Bieberbach groups with finite commutator quotient and point-group $C_{p^n} \times C_{p^m}$

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1 Introduction

A Bieberbach group G of dimension n is a torsion-free group satisfying the exact short sequence

$$0 \to V \to G \to P \to 1,$$

where P is a finite group acting faithfully on a free abelian subgroup V of G of rank n. The groups P and V are called *point-group (or holonomy group)* and *translation subgroup* of G, respectively. Bieberbach groups appear as fundamental groups of compact, connected, flat Riemannian manifolds (flat manifolds for short). If X is a flat manifold of dimension n, then its fundamental group G is a Bieberbach group of dimension n, and determines X up to affine equivalence [3]. It is well known that the first Betti number of the manifold X is zero if and only if the commutator quotient of G is finite, which is also equivalent to the triviality of the centre of G [7]. This paper is concerned with metabelian Bieberbach groups having finite commutator quotient and point-group isomorphic to $C_{p^n} \times C_{p^m}$, where p is prime and $n, m \in \mathbb{N}$.

If P is a finite group, Auslander and Kuranishi [1] have shown that there is a Bieberbach group having point-group isomorphic to P. Following the terminology in Hiller and Sah [7], we will call a finite group P *primitive* if it can be realized as point-group of a Bieberbach group with finite commutator quotient. In contrast to the result of Auslander and Kuranishi, not every finite group is primitive. In [7], Hiller and Sah prove that a finite group P is primitive if and only if no cyclic Sylow p-subgroup of P has a normal complement. Thus, any non-cyclic p-group is primitive. In [5], Gupta and Sidki proved that a Bieberbach group with finite commutator quotient and point-group isomorphic to $C_p \times C_p$ contains a subgroup isomorphic to

$$K(p) = \left\langle a, b \mid (a^p)^{t(p,b)}, (b^p)^{t(p,a)}, [[a,b], a^p], [[a,b], b^p],\ metabelian \right\rangle,$$

where $t(s,x) = \sum_{i=0}^{s-1} x^i$ and the presentation is written relative to the variety of metabelian groups. The group $K(2)$ is isomorphic to the fundamental group of the Hantzsche-Wendt manifold [6].

[1]Supported by the Brazilian scientific agency CNPq

We generalize the class $K(p)$ and define a family of metabelian groups with finite commutator quotient, given by the presentation

$$K(p^n, p^m) = \langle a, b \mid (a^{p^n})^{t(p^m,b)}, (b^{p^m})^{t(p^n,a)}, \left[[a,b], a^{p^n}\right], \left[[a,b], b^{p^m}\right], \ metabelian \rangle ,$$

where $n, m \in \mathbb{N}$. In Section 2, we prove

Theorem A. *Let G be a Bieberbach group with finite commutator quotient and point-group isomorphic to $C_{p^n} \times C_{p^m}$. Then G contains a subgroup isomorphic to a torsion-free quotient of $K(p^n, p^m)$. Furthermore, $K(p^n, p^m)$ itself is a Bieberbach group of this type.*

In Section 3, we describe the group of automorphisms of $K(p^n, p^m)$. Since $K(p^n, p^m)$ has trivial centre, we concentrate on finding the group of outer automorphisms of $K(p^n, p^m)$.

Theorem B. *Let G denote the group $K(p^n, p^m)$.*
a) If $(p, n, m) = (2, 1, 1), (3, 1, 1), (2, 1, 2)$ or $(2, 2, 2)$, then $Out(G)$ is isomorphic to a subgroup of $GL(2, \mathbb{Z}_{p^{n+m}})$ and is therefore finite;
b) In the remaining cases, $Out(G)$ is the extension of an abelian subgroup of $GL(2, \mathbb{Z}_{p^{n+m}})$ by a crystallographic group.

Gupta and Sidki have shown in [5] that $K(p, p)$ has no non-trivial torsion-free quotients. We study the torsion-free quotients of $K(p^n, p^m)$ in a forthcoming paper.

2 The group $K(p^n, p^m)$

Let

$$F_n = \langle x_1, \ldots, x_n | metabelian \rangle$$

denote the free group of rank n in the variety of finitely generated metabelian groups. A finitely generated metabelian group G is presented as

$$\langle x_1, \ldots, x_n | R_1, R_2, \ldots, R_s, metabelian \rangle \cong F_n / \langle R_1, R_2, \ldots, R_s \rangle^{F_n}.$$

We define the following polynomials, for $s \in \mathbb{N}$:

$$t(s, x) = 1 + x + \ldots + x^{s-1}$$
$$d(x) = x - 1$$
$$l(s, x) = (t(s, x) - s)/d(x) = \sum_{i=1}^{s-1} t(i, x) = \sum_{i=0}^{s-2} (s - i - 1) x^i.$$

If $g, x_1, \ldots, x_n$ are elements of a group G, $s_1, \ldots, s_n \in \mathbb{Z}$, and $[(g^{s_i})^{x_i}, (g^{s_j})^{x_j}] = e$ for $1 \le i, j \le n$, then we write $g^{s_1 x_1 + s_2 x_2 + \ldots + s_n x_n}$ for $(g^{s_1})^{x_1} (g^{s_2})^{x_2} \ldots (g^{s_n})^{x_n}$.

Whenever it is convenient, we will write additively in abelian subgroups of G. Notice that when the commutator subgroup G' of G is abelian, using the module notation, we write

$$[x_1, x_2^s] = [x_1, x_2].t(s, x_2).$$

Proposition 2.1 *Let G be a Bieberbach group with finite commutator quotient and point-group isomorphic to $C_{p^n} \times C_{p^m}$. Then G contains a subgroup isomorphic to a torsion-free quotient of $K(p^n, p^m)$.*

Proof. Let V be the translation subgroup of G. If $a, b \in G$ generate G modulo V, let H be the subgroup of G generated by a and b. It is clear that $a^{p^n}, b^{p^m} \in V$. Because G has an abelian point-group, then $G' \subseteq V$. Thus both G and H are metabelian and $[[a,b], a^{p^n}] = [[a,b], b^{p^m}] = e$.

On working with the transfer homomorphism from G on G', we can show that centre of G is trivial. Therefore

$$(a^{p^n})^{t(p^m,b)} = (b^{p^m})^{t(p^n,a)} = e,$$

since both elements are centralized by a, b and V. Thus H is homomorphic image of $K(p^n, p^m)$. ∎

In particular, every 2-generated Bieberbach group with finite commutator quotient and point-group isomorphic to $C_{p^n} \times C_{p^m}$ must be isomorphic to a torsion-free quotient of $K(p^n, p^m)$. Next we show that $K(p^n, p^m)$ is itself a Bieberbach group of this type.

Proposition 2.2 *Let*

$$K(p^n, p^m) = \left\langle a, b \mid (a^{p^n})^{t(p^m,b)}, (b^{p^m})^{t(p^n,a)}, \left[[a,b], a^{p^n}\right], \left[[a,b], b^{p^m}\right], \ \textit{metabelian} \right\rangle,$$

where $n, m \in \mathbb{N}$ and p is prime. Then $K(p^n, p^m)$ is a Bieberbach group, with point-group isomorphic to $C_{p^n} \times C_{p^m}$ and commutator quotient $C_{p^{n+m}} \times C_{p^{n+m}}$.

Proof. Let $G = K(p^n, p^m)$ and denote $[a, b]$ by c. We can suppose that $m \geq n$. It follows from the presentation above that $G/G' \cong C_{p^{n+m}} \times C_{p^{n+m}}$, and as a $\frac{G}{G'}$−module, G' is cyclic generated by c. If we denote the action of a and b on G' by A and B, respectively, then G' is generated by $c.A^iB^j$, $0 \leq i \leq p^{n+m}$ and $0 \leq j \leq p^{n+m}$.

Because G is metabelian, the relations of G' can be obtained by the commutation of the generators of G with the relations of G [5]. We already have

$$c.(A^{p^n} - 1) = c.(B^{p^m} - 1) = 0.$$

Then

$$\begin{aligned}\left[(a^{p^n})^{t(p^m,b)}, a\right] &= \left[a^{p^n}.(a^{p^n})^b \dots (a^{p^n})^{b^{p^m-1}}, a\right] \\ &= \left[a^{p^n}, a\right] + \left[(a^{p^n})^b, a\right] + \dots + \left[(a^p)^{b^{p^m-1}}, a\right].\end{aligned}$$

But $\left[(a^{p^n})^{b^i}, a\right] = \left[a^{p^n}\left[a^{p^n}, b^i\right], a\right] = \left[\left[a^{p^n}, b^i\right], a\right] = \left[a^{p^n}, b^i\right].d(A)$ and

$$\left[a^{p^n}, b\right] = [a, b].t(p^n, A).$$

Therefore

$$\begin{aligned}\left[(a^{p^n})^{t(p^m,b)}, a\right] &= \left(\left[a^{p^n}, b\right] + \left[a^{p^n}, b^2\right] + \dots + \left[a^{p^n}, b^{p^m-1}\right]\right).d(A) \\ &= \left(\left[a, b\right] + \left[a, b^2\right] + \dots + \left[a, b^{p^m-1}\right]\right).t(p^n, A)d(A) \\ &= c.l(p^m, B)(A^{p^n} - 1) = 0.\end{aligned}$$

We can do the same with $\left[(b^{p^m})^{t(p^n,a)}, b\right]$, which gives us $-c.l(p^n, A)(B^{p^m}-1) = 0$. On working with $\left[(a^{p^n})^{t(p^m,b)}, b\right] = 0$, we get

$$\begin{aligned} \left[(a^{p^n})^{t(p^m,b)}, b\right] &= \left[a^{p^n}, b\right] + \left[(a^{p^n})^b, b\right] + \ldots + \left[(a^{p^n})^{b^{p^m-1}}, b\right] \\ &= \left[a^{p^n}, b\right] + \left[a^{p^n}, b\right]^b + \ldots + \left[a^{p^n}, b\right]^{b^{p^m-1}} \\ &= \left[a^{p^n}, b\right].t(p^m, B) = [a,b].t(p^n, A)t(p^m, B), \end{aligned}$$

i.e., $c.t(p^n, A)t(p^m, B) = 0$. This expression tells us that $[a^{p^n}, b^{p^m}] = e$. Similarly, $\left[(b^{p^m})^{t(p^n,a)}, a\right]$ gives us the same relation. No new relations are obtained from the commutators $[c, a^{p^n}]$, $[c, b^{p^m}]$ and from the *metabelian* condition. Therefore G' is an abelian subgroup of G, generated by the set $\{c.A^iB^j, 0 \le i \le p^n, 0 \le j \le p^m\}$ and with relations

$$\begin{aligned} &c.t(p^n, A)t(p^m, B) = 0 \\ &c.(A^{p^n} - 1) = 0 \\ &c.(B^{p^m} - 1) = 0. \end{aligned}$$

With the application of the *Diamond Lemma* of Newman and the strategy of *ambiguity resolution* of Bergman [2], we conclude that G' is free abelian of rank $p^{n+m} - 1$, freely generated by the set

$$S = \{c.A^iB^j, 0 \le i < p^n, 0 \le j < p^m, (i,j) \ne (p^n - 1, p^m - 1)\}.$$

Let V be the subgroup $\left\langle a^{p^n}, b^{p^m}, G' \right\rangle$. It is easy to verify that V is a normal, maximal abelian subgroup of G and that $G/V \cong C_{p^n} \times C_{p^m}$.

Because $a^{p^{n+m}}, b^{p^{n+m}} \in G'$, we can express them using the basis S of G' as follows :

$$\begin{aligned} (a^{p^n})^{t(p^m,b)} &= (a^{p^n})(a^{p^n})^b \ldots (a^{p^n})^{b^{p^m-1}} \\ &= a^{p^n} a^{p^n}\left[a^{p^n}, b\right] a^{p^n}\left[a^{p^n}, b^2\right] \ldots a^{p^n}\left[a^{p^n}, b^{p^m-1}\right] \\ &= p^m a^{p^n} + \left[a^{p^n}, b\right].l(p^m, B) \\ &= p^m a^{p^n} + c.t(p^n, A)l(p^m, B). \end{aligned}$$

Therefore

$$p^m a^{p^n} = -c.t(p^n, A)l(p^m, B)$$

and, similarly,

$$p^n b^{p^m} = c.t(p^m, B)l(p^n, A).$$

Thus V is generated by the set $\{a^{p^n}, b^{p^m}, c.A^iB^j, 0 \le i < p^n, 0 \le j < p^m, i + j \le p^n + p^m - 4\}$. But then V is an abelian group generated by $(p^{n+m} - 1)$ elements and contains the free abelian subgroup G' with same rank. We conclude that V is also free abelian, of rank $p^{n+m} - 1$, and is freely generated by this set.

It remains to show that G is torsion-free. We will use induction on $n + m$. First we show that $K(p,p)$ is torsion-free. If $x \in G(= K(p,p))$, then it can be written as $x = a^ib^jg'$, where $g' \in G'$ and $0 \le i, j \le p^2 - 1$. Suppose that x has finite order. Then $x^p \in V$, and we must have $x^p = (a^ib^jg')^p = e$. Now

$$x^p = (a^ib^jg')^p = (a^p)^i(b^p)^jg'',$$

where $g'' \in G'$. In additive notation,

$$x^p = ia^p + jb^p + g'' = 0.$$

As we can write $g'' = \alpha pa^p + \beta pb^p + c.r(A,B)$, where $\alpha, \beta \in \mathbb{Z}$ and $r(A,B) \in \mathbb{Z}[A,B]$, we conclude that p divides i, j. But then $x \in V$ and $x = e$. Now suppose $n + m \geq 3$. Because $m \geq n$, then we have $m \geq 2$ and $t(p^m, x) = t(p^{m-1}, x)t(p, x^{p^{m-1}})$. Therefore $K(p^n, p^{m-1})$ is homomorphic image of $K(p^n, p^m)$. The kernel of the homomorphism is the normal closure in $K(p^n, p^m)$ of the subgroup

$$\langle (a^{p^n})^{t(p^{m-1},b)}, (b^{p^{m-1}})^{t(p^n,a)}, [c, b^{p^{m-1}}] \rangle.$$

It is clear that $(a^{p^n})^{t(p^{m-1},b)}, [c, b^{p^{m-1}}] \in V$. Furthermore

$$\begin{aligned}(b^{p^{m-1}})(b^{p^{m-1}})^a \ldots (b^{p^{m-1}})^{a^{p^{n-1}}} &= b^{p^{m-1}} b^{p^{m-1}} [b^{p^{m-1}}, a] \ldots b^{p^{m-1}} [b^{p^{m-1}}, a^{p^{n-1}}] \\ &= b^{p^{m+n-1}} g',\end{aligned}$$

for some $g' \in G'$. Thus the kernel is contained in V, and therefore is torsion-free. By induction, $K(p^n, p^{m-1})$ is torsion-free. Then $K(p^n, p^m)$, being an extension of a torsion-free group by a torsion-free group, must be torsion-free as well. We conclude that $K(p^n, p^m)$ is a Bieberbach group of dimension $p^{n+m} - 1$, has point-group isomorphic to $C_{p^n} \times C_{p^m}$ and commutator quotient $C_{p^{n+m}} \times C_{p^{n+m}}$. ■

The Propositions 2.1 and 2.2 cover the proof of Theorem A. Furthermore, we can apply the Proposition 2.1 for the group $K(p^n, p^m)$. In this particular case, we have

Proposition 2.3 *Let $G = K(p^n, p^m)$ and V the translation subgroup of G. If $x, y \in G$ generate G* modulo V, *then $H = \langle x, y \rangle \cong G$.*

Proof. It follows from Proposition 2.1 that H is isomorphic to a torsion-free quotient of G. Because every non-trivial quotient must have smaller dimension than that of G, and since G' has finite index in V, it is sufficient to show that $rk(H') = rk(G')$. Now $V/G' \leq \Phi(G/G')$, the Frattini subgroup of G/G'. Then x, y also generate G modulo G', and so $G = G'H$. Because G is metabelian, $H' \trianglelefteq G$ and it follows that

$$G' = [G'H, G'H] = G''[G', H]H' = [G', H]H'.$$

Let $N = (G')^p H'$. Then $N \trianglelefteq G$, and because G is finitely generated, $\frac{G}{N}$ is a finite p-group. Now, following part of the proof of Theorem 2 of [5], we calculate the second and third terms of the lower central series of $\frac{G}{N}$.

$$\Gamma_2\left(\frac{G}{N}\right) = \left[\frac{G}{N}, \frac{G}{N}\right] = \frac{G'N}{N} = \frac{G'}{N}$$

and

$$\Gamma_3\left(\frac{G}{N}\right) = \left[\frac{G'}{N}, \frac{G}{N}\right] = \frac{[G', G]N}{N} = \frac{[G', G'H]N}{N} = \frac{[G', H]H'(G')^p}{N} = \frac{G'}{N}.$$

Thus $\Gamma_2(\frac{G}{N}) = \Gamma_3(\frac{G}{N})$, and because $\frac{G}{N}$ is nilpotent, $G' = N = (G')^p H'$.

Since G' is torsion-free, H' is subgroup of a direct summand U of G', with same rank as that of H'. Then $G' = U \oplus W$,

$$(G')^p = U^p \oplus W^p$$

and

$$(G')^p H' = G' \leq U \oplus W^p \leq G'.$$

Thus W is trivial, $G' = U$ and $rk(H') = rk(G')$. ■

3 Automorphisms

We now study the group of automorphisms of $K(p^n, p^m)$. We will denote the group of automorphisms and the group of inner automorphisms of G by $Aut(G)$ and $I(G)$, respectively. If X is a flat manifold with fundamental group G, then the set $Aff(X)$ of affinities of X is a Lie group. Its identity component $Aff_0(X)$ is a torus of dimension equal to the first Betti number of X, and $Aff(X)/Aff_0(X)$ is isomorphic to $Out(G)$ [3]. Therefore the group of affinities $Aff(X)$ is finite if and only if $Out(G)$ is finite and $b_1(X) = 0$. The problem of finiteness of $Aff(X)$ is studied in [9], [10]. It follows from [9] that when $(p, n, m) \neq (2, 1, 1), (3, 1, 1), (2, 1, 2)$ or $(2, 2, 2)$, then $Out(K(p^n, p^m))$ is infinite. We will show that for these four cases, $Out(K(p^n, p^m))$ is isomorphic to a subgroup of $GL(2, \mathbb{Z}_{p^{n+m}})$.

For every group G there exists a natural homomorphism from $Aut(G)$ on $Aut(\frac{G}{G'})$, whose kernel is denoted by $IA(G)$, the elements of which are called *IA-automorphisms*. It is clear that

$$I(G) \trianglelefteq IA(G) \trianglelefteq Aut(G).$$

When G is a 2-generated metabelian group, then by C. K. Gupta's [4], $IA(G)$ is also metabelian.

If H is a subgroup of G, we will denote by $I_H(G)$ the subgroup of $I(G)$ induced by conjugation by elements of H. Also, we denote by $\mathbb{Z}G$ the group ring of G, $U(\mathbb{Z}G)$ the group of units of $\mathbb{Z}G$, ϵ the augmentation map, $\Delta(\mathbb{Z}G)$ the augmentation ideal of $\mathbb{Z}G$, and $U_1(\mathbb{Z}G) = \{u \in U(\mathbb{Z}G) | \epsilon(u) = 1\}$.

Now let $G = K(p^n, p^m)$ and V the translation subgroup of G. Since the centre of G is trivial, we have $I_V(G) \cong V$.

Proposition 3.1 *Let $G = K(p^n, p^m)$ and $P \cong C_{p^n} \times C_{p^m}$ the point-group of G. Then $\frac{IA(G)}{I_V(G)} \cong U_1(\mathbb{Z}P)$.*

Proof. Let σ be a map from G on G given by $\sigma : a \mapsto ag_1,\ b \mapsto bg_2$, where $g_1, g_2 \in G'$. It follows from the Proposition 2.3 that σ is an endomorphism of G, and that if $H = \langle ag_1, bg_2 \rangle$, then $H \cong G$. Let $\overline{a} = Va$ and $\overline{b} = Vb$ be the generators of P. By abuse of notation, it is convenient to remove the *bar* notation. Now, if $c = [a, b]$, then $g_1 = c.r_1$ and $g_2 = c.r_2$, where $r_1, r_2 \in \mathbb{Z}P$. Since $G = HG'$, then σ is an IA-automorphism if and only if $H' = G'$. But H' is a cyclic $\frac{G}{V}$-module generated by

$$c_H = [ag_1, bg_2] = [a, b] + [g_1, b] + [a, g_2] = c.(1 + r_1(b-1) - r_2(a-1)).$$

Thus, $H' = G'$ if and only if $(1 + r_1(b-1) - r_2(a-1))$ is a unit in the ring $\mathbb{Z}P/\Im$, where $\Im$ is the ideal generated by $z = (t(p^n, a)t(p^m, b))$. Furthermore, considering a unit of $\mathbb{Z}P/\Im$ given by $(1 + r_1(b-1) - r_2(a-1))$ as an element of $\mathbb{Z}P$, then it must have augmentation equal to 1.

Let $\overline{u} = u + \Im$ be a unit of $\mathbb{Z}P/\Im$ as above. If $\overline{w} = w + \Im$ is the inverse of $\overline{u}$ in $\mathbb{Z}P/\Im$, one can show that we can obtain u and w such that $\epsilon(u) = \epsilon(w) = 1$. Now every element of $\Im$ can be written as $k.z$, $k \in \mathbb{Z}$. Therefore $uw = 1 + k.z$, and using the map ϵ, we will have $1 = 1 + k.|P|$, i.e., $k = 0$ and $u, w \in U_1(\mathbb{Z}P)$.

Therefore we can define the map $\tau : IA(G) \to U_1(\mathbb{Z}P)$, that maps the IA-automorphism $\sigma : a \mapsto ag_1,\ b \mapsto bg_2$ into the element $(1 + r_1(b-1) - r_2(a-1))$, where $g_1 = c.r_1$ and $g_2 = c.r_2$. By straightforward verification, one can see that τ is a homomorphism. Now if $u \in U_1(\mathbb{Z}P)$, then $u - 1 \in \Delta(\mathbb{Z}P)$, that is generated by $(a-1)$ and $(b-1)$. Therefore $\exists\ r, s \in \mathbb{Z}P$, such that $r(b-1) - s(a-1) = u - 1$, and τ is onto.

It remains to show that $Ker(\tau) = I_V(G)$. It is clear that $I_V(G) \leq Ker(\tau)$. We must find $r_1, r_2 \in \mathbb{Z}P$, such that

$$r_1(b-1) - r_2(a-1) \equiv 0\ (\Im).$$

Let $r_1 = \sum_{i,j} \alpha_{ij} a^i b^j$ and $r_2 = \sum_{i,j} \beta_{ij} a^i b^j$, $(i, j) \neq (p^n - 1, p^m - 1)$. On calculating $r_1(b-1) = r_2(a-1)$, modulo $t(p^n, a)t(p^m, b)$, we get

$$r_1 = \gamma t(p^m, b) + s(a-1)$$

and

$$r_2 = \delta t(p^n, a) + s(b-1),$$

where $\gamma, \delta \in \mathbb{Z}$, and $s \in \mathbb{Z}P$. Therefore

$$\begin{aligned} g_1 &= c.(\gamma t(p^m, b) + s(a-1)) = [a, (b^{p^m})^\gamma g'] \\ g_2 &= c.(\delta t(p^n, a) + s(b-1)) = [b, (a^{p^n})^{-\delta} g'], \end{aligned}$$

where $g' = c^{-s}$. Let $v = (a^{p^n})^{-\delta}(b^{p^m})^\gamma g'$. Then

$$\begin{aligned} a^v &= a[a, v] = ag_1 \\ b^v &= b[b, v] = bg_2 \end{aligned}$$

and it follows that $Ker(\tau) = I_V(G)$. ∎

To describe the structure of $U_1(\mathbb{Z}(C_{p^n} \times C_{p^m}))$, we recall

Theorem 3.2 (G. Higman [11]) *Let P be a finite abelian group of order n. Then*

$$U(\mathbb{Z}P) = \pm P \times F,$$

where F is a free abelian group of rank $\frac{1}{2}(n+1+n_2-2\ell)$. Here, n_2 denotes the number of subgroups of P of order 2 and ℓ is the number of cyclic subgroups of P.

Lemma 3.3 *Let $P \cong C_{p^n} \times C_{p^m}$, where $m \geq n$. Then F has rank*

$$(2^{m-1} + n - m - 3)2^n + 4$$

if $p = 2$, and

$$\frac{1}{2}\left(p^{n+m} + 2(n-m)p^n - 2(p+1)\frac{p^n-1}{p-1} - 1\right)$$

if p is odd. In particular, $rk(F) = 0$ if and only if $(p, n, m) = (2, 1, 1), (3, 1, 1), (2, 1, 2)$ or $(2, 2, 2)$.

Proof. By induction on $(m+n)$, we can show that $C_{p^n} \times C_{p^m}$ has

$$(m-n)p^n + (p+1)\frac{p^n-1}{p-1} + 1$$

cyclic subgroups (including the trivial one). Furthermore, $n_2 = 3$ when $p = 2$, and $n_2 = 0$ when p is odd. Thus, when $p = 2$, F has rank

$$\begin{aligned}\tfrac{1}{2}(2^{n+m} + 4 - 2((m-n)2^n + 3(2^n - 1) + 1)) &= 2^{n+m-1} + 2 - (m-n)2^n - 3(2^n) + 2 \\ &= (2^{m-1} + n - m - 3)2^n + 4.\end{aligned}$$

Similarly, we obtain the expression for p odd. Now it is straightforward to verify the cases for which $rk(F) = 0$. ■

Theorem 3.2 tells us that $U_1(\mathbb{Z}P) = P \times F$. P is called the set of trivial units. On working with the trivial units $a^i b^j$, then

$$1 + r_1(b-1) - r_2(a-1) = a^i b^j$$

is satisfied by

$$r_1 = \frac{(b^j - 1)}{(b-1)} = 1 + b + \ldots b^{j-1}$$

and

$$r_2 = -b^j \frac{(a^i - 1)}{(a-1)} = -b^j(1 + a + \ldots + a^{i-1}),$$

which correspond to the inner automorphism induced by $a^i b^j$.

Corollary 3.4 *Let $G = K(p^n, p^m)$. Then $\frac{IA(G)}{I(G)}$ is isomorphic to the free abelian group F given in Lemma 3.3.*

Let $G = K(p^n, p^m)$ and Q the image of the natural homomorphism from $Aut(G)$ on $Aut(\frac{G}{G'})$. Then $Out(G)$ is an extension of the free abelian group F by a group isomorphic to Q. For $G = K(p^n, p^m)$, we have $\frac{G}{G'} \cong C_{p^{n+m}} \times C_{p^{n+m}}$. Therefore $Aut(\frac{G}{G'}) \cong GL(2, \mathbb{Z}_{p^{n+m}})$, which is generated by the set

$$\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 1 & 0 \\ 0 & U(\mathbb{Z}_{p^{n+m}}) \end{pmatrix} \right\}.$$

It is well known that $U(\mathbb{Z}_{p^{n+m}})$ is the direct product of two cyclic groups when $p = 2$ and $n+m \geq 3$, and is cyclic in the remaining cases. The natural homomorphism from $\mathbb{Z}_{p^{n+m}}$ onto $\mathbb{Z}_p$ induces an epimorphism from $GL(2, \mathbb{Z}_{p^{n+m}})$ onto $GL(2, \mathbb{Z}_p)$, whose kernel N is generated by

$$\begin{pmatrix} 1+p & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1+p \end{pmatrix}, \quad \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}.$$

Therefore the group N is abelian and isomorphic to $(C_{p^{n+m-1}})^4$, and $GL(2, \mathbb{Z}_{p^{n+m}})$ has order $p^{4(n+m)-3}(p^2-1)(p-1)$.

The elements of Q are induced by the elements of $Aut(G)$ *mod* $IA(G)$. We can show by direct calculations that there are $g_1, g_2 \in G'$ such that $\gamma : a \mapsto ag_1, b \mapsto b^d g_2$ is an automorphism of G if and only if $d = \pm 1$. Therefore Q is in fact isomorphic to a subgroup of

$$SL(2, \mathbb{Z}_{p^{n+m}}) \rtimes \langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rangle.$$

When $n = m$, we have

$$((ab)^{p^n})^{t(p^n,a)} = ((ab)^{p^n})^{t(p^n,b)} = e,$$

and therefore

$$\sigma : \begin{array}{l} a \mapsto ab \\ b \mapsto b \end{array} \quad \textit{and} \quad \omega : \begin{array}{l} a \mapsto a \\ b \mapsto ab \end{array}$$

are automorphisms of G. Thus, in this case

$$Q \cong SL(2, \mathbb{Z}_{p^{2n}}) \rtimes C_2$$

and $|Q| = 2p^{6n-2}(p^2-1)$.

Theorem 3.5 *If* $(p, n, m) = (2, 1, 1), (3, 1, 1), (2, 1, 2)$ *or* $(2, 2, 2)$, *then* $Out(K(p^n, p^m))$ *is isomorphic to a subgroup of* $GL(2, \mathbb{Z}_{p^{n+m}})$ *and is therefore finite.*

Proof. It follows from Lemma 3.3 that for these groups, $rk(F) = 0$. Therefore $IA(G) = I(G)$, and $Out(G)$ is isomorphic to Q, a subgroup of $GL(2, \mathbb{Z}_{p^{n+m}})$.

The group $Out(K(2,2)) \cong SL(2, \mathbb{Z}_4) \rtimes C_2 \cong GL(2, \mathbb{Z}_4)$, of order 96, is isomorphic to the group of affinities of the Hantzsche-Wendt manifold ([6], [3], chapter 5). The groups $Out(K(3,3)) \cong SL(2, \mathbb{Z}_9) \rtimes C_2$ and $Out(K(4,4)) \cong SL(2, \mathbb{Z}_{16}) \rtimes C_2$ have order 6^4 and 3.2^{11}, respectively.

For $K(2,4)$, we can show by direct calculations that there are no $g_1, g_2 \in G'$ such that $\gamma : a \mapsto abg_1, b \mapsto bg_2$ is an automorphism of $K(2,4)$, and that $\delta : a \mapsto ab^2, b \mapsto b$ is an automorphism of $K(2,4)$. Thus $Out(K(2,4))$ is isomorphic to a proper subgroup of $SL(2,\mathbb{Z}_8) \rtimes C_2$, containing the subgroup L of $SL(2,\mathbb{Z}_8) \rtimes C_2$ generated by

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since L has index 3 in $SL(2,\mathbb{Z}_8) \rtimes C_2$, and is therefore maximal, we conclude that $Out(K(2,4)) \cong L$, of order 2^8. ■

For the remaining cases, we have $Out(K(p^n, p^m))$ as an extension of a free abelian group F of finite rank by a finite group Q, isomorphic to a subgroup of $SL(2,\mathbb{Z}_{p^{n+m}}) \rtimes C_2$. We notice that the difficulty in explicitly describing $Out(K(p^n, p^m))$ for these cases is due to the difficulty in describing the generators of F [11]. (But anyhow, if F is maximal abelian, $Out(K(p^n, p^m))$ will be a crystallographic group; we must then verify if the action of Q on F is faithful).

Let G be a Bieberbach group, with translation subgroup V and point-group P. Since V is a characteristic subgroup of G, there is a natural homomorphism from $Aut(G)$ on $Aut(V) \cong GL(n,\mathbb{Z})$, whose kernel we will denote by $C(V)$, the centralizer of V in $Aut(G)$. It is clear that $I_V(G)$, the subgroup of inner automorphisms induced by V, is a normal subgroup of $C(V)$. It is known that the group of outer automorphisms $Out(G)$ contains a finite normal subgroup isomorphic to $C(V)/I_V(G)$. Furthermore, $C(V)/I_V(G)$ is isomorphic to $H^1(P,V)$, the first cohomology group [3].

For any abelian group P, the Bass cyclic units generate a subgroup of finite index in $U(\mathbb{Z}P)$ [11]. Furthermore, we can show that in our case, the action of Q on F, by conjugation, is given by extending the action of the image of Q in $SL(2,\mathbb{Z}_{p^{n+m}}) \rtimes C_2$ to the subgroup of $U(\mathbb{Z}P)$ corresponding to F. Therefore, to test if F is maximal abelian, it is enough to check this action on the Bass cyclic units.

Now, since F is free abelian, Q contains an abelian subgroup N isomorphic to $H^1(P,V)$, and if we consider the action of Q on F as described above, we can see that N acts trivially on F, while Q/N acts faithfully on FN/N. Thus we have

Theorem 3.6 *Let $G = K(p^n, p^m)$, with $(p,n,m) \neq (2,1,1), (3,1,1), (2,1,2), (2,2,2)$. Then $Out(G)$ satisfies the short exact sequence*

$$0 \to N \to Out(G) \to R \to 1,$$

where N is isomorphic to an abelian subgroup of $SL(2,\mathbb{Z}_{p^{n+m}}) \rtimes C_2$ and R is a crystallographic group with point-group isomorphic to Q/N.

Theorems 3.5 and 3.6 cover the items a) and b) of Theorem B, respectively. In particular, when $m = n$, we have seen that $Q \cong SL(2,\mathbb{Z}_{p^{2n}}) \rtimes C_2$. There exists a natural epimorphism from Q onto $SL(2,\mathbb{Z}_{p^n}) \rtimes C_2$, whose kernel K is generated by

$$\begin{pmatrix} 1 & p^n \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ p^n & 1 \end{pmatrix}, \quad \begin{pmatrix} 1+p^n & 0 \\ 0 & 1-p^n \end{pmatrix}.$$

Thus K is isomorphic to $C_{p^n} \times C_{p^n} \times C_{p^n}$, and in this particular case, $K = N$ and $Q/N \cong SL(2, \mathbb{Z}_{p^n}) \rtimes C_2$. Thus, for $(p, n) \neq (2, 1), (3, 1), (2, 2)$, there exists the following normal series for $Aut(K(p^n, p^n))$

$$\{1\} = N_0 \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq N_3 \trianglelefteq N_4 = Aut(K(p^n, p^n)),$$

where $N_1 = I(K(p^n, p^n)) \cong K(p^n, p^n)$, $N_2/N_1 \cong H^1(P, V) \cong C_{p^n} \times C_{p^n} \times C_{p^n}$, $N_3/N_2 \cong F$, free abelian group of finite rank, and $N_4/N_3 \cong SL(2, \mathbb{Z}_{p^n}) \rtimes C_2$. Furthermore, N_4/N_2 is a crystallographic group with point-group isomorphic to $SL(2, \mathbb{Z}_{p^n}) \rtimes C_2$.

Acknowledgements

The author would like to thank Professor Said Sidki for the discussions, suggestions and help in the elaboration of this paper.

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