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GRAPH IMPERFECTION WITH A CO-SITE CONSTRAINT*

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Abstract. We are interested in a version of graph coloring where there is a "co-site" constraint value k. Given a graph G with a nonnegative integral demand x_v at each node v, we must assign x_v positive integers (colors) to each node v such that the same integer is never assigned to adjacent nodes, and two distinct integers assigned to a single node differ by at least k. The aim is to minimize the span, that is, the largest integer assigned to a node. This problem is motivated by radio channel assignment where one has to assign frequencies to transmitters so as to avoid interference. We compare the span with a clique-based lower bound when some of the demands are large. We introduce the relevant graph invariant, the k-imperfection ratio, give equivalent definitions, and investigate some of its properties. The k-imperfection ratio is always at least 1: we call a graph k-perfect when it equals 1. Then 1-perfect is the same as perfect, and we see that for many classes of perfect graphs, each graph in the class is k-perfect for all k. These classes include comparability graphs, co-comparability graphs, and line-graphs of bipartite graphs.

Key words. imperfection ratio, generalized graph coloring, perfect graphs, channel assignment

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1. Introduction.

We are interested in a problem motivated by radio channel assignment in cellular networks, where one has to assign sets of frequencies or channels to transmitters so as to satisfy the local demand for channels at every transmitter, to avoid unacceptable interference, and to use the minimum amount of the spectrum; see, for example, [13], [14], or [17]. We assume that the interference is acceptable if any two channels assigned to a pair of potentially interfering transmitters are different and the distance (in the spectrum) between two distinct channels assigned to the same transmitter is at least k, where the positive integer k is a given constant which is called the *co-site constraint value*. Typically k will be a small positive integer. We are particularly interested in this problem when the demand for channels at some of the transmitters is large. This is not only because this case is important in practical situations, but even more since it leads to significant simplifications that reveal interesting structure.

If we represent colors by positive integers $1, 2, \ldots$, then this problem translates to coloring the nodes of a weighted graph $G = (V, E)$ with nonnegative integral weight vector $\mathbf{x} = (x_v : v \in V)$ in such a way that x_v colors are assigned to each node v, two colors assigned to adjacent nodes are different, and two distinct colors assigned to the same node differ by at least the co-site constraint value k. Such a coloring is called *k-feasible* for G and $\mathbf{x}$. The objective is to minimize the largest number used. We define $\mathrm{span}_k(G, \mathbf{x})$ to be the minimum value of the largest number used, over all k-feasible assignments for G and $\mathbf{x}$. Observe that $\mathrm{span}_1(G, \mathbf{1})$ equals the chromatic number $\chi(G)$ of the graph G.

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We want to compare $\mathrm{span}_k(G, \mathbf{x})$ with a clique-based lower bound, when some of the demands are large. To do this we set

$$\omega_k(G, \mathbf{x}) = \max \{\mathrm{span}_k(K, \mathbf{x}) : K \text{ is a clique in } G\},$$

where we abuse notation and use $\mathbf{x}$ also for its restriction to subgraphs of G. It is known [9] that for a clique K, there is a simple formula for $\mathrm{span}_k(K, \mathbf{x})$; see (2.1) below. Observe that $\omega_k(G, \mathbf{x})$ is always at least $\omega(G, \mathbf{x})$, where $\omega(G, \mathbf{x})$ is the maximum of $\sum_{v \in V(K)} x_v$ over all cliques K of G.

Given a weight vector $\mathbf{x}$ of a graph G, that is, a nonnegative vector indexed by the nodes of G, we let $x_{\max}$ denote the maximum value of x_v over all the nodes v. We set

$$s_k^j(G) = \max \left\{ \frac{\mathrm{span}_k(G, \mathbf{x})}{\omega_k(G, \mathbf{x})} : x_{\max} = j \right\}, \tag{1.1}$$

where the maximum is over all integral weight vectors with $x_{\max} = j$. Observe that $s_k^j(G) \geq 1$ by definition, and $s_k^j(G) = 1$ if G is a complete graph. Consider the case $k = 1$: it is known [7] that $\lim_{j \to \infty} s_1^j(G)$ exists and is the imperfection ratio, which we discuss below. We will see that the corresponding result holds for each positive integral k, namely $s_k^j(G)$ tends to a limit as $j \to \infty$. This limit is the "k-imperfection ratio" and is the subject of this paper. In order to give a convenient definition of it, we first introduce the fractional k-clique-bound and the fractional k-span.

For a fixed positive integer k, the *fractional k-clique-bound* $\omega_k^f(G, \mathbf{x})$ of a graph G with weight vector $\mathbf{x}$ is

$$\omega_k^f(G, \mathbf{x}) = \max \{k x_{\max}, \omega(G, \mathbf{x})\},$$

and the *fractional k-span* $\mathrm{span}_k^f(G, \mathbf{x})$ is the value of the following linear program (LP) which has a variable y_S for each induced k-colorable subgraph S of G: $\min k \sum_S y_S$ subject to $\sum_{S \ni v} y_S \geq x_v$ for each node v, and $y_S \geq 0$ for each k-colorable induced subgraph S of G. Observe that for any graph G, $\mathrm{span}_1^f(G, \mathbf{x})$ is the weighted fractional chromatic number $\chi_f(G, \mathbf{x})$, and, in particular, $\mathrm{span}_1^f(G, \mathbf{1})$ is the fractional chromatic number $\chi_f(G)$. It is easy to check that

$$\mathrm{span}_k^f(G, \mathbf{x}) \geq \omega_k^f(G, \mathbf{x}), \tag{1.2}$$

and we do so toward the end of this introductory section. Observe that one can easily extend the definitions of $\omega_k^f(G, \mathbf{x})$ and $\mathrm{span}_k^f(G, \mathbf{x})$ to rational or real weight vectors, and we will use them in this way later, whereas we will use $\omega_k(G, \mathbf{x})$ and $\mathrm{span}_k(G, \mathbf{x})$ for integral weight vectors only.

The *k-imperfection ratio* $\mathrm{imp}_k(G)$ of a graph G is defined by setting

$$\mathrm{imp}_k(G) = \sup_{\mathbf{x}} \frac{\mathrm{span}_k^f(G, \mathbf{x})}{\omega_k^f(G, \mathbf{x})}, \tag{1.3}$$

where the supremum is over all nonzero integral weight vectors $\mathbf{x}$. It turns out that there always exists such a vector $\mathbf{x}$ with $\mathrm{imp}_k(G) = \mathrm{span}_k^f(G, \mathbf{x}) / \omega_k^f(G, \mathbf{x})$ (see (2.7) below), and thus the supremum in the definition (1.3) may be replaced by the maximum. Observe that if H is an induced subgraph of G, then $\mathrm{imp}_k(H) \leq \mathrm{imp}_k(G)$. By (1.2) we have

$$\mathrm{imp}_k(G) \geq 1. \tag{1.4}$$

We say that a graph G with $\mathrm{imp}_k(G) = 1$ is *k-perfect*. Observe that if $\chi(G) \leq k$, then trivially G is k-perfect, since then $\mathrm{span}_k^f(G, \mathbf{x}) = k x_{\max}$ (take $y_V = x_{\max}$).

The 1-imperfection-ratio has been studied in [7], [8]. It is called the *imperfection ratio* and is denoted by $\mathrm{imp}(G)$. Its name was motivated by the fact that $\mathrm{imp}(G) \geq 1$ for all graphs G and that $\mathrm{imp}(G) = 1$ if and only if G is perfect. We shall see that not every perfect graph is k-perfect when $k \geq 2$, but, for example, this does hold for comparability graphs and some other classes of graphs, and there are many interesting properties of the imperfection ratio which have their equivalents in the more general case.

The plan of the paper is as follows. After giving three introductory results at the end of this section, we see in section 2 that for any graph G and any positive integer k, the quantity s_k^j defined in (1.1) above satisfies

$$s_k^j(G) \to \mathrm{imp}_k(G) \text{ as } j \to \infty, \tag{1.5}$$

and we present equivalent alternative polyhedral definitions of $\mathrm{imp}_k(G)$.

In section 3 we find upper and lower bounds on the k-imperfection ratio, including the result that $\mathrm{imp}_k(G)/\mathrm{imp}(G) \leq \frac{1}{1+1/e} \sim 1.6$. These bounds yield some extremal results. We also see, for example, that the Petersen graph P satisfies $\mathrm{imp}_2(P) = 10/7$.

In section 4 we see that the class of 2-perfect graphs is a proper subclass of the class of perfect graphs. In contrast, it is easy to find nonperfect graphs which are k-perfect for each $k \geq 3$, for example, the odd cycles C_n on $n \geq 5$ nodes (the *odd holes*). We then consider some classes of perfect graphs, where each graph G in the class is k-perfect for each positive integer k. We call such a graph G *all-perfect*. We already know that this holds for bipartite graphs (since $\chi(G) \leq k$ for each $k \geq 2$), and we shall see shortly that it is true also for complete graphs. In section 4 we shall see that it is also true for comparability graphs, co-comparability graphs, and line-graphs of bipartite graphs.

In section 5 we see that, in contrast to the nice behavior for perfect graphs, for each $k \geq 2$ there are many nonisomorphic node-minimal non-k-perfect graphs: indeed, the number on at most n nodes grows at least exponentially with n. We see that an odd hole on n nodes is node-minimal non-2-perfect and that its complement (an *odd antihole*) is node-minimal non-k-perfect for all $k \leq (n-1)/2$. We also determine imp_k for all odd holes and antiholes.

In section 6 we consider disk graphs, which crop up naturally in models for radio channel assignment, and give bounds for their k-imperfection ratio.

In section 7 we see that for the random graph $G_{n,\frac{1}{2}}$, the k-imperfection ratio is about $n/(4\log_2^2 n)$ (which is independent of k), and also we obtain corresponding results for sparse random graphs and random regular graphs (which do depend on k).

Let us finish this section by giving three simple introductory results, as mentioned above. The first task is to prove (1.2). Let (y_S) be a feasible solution to the LP defining $\mathrm{span}_k^f(G, \mathbf{x})$. If x_v is $x_{\max}$, then

$$k\sum_S y_S \geq k \sum_{S:v\in S} y_S \geq kx_v = kx_{\max}.$$

Also, if the set K of nodes forms a complete subgraph of G with $\omega(G, \mathbf{x}) = \sum_{v\in K} x_v$, then, since $|K \cap S| \leq k$ for each k-colorable subset S, we have

$$k\sum_S y_S \geq \sum_S \sum_{v\in K\cap S} y_S = \sum_{v\in K} \sum_{S:v\in S} y_S \geq \sum_{v\in K} x_v = \omega(G, \mathbf{x}).$$

Hence $k\sum_S y_S \geq \omega_k^f(G, \mathbf{x})$, which yields (1.2).

The second of our three introductory results shows that (not unexpectedly) we may restrict our attention to connected graphs.

PROPOSITION 1.1. *For any positive integer k and any graph G, if G consists of the disjoint union of graphs $G_1, \ldots, G_t$, then*

$$\mathrm{imp}_k(G) = \max\{\mathrm{imp}_k(G_1), \ldots, \mathrm{imp}_k(G_t)\}.$$

Proof. Directly from the definitions, for any weight vector $\mathbf{x}$

$$\begin{aligned}\mathrm{span}_k^f(G,\mathbf{x}) &= \max_i\{\mathrm{span}_k^f(G_i,\mathbf{x})\}\\ &\le \max_i\{\mathrm{imp}_k(G_i)\ \omega_k^f(G_i,\mathbf{x})\}\\ &\le (\max_i \mathrm{imp}_k(G_i))\ \omega_k^f(G,\mathbf{x}),\end{aligned}$$

and so $\mathrm{imp}_k(G) \le \max_i \mathrm{imp}_k(G_i)$. The lower bound follows immediately from the earlier remark that the k-imperfection ratio of an induced subgraph of G is always at most the k-imperfection ratio of G. ∎

Next we meet a connection with scheduling theory. Recall that we say that a graph is all-perfect if it is k-perfect for each positive integer k.

PROPOSITION 1.2. *Each complete graph K is all-perfect.*

Proof. This result will follow directly from the fact we noted above (see (2.6) below) that $1 = s_k^j(K) \to \mathrm{imp}_k(K)$ as $j \to \infty$, but it is interesting to note that it is a disguised form of a standard basic result in scheduling theory. Suppose that we have k identical machines in parallel, a collection V of jobs v with processing time x_v, and pre-emptions are allowed (we need at most 1 per job). It is well known [19] and not hard to see that the makespan m (the minimum completion time) is given by

$$m = \max\left\{x_{\max}, \left(\sum_v x_v\right)/k\right\} = \omega_k^f(K,\mathbf{x})/k.$$

Given a schedule with makespan m, for each set $S \subseteq V$ let y_S be the total time that S is the set of jobs being processed. Then $\sum_{S:v\in S} y_S = x_v$ for each $v \in V$, and $\sum_S y_S = m$. ∎

2. Equivalent descriptions. In this section we introduce equivalent polyhedral descriptions for $\mathrm{imp}_k(G)$; see Theorem 2.3. We also show that for any graph G there is an integral weight vector $\mathbf{x}$ with $\mathrm{imp}_k(G) = \mathrm{span}_k^f(G,\mathbf{x})/\omega_k^f(G,\mathbf{x})$ and each coordinate at most $2^{-n}(n+1)^{(n+1)/2}$ (where $n = |V(G)|$). It was shown in [7] that we may need coordinates as large as $2^{(n-5)/4}$ if $k = 1$.

To prove Theorem 2.3 we need one preliminary lemma, some more notation, and a result of [9], which says that for any clique K and any integral weight vector $\mathbf{x}$,

$$\mathrm{span}_k(K,\mathbf{x}) = \max\left\{(x_{\max}-1)k + |\{v \in V(K) : x_v = x_{\max}\}|, \sum_{v\in V(K)} x_v\right\}. \tag{2.1}$$

By this result,

$$\omega_k^f(G,\mathbf{x}) - k + 1 \le \omega_k(G,\mathbf{x}) \le \omega_k^f(G,\mathbf{x}), \tag{2.2}$$

and since $\omega_k^f(G,a\mathbf{x}) = a\,\omega_k^f(G,\mathbf{x})$,

$$\omega_k^f(G,\mathbf{x}) = \lim_{a\to\infty} \frac{\omega_k(G,a\mathbf{x})}{a}.$$

The latter equality partially motivated the notation $\omega_k^f(G, \mathbf{x})$ since many fractional versions of graph parameters can be defined in this way [21]—see also Corollary 2.2 below.

LEMMA 2.1. *For any graph G on n nodes with integral weight vector $\mathbf{x}$,*

$$\operatorname{span}_k^f(G, \mathbf{x}) - k \le \operatorname{span}_k(G, \mathbf{x}) \le \operatorname{span}_k^f(G, \mathbf{x}) + 2nk.$$

Proof. In any k-feasible assignment each node belongs to at most one of any k consecutive color classes, and the graph induced by the nodes of k consecutive color classes is k-colorable. Hence each node v can be covered x_v times by at most $\lceil \operatorname{span}_k(G, \mathbf{x})/k \rceil$ k-colorable graphs, which yields

$$\frac{\operatorname{span}_k^f(G, \mathbf{x})}{k} \le \left\lceil \frac{\operatorname{span}_k(G, \mathbf{x})}{k} \right\rceil \le \frac{\operatorname{span}_k(G, \mathbf{x}) + k - 1}{k},$$

and so

$$\operatorname{span}_k^f(G, \mathbf{x}) - k + 1 \le \operatorname{span}_k(G, \mathbf{x}).$$

To prove that $\operatorname{span}_k(G, \mathbf{x}) \le \operatorname{span}_k^f(G, \mathbf{x}) + 2nk$, consider an optimal basic feasible solution $\mathbf{y}$ of the LP determining $\operatorname{span}_k^f(G, \mathbf{x})$. Since $\mathbf{y}$ is a basic feasible solution, at most n values y_S are nonzero. Hence by rounding up $\mathbf{y}$ one obtains an integral feasible solution $\mathbf{z}$ with value less than $\operatorname{span}_k^f(G, \mathbf{x}) + nk$. Now we can color a k-colorable subgraph S of G in a k-feasible way z_S times using $z_S k$ consecutive colors. To put these colorings together for a k-feasible assignment one can introduce gaps of size $k - 1$ to ensure that two distinct colors assigned to a node are at least k apart. Hence

$$\operatorname{span}_k(G, \mathbf{x}) < \operatorname{span}_k^f(G, \mathbf{x}) + nk + (n-1)(k-1) \le \operatorname{span}_k^f(G, \mathbf{x}) + 2nk$$

as claimed. ☐

Since $\operatorname{span}_k^f(G, a\mathbf{x}) = a \operatorname{span}_k^f(G, \mathbf{x})$, Lemma 2.1 yields as a corollary the following result, which motivated the choice of the name "fractional k-span."

PROPOSITION 2.2. $\operatorname{span}_k^f(G, \mathbf{x}) = \lim_{a\to\infty} \operatorname{span}_k(G, a\mathbf{x})/a$.

We denote the set of all real weight vectors $\mathbf{x}$ with $\omega_k^f(G, \mathbf{x}) \le 1$ by $QSTAB_k(G)$, or equivalently

$$QSTAB_k(G) = QSTAB(G) \cap [0, 1/k]^n,$$

where $QSTAB(G) = QSTAB_1(G)$ is the *fractional node-packing polytope*; see [12] for further discussion. The convex hull of the incidence vectors of the k-colorable induced subgraphs of G scaled by $1/k$ is denoted by $STAB_k(G)$. Thus $STAB_1(G)$ is the familiar stable set polytope; again see [12] for further discussion. Note that

$$\operatorname{span}_k^f(G, \mathbf{x}) \le t \text{ if and only if } \mathbf{x} \in t\, STAB_k(G). \tag{2.3}$$

Here $t\,P$ denotes the scaled set $\{t\mathbf{x} : \mathbf{x} \in P\}$. We are now able to state and prove the main theorem of this section.

THEOREM 2.3. *For any graph G,*

$$\operatorname{imp}_k(G) = \min\{t : QSTAB_k(G) \subseteq t\, STAB_k(G)\} \tag{2.4}$$

$$= \max\{\operatorname{span}_k^f(G, \mathbf{x}) : \mathbf{x} \textit{ is a vertex of } QSTAB_k(G)\} \tag{2.5}$$

$$= \lim_{j\to\infty} s_k^j(G). \tag{2.6}$$

In addition, there exists an integral weight vector $\mathbf{x}$ *with*

$$\mathrm{imp}_k(G) = \frac{\mathrm{span}_k^f(G,\mathbf{x})}{\omega_k^f(G,\mathbf{x})}, \tag{2.7}$$

and if G *has* n *nodes, then there is such a vector* $\mathbf{x}$ *with each coordinate at most* $2^{-n}(n+1)^{(n+1)/2}$.

Proof. Let $s(G)$ denote the right-hand side of (2.4). Observe that

$$s(G) = \min\{t : \mathbf{x} \in t\, STAB_k(G) \text{ for all vertices } \mathbf{x} \in QSTAB_k(G)\},$$

which equals (2.5) because of (2.3). Thus $s(G)$ is rational, and $QSTAB_k(G) \subseteq s(G)\, STAB_k(G)$. Consider a weight vector $\mathbf{x}$, with $\omega_k^f(G,\mathbf{x}) = l$, which implies that $\mathbf{x} \in l\, QSTAB_k(G)$, so $\mathbf{x} \in ls(G)\, STAB_k(G)$, and hence $\mathrm{span}_k^f(G,\mathbf{x}) \le ls(G)$. Thus $\mathrm{span}_k^f(G,\mathbf{x})/\omega_k^f(G,\mathbf{x}) \le s(G)$, and it follows that $\mathrm{imp}_k(G) \le s(G)$.

Now we show that $\mathrm{imp}_k(G) \ge s(G)$. Let $\mathbf{x}$ be a vertex of $QSTAB_k(G)$ such that $s(G) = \mathrm{span}_k^f(G,\mathbf{x})$. Since $\mathbf{x}$ is rational, we may choose a positive integer N such that the vector $\tilde{\mathbf{x}} = N\mathbf{x}$ is integral. Then $\mathrm{span}_k^f(G,\tilde{\mathbf{x}}) = Ns(G)$, and $\omega_k^f(G,\tilde{\mathbf{x}}) \le N$. Hence $\mathrm{imp}_k(G) \ge \mathrm{span}_k^f(G,\tilde{\mathbf{x}})/\omega_k^f(G,\tilde{\mathbf{x}}) \ge s(G)$. Thus $s(G) = \mathrm{imp}_k(G)$, and further the supremum of the ratios $\mathrm{span}_k^f(G,\mathbf{x})/\omega_k^f(G,\mathbf{x})$ over all weight vectors $\mathbf{x}$ as in the definition of $\mathrm{imp}_k(G)$ is attained at $\tilde{\mathbf{x}}$ (and thus at all integer multiples of $\tilde{\mathbf{x}}$).

Next we prove (2.6). Let $\tilde{\mathbf{x}}$ be an integral weight vector as above such that $\mathrm{span}_k^f(G,\tilde{\mathbf{x}})/\omega_k^f(G,\tilde{\mathbf{x}}) = \mathrm{imp}_k(G)$. Let u be a node of maximal demand, and let $\tilde{l} = \tilde{x}_{\max} = \tilde{x}_u$. For any $l \ge \tilde{l}$, write $l = q\tilde{l} + r$ with $0 \le r < \tilde{l}$, and define $\mathbf{y}_u^l = l = q\tilde{x}_u + r$ and $\mathbf{y}_v^l = q\tilde{x}_v$ for all $v \ne u$. We have $y_{\max}^l = l$ and thus

$$\begin{aligned} s_k^l(G) &\ge \frac{\mathrm{span}_k(G,\mathbf{y}^l)}{\omega_k(G,\mathbf{y}^l)} \ge \frac{\mathrm{span}_k(G,q\tilde{\mathbf{x}})}{\omega_k(G,q\tilde{\mathbf{x}})+rk} \ge \frac{\mathrm{span}_k^f(G,q\tilde{\mathbf{x}})-k}{\omega_k^f(G,q\tilde{\mathbf{x}})+rk+2nk} \\ &\ge \frac{\mathrm{span}_k^f(G,q\tilde{\mathbf{x}})}{\omega_k^f(G,q\tilde{\mathbf{x}})} \frac{\omega_k^f(G,q\tilde{\mathbf{x}})}{\omega_k^f(G,q\tilde{\mathbf{x}})+\tilde{l}k+2nk} - \frac{k}{\omega_k^f(G,q\tilde{\mathbf{x}})} \\ &\ge \mathrm{imp}_k(G) \frac{k(l-\tilde{l})}{kl+2nk} - \frac{k}{k(l-\tilde{l})}. \end{aligned}$$

Now, let $\mathbf{x}^l$ be a weight vector such that $x_{\max}^l = l$ and $s_k^l(G) = \mathrm{span}_k(G,\mathbf{x}^l)/\omega_k(G,\mathbf{x}^l)$. We obtain

$$\begin{aligned} s_k^l &= \frac{\mathrm{span}_k(G,\mathbf{x}^l)}{\omega_k(G,\mathbf{x}^l)} \le \frac{\mathrm{span}_k^f(G,\mathbf{x}^l)+2nk}{\omega_k^f(G,\mathbf{x}^l)-k} \\ &\le \frac{\mathrm{span}_k^f(G,\mathbf{x}^l)}{\omega_k^f(G,\mathbf{x}^l)} \frac{\omega_k^f(G,\mathbf{x}^l)}{\omega_k^f(G,\mathbf{x}^l)-k} + \frac{2nk}{\omega_k^f(G,\mathbf{x}^l)-k} \\ &\le \mathrm{imp}(G) \frac{kl}{kl-k} + \frac{2nk}{kl-k}, \end{aligned}$$

and the result follows.

It remains to show that there is weight vector $\mathbf{x}$ as in (2.7) with "small" coordinates. Any vertex $\mathbf{y}$ of $QSTAB_k(G)$ is the unique solution of $A\mathbf{z} = \mathbf{b}$ for some $n \times n$ matrix A with $0,1$ entries and some vector $\mathbf{b}$ the entries of which equal $0,1$

or $1/k$. Therefore and because $y_i \leq 1/k$, Cramer's rule implies that $\mathbf{y}$ has entries of the form $a_i/(k \det(A))$ for integers $0 \leq a_i \leq \det(A)$, $i = 1, \ldots, n$. But since A is a 0, 1-matrix, $\det(A) \leq 2^{-n}(n+1)^{(n+1)/2}$ [1]. Considering a vertex $\mathbf{y}$ of $QSTAB_k(G)$ with $\mathrm{imp}_k(G) = \mathrm{span}_k^f(G, \mathbf{y})$ and setting $\mathbf{x} = k \det(A)\mathbf{y}$ yields the result. ∎

Let us note one more equivalent definition of the k-imperfection ratio, which follows easily from the work above. We could define $\mathrm{imp}_k(G)$ as the least a such that (2.8) below holds for some choice of b.

PROPOSITION 2.4. *Consider a graph G and a positive integer k. Let A be the set of values a such that, for some b,*

$$\mathrm{span}_k(G, \mathbf{x}) \leq a\, \omega_k(G, \mathbf{x}) + b \quad \textit{for each integral weight vector } \mathbf{x}. \tag{2.8}$$

Then $a \in A$ if and only if $a \geq \mathrm{imp}_k(G)$.

Proof. If G has n nodes, by Lemma 2.1 and (2.2)

$$\begin{aligned}\mathrm{span}_k(G, \mathbf{x}) &\leq \mathrm{span}_k^f(G, \mathbf{x}) + 2nk \\ &\leq \mathrm{imp}_k(G)\, \omega_k^f(G, \mathbf{x}) + 2nk \\ &\leq \mathrm{imp}_k(G)\, \omega_k(G, \mathbf{x}) + \mathrm{imp}_k(G)(k-1) + 2nk.\end{aligned}$$

Thus there is a constant b such that (2.8) holds. Conversely, suppose that a and b are such that (2.8) holds. Then $s_k^j(G) \leq a + b/j$, and so by (2.6) it follows that $\mathrm{imp}_k(G) \leq a$. ∎

3. Bounds. In this section, we first give bounds on $\mathrm{imp}_k(G)$ in terms of the $\chi(G)$, $\chi_f(G)$, $\omega(G)$, and so on. From these bounds we make various deductions, including determining the value of $\mathrm{imp}_2(P)$ for the Petersen graph P. Next we give an upper bound on $\mathrm{imp}_k(G)$ in terms of $\mathrm{imp}(G)$. These results, together with results from [8], yield various extremal results.

LEMMA 3.1. *For any positive integer k and any graph G,*

(a) $\mathrm{imp}_k(G) \geq \min\{\chi_f(G)/k, \chi_f(G)/\omega(G)\}$,

(b) $\mathrm{imp}_k(G) \leq \mathrm{span}_k^f(G, \mathbf{1})/k \leq \max\{1, \chi(G)/k\}$.

Proof. We have

$$\mathrm{imp}_k(G) \geq \frac{\mathrm{span}_k^f(G, \mathbf{1})}{\omega_k^f(G, \mathbf{1})} \geq \frac{\chi_f(G)}{\max\{\omega(G), k\}} = \min\left\{\frac{\chi_f(G)}{k}, \frac{\chi_f(G)}{\omega(G)}\right\}$$

as required for (a).

For every $\mathbf{x}$ in $QSTAB_k(G)$, we have $\mathbf{x} \leq (1/k)\mathbf{1}$ and therefore

$$\mathrm{span}_k^f(G, \mathbf{x}) \leq \mathrm{span}_k^f(G, (1/k)\mathbf{1}) = \frac{\mathrm{span}_k^f(G, \mathbf{1})}{k}.$$

The first inequality of (b) now follows by (2.5). For the second inequality observe that $\mathrm{span}_k^f(G, \mathbf{1}) = k$ if $\chi(G) \leq k$. If $\chi(G) > k$, then we can partition G into $\chi(G) = \chi$ color classes. With the $\binom{\chi}{k}$ k-colorable subgraphs each consisting of a different set of k color classes, we can cover every node $\binom{\chi-1}{k-1}$ times. Therefore, we have $\mathrm{span}_k^f(G, \mathbf{x}) \leq k\binom{\chi}{k}/\binom{\chi-1}{k-1} = \chi$, and so $\mathrm{span}_k^f(G, \mathbf{1})/k \leq \chi(G)/k$. ∎

Observe that part (b) extends the result noted earlier that G is k-perfect if $\chi(G) \leq k$. It follows directly from the definition of the fractional k-span that

$$\frac{1}{k}\, \mathrm{span}_k^f(G, \mathbf{x}) \geq \frac{1}{k+1}\, \mathrm{span}_{k+1}^f(G, \mathbf{x}).$$

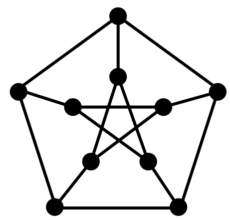

FIG. 3.1. *The Petersen graph.*

Also, if $k \geq \omega(G)$, then $\mathrm{imp}_k(G) \geq \mathrm{span}_k^f(G, \mathbf{1})/\omega_k^f(G, \mathbf{1}) = \mathrm{span}_k^f(G, \mathbf{1})/k$. Hence from part (b) above we obtain the following result.

LEMMA 3.2. *If* $k \geq \omega(G)$, *then* $\mathrm{imp}_k(G) = \mathrm{span}_k^f(G, \mathbf{1})/k$, *and* $\mathrm{imp}_k(G) \geq \mathrm{imp}_{k+1}(G)$.

In particular, the last result implies that for any $k \geq 2$ and for any triangle-free graph G, we have $\mathrm{imp}_k(G) = \mathrm{span}_k^f(G, \mathbf{1})/k$. The case $k = 1$ is different [7]: if G is a triangle-free graph which contains at least one edge, then $\mathrm{imp}(G) = \chi_f(G)/2 = \mathrm{span}_1^f(G, \mathbf{1})/2$.

If $\omega(G) \leq k \leq \chi(G)$, then by Lemma 3.1 and [15]

$$\chi_f(G) \leq k\ \mathrm{imp}_k(G) \leq \chi(G) \leq (1 + \log_2 n)\chi_f(G)$$

if G has n nodes. Hence if we could prove that it is hard to approximate the chromatic number of a triangle-free graph G up to some factor $f(n) \geq (1 + \log_2 n)$, then this would show that it is hard to approximate $\mathrm{imp}_k(G)$ up to the factor $f(n)/(1+\log_2 n)$. It is NP-hard to approximate the chromatic number up to a factor of $n^{\frac{1}{7}-\varepsilon}$ for general graphs [3]. Also, it is NP-hard to determine $\chi_f(G)$ exactly for triangle-free graphs, and hence it is NP-hard to determine $\mathrm{imp}(G)$ exactly [7].

We cannot replace $\chi(G)$ by $\chi_f(G)$ in part (b) of Lemma 3.1 as we might hope, by analogy with the case $k = 1$ (recall from [7] that $\mathrm{imp}(G) \leq \chi_f(G)/2$ if G has at least one edge), as the following example shows.

Example 3.1. The Petersen graph P, shown in Figure 3.1, satisfies

$$\mathrm{imp}_2(P) = 10/7 > 5/4 = \chi_f(P)/2.$$

For, observe that P is node-transitive, and the maximal number of nodes in a bipartite induced subgraph is 7. Thus we obtain $\mathrm{span}_2^f(P, \mathbf{1}) = 20/7$ by considering the hypergraph which has a hyperedge for each 2-colorable graph and applying, for example, Proposition 1.3.4 of [21, p. 7]. But Lemma 3.2 shows that $\mathrm{imp}_2(P) = \mathrm{span}_2^f(P, \mathbf{1})/2 = 10/7$.

Lemma 3.1 also implies that if $\omega(G) \leq k < \chi_f(G)$, then $\mathrm{imp}_k(G) > 1$: the next lemma extends this result, and will be useful in the next section.

LEMMA 3.3. *For any graph* G, *if* $\omega(G) \leq k < \chi(G)$, *then* G *is not* k*-perfect.*

Proof. Since $\omega(G) \leq k$, $(1/k)\mathbf{1} \in QSTAB_k(G)$, but since $k < \chi(G)$, $(1/k)\mathbf{1} \notin STAB_k(G)$. The result now follows from (2.4). ◻

The next result gives a bound on the k-imperfection ratio in terms of the imperfection ratio. It will allow us to extend the known extremal results for the case $k = 1$ [8] to cover each $k \geq 1$.

THEOREM 3.4. *For any graph G,*

$$\operatorname{imp}_k(G) \le \frac{1}{1-(1-1/k)^k}\operatorname{imp}(G) \le \frac{1}{1-1/e}\operatorname{imp}(G).$$

Note that $(1-1/e)^{-1} < 1.582$. To prove the theorem we need one auxiliary lemma.

LEMMA 3.5. *Let G be a graph with weight vector $\mathbf{x}$. For any $\rho \ge x_{\max}/\chi_f(G,\mathbf{x})$, we have*

$$\operatorname{span}_k^f(G,\mathbf{x}) \le \frac{k\rho}{1-(1-\rho)^k}\chi_f(G,\mathbf{x}).$$

Proof. Let $\mathbf{y}$ be an optimal feasible solution for the LP defining $\chi_f(G,\mathbf{x})$, and let $\chi_f(G,\mathbf{x}) = \gamma$. Set $y'_S = y_S/\gamma$ for each stable set S, so $\sum_S y'_S = 1$. For each k-colorable set T of nodes in G, let

$$z_T = \sum_{\substack{S_1,S_2,\ldots,S_k\\ S_1\cup S_2\cup\ldots\cup S_k=T}} y'_{S_1}y'_{S_2}\ldots y'_{S_k}.$$

Then

$$\sum_T z_T = \sum_{S_1}\sum_{S_2}\ldots\sum_{S_k} y'_{S_1}y'_{S_2}\ldots y'_{S_k} = 1.$$

(Indeed, z_T is the probability that we obtain T if we form the union of k (not necessarily distinct) stable sets picked independently at random where the stable set S has probability y'_S.) For a node $v \in V(G)$, we have

$$\begin{aligned}\sum_{T\ni v} z_T &= 1 - \sum_{S_1\not\ni v}\sum_{S_2\not\ni v}\ldots\sum_{S_k\not\ni v} y'_{S_1}y'_{S_2}\ldots y'_{S_k}\\ &= 1 - \left(\sum_{S_1\not\ni v} y'_{S_1}\right)\left(\sum_{S_2\not\ni v} y'_{S_2}\right)\ldots\left(\sum_{S_k\not\ni v} y'_{S_k}\right)\\ &= 1 - \left(\sum_{S\not\ni v} y'_S\right)^k \ge 1-(1-x_v/\gamma)^k.\end{aligned}$$

It is easily verified that the function $f(x) = (1-(1-x)^k)/x$ is decreasing for $0 \le x \le 1$. Hence

$$\sum_{T\ni v} z_T \ge 1-(1-x_v/\gamma)^k = \frac{x_v}{\gamma}f(x_v/\gamma) \ge \frac{x_v}{\gamma}f(\rho).$$

Therefore $\operatorname{span}_k^f(G,\mathbf{x}) \le k\gamma/f(p) = (k\rho/1-(1-\rho)^k)\chi_f(G,\mathbf{x})$. ∎

Proof of Theorem 3.4. For any $k \ge \omega(G)$ we have by Lemma 3.2 that $\operatorname{imp}_k(G) \ge \operatorname{imp}_{k+1}(G)$. Therefore it suffices to consider the case $k \le \omega(G)$. Let $\mathbf{x}$ be a weight vector such that

$$\operatorname{span}_k^f(G,\mathbf{x}) = \operatorname{imp}_k(G) \quad \text{and} \quad \omega(G,\mathbf{x}) = 1 \ge kx_{\max}.$$

Since $\chi_f(G,\mathbf{x}) \ge \omega(G,\mathbf{x}) = 1 \ge kx_{\max}$, we have $x_{\max}/\chi_f(G,\mathbf{x}) \le 1/k$ and $\operatorname{imp}(G) \ge \chi_f(G,\mathbf{x})$. Hence by Lemma 3.5

$$\operatorname{imp}_k(G) = \operatorname{span}_k^f(G,\mathbf{x}) \le \frac{\chi_f(G,\mathbf{x})}{1-(1-1/k)^k} \le \frac{\operatorname{imp}(G)}{1-(1-1/k)^k}.$$

Finally, note that $(1-1/k)^k \le e^{-1}$. ∎

Theorem 3.4 together with Theorem 3.1 of [8] (which says that there exists a constant c' such that for all graphs G with $n \geq 3$ nodes $\mathrm{imp}(G) \leq c' n \log\log n/\log^2 n)$ implies the following extension of the latter result.

PROPOSITION 3.6. *There exists a constant c such that for each graph G with $n \geq 3$ nodes, and each positive integer k,*

$$\mathrm{imp}_k(G) \leq c\frac{n(\log\log n)}{\log^2 n}.$$

The upper bound here is at most a factor $\log\log n$ too generous; see Theorem 7.2 below. We can also extend a result from [8] concerning graphs G with bounded maximum degree.

PROPOSITION 3.7. *For each $\varepsilon > 0$, there exists a constant d_0 such that, for each positive integer k, for each $d \geq d_0$, and for each graph G with maximum degree $\Delta(G) \leq d$,*

$$\mathrm{imp}_k(G) \leq \varepsilon d.$$

This result shows that the k-imperfection ratio grows more slowly than the maximum degree. It may be proved along similar lines to the proof of Theorem 3.2 in [8].

4. Some classes of perfect graphs. Which perfect graphs are k-perfect? In this section, we first give a polyhedral characterization. We then investigate whether a nonperfect graph can be k-perfect. This question is easily answered for $k \geq 3$, since in this case there are indeed graphs which are nonperfect but are k-perfect—just take any nonperfect graph G with $\chi(G) \leq k$. We will see that this is not true for $k = 2$: the 2-perfect graphs form a subclass of the perfect graphs, and we shall see that it is in fact a proper subclass, by considering a class of perfect graphs G (the split graphs) such that G need not be k-perfect when $k \geq 2$. Finally, we consider three standard classes of perfect graphs, namely comparability graphs, line graphs of bipartite graphs, and co-comparability graphs, and show that each graph in these classes is all-perfect (that is, k-perfect for each k).

PROPOSITION 4.1. *Let G be a perfect graph with n nodes, and let k be a positive integer. Then G is k-perfect if and only if the polytope*

$$\{\mathbf{x} \geq \mathbf{0} : \omega(G, \mathbf{x}) \leq k\} \cap [0,1]^n$$

has only integral extreme points. If the polytope has a unique nonintegral extreme point $\mathbf{z}$, then $\mathrm{imp}_k(G) = \frac{1}{k}\,\mathrm{span}_k^f(G, \mathbf{z})$.

Proof. Let A denote the polytope in the proposition, and let B denote the convex hull of the incidence vectors of the k-colorable sets of nodes (so that $A = k\,QSTAB_k(G)$ and $B = k\,STAB_k(G)$). Then $A \supseteq B$, and by (2.4) in Theorem 2.3, G is k-perfect if and only if $A = B$. So, if G is k-perfect, then of course the extreme points of A are $0, 1$-valued. For the converse, let $\mathbf{z}$ be any integral point in A. Then $\mathbf{z}$ is the incidence vector of the nodes in a subgraph H of G with $\omega(H) \leq k$, and so with $\chi(H) \leq k$: hence $\mathbf{z} \in B$. Hence if each extreme point of A is integral, then $A \subseteq B$, and it follows that G is k-perfect. This completes the proof of the first part of the proposition.

Further, it now follows using (2.5) in Theorem 2.3 that if A has a nonintegral extreme point $\mathbf{z}$, then $\mathrm{imp}_k(G)$ is the maximum value of $\frac{1}{k}\,\mathrm{span}_k^f(G, \mathbf{z})$ over such points $\mathbf{z}$. $\square$

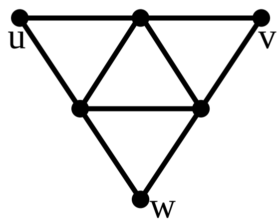

FIG. 4.1. *The "Hajos" graph G_2 as in Proposition* 4.3.

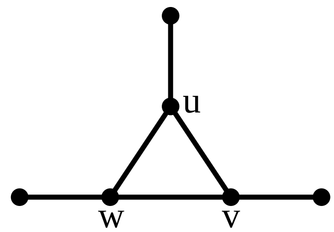

FIG. 4.2. *The complement of the "Hajos" graph.*

The next two propositions show that the 2-perfect graphs form a proper subclass of the perfect graphs.

PROPOSITION 4.2. *Each* 2*-perfect graph is perfect.*

Proof. Recall from [7] that the *binary imperfection ratio* $\mathrm{imp^b}(G)$ is the maximum value of $\mathrm{span}_1(G,\mathbf{x})/\omega(G,\mathbf{x})$ over all nonzero $0,1$ weight vectors $\mathbf{x}$: also $\mathrm{imp^b}(G) \leq \mathrm{imp}(G)$, and $\mathrm{imp^b}(G) = 1$ if and only if G is perfect. We claim that $\mathrm{imp}_2(G) \geq \mathrm{imp^b}(G)$ for each graph G. But then, if G is not perfect, we have $\mathrm{imp}_2(G) \geq \mathrm{imp^b}(G) > 1$ and so G is not 2-perfect, and the proposition follows.

To prove the claim, note first that if G consists of isolated nodes, then $\mathrm{imp}_2(G) = 1 = \mathrm{imp^b}(G)$, so we may assume that G has at least one edge. Let $\mathbf{x}$ be a 0,1 weight vector of G with $\mathrm{imp^b}(G) = \chi_f(G,\mathbf{x})/\omega(G,\mathbf{x})$ and $\omega(G,\mathbf{x}) \geq 2$. Then $\omega_2^f(G,\mathbf{x}) = \omega(G,\mathbf{x})$, and hence

$$\mathrm{imp}_2(G) \geq \frac{\mathrm{span}_2^f(G,\mathbf{x})}{\omega_2^f(G,\mathbf{x})} \geq \frac{\chi_f(G,\mathbf{x})}{\omega(G,\mathbf{x})} = \mathrm{imp^b}(G)$$

as claimed. ∎

The next proposition shows that for each $k \geq 2$ there are perfect graphs which are not k-perfect. Recall that a *split graph* is a graph the nodes of which can be covered by a clique and a stable set. It is well known and easy to see that such graphs are perfect; see, for example, [12].

PROPOSITION 4.3. *For each $k \geq 2$, there exists a split graph G_k which is not k-perfect.*

Proof. Consider the graph G_k which consists of a clique of size $2k-1$ and a stable set of size $\binom{2k-1}{k}$ such that every k-subset of nodes of the clique is adjacent to exactly one node of the stable set. For the graph G_2, the "Hajos graph," see Figure 4.1. Let $\mathbf{x}^k$ be the weighting of G_k with $x_u^k = 1$ for each node u of the clique, and $x_v^k = 2$ for each node v of the stable set. Since $k \geq 2$, we have $\omega_k^f(G_k,\mathbf{x}^k) = 2k$ but $\mathrm{span}_k^f(G_k,\mathbf{x}^k) > 2k$. For suppose that there is a solution $\mathbf{y}$ to the LP for $\mathrm{span}_k^f(G_k,\mathbf{x}^k)$ with value $2k$. Then any k-colorable graph S actually used (that is, with $y_S > 0$) must contain all the nodes of the stable set, and so can contain at most $k-1$ nodes of the clique. Hence the total weight covered on the nodes of the clique is at most $2k-2 < 2k-1$, and so the covering is not feasible for G_k and $\mathbf{x}^k$. In summary, $\mathrm{imp}_k(G_k) \geq \mathrm{span}_k^f(G_k,\mathbf{x}^k)/\omega_k^f(G_k,\mathbf{x}^k) > 1$. ∎

Example 4.1. Let us consider more carefully the Hajos graph H shown in Figure 4.1, which we have already noted is perfect. We shall see that H is a minimal non-2-perfect graph and that $\mathrm{imp}_2(H) = \frac{9}{8}$.

It is easy to check that any proper induced subgraph of H is an interval graph, and hence it is a co-comparability graph and thus is all-perfect by Proposition 4.8

below. Let $\mathbf{z}$ be the demand vector with $z_v = 1$ on the three degree 2 nodes and $z_v = \frac{1}{2}$ on the other three nodes. We claim that $\mathbf{z}$ is the unique nonintegral vertex of $2\,QSTAB_2(H)$. Then by Proposition 4.1, $\mathrm{imp}_2(H) = \frac{1}{2}\mathrm{span}_2(H, \mathbf{z})$. Since the maximum number of nodes in a bipartite subgraph of H is 4, and $\sum_v z_v = \frac{9}{2}$, it follows that $\mathrm{span}_2(H, \mathbf{z}) \geq \frac{9}{4}$. But it is straightforward to find an appropriate covering which shows that equality holds.

It remains to establish the claim. Let $\mathbf{x}$ be a nonintegral vertex of $2\,QTAB_2(H) = 2\,QSTAB(H) \cap [0,1]^V$. Since each proper subgraph of H is 2-perfect, we must have that each $x_v > 0$. Also, since each vertex corresponds to a basic feasible solution and H has 6 nodes, there must be at least 6 tight constraints.

Suppose that all 4 triangles yield a tight constraint. Then opposite pairs of nodes must have the same value x_v. (An opposite pair consists of a degree-2 node and the nonadjacent degree-3 node.) Also, at least 6–4=2 coordinates x_v equal 1. Let the values on the opposite pairs be 1, x, and y, where $0 < x \leq y < 1$. (Note that $y < 1$ since $x + y \leq 1$.) But then $\mathbf{x}$ is not a vertex, since we could replace x, y by $x \pm \delta$ and $y \mp \delta$, where $\delta = \min\{x, 1 - y\} > 0$.

Hence at most 3 triangles yield a tight constraint, and so $x_v = 1$ for at least 3 nodes v. But no two of these nodes can lie on a triangle, so they must be the three degree-2 nodes. Now we are forced to put value $\frac{1}{2}$ on the other nodes. Thus indeed $\mathbf{z}$ is the unique nonintegral vertex of $2\,QSTAB(H) \cap [0,1]^V$, as claimed.

Now we consider three classes of perfect graphs G such that each G is all-perfect, namely comparability graphs, line graphs of bipartite graphs, and co-comparability graphs. Before we proceed further, let us remark that in contrast to the case $k = 1$, when $k \geq 2$ the complement $\overline{G}$ of a k-perfect graph G need not be k-perfect. Consider, for example, the odd holes and antiholes; see Proposition 5.3 below. Also, the Hajos graph G_2 shown in Figure 4.1 is not 2-perfect, but its complement shown in Figure 4.2 is the line-graph of a bipartite graph and so is 2-perfect, indeed all-perfect; see Proposition 4.7 below.

A graph is a *comparability graph* if there exists a partial order of the nodes such that distinct nodes u and v are adjacent exactly when they are comparable in the partial order. We use one lemma to prove that comparability graphs are all-perfect (and indeed we use this lemma again in section 6). The lemma involves "circular" (or "cyclic") interval colorings. An *m-circular interval coloring* of G with integral weight vector $\mathbf{x}$ is a multicoloring of the nodes of G using the colors $0, 1, \ldots, m-1$ such that for each node v of G, the set $\{i : v \text{ has color i}\}$ has cardinality x_v and forms an interval in the cyclic order $(0, 1, \ldots, m-1)$.

LEMMA 4.4. *Let G be a graph with integral weight vector $\mathbf{x}$. Suppose that there is an m-circular interval coloring of the graph $G, \mathbf{x}$, where m satisfies $m \geq k x_{\max}$ and $m \equiv 1 \ (mod\, k)$. Then* $\mathrm{span}_k(G, \mathbf{x}) \leq m$.

Proof. Consider the assignment $\phi(v) = \{ki (\mathrm{mod}\, m) : v \text{ has color } i\}$. Then for $i, j \in \{0, 1, \ldots, m-1\}$, $ki = kj$ is equivalent to $i = j$ since $m \equiv 1 (\mathrm{mod}\, k)$. It follows that $|\phi(v)| = x_v$ for each node v and $\phi(u) \cap \phi(v) = \emptyset$ for adjacent nodes u and v.

It remains to show that for each node $v \in V(G)$ and any two distinct elements $c_1, c_2 \in \phi(v)$, we have $|c_1 - c_2| \geq k$. For each node $v \in V(G)$, two distinct elements of $\phi(v)$ are of the form $kc + ik (\mathrm{mod}\, m)$ and $kc + jk (\mathrm{mod}\, m)$ with $0 \leq i < j \leq x_v - 1$ by the definition of ϕ and the fact that the colors of v form an interval in the cyclic order $0, 1, \ldots, m-1$. But $|k(j-i) - 0| = k(j-i) \geq k$ and $|m - k(j-i)| \geq m - (x_{\max} - 1)k \geq k$, and the assignment is therefore k-feasible and uses only the colors $0, \ldots, m-1$. ∎

PROPOSITION 4.5. *Each comparability graph is all-perfect.*

Proof. Let k be a positive integer, and let $G = (V, E)$ be a comparability graph. Let $\prec$ be a partial order on V such that distinct u and v are comparable if and only if $\{u, v\}$ is an edge of G. Let D be the corresponding acyclic (transitive) orientation of G, where we orient the edge $\{u, v\}$ from u to v if $u \prec v$. Let $\mathbf{x}$ be a integral weight vector. Form an acyclic directed graph D' from D by replacing each node v by a directed path of x_v nodes. Thus D' has nodes $v_1, \ldots, v_{x_v}$ for each $v \in V$, and there is an arc $u_i v_j$ in D' if and only if either $u = v$ and $i < j$, or $u \neq v$ and uv is an arc in D.

For each node v_i in D', let $\phi(v_i)$ be the maximum length (i.e., number of arcs) in a path in D' ending at v_i. Then ϕ takes values in $\{0, 1, \ldots, \omega(G, \mathbf{x}) - 1\}$; for each node $v \in V$, ϕ takes distinct consecutive values on the x_v nodes $v_1, \ldots, v_{x_v}$ of D'; and if uv is an arc of D, then $\phi(u_i) < \phi(v_j)$ for each i, j. Thus ϕ gives a proper coloring of $(G, \mathbf{x})$, using colors $\{0, 1, \ldots, \omega(G, \mathbf{x}) - 1\}$, such that for each node $v \in V$ the colors on v are consecutive. Hence there is an m-cyclic interval coloring of $G, \mathbf{x}$ with $m \leq \omega_k(G, \mathbf{x}) + k - 1$. So by Lemma 4.4, $\text{span}_k(G, \mathbf{x}) \leq \omega_k(G, \mathbf{x}) + k - 1$, and the result now follows by (2.6). $\square$

The first part of the above proof is not new. We defined circular interval colorings above, and in a similar way, when we use ordinary linear channels, we may define an *interval coloring* of $G, \mathbf{x}$. The *interval span* $\text{ispan}(G, \mathbf{x})$ is the smallest number for which an interval coloring exists. A graph G is called *superperfect* in [10] if $\text{ispan}(G, \mathbf{x}) = \omega(G, \mathbf{x})$ for each integral weight vector $\mathbf{x}$. Observe that by Lemma 4.4, any superperfect graph is all-perfect. Hoffman showed that any comparability graph is superperfect; see [10]. It is also shown there that any graph in a certain class is superperfect, where this class contains the complements of the even cycles.

We shall consider two more classes of perfect graphs and show that each graph in these classes is all-perfect. These classes are the line-graphs of bipartite graphs and the co-comparability graphs (that is, complements of comparability graphs). We give a unified proof treatment, based on the polyhedral characterization in Proposition 4.1.

Recall that a matrix is *totally unimodular* if each square submatrix has determinant 0 or ± 1. Let us call a polyhedron *totally unimodular* if it may be expressed as $\{\mathbf{x} : A\mathbf{x} \leq \mathbf{b}\}$ for some totally unimodular matrix A and integral vector $\mathbf{b}$. It is well known that such a polyhedron is *integral*; that is, it has the property that each face contains an integral vector; see, for example, [20]. If $S \subseteq R^n$ and I is a nonempty subset of the indices $\{1, \ldots, n\}$, we call $\{\mathbf{x} \in R^I : (\mathbf{x}, \mathbf{y}) \in S \text{ for some } \mathbf{y}\}$ the *projection* of S onto the coordinates I. If we start with an integral polyhedron and project onto some set of coordinates, then the resulting polyhedron is again integral.

LEMMA 4.6. *Let G be a perfect graph, and let $P = STAB(G)$ $(= QSTAB(G))$. If P is a totally unimodular polyhedron, or more generally the projection of such a polyhedron onto some set of coordinates, then G is all-perfect.*

Proof. Note first that if the $(m \times n)$ matrix A is totally unimodular, then so is any submatrix of the $((m + 2n) \times n)$ matrix obtained by stacking the matrices $A, I_n, -I_n$ above one another (where I_n denotes the $(n \times n)$ identity matrix). It follows that if P satisfies the condition in the lemma, then so does $Q = kP \cap [0, 1]^V$ for any positive integer k. Then Q is the projection of an integral polyhedron onto some set of co-ordinates, and so Q is integral. Hence the result follows from Proposition 4.1. $\square$

Now let us consider the line-graphs of bipartite graphs and use Lemma 4.6 to show that such graphs are all-perfect. The complements of these graphs need not be all-perfect: we have already seen that the Hajos graph is not 2-perfect, and it is the complement of the line-graph of a bipartite graph.

PROPOSITION 4.7. *Let $G = (V, E)$ be a bipartite graph. Then the line-graph $L(G)$ is all-perfect.*

Proof. Observe that

$$QSTAB(L(G)) = \left\{ \mathbf{x} \in R_+^E : \sum_{e: v \in e} x_e \leq 1 \ (\forall v \in V) \right\},$$

which is a totally unimodular system (see, for example, [20]); so the result follows from Lemma 4.6. $\square$

Finally in this section, we use Lemma 4.6 to show that co-comparability graphs are all-perfect.

PROPOSITION 4.8. *Each co-comparability graph is all-perfect.*

Proof. Let $\prec$ be a partial order on V such that distinct nodes u and v are adjacent in G if and only if they are incomparable under $\prec$. We construct a directed graph D as follows. There are nodes v^- and v^+ for each node v in V, together with a new source node s and sink node t. There is an arc st, there are arcs sv^- and v^+t for each $v \in V$, and there are arcs u^+v^- for each pair of nodes $u, v \in V$ with $u \prec v$. Also, there is an arc v^-v^+ for each $v \in V$. We shall identify in the obvious way a vector indexed by the arcs v^-v^+ with a vector indexed by V.

Note that a stable set in G corresponds to an $s - t$ path in D, and a convex combination of incidence vectors of stable sets of G corresponds to a unit volume $s-t$ flow in D. Now $\mathbf{x} \in STAB(G)$ if and only if $\mathbf{x}$ is a convex combination of incidence vectors of stable sets of G. Thus $\mathbf{x} \in STAB(G)$ if and only if $\mathbf{x}$ "is" the projection onto the arcs v^-v^+ of a unit volume $s - t$ flow in D. But such flows in D form a totally unimodular polyhedron (since the node-arc incidence matrix of D is totally unimodular), so the result follows from Lemma 4.6. $\square$

5. Minimal non-*k*-perfect graphs. In this section we consider minimal non-k-perfect graphs, in other words, graphs G which are not k-perfect, but deleting any node yields a k-perfect graph. The strong perfect graph theorem [2] asserts that the only minimal non-1-perfect graphs are the odd holes and antiholes, that is, the odd cycles C_n for $n \geq 5$ and their complements; so there would be a "small" list of excluded induced subgraphs for perfection. Can we hope for such a concise result for k-perfect graphs? Regrettably, the answer is no, at least not in this form; see Theorem 5.4.

Before we prove this theorem, we consider the k-imperfection ratio of minimal non-k-perfect graphs and of the odd holes and antiholes. We show first that for any minimal non-k-perfect graph G on n nodes, $\mathrm{imp}_k(G) \leq n/(n-1)$. To do this we need the following lemma.

LEMMA 5.1. *If the nodes of a graph G can be covered q times by p induced subgraphs $H_1, \ldots, H_p$, then*

$$\mathrm{imp}_k(G) \leq \frac{1}{q} \sum_{i=1}^{p} \mathrm{imp}_k(H_i).$$

Proof. For every weight vector $\mathbf{x}$ of G, we have

$$q \,\mathrm{span}_k^f(G, \mathbf{x}) = \mathrm{span}_k^f(G, q\mathbf{x}) \leq \sum_{i=1}^{p} \mathrm{span}_k^f(H_i, \mathbf{x})$$

$$\leq \sum_{i=1}^{p} \mathrm{imp}_k(H_i)\, \omega_k^f(H_i, \mathbf{x}) \leq \omega_k^f(G, \mathbf{x}) \sum_{i=1}^{p} \mathrm{imp}_k(H_i),$$

and the result now follows by the definition of $\mathrm{imp}_k(G)$. □

PROPOSITION 5.2. *For each $k \geq 1$, if G is a minimal non-k-perfect graph on n nodes, then*

$$\mathrm{imp}_k(G) \leq \frac{n}{n-1}.$$

Proof. The removal of any node v yields a k-perfect graph, and hence G can be covered $n-1$ times by n k-perfect graphs. Lemma 5.1 now yields the result. □

We can now determine the k-imperfection ratios of the odd holes and antiholes. Recall that we already know that even cycles and their complements are k-perfect for all k, since even cycles are bipartite, and thus are comparability graphs. Let n be odd and at least 5. Then C_n is k-perfect for $k \geq 3$, since $\chi(C_n) \leq 3$, and $\overline{C}_n$ is k-perfect for all $k \geq (n+1)/2$ since $\chi(\overline{C}_n) \leq (n+1)/2$. The following proposition completes the picture.

PROPOSITION 5.3. *Let $n \geq 5$ be an odd integer. Then the odd hole C_n is minimal non-2-perfect, and* $\mathrm{imp}_2(C_n) = \frac{n}{n-1}$. *Also, for each $k = 1, \ldots, \frac{n-1}{2}$, the odd antihole $\overline{C}_n$ is minimal non-k-perfect, and* $\mathrm{imp}_k(\overline{C}_n) = \frac{n}{n-1}$.

Proof. Since $\omega(C_n) = 2$ and $\chi_f(C_n) = \frac{2n}{n-1}$, Lemma 3.1 shows that $\mathrm{imp}_k(C_n) \geq \frac{n}{n-1}$. Since bipartite graphs are 2-perfect, it follows that C_n is minimal non-2-perfect, and so $\mathrm{imp}_k(C_n) \leq \frac{n}{n-1}$ by the last proposition.

Since $\omega(\overline{C_n}) = \frac{n-1}{2}$ and $\chi_f(\overline{C_n}) = \frac{n}{2}$, Lemma 3.1 shows that $\mathrm{imp}_k(\overline{C_n}) \geq \frac{n}{n-1}$. Since co-bipartite graphs are 2-perfect, it follows that $\overline{C_n}$ is minimal non-k-perfect, and so $\mathrm{imp}_k(\overline{C_n}) \leq \frac{n}{n-1}$ by the last proposition. □

We saw at the end of the introduction that cliques and stable sets always form k-perfect graphs. Hence by Lemma 5.1, the cochromatic number $z(G)$ of G is an upper bound on $\mathrm{imp}_k(G)$. (Recall that the cochromatic number $z(G)$ is the least number of stable sets and cliques needed to cover the graph G.) For the case $k = 1$, one can strengthen this bound and obtain $\mathrm{imp}_k(G) \leq z(G)/2$ for any nontrivial graph G [7]. Proposition 4.3 shows that this is not true when $k \geq 2$, but it is easy to see that $\mathrm{imp}_k(G) < z(G)/2 + 1$ by noting that any two stable sets and any two cliques induce an all-perfect graph.

We now consider the number of (node-)minimal non-k-perfect graphs on n nodes and show that when $k \geq 2$ there are many such graphs.

THEOREM 5.4. *For each integer $k \geq 2$, let $f_k(n)$ be the number of nonisomorphic minimal non-k-perfect graphs on at most n nodes. Then $f_k(n)$ grows at least exponentially with n.*

The rest of this section is devoted to proving this result. We first consider the easier case when $k \geq 3$. After that, we start the proof for the case $k = 2$, then break to state and prove four lemmas, and then complete the proof.

Proof. Consider first the case $k \geq 3$. We call a graph G k-critical if $\chi(G) = k$ and deleting any node yields a graph with chromatic number $k-1$. Note that each $(k+1)$-critical graph G other than K_{k+1} has $\omega(G) \leq k$, and so G is node-minimal non-k-perfect by Proposition 3.3. It is known [23] that the number of nonisomorphic 4-critical graphs on at most n nodes is at least $c^{(n^2)}$ for some $c > 1$. By adding to a 4-critical graph G a clique with l nodes, each of which is adjacent to each of the nodes of G, one obtains a $(4+l)$-critical graph. This completes the proof for the case

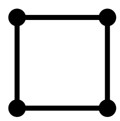

FIG. 5.1. *The seed graph H.*

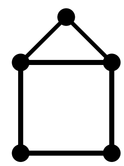

FIG. 5.2. *The graph H′.*

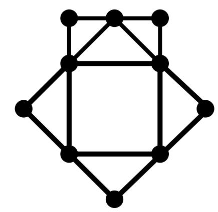

FIG. 5.3. *The graph $G = G(H, e)$.*

$k \geq 3$, but this approach will not work for $k = 2$, as the only 3-critical graphs are the odd cycles.

We now consider the case $k = 2$. We shall show that each graph G formed as below is node-minimal non-2-perfect (and outerplanar and perfect). The smallest graph G we shall construct is the Hajos graph, as shown in Figure 4.1.

Construct the graph G as follows. Start with any 2-node-connected outerplanar bipartite graph H with at least one edge, the "seed" graph. It has a unique outerplanar embedding with each node on the infinite face. Pick an edge $e = uv$ on the infinite face and add a new degree-2 node v^* adjacent to u and v. The new graph H' is 2-node-connected and outerplanar, with a unique outerplanar embedding such that each node is on the infinite face. Let C be the Hamilton cycle bounding the infinite face, which is in fact the unique Hamilton cycle in H. Note that C has an odd number of nodes. For each edge $f = ab$ on C, add a new degree-2 node v_f adjacent to a and b. This gives the desired graph $G = G(H, e)$; see Figures 5.1–5.3. The graph G is outerplanar, and as it has no odd holes, it is perfect [24]. It remains to show three things.

1. It is easy to see that the number of nonisomorphic graphs G as above on at most n nodes grows at least exponentially with n, since this holds for the seed graphs H.
2. The graph G is not 2-perfect. For we may give weight 2 to each of the degree-2 nodes added at the last step when we formed G from H', and weight 1 to each of the other nodes (which came from H'). It is easy to see that $\omega_2(G, \mathbf{x}) = 4$. But $\text{span}_2^f(G, \mathbf{x}) > 4$. For suppose that there is a solution $\mathbf{y}$ to the LP for $\text{span}_2^f(G, \mathbf{x})$ with value 4. Then any 2-colorable graph S actually used (that is, with $y_S > 0$) must contain all the nodes v with $x_v = 2$ and so can contain at most $\frac{|C|-1}{2}$ nodes of the circuit C. Hence the total weight covered on the nodes of C is at most $|C| - 1$, and so the covering is not feasible for $G, \mathbf{x}$.
3. Finally, we must check that any proper induced subgraph of G is 2-perfect. We shall complete this last requirement, and thus complete the proof of the theorem, after stating and proving four lemmas.

The next lemma (together with Proposition 1.1) shows that each node-minimal non-k-perfect graph is 2-node-connected.

LEMMA 5.5. *Let the graph G be connected, with a cut node v. Let $G_1, G_2, \ldots$ be the components formed when the node v is deleted, and for $i = 1, 2, \ldots$ let H_i be the graph formed by adding back the node v to G_i (that is, H_i is the subgraph of G induced by $V(H_i) \cup \{v\}$). Then for any positive integer k,*

$$\text{imp}_k(G) = \max_i \{\text{imp}_k(H_i)\}.$$

Proof. It suffices to note that if $A_i \subseteq V(H_i)$ is k-colorable for each $i = 1, 2, \ldots$

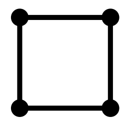

FIG. 5.4. *A bipartite graph H.*

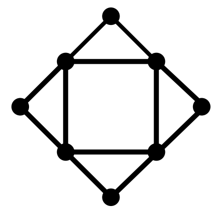

FIG. 5.5. *The graph $\hat{H}$.*

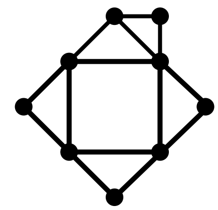

FIG. 5.6. *The all-perfect graph G.*

(that is, each induced subgraph $G[A_i]$ is k-colorable) and either $v \in \cap_i A_i$ or $v \notin \cup_i A_i$, then $\cup_i A_i$ is k-colorable. ∎

LEMMA 5.6. *Let the 2-node-connected graph G have no odd holes. Suppose that there is a separating set consisting of a node u and an edge vw, where it is not the case that u is adjacent to both v and w. Let G_1 and G_2 be the components formed when the separating set is deleted, and let H_1 and H_2 be the graphs formed by adding back the node u to G_1 and G_2, respectively. Then*

$$\mathrm{imp}_2(G) = \max\{\mathrm{imp}_2(H_1), \mathrm{imp}_2(H_2)\}.$$

Proof. It suffices to show that if $A_1 \subseteq V(H_1)$ and $A_2 \subseteq V(H_2)$ are 2-colorable and either $u \in A_1 \cap A_2$ or $u \notin A_1 \cup A_2$, then $A_1 \cup A_2$ is 2-colorable. This is obvious if $u \notin A_1 \cup A_2$, so assume that $u \in A_1 \cap A_2$. Suppose that there is an odd cycle contained in $A_1 \cup A_2$. This cycle must go through both the node u and the edge vw. Without loss of generality, we may assume that v is in H_1 and w is in H_2. There is a u-v path in H_1: consider a shortest such path Q_1. Similarly, there is a u-w path in H_2: consider a shortest such path Q_2. Then the cycle formed from Q_1, Q_2, and the edge vw is an odd hole in G, a contradiction. ∎

LEMMA 5.7. (a) *Start with a bipartite graph H. For each edge $e = ab$, add a new degree-2 node v_e adjacent to a and b. Then the graph $\hat{H}$ formed is all-perfect.*

(b) *Now take an edge a_0b_0 in $\hat{H}$, where the node a_0 is in H and the node b_0 is in $\hat{H}$ but not in H, and add a new degree-2 node v^* adjacent to a_0 and b_0. Then the graph G formed is all-perfect.*

See Figures 5.4–5.6 for an illustration of the construction.

Proof. It suffices to prove (b). Let $\mathbf{x}$ be a weight vector for G. Denote $\omega(G, \mathbf{x})$ by ω. We shall show that there is an interval coloring of $G, \mathbf{x}$ using colors $1, \ldots, \omega$. The result will then follow by Lemma 4.4.

Properly color the nodes of H with the two labels "low" and "high," where a_0 is "low." Give each "low" node v the "low" colors $1, \ldots, x_v$; give each "high" node v the "high" colors $\omega - x_v + 1, \ldots, \omega$. For each edge e of H, assign node v_e in $\hat{H}$ the interval $x_a + 1, \ldots, x_a + x_{v_e}$, where a is the "low" node incident with e. This gives an interval coloring of $\hat{H}, \mathbf{x}$ with colors $1, \ldots, \omega$.

Finally we handle the node v^* formed at the last stage. Note that node a_0 has been assigned the "low" interval of colors $1, \ldots, x_{a_0}$, and node b_0 has been assigned the "next" interval of colors $x_{a_0} + 1, \ldots, x_{a_0} + x_{b_0}$. Thus we may assign to v^* the "high" interval of colors $\omega - x_{v^*} + 1, \ldots, \omega$. ∎

Completion of the proof of Theorem 5.4. Recall that we must check that any proper induced subgraph of the graph $G = G(H, e)$ is 2-perfect. We use induction on the number of nodes of the seed graph H. The base case is the Hajos graph, which we have already handled. Now let $G = G(H, e)$, where H has Hamilton circuit C, and suppose that we know the result for any smaller seed graph. Let v be a node in G, and let G' be the graph $G - v$ obtained from G by deleting v. We must show that G' is 2-perfect. Let T denote the unique triangle in H'. We consider four cases. The first two cover the possibilities when v is in H', and the second two when v is not in H' (and so v has degree 2).

1. Suppose that v is in T. Then $H' - v$ is bipartite, so G' is 2-perfect by Lemma 5.7(a).
2. Suppose that v is in H' and not in T, and so v is in H. Consider the outerplanar embedding of H'. Let F be a bounded face such that its boundary cycle D contains v. Let x be a node on D not adjacent to v. Then x is a cut-node for G'. Let $\tilde{G}$ be the graph obtained by adding x back to the component of $G' - v$ which contains v^*. Then $\mathrm{imp}_2(G') = \mathrm{imp}_2(\tilde{G})$ by Lemmas 5.5 and 5.7(a). But $\tilde{G}$ is 2-perfect by Lemma 5.5 and the induction hypothesis.
3. Suppose that v is v_f for some edge $f = ab$ in C and not in T. Consider the outerplanar embedding of H'. Let F be the bounded face containing f on its boundary. There is a node x on F other than a and b such that x and the edge f form a separating set S for G'. Let $\tilde{G}$ be the graph obtained by adding x back to the component of $G' - S$ which contains v^*. Then $\mathrm{imp}_2(G') = \mathrm{imp}_2(\tilde{G})$ by Lemmas 5.6 and 5.7(a). But $\tilde{G}$ is 2-perfect by Lemma 5.5 and the induction hypothesis.
4. The remaining case is when v is v_f for one of the two edges f of C in T, and this is exactly the case covered by Lemma 5.7(b). ∎

6. Disk graphs. In this section we bound the k-imperfection ratio of unit disk graphs, general disk graphs, and induced subgraphs of the triangular lattice (which are a subclass of unit disk graphs).

In a *unit disk graph* the nodes can be represented by unit diameter (closed) disks in the plane such that two distinct nodes are adjacent if and only if the corresponding disks intersect. These graphs are important in radio channel assignment, since we obtain a unit disk graph as an interference graph if we assume that the service area of a transmitter corresponds to a unit size disk. It is known [21, pp. 60–63] that we can fractionally cover the nodes of a unit disk graph G d times by about $4.36d$ graphs, which are disjoint unions of cliques, and hence by Lemma 5.1 we have $\mathrm{imp}_k(G) \leq 4.36$. The next result improves this bound: it extends Proposition 3.3 of [7], which is the case $k = 1$.

PROPOSITION 6.1. *For each unit disk graph G and each positive integer k,*

$$\mathrm{imp}_k(G) \leq 1 + 2/\sqrt{3} \sim 2.155.$$

Proof. If the center of each disk lies in a stripe of width $\sqrt{3}/2$, then the corresponding unit disk graph is a co-comparability graph [11], and so is all-perfect by Proposition 4.8. If t is sufficiently large, then with t such graphs we can cover a given finite unit disk graph at least $\frac{\sqrt{3}}{2+\sqrt{3}}\,t$ times; see the proof of Proposition 3.3 of [7]. The result now follows by Lemma 5.1. ∎

A generalization of a unit disk graph is a disk graph. A *disk graph* is a graph the nodes of which can be represented by (closed) disks in the plane such that two nodes are adjacent if and only if the corresponding disks intersect. (The nodes may correspond to transmitters with different powers.) It is easy to verify that the neighborhood of the node represented by a smallest size disk can be covered by 6 cliques. Hence the bound below follows from the lemma after it.

Proposition 6.2. *For each disk graph G and each positive integer k,*

$$\mathrm{imp}_k(G) \le \begin{cases} 6 & \text{if} \quad k \le 6, \\ 6 - \frac{6}{k} & \text{if} \quad k \ge 6. \end{cases}$$

Lemma 6.3. *For each graph G and $t \ge 1$, if each induced subgraph of G contains a node the neighborhood of which can be covered at least p/t times by a family of p cliques, then $\mathrm{imp}_k(G) \le t + \max\{0, 1 - t/k\} < t + 1$.*

Proof. Let G have n nodes. We can order the nodes of G in such a way that, for each $i = 2, \ldots, n$, the nodes of $\{v_1, \ldots, v_{i-1}\}$ which are adjacent to v_i can be covered q_i times by a family of p_i cliques, where $p_i/q_i \le t$. Consider any integral weight vector $\mathbf{x}$ for G. Now, we greedily color the nodes of G in the order above, i.e., when we come to color the node v we assign to it the lowest color, say c, which is not already assigned to a neighbor of v, then the lowest color available which is at least $c + k$, and so on until the node v is colored x_v times. Clearly, we obtain a k-feasible assignment. We claim that this procedure uses only the colors up to $t\omega_k(G, \mathbf{x}) + (t - k)$ if $k \le t$ and the colors up to $(t + 1 - t/k)\omega_k(G, \mathbf{x})$ if $t < k$. To see this, for each $i = 2, \ldots, n$, let $w(i)$ be the sum of the values x_{v_j} over all neighbors v_j of v_i with $j < i$. Observe that we have $p_i(\omega(G, \mathbf{x}) - x_{v_i}) \ge q_i w(i)$, and so $w(i) \le t(\omega(G, \mathbf{x}) - x_{v_i})$. When we come to color v_i, at most $w(i)$ colors are already used for the neighbors of v_i. Thus we can color v_i with x_{v_i} colors using only colors up to $t(\omega(G, \mathbf{x}) - x_{v_i}) + k(x_{v_i} - 1) = t\omega(G, \mathbf{x}) + (k - t)x_{v_i} - k$. Therefore we use only the colors up to $t\omega(G, \mathbf{x}) - k \le t\omega_k(G, \mathbf{x}) + (t - k)$ if $k - t \le 0$, and only the colors up to $t(\omega(G, \mathbf{x}) - 1) + (1 - t/k)(x_{\max} - 1)k \le (t + 1 - t/k)\omega_k(G, \mathbf{x})$ if $k - t > 0$. The result now follows by (2.6). ∎

A subclass of unit disk graphs, the class of finite induced subgraphs G of the triangular lattice, has attracted considerable attention from researchers interested in the channel assignment problem. The reason for this interest is the fact that when the potential service area for each transmitter is a unit diameter disk in the plane, arranging the transmitters on a triangular lattice is most efficient, in the sense of achieving universal coverage with as few transmitters as possible. Observe that such a graph G is k-perfect for each $k \ge 3$, since $\chi(G) \le 3$.

Proposition 6.4. *Let G be a finite induced subgraph of the triangular lattice. Then $\mathrm{imp}_k(G) \le \frac{4}{3}$ for $k = 1$ and $k = 2$.*

Proof. For $k = 1$ we can use a result obtained in [18], which says that $\mathrm{span}_1(G, \mathbf{x}) \le (4\omega_1(G, \mathbf{x}) + 1)/3$ and thus implies that $\mathrm{imp}(G) \le 4/3$ by (2.6); see also [7]. To prove the result for $k = 2$ consider a weight vector $\mathbf{x}$ of G. We will show that

$$\mathrm{span}_2(G, \mathbf{x}) \le \frac{4\omega_2(G, \mathbf{x}) + 8}{3}. \tag{6.1}$$

The result will then follow by (2.6). This result appears in [22] in a slightly weaker form, but since the improvement is easy from [18] once you know Lemma 4.4, we spell out a proof.

We may assume without loss of generality that $\omega(G, \mathbf{x}) \ge 2x_{\max}$, since for graphs with no edges the result is trivial, and for all other graphs if $\omega(G, \mathbf{x})$ is strictly less than $2x_{\max}$, then we could increase the weights at some nodes without increasing $\omega_2(G, \mathbf{x})$. Let us denote $\omega(G, \mathbf{x})$ simply by ω. By the proof of the main result in [18] we can find weight vectors $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ such that $\mathbf{x} = \mathbf{x}^{(1)} + \mathbf{x}^{(2)}$, and the following holds:

- there is an ω-cyclic interval coloring of $G, \mathbf{x}^{(1)}$, and
- the subgraph H of G induced by the nodes v with $x_v^{(2)} > 0$ is bipartite (indeed acyclic), and $x_v^{(2)} \le (\omega + 2)/6$ for each v.

Hence by Lemma 4.4, $\text{span}_2(G, \mathbf{x}^{(1)}) \leq \omega + 1$, and since the graph H is bipartite, $\text{span}_2(G, \mathbf{x}^{(2)}) \leq 2x^{(2)}_{\max} \leq (\omega + 2)/3$. Hence

$$\text{span}_2(G, \mathbf{x}) \leq \text{span}_2(G, \mathbf{x}^{(1)}) + \text{span}_2(G, \mathbf{x}^{(2)}) + 1 \leq (4\omega + 8)/3,$$

as required. ∎

For $k = 1$ and $k = 2$ we can have $\text{imp}_k(G) > 1$ for a finite induced subgraph of the triangular lattice, since the cycle C_9 on 9 nodes is such a graph and we have already seen that $\text{imp}(C_9) = \text{imp}_2(C_9) = 9/8$. It would be interesting to determine whether in fact $\text{imp}_2(G) \leq 9/8$ for all induced subgraphs of the triangular lattice; see [7] for a discussion on the corresponding question for $\text{imp}(G)$.

7. Random graphs. In this section we use results on the imperfection ratio of a random graph from [8] to prove corresponding results for the k-imperfection ratio. First we need one deterministic lemma, which gives a bound on the k-imperfection ratio of a graph G in terms of the imperfection ratios of k induced subgraphs of G.

LEMMA 7.1. *Let $V_0, \ldots, V_{k-1}$ be a partition of the node set of a graph G, and let G^i be the subgraph induced by V_i for $i = 0, \ldots, k-1$. Then*

$$\text{imp}_k(G) \leq k \max_i \{\text{imp}(G^i)\},$$

where the maximum is over $i = 0, \ldots, k-1$.

Proof. Suppose that G has n nodes. Let $\mathbf{x}$ be a nonzero integral weight vector for G. Let $G^i_{\mathbf{x}}$ denote the graph where each node v of $V(G^i)$ is replaced by a clique of x_v nodes. Recall that

$$\chi(G^i_{\mathbf{x}}) \leq \chi_f(G^i, \mathbf{x}) + |V(G^i)| \leq \chi_f(G^i, \mathbf{x}) + n; \tag{7.1}$$

see, for example, the proof of Lemma 2.1.

For each $i = 0, \ldots, k-1$ consider a coloring of $G^i, \mathbf{x}$ which uses the colors $\{1, 2, \ldots, \chi(G^i_{\mathbf{x}})\}$. To a node $v \in V_i$ which is colored with the x_v colors $\{c_1, \ldots, c_{x_v}\}$, assign the x_v new colors $\{kc_1 - i, \ldots, kc_{x_v} - i\}$. This yields a k-feasible assignment for $G, \mathbf{x}$, and so

$$\text{span}_k(G, \mathbf{x}) \leq k \max_i \{\chi(G^i_{\mathbf{x}})\} \leq k \max_i \{\chi_f(G^i, \mathbf{x})\} + kn$$

by (7.1). Hence since $\omega(G^i, \mathbf{x}) \leq \omega_k(G, \mathbf{x})$ for each i,

$$\begin{aligned}\frac{\text{span}_k(G, \mathbf{x})}{\omega_k(G, \mathbf{x})} &\leq k \max_i \left\{ \frac{\chi_f(G^i, \mathbf{x})}{\omega(G^i, \mathbf{x})} \right\} + \frac{kn}{\omega_k(G, \mathbf{x})} \\ &\leq k \max_i \{\text{imp}(G^i)\} + \frac{n}{x_{\max}},\end{aligned}$$

and the result follows by (2.6). ∎

The first theorem in this section shows that for dense random graphs the k-imperfection ratio is asymptotically independent of k.

THEOREM 7.2. *Let k be a positive integer, and let $0 < p < 1$. Then for any $\eta > 0$, a.s.*

$$\frac{n}{4 \log_{\frac{1}{p}} n \log_{\frac{1}{q}} n} \leq \text{imp}_k(G_{n,p}) \leq (1 + \eta) \frac{n}{4 \log_{\frac{1}{p}} n \log_{\frac{1}{q}} n},$$

where $q = 1 - p$.

Proof. First we consider the lower bound. A simple first moment argument shows that the following two conditions on $G_{n,p}$ are a.s. satisfied (see, for example, [4]):

$$\alpha(G_{n,p}) \le 2\log_{\frac{1}{q}} n \qquad \text{and} \qquad \omega(G_{n,p}) \le 2\log_{\frac{1}{p}} n.$$

In addition, it is easy to verify that a.s. $\omega(G_{n,p}) \ge k$, and hence by Lemma 3.1, we have a.s.

$$\mathrm{imp}_k(G) \ge \frac{\chi_f(G_{n,p})}{\omega(G_{n,p})} \ge \frac{n}{\alpha(G_{n,p})\omega(G_{n,p})} \ge \frac{n}{4\log_{\frac{1}{p}} n \log_{\frac{1}{q}} n}.$$

Now we consider the upper bound. By Lemma 7.1, for any $t \ge 0$

$$P(\mathrm{imp}_k(G_{n,p}) > tk) \le kP(\mathrm{imp}(G_{\lceil n/k\rceil,p}) > t),$$

and hence by Theorem 3.3 of [8], we have a.s.

$$\begin{aligned}\mathrm{imp}_k(G_{n,p}) &\le k\,(1+\eta/2)\frac{\lceil n/k\rceil}{4\log_{\frac{1}{p}}\lceil n/k\rceil \log_{\frac{1}{q}}\lceil n/k\rceil}\\ &\le (1+\eta)\frac{n}{4\log_{\frac{1}{p}} n\log_{\frac{1}{q}} n} \qquad \text{for sufficiently large } n. \qquad \square\end{aligned}$$

The next result corresponds to Theorem 3.5 of [8], which shows that for suitable sparse random graphs $G_{n,p}$, $\mathrm{imp}(G_{n,p})$ is about $np/(4\ln np)$. Now we may allow slightly denser graphs and see that $\mathrm{imp}_k(G_{n,p})$ is about $np/(2k\ln np)$ when $k \ge 2$. Note that this formula depends on k, in contrast to the dense case, and it does not give the correct answer for $k = 1$.

THEOREM 7.3. *Let $k \ge 2$, and suppose that $p = p(n)$ satisfies $np \to \infty$ as $n \to \infty$ but $p = o(n^{-2/(k+1)})$. Then for any $\varepsilon > 0$, a.s.*

$$(1-\varepsilon)\frac{np}{2k\ln np} \le \mathrm{imp}_k(G_{n,p}) \le (1+\varepsilon)\frac{np}{2k\ln np}.$$

Proof. Since $p = o(1)$ and $np \to \infty$ as $n \to \infty$, for any $\varepsilon > 0$ we have a.s.

$$\chi(G_{n,p}) \le (1+\varepsilon)\frac{np}{2\ln np}; \tag{7.2}$$

see [16]. The required upper bound on the k-imperfection ratio now follows by Lemma 3.1(b).

For the lower bound, assume that $0 < \varepsilon < 1$, and let $\delta > 0$ satisfy $(1-\delta)/(1+\delta) \ge 1-\varepsilon$. By [5], a.s.

$$\alpha(G_{n,p}) \le (1+\delta)\frac{2\ln np}{p}.$$

Also, the expected number of cliques with $k+1$ nodes is $\binom{n}{k+1}p^{\binom{k+1}{2}}$, which is at most $n^{k+1}p^{\frac{k(k+1)}{2}}$. Hence the probability that the number of cliques with $k+1$ nodes in $G_{n,p}$ is at least δn is at most $(np^{\frac{k+1}{2}})^k/\delta$. Since $np^{\frac{k+1}{2}} = o(1)$, there is a.s. an induced subgraph H of $G_{n,p}$ on at least $n - \delta n$ nodes with $\omega(H) \le k$. But then a.s.

$$\chi_f(H) \ge \frac{n-\delta n}{\alpha(H)} \ge \frac{1-\delta}{1+\delta}\frac{np}{2\ln np} \ge (1-\varepsilon)\frac{np}{2\ln np}.$$

Hence by Lemma 3.1(a), a.s.

$$\mathrm{imp}_k(G_{n,p}) \geq \mathrm{imp}_k(H) \geq (1-\varepsilon)\frac{np}{2k\ln np},$$

as required. ☐

There is a similar result for random r-regular graphs $G_{n,r}$, which are graphs taken uniformly at random from the set of all r-regular graphs on the n nodes $\{1,2,\ldots,n\}$ (where rn is even). The limit in the following theorem refers to $n \to \infty$ with n restricted to even integers if r is odd.

THEOREM 7.4. *Let $k \geq 2$. For each integer $r \geq 2$, there exists $\varepsilon = \varepsilon(r) > 0$ such that $\varepsilon(r) \to 0$ as $r \to \infty$ and such that for each fixed $r \geq 2$, a.s.*

$$\frac{r}{2k\ln r} \leq \mathrm{imp}_k(G_{n,r}) \leq (1+\varepsilon)\frac{r}{2k\ln r}.$$

Proof. We may argue much as in the proof of Theorem 7.3. To do this, the upper bound (7.2) has to be replaced by the following result from [6]: for each $r \geq 2$, there exists $\varepsilon = \varepsilon(r) > 0$ with $\varepsilon(r) \to 0$ as $r \to \infty$, such that

$$\chi(G_{n,r}) \leq (1+\varepsilon)\frac{r}{2\ln r}.$$

The lower bound follows from the result that $G_{n,r}$ a.s. contains a triangle-free induced subgraph H with $\chi_f(H) \geq r/2\ln r$; see the proof of Theorem 3.6 of [8]. ☐

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