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On the average number of divisors of quadratic polynomials

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1. *Introduction*

Let $d(n)$ denote the number of positive divisors of n, and let $f(x)$ be a polynomial in x with integer coefficients, irreducible over $\mathbb{Z}$. Erdös [**3**] showed that there exist constants λ_1, λ_2 (depending on f) such that

$$\lambda_1 x \log x \leqslant \sum_{n \leqslant x} d(f(n)) \leqslant \lambda_2 x \log x.$$

For the case where $f(x) = ax^2+bx+c$ is a quadratic polynomial, one has in fact

$$\sum_{n \leqslant x} d(f(n)) \sim \lambda x \log x, \tag{1}$$

for some constant λ (depending on a, b and c). Apparently this is due to Bellman and Shapiro (unpublished), and Bellman describes the proof as 'not elementary, although not difficult' [**1**]. The first published proof seems to be that of Scourfield [**7**]. For the case $a = 1, b = 0$, Hooley [**5**] gives an excellent description of the error. His expression for λ in (1) is

$$\lambda = \frac{8}{\pi^2} \sum_{\alpha=0}^{\infty} \frac{\rho(2^\alpha)}{2^\alpha} \sum_{\substack{d^2 \mid c \\ (d,2)=1}} \frac{1}{d} \sum_{\substack{l=1 \\ (l,2)=1}}^{\infty} \left(\frac{-c/d^2}{l}\right) \frac{1}{l}, \tag{2}$$

where ρ is defined below and (p/q) is the Legendre symbol. In this paper, we consider the case $a = 1, b^2-4c = \Delta < 0$, and give a more compact expression for λ, namely

$$\lambda = \frac{12H^*(\Delta)}{\pi\sqrt{-\Delta}}, \tag{3}$$

where $H^*(\Delta)$ is the Hurwitz class number, defined below. Using the analytic class number formula, it is not difficult to check that these two expressions for λ agree when $b = 0, c > 0$. The proof of (3) is completely elementary. The appearance of a class number is not surprising (the connection with class numbers was noted by Hooley in [**4**] and [**6**] (p. 32)), but the precise relationship (3) seems not to have been formulated before.

The proof makes use of binary quadratic forms, so in Section 2 we recall the results which are needed. The proof of (3) is given in Section 3. More precisely, we show

THEOREM. *If $b, c \in \mathbb{Z}$ with $\Delta = b^2-4c < 0$, then*

$$\sum_{n \leqslant x} d(n^2+bn+c) = \frac{12H^*(\Delta)}{\pi\sqrt{-\Delta}} x \log x + O(x),$$

where the implied constant in the $O(x)$ depends on Δ.

2. ρ and representations by quadratic forms

Let b, c be integers, with $\Delta = b^2 - 4c < 0$. For positive integers d, let $\rho(d)$ be the number of solutions to the congruence

$$n^2 + bn + c \equiv 0 \pmod{d}, \quad 0 \leqslant n < d. \tag{4}$$

ρ is multiplicative, but not totally multiplicative. Multiplying (4) by 4, and writing $m = 2n + b$, we see that $\rho(d)$ is the number of solutions to the congruence

$$m^2 \equiv \Delta \pmod{4d}, \quad b \leqslant m < b + 2d. \tag{5}$$

The proof of (3) will involve binary quadratic forms, so we now recall the essential facts (cf. [2]).

Let $f(x, y) = Ax^2 + Bxy + Cy^2$ be a positive definite binary quadratic form, so that $A, B, C \in \mathbb{Z}$ with the discriminant $\Delta = B^2 - 4AC < 0$, and $A > 0$. Given

$$\alpha = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL_2(\mathbb{Z}),$$

define $f^{\alpha}(x, y) = f(px + ry, qx + sy)$. Then f^{α} has the same discriminant as f. Forms related in this way are called equivalent. The form f is called *reduced* if $|B| \leqslant A \leqslant C$, with $B \geqslant 0$ if either $A = |B|$ or $A = C$. Each positive definite binary quadratic form is equivalent to precisely one reduced form. The number of reduced forms with discriminant Δ is finite, and is denoted $H(\Delta)$. The subgroup of $SL_2(\mathbb{Z})$ consisting of those α such that $f^{\alpha} = f$ is called the automorphism group of f.

We shall write (A, B, C) as a shorthand for $Ax^2 + Bxy + Cy^2$. Let $w(A, B, C)$ denote the size of the automorphism group of (A, B, C). Then $w(A, A, A) = 6$, $w(A, 0, A) = 4$, and otherwise, if (A, B, C) is reduced, we have $w(A, B, C) = 2$. Define

$$H^*(\Delta) = \sum_{\substack{(A, B, C) \text{ reduced}, \\ B^2 - 4AC = \Delta}} 2/w(A, B, C). \tag{6}$$

Then $H^*(\Delta)$ usually equals $H(\Delta)$, and the two never differ by more than $\frac{2}{3}$.

We say that (A, B, C) properly represents d if there exist co-prime integers p and q with $Ap^2 + Bpq + Cq^2 = d$. Given such p and q, one can find $r, s \in \mathbb{Z}$ with $ps - qr = 1$. Setting $\alpha = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, one checks that $(A, B, C)^{\alpha} = (d, m, l)$ for some m and l depending on α. Then $m^2 - 4dl = B^2 - 4AC = \Delta$, so that

$$m^2 \equiv \Delta \pmod{4d}. \tag{7}$$

Given p and q, there is some choice for r and s. Indeed if $ps_0 - qr_0 = 1$, then the general solution to $ps - qr = 1$ is $s = s_0 + \beta q, r = r_0 + \beta p$, for $\beta \in \mathbb{Z}$. If $(r, s) = (r_0, s_0)$ leads to $m = m_0$ in (7), then $(r, s) = (r_0 + \beta p, s_0 + \beta q)$ leads to $m = m_0 + 2\beta d$. Thus given any integers b and d, and any co-prime p and q with $Ap^2 + Bpq + Cq^2 = d$, we have defined a unique $m \in \mathbb{Z}$ satisfying

$$m^2 \equiv \Delta \pmod{4d}, \quad b \leqslant m < b + 2d. \tag{8}$$

Conversely, a solution $m^2-4dl = \Delta$ satisfying (8) implies (d,m,l) properly represents $d(x=1, y=0)$. If (A,B,C) is the unique reduced form equivalent to (d,m,l) then (A,B,C) also properly represents d, and if $(A,B,C) = (d,m,l)^\alpha$ then α determines the representation of d by (A,B,C). α is unique up to multiplication by an automorphism of (A,B,C). Hence this solution to (8) defines $w(A,B,C)$ distinct ways in which (A,B,C) properly represents d. If we count each representation with weight $1/w(A,B,C)$ then the total number of ways of representing d by some reduced form of discriminant Δ is precisely the number of solutions to (8).

If $\Delta = b^2-4c < 0$, then comparing (5) and (8), the above discussion implies

$$\rho(d) = \sum_{\substack{(A,B,C)\text{ reduced},\\ B^2-4AC=\Delta}} \frac{1}{w(A,B,C)} \sum_{\substack{Ap^2+Bpq+Cq^2=d,\\ \gcd(p,q)=1}} 1. \tag{9}$$

This expression for $\rho(d)$ is the main ingredient in the proof of (3).

3. *Proof of Theorem*

Suppose $b, c \in \mathbb{Z}$ with $\Delta = b^2-4c < 0$. In what follows, the implied constant in all O estimates may depend on Δ.

Positive divisors of n^2+bn+c generally pair off, by pairing a factor less than $\sqrt{(n^2+bn+c)}$ with its co-factor (which is greater than $\sqrt{(n^2+bn+c)}$). The exception is if n^2+bn+c is a square, when its square root is not paired off. Hence

$$\sum_{n \leqslant x} d(n^2+bn+c) = 2\sum_{n\leqslant x} \sum_{\substack{d \mid n^2+bn+c,\\ d \leqslant \sqrt{(n^2+bn+c)}}} 1 + O(x). \tag{10}$$

Now reverse the order of summation in (10):

$$\begin{aligned}\sum_{n \leqslant x} d(n^2+bn+c) &= 2 \sum_{d \leqslant x+O(1)} \sum_{\substack{d+O(1) \leqslant n \leqslant x,\\ d \mid n^2+bn+c}} 1 + O(x)\\ &= 2\sum_{d\leqslant x} \sum_{\substack{d+O(1)\leqslant n \leqslant x,\\ d \mid n^2+bn+c}} 1 + O(x).\end{aligned} \tag{11}$$

Consider the inner sum in (11). We know that $\rho(d)$ out of every d consecutive values of n satisfy $d \mid n^2+bn+c$. Hence the inner sum is $x\rho(d)/d + O(\rho(d))$. Therefore (11) gives

$$\sum_{n\leqslant x} d(n^2+bn+c) = 2x \sum_{d \leqslant x} \rho(d)/d + O\Big(\sum_{d\leqslant x} \rho(d)\Big) + O(x). \tag{12}$$

To evaluate (12), we shall show that

$$\sum_{d \leqslant x} \rho(d) = \frac{6H^*(\Delta)}{\pi\sqrt{-\Delta}} x + O(\sqrt{x}\log x). \tag{13}$$

Then summation by parts gives

$$\sum_{d\leqslant x} \rho(d)/d = \frac{6H^*(\Delta)}{\pi\sqrt{-\Delta}} \log x + O(1). \tag{14}$$

Substituting (13) and (14) into (12) proves the Theorem.

It remains to prove (13). For this we use the expression for $\rho(d)$ from Section 2, namely (9).

$$\sum_{d\leqslant x}\rho(d)=\sum_{d\leqslant x}\sum_{\substack{(A,B,C)\text{ reduced},\\ B^2-4AC=\Delta}}\frac{1}{w(A,B,C)}\sum_{\substack{Ap^2+Bpq+Cq^2=d,\\ gcd(p,q)=1}}1$$

$$=\sum_{(A,B,C)}\frac{1}{w(A,B,C)}\sum_{\substack{Ap^2+Bpq+Cq^2\leqslant x,\\ gcd(p,q)=1}}1$$

$$=\sum_{(A,B,C)}\frac{1}{w(A,B,C)}\sum_{1\leqslant Ap^2+Bpq+Cq^2\leqslant x}\ \sum_{f\mid gcd(p,q)}\mu(f), \tag{15}$$

where μ is the Möbius function. Here p and q can be any integers, but f is always positive.

Now we interchange the inner two sums in (15). Given $f\mid gcd(p,q)$, write $p=fs$, $q=ft$. Then $As^2+Bst+Ct^2\leqslant x/f^2$. We have $f\leqslant\sqrt{x}$ (actually $f\leqslant\sqrt{(x/A)}$), so that

$$\sum_{d\leqslant x}\rho(d)=\sum_{(A,B,C)}\frac{1}{w(A,B,C)}\sum_{f\leqslant\sqrt{x}}\mu(f)\sum_{1\leqslant As^2+Bst+Ct^2\leqslant x/f^2}1. \tag{16}$$

The inner sum is the number of non-trivial integer points within an ellipse of area $2\pi x/f^2\sqrt{-\Delta}$, circumference $O(\sqrt{x}/f)$. Thus (16) gives

$$\sum_{d\leqslant x}\rho(d)=\sum_{(A,B,C)}\frac{1}{w(A,B,C)}\sum_{f\leqslant\sqrt{x}}\mu(f)\left\{\frac{2\pi x}{f^2\sqrt{-\Delta}}+O(\sqrt{x}/f)\right\}$$

$$=\sum_{(A,B,C)}\frac{1}{w(A,B,C)}\left\{\frac{2\pi x}{\sqrt{-\Delta}}(6/\pi^2+O(1/\sqrt{x}))+O(\sqrt{x}\log x)\right\}$$

$$=\frac{6H^*(\Delta)}{\pi\sqrt{-\Delta}}x+O(\sqrt{x}\log x),$$

from (6). This proves (13), as desired.

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