{2}-1} + \int_{\sqrt{2}-1}^{y} - \int_y^1\right) - \frac{\partial^2 W}{\partial r\,\partial t}\,dr$$

$$= \frac{4y(1-y^2)\sin t}{(1-2y\cos t+y^2)^2}.$$

Since $1-2y\cos t+y^2 = \frac{2(1-y^2)^2}{(1+y^2)}$, it follows that

$$\int_0^{r_1}\left|\frac{\partial^2 W}{\partial t\,\partial r}\right| dr \leq \frac{4y(1-y^2)\sin t}{\left\{(1-y)^2+4y\sin^2\frac{t}{2}\right\}^{\frac{3}{2}}}\;\frac{\sqrt{1+y^2}}{\sqrt{2}(1-y^2)}$$

$$\leq \frac{8\sin\frac{t}{2}\cos\frac{t}{2}}{8y^{\frac{3}{2}}\sin^3\frac{t}{2}}\cdot\frac{y\sqrt{1+y^2}}{\sqrt{2}}$$

$$< D\,\frac{\cos\frac{t}{2}}{\sin^2\frac{t}{2}}$$, (where D is a positive constant)

$$< D\,\frac{\pi^2}{t^2}$$, (by Jordan's inequality).

Let $K = \max\left(\frac{\pi^2}{2}, D\pi^2\right)$. Then, combining these results we have

$$\int_0^{r_1}|Q_1'(r)|\,dr \leq A + \left(\int_0^{t_1} + \int_{t_1}^{\frac{\pi}{2}}\right)|\Phi_1(t)|\,dt \int_0^{r_1}\left|\frac{\partial^2 W}{\partial t\,\partial r}\right| dr$$

$$< A + K\int_0^{\frac{\pi}{2}}\left|\frac{\Phi_1(t)}{t^2}\right| dt$$

By hypothesis $\int_0^{\delta}\frac{|\Phi_1(t)|}{t^2}\,dt$ exists for some $\delta > 0$. It follows that $\int_0^{\frac{\pi}{2}}\frac{|\Phi_1(t)|}{t^2}\,dt$ exists and hence that $Q_1(r)$ is of bounded variation in $(0,1)$, and the condition for summability $|A|$ of the Fourier series is again satisfied.

The next group of theorems, due to Bosanquet, represents a substantial development in the theory of absolute summability as applied to Fourier series. They actually include as special cases the three preceding theorems.

THEOREM 13[1]. If $\phi_\alpha(t)$ is of bounded variation in $(0,\pi)$, then the Fourier series of $f(t)$ is summable $|C,\beta|$ at the point x where $\beta > \alpha \geqslant 0$.

THEOREM 14.[2] If $\alpha \geqslant 0$ and the Fourier series of $f(t)$ is summable $|C,\alpha|$ at the point x, then $\phi_\beta(t)$ is of bounded variation in $(0,\pi)$ where $\beta > \alpha+1$.

Combining these two statements we see that in order that a Fourier series be summable $|C|$ at the point x it is necessary and sufficient that $\phi_\alpha(t)$ should be of bounded variation in $(0,\pi)$ for some $\alpha \geqslant 0$. The proofs of these theorems in the general case were established by Bosanquet in 1936, and are here omitted. In the special case where $\alpha = 0$ the theorems take the simpler form:-

THEOREM 13'[3] If $\phi(t)$ is of bounded variation in $(0,\pi)$, then the Fourier series of $f(t)$ is summable $|C,\beta|$ at the point x for every $\beta > 0$.

THEOREM 14'[4] If the Fourier series of $f(t)$ is absolutely convergent at the point x, then $\phi_{1+\delta}(t)$ is of bounded variation in $(0,\pi)$ for every $\delta > 0$.

(1) Bosanquet, 2.
(2) Bosanquet, 2.
(3) Bosanquet, 1.
(4) Bosanquet, 1.

The following lemma is required for the proofs of Theorem 13' and Theorem 14' :-

LEMMA 1. If $\alpha \geq 0$ and $\phi_\alpha(t)$ is of bounded variation in an interval $(0, \eta)$, then $\phi_\beta(t)$ is also of bounded variation in $(0,\eta)$ for every $\beta > \alpha$.

Since any function of bounded variation may be expressed as the difference of two positive non-decreasing functions it is sufficient to consider the case when $\phi_\alpha(t)$ is positive and non-decreasing. Further it is sufficient to suppose that $0 \leq \alpha < \beta \leq \alpha + 1$; the result for the general case will then follow by induction.

Choose t and $t+h$ so that $0 < t < t+h < \eta$. We shall show that $\phi_\beta(t+h) - \phi_\beta(t) \geq 0$.

$$\begin{aligned}\phi_\beta(t) &= \frac{\Gamma(\beta+1)}{t^\beta} \cdot \Phi_\beta(t) \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha)} \cdot \frac{1}{t^\beta} \int_0^t (t-u)^{\beta-\alpha-1} \Phi_\alpha(u)\, du \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta-\alpha)} \cdot \frac{1}{t^\beta} \int_0^t (t-u)^{\beta-\alpha-1} u^\alpha \phi_\alpha(u)\, du.\end{aligned}$$

Then

$$\frac{\Gamma(\alpha+1)\Gamma(\beta-\alpha)}{\Gamma(\beta+1)} \left\{ \phi_\beta(t+h) - \phi_\beta(t) \right\} = T_1 + T_2$$

where

$$T_1 = \int_0^t \left\{ \frac{(t+h-u)^{\beta-\alpha-1} u^\alpha}{(t+h)^\beta} - \frac{(t-u)^{\beta-\alpha-1} u^\alpha}{t^\beta} \right\} \phi_\alpha(u)\, du$$

and

$$T_2 = \int_t^{t+h} \frac{(t+h-u)^{\beta-\alpha-1} u^\alpha}{(t+h)^\beta} \cdot \phi_\alpha(u)\, du .$$

Applying the second Mean Value Theorem to the first of these integrals,

$$T_1 = \phi_\alpha(t) \int_\xi^t \left\{ \frac{(t+h-u)^{\beta-\alpha-1}}{(t+h)^\beta} - \frac{(t-u)^{\beta-\alpha-1}}{t^\beta} \right\} u^\alpha\, du ,$$

where $0 < \xi < t$.

Also

$$T_2 \geqslant \phi_\alpha(t)\int_t^{t+h} \frac{(t+h-u)^{\beta-\alpha-1}}{(t+h)^\beta} u^\alpha du$$

Hence

$$T_1 + T_2 \geqslant \phi_\alpha(t)\left\{\int_\xi^{t+h} \frac{(t+h-u)^{\beta-\alpha-1}}{(t+h)^\beta} u^\alpha du - \int_\xi^t \frac{(t-u)^{\beta-\alpha-1}}{t^\beta} u^\alpha du\right\}$$

$$= \phi_\alpha(t)\left\{\int_0^{t+h} \frac{(t+h-u)^{\beta-\alpha-1}}{(t+h)^\beta} u^\alpha du - \int_0^t \frac{(t-u)^{\beta-\alpha-1}}{t^\beta} u^\alpha du\right\}$$

$$+ \phi_\alpha(t)\int_0^\xi u^\alpha\left\{\frac{(t-u)^{\beta-\alpha-1}}{t^\beta} - \frac{(t+h-u)^{\beta-\alpha-1}}{(t+h)^\beta}\right\} du$$

The first two of these integrals are equal and so cancel out.

Also

$\left(\frac{t-u}{t}\right)^{\beta-\alpha-1} \geqslant \left(\frac{t+h-u}{t+h}\right)^{\beta-\alpha-1}$ and $t^{-\alpha-1} \geqslant (t+h)^{-\alpha-1}$ since $\beta-\alpha-1 < 0$ and $0 \leq u \leq \xi < t < t+h$. Hence the integrand in the last integral is non-negative. It follows that

$$T_1 + T_2 \geqslant 0$$

Thus $\phi_\beta(t+h) - \phi_\beta(t) \geqslant 0$ and $\phi_\beta(t)$ is a non-decreasing bounded function. The result stated in the lemma is immediately deducible.

<u>Proof of Theorem 13'</u>.

Let $A_n = a_n \cos nx + b_n \sin nx$, $(n = 1, 2, \ldots\ldots)$, $A_0 = \frac{1}{2}a_0$, and let S_n^α, T_n^α, $\sigma^\alpha(n,t)$ denote the n'th Cesaro means of order α of the sequences $S_n = A_0 + A_1 + \ldots\ldots + A_n$, $T_n = nA_n$, and $\sigma(n,t) = \frac{2}{\pi}\sin nt$. Then

$$nA_n = \frac{n}{\pi}\int_{-\pi}^{\pi} f(t)\cos n(t-x)\,dt$$

$$= \frac{n}{\pi}\int_0^{\pi}\{f(x+t) + f(x-t)\}\cos nt\,dt \text{ , by a suitable}$$

change of variable and the periodicity of $f(t)$. Thus

$$nA_n = \frac{2}{\pi}\Big[(\phi(t)+s)\sin nt\Big]_0^\pi - \frac{2}{\pi}\int_0^\pi \sin nt\,d\phi(t)$$

$$= \int_0^\pi -\frac{2\sin nt}{\pi}\,d\phi(t).$$

Taking the Cesaro mean of order β, it follows that

$$\tau_n^\beta = \frac{1}{A_n^\beta}\sum_{r=0}^{n}\left\{A_{n-r}^{\beta-1}\int_0^\pi -\frac{2\sin rt}{\pi}\,d\phi(t)\right\}$$
$$= -\int_0^\pi \frac{1}{A_n^\beta}\left\{\sum_{r=0}^{n} A_{n-r}^{\beta-1}\cdot\frac{2\sin rt}{\pi}\right\}d\phi(t)$$
$$= -\int_0^\pi \sigma^\beta(n,t)\,d\phi(t).$$

We now proceed to prove the convergence of the series $\sum_1^\infty \frac{|\tau_n^\beta|}{n}$ for $0<\beta<1$ and to deduce from this the summability $|C,\beta|$ of the original Fourier series. We have

$$\sum_1^\infty \frac{|\tau_n^\beta|}{n} \leq \sum_1^\infty \int_0^\pi \frac{|\sigma^\beta(n,t)|}{n}\,|d\phi(t)|$$
$$= \int_0^\pi \sum_1^\infty \frac{1}{n}\left|\sigma^\beta(n,t)\right|\,|d\phi(t)|$$

We require to show that the integral on the right is finite. This follows from the following inequalities, true for $n \geq 1$, $0<t<\pi$, $0<\beta<1$, and proved at the end of the theorem:-

$$(7.10)\qquad \begin{aligned} |\sigma^\beta(n,t)| &\leq Bnt, \\ |\sigma^\beta(n,t)| &\leq Bn^{-\beta}t^{-\beta}, \end{aligned}$$

where B is positive and independent of n and t. Thus

$$\sum_1^\infty \frac{1}{n}\left|\sigma^\beta(n,t)\right| = \sum_{n=1}^{[t^{-1}]}\frac{1}{n}O(nt) + \sum_{n=[t^{-1}]+1}^{\infty}\frac{1}{n}O(n^{-\beta}t^{-\beta})$$
$$= \sum_{n=1}^{[t^{-1}]} O(t) + t\sum_{n=[t^{-1}]+1}^{\infty} O\left\{\frac{1}{(nt)^{1+\beta}}\right\}$$
$$= O(1) + O(1)$$

uniformly for $0 < t < \pi$. It follows that

$$\sum_{1}^{\infty} \frac{|\tau_n^\beta|}{n} = O(1) \int_0^\pi |d\phi(t)|.$$

Since $\phi(t)$ is of bounded variation $\int_0^\pi |d\phi(t)| < \infty$.
Thus $\sum_{1}^{\infty} \frac{|\tau_n^\beta|}{n}$ converges to a finite limit and the Fourier series is summable $|C,\beta|$, $0 < \beta < 1$. Since summability $|C,\beta|$ implies summability $|C,\gamma|$ for all $\gamma > \beta$, the complete result stated in the enunciation of the theorem is true.

It remains to prove the inequalities of (7.10).

Now

$$\sigma^\beta(n,t) = \sum_{r=0}^{n} \frac{A_{n-r}^{\beta-1} \cdot \frac{2}{\pi} \sin rt}{A_n^\beta} \qquad (0 > \beta-1 > -1)$$

$$= \frac{2}{\pi A_n^\beta} \sum_{r=0}^{n} A_{n-r}^{\beta-1}\, I(e^{irt})$$

$$= I\left[\frac{2e^{int}}{\pi A_n^\beta} \sum_{r=0}^{n} A_r^{\beta-1} e^{-irt}\right]$$

$$= I\left[\frac{2e^{int}}{\pi A_n^\beta}\left\{(1-e^{-it})^{-\beta} - \sum_{r=n+1}^{\infty} A_r^{\beta-1} e^{-irt}\right\}\right].$$

Since $A_n^{\beta-1}$ decreases steadily to zero when $0 > \beta-1 > -1$, the infinite series converges for $t \neq 0$, by Dirichlet's test, and the modulus of its sum does not exceed $4A_{n+1}^{\beta-1}/|1-e^{-it}|$.
Thus, when $0 < t \leq \pi$,

$$|\sigma^\beta(n,t)| \leq \frac{2}{\pi A_n^\beta}\left\{\left(2\sin\frac{t}{2}\right)^{-\beta} + \frac{4A_{n+1}^{\beta-1}}{2\sin\frac{t}{2}}\right\}$$

$$\leq \frac{2}{\pi A_n^\beta}\left\{2^{-\beta}\left(\frac{\pi}{t}\right)^{\beta} + \frac{4A_{n+1}^{\beta-1}\cdot\pi}{2t}\right\},$$

using Jordan's inequality. Thus

$$|\sigma^\beta(n,t)| \leq D_1 n^{-\beta} t^{-\beta} + D_2 n^{-1} t^{-1}$$

where D_1 and D_2 are independent of n and t. Now for $nt \geq 1$, $\frac{1}{nt} \leq \frac{1}{n^\beta t^\beta}$, so that

$$|\sigma^\beta(n,t)| \leq (D_1 + D_2)\, n^{-\beta} t^{-\beta}$$
$$\leq (D_1 + D_2)\, nt.$$

To obtain the inequalities for $nt < 1$ we observe that $\sin\theta \leq \theta$ whenever $0 \leq \theta \leq \frac{\pi}{2}$, so that

$$|\sigma^\beta(n,t)| \leq \sum_{r=0}^{n} \frac{A^{\beta-1}_{n-r}}{A^\beta_n} \cdot \frac{2}{\pi} rt \quad (nt < 1)$$
$$\leq \frac{2}{\pi} nt$$
$$\leq \frac{2}{\pi} n^{-\beta} t^{-\beta}.$$

Writing $B = \max\left(\overline{D_1 + D_2}, \frac{2}{\pi}\right)$ we obtain the complete result stated in (7.10).

Proof of Theorem 14'.

Using the formula (7.5) with α replaced by $(\alpha - 1)$ and β replaced by 1,

$$\overline{\Phi}_\alpha(t) = \int_0^t \overline{\Phi}_{\alpha-1}(u)\,du, \quad (\alpha - 1 \geq 0,\ 0 < t \leq \pi).$$

Thus $\overline{\Phi}_\alpha(t)$ is a Lebesgue integral if $\alpha \geq 1$ and must be absolutely continuous in $(0, \pi)$. Also $\frac{1}{t^\alpha}$ is absolutely continuous in any interval which does not include the origin, for $\alpha > 0$. Since the product of two such functions is itself absolutely continuous, it follows that $\phi_\alpha(t) = \frac{\Gamma(\alpha+1)}{t^\alpha} \cdot \overline{\Phi}_\alpha(t)$ is absolutely continuous in (ε, π) where $\varepsilon > 0$. We require to show that $\int_0^\pi |d\phi_{1+\delta}(t)|$ is finite for $\delta > 0$; appealing to the absolute continuity of $\phi_\alpha(t)$ this is equivalent to proving that $\int_0^\pi |\phi'_{1+\delta}(t)|\,dt < \infty$. In view of Lemma 1, it is sufficient to consider the case when $0 < \delta < 1$.

It is convenient here to replace the variable u in the formula (7.4) by $\frac{u}{t}$, obtaining $\phi_\alpha(t) = \alpha\int_0^1 (1-u)^{\alpha-1}\phi(tu)\,du$

Let $\gamma_\alpha(n,t)$ denote $\alpha\int_0^1 (1-u)^{\alpha-1}\cos(ntu)\,du$: that is the fractional mean of order α of $\cos nt$. Now

$$f(x+t) \sim \tfrac{1}{2}a_0 + \sum_1^\infty \{a_n \cos n(x+t) + b_n \sin n(x+t)\},$$

and

$$f(x-t) \sim \tfrac{1}{2}a_0 + \sum_1^\infty \{a_n \cos n(x-t) + b_n \sin n(x-t)\},$$

so by a suitable choice of s we can obtain

$$\phi(t) \sim \sum_1^\infty A_n \cos nt \quad \text{where} \quad A_n = a_n \cos nx + b_n \sin nx.$$

Since $\sum A_n$ converges absolutely,

$$\begin{aligned}\phi_{1+\delta}(t) &= (1+\delta)\int_0^1 (1-u)^\delta \sum_1^\infty A_n \cos(ntu)\,du \\ &= \sum_1^\infty A_n (1+\delta)\int_0^1 (1-u)^\delta \cos(ntu)\,du \\ &= \sum_1^\infty A_n . \gamma_{1+\delta}(nt).\end{aligned}$$

Assuming that term-by-term differentiation of this series is valid, it follows that

(7.11) $$\phi'_{1+\delta}(t) = \sum_1^\infty A_n . n\gamma'_{1+\delta}(nt).$$

Hence

$$\begin{aligned}\int_0^\pi |\phi'_{1+\delta}(t)|\,dt &= \int_0^\pi \Big|\sum_1^\infty A_n . n\gamma'_{1+\delta}(nt)\Big|\,dt \\ &\leq \int_0^\pi \sum_1^\infty |A_n| . n\,|\gamma'_{1+\delta}(nt)|\,dt \\ &= \sum_1^\infty |A_n| . n\int_0^\pi |\gamma'_{1+\delta}(nt)|\,dt.\end{aligned}$$

Since[1]

(7.12) $$|\gamma'_{1+\delta}(nt)| \leq B_1, \quad \text{and} \quad |\gamma'_{1+\delta}(nt)| \leq B_2 (nt)^{-1-\delta},$$

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(1) see Hobson, Theory of functions of a real variable, p.565, for discussion on Young's function.

where B_1 and B_2 are independent of n and t,

$$n\int_0^\pi |\gamma'_{1+\delta}(nt)|\,dt = n\left(\int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\pi}\right)|\gamma'_{1+\delta}(nt)|\,dt$$

$$\leq nB_1 . \frac{1}{n} + \frac{nB_2}{n^{1+\delta}}\left[\frac{1}{\delta t^\delta}\right]_\pi^{\frac{1}{n}}$$

$$= O(1) + O(1)$$

uniformly for $n \geq 1$ since $\delta > 0$. Hence, since $\sum_1^\infty |A_n| < \infty$, $\int_0^\pi |\phi'_{1+\delta}(t)|\,dt < \infty$ and the result follows.

It remains to justify the differentiation in (7.11). This is permissible since, from the convergence of $\sum |A_n|$ and the second inequality of (7.12) the derived series is uniformly convergent for $t \geq \epsilon > 0$.

It would appear from the statements of Theorems 13 and 14 that summability $|C, \beta|$ of a Fourier series depends on the behaviour of the function throughout the whole interval $(-\pi, \pi)$. Actually, for $\beta > 1$ this is not true; for it can be shown that, for $\alpha \geq 1$, $\phi_\alpha(t)$ which is the Lebesgue integral of an absolutely integrable function is necessarily of bounded variation in any interval (η, π), so the sufficient condition for summability $|C, \beta|$ where $\beta > 1$ can be reduced to the condition that $\phi_\alpha(t)$ should be of bounded variation in $(0, \eta)$ where $\beta > \alpha \geq 1$ and $0 < \eta \leq \pi$. Thus, for $\beta > 1$, summability $|C, \beta|$ is a local property. It has been shown by means of an example (1) that summability $|C, 1|$ of a Fourier series is not a local property and it may be inferred that

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(1) Bosanquet and Kestelman, 16.

summability $|C,\beta|$ where $0<\beta\leq 1$ is not a local property.

It has been shown earlier in this dissertation that summability $|C,\delta|$ for any $\delta>0$ implies summability $|A|$. From Theorems 13 and 14 we can therefore deduce that a necessary and sufficient condition for the summability $|A|$ of a Fourier series is that $\phi_\alpha(t)$ should be of bounded variation in $(0,\pi)$ for some $\alpha\geq 0$. The question now arises as to how far the sufficient conditions given in Theorems 10 to 12 and the deduction from Theorem 13 are necessary for summability $|A|$, and also how far the conditions are independent of each other.

If $\phi(t)$ is absolutely continuous in $(0,\eta)$ where $\eta>0$, then by Lemma 1, $\phi_1(t)$ is of bounded variation in $(0,\eta)$; also $\phi_1(t)$ is of bounded variation in (η,π). Thus, if the conditions for Prasad's first theorem are satisfied the conditions for Bosanquet's first result are necessarily satisfied: that is, Prasad's result is included in Bosanquet's result.

If $\phi(t)$ satisfies the condition $\int_0^\delta \frac{|\Phi_1(t)|}{t^2}dt<\infty$ given in Theorem 12, we can show that $\phi_2(t)$ is necessarily of bounded variation in $(0,\delta)$. For, let

$$\Psi(t)=\int_0^t \frac{\Phi_1(u)}{u^2}du,\ (0<t\leq\delta).$$

Then $|\Psi(t)|<\infty$ and $\Psi(t)$ which is the Lebesgue integral of an absolutely integrable function is of bounded variation in $(0,\delta)$. By Lemma 1, $\phi_1(t)=\frac{\Phi_1(t)}{t}$ and $\phi_2(t)=2\frac{\Phi_2(t)}{t^2}$ are also of bounded variation in $(0,\delta)$.

Now

$$\phi_2(t) = \frac{2}{t^2}\int_0^t \frac{\Phi_1(u)}{u^2}\cdot u^2\,du$$

$$= \frac{2}{t^2}\Big[\psi(u).u^2\Big]_0^t - \frac{4}{t^2}\int_0^t \psi(u).u\,du$$

$$= 2\psi(t) - \frac{4}{t^2}\int_0^t \psi(u).u\,du.$$

Also

$$\frac{4}{t^2}\int_0^t \psi(u).u\,du = \frac{4}{t^2}\Big[u\,\Psi_1(u)\Big]_0^t - \frac{4}{t^2}\int_0^t \Psi_1(u)\,du$$

$$= 4\psi_1(t) - 2\psi_2(t).$$

Hence

$$\phi_2(t) = 2\psi(t) - 4\psi_1(t) + 2\psi_2(t).$$

It follows that $\phi_2(t)$ is of bounded variation in $(0,\delta)$. Thus a function which satisfies Prasad's second condition for summability $|A|$ also satisfies the condition given in Bosanquet's first theorem.

if $\phi(t)$ satisfies Whittaker's condition $\int_0^\delta \frac{|\phi(t)|}{t}\,dt < \infty$ for summability $|A|$, then it also satisfies Prasad's condition $\int_0^\delta \frac{|\Phi_1(t)|}{t^2}\,dt < \infty$. For, since

$$\frac{1}{t}\int_0^t |\phi(u)|\,du \leq \int_0^t \frac{|\phi(u)|}{u}\,du, \quad (0 < t \leq \delta)$$

$$\to 0 \text{ as } t \to 0,$$

$$\int_0^t |\phi(u)|\,du = o(t) \text{ as } t \to 0.$$

If we write

$$\overline{\Phi}_1(t) = \int_0^t |\phi(u)|\,du,$$

Then

$$\int_\varepsilon^\delta \frac{|\phi(u)|}{u}\,du = \Big[\frac{1}{u}\,\overline{\Phi}_1(u)\Big]_\varepsilon^\delta + \int_\varepsilon^\delta \frac{\overline{\Phi}_1(u)}{u^2}\,du.$$

Thus, letting $\varepsilon \to 0$,

$$\int_0^\delta \frac{\overline{\Phi}_1(u)}{u^2}\,du = \int_0^\delta \frac{|\phi(u)|}{u}\,du - \frac{1}{\delta}\,\overline{\Phi}_1(\delta)$$

$$< \infty.$$

Now

$$|\Phi_1(t)| \leq \overline{\Phi}_1(t).$$

Hence

$$\int_0^\delta \frac{|\Phi_1(u)|}{u^2}\,du < \infty.$$

It follows that Whittaker's result is included in Prasad's second result.

If $\phi(t)$ satisfies the condition $\int_0^\delta \frac{|\phi(t)|}{t}\,dt < \infty$, we can show that $\phi_1(t)$ is of bounded variation in $(0,\delta)$. Let $\theta(t)$ denote $\int_0^t \frac{\phi(u)}{u}\,du$ and let $\theta_1(t) = \frac{1}{t}\int_0^t \theta(u)\,du$. Then $\theta(t)$ which is the Lebesgue integral of an absolutely integrable function is of bounded variation in $(0,\delta)$, and by Lemma 1 the mean value $\theta_1(t)$ is also of bounded variation in $(0,\delta)$. Now

$$\phi_1(t) = \frac{1}{t}\int_0^t \phi(u)\,du$$

$$= \frac{1}{t}\Big[u\,\theta(u)\Big]_0^t - \frac{1}{t}\int_0^t \theta(u)\,du$$

$$= \theta(t) - \theta_1(t).$$

Hence $\phi_1(t)$ is of bounded variation in $(0,\delta)$ and Whittaker's result is included in Bosanquet's first theorem.

The following example[1] of a Function whose

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(1) see Titchmarsh, Theory of Functions, p.409, where a similar example is used in the comparison of convergence tests for Fourier series.

Fourier series is summable $|A|$ at the origin but for which Dini's condition is not satisfied illustrates the fact that Whittaker's condition for summability $|A|$ is sufficient but not necessary. Let

$$\begin{cases} f(x) = \dfrac{1}{\log|x|}, & |x| \le \tfrac{1}{2},\ x \neq 0, \\ f(x) = 0, & \tfrac{1}{2} < |x| \le \pi, \\ f(0) = 0, & \end{cases}$$

and consider the behaviour of the Fourier series of $f(x)$ at the origin. Here $\phi(u) = \dfrac{1}{\log|u|}$, $(0 < u \le \tfrac{1}{2})$ and $\phi(u) = 0$, $(\tfrac{1}{2} < u \le \pi)$ so that $\phi(u)$ is of bounded variation in $(0, \pi)$ and, by Theorem 13' the Fourier series is summable $|C|$ and hence summable $|A|$.

But Dini's condition is not satisfied; for, if $0 < \varepsilon < \delta$,

$$\begin{aligned} \int_\varepsilon^\delta \frac{|\phi(u)|}{u}\,du &= \int_\varepsilon^\delta \frac{1}{u\log u}\,du, \quad (\delta \le \tfrac{1}{2}) \\ &= \Big[\log|\log u|\Big]_\varepsilon^\delta \\ &= \log|\log\delta| - \log|\log\varepsilon| \\ &\not\to \text{ a finite limit as } \varepsilon \to 0. \end{aligned}$$

We now give an example of a function for which the Fourier series is summable $|A|$ at the origin but $\phi(t)$ is not absolutely continuous in any interval $(0, \delta)$. Consider

$$\begin{cases} f(x) = x \sin \dfrac{1}{x}, \quad x \neq 0,\ |x| \le \pi, \\ f(0) = 0. \end{cases}$$

At $x = 0$, $\phi(u) = u \sin \frac{1}{u}$ and it is well-known that $\phi(u)$ is not of bounded variation and so cannot be absolutely continuous in $(0, \delta)$. On the other hand

$$\int_0^\delta \frac{|\phi(u)|}{u}\,du \le \int_0^\delta 1\,du = \delta,$$

so that, by Whittaker's theorem the series is summable $|A|$. Thus the condition given in Prasad's first theorem is sufficient but not necessary for summability $|A|$.

Finally, we give an example of a function which does not satisfy the condition $\int_0^{\delta} \frac{|\Phi_1(t)|}{t^2}\,dt < \infty$ but for which the Fourier series is summable $|A|$. Consider the function

$$\begin{cases} f(x) = \dfrac{1}{\log|x|} - \dfrac{1}{(\log|x|)^2}, & |x| \le \frac{1}{2},\ x \ne 0, \\ f(x) = 0, & \frac{1}{2} < |x| \le \pi, \\ f(0) = 0, & \end{cases}$$

and consider the behaviourof the Fourier serie s at the origin. Here $\phi(u) = \dfrac{1}{\log|u|} - \dfrac{1}{(\log|u|)^2}$, $(0 < u \le \frac{1}{2})$ and $\phi(u) = 0$, $(\frac{1}{2} < u \le \pi)$ so that $\phi(u)$ is of bounded variation in $(0, \pi)$ and by Theorem 13' the Fourier series is summable $|C|$ and hence summable $|A|$. Now

$$\frac{\Phi_1(t)}{t^2} = \frac{1}{t^2}\int_0^t \phi(u)\,du$$

$$= \frac{1}{t^2}\left[\frac{u}{\log|u|}\right]_0^t$$

$$= \frac{1}{t\log|t|},$$

and this function has not a finite Lebesgue integral in $(0,\delta)$. We conclude, then, that the condition given in Prasad's second theorem is sufficient but not necessary for summability $|A|$.

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