FUNDAMENTAL SEQUENCES IN THE SECOND AND THIRD ORDINAL NUMBER CLASSES

a thesis by

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The text is concerned with the definition and investigation of some classes of ordinal valued-functions; and with the determining of a particular assignment function $\Omega$. This function assigns to each limit ordinal $\alpha$ smaller than a constant $\theta$ and belonging to the second number class a fundamental sequence $\{\Omega_{\alpha}(\omega)\}_{\omega}$; that is, a strictly increasing sequence satisfying $\alpha = \lim_{n<\omega} \Omega_{\alpha}(n)$.

The operations used to determine $\Omega$ are derived from a class $\mathfrak{D}^{(n)}$ of functions $\mathfrak{D}_{\alpha}^{(0)}(\alpha), \alpha<\omega$. Since $\mathfrak{D}^{(n)}$ generalizes the standard ordinal arithmetic operations, some of the properties of the $\mathfrak{D}_{\alpha}$ are studied, together with the operations $\mathfrak{T}$ which are generalizations of transfinite sum and product. Also investigated is a related class $\mathfrak{E}^{(n)}$ of number-theoretic functions.

In this context $\Omega$ is an arbitrary assignment bounded by $\omega$, in place of $\theta<\omega$.

It is proved that the functions $\mathfrak{D}_{\alpha}$ are normal in the second argument, and these functions are compared with a hierarchy of normal functions obtained by Veblen's process of iteration. The $\mathfrak{D}_{\alpha}$ provide a natural means of extending the notion of epsilon number, and some of the properties of the generalized epsilon numbers are presented.
The function of \( \alpha \overset{\infty}{\mathcal{W}} \) is normal, and the notation \( \{q^\alpha \}_{q=1}^\infty \) is adopted for the sequence of countable fixed points of the function. The generalized epsilon numbers determine a hierarchical classification of limit numbers \( < \xi \), and on this basis a normal form is determined for each, and thence the function \( \Omega \) is defined by transfinite recursion.
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## CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>§ 1</td>
<td>INTRODUCTION</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Order types and ordinal numbers</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>Segments and number classes</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>Sequences and limits</td>
<td>9</td>
</tr>
<tr>
<td></td>
<td>Least number operation ( \mu ) and ( \pi )-notation</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>Functions</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>Elementary functions</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>Schemata ( R ) for nested course-of-values transfinite recursion</td>
<td>17</td>
</tr>
<tr>
<td>§ 2</td>
<td>FUNDAMENTAL SEQUENCES</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Fundamental sequence assignments</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>Epsilon numbers</td>
<td>31</td>
</tr>
<tr>
<td></td>
<td>A problem concerning fundamental sequences</td>
<td>32</td>
</tr>
<tr>
<td>§ 3</td>
<td>THE CLASS ( \mathbb{W}^\omega ) OF FUNCTIONS ( \mathbb{W}^\omega : \omega \to \omega )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Definition of ( \mathbb{W}^\omega )</td>
<td>35</td>
</tr>
<tr>
<td></td>
<td>Definition of operations ( \Gamma )</td>
<td>35</td>
</tr>
<tr>
<td></td>
<td>Examples</td>
<td>37</td>
</tr>
<tr>
<td></td>
<td>The classes ( \mathcal{L}_n, \mathcal{V}_n )</td>
<td>40</td>
</tr>
<tr>
<td></td>
<td>( \mathcal{W} ) defined by schemata ( R )</td>
<td>44</td>
</tr>
<tr>
<td></td>
<td>The functions ( [P^\lambda]_{\omega_n \to \omega_m} = #^\alpha_n )</td>
<td>47</td>
</tr>
<tr>
<td></td>
<td>Majorization theorems</td>
<td>52</td>
</tr>
<tr>
<td>§ 4</td>
<td>SOME PROPERTIES OF FUNCTIONS IN THE CLASS</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Strictly increasing property of ( \lambda f \mathbb{W}^f (\eta) )</td>
<td>71</td>
</tr>
<tr>
<td></td>
<td>A theorem concerning limit expressions</td>
<td>72</td>
</tr>
<tr>
<td></td>
<td>Simplification of definition ( 3.1 )</td>
<td>80</td>
</tr>
<tr>
<td></td>
<td>The leading term</td>
<td>109</td>
</tr>
<tr>
<td></td>
<td>Some functions of a finite operand</td>
<td>112</td>
</tr>
<tr>
<td></td>
<td>Some arithmetic properties of the operations ( \circ )</td>
<td>119</td>
</tr>
<tr>
<td>§ 5</td>
<td>GENERALIZED EPSILON NUMBERS</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Normal functions</td>
<td>124</td>
</tr>
<tr>
<td></td>
<td>Some Veblen functions ( \Phi )</td>
<td>127</td>
</tr>
<tr>
<td></td>
<td>Epsilon numbers derived from generalized arithmetic operations</td>
<td>142</td>
</tr>
<tr>
<td></td>
<td>( \varepsilon^\alpha )-numbers</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Characterization of the ( \varepsilon^\alpha )-numbers:</td>
<td></td>
</tr>
<tr>
<td></td>
<td>for successor ( \alpha )</td>
<td>146</td>
</tr>
<tr>
<td></td>
<td>for limit ( \alpha )</td>
<td>161</td>
</tr>
</tbody>
</table>
CONTENTS

§ 6  NORMAL FORM THEOREMS  
Normal form theorem for ordinals belonging to a segment of \( \omega_\alpha \)  
The number \( \delta^\alpha \)  
Exact \( \varepsilon^\alpha \)-numbers  
Normal form theorem for ordinals < \( \delta^\alpha \)  

171
171
172
175
179

CONCLUSION  

182

APPENDIX I  

182a

APPENDIX II  

183

BIBLIOGRAPHY  

184

INDEX  

186
§1. INTRODUCTION.

ORDER TYPES AND ORDINAL NUMBERS

A binary relation $\prec$ on a set $S$ is an ordering relation on $S$ iff

(i) $(\forall x, y, z \in S) \left[ x \prec y \land y \prec z \Rightarrow x \prec z \right]$  
    (transitivity)

(ii) $(\forall x, y \in S) \left[ x \not\prec y \Rightarrow x \prec y \lor y \prec x \right]$  
    (connectivity)

(iii) $(\forall x, y \in S) \left[ x \prec y \Rightarrow \neg y \prec x \right]$  
    (antisymmetry)

$(S, \prec)$ denotes the set $S$ ordered by $\prec$.

The relation $\prec$ is a well-ordering on the set $S$ iff the following condition is also satisfied:

(iv) $(\forall T \subseteq S) \left( \exists x \in T \right) (\forall y \in T) \left[ y \not\prec x \Rightarrow x \prec y \right]$  

Two ordered sets $(S, \prec), (S', \prec')$ are said to be similar if there exists a bijective function $f$ such that

(i) $f : S \to S'$

(ii) $(\forall x, y \in S) \left[ x \prec y \Rightarrow f(x) \prec' f(y) \right]$  

An order type is the class of ordered sets similar to a given set. An ordinal number is an order type of well-ordered sets.
Ordinal numbers are characterized in set theory by regarding them as certain canonical sets $x$ satisfying:

(i) $\forall y \left[ y \in x \Rightarrow y \subseteq x \right]$ or equivalently

$\forall y,z \left[ y \in x \land z \in y \Rightarrow z \in x \right]$. This property of $x$ is written $\text{From}(x)$.

(ii) $\forall y \left[ y \in x \Rightarrow \text{From}(y) \right]$

These sets are each well-ordered by the membership relation $\in$, and constitute a sequence well ordered by $\in$. (c.f. Schoenfield [17], p 246).

Zero is the smallest ordinal number, viz the empty set $\emptyset$.

Suppose $S$ is a canonical set determining the ordinal $\alpha$. Then the set $\{ x | x = S \lor x \in S \}$ is the canonical set determining the least ordinal greater than $\alpha$, and this number is written $\alpha + 1$, and is called a successor number, or the successor of $\alpha$. Non-zero ordinals which are not successor are called limit numbers.

The finite numbers $0, 1, 2, \ldots$ start the sequence of ordinal numbers.
The least transfinite ordinal is denoted by the symbol \( \omega \), or by \( \omega^0 \).

Some of the notations adopted in this text for ordinals are as follows:

\[(\forall \alpha) \text{ Suc } \alpha \iff 0 < \alpha \land (\exists \beta < \alpha) \\alpha = \beta + 1\]

\[\text{Lim } \alpha \iff 0 < \alpha \land (\forall \beta < \alpha) \beta + 1 < \alpha\]

The notation \( \text{ Lim } (\alpha) \) is also used as an alternative to \( \text{ Lim } (\alpha) \). Furthermore, \( \text{ Lim } (\alpha, \beta, \ldots) \), \( \text{ Suc } (\alpha, \beta, \ldots) \) are used to denote \( \text{ Lim } (\alpha) \& \text{ Lim } (\beta) \& \ldots \), and \( \text{ Suc } (\alpha) \& \text{ Suc } (\beta) \& \ldots \) respectively.

For \( \text{ Lim } (\alpha) \) (\( \forall m, n < \omega \)) \( \text{ P }^m (\alpha + n) = \text{ if } \alpha + (n - m) \),

where \( n - m = \begin{cases} n - m & \text{if } m \leq n \\ 0 & \text{otherwise} \end{cases} \)

Also, for each \( \beta \), \( \text{ P }^1 (\beta) = \text{ if } \text{ P } (\beta) \).

**SEGMENTS AND NUMBER CLASSES**

\( W \) denotes the class of ordinal numbers.

For each ordinal \( \alpha \), \( W_\alpha \) denotes the set of ordinals smaller than \( \alpha \). This set is called a segment (of type \( \alpha \)) of \( W \).
$N$ denotes the set consisting of zero and the natural numbers, and is called the first number class.

$\mathbb{Z}(\aleph_0)$ denotes the set of denumerable ordinals (that is, ordinals $\alpha$ for which there exists a bijection from $\omega$ onto $\alpha$). $\mathbb{Z}(\aleph_0)$ is called the second number class.

For each ordinal $\mu > 0$, $\mathbb{Z}(\beth_\mu)$ denotes the set of ordinals of cardinality or power equal to $\beth_\mu$, and $\omega_\mu$ denotes the first ordinal of $\mathbb{Z}(\beth_\mu)$. $\mathbb{Z}(\beth_\mu)$ is called the $(2+\mu)$-th number class.

**SEQUENCES AND LIMITS**

Five classes of ordinal sequences are characterized here. Suppose $\{\alpha_\xi\}_{\xi < \omega}$ is a sequence of ordinals $\alpha_0, \alpha_1, \ldots, \alpha_\xi, \alpha_{\xi+1}, \ldots$ of limit type $\alpha$. Then

1. $\{\alpha_\xi\}_{\xi < \omega}$ is strictly increasing if

   $$(\forall \xi, \xi' < \omega) \ \xi' < \xi \Rightarrow \alpha_{\xi'} < \alpha_\xi$$

2. $\{\alpha_\xi\}_{\xi < \omega}$ is internally unbounded if

   $$(\forall \xi < \omega) (\exists \xi' < \omega) \ \xi' < \xi \land \alpha_{\xi'} < \alpha_\xi$$

3. $\{\alpha_\xi\}_{\xi < \omega}$ is internally bounded if

   $$(\exists \xi < \omega) (\forall \xi < \omega) \ \xi < \xi' \Rightarrow \alpha_{\xi'} \leq \alpha_\xi$$
4. \( \alpha, \beta \leq \gamma \) is non-decreasing if
\[
(\forall \gamma, \delta \leq \gamma) \quad \delta' \leq \gamma \Rightarrow \alpha_{\delta'} \leq \alpha_{\gamma}.
\]

5. \( \alpha, \beta \leq \gamma \) is ultimately increasing if it is non-decreasing and is internally unbounded.

Now suppose \( \{\alpha_{\gamma}\}_{\gamma} \) is an arbitrary sequence of limit type \( \gamma \).

**Definition 1.1.** (Kline, "The Foundations of Mathematics," p. 87)

The sequence \( \{\alpha_{\gamma}\}_{\gamma} \) is said to have an ordinal \( \alpha \) as a limit if
\[
(\forall \gamma < \alpha)(\exists \mu = \gamma)(\forall \delta < \gamma) \quad \delta' \leq \gamma \Rightarrow \mu < \alpha_{\delta'} \leq \alpha,
\]
and this property of \( \alpha \) is expressed by
\[
\alpha = \lim_{\gamma \to \alpha} \alpha_{\gamma}.
\]

Symbols \( \rightarrow, \Rightarrow, \ldots \) are used to denote finite sequences of ordinals which arise as function parameters.

**Least Number Operation** \( \mu \), and \( \gamma \)-Notation.\[\text{Suppose } \mu \text{ is a unary predicate defined on } W. \text{ Then,}
\]
\[
(\mu)(P(\mu)) = \begin{cases} \text{the least } \mu \text{ such that } P(\mu), & \text{if there is such a } \mu, \text{ or} \\ 0 & \text{otherwise.} \end{cases}
\]
In particular, for all ordinals \( \alpha, \beta \) such that \( \alpha \geq \beta \), \( \alpha - \beta = \omega (\mu) \left[ \alpha = \beta + \delta \right] \).

The \( \gamma \)-notation adopted is as follows:

Suppose \( f : A_1 \times A_2 \times \ldots \times A_n \rightarrow B \), for some \( n \) such that \( 0 < n < \omega \).

Suppose \( \left( \forall i \in n \right)_i \alpha_i \) is any element of \( A_i \).

Then \( \left( \forall m \in n \right)_m \left( \forall a_1, a_2, \ldots, a_m \right) \leq_n \alpha_1 < a_2 < \ldots < a_m \Rightarrow \nabla a_1, a_2, \ldots, a_m \cdot f \left( \overset{\phi_1}{\beta_1}, a_2, \overset{\phi_2}{\beta_2}, a_3, \overset{\phi_3}{\beta_3}, \ldots, \overset{\phi_m}{\beta_m}, a_m, a_{m+1} \right) \)

denotes the function \( g \) of \( m \) arguments obtained from the function \( f \) of \( n \) arguments by fixing the values of arguments

\( \alpha_1, \alpha_2, \ldots, \alpha_{a-1} \) of \( f \) with the values \( \overset{\phi_1}{\beta_1} \), when \( 1 \leq a \),

\( \alpha_{a+1}, \alpha_{a+2}, \ldots, \alpha_{a-1} \) of \( f \) with the values \( \overset{\phi_2}{\beta_2} \), when \( a+1 \leq a \),

\( \vdots \)

\( \alpha_{a^*+1}, \alpha_{a^*+2}, \ldots, \alpha_{a^*} \) of \( f \) with the values \( \overset{\phi_{a^*+1}}{\beta_{a^*+1}} \), when \( a^*+1 \leq n \).

The parameter strings \( \overset{\phi}{\beta} \) are empty in these cases where the inequalities are not satisfied.

Thus \( g : A_{a_1} \times A_{a_2} \times \ldots \times A_{a_m} \rightarrow B \).
In particular, for \( \alpha_i = i \), for each \( i \) such that \( 1 \leq i \leq n \),
\[
g = f = \tau_{\alpha_1, \alpha_2, \ldots, \alpha_n} \cdot f(\alpha_1, \alpha_2, \ldots, \alpha_n).
\]

Although the symbols 'μ', 'λ' are also used to denote ordinals, this is only in contexts where confusion with the least number operator or with the \( \lambda \)-notation symbol will not arise.

**FUNCTIONS**

A function \( f : W \rightarrow W \) is non-decreasing if \((\forall \alpha, \alpha')\)
\[
\alpha' < \alpha \Rightarrow f(\alpha') \leq f(\alpha).
\]

A function \( f : W \rightarrow W \) is strictly increasing if \((\forall \alpha, \alpha')\)
\[
\alpha' < \alpha \Rightarrow f(\alpha') < f(\alpha).
\]

A function \( f : W \rightarrow W \) is continuous if for every strictly increasing sequence \( \{ \alpha_n \}_n \rightarrow \alpha \) of limit type
\[
f(\lim_{n \to \infty} \alpha_n) = \lim_{n \to \infty} f(\alpha_n).
\]

A function \( f : W \rightarrow W \) is normal if \( f \) is both strictly increasing and continuous. An ordinal \( \alpha \) is a fixed point of \( f \) if \( f(\alpha) = \alpha \).

**LEMMA 1.2** If \( f \) is a normal function, then \((\forall \alpha)\)
\[
\alpha \leq f(\alpha).
\]
PROOF. Suppose otherwise, such that $\beta$ is the least ordinal such that $f(\rho) = \beta$. Now $f(0) = 0$, therefore $\rho > 0$ and $(\forall \xi < \rho) \xi \leq f(\xi)$.

If $\text{Succ} (\rho)$, then $f(\rho(\rho)) < f(\rho)$ as $f$ is strictly increasing, therefore $f(\rho) \geq \rho(\rho) + 1$ by hypothesis

$= \beta$.

If $\text{Lim} (\rho)$, then $f(\rho) = \lim_{\rho' < \rho} f(\rho')$, as $f$ is continuous

$= f(\rho)$ for each $\rho' < \rho$, as $f$ is strictly increasing

$= \beta$ by hypothesis.

Therefore $f(\rho) \geq \beta$, which contradicts hypothesis $\square$ 1.2.

Transfinite iteration of any normal function $f$ is defined as follows:

$(\forall \alpha) \left[ f'(\alpha) = f(\alpha) \& (\forall \rho) \geq 1 \left[ \left[ \text{Succ} (\rho) \Rightarrow f^\rho(\alpha) = f \left\{ f^\rho(\alpha) \right\} \right] \& [ \text{Lim} (\rho) \Rightarrow f^\rho(\alpha) = \lim_{\rho' < \rho} f^{\rho'}(\alpha) ] \right] \right]$

THEOREM 1.3. If $f$ is a normal function, then

a) the least fixed point of $f$ is $f^\omega (\alpha)$

b) $\left( \forall \beta \right) \left( \exists \alpha \right) [ \beta < \alpha \& f(\alpha) = \alpha ]$
PROOF \( \alpha \):

\[ f\left(f^\omega(0)\right) = f\left\{ \lim_{n \to \omega} f^n(0) \right\} \]
\[ = \lim_{n \to \omega} f f^n(0) \]
\[ = \lim_{n \to \omega} f^{n+1}(0) \]
\[ = f(\omega)(0), \text{ therefore } f^\omega(0) \text{ is a fixed point of } f. \]

Suppose \( p \leq f^\omega(0) \) is a fixed point of \( f \). Then if \( f(0) = 0 \), \( f^\omega(0) = 0 \), therefore \( p \neq f^\omega(0) \). Thus suppose \( f(0) > 0 \). Therefore \( f(0) \neq p \), since \( f \) is strictly increasing, and so \( f(f(0)) = f^2(0) > f(0) \), and as \( p \) belongs to \( f[\text{domain of } f] \), \( f(0) \leq p \), therefore \( f(0) \leq p \). And if \( f^n(0) \leq p, n > 1 \), then \( f(f^n(0)) = f^{n+1}(0) > f^n(0) \) therefore \( f^{n+1}(0) = f(f^n(0)) < f(f^{n+1}(0)) = f^{n+2}(0) \), and therefore \( f^{n+1}(0) \neq p \), and as \( p \in f[\text{domain of } f] \), by hypothesis \( f(f^n(0)) = f^{n+1}(0) \leq p \), therefore \( f^{n+1}(0) \leq p \), and so by induction \( (\forall n < \omega) f^n(0) \leq p \), therefore \( f^\omega(0) \leq p \), which contradicts hypothesis.
b) For some ordinal \( \beta \), take \( \alpha = f^\omega(\beta+1) \geq \beta+1 > \beta \).

Then \( f(\alpha) = f(f^\omega(\beta+1)) \)

\[ = f(\lim_{n<\omega} f^n(\beta+1)) \]

\[ = \lim_{n<\omega} f^{n+1}(\beta+1) = \alpha . \]

**THEOREM 1.3**

Note that \( \alpha \) is the least fixed point of \( f \) such that \( \alpha > \beta \). For suppose not, and let there be a \( \xi \) such that \( \beta+1 < \xi < \alpha \) and \( f(\xi) = \xi \). Then \( f(\beta+1) \leq \xi \), and \( (\forall n<\omega) \) if \( f^n(\beta+1) \leq \xi \), then \( f^{n+1}(\beta+1) \leq \xi \), so \( (\forall n<\omega) \) \( f^n(\beta+1) \leq \xi \), therefore \( \alpha = f^\omega(\beta+1) \leq \xi \) which contradicts the hypothesis.

**LEMMA 1.4**. If \( f \) is normal, then the fixed points of \( f \) form a set closed under the operation of taking limits.

**PROOF**. For some limit number \( \mu \), let \( \{ \alpha_\xi \}_{\xi<\mu} \) be a strictly increasing sequence of fixed points of a normal function \( f \). Then

\[ f(\lim_{\xi<\mu} \alpha_\xi) = \lim_{\xi<\mu} f(\alpha_\xi) = \lim_{\xi<\mu} \alpha_\xi . \]
DEFINITION 1.5. The operation 'F' upon normal functions \( f \) is defined by \( f^F(a) = a \) th fixed point of \( f \). Thus by theorem 1.3:
\[
\begin{align*}
f^F(0) &= f^0(0) \\
f^F(a+1) &= f^\omega(f^F(a) + 1), \text{ and } \lim(a) \Rightarrow f^F(a) &= \lim_{a'' \leq a} f^F(a').
\end{align*}
\]
Thus \( f^F \) is continuous, and if \( p < a \), then \( f^F(p) = f^F(a) \), so \( f^F \) is normal. (c.f. Veblen, [20], p 285).

THEOREM 1.5a. Suppose \( f, g \) are normal functions. Then \( f \circ g \) is normal.

PROOF. Suppose \( a' < a \). Then \( g(a') = g(a) \), therefore \( f(g(a')) = f(g(a)) \). Now suppose \( \lim(a) \). Then \( f(g(a)) = \lim_{a'' \leq a} f(g(a')) \)
\[
= \lim_{a' \leq a} f(g(a')) \quad \square 1.5a
\]

ELEMENTARY FUNCTIONS.

The class \( \mathcal{E} \) of elementary functions on the ordinals \( \leq \omega \), is now defined. \( \mathcal{E} \) contains many
of the simpler operations found in ordinal arithmetic, and functions taken from this class are used to form an initial basis for the equations of nested case-of-values transfinite recursion called Schmata R.

**DEFINITION 1.5b1**

The class \( \mathcal{E} \) is the smallest class satisfying:

a. Initial functions:

\[ \forall \alpha. \, S \alpha, I \alpha, I' \alpha, Z \alpha \in \mathcal{E} \]  (successor, predecessor, identity, and zero functions)

\[ \forall \alpha_0, \alpha_1. \, U_0(\alpha_0, \alpha_1), U_1(\alpha_0, \alpha_1), \alpha_0 \alpha_1, \alpha_0^\alpha_1 \in \mathcal{E} \]  (first and second projection functions, multiplication and exponentiation)

b. Closure:

I Superposition:

Suppose \( \alpha \vdash \overrightarrow{x} \Delta(\alpha_0, \ldots, \alpha_i, \alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_n) \) and \( \forall \beta. \, \Delta(\beta_0, \ldots, \beta_m) \in \mathcal{E} \)

Then \( \forall \alpha_0, \ldots, \alpha_i, \beta, \alpha_{i+2}, \ldots, \alpha_n \).

\[ \alpha \vdash \overrightarrow{x} \Delta(\alpha_0, \ldots, \alpha_i, \overrightarrow{\beta}, \alpha_{i+2}, \ldots, \alpha_n) \in \mathcal{E}. \]

II Bounded sum and product:

Suppose \( \forall \beta, \alpha. \, \Delta(\beta, \alpha) \in \mathcal{E} \). Then each of
the functions \( \overline{\mathcal{P}}^{(o)} (\beta, \alpha) = \begin{cases} \overline{\mathcal{P}} (\beta', \alpha) \\ \beta < \beta' \end{cases} \)
\( \overline{\mathcal{P}}^{(o)} (\beta, \alpha) = \prod_{\beta' < \beta} \overline{\mathcal{P}} (\beta', \alpha) \)
belong to \( \mathcal{E} \).

(The operations \( \Sigma \) and \( \Pi \) are defined on page ,
and in Bachmann, [43, pp. 53, 56].)

SCHEMATA R FOR NESTED TRANSFINITE COURSE-OF-VALUES RECURSION

The schemata provide the definition for certain ordinal valued functions of finitely many (\( \geq 2 \)) arguments, and are used in \( \S 3 \) to obtain the higher arithmetic operations on the ordinal numbers \( < \omega \).

The functions of Schemata R are defined inductively.

The recursive scheme for functions \( \mathcal{P} \) of three arguments is given, then for some integer \( n \geq 3 \), the recursive scheme for functions \( \mathcal{P} \) of \( n+1 \) arguments is defined, upon the basis of a function \( \mathcal{P} \), which is itself defined from the recursive scheme for functions of \( n \) arguments.
DEFINITION 1.5c

Suppose the function \( \mu \) of three arguments is defined by:
\[
\mu(i, \alpha_1, \alpha_2) = \mu_i(\alpha_1, \alpha_2), \quad i \leq 2, \quad \mu \in E
\]
and for \( \alpha_0 \geq 3 \):
\[
\mu(\alpha_0, 0, \alpha_2) = \tau, \quad \text{for some } \tau \leq \omega;
\]
\[
\mu(\alpha_0, 1, \alpha_2) = \alpha_2
\]
\[
\mu(\alpha_0, 2, \alpha_2) = \text{ON}^{(0)}(\{\mu(\alpha', \alpha_2, \alpha_2)_{\alpha' < \alpha_0}\})
\]
and for \( \alpha_0 \geq 3 \) and \( \alpha_1 \geq 3 \):
\[
\mu(3, \alpha_1, \alpha_2) =
\]
\[
\text{ON}^{(2)}(\{\mu(h, \alpha_1, \text{ON}^{(0)}(\{\mu(3, \alpha, \alpha_2)_{\alpha < \alpha_i}\})_{\alpha' < \alpha_i}\})\}_{h < 3})
\]
and for \( \alpha_0 \geq 4 \), \( \alpha_1 \geq 3 \):
\[
\mu(\alpha_0, \alpha_1, \alpha_2) =
\]
\[
\text{ON}^{(3)}(\{\mu(d, \text{ON}^{(0)}(\{\mu(\alpha_0, \alpha', \alpha_2)_{\alpha' < \alpha_0}, \alpha_2)_{\alpha < \alpha_0}\})\}_{d < \alpha_0})
\]
where \( \text{ON}^{(j)} \), \( 0 \leq j \leq 6 \), are given operations, defined on each sequence of countable ordinals \( \{\alpha_0, \beta_0, \beta_1, \ldots, \beta_n, \ldots, \beta = \mu \} \),
for \( \mu \leq \omega \), and such that for each such sequence \( \text{ON}^{(j)}(\{\alpha_0, \beta_0, \beta_1, \ldots, \beta_n, \ldots, \beta = \mu \}) \leq \omega \).

In many instances, these operations are derived from the operation of limit, and in some cases reduce to:
\[
\text{ON}^{(j)}(\{\alpha_0, \beta_0, \beta_1, \ldots, \beta_n, \ldots, \beta = \mu \}) = \lim_{n \to \omega} \alpha_n
\]

Then the function \( \mu \) is said to be defined by scheme \( \text{R} \).
from the functions $f_i$, $i \leq 2$, and from the constant $\gamma$, and from the operations $\text{ON}_j$, $0 < j < 6$.

Now let $\square$ be a function of $n$ ordinal arguments, where $3 \leq n < \omega$, which is defined by schemata $R$ from certain functions. Suppose the function $\square$ of $n+1$ arguments is defined by: $(\forall \alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \gamma)(\square(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \gamma) \leq \omega,$

(i) $\square(0, \alpha_1, \ldots, \alpha_{n-1}, \gamma) = \square(\alpha_1, \ldots, \alpha_{n-1}, \gamma)$

(ii) for $\alpha_0 > 0$ ($\forall i < n-1$)

$\square(\alpha_0, \ldots, \alpha_i, 0, \alpha_{i+2}, \ldots, \alpha_{n-1}, \gamma) = 1$

$\square(\alpha_0, \ldots, \alpha_i, 1, \alpha_{i+2}, \ldots, \alpha_{n-1}, \gamma) = \gamma$

$\square(\alpha_0, \ldots, \alpha_i, 2, \alpha_{i+2}, \ldots, \alpha_{n-1}, \gamma) =$

$\text{ON}_i(\{\square(\alpha_0, \ldots, \alpha_{i-1}, \alpha, \alpha_{i+2}, \alpha_{i+3}, \ldots, \alpha_{n-1}, \gamma)\} \alpha < \alpha_i)$

where $\alpha_{i+2} = \gamma$, when $i = n-2$, and where $\text{ON}_i$ are given operations satisfying the conditions specified in the definition of the operations $\text{ON}_j$ above.

(iii) $\square(\alpha_0, \ldots, \alpha_{n-1}, 0) = \square(\alpha_0, \ldots, \alpha_{n-1}, 1) = 1$

(iv) The final inductive clause is given in equations containing function symbols $\Lambda_i$, $i \leq n-1$, from which the parameters have been omitted.
for $\alpha_0 > 0$, $\alpha_i \geq 2$ for $0 < i \leq n-1$, $\gamma > 1$:

$$\mathbb{E}(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \gamma) = \Lambda_{n-1},$$

where

$$\Lambda_0 = \text{ON}_0(\mathbb{E}(\alpha_0, \alpha_1, \ldots, \alpha_{n-2}, \alpha, \gamma)^{\alpha < \alpha_{n-1}}),$$

and for all $i < n-1$

$$\Lambda_{i+1} = \text{ON}_{i+1}(\mathbb{E}(\alpha_0, \alpha_1, \ldots, \alpha_{n-2}, \alpha, \gamma, \Lambda_i)^{\alpha < \alpha_{n-1}},$$

and where $\text{ON}_i$, $i < n$, are operations satisfying the conditions specified in the definition of the operations $\text{ON}^{(j)}$ above, and where $\alpha_{n-1} = \text{the empty sequence}$, when $i = n-2$.

Then $\mathbb{E}$ is said to be defined by Schanita $R$ from the function $\mathbb{E}$ and from the operations $\text{ON}_i$, $0 < i < n$; $\text{ON}_i$, $i < n$.

**DEFINITION 1.5c.**

Schanita $R$ are based on a generalization of the Ackermann number-theoretic function, (*) and this has determined the choice of initial and inductive clauses.

**THEOREM 1.5d.** Suppose the function $\mathbb{E}$ is defined by Schanita $R$. Then $\mathbb{E}$ is uniquely defined.

(*) See appendix I.
PROOF. This is by induction on the number of arguments of $\square$.

Firstly, there cannot be more than one such function $\square$ of three arguments. For suppose $\square^{(1)}$, $\square^{(2)}$ are such distinct functions, satisfying identical equations of schema $R$. Then $(\forall \alpha_1, \alpha_2)$

$$\square^{(1)}(\alpha_0, \alpha_1, \alpha_2) = \square^{(2)}(\alpha_0, \alpha_1, \alpha_2), \quad \alpha_0 \leq 2.$$  

Let $\alpha_0$ be the least number $\geq 2$ for which $(\exists \alpha_1, \alpha_2)$

$$\square^{(1)}(\alpha_0, \alpha_1, \alpha_2) \neq \square^{(2)}(\alpha_0, \alpha_1, \alpha_2).$$  

Thus $(\forall \alpha_0' \prec \alpha_0)$

$$\square^{(1)}(\alpha_0', \alpha_1, \alpha_2) = \square^{(2)}(\alpha_0', \alpha_1, \alpha_2).$$

From the definition of schema $R$, the two functions clearly coincide at $\alpha_1 = 0, 1$.

Suppose $\alpha_1 = 2$. Then

$$\square^{(1)}(\alpha_0, 2, \alpha_2) = ON^{(1)}(\square^{(1)}(\alpha_0', 2, \alpha_2)_{\alpha_0' \prec \alpha_0})$$

$$= ON^{(1)}(\square^{(2)}(\alpha_0', 2, \alpha_2)_{\alpha_0' \prec \alpha_0})$$

$$= \square^{(2)}(\alpha_0, 2, \alpha_2).$$

Thus, $\alpha_1 \neq 2$.  


Suppose $\alpha_0 = 3$.
Then let $\alpha_1$ be the least number greater than 2 and such that
\[
H^{(3)}(3, \alpha_1, \alpha_2) \neq H^{(2)}(3, \alpha_1, \alpha_2).
\]
\[
H^{(1)}(3, \alpha_1, \alpha_2) = \text{ON}^{(2)}\left\{ \{ H^{(1)}(3, \alpha_1, \text{ON}^{(3)}(\{ \right.
\]
\[
H^{(1)}(3, \alpha_1, \alpha_2) \mid \alpha_i < \alpha_i \} \}_n \leq 3 ) \right\}
\]
\[
= \text{ON}^{(2)}\left\{ \{ H^{(2)}(3, \alpha_1, \text{ON}^{(3)}(\{ \right.
\]
\[
H^{(2)}(3, \alpha_1, \alpha_2) \mid \alpha_i < \alpha_i \} \}_n \leq 3 ) \right\}
\]
\[
= \{ H^{(2)}(3, \alpha_1, \alpha_2) \}.
\]

Thus $\alpha_0 \neq 3$.
Then suppose $\alpha_0 > 3$. Then, as before, $\alpha_1 \neq 2$.
Let $\alpha_1$ be the smallest number such that
\[
H^{(n)}(\alpha_0, \alpha_1, \alpha_2) \neq H^{(2)}(\alpha_0, \alpha_1, \alpha_2).
\]
Then
\[
H^{(n)}(\alpha_0, \alpha_1, \alpha_2) = \text{ON}^{(5)}(\{ H^{(n)}(\alpha_0, \alpha_1, \alpha_2) \} \alpha_i < \alpha_i),
\]
\[
(\alpha_2) \mid \alpha_i < \alpha_0 \}
\]
\[ = \text{ON}^{(5)}(\{ \{ \mathbb{E}^{(1)}(a'_0, \text{ON}^{(4)}(\{ \mathbb{E}^{(2)}(a'_0, a'_1, a'_2)

\{ \max_{i,j} \leq a'_0 \}) \}, a'_1, a'_2 \}\}) \]

\[ = \text{ON}^{(5)}(\{ \{ \mathbb{E}^{(2)}(a'_0, \text{ON}^{(4)}(\{ \mathbb{E}^{(2)}(a_0, a'_1, a'_2)

\{ \max_{i,j} \leq a'_0 \}) \}, a'_1, a'_2 \}\}) \]

\[ = \mathbb{E}^{(2)}(a_0, a_1, a_2). \]

Thus, the assumption of two different functions \( \mathbb{E}^{(1)} \)
\( \mathbb{E}^{(2)} \), each on three arguments and satisfying identical
equations of schemata \( R \), leads to a contradiction.

Now it is proved that for every constant \( \gamma \), and pair
\( \{ \mu'_j : j \leq 3 \} \), \( \{ \text{ON}(j) : 0 < j < 6 \} \), there exists a function
\( \mathbb{E} \) satisfying the equations of schemata \( R \). Suppose
there is no such function. Now \( \mathbb{E} \) satisfies \( R \)
for each \( \alpha_1, \alpha_2 \), and each \( j \leq 2 \), so suppose \( \alpha_0 \) is the
least number for which \( (\exists \alpha_1, \alpha_2) \mathbb{E}(\alpha_0, \alpha_1, \alpha_2) \) is
not defined. Clearly \( \alpha_1 \neq 0, 1. \)

Suppose \( \alpha_1 = 2 \). Then \((\alpha'_0, \leq a'_0) \mathbb{E}(a'_0, 2, \alpha'_2) \) is,
by hypothesis, defined, and so by \( R \), \( \mathbb{E}(a'_0, 2, \alpha'_2) \) is
defined, thus \( \alpha_1 \neq 2. \)

Suppose \( \alpha_0 = 3 \). Then let \( \alpha_1 \) be the least number \( > 2 \)
such that \((\exists \alpha_2) \mathbb{E}(2, \alpha_1, \alpha_2) \) is not defined.
Then \((\forall \alpha \leq \alpha' \forall \alpha_i < \alpha_i') (\forall \alpha_2')
\(\mathbb{E}(\alpha_0, \alpha_1, \alpha_2') \) are defined, and thus
by \(R \mathbb{E}(\alpha_0, \alpha_1, \alpha_2') \) too is defined.

Thus the assumption that there is no function \(\mathbb{E} \)
satisfying, for given \(\gamma \), \(\exists \mu \exists j \leq \gamma \), \(\{\text{ON}^j\}_{i < \gamma} \)
leads to a contradiction, so theorem 1.5D is now proved for functions of three arguments.

Suppose, for induction hypothesis, that it has been
proved that, for some \(n \geq 3 \), theorem 1.5D is
satisfied for all functions \(\mathbb{E} \) of \(n \) ordinal arguments
defined by application of schemata \(R \).

Let \(\mathbb{E} \) be such a function, and let \(\{\text{ON}^j\}_{i < \gamma} \)
\(\{\text{ON}^j\}_{i < \gamma} \) be sequences of ordinal valued \((\leq \alpha_i) \)
operations defined on countable sequences of countable ordinals. Further, let \(\mathbb{E} \) be a function of \(n+1 \)
arguments defined from these functions by schemata \(R \).

Suppose that for certain sets of argument values, \(\mathbb{E} \) is
not uniquely defined. Then

**DEFINITION.** Let the sequence of numbers \(\bar{\alpha}_0, \bar{\alpha}_1, \ldots \bar{\alpha}_{n-1}, \bar{\delta} \) be defined by induction as follows:
\(\bar{\alpha}_0 = \alpha \) of the least ordinal number such that
\((\exists \alpha_i, \alpha_{i+1}, \ldots, \alpha_{n-1}, \delta) \) is
uniquely defined (u.d.).
Suppose for some \( i \) such that \( 0 \leq i < n-2 \), numbers
\( \overline{\alpha}_0, \overline{\alpha}_1, \ldots, \overline{\alpha}_i \) have been defined. Then
\[
\overline{\alpha}_{i+1} = \text{the least ordinal number such that }
(\exists \overline{\alpha}_{i+2}, \overline{\alpha}_{i+3}, \ldots, \overline{\alpha}_{n-1}, \overline{\delta}) \equiv (\overline{\alpha}_0, \overline{\alpha}_1, \ldots, \overline{\alpha}_i, \overline{\alpha}_{i+1}, \overline{\alpha}_{i+2}, \ldots, \overline{\alpha}_{n-1}, \overline{\delta}) \text{ is not u.d.}
\]
and
\[
\overline{\alpha}_{n-1} = \text{the least ordinal number such that }
(\exists \overline{\alpha}_n, \overline{\delta}) \equiv (\overline{\alpha}_0, \overline{\alpha}_1, \ldots, \overline{\alpha}_{n-1}, \overline{\delta}) \text{ is not u.d.}
\]
and
\[
\overline{\delta} = \text{the least ordinal number such that }
(\overline{\alpha}_0, \overline{\alpha}_1, \ldots, \overline{\alpha}_{n-1}, \overline{\delta}) \text{ is not u.d.}
\]

**LEMMA**. Let \( \overline{\alpha}_0, \ldots, \overline{\delta} \) be as defined above. Then
\( \overline{\alpha}_0 > 0 \), \((\forall i), 0 < i < n-1 \), \( \overline{\alpha}_i > 2 \) and \( \overline{\delta} > 1 \).

**PROOF**. By definition \((\forall \overline{\alpha}_0, \overline{\alpha}_1, \ldots, \overline{\alpha}_{n-1}, \overline{\delta}) \equiv (\overline{\alpha}_0, \overline{\alpha}_1, \ldots, \overline{\alpha}_{n-1}, \overline{\delta}) \) is u.d. Thus \( \overline{\alpha}_0 > 0 \), and thus proves the first clause of the Lemma. Furthermore \((\forall \overline{\alpha}_0 < \overline{\alpha}_0) \)
\((\forall \overline{\alpha}_0, \overline{\alpha}_1, \ldots, \overline{\alpha}_{n-1}, \overline{\delta}) \equiv (\overline{\alpha}_0, \overline{\alpha}_1, \ldots, \overline{\alpha}_{n-1}, \overline{\delta}) \) is u.d., and thus \((\forall \overline{\alpha}_0, \overline{\alpha}_1, \ldots, \overline{\alpha}_{n-1}, \overline{\delta}) \equiv (\overline{\alpha}_0, 2, \overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}_3, \ldots, \overline{\alpha}_{n-1}, \overline{\delta}) \) is u.d. too.

Therefore \( \overline{\alpha}_i > 2 \).
Now suppose for induction hypothesis that it has been shown for some \( i \) such that \( 0 < i < n-1 \) that
(\forall k \in I)_0 \overline{\alpha}_k > 2. \text{ Then } (\forall \alpha'_k < \overline{\alpha}_k) \forall \alpha'_{k+2}, ... \overline{\alpha}_{n-1}, \overline{\alpha} \in \alpha_{n-1}, \overline{\alpha} \in \text{u.d.}, \text{ and thus } (\forall \alpha'_{k+2}, \alpha'_{k+3}, ..., \alpha'_{n-1}, \overline{\alpha}) \in \text{u.d.}, \text{ and thus } (\forall \alpha'_{k+2}, \alpha'_{k+3}, ..., \alpha'_{n-1}, \overline{\alpha}) \text{ too is u.d. Thus the second clause of the lemma is proved.}

Furthermore, by definition of schema \( R \), \( H \) is uniquely defined for \( \overline{\alpha} = 0, 1 \).

Thus for \( H (\alpha_0, \alpha_1, ..., \alpha_{n-1}, \overline{\alpha}) \) not u.d., \( \alpha_0 > 0, \alpha_1 > 2, ..., \alpha_{n-1} > 2, \overline{\alpha} > 1. \) \( \square \text{ LEMMA} \).

By the hypothesis that \( (\exists \alpha_0, ..., \alpha_{n-1}, \overline{\alpha}) H (\alpha_0, ..., \alpha_{n-1}, \overline{\alpha}) \) is not u.d., and by the definition of numbers \( \overline{\alpha}_0, \overline{\alpha}_1, ..., \overline{\alpha}_{n-1}, \overline{\alpha} \), it follows that \( (\forall \alpha'_{n-1} < \overline{\alpha}_{n-1} (\forall \overline{\alpha}) \)

\( H (\alpha_0, \alpha_1, ..., \overline{\alpha}_n, \alpha'_{n-1}, \overline{\alpha}) \) is u.d., and thus so is \( \Lambda_0 \).

Suppose, for some \( i < n-1 \), that \( \Lambda_i \) is u.d. Then

\( (\forall \alpha'_{n-1-2} < \alpha'_{n-2},(\forall \alpha'_{n-2}, ..., \alpha'_{n-1}, \overline{\alpha}) H (\alpha_0, \alpha_1, ..., \overline{\alpha}_{n-1}, \overline{\alpha}) \) is u.d., ... \overline{\alpha}_{n-3}, \alpha'_{n-2}, \Lambda_i, \alpha'_{n-2}, ..., \alpha'_{n-1}, \overline{\alpha} \) is u.d.,

and thus so is \( \Lambda_{i+1} \).

Hence it follows by induction that \( \Lambda_{n-1} = \)

\( H (\alpha_0, ..., \overline{\alpha}_{n-1}, \overline{\alpha}) \) is uniquely defined, contradicting the hypothesis.
Thus it is proved by contradiction that if theorem 1.5D is satisfied for all functions of \( n \geq 3 \) ordinal arguments defined by application of schemata \( R \), then the theorem is satisfied for all such functions of \( n+1 \) arguments, and so by induction it follows that theorem 1.5D is proved.
§2. FUNDAMENTAL SEQUENCES.

In this section, some standard definitions and results are outlined, and an unsolved problem arising in connection with a scheme of G.H. Hardy is stated.

If \( \alpha \) is an ordinal such that \( \text{Lim}(\alpha) \) and \( \alpha \in \mathbb{Z}(\frac{1}{\alpha}) \), then a fundamental sequence for \( \alpha \) is an infinite sequence of countable ordinals \( \{\alpha_n\}_{n<\omega} \) such that

1) \((\forall n<\omega) \alpha_n < \alpha_{n+1}
2) \lim_{n<\omega} \alpha_n = \alpha

THEOREM 2.1. For every limit number belonging to \( \mathbb{Z}(\frac{1}{\alpha}) \), there exists a fundamental sequence.

PROOF (Scipinowski, [18], p 384). Suppose \( \alpha < \omega \), and \( \text{Lim}(\alpha) \). Now the set \( W_\alpha \) is thus denumerable, and does not contain a greatest element. So let \( S = \{\xi_n\}_{n<\omega} \) be an infinite sequence containing exactly the ordinals in the set \( W_\alpha \). Let \( \alpha_0 = \xi_0 \), and for every \( n<\omega \), let \( \alpha_{n+1} = (\mu_\xi)[\xi \in S \& \alpha_n < \xi] \).
Thus the sequence $\{\alpha_n\}_{n<\omega}$ is strictly increasing.

Now $(\forall \beta<\omega)(\exists n<\omega) \beta < \alpha_n$, thus no element of $S$ is greater than every element of $\{\alpha_n\}_{n<\omega}$. So $\alpha$ is the least ordinal greater than each element of $\{\alpha_n\}_{n<\omega}$, therefore $\alpha = \lim_{n<\omega} \alpha_n$.

\[ \text{THEOREM 2.1} \]

Theorem 2.1 does not enable a function to be exhibited such that the function assigns to each countable limit number one and only one fundamental sequence. On the assumption of the resistance of such a function, Alonzo Church [8] theorem A4, § 191 has proved:

$$\alpha_1 \leq 2$$

without the aid of the axiom of choice.

**DEFINITION 2.2** $\alpha, n, \Omega_{\alpha}(n)$ is called a fundamental sequence assignment (f.s.a.) if $(\forall n<\omega)$

$(\forall n<\omega)$

i) $\Omega_1: \omega \times \omega \rightarrow \omega$

ii) $(\alpha = 0 \vee \text{Succ}(\omega)) \Rightarrow \Omega_{\alpha}(0) = 0$

iii) $\text{Lim} (\alpha) \Rightarrow \Omega_{\alpha}(n) < \Omega_{\alpha}(n+1)$

iv) $\text{Lim} (\alpha) \Rightarrow \lim_{n<\omega} \Omega_{\alpha}(n) = \alpha$
Thus, $\Omega_\alpha$ assigns to each countable limit ordinal $\alpha$, a fundamental sequence $\{\Omega_\alpha(n)\}_{n=0}^\infty$.

**EPSILON NUMBERS**

Let $p_0 = 0$, and $(\forall n < \omega) \ p_{n+1} = \omega^{p_n}$.

Let $\varepsilon_0 = \lim_{n \to \omega} p_n$.

Then $\omega^{\varepsilon_0} = \lim_{n \to \omega} \omega^{p_n}$, as $\forall p. \omega^p$ is continuous.

$$= \lim_{n \to \omega} p_{n+1} = \varepsilon_0$$

By theorem 1.3, and since $\forall p. \omega^p$ is strictly increasing, it follows that $\varepsilon_0$ is the least number with this property.

The solutions $\varepsilon_i$ to the equation

$$\omega^{\varepsilon_i} = \varepsilon_i$$

are called by Cantor epsilon numbers, $\{\varepsilon_i\}_{i=0}^\omega$, denoting the sequence of countable epsilon numbers, ordered according to magnitude.

According to the normal form theorem of Cantor, every limit number $\alpha < \varepsilon_0$ can be expressed uniquely as

$$\alpha = \sum_{i=1}^\omega \omega^{\varepsilon_i} \alpha_i$$

where $0 < \alpha_0 < \alpha_{n+1} < \ldots < \alpha_n < \alpha$, and
where $r$ and $a_1, a_2, \ldots, a_r$ are natural numbers.

It is thus possible to define a fundamental sequence $\Omega_\alpha(n)$ for limit $\alpha \leq \varepsilon_0$ by transfinite induction: $(\forall n < \omega)$

(i) If $\alpha = \omega$, then $\Omega_\alpha(n) = n$

(ii) If $\text{Succ}(\alpha)$, then

$$\Omega_\alpha(n) = \sum_{i=1}^{\omega^\alpha} a_i + \omega^\alpha (a_r - 1) + \omega^{\alpha^{r-1}} n$$

(iii) If $\text{Lim}(\alpha)$, and $\forall m. \Omega_\alpha(m)$ has already been defined, then

$$\Omega_\alpha(n) = \sum_{i=1}^{\omega^\alpha} a_i + \omega^\alpha (a_r - 1) + \omega^{\Omega_\alpha(m)}$$

A PROBLEM CONCERNING FUNDAMENTAL SEQUENCES.

There is an unsolved problem in the theory of ordinal numbers which arises in connection with C.H. Hoddy's scheme [12] for determining a set of points belonging to the continuum whose cardinal number is $\delta_1$.

Stated in its general form, the problem is: find a
function of two arguments $\alpha, \beta \cdot \Omega_\alpha(\beta)$ such that for every ordinal number $\mu$, and for every limit ordinal $\gamma$ such that

$$\gamma \in \mathbb{Z}(\beta, \mu)$$

the function $\Omega$ determines a unique strictly increasing sequence \(\{\Omega_\alpha(\gamma)\}_{\gamma < \omega_\alpha}\), for some $\alpha \leq \mu$, such that

$$\gamma = \lim_{\gamma < \omega_\alpha} \Omega_\alpha(\gamma).$$

The sequence \(\{\Omega_\alpha(\gamma)\}_{\gamma < \omega_\alpha}\) is called a fundamental sequence for the ordinal $\gamma$. (c.f. Bachmann [3], p. 141).

In connection with this problem, as it applies to the second number class, several classes of ordinal-valued functions are presented, and some of their properties investigated.
§3. THE CLASS $\mathcal{D}_\alpha^{(\omega)}$ OF
FUNCTIONS $\mathcal{D}_\alpha^\beta: \omega_1 \to \omega_1$.

Certain arithmetic operations on the countable ordinals, including the familiar operations of addition, multiplication and exponentiation, generate on $\omega \times \omega$, an array of functions:

\[
\begin{array}{cccccc}
\mathcal{D}_0 & \mathcal{D}_1 & \mathcal{D}_2 & \mathcal{D}_3 & \ldots & \mathcal{D}_\alpha \\
\mathcal{D}_0^2 & \mathcal{D}_1^2 & \mathcal{D}_2^2 & \mathcal{D}_3^2 & \ldots & \mathcal{D}_\alpha^2 \\
\mathcal{D}_0^3 & \mathcal{D}_1^3 & \mathcal{D}_2^3 & \mathcal{D}_3^3 & \ldots & \mathcal{D}_\alpha^3 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\end{array}
\]

[1] $\mathcal{D}_0^\beta, \mathcal{D}_1^\beta, \mathcal{D}_2^\beta, \ldots, \mathcal{D}_\alpha^\beta$...

In the following definition, these functions in region [1] are defined explicitly (Case (i) below) and those in region [2] are defined by transfinite recursion (Cases (iii), (iii'), (iv')).

DEFINITION 3.1. For each countable ordinal $\xi$, and some fundamental sequence assignment $\Omega_1$ (such that for each $n < \xi$, $\Omega_1(n) = n$; cf. definition 2.2).
(i) \( (\forall \alpha < \omega_1) \left[ \left( \mathfrak{I}^\alpha_\omega (x) = 1 \right) \land \mathfrak{I}^\alpha_\omega (x) = 0 \right. \land \mathfrak{I}^\alpha_\omega (x) = \gamma + \alpha \right] \land \\
\left( \forall \alpha < \omega \right) \left[ \left( \mathfrak{I}^\alpha_\alpha (x) = \gamma \land \mathfrak{I}^\alpha_\alpha (x) = 0 \right) \right]

(ii) \( (\forall \alpha < \omega_1) \left[ \text{Suc}(x) \Rightarrow \mathfrak{I}^\alpha_\alpha (x) = \mathfrak{I}_\alpha^\alpha (x) \land \right. \left. \lim_{n \to \omega} \mathfrak{I}^\alpha_n (x) = \lim_{n \to \omega} \mathfrak{I}^{\alpha_2}_n (x) \right] \)

(iii) For \( \alpha = x = 1, 2 \), and \( 2 < \beta \):

\[
\mathfrak{I}^\beta_{\alpha + 1} (x) = \begin{cases} \\
\mathfrak{I}^\beta_{\alpha + 1} (x, \alpha) \text{ if } \text{Suc}(\beta) \text{ and } \lim_{n \to \omega} \mathfrak{I}^\beta_n (x, \alpha) = \lim_{n \to \omega} \mathfrak{I}^{\alpha_2}_n (x) \text{ if } \text{Lui}(\beta) 
\end{cases}
\]

(iv) \( (\forall \alpha, \beta < \omega_1)_{2 < \beta, \alpha} \)

\[
\mathfrak{I}^\beta_{\alpha} (x) = \begin{cases} \\
\mathfrak{I}^\beta_{\alpha} (x, \beta) + 1 \text{ if } \text{Suc}(\alpha, \beta) \land x = 1 \\
\mathfrak{I}^\beta_{\alpha} (x, \beta) + 1 \text{ if } \text{Suc}(\alpha, \beta) \land 1 < \alpha \\
\lim_{n \to \omega} \mathfrak{I}^\beta_n (x, \alpha) \text{ if } \text{Lui}(\beta) \\
\lim_{n \to \omega} \mathfrak{I}^\beta_n (x, \alpha) \text{ if } \text{Lui}(\alpha) \land \text{Suc}(\beta).
\end{cases}
\]

In each case, \( \beta \) is expressed in the unique form \( \beta = \omega \beta' + \gamma, \beta' < \beta, \gamma < \omega \).

\( \square \) Definition 3.1

This class of functions is denoted by the symbol \( \mathfrak{I}^\beta_{\alpha} (x) \).
\( \gamma \) is called here the operand, \( p \) the exponent and \( a \) the index of the expression \( \Delta_a^\gamma (x) \). Cantor [6] shows that each countable limit ordinal \( \gamma \) can be expressed uniquely in the form \( \gamma = s + \omega^p \), the number \( p \) being called the exponent of \( \gamma \). Definition 3.1 with respect to this terminology thus conforms to Cantor in the case \( \gamma = \Delta_a^\gamma (\omega) = \omega^p \).

The following notation is sometimes useful: \( (\forall \alpha, p, s < \omega) \)

\[ \gamma \ast p = \rho \Delta_a^\gamma (x) \]

**Definition 3.2. The Operations \( \Sigma_1 \), \( \Pi \) and \( \Gamma_\alpha \), \( \alpha = \omega_1 \).**

These follow standard definitions for sum and product of finite and transfinite sequences of ordinals (c.f. Bachman [4], pp. 53, 56): for each \( f: \omega \rightarrow (\omega, n+1) \), \( n > 0 \), and each string \( s \) of parameters of length \( n-1 \), and each \( p < \omega_1 \):

\[ \Sigma_1 \ f(p, s) = 0 \quad \Pi \ f(p, s) = 1 \quad p < 0 \]

---

(*) See afkaufd II, p. 183.
\[ \sum_{p' < p+1} f(p', \vec{s}) = \sum_{p' < p} f(p', \vec{s}) + f(p, \vec{s}) \]
\[ \prod_{p' < p+1} f(p', \vec{s}) = \left\{ \prod_{p' < p} f(p', \vec{s}) \right\} \cdot f(p, \vec{s}) \]

and for \( \lambda \) a countable limit number
\[ \sum_{p' < \lambda} f(p', \vec{s}) = \lim_{\mu \to \lambda} \sum_{p' < \mu} f(p', \vec{s}) \]
\[ \prod_{p' < \lambda} f(p', \vec{s}) = \lim_{\mu \to \lambda} \prod_{p' < \mu} f(p', \vec{s}) \]

The notions \( \Gamma_\alpha \), \( \alpha \in \omega \), map 1-1 from \( \omega_i^{[\omega^{(n)}]} \) into \( \omega_i^{[\omega^{(n)}]} \), for each \( n \in \omega \), and are defined as follows from elements of the class \( \omega_i^{[\omega^{(1)}]} \) (the square brackets denote set-theoretic exponentiation). For each \( n \) such that \( 0 < n < \omega \), and for each \( f : \omega_i^{[n]} \rightarrow \omega_i \),

\( \forall \vec{x} \in \omega_i^{[n-1]} \) \( \forall \vec{y}, \vec{p} \in [\omega_i, 0 < p] \)

\[ \left[ (i) \right] \Gamma_\alpha f(p', \vec{s}) = 1, \text{ for } \alpha > 3. \]
(ii) for $\alpha = 0$, $\Gamma_{p' = p} f(p', \hat{s}) = f(0, \hat{s}) + \Gamma_p$

\[
\alpha = 1, \quad \Gamma_{p' = p} f(p', \hat{s}) = \frac{1}{\alpha} f(p', \hat{s}) \quad p' = p
\]

\[
\alpha = 2, \quad \Gamma_{p' = p} f(p', \hat{s}) = \prod_{p' = p} f(p', \hat{s})
\]

(iii) $3 \leq \alpha$ and $\text{Suc}(p) \Rightarrow$

\[
\Gamma_{p' = p} f(p', \hat{s}) = \begin{cases} 
\mathbb{D}_u^{- \alpha} \{ f(\Gamma_p, \hat{s}) \} & \text{if } \alpha < \omega, \ x = 1 \\
\mathbb{D}_u^{\alpha-1} \{ f(\Gamma_p, \hat{s}) \} & \text{if } \alpha < \omega, \ 1 < x \\
\mathbb{D}_{P_d}^{\alpha-1} \{ f(\Gamma_p, \hat{s}) \} & \text{if } \omega < \alpha, \ 1 < x \\
\mathbb{D}_{P_d}^{\alpha} \{ f(\Gamma_p, \hat{s}) \} & \text{if } \omega < \alpha, \ 1 < x
\end{cases}
\]

where $u = \Gamma_{p' = p} f(p', \hat{s})$. The parameters for $u$ have been omitted.

\[
\sup_{\alpha' < \alpha} \Gamma_{p' = p} f(p', \hat{s}) \text{ if } \text{Lim}(\alpha).
\]

(iv) for $3 \leq \alpha$ and $\text{Lim}(p)$

\[
\Gamma_{p' = p} f(p', \hat{s}) = \sup_{p' = p} \Gamma_{p' = p} f(p', \hat{s}) \quad p' = p \quad p^* = p^*
\]

□ DEFINITION 3.2
The operations $\mathcal{Y}$, $\Upsilon$ form special cases on account of the property of the arithmetic operations $\odot$ on the countable ordinals, which is discussed on pages 121-123 of § 4.

**EXAMPLES.**

a) $\mathcal{Y}_1^n(\gamma) = \gamma + \beta$ by definition

b) $\Upsilon_2^n(\gamma) = \gamma \cdot \beta$. For suppose $\beta = 0, 1, 2$. Then $\Upsilon_2^n(\gamma)$ equals $0 = \gamma \cdot 0$; $\gamma = \gamma \cdot 1$; $\Upsilon_2^n(\gamma) = \gamma + \gamma = \gamma \cdot 2$ respectively.

Now suppose $\gamma = \beta$, and that the equality holds for all numbers smaller than $\beta$. Then $\text{Suc}(\gamma) \Rightarrow$

$$\Upsilon_2^n(\gamma) = \mathcal{Y}_2^n \{ \Upsilon_2^n(\gamma) \} \text{ by definition } 3.1(\text{iii})$$

$$= \Upsilon_2^n(\gamma \cdot \beta) \text{ by hypothesis}$$

$$= \gamma \cdot \beta + \gamma = \gamma \cdot \beta$$

Furthermore, $\text{Lim}(\beta) \Rightarrow$

$$\Upsilon_2^n(\gamma) = \text{Lim}_{\gamma < \omega} \Upsilon_2^n \{ \Upsilon_2^n(\gamma) \} \text{ by definition } 3.1(\text{iii})$$
Therefore \( \Omega_2^\beta(\gamma) = \lim_{n \to \omega} \Omega_\beta^\gamma(n) \) by hypothesis

\[
= \gamma \cdot \beta
\]

Thus by transfinite induction the equality is proved to hold for all numbers \( \beta \).

Similarly, \( \Omega_3^\beta(\gamma) = \gamma^\beta \)

\[
\Omega_4^\beta(\gamma) = \sqrt[p]{\gamma}
\]

In particular, \( \Omega_1^2(\gamma) = \gamma + 2 \), \( \Omega_2^2(\gamma) = \gamma^2 \), \( \Omega_3^2(\gamma) = \gamma^2 \), \( \Omega_4^2(\gamma) = \gamma^\gamma \). Also, since

\[
\varepsilon_0 = \lim \{ \omega, \omega^\omega, \omega^{\omega^\omega}, \ldots \}
\]

\[
\varepsilon_0 = \lim_{n \to \omega} \Omega_4^{\Omega_2^n(\omega)}(\omega) = \Omega_4^\omega(\omega) = \Omega_5^2(\omega)
\]

Certain functions on the natural numbers, of interest in recursion theory, can be obtained from elements of the class \( \Omega^{(n)}_\omega \). Thus, by writing

\[
(\forall n \in \omega) \phi(n) = \Omega_4^n(2),
\]

\[
\phi(0) = \Omega_4^0(2) = 1
\]

\[
\phi(n+1) = \Omega_3^{\phi(n)}(2) = 2^{\phi(n)}, \text{ so that}
\]

\[
\text{ etc.}
\]
\[
\phi(n) = \sqrt[n]{2^{2^n}}
\]

\(\phi\) is considered by Posters [14] to be, according to a suitable set of criteria 'the simplest function which is not elementary'.

It can be shown that for each \(m, n < \omega\)
\(\forall x. \, \phi^m(x) \leq \omega, (1)\), and for each \(n < \omega, \forall x, y. \, \phi^n(x) \leq \omega\) are primitive recursive number theoretic functions; but \(\forall x. \, \phi^n(x) \leq \omega, \) although it is a restriction of \(\phi\) the first 'new' function to be obtained from the above functions in definition 3.1, it is not a 'majorizing function' of the Ackermann type, since it is always the limit of a transfinite increasing sequence of ordinals.

Such majorizing functions are obtained in the array \(\{\phi^n\}_{n=\omega, \alpha \in \omega}\), defined on p. 54.

Two important examples of the operations \(\Gamma^\alpha\) are already defined explicitly: \(\Gamma_0 = \Sigma^1, \Gamma_1 = \Pi^1\). Hausdorff, [13] p. 119, has defined the transfinite product, for functions \(f\) such that \((\forall \beta)\, 0 < f(\beta)\), as follows:

\[
\prod_{\beta < \alpha}^{(\Pi)} f(\beta) = 1, \text{ and for each } \beta > 0,
\]

\((1)\) The restriction of a function \(f\) to \(A'\), written \(f|_{A'}\), is defined as follows: if \(f: A \to B\), and \(A' \subseteq A\), then \(f|_{A'}\) denotes the function \(g: A' \to B\) such that \((\forall x \in A')\, g(x) = f(x)\).
\[ \prod^{(H)}_{\beta' < \beta} f(\beta') = (\mu \alpha) \left[ (\forall \theta < \beta) \prod^{(H)}_{\beta' < \theta} f(\beta') \right. \cdot f(\theta) \leq \alpha \bigg] \]

\[ = \sup_{\theta < \beta} \left\{ \prod^{(H)}_{\beta' < \theta} f(\beta') \right\} \cdot f(\theta). \]

**Lemma.** If \((\forall \beta') f(\beta') > 0\), then both
\[ \forall \beta, \prod^{(H)}_{\beta' < \beta} f(\beta') \text{ and } \forall \beta, \prod^{(H)}_{\beta' < \beta} f(\beta') \text{ are monotone.} \]

**Proof.** By transfinite induction. Immediate for \(\prod^{(H)}_{\theta < \beta} f(\beta') \leq \prod^{(H)}_{\beta_0 < \beta} f(\beta'), f(\beta_0) \)

For \(\prod^{(H)}_{\beta' < \beta} f(\beta') \), if \(\text{suc}(\beta)\) then \(\prod^{(H)}_{\beta' < \beta} f(\beta') \geq \prod^{(H)}_{\beta_0 < \beta} f(\beta_0) \cdot f(\beta_0) \)

\[ \geq \prod^{(H)}_{\beta_0 < \beta} f(\beta_0), \text{ as } f(\beta_0) \text{ is positive.} \]

**Corollary 3.3.** \(\prod^{(H)}_{\beta' < \beta} f(\beta') = \prod^{(H)}_{\beta' < \beta} f(\beta') \)

**Proof.** By transfinite induction. From the Lemma it follows that for \(\text{suc}(\beta)\), \(\prod^{(H)}_{\beta' < \beta} f(\beta') = \prod^{(H)}_{\beta' < \beta} f(\beta') \)

\[ \Box \]
= \{ \prod_{p' < p} f(p') \} \cdot f(p), \quad \text{and for} \quad \text{Lim}(p),

\prod_{p' = p} f(p') = \sup_{p^* = p} \prod_{p' < p} f(p^*)

\tag*{\square} \text{COROLLARY 3.3}

The operations \( \Gamma_{n^+} \) for \( n \in \omega \), when applied to number theoretic functions, are operations under which one closed the classes \( \mathcal{E}^{n^+} \), but not the classes \( \mathcal{E}^n \), where those classes are members of A. Grzelczyk's hierarchy of primitive recursive functions.

Two classes of maps are now defined, using definitions 3.1, 3.2.

DEFINITION 3.4

Some of the defining equations of definition 3.1 contain limit operations in their right hand sides. Let the functions \( D^2\alpha \) be those obtained in place of the functions \( T^2\alpha \) by restriction to finite values \( x, n \) of ordinals \( x \) and \( n \) respectively, and by making the following substitutions within two of these equations, and by leaving the remaining equations unchanged.

In definition 3.1(iii):

\[ \text{Lim} (d) \Rightarrow D^2\alpha (x) = D^2\alpha \{ D^2\alpha (x) \} (x) \]

In definition 3.1(iv):

\[ \text{Lim} (d) \Rightarrow D^{n+1}\alpha (x) = D^2\alpha \{ D^{n}\alpha (x) \} (x) \]

\[ \{ D^2\alpha \}_{n < \gamma, \alpha < \omega} = \text{of } D^2\alpha \]  

\( \square \) 3.4
Thus, number-theoretic functions have been obtained from some of the \( \Xi^p \) by replacing the taking of limits by a process of diagonalization, and restricting the domain of arguments \( p, \xi \) to finite ordinal values. Not all these functions are general recursive.

**Definition 3.5.** Some of the defining equations of 3.2 contain in their r.h.s's expressions built up from \( \Xi \), and also limit operations. Let the operations \( \mathcal{U}_\alpha \) which map 1-1 from \( \omega^{\omega^{\omega^p}} \) into \( \omega^{\omega^{\omega^p}} \) for \( n < \omega \), be those obtained by considering only finite values \( t > 0 \) of the index \( p \) of 3.2, and by making the following substitutions within two of these equations, and leaving the remaining equations unchanged:

In 3.2 (iii): \( t > 0, 3 \leq \alpha \) and \( \text{Suc}(\alpha) \Rightarrow \\
\mathcal{U}_\alpha f(x, y) = \begin{cases} \\
\text{Def} \{ f(t-1, y) \} & \text{if } \alpha < \omega \\
\text{Def}_\alpha \{ f(t-1, y) \} & \text{if } \omega < \alpha \\
\end{cases} \\
\text{where } \text{Def} = \mathcal{U}_\omega f(x, y) \ \ \text{if } x < t-1 \\
\text{Def}_\alpha f(x, y) \ \ \text{if } x = t-1 \\
\text{In 3.2 (iv)}:
\[
U_\alpha f(x, y) = U_{\alpha f(x, y)}^x
\]

\{U_\alpha\}_{\alpha \in \omega} = \mathcal{U}^{(\omega)}_{\omega}

\[\square\] DEFINITION 3.5

It is clear that the classes obtained in definitions 3.4 and 3.5 depend on the choice of the fundamental sequence assignment \(\Omega\). (However, it is shown in §4 that the class \(\Gamma^{(\omega)} = \Gamma^{(\omega)}_\omega\) is independent of the choice of \(\Omega\).)

Note that given an f.s.a. \(\Omega\) on f.s.a. \(\Omega'\neq \Omega\), can readily be found. For example, let \(\Omega, \Omega'\) satisfy \((\forall \alpha < \omega) \Omega_\alpha(n \cdot 2) = \Omega'_\alpha(n)\), for each \(n < \omega\) and for \(\lim(\alpha)\). Then \(\Omega'\) is an f.s.a. iff \(\Omega\) is an f.s.a., but since both sequences \(\{\Omega_\alpha(n)\}_{n < \omega}\), \(\{\Omega'_\alpha(n)\}_{n < \omega}\) are strictly increasing, for each \(n\) the term \(\Omega_\alpha(n+1)\) will not occur in \(\{\Omega'_\alpha(n)\}_{n < \omega}\).
It is now shown that the function $\mathcal{W}$ is an instance of definition by application of schemata $R$, definition 1.5C, p.20.

**Theorem 3.6.** The functions $\mathcal{W}_{\alpha, \beta}^\gamma$, $\alpha, \beta < \omega_1$, of definition 3.1 are well-defined.

**Proof.** By theorem 1.5D, every function $\mathcal{W}$ defined by schemata $R$ of definition 1.5C is well-defined.

Let $\mathcal{W}$ be the function of three arguments of 3.1, and put

$$ (\forall \alpha, \beta, \gamma) \; \mathcal{W}_{\alpha}^\beta(\gamma) = \mathcal{W}(\alpha, \beta, \gamma) $$

Then $$(\forall \beta, \gamma) \; \mathcal{W}(0, \beta, \gamma) = 0 = \mathcal{W}(\mathcal{U}(\beta, \gamma))$$

$$ \mathcal{W}(1, \beta, \gamma) = \gamma + \beta = \bigcup_{\eta \leq \beta} \mathcal{U}(\eta, \gamma) $$

Thus, by definition 1.5D1, $$(\forall i, \beta, \gamma) \; \mathcal{W}(i, \beta, \gamma) \in \mathcal{E}$$

Furthermore, for each $\alpha \geq 3$, and each $\gamma$

$$ \mathcal{W}(\alpha, 0, \gamma) = 1 $$
$\varPi (\alpha, 1, \tau) = \begin{cases} \tau & \text{if } \alpha > 1 \\ \tau + 1 & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha = 0 \end{cases}$

In which cases $\varPi$ corresponds to definition $\varOmega$ of definition 1.5C.

For $p = 2$, and each $\sigma$

$\varPi (\alpha, 2, \sigma) = \begin{cases} \varPi (P\alpha, \sigma, \sigma) & \text{if } \text{Suc} (\alpha) \\ \lim_{n \to \omega} \varPi (\varOmega_\alpha (n) + 1, 2, \sigma) & \text{if } \text{Lim} (\alpha) \end{cases}$

Thus, by putting $\text{ON}^\alpha (\{ \varPi (\alpha', \sigma, \sigma) \}_{\alpha' < \alpha})$

$$= \begin{cases} \varPi (P\alpha, \sigma, \sigma) & \text{if } \text{Suc} (\alpha) \\ \lim_{n \to \omega} \varPi (\varOmega_\alpha (n), \sigma, \sigma) & \text{if } \text{Lim} (\alpha) \end{cases}$$

$\varPi$ and $\varOmega$ again correspond in definition.

For $\alpha = 3$, $p > 2$, and each $\sigma$

$\varPi (3, p, \tau) = \begin{cases} \varPi (2, \tau, \varPi (3, p, \tau)) & \text{if } \text{Suc} (p) \\ \lim_{n \to \omega} \varPi (2, \varOmega_p (n), \sigma) & \text{if } \text{Lim} (p) \end{cases}$
Thus by putting \( ON^{(3)}(\{ III (\alpha, \beta, \gamma) \} \beta^\prime < \beta) \), \( n < 3 \)

\[
= \begin{cases} 
\text{} & \text{if } \text{Succ}(\beta) \\
ON^{(3)}(\{ III (\alpha, \beta, \gamma) \} \beta^\prime < \beta) & \text{if } \text{Limi}(\beta) 
\end{cases}
\]

and

\[
ON^{(3)}(\{ III (\alpha, \beta, \gamma) \} \beta^\prime < \beta) 
\]

\[
= \begin{cases} 
\text{III}(\alpha, \beta, \gamma) & \text{if } \text{Succ}(\beta) \\
\text{Limi } \text{III}(\alpha, \omega \beta(n), \gamma) & \text{if } \text{Limi}(\beta) \text{ and } n < \omega
\end{cases}
\]

Again \( \text{III} \) and \( \text{III}^\prime \) correspond in definition.

Finally, for \( \alpha > 3 \), \( \beta > 2 \) and each \( \gamma \)

\[
\text{III}(\alpha, \beta, \gamma) = \begin{cases} 
\text{III}(\alpha, \omega \beta^\prime + 1, \gamma) & \text{if } \text{Succ}(\alpha, \beta) \text{ and } x = 1 \\
\text{III}(\alpha, \omega \beta^\prime + x - 1, \gamma) & \text{if } \text{Succ}(\alpha, \beta) \text{ and } 1 < x \\
\text{Limi } \text{III}(\alpha, \omega \beta(n), \gamma) & \text{if } \text{Limi}(\beta) \text{ and } n < \omega \\
\text{Limi } \text{III}(\omega \beta(n), \text{III}(\gamma, \beta, \gamma) + 1, \gamma) & \text{if } \text{Limi}(\alpha) \text{ and } \text{Succ}(\beta)
\end{cases}
\]
where $p$ is expressed uniquely as $p = \omega p' + x$, $x \leq \omega$.

Thus, by putting \( ON(4) (\{ \overline{\Pi} (\alpha, p', \xi) \}_{p' < p} ) \)

\[
\begin{align*}
\overline{\Pi} (\alpha, p, \xi) & = \begin{cases} \\
\lim \overline{\Pi} (\alpha, \Omega_p (n), \xi) & \text{if } \lim (\alpha) \\
\Pi (\alpha, p, \xi) + 1 & \text{if } \lim (\alpha) \text{ and } \text{Suc}(\alpha) \\
\Pi (\alpha, \omega p' + x - 1, \xi) & \text{if } \text{Suc}(\alpha, p) \text{ and } 1 < x \\
\Pi (\alpha, \omega p', \xi) & \text{if } \text{Suc}(\alpha, p) \text{ and } x = 1
\end{cases}
\end{align*}
\]

and by putting \( ON(5) (\{ \overline{\Pi} (\alpha', \xi), ON(4) (\{ \overline{\Pi} (\alpha, p', \xi) \}_{p' < p} ) \}_{\omega < \alpha} ) \)

\[
\begin{align*}
\overline{\Pi} (\alpha, p, \xi) & = \begin{cases} \\
ON(4) (\{ \overline{\Pi} (\alpha, p', \xi) \}_{p' < p} ) & \text{if } \lim (\alpha) \\
\lim \overline{\Pi} (\alpha, \Omega_p (n), ON(4) (\{ \overline{\Pi} (\alpha, p', \xi) \}_{p' < p} ) \\
+ 1, \xi) & \text{if } \lim (\alpha) \text{ and } \text{Suc}(\alpha) \\
\Pi (\alpha, \Omega_p (n), ON(4) (\{ \overline{\Pi} (\alpha, p', \xi) \}_{p' < p} ) , \xi) & \text{if } \text{Suc}(\alpha, p)
\end{cases}
\end{align*}
\]

Again, \( \overline{\Pi} \) and \( \overline{\Pi} \) correspond in definition.

Thus it is proved that the function \( \overline{\Pi} \) is definition.
3.1 is an instance of a function defined from certain elementary functions \( \mu_i, i \leq 2 \), a constant \( r (=1) \), and operations \( \text{ON}_j \), \( 0 < j < 6 \), by schemata \( R \). Thus, by theorem 1.5D it follows that \( \mathcal{D} \) is well-defined.
THE FUNCTIONS \( [D^\alpha_n]_{n=0}^{\infty} = \Omega''\)

These functions were characterized in definition 3.4: they were obtained by a certain transformation from the operations \( \Pi^\alpha_n \) of definition 3.1.

The recursive definition of these functions is now given explicitly, with further constants upon the f.s.a. \( \Omega \). And this is followed by a brief investigation of the class \( \Omega'' \) upon the basis of two unproved hypotheses (1) concerning a property of \( \Omega \) and a consequent property of functions in the class. There follow some preliminary definitions.

THE FUNCTION \( M_\Omega \).

Suppose \( \text{Lim} (\alpha, \beta), \alpha, \beta < \omega_1, \) and \( \beta < \alpha \).

Then there is a least \( t \) such that \( \beta \in \Omega_\alpha (t) \).

So for a given f.s.a. \( \Omega \) it is possible to define a function \( M_\Omega : \omega^{\omega} \rightarrow \omega \) such that

\[
M_\Omega (\alpha, \beta) = 0 \text{ if } (7 \text{ Lim} (\alpha, \beta) \vee \alpha \leq \beta), \text{ and } \\
M_\Omega (\alpha, \beta) = (\mu y)_{\omega} \left[ (\forall x < \omega) y < x \Rightarrow \Omega_\beta (y) < \Omega_\alpha (x) \right] \text{ o.w.}
\]

(1) These hypotheses are discussed on p. 70.
DEFINITION 3.7
The fundamental sequence \( \{\alpha_n\}_{n=0}^{\infty} \) for the countable limit ordinal \( \alpha \) is \textit{perfect} if
\[
(\forall x, y < \omega) \quad D^2_{\alpha_n}(y) = D^2_{\alpha_{n+1}}(y)
\]

DEFINITION 3.7a
Let \( \Omega \) be an f.s.a. such that
(i) \( (\forall \alpha < \omega, \Omega_{\alpha}(\omega))_{\alpha<\omega} \) is a perfect fundamental sequence.
(ii) it is the f.s.a. \( \Omega \) upon which is based the definition of the \( D^\alpha \)
(iii) \( (\forall \alpha < \omega, \{\text{Lim} \, (\alpha) \Rightarrow \}) \)

\[
(\exists \beta < \alpha)[\alpha = \beta + \omega] \Rightarrow (\forall \eta < \omega) \quad \text{Lim} \, (\Omega_{\alpha}(\eta)) \text{ and }
\]
\[
(\exists \beta < \alpha)[\alpha = \beta + \omega] \Rightarrow (\forall \eta < \omega) \quad \Omega_{\alpha}(\eta) = \beta + \eta]
\]
Such an f.s.a. \( \Omega \) is called a \textit{perfect fundamental sequence assignment} (p.f.s.a.)
\( \square \) 3.7a.

There follows an explicit definition of the functions \( D^\alpha \), and an inductive proof of certain of their properties, that of theorem 3.8.
This contains parts a) (i), (ii), (iii), b) (i), (ii), (iii), and c). The transfinite induction is on the index \( \alpha \), and thus is in two major steps, for \( \text{Suc}(\alpha) \) and \( \text{Lim}(\alpha) \) respectively. The expression 1.1. b (ii), for example, denotes the transfinite induction hypothesis, as it applies to part b (ii) of theorem 3.8.

As each part of the theorem is proved, within each of the two steps, there is, in the cases where necessary, an inner inductive hypothesis upon the index \( n \).

**HYPOTHESIS 3.7 b.** There exists a p.f.s.a.

**DEFINITION 3.7 c.**

Let \( \Omega \) be a p.f.s.a.

\[
(\forall x, y < \omega) \quad D_0^\alpha(x) = 0 ; \quad D_1^\alpha(x) = x + y ;
\]

\[
D_2^\alpha(x) = x \cdot y
\]

and for every countable ordinal \( \alpha > 2 \) :

\[
D_0^\alpha(x) = 1
\]

\[
D_1^\alpha(x) = x
\]
\[
\begin{align*}
D_\alpha^2(x) &= D_{\Omega_\alpha}^2(x)(x) \\
D_\alpha^{y+1}(x) &= D_{\Omega_\alpha}^2 \{ D_\alpha^y(x) \}(x) \\
D_\alpha^2(x) &= D_{\Pi_\alpha}(x) \\
D_\alpha^{y+1}(x) &= D_{\Pi_\alpha}^{D_\alpha^y(x)}(x)
\end{align*}
\]

\text{THEOREM 3.8}

a) \( \forall \alpha < \omega_1 \) \( \forall n, m, x, y < \omega \)

\( \begin{align*}
(i) & \quad x < y \Rightarrow D_\alpha^y(x) < D_\alpha^y(y) \\
(ii) & \quad m < n \Rightarrow D_\alpha^m(x) < D_\alpha^n(x) \\
(iii) & \quad D_\alpha^n(x) < D_{\alpha+1}(x)
\end{align*} \)

b) \( \forall \alpha < \omega_1 \) \( \forall n, x, y < \omega \)

\( \begin{align*}
(i) & \quad - \\
(ii) & \quad \text{Succ}(\alpha) \Rightarrow D_{\Pi_\alpha}^{n+1}(x) < D_\alpha^n(x) \\
(iii) & \quad \text{Luni}(\alpha) \Rightarrow D_{\Omega_\alpha(\eta)}^{n+1}(x) < D_{\Omega_\alpha(\eta+1)}^n(x)
\end{align*} \)
3.8c) \((\forall x < \omega_1)(\forall \alpha) (\forall n, x < \omega) > 2 \max(n,x)^2 \leq D_3^n(x),\)

where in cases a) - c) for each \(n, \alpha, \) \(D_3^n \in D_{\omega \alpha}^{(n)},\)

such that \(\Omega \) is a p.f.s.a., and such that

\((\forall x < \omega_1)(\forall x) \Rightarrow \Omega_{\alpha}(\omega) > 0.\)

**Proof** by transfinite induction on \(\alpha.\)

Let \(\alpha = 3.\) Then \(D_3^n(x) = x^n.\) For \(n, x > 2,\)

\(x < y \Rightarrow x^n < y^n,\) and \(n < m \Rightarrow\)

\(x^m < x^m;\) and since \(x^2 < x^x,\) and for each \(n \geq 3,\)

\[x^n < \left(\frac{x}{x^{\ldots x}}\right) \Rightarrow x^{n+1} = x^\mu x < \left(\frac{x^{\ldots x}}{x^{\ldots x}}\right).\]

It follows \((\forall x < \omega) x^n < x.\)

With regard to part b(ii), to prove that \(x^\mu < x^{\ldots x},\)

it is sufficient to prove \(n+1 = D_4^{n+1}(x).\) Suppose

\(x \geq 2\) and \(n > 5.\) Let \(n = 6.\) Then

\[D_4^{n+1}(x) > \left(\frac{5}{2n+2}\right) > 7.\]
Suppose $D_{\alpha}^{n-1}(x) > n+1$, for some $n > 5$.
Then $D_{\alpha}^{n}(x) = x D_{\alpha}^{n-1}(x)$

\[ \geq 2 D_{\alpha}^{n-1}(x) > D_{\alpha}^{n-1}(x) + 2 \]

\[ > (n-1) + 2 > n+1 > n+2, \] therefore for all $n > 5$

3.8 a), b) hold for $\alpha = 3$. It is easy to show that 3.8 a), b) hold for $n = 3, 4, 5$.
3.8 c) follows in this case as $x^n > x+2$;
$a^n > n+2$, for $x, n > 2$.

Now suppose that for some $\alpha$ such that $3 < \alpha < \omega$, theorem 3.8 has been proved for all $\alpha < \alpha$.
Then if $\text{Succ}(\alpha)$, suppose $\alpha < \beta$, $\alpha, \beta > 2$.
For 3.8 a):

(a)(i) $D_{\alpha}^{2}(y+1) = D_{\alpha}^{2}(y+1)$ by 3.7 c

\[ > D_{\alpha}^{2}(y+1) \] by 1.H. a)(ii)

\[ > D_{\alpha}^{2}(y) \] by 1.H. a)(i)

\[ = D_{\alpha}^{2}(y) \]

Suppose for minor inductive hypothesis (i.e., h.)
that $D_{\alpha}^{n}(y+1) > D_{\alpha}^{n}(y)$.
Then $D_{\alpha}^{n+1}(y+1) = D_{\alpha}^{n+2}(y+1)$.
\[
\begin{align*}
> D_{\mathcal{B}_1}^{d_0(n)}(y+1) & \text{ by i. i. h. } \\
= D_{\mathcal{B}_1}^{d_0(n)}(y) & \text{ by i. H. a) (i). }
\end{align*}
\]
Therefore \((\forall n > 2) D_{\alpha}(y+1) > D_{\alpha}(y).
\]

a) (ii).
For every \(y\), \(D_{\alpha}^{y+1}(x) = D_{\mathcal{B}_1}^{y}(x) > D_{\alpha}^{y}(x)\) by i. H. c)

a) (iii).
\(D_{\alpha+1}(y) = D_{\alpha}^{y}(y) = D_{\mathcal{B}_1}^{d_0(n)}(y)
\]
Now \(D_{\alpha}^{y-1}(y) > D_{\mathcal{B}_1}^{d_0(n)}(y)\) by i. H. a) (iii)
\[
= y \text{ by i. H. c) }
\]
Therefore
\[
D_{\mathcal{B}_1}^{d_0(n)}(y) > D_{\mathcal{B}_1}^{y}(y) \text{ by i. H. a) (ii) }
\]
\[
= D_{\alpha}^{2}(y) .
\]
Suppose for minor inductive hypothesis that for some \(n > 2\) it has been demonstrated that for every \(n' < n\), that
\(D_{\alpha}^{n'}(y) > D_{\alpha}^{n'}(y)\). Then
\(D_{\alpha+1}(y) = D_{\mathcal{B}_1}^{d_0(n)}(y)(y)\) .
But \( D_{a+1}^{n-1}(y) > D_a^{n-1}(y) \) by \( i.i.d. \).

\( > D_{Pa}^{n-1}(y) \) by \( 1.H. a)(ii) \)

\( > n \) by \( 1.H. c) \)

Thus it follows that \( D_{a+1}^{n-1}(y) > n. \)

From what has been shown for case \( a)(ii) \), it thus follows that \( D_a^n(y) > D_a(y) \).

Now for \( 3.8\, b) \):

b \( (ii) \)

\( D_a^2(x) = D_{Pa}^2(x) > \maxc|x, x^*| + 2 \) by \( 1.H. c) \)
New suppose for some $n \geq 3$ that $(\forall k < n-1) \quad D^k_\alpha(x) \geq \text{max}(k, x) + 2$.

Then $D^{n-1}_\alpha(x) = D^{D^{n-2}_\alpha(x)}_{D^2_\alpha}(x)$ by 3.7c

$\geq D^n_{D^2_\alpha}(x)$ by i.i.h. and

$\geq \text{max}(n, x) + 2$ by 1.H. c)

Then since $D^n_\alpha(x) = D^{D^{n-1}_\alpha(x)}_{D^2_\alpha}(x)$, it follows

$\quad \geq D^{n+2}_{D^2_\alpha}(x)$ by 1.H. a)(ii)

$\quad \geq D^{n+1}_{D^2_\alpha}(x)$.

Therefore $(\forall n < \omega) \quad D^n_\alpha(x) \geq D^{n+1}_{D^2_\alpha}(x)$

For 3.8 c), it is required to show that $(\forall n, x < \omega) \geq \text{max}(n, x) + 2 < D^2_\alpha(x)$. This is proved by induction on $n$. $D^2_\alpha(x) = D^{D^2_\alpha(x)}_{D^2_\alpha}(x)$ by 3.7c

$\quad \geq \text{max}(x, x) + 2$ by 1.H. c)

$\quad \geq \text{max}(2, x) + 2$.

Now suppose for i.i.h. that the inequality has been proven $(\forall k < n)$, for some $n \geq 2$. Then
$D_\alpha^n(x) = D_{\alpha}^{D_\alpha^{n-1}(x)}(x)$ by 3.7c

$> D_{\alpha}^{\max(n-1,x)+2}(x)$ by i.i.d.

$> \max\{\max(n-1,x)+x\}+2$ by 1.H. c

$> \max(n,x)+2$. Thus the inequality holds for every $n < \omega$, $n \geq 2$, and this concludes the cases a) - c) of 3.8 for $\alpha$ a successor.

Now suppose $\alpha$ is a limit ordinal. Then for 3.8 a):

a) (i)

$D_\alpha^2(x+1) = D_{\omega_\alpha^2}(x+1)(x+1)$ by 3.7c

$> D_{\omega_\alpha^2}(x)(x+1)$ since $\omega_\alpha$ is a perfect f.s.a.

$> D_{\omega_\alpha^2}(x)(x)$ by 1.H. a)(i)

$= D_\alpha^2(x)$

Now suppose for minor inductive hypothesis that there is an $n \geq 2$ such that $(\forall k < n)(\forall x) D_\alpha^k(x) < D_\alpha^k(x+1)$. 
Then \( D_\alpha^2(x+1) = D_\alpha^2 \{ D_\alpha^{\alpha-1}(x+1) \} \), by 3.7(c)

\[ D_\alpha^2 \{ D_\alpha^{\alpha-1}(x+1) \}, \text{ by} \]

I.H. a) (i), since \( D_\alpha \{ D_\alpha^{\alpha-1}(x+1) \} \leq \alpha \)

\[ D_\alpha^2 \{ D_\alpha^{\alpha-1}(x) \}, \text{ since} \]

\( D_\alpha^{\alpha-1}(x) < D_\alpha^{\alpha-1}(x+1) \) by \text{i. i. l.}

and since \( \Omega \) is perfect

\[ = D_\alpha(x), \text{ therefore } (\forall \eta < \omega) \eta \geq 2 \]

\( D_\alpha^n(x) < D_\alpha(x+1) \).

a) (ii)

By definition of \( \Omega \), \( \Omega_\alpha(0) > 0 \), therefore

\( D_\alpha^2(0)(x) > 0 \), by I.H. c). Furthermore, since

\( \Omega \) is perfect, it follows that \( (\forall \eta < \omega) \)

\( D_\alpha^2(\eta)(x) < D_\alpha^2(\eta+1)(x) \). Therefore \( (\forall \eta < \omega) \)

\( D_\alpha^{\eta+1}(x) > n. \)

Now \( D_\alpha^{\eta+1}(x) = D_\alpha^2 \{ D_\alpha^\eta (x) \} \),

\[ > D_\alpha^2(x) \text{ by above argument.} \]
\( \alpha^{(iii)} \)

\[
D_{\alpha+1}^2(y) = D_{\alpha}^2(y) \\
= D_{\alpha-1}^2 D_{\alpha}^{y-1}(y) / (y) \\
> D_{\alpha}^{y-1}(y)
\]

Repetition of this step proves \( D_{\alpha+1}^2(y) > D_{\alpha}^{y-2}(y) \); and finite iteration of this step proves \( D_{\alpha+1}^2(y) > D_{\alpha}^{y}(y) \).

Suppose for i.i.h. that for some \( n > 2 \) it has been demonstrated, for every \( n' \leq n \), that \( D_{\alpha}^{n'}(y) > D_{\alpha}^{n'}(y) \). Then

\[
D_{\alpha+1}^n(y) = D_{\alpha}^{D_{\alpha+1}^y}(y) \quad \text{by 3.7c. Now}
\]

\( D_{\alpha}^{n'}(y) > D_{\alpha}^{n'}(y) \) by i.i.h., and so by the proof already given that \( (\forall x, x') \ D_{\alpha}^{x+y} D_{\alpha}^{x'}(x) > D_{\alpha}^{x'}(x) \), it follows \( D_{\alpha+1}^n(y) > D_{\alpha}^{D_{\alpha+1}^y}(y) \), therefore \( D_{\alpha}^{n-1}(y) > n+1 > n \), since \( D_{\alpha}^{n}(y) = D_{\alpha}^{D_{\alpha}^y}(y) > \max(2, y) + 2 \) by i.i.h. c).

Thus \( D_{\alpha+1}^n(y) = D_{\alpha}^y(y) \), and so it is proved by induction that \( (\forall n, y < \omega) \ D_{\alpha+1}^n(y) > D_{\alpha}^n(y) \).
For 3.8 b):

b)(iii). This case constitutes the 2-nd hypothesis of this section.

**HYPOTHESIS 3.8 b) (iii).**

\[(\forall n, x, y < \omega) \exists \epsilon_x > \epsilon_y \text{ s.t. } D_{\omega}^{\epsilon_x}(y) < D_{\omega}^{\epsilon_y}(x)\]

This seems likely for \(\Omega\) projects, but difficult to prove without knowing more about \(\Omega\).

**EXAMPLE.** Let \(\alpha = \omega, y = 0, n = 2\), \([\Omega, \omega]_{\omega} = [\omega + 1, \omega]_{\omega}\). Then \(D_{\omega}^{3}(n) = D_{\omega}^{2}(x)\), the 3-nd power of the Ackermann function applied to \(x\). Since it can readily be shown by induction that for \([\Omega, \omega]_{\omega} = [\omega + 1, \omega]_{\omega}\), \((\forall x) x > 3 \text{ in } D_{\omega}^{n}(x)\) is strictly increasing, and since \(x > 3\), it follows:

\[D_{\omega}^{3}(n) = D_{\omega}^{2}(y+1)(x)\]

b)(ii). In this case there is nothing to prove, \(\alpha = 3 \text{ in } (x)\).

For 3.8 c): this also is proved by induction.

\[D_{\omega}^{2}(x) = D_{\omega}^{3}(x)\]

\[> \text{ max } (2, x) + 2 \text{ by 1.4. c)}\]

Suppose the inequality of theorem 3.8 c) has been proved to hold for each number \(n'\) smaller than some \(n > 2\), so that \(n' > 2\). Then:

\[D_{\omega}^{n}(x) = D_{\omega}^{2}[D_{\omega}^{n-1}(x)](x)\]

\[> D_{\omega}^{2}[\text{ max } (n-1, x) + 2](x) \text{ by 2. 3. 4.} \]
and by $\Omega$ being a p.f.s. Also since $\Omega$ is perfect and
\[ D_{\Omega}(x) > 1, \quad (\forall k) \quad D_{\Omega}(k)(x) > k + 1 \]
therefore
\[ D_\Omega(x) = \left\lfloor \max\left(n - 1, x\right) + 2 \right\rfloor + 1 \]
\[ \geq \max(n, x) + 2 \]
Thus it follows by transfinite induction that
\( (\forall x < \omega, 1) d > 2 \), the inequalities of theorem 3.8
are proved to hold. \( \square \)

**MAJORIZATION THEOREMS.** From theorem 3.8, several majorization properties of the $D_\alpha^n$ can be deduced.

**DEFINITION** \( (\forall f, g : N \to N) \) \( f \) **majorizes** \( g \) iff
\[ (\exists n)(\forall x) \left[ y < x \Rightarrow g(y) < f(x) \right] \] The relation is written \( g \leq_m f \).

**COROLLARY 3.9** \( (\forall n < \omega, 1) (\forall x) D_\alpha^n(x) = 0 \)
\( \forall x \cdot D_\alpha^n(x) \leq_m \forall x \cdot D_\alpha^{n+1}(x) \)
\( \leq_m \forall x \cdot D_\alpha^\omega(x) \)

**PROOF** follows from theorem 3.8 a(iii) and b(ii).

**THEOREM 3.10** \( (\forall n < \omega, 1) (\forall x < \omega) (\forall \beta < \omega, 1) \)
\( \forall x \cdot D_\beta^n(x) \leq_m \forall x \cdot D_\alpha^n(x) \)

**PROOF** by transfinite induction on \( \alpha \).
Let $\alpha$ be any finite ordinal $> 2$. Then 3.10 is a standard result for primitive recursive functions. Now suppose for induction hypothesis that 3.10 has been proved true for all ordinals $\alpha' < \alpha$, for some countable $\alpha > \omega$.

Then if $\text{Suc}(\alpha), (\forall n \geq 2)(\forall x) \exists x, D_{\omega}^{n+1}(x) = D_{\omega}^{n}(x)$ by 3.7 c

$> D_{\omega}^{n+1}(x) = D_{\omega}^{n+1}(x)$ by 3.8 c

$> D_{\omega}^{n+1}(x)$ by 3.8 a) (ii)

Now suppose $\text{Lim}(\alpha)$. This case is proved by induction on the power $n$. For $n = 2$,

$D_{\omega}^{2}(x) = D_{\omega}^{2}(x)$ by 3.7 c. Now suppose $p < \alpha, p > 0$. Then let $y_{p}$ be the smallest integer such that $p < \omega_{p}(y_{p})$. Then by I.H.

$\forall x. D_{\omega}^{2}(x) \leq \wedge x. D_{\omega}^{2}(y_{p})(x)$.

But as $\omega_{p}$ is a p.f.s. a., $(\forall x < \omega) x > y_{p} \Rightarrow D_{\omega}^{2}(y_{p})(x) < D_{\omega}^{2}(x)(\alpha) = D_{\omega}^{2}(x)$

Therefore $\forall x. D_{\omega}^{2}(y_{p})(x) \leq m \forall x. D_{\omega}^{2}(x)$,
and so \( \lambda x . D_p^2(x) \leq m \Rightarrow \alpha . D_\alpha^2(x) \), for each \( \alpha < \alpha \). Let \( n \geq 2 \), and suppose for i.i.h. that the case has been proved (for \( m \leq n \)), and let \( p \) be any ordinal \( \leq \alpha \).

Let \( \alpha \) be the smallest limit ordinal such that \( \alpha \leq \alpha < \alpha \), if there is such an ordinal.

Then by i.i.h. \((\exists y)(\forall x)\ x > y \Rightarrow D_{p}^{n+1}(x) \leq D_{p}^{n+1}(y)\)

Now let \( x = \max(y, z(1), z(2), z(3)) \), where

\[
z(1) = \inf_{t} (\forall t) \in \omega \left[ D_{p}^{n}(t) \leq D_{\alpha}^{n}(t) \right] \quad \text{(there is such a } t \text{ by i.i.h.)}
\]

\[
z(2) = \inf_{t} (\forall t) \in \omega \left[ p \leq \omega \left( D_{p}^{n}(t) \right) \right]
\]

\[
z(3) = \inf_{t} (\forall t) \in \omega \left[ \forall \gamma \ t < \gamma \Rightarrow D_{p}^{n+1}(x) \leq D_{\alpha}^{n+1}(x) \right]
\]

\[
D_{p}^{n+1}(x) = D_{p}^{n+1}(x) \leq D_{p}^{n+1}(x) \text{ by } 3.7 \text{c}
\]

\[
D_{p}^{n+1}(x) < D_{p}^{n+1}(x) \text{, since } D_{p}^{n+1}(x) < D_{\alpha}^{n+1}(x) \text{ by lower bound on } \alpha \text{, and by i.i.h. } \alpha \text{ and by } \Omega \text{ being a } p.t.s.a. \text{. Therefore}
\]

\[
D_{p}^{n+1}(x) < D_{p}^{n+1}(x) \text{, since } x \text{ is such that}
\]
$M_\alpha(\alpha, \beta^*) < D^\alpha_\beta(x)$, and by l. b. on $x$, and by 1. H. Thus $D^{\alpha+1}_\beta(x) < D^{\alpha+1}_\alpha(x)$ by 3.7c.

Alternatively, if each $\beta^*$ such that $\beta \leq \beta^* < \alpha$ is a successor ordinal, then $\alpha$ is of the form $\alpha = \alpha' + \omega$, and $\beta$ of the form $\beta = \alpha' + k$, some $k \leq \omega$, $k > 0$. Thus

$D^{\alpha+1}_\alpha(x) = D^{\alpha+1}_{\alpha + \omega}(x)$

$= D^{\alpha + \omega}_{\alpha + \omega}(x)(x)$, by 3.7c, and

where it is assumed that $(\forall k \leq \omega) \quad \Omega_{\alpha}(k) = \alpha' + k$

$> D^{\alpha + k - 1}_{\alpha + \omega} + D^{\alpha + k}_{\alpha + \omega}(x)(x)$ by lower bound

on $x$, and 3.8 a)(iii), where $x$

satisfies $x \geq \max(2^{(1)}, 2^{(2)})$, for

$Z^{(1)} = \{ (\mu, t) < \omega \mid (\forall r)(r > t \Rightarrow (k-1) + D^{\alpha + k}_{\alpha + \omega}(r) < D^{\alpha + k}_{\alpha + \omega}(r)) \}$

(such a $t$ can be found by iterating $\nu + \nu$.

$k$ times for ordinals $\nu = \alpha' + k, \alpha' + k + 1, \ldots, \alpha' + k + (k-1)$)

$Z^{(2)} = \{ (\mu, t) < \omega \mid (\forall r)(r > t \Rightarrow D^{\alpha + k - 1}_{\alpha + \omega} + D^{\alpha + k}_{\alpha + \omega}(z^{(2)})(r))$ $< D^{\alpha + k + 1}_{\alpha + \omega}(z^{(2)})(r)) \}$

(by 1. H., there exists such a $t$)

Thus

$D^{\alpha+1}_\alpha(x) > D^{\alpha + k - 1}_{\alpha + \omega}(x)(x)$ by 3.8 b)(iii)
> \[ D_{\alpha+k-1}^{\beta+n}(x) \] by 3.8 a)(ii)

= \[ D_{\alpha+k}^{n+1}(x) \] by 3.7 c

= \[ D_{\beta}^{n+1}(x) \]

Hence it follows by transfinite induction that the theorem is proved. \[ \square \] 3.10

**INVERSES.** Integer-valued inverses can readily be defined for the operations \( D_\alpha \):

**DEFINITION 3.11** \((\forall \alpha < \omega, \forall n, x < \omega) x, n > 0\)

\[
rt_{\alpha n}(x) = (\max y) [D_{\alpha}^n(y) \leq x]
\]

\[
log_{\alpha n}(x) = (\max y) [D_{\alpha}^y(n) \leq x].
\]

Thus, for example, \(rt_{3n}(x) = [\sqrt[n]{x}]\);

\(log_{3n}(x) = [\log_{n}(x)]\). Also, \(rt_{2,3n}(x)\)

= \((\max y) y \cdot n \leq x\)

= \([\frac{\alpha}{n}]\)

= \((\max y) n \cdot y \leq x\)

= \(log_{2,3n}(x)\).

In these examples, square bracket expressions denote the integral part of the value denoted by their contents.
EVIDENCE IN SUPPORT OF THE HYPOTHESES

In the course of his construction of a set of points, belonging to the continuum of cardinal number $\aleph_1$, G. H. Hardy remarks:

In the case of the comparatively early members of the second (number) class, it is generally evident that any which has no predecessor is most naturally regarded as the limit of one particular sequence $p_1, p_2, \ldots$. Thus it is natural to regard $\omega^0$ as the limit of

$$1, \omega, \omega^2, \omega^3, \ldots$$

$\omega^0$ as the limit of

$$\omega, \omega^0, \omega^2, \omega^3, \ldots$$

and $\varepsilon_0(\cdot)$, the first of Cantor $\varepsilon$-numbers, as the limit of

$$\omega, \omega^0, \omega^2, \omega^3, \ldots$$

S. S. Wainer [21] has presented a class of functions $F_a^n : \omega \to \omega$, $n \leq \omega$, $a \leq \omega$, defined by iteration and diagonalization at successor and limit ordinals respectively. For a choice of fundamental sequence assignment $\alpha, \eta, n, \eta \delta_3(n)$ of the kind envisaged by Hardy, restricted to limit ordinals $\alpha < \varepsilon_0$, Wainer has shown that $(\forall x < \omega) \quad F_{\omega \delta_3(\eta)}^\alpha(x+1) = F_{\omega \delta_1(\eta)(x+1)}^\alpha(x+1)$, which gives support to hypothesis 3.7.6.

With regard to hypotheses 3.8.6(iii), let $\alpha = \omega$. Then

$$\forall y, n, x. \quad D^n_{\omega, \omega}(y) (x) = \forall y, n, x. \quad D^n_x (x),$$

and this is the Ackermann function, which has the property of hypothesis 3.8.6(ii).

(1) In his paper, Hardy uses $\varepsilon_1$ to denote this number.
4. SOME PROPERTIES OF FUNCTIONS IN THE CLASS $\mathcal{D}_n^{(2)}$.

Some normal functions in the class $\mathcal{D}_{\omega}^{(2)}$ are singled out and examined, and then the notion of leading term is developed.

Not every function in the class $\mathcal{D}_{\omega}^{(2)}$ is strictly increasing (s.i.); it can readily be shown that the statement $(\forall p, s < \omega) p > 1, s \geq \omega \Rightarrow \exists \alpha \in \mathcal{D}_{\omega}^{(2)}$ is s.i. is false, by considering the following counter example: For each positive $n < \omega$ let $\lambda_n$ be the principal number of multiplication expressed by $\underbrace{\omega \cdots \omega}_{n}$. Then

$$\mathcal{D}_{\omega}^{2n}(\omega) = \lim_{n \to \omega} \mathcal{D}_{\omega}^{2n}(\omega) = \lim_{n \to \omega} \omega^{\lambda_n}$$

$$= \lim_{n \to \omega} \lambda_n = \varepsilon_0,$$

and

$$\mathcal{D}_{\omega}^{3n}(\omega) = \lim_{n \to \omega} \mathcal{D}_{\omega}^{3n}(\omega) = \lim_{n \to \omega} \omega^{\lambda_n}$$

$$= \lim_{n \to \omega} \lambda_n = \varepsilon_0.$$

Thus $f = \lambda\alpha$. $\mathcal{D}_{\omega}^{\varepsilon_0}(\omega)$ is not s.i., as $f(2) = f(3)$.

This property does not depend on any arbitrary...
features of definition 3.1, since only the standard operations of multiplication and exponentiation are used. And it is clear that similar equal functional values can be obtained by choosing an exponent suitably larger than $\varepsilon_0$. On the other hand, it is shown below that $(\forall \gamma, \beta < \omega_1). \beta > 1, \gamma \geq \omega \Rightarrow \gamma \beta. \mathcal{F}_\alpha^{\beta}(\gamma)$ is ultimately increasing.

**THEOREM 4.1** $(\forall \alpha, \beta < \omega_1). \alpha > 0, \gamma \geq \omega \Rightarrow \gamma \beta. \mathcal{F}_\alpha^{\beta}(\gamma) = \gamma \beta. \mathcal{F}_\alpha^{\beta}(\gamma) \text{ is strictly increasing}.$

**PROOF.** This is by nested transfinite induction on the indices $\alpha, \beta$. By definition 3.1, for $\alpha = 1$

$\gamma \beta. \mathcal{F}_1^{\beta}(\gamma) = \gamma \beta. \gamma + \beta$, which is strictly increasing.

For $\alpha = 2, 3$

$\gamma \beta. \mathcal{F}_2^{\beta}(\gamma) = \gamma \beta. \gamma \beta$

$\gamma \beta. \mathcal{F}_3^{\beta}(\gamma) = \gamma \beta. \gamma \beta$, both of which are strictly increasing.

For induction hypothesis (I.H.) suppose $(\forall \alpha < \omega_1). \alpha > 0, \gamma \beta. \mathcal{F}_\alpha^{\beta}(\gamma)$ has been proved to be strictly increasing for some index $\alpha > 3$. 


The proof is now in two steps. The initial step deals with the cases where \( p \) is finite in value, or is equal to \( \omega \), for a successor or limit number. The second step deals with the remaining cases.

Suppose \( \alpha \leq \beta \). Then

\[ \Omega^0_\alpha (\beta) = 1 < \gamma = \Omega^1_\alpha (\beta) \]

\[ = \Omega^1_{\omega\alpha} (\beta) \]

\[ < \Omega^2_{\omega\alpha} (\beta) \]

\[ = \Omega^2_\alpha (\beta) \], by definition

3.1, and by inductive hypothesis. Also

\[ \Omega^3_\alpha (\beta) = \Omega^{\Omega^2_\alpha (\beta)}_{\omega\alpha} (\beta) \]

\[ > \Omega^2_{\omega\alpha} (\beta) \] by I.H., and by theorem

4.1a, 3a below. And if

\[ \Omega^{\alpha+1}_\alpha (\beta) > \Omega^\alpha_\alpha (\beta) \], then by definition 3.1 and

by I.H.

\[ \Omega^{\alpha+2}_\alpha (\beta) = \Omega^{\Omega^{\alpha+1}_\alpha (\beta)}_{\omega\alpha} (\beta) > \Omega^{\Omega^\alpha_\alpha (\beta)}_{\omega\alpha} (\beta) \]

\[ = \Omega^{\alpha+1}_\alpha (\beta) \], for

\( \alpha \geq 2 \), so \( \forall x < \omega \) \( \Omega^{x+1}_\alpha (\beta) > \Omega^x_\alpha (\beta) \).
Suppose $\lim (\alpha)$. Then for $p = 0, 1$:

$$\Xi^0_\alpha (\delta) = 1 < \delta = \Xi^1_\alpha (\delta)$$

$$\leq \lim_{n \to \omega} \Xi^2_{\omega \alpha} (\delta)$$

$$\leq \lim_{n \to \omega} \Xi^2_{\omega \alpha} (\delta) \text{ by 1. H.}$$

$$= \Xi^2_\alpha (\delta), \text{ by definition 3.1.}$$

Now suppose $p = \alpha$ such that $2 \leq \alpha < \omega$. Then

$$\Xi^{\alpha+1}_\alpha (\delta) = \lim_{n \to \omega} \Xi^2_{\omega \alpha} (\delta) + 1 \text{ by definition 3.1}$$

$$\geq \Xi^2_\alpha (\delta) + 1 \text{ by 1. H.}$$

$$> \Xi^2_\alpha (\delta).$$

Thus it is proved that for $\alpha$ a successor or a limit number, $(\forall p < \omega)$

$$\Xi^{\alpha+1}_\alpha (\delta) > \Xi^p_\alpha (\delta).$$

And since $\Xi^\omega_\alpha (\delta) = \lim_{n \to \omega} \Xi^n_\alpha (\delta)$, it thus follows that $(\forall p < \omega)$

$$\Xi^p_\alpha (\delta) < \Xi^\omega_\alpha (\delta).$$

This completes the initial step. Proceeding now to the second step, for inner inductive hypothesis (i.i.d.)
Suppose that for some \( p \) such that \( \omega < p < \omega^2 \),

\( (\forall \gamma < \omega) \) \( \gamma \notin \beta \). \( \mathfrak{D}^2_\alpha (\gamma) \upharpoonrightarrow p \) is s.t.

In the first instance, suppose \( \text{Succ} (\alpha, p) \) and \( x = 1 \),

where \( p \) is expressed in the unique form \( p = p' + x, 0 < p', \)

\( p' < p \), \( x < \omega \). Then by definition 3.1,

\[
\mathfrak{D}^2_\alpha (\gamma) = \mathfrak{D}^2_{\beta_x} \mathfrak{D}^2_{\alpha} (\gamma) + 1 (\gamma)
\]

\[
= \mathfrak{D}^2_{\beta_x} \mathfrak{D}^2_{\alpha} (\gamma) + 1 (\gamma) \text{ by definition of } p'
\]

\[
\geq \mathfrak{D}^2_{\alpha} (\gamma) + 1 \text{ by } 1.H.
\]

\[
> \mathfrak{D}^2_{\alpha} (\gamma)
\]

Thus \( \gamma \notin \beta \). \( \mathfrak{D}^2_\alpha (\gamma) \) is s.t. at \( \gamma = p \) for \( \text{Succ} (\alpha, p) \)

& \( x = 1 \).

In the second instance, suppose \( \text{Succ} (\alpha, p) \) & \( 1 < x \),

and in this instance, cases A, B are distinguished.

In case A, suppose \( x = 2 \). Then by definition 3.1:

\[
\mathfrak{D}^2_\alpha (\gamma) = \mathfrak{D}^2_{\beta_x} \mathfrak{D}^2_{\alpha} (\gamma)
\]

Now \( \mathfrak{D}^2_{\alpha} (\gamma) = \mathfrak{D}^2_{\beta_x} \mathfrak{D}^2_{\alpha} (\gamma) + 1 (\gamma)
\]

\[
> \mathfrak{D}^2_{\alpha} (\gamma) + 1 \text{ by } 1.H.
\]

But \( \min (l.h.s.) \) and \( \text{Succ} (r.h.s.) \) by corollary 4.1a.4
below, therefore \( D_{\alpha}^{\alpha + x - 1}(\delta) > D_{\alpha}^{\alpha + x}(\delta) + 1 \)

therefore \( D_{\alpha}^{\alpha}(\delta) > D_{\alpha}^{\alpha + x}(\delta) + 1(\delta) \) since \( \lambda \delta, D_{\alpha}^{\alpha}(\delta) \)

is s.i. by 1.H.

\[ = D_{\alpha}^{\alpha + x}(\delta) \]

In case B, suppose \( \alpha \geq 2 \). Then

\[ D_{\alpha}^{\alpha}(\delta) = D_{\alpha}^{\alpha + x - 1}(\delta) \]

Next \( D_{\alpha}^{\alpha + x - 1}(\delta) > D_{\alpha}^{\alpha + x - 2}(\delta) \) by i.i.h.

Also \( \lambda \delta, D_{\alpha}^{\alpha}(\delta) \) is s.i. by 1.H., therefore

\[ D_{\alpha}^{\alpha + x - 1}(\delta)(\delta) > D_{\alpha}^{\alpha + x - 2}(\delta)(\delta) \]

\[ = D_{\alpha}^{\alpha + x}(\delta) \]

\[ = D_{\alpha}^{\alpha}(\delta) \]

Thus \( \lambda \delta, D_{\alpha}^{\alpha}(\delta) \) is strictly increasing at \( \delta = \beta \)

for some \((\alpha, \beta)\) and \( \alpha > 1 \).

In the third instance which arises in the proof of theorem 4.1, suppose \( \text{Lim} (\beta) \). Then by definition 3.1:

\[ D_{\alpha}^{\alpha}(\delta) = \lim_{n < \omega} D_{\alpha}^{\alpha + x}(\delta), \quad \text{since } S \text{ is an f.s.a.} \]

\[ (\forall \beta' < \mu)(\exists n < \omega) \beta' < \mu n (\delta), \quad \text{Therefore} \]

\[ \]
\((\forall \beta < \gamma)(\exists \eta < \omega) \, \omega^\alpha p^\beta(\eta) < \omega^\alpha p^{\eta}(\eta)\) by i.i.h.

But \((\forall \eta < \omega) \, \omega^\alpha p^{\eta}(\eta) < \omega^\alpha p^{\eta+1}(\eta)\) since \(\omega\) is an f.s.a., and by i.i.h.

therefore \((\forall \eta < \omega) \, \omega^\alpha p^{\eta}(\eta) < \omega^\alpha p(\eta)\).

therefore \((\forall \beta < \gamma) \, \omega^\alpha p(\beta) < \omega^\alpha p(\eta)\); that is,

the function denoted by \(\chi\) : \(\omega^\alpha p(\beta)\) is strictly increasing at \(\beta\).

In the fourth and final instance, suppose \(\lim\ \omega^\alpha(\gamma)\) and \(\text{Succ}\ (\beta)\). Then by definition 3.1,

\[
\omega^\alpha p(\gamma) = \lim_{n<\omega} \omega^\alpha p^{\omega^\alpha p(\beta)+1}(\beta)
\]

\[
\geq \omega^\alpha p^{\omega^\alpha p(\beta)+1}(\beta) \text{ for every } n < \omega
\]

\[
> \omega^\alpha p^{\omega^\alpha p(\beta)}(\beta),
\]

therefore \(\omega^\alpha p(\beta)\) is s.i. at \(\beta\).

Thus it follows by transfinite induction that \((\forall \beta < \omega)\) : \(\gamma^\alpha p(\beta) > 1\); \(\omega^\alpha p(\beta)\) \(\beta\) in s.i., therefore
for every \( \alpha \) belonging to the second number class, \( \forall \beta . \, \Pi_2^B(\alpha) \in S \). 

Thus it follows by nested transfinite induction that 
\[
(\forall \alpha, \beta < \omega_1)_{\alpha, \beta \geq \omega} \implies \forall \beta . \, \Pi_2^B(\alpha) \in S. \]

Hence follow results proving that in 'most' cases, the expression \( \Pi_2^B(\alpha) \) evaluates to a limit ordinal.

**THEOREM 4.1.**2 (\( \forall \alpha, \beta, \delta < \omega_1 \)) \( \alpha \geq \omega \implies \text{Lim}(\delta) \implies \text{Lim}(\{\Pi_2^B(\delta)\}) \).

**PROOF by nested transfinite induction.**

For \( \alpha = 2, \beta > 0 \)

\[
\Pi_2^B(\alpha) = \alpha \cdot \beta = \begin{cases} 
\delta \cdot \beta + \delta \cdot \text{Succ}(\beta) & \text{if } \delta \cdot \beta \in \text{Lim}(\beta), \\
\lim_{n \to \omega} n \cdot \beta & \text{if } \delta \cdot \beta \notin \text{Lim}(\beta), 
\end{cases}
\]

Therefore \( \text{Lim}(\Pi_2^B(\alpha)) \).

Suppose for inductive hypothesis that the theorem has been proved \( (\forall \alpha' < \alpha)_{\alpha' \geq 1} (\forall \beta, \delta < \omega_1)_{\beta > 0} \)

for some \( \alpha > 2 \). Then for \( \beta = 2 \), \( \Pi_2^B(\alpha) = \Pi_2^B(\delta) \) if \( \text{Succ}(\alpha) \) or \( \lim_{n \to \omega} \Pi_2^B(\alpha(\alpha \cdot \beta))(\delta) \)

if \( \text{Lim}(\alpha) \), therefore in either case by I.H.,

\( \text{Lim}(\Pi_2^B(\alpha)) \).
Now suppose for i.e.h. that for some $p > 2$, it has been proved that $(\forall \delta < \omega_1)(\forall \gamma < p) x > 0$.

$L \subseteq (x) \Rightarrow \lim \{ \mathcal{D}_a^p(x) \}$. Then if $\text{Sue}(x, p)$, by definition 3.1

$$\mathcal{D}_a^p(x) = \mathcal{D}_a^{\mathcal{D}_a^p(x)}(x) \text{ or } \mathcal{D}_a^{\mathcal{D}_a^p(x)+1}(x)$$

or $\mathcal{D}_a^p(x) = x.$

Thus in every case the i.e.h. gives $L \subseteq \lim \{ \mathcal{D}_a^p(x) \}$.

If $L \subseteq (x)$ and $\text{Sue}(p)$:

$$\mathcal{D}_a^p(x) = \lim_{n < \omega} \mathcal{D}_a^{\mathcal{D}_a^p(x)+1}(x), \text{ and thus the i.e.h. gives } L \subseteq \lim \{ \mathcal{D}_a^p(x) \}.$$

If $L \subseteq (p)$:

$$\mathcal{D}_a^p(x) = \lim_{n < \omega} \mathcal{D}_a^{\mathcal{D}_a^p(x)}(x), \text{ and thus the i.e.h. gives } L \subseteq \lim \{ \mathcal{D}_a^p(x) \}. \text{ Thus it follows by transfinite induction that } (\forall \beta \delta < \omega_1) p x > 0.$$

$L \subseteq (x) \Rightarrow \lim \{ \mathcal{D}_a^p(x) \}$, therefore by restricted transfinite induction theorem 4.1a.2 is proved. □

THEOREM 4.1a.3. $(\forall \delta p \delta < \omega_1 \alpha > 2, \beta, \delta > \omega) \text{Sue}(x) \Rightarrow \lim \{ \mathcal{D}_a^p(x) \}$.

PROOF. Theorem 4.1a.3 is a special case of theorem 4.1a.3a which follows. □
THEOREM 9.13a

\((\forall \alpha, \beta, \gamma < \omega, \delta \geq \omega, \omega \geq 2) \lim (\Pi^\alpha_\beta(\gamma))\)

PROOF, by nested transfinite induction.

In addition to considering successor and limit cases for \(\alpha, \beta, \gamma\), there are separate cases for each of \(\omega \geq 2, \gamma < \omega, \omega \leq \delta = \omega, \omega = \omega\).

Furthermore, the cases \(3 \leq \alpha < \omega, \alpha = \omega, \omega < \omega, \omega < \omega\) each have to be distinguished from the case \(\omega = \omega < \omega\). Thus the proof of this theorem is complicated by a multiplicity of cases.

The proof is given in three steps: the primary proof deals with finite values of indices \(\alpha, \beta, \gamma\). The initial proof, which includes the primary proof, deals with all cases such that \(\alpha\) lies in the range \(3 \leq \alpha \leq \omega\).

In the initial proof, suppose that \(\alpha\) is such that \(3 \leq \alpha \leq \omega\).

Suppose firstly that \(\alpha\) is such that \(3 \leq \alpha < \omega\).

Suppose \(\gamma\) is finite, with \(2 \leq \gamma\).

Suppose \(\beta = \omega\).

Then \(\Pi^\omega_\beta(\gamma) = \lim_{n<\omega} \Pi^\omega_\beta(\gamma)

Now \((\forall \alpha, \beta, \gamma < \omega, \delta \geq \omega, \omega \geq 3, n \geq 3, \delta \geq 2)\)
\( n, m, s \cdot \mathcal{A}_2(8) = \omega, n, s \cdot g(n, s) \), the Ackermann function.

In the stated ranges of values of each of it's arguments, this number - theoretic function is strictly increasing. Therefore \((\forall \alpha, \gamma < \omega) \alpha \geq 3, \gamma \geq 2\)

\[ \mathcal{A}_\alpha^\omega(\gamma) = \omega, \text{ i.e. } \lim \left( \mathcal{A}_\alpha^\omega(\gamma) \right) . \]

This is called the primary proof.

Suppose now \( \beta \) is such that \( \omega = \beta < \omega \).

Then \( \lim \left( \mathcal{A}_\beta(\gamma) \right) \) or \( \lim \left( \mathcal{A}_\beta^\omega(\gamma) \right) \), because suppose \( \delta = x_\delta < \omega, \gamma \geq 2 \). Then if \( \beta \) is such that \( \omega < \beta < \omega_1 \), and \( \beta = \omega \beta' + x_\beta \), then

\[ \delta' = x_\delta^{\omega \beta'} \cdot x_\delta^{x_\beta} \], which is a limit number.

Suppose for 0.1. H. that for some \( \alpha \) such that \( 3 \leq \alpha < \omega \) (\( \forall \beta < \omega \)), \( \forall \alpha, \gamma < \omega, \gamma \geq 2 \lim \left( \mathcal{A}_\alpha^\beta(\gamma) \right) \).

Suppose for i. i. h. that for some \( \beta < \omega \), such that \( \beta \geq \omega \) (\( \forall \beta' < \beta \)) \( \lim \left( \mathcal{A}_\beta^{\beta'}(\gamma) \right) \).

Then if \( \text{Suc}(\beta) \) and \( \text{Lim}(\beta \beta') \):

\[ \mathcal{A}_\alpha^{\beta}(\gamma) = \mathcal{A}_\alpha^{\beta_1+1}(\gamma)+1(\gamma) \text{ by definition 3.1} \]

\[ = \mathcal{A}_\alpha^{\beta_1+1}(\gamma) \text{ such that } \text{Lim}(\gamma) \]

by i. i. h., therefore \( \lim \left( \mathcal{A}_\alpha^{\beta}(\gamma) \right) \) by 0.1. H.
If \( \text{Suc}(p, p) \):

\[
\mathbb{D}^{p}_{\alpha + 1}(\xi) = \mathbb{D}^p_\alpha(\xi) \quad \text{by definition 3.1}
\]

\[
= \mathbb{D}_\alpha(\xi) \quad \text{such that } \lim(\alpha) \text{ by } \text{i.i.h.}
\]

therefore \( \text{LIM}(\mathbb{D}^p_{\alpha + 1}(\xi)) \) by 0.1.H.

If \( \text{Limi}(p) \),

\[
\mathbb{D}^{p}_{\alpha + 1}(\xi) = \text{lim}_n \mathbb{D}^{\omega p}_n(\xi) \quad \text{by definition 3.1}
\]

= the limit of an increasing sequence of finite numbers, or a limit of finite numbers and limit numbers by i.i.h., therefore \( \text{LIM}(\mathbb{D}^p_{\alpha + 1}(\xi)) \).

Therefore \( \forall p < \omega \exists \omega \quad \text{LIM}(\mathbb{D}^p_{\alpha + 1}(\xi)) \) by transfinite induction and by strong proof.

Therefore \( \forall \alpha < \omega \forall p < \omega \exists \omega \quad \text{LIM}(\mathbb{D}^p_{\alpha}(\xi)) \) by transfinite induction and by strong proof.

Now suppose \( \xi \) is transfinite.

Then \( \forall p < \omega_1 \exists \omega \quad \text{LIM}(\mathbb{D}^p_\xi(\xi)) \) or \( \text{LIM}(\mathbb{D}^{\omega}_\xi(\xi)) \), because suppose the ordinals \( \alpha, \beta \) are expressed in the unique form

\[
\xi \in \omega \beta + x_\beta
\]

\[
\beta = \omega \beta + x_{\beta}, \quad x_{\beta}, x_{\beta} < \omega
\]
Then $\gamma^p = (\omega x + x_x)^{\omega p + x_p} = (\omega x + x_x)^{\omega p} \cdot (\omega x + x_x)^{x_p}$

Now either $\text{LIM}\{(\omega x + x_x)^{x_p}\}$ or $\text{SUC}\{(\omega x + x_x)^{x_p}\}$.

The former implies $\text{LIM}\{(\omega x + x_x)^{x_p}\}$.

The latter implies $\gamma^p = (\omega x + x_x)^{\omega p} \cdot \text{LIM}\{(\omega x + x_x)^{x_p}\} + 1$

$$= \lim_{\mu \leq (\omega x + x_x)^{\omega p}} \left[ (\omega x + x_x)^{\omega p} \cdot \text{LIM}\{(\omega x + x_x)^{x_p}\} \right] + 1, $$

since $0 < \mu$, and thus $\text{LIM}(\omega x + x_x)^{\omega p}$. Therefore $\text{LIM}(\omega x + x_x)^{\omega p}$.

Thus, Suppose for O.I.H. that for some $a$ such that $3 \leq a \leq \omega$, it has been proved that

$(\forall x \leq \omega) \Rightarrow (\forall p \leq \omega) \text{LIM}(\omega x + x_x)^{\omega p}$. Then

$$\text{LIM}_{a+1}(\omega) = \lim_{n \to \omega} \text{LIM}_{a+1}^{\omega n}(\omega)$$

$$= \lim \left\{ \text{LIM}_{a}^{\omega n}(\omega), \text{LIM}_{a}^{\omega n}(\omega), \ldots \right\} \ldots (0)$$

Now $\omega \leq x$, therefore $\text{LIM}(\omega x + x_x)^{\omega p}$ by O.I.H.

Therefore $\text{LIM}_{a+1}(\omega)$ is the limit of a sequence $\delta$-limit numbers by O.I.H.; therefore $\text{LIM}(\omega x + x_x)^{\omega p}$.

Now suppose for O.I.H. that for some $p$ such that $\omega \leq p \leq \omega$, $(\forall x \leq \omega) \Rightarrow (\forall p \leq \omega) \text{LIM}(\omega x + x_x)^{\omega p}$.
Then if \( \text{Suc}(p) \) and \( \text{Lim}(p) \), \( (\forall \delta < \omega) \Rightarrow \),
\[
\Xi_{a+1}^p(\delta) = \Xi_a^{\Xi_{a+1}^p(\delta)+1}(\delta) \quad \text{by definition 3.1}
\]
\[
= \Xi_a^{\omega+1}(\delta) \quad \text{such that} \quad \text{Lim} \left( \Xi_a^{\omega+1}(\delta) \right) \quad \text{by 0.1.1.}
\]
If \( \text{Suc}(p, p) \), \( (\forall \delta < \omega) \Rightarrow \omega \),
\[
\Xi_{a+1}^p(\delta) = \Xi_a^{\Xi_{a+1}^p(\delta)}(\delta) \quad \text{by definition 3.1}
\]
\[
= \Xi_a^\omega(\delta) \quad \text{such that} \quad \text{Lim} \left( \Xi_a^\omega(\delta) \right) \quad \text{by 0.1.1.}
\]
therefore \( \text{Lim} \left( \Xi_{a+1}^p(\delta) \right) \quad \text{by 0.1.1.}\)

If \( \text{Lim}(p) \),
\[
\Xi_{a+1}^p(\delta) = \lim_{n \to \omega} \Xi_{a+1}^{\delta(n)}(\delta) \quad \text{which is a limit}
\]
of limit numbers by property 2 of sequence \( (\delta) \), and by
\( 3 : 1 : \), therefore \( \text{Lim} \left( \Xi_{a+1}^p(\delta) \right) \).
Thus it follows by transfinite induction that \( (\forall \delta < \omega) \Rightarrow \omega \leq p, \delta \)
\( \text{Lim} \left( \Xi_{a+1}^p(\delta) \right) \).
Thus it follows by induction that \( (\forall a < \omega) \Rightarrow_3 \)
\( (\forall \delta, \rho < \omega) \Rightarrow_3 \text{Lim} \left( \Xi_a^\rho(\delta) \right) \).
Thus by primary proof, and proof for the case
\( 2 \leq \delta < \omega, \omega < \rho < \omega, \), \( 3 \leq a < \omega \), it follows:
\( (\forall a < \omega) \Rightarrow_3 (\forall \delta, \rho < \omega) \Rightarrow_3 \text{Lim} \left( \Xi_a^\rho(\delta) \right) \).
To complete the initial proof, suppose now $\alpha = \omega$.
Suppose $\beta = \omega$.
Suppose $\gamma = 2$. Then

$$\Xi^{\omega}(2) = \lim_{n \to \omega} \Xi_{\omega}^{\gamma}(2)$$

$$= \lim_{n \to \omega} \{\Xi_{\omega}^{2}(2), \lim_{n \to \omega} \Xi_{\omega}^{\omega}(2) + 1(2), \ldots\} \tag{2}$$

Now $\Xi_{\omega}^{2}(2) = \lim_{n \to \omega} \Xi_{0}^{2}(2) = 4$ by theorem 4.15(1) (This theorem relies only upon definition 3.1)

$$\lim_{n \to \omega} \Xi_{\omega}^{5}(2) = \lim_{n \to \omega} \{\Xi_{\omega}^{5}(2), \Xi_{\omega}^{\delta}(2), \Xi_{\omega}^{\gamma}(2), \ldots\} \tag{3}$$

$$= \omega$$ by the strictly increasing property of the Ackermann function, and by the identity of the finite restriction of $\Xi$ with this function.

Therefore (2) is the limit of limit numbers by the proofs for the cases previously considered where $\alpha$ is such that $3 \leq \alpha \leq \omega$, therefore $\lim(\Xi^{\omega}(2))$.

Now suppose $\gamma > 2$ but still finite. Then

$$\Xi^{\omega}(\gamma) = \lim_{n \to \omega} \Xi_{\omega}^{\gamma}(\gamma)$$

$$= \lim_{n \to \omega} \{\Xi_{\omega}^{\gamma}(\gamma), \lim_{n \to \omega} \Xi_{\omega}^{\omega}(\gamma) + 1(\gamma), \ldots\} \tag{3}$$
Now \( \prod^2_\alpha (\delta) = \lim_{n \to \omega} \prod^n_\alpha (\delta) = \lim_{n \to \omega} \prod_{\alpha+1}^n (\delta) \)

\[ = \lim_{n \to \omega} \prod^n_\alpha (\delta), \text{ which is a limit of limit numbers by proof for cases already considered where } \alpha \text{ is such that } 3 \leq \alpha \leq \omega. \]

Therefore \( \text{LIM} (\prod^2_\alpha (\delta)) \), therefore \((3)\) is the limit of limit numbers by the same proof, therefore \( \text{LIM} (\prod^\omega_\alpha (\delta)) \).

Now suppose \( \delta \) is transfinite. Then

\[ \prod^\omega_\alpha (\delta) = \lim_{n \to \omega} \prod^2_\alpha (\delta) \]

\[ = \lim \{ \lim_{n \to \omega} \prod^n_\alpha (\delta), \lim_{n \to \omega} \prod_{\alpha+1}^n (\delta), \ldots \} \]

the first term, and therefore the whole sequence is a limit of limit numbers by the proof for finite \( \alpha \geq 3 \), therefore \( \text{LIM} (\prod^\omega_\alpha (\delta)) \).

Suppose for i.e. h. that for some \( p > \omega \) it has been proved that \((\forall \gamma < p) (\forall \delta < \omega) \Rightarrow \text{LIM} (\prod^\gamma_\delta (\delta)) \).

Suppose \( \delta \) is such that \( 2 \leq \delta < \omega \), then if \( \text{SUC} (p) \):
$$\omega^P(\alpha) = \lim_{n \to \omega} \omega^{P(n)}(\alpha)$$

$$= \lim_{n \to \omega} \omega^{P(n+1)}(\alpha),$$ which is a limit number by proof for cases where \( \alpha \) is such that \( 3 \leq \alpha < \omega \), as \( \alpha \) is a limit number by \( \omega \). If \( \lim(\omega) \), \( \omega^P(\alpha) = \lim_{n \to \omega} \omega^{P(n)}(\alpha) \), which is a limit of limit numbers by proof of sequences (2), (3), (3'), and by \( \omega \), therefore

$$\lim(\omega^P(\alpha)).$$

Therefore by transfinite induction it follows that

\[(\forall \beta, \delta < \omega_1) \beta \geq \omega, \delta \geq 2 \lim(\omega^P(\alpha)).\]

Thus so far it has been shown that

\[(\forall \alpha < \omega_3 \exists \omega) (\forall \beta, \delta < \omega_1) \beta \geq \omega, \delta \geq 2 \lim(\omega^P(\alpha)).\]

This part of the proof of theorem 4.1a 3 a will be referred to as the initial proof.

The remaining part of the proof concerns the cases for which \( \alpha \) is such that \( \omega < \alpha < \omega_1 \). The argument proceeds by nested transfinite induction, upon indices \( \alpha, \beta \), where \( \beta \) lies in the range \( \omega \leq \beta < \omega_1 \).
Suppose for O.I.H. that for some \( x \) such that \( \omega = x < \omega \), it has been shown \((\forall x' < x) \geq \omega \)
\((\forall \beta, \delta < \omega, \delta \geq \omega \), \( \lim (\Xi^\beta_x (\delta)) \).

Suppose \( \text{Suc}(\alpha) \) such that \( \alpha = \omega x + k, k \in \omega \).
Suppose \( p = \omega \).
Suppose \( \delta = 2 \).

Then \( \Xi x^\omega (2) = \lim_{n < \omega} \Xi x^\alpha (2) \)
\( = \lim_{n < \omega} \left\{ \ldots \frac{\Xi x^\alpha (2)}{\Xi x^\alpha (2)} \ldots \right\} \)
and \( \Xi x^\alpha (2)(n) = \Xi x^{\mu} (2) \) by theorem 4.15(i)
\( = \Xi x^{\mu} (2)(n) \).

Repeating this decomposition \((k-1)\) times to the topmost term leads to \( \Xi x^{\mu} (2) \)
\( = \lim_{\alpha' < \omega} \Xi x^{\mu} (2) + 1 (2) \).

\( \Xi x^{\mu} (2) = \lim_{n < \omega} \Xi x^{\mu} (2) + 1 (2) \)
\( = \lim_{n < \omega} \Xi x^{5} (2) \).

Let \( t_1(\mu(n)) \omega = \Xi x^{\mu} (n) \). Then
\[ \Omega_{E_{\omega}}(t) \geq \Omega_{E_{\omega}}(t) \]

Let \( \Omega_{E_{\omega}}(t) = \omega_\xi + a \), \( a > 0 \). Then for \( a > 0 \):

\[ \Omega_{E_{\omega}}(t) = \begin{cases} \Omega_{E_{\omega}}(t) & \text{if } a < \omega_\xi \\ \Omega_{E_{\omega}}(t) + 1 & \text{if } a = \omega_\xi \end{cases} \]

Now

\[ \Omega_{E_{\omega}}(t) \geq \Omega_{E_{\omega}}(t) \]

by theorem 4.15 (1)

and

\[ \Omega_{E_{\omega}}(t) = \Omega_{E_{\omega}}(t) \]

by this theorem.

Thus, by repeating this decomposition \( n \) times, the number \( \Omega_{E_{\omega}}(t) \) is obtained from \( \Omega_{E_{\omega}}(t) \).

Alternatively, if \( a = 0 \), then \( \Omega_{E_{\omega}}(t) = \omega_\xi \).

Now \( \Omega_{E_{\omega}}(t) = \lim_{n \to \infty} \Omega_{E_{\omega}}(t) \)

by definition 3.1 and by theorem 4.15 (1).

Now let \( t_2 = (\mu_n \xi) \) \( \omega \in \Omega_{E_{\omega}}(t) \).

Then \( \Omega_{E_{\omega}}(t) \geq \Omega_{E_{\omega}}(t_2) \)

Thus, the process is by which number \( \omega_\xi(t) \) was obtained from number \( \Omega_{E_{\omega}}(t) \).
Then \((\exists r, \nu < \omega)\) after \(r\) repetitions of process \(P\), a strictly decreasing sequence \(\{\Omega_{\alpha_{\nu}}(t_r)\}_{1 \leq \nu < r}\)

is obtained, for certain ordinals \(\{\alpha_{\nu}\}_{1 \leq \nu < r}\), such that \(\Omega_{\alpha_{\nu}}(t_r) = \omega + \nu\).

Note

\[
\text{II}^{5+\nu}(z) = \underbrace{\text{II}^{5+\nu}(z) \ldots (z)}_{1 \leq \nu < r},
\]

and by repeating the decomposition used above \(a\) times, the number \(\text{II}^4(z)\) is obtained.

\[
\text{II}^4(z) = \lim_{m \to \omega} \text{II}^{5+1}(z) \quad \text{by theorem 4.150}
\]

\[
= \lim_{m \to \omega} \text{II}^{5+1}(z) \quad \text{by initial proof}
\]

Therefore \(\text{LIM} (\text{II}^4(z)) \quad \text{by initial proof}\).

Therefore \(\text{LIM} (\text{II}^{5+\nu}(z)) \quad \text{by 0.1.1.}\).

Therefore \((\forall \nu' < r) \quad r \geq 0 \quad \text{LIM} (\Omega_{\alpha_{\nu'}}(t_{\nu})(z)) \quad \text{by 0.1.1.}\).

In particular, for \(\alpha_{\nu} = \text{E}_k\)

\[
\text{LIM} (\Omega_{\text{E}_k}(t_{\nu})(z)),
\]

therefore \(\text{II}^{3}_{\text{E}_k}(z)\) is transfinite.
\[
\therefore \Omega^3_{\omega \alpha}(2) = \lim_{n \to \omega} \Omega^2_{\omega \alpha}(2) + 1, \tag{2}
\]
as a limit of limit numbers by O.I.H., is itself a limit number.

**DEFINITION.** The route taken from the number \(\Omega^3_{\omega \alpha}(2)\) to the number \(\Omega^\omega(2)\) is called *route RR*, for subsequent reference.

Therefore, by O.I.H., at each step in the \((k-1)\)
deconstructions (that is, the deconstructions starting with the ordinal \(\Omega^4_{\omega \alpha}(2)\)), a limit number is obtained, therefore \(\text{LIM}(\Omega^3_{\omega \alpha}(2))\), and therefore by O.I.H., \(\Omega^\omega(2)\) is the limit of limit numbers, therefore \(\text{LIM}(\Omega^\omega(2))\).

Suppose now \(\omega\) is finite and \(\omega > 2\). Then

\[
\Omega^\omega(\omega) = \lim_{n \to \omega} \Omega^\omega(n) = \lim_{n \to \omega} \left( \lim_{m \to \omega} \Omega^{m+1}(n) \right) \tag{5}
\]
Now \( \varPi_{\alpha}^{2}(\delta) = \varPi_{\beta_{\alpha}}^{\delta}(\delta) \)

\[
\begin{align*}
\varPi_{\alpha}^{2}(\delta) &= \varPi_{\beta_{\alpha}}^{\delta}(\delta) \\
&\xrightarrow{(\delta-2)} \varPi_{\beta_{\alpha}}^{2}(\delta) \\
&\xrightarrow{\varPi_{\beta_{\alpha}}^{2}(\delta)} \varPi_{\beta_{\alpha}}^{\delta}(\delta)
\end{align*}
\]

Repeating this decomposition \(k\) times to the topmost term leads to \( \varPi_{\beta_{\alpha}}^{2}(\delta) \), and \( \lim (\beta_{\alpha}) \).

Therefore \( \varPi_{\beta_{\alpha}}^{2}(\delta) = \lim_{n<\omega} \varPi_{\beta_{\alpha}}^{2}(\delta) \),

\[
\geq \varPi_{\beta_{\alpha}}^{\delta}(\delta) \text{ by the argument of route RR, taken in the case } \delta=2 (\text{the case for argument } \delta \text{ prior to the current case}),\text{ but by replacing the substitution}
\]

\[
\varPi_{\gamma}^{\varPi_{\gamma}^{2}(\delta)}(2) = \varPi_{\gamma}^{4}(2), \text{ for some ordinal } \eta \text{ (by theorem 4.15(ii)), by the substitution:}
\]

\[
\varPi_{\gamma_{1}}^{2}(\delta) = \varPi_{\gamma_{1}}^{\delta}(\delta), \text{ for some ordinal } \gamma
\]

(by definition 3.1).

Thus

\[
\varPi_{\beta_{\alpha}}^{2}(\delta) \geq \lim_{n<\omega} \varPi_{\beta_{\alpha}}^{2}(\delta+1) \]

But \( \lim (\varPi_{\beta_{\alpha}}^{2}(\delta)) \) by initial proof, therefore \( \lim (r.h.s.) \)
by initial proof, therefore $\Xi^2_{\alpha} (\delta)$ is transfinite.

Therefore, by O.H., at each step in the k decomposition, a limit number is obtained, therefore $\text{LIM}(\Xi^2_{\alpha} (\delta))$.

Therefore by O.H. $\Xi^\omega_{\alpha} (\delta)$ is the limit of limit numbers, therefore $\text{LIM}(\Xi^\omega_{\alpha} (\delta))$.

Suppose now $\delta$ is transfinite. Then

$$\Xi^\omega_{\alpha} (\delta) = \lim_{n \to \omega} \Xi^n_{\alpha} (\delta)$$

$$= \lim_{n \to \omega} \left\{ \Xi^n_{\alpha} (\delta), \Xi^n_{\alpha} (\delta), \Xi^n_{\alpha} (\delta), \ldots \right\}$$

Now $\Xi^n_{\alpha} (\delta) = \Xi^n_{\alpha} (\delta)$, which is a limit number by O.H., therefore $\text{LIM}(\Xi^\omega_{\alpha} (\delta))$.

Suppose for i.i.h. that for some $\beta < \omega$, such that $\omega < \beta$ it has been shown $(\forall \delta < \omega), \exists$

$$(\forall \beta' < \beta) \geq \omega \quad \text{LIM}(\Xi^\beta'_{\alpha} (\delta))$$

Then if $\text{Suc} (\beta)$ and $\text{LIM} (\Pi^\beta_{\alpha} (\delta))$, $(\forall \delta < \omega), \exists$

$$\Xi^\beta_{\alpha} (\delta) = \Xi^\beta_{\alpha} (\delta)$$

$$= \Xi^{\beta+1}_{\alpha} (\delta), \text{where LIM} (\delta) \text{ by i.i.h.}$$

Therefore $\text{LIM}(\Xi^\beta_{\alpha} (\delta))$ by O.H.
If $\exists (\beta, \Delta \beta)$, $(\forall \theta < \omega_1) \Rightarrow$

$$I_\alpha^\beta(\theta) = I_{\Delta \beta}^{\Delta \alpha}(\theta)(\theta)$$

$$= I_{\Delta \alpha}^\beta(\theta), \text{ where } \lim(\theta) \text{ by i.ii. i.ii.}$$

therefore $\text{LIM}(I_\alpha^\beta(\theta))$ by O.H.

If $\text{Lim}(\beta)$, $(\forall \theta < \omega_1) \Rightarrow$

$$I_\alpha^\beta(\theta) = \lim_{n \to \omega} I_{\alpha}^{\Delta^2 \beta(\theta)}(\theta), \text{ which is a limit of limit numbers by properties of sequences (4), (5), (6) and by i.ii. i.ii., therefore } \text{LIM}(I_\alpha^\beta(\theta)).$$

Suppose now Lim(\alpha).

Suppose $\beta = \omega$

Suppose first that $\theta = 2$. Then

$$I_{\alpha}^{\Delta \alpha}(2) = \lim_{n \to \omega} I_{\alpha}^{\Delta \alpha}(2)$$

$$= \lim_{n \to \omega} \{ I_{\alpha}^{\Delta \alpha}(2), \lim_{n \to \omega} I_{\alpha}^{\Delta \alpha}(2) + 1(2), \ldots \}$$

Now $I_{\alpha}^{\Delta \alpha}(2) = 4$ by theorem 4.15(1), and

$$\lim_{n \to \omega} I_{\alpha}^{\Delta \alpha}(2)$$

is related to limit RR to number $I_{\alpha}^{\Delta \alpha}(2)$ which is a limit, therefore

$$\text{LIM}\left(\lim_{n \to \omega} I_{\alpha}^{\Delta \alpha}(2)\right),$$
So by O.I.H. \( \text{LIM} (\omega \alpha (\aleph)) \).

Now suppose \( \gamma \) is finite and greater than 2. Then

\[
\omega \alpha (\gamma) = \lim_{n \to \infty} \omega \alpha (\gamma)
\]

\[
= \lim \{ \omega \alpha (\gamma), \lim_{n \to \infty} \omega \alpha (\gamma), \ldots \}
\]

Now \( \omega \alpha (\gamma) = \lim_{n \to \infty} \omega \alpha (\gamma) \), and this number is related to \( \omega \alpha (\gamma) \) by some RR, in which it is made, for some ordinal \( \eta \), the substitution

\[
\omega \alpha (\gamma) = \omega \alpha (\gamma) \quad (\text{by definition 3.1})
\]

in place of the substitution

\[
\omega \alpha (\gamma + 1) = \omega \alpha (\gamma) \quad (\text{by theorem 4.15(1)})
\]

Therefore \( \text{LIM} (\omega \alpha (\gamma)) \), therefore by O.I.H.

\( \omega \alpha (\gamma) \) is the limit of limit numbers, and therefore itself a limit.

Now suppose \( \gamma \) is transfinite. Then

\[
\omega \alpha (\gamma) = \lim_{n \to \infty} \omega \alpha (\gamma)
\]

\[
= \lim \{ \omega \alpha (\gamma), \lim_{n \to \infty} \omega \alpha (\gamma), \ldots \}
\]

\[\ldots (9)\]
Now \( \mathcal{D}_\alpha^2 (\gamma) = \lim_{n \to \omega} \mathcal{D}_\alpha^2 (\gamma) \ldots \ldots \) (10)

Then \((\forall n < \omega), \text{ if } 3n < (\Omega_\alpha (n)), \text{ then}\)

\[ \mathcal{D}_{\Omega_\alpha (n)}^2 (\gamma) = \mathcal{D}_{P_\alpha (n)}^5 (\gamma) \ldots \ldots \] (11)

which is a limit number by O.I.H., as the exponent \( \gamma \) is greater than or equal to \( \omega \).

If \( \text{LIM} (\Omega_\alpha (n)) \), then

\[ \mathcal{D}_{\Omega_\alpha (n)}^2 (\gamma) = \lim_{m \to \omega} \mathcal{D}_{\Omega_\alpha (m)}^2 (\gamma) , \text{ and the decomposition following expression (10) is repeated.} \]

Always after a finite number of such repetitions, a successor fundamental sequence term is reached, which gives a limit number of the form of expression (11) .

Thus, by O.I.H., (10) is the limit of limit numbers, therefore \( \text{LIM} (\mathcal{D}_\alpha^2 (\gamma)) \), and thus by O.I.H. (9) is the limit of limit numbers, therefore

\( \text{LIM} (\mathcal{D}_\alpha^2 (\gamma)) \).

Suppose for \( \forall \cdot \cdot \cdot \cdot \cdot \), for some \( \beta < \omega \), such that \( \omega < \beta \) it has been shown \((\forall \gamma \in \omega), \forall z \).
\[ (\forall \beta < \omega) \Rightarrow \omega \quad \text{LIM} \left( \Delta^P_\alpha (\beta) \right) \]

Then if \( \text{Suc} (\beta) \), \( (\forall \delta < \omega_1) \Rightarrow \omega \left( \forall \alpha \in \delta \right) \\Rightarrow \omega \),

\[ \Delta^P_\alpha (\beta) = \lim_{n \to \omega} \Delta^P_{\omega \alpha} (\beta) \]

\[ = \lim_{n \to \omega} \Delta^{\alpha+1}_{\omega \alpha} (\beta), \quad \text{where LIM} (\alpha) \text{ k, i.e. h.} \]

Therefore \( \text{LIM} \left( \Delta^P_\alpha (\beta) \right) \) by O.I.H.

If \( \text{LIM} (\alpha) \), \( (\forall \delta < \omega_1) \Rightarrow \omega \)

\[ \Delta^P_\alpha (\beta) = \lim_{n \to \omega} \Delta^P_{\omega \alpha} (\beta), \quad \text{which is a limit of limit numbers, by properties of sequences (7), (8), (9)} \]

and by i.e. h., therefore \( \text{LIM} \left( \Delta^P_\alpha (\beta) \right) \).

Thus it follows by transfinite induction on \( \beta \) that

\[ (\forall \beta \in \omega_1) \Rightarrow \omega \quad (\forall \delta < \omega_1) \Rightarrow \omega \text{ \( \text{LIM} \left( \Delta^P_\alpha (\beta) \right) \)} \]

Thus it follows by transfinite induction on \( \alpha \), and by the initial proof, that \( (\forall \alpha < \omega_1)_{\omega \geq 3} \quad (\forall \beta < \omega_1)_{\omega \geq 3} \quad (\forall \delta < \omega_1)_{\omega \geq 3} \text{ \( \text{LIM} \left( \Delta^P_\alpha (\beta) \right) \)} \)

\[ \square \text{THEOREM 4.1a.3a} \]
Note that for $\alpha = 2$, $\text{Suc}(\beta) \Rightarrow \text{Suc}(\Pi_\alpha^\beta(\beta))$, since $\Pi_\alpha^\beta(\beta) = \beta \cdot (\omega \beta + \omega \beta)$, by the left distributive property of ordinals, and $\text{Suc}(\beta \cdot \omega \beta) = \omega$. Combining theorems 4.1a, 2 and 3 gives:

**COROLLARY 4.1a4** ($\forall \alpha, \beta, \gamma < \omega, \alpha > 2, \beta, \gamma \geq 0$

\[\text{Lim} \{ \Pi_\alpha^\beta(\beta) \} \]

Of course, since ($\forall \gamma, \gamma < \omega$), $\Pi_\alpha^\gamma(\gamma) = \omega$, it follows that $\text{Suc} \{ \Pi_\alpha^\gamma(\gamma) \}$.

It is now a simple matter to show that the exponent functions are normal:

**COROLLARY 4.1a1** ($\forall \alpha, \beta < \omega, \beta \geq 0$)

$\forall \beta . \Pi_\alpha^\beta(\beta)$ is normal.

**PROOF**. By theorem 4.1 the functions in question are strictly increasing. It remains to be shown that they are continuous.

\[\forall \alpha, \beta < \omega, \beta \geq 0 \ (\forall \beta < \omega) \text{ Lim} (\beta) \Rightarrow \]

\[\Pi_\alpha^\beta(\beta) = \lim_{n \to \omega} \Pi_\alpha^{S_2 \beta(\beta)}(\beta) \text{ by definition 3.1} \]

\[= \lim_{\beta' \to \beta} \Pi_\alpha^{\beta'}(\beta), \text{ since } \Pi_\alpha^{\beta'}(\beta) \text{ is s.i.} \]

by theorem 4.1, $\{S_2^{\beta(\beta)}\}_{n \to \omega}$ is s.i. as $S_2$ is an f.s.a. and thus ($\forall \beta' < \beta)(\exists n < \omega) \Pi_\alpha^{\beta'}(\beta) < \Pi_\alpha^{S_2^{\beta(\beta)}}(\beta)$

\[\square 4.1a1\]
**THEOREM 4.16.** \( \forall \alpha . \ \mathbb{D}_\alpha^2(\omega) \) is normal.

**PROOF** by transfinite induction. The function is strictly increasing, since

\[
\mathbb{D}_0^2(\omega) = 0 < \mathbb{D}_1^2(\omega) = \omega + 2 ;
\]

and suppose for some \( \alpha > 4^+ \) that the function is strictly increasing at non-zero ordinals smaller than \( \alpha \). Then \( \exists \gamma < (\alpha) \Rightarrow \)

\[
\mathbb{D}_\alpha^2(\omega) = \mathbb{D}_\gamma^2(\omega)
\]

\[
> \mathbb{D}_{\Omega_\gamma(\omega)}^2(\omega), \text{ by theorem 4.1}
\]

\( \lambda \in (\alpha) \Rightarrow \)

\[
\mathbb{D}_\alpha^2(\omega) = \lim_{n < \omega} \mathbb{D}_{\Omega_\alpha(n)}^2(\omega)
\]

\[
> \mathbb{D}_{\Omega_\alpha(n)}^2(\omega), \ (\forall n < \omega) > 0 ,
\]

since \( \{ \Omega_\alpha(n) \}_{n < \omega} \) is strictly increasing as \( \Omega \) is an f.s.a., and (\( \forall n \))

\[
\mathbb{D}_{\Omega_\alpha(n)}^2(\omega) < \mathbb{D}_{\Omega_\alpha(n+1)}^2(\omega) \text{ by 1.11}.
\]

Since \( (\forall \delta < \alpha)(\exists n < \omega) \mathbb{D}_\delta^2(\omega) < \mathbb{D}_{\Omega_\delta(n)}^2(\omega) \), it follows \( (\forall \delta < \alpha) \mathbb{D}_\alpha^2(\omega) > \mathbb{D}_{\gamma'}^2(\omega) \).
Therefore the function is strictly increasing at $\alpha$, and so by transfinite induction, $\alpha$. $\Delta^2 \alpha(w)$ is strictly increasing.

Furthermore, this property implies

$$(\forall \alpha) \lim (\alpha) \Rightarrow \Delta^2 \alpha(w) = \lim_{n \leq \omega} \Delta^2 \alpha(n)(w) \text{ by 3.1}$$

$$= \lim_{\alpha' \leq \alpha} \Delta^2 \alpha'(w)$$

Thus $\forall \alpha$, $\Delta^2 \alpha(w)$ is normal. \quad \Box 4.1b

**THEOREM 4.1c** $(\forall \alpha, \beta, \gamma < \omega), \alpha > \beta, \gamma \geq \omega$

$\max(\alpha, \beta, \gamma) \leq \Delta^2 \gamma(\beta)$

**PROOF** by transfinite induction. 4.1c holds for given values of the index $\beta$, by theorem 4.1. The proof for index $\alpha$ follows.

Let $\gamma = \omega$, $\beta = 2$. Then $\Delta^2 \gamma(\omega) = \omega + 2 > \alpha = 1$.

Suppose for i.i.h. for some $\alpha < \omega$, such that $\alpha > 1$ that $(\forall \alpha' < \alpha), \alpha' \leq \Delta^2 \alpha'(w)$

Then if $\text{succ} (\alpha)$, $\Delta^2 \alpha(w) = \Delta^2 \alpha'(w)$

$$> \Delta^2 \alpha(\omega) \text{ by theorem 4.1}$$

$$\geq \alpha \alpha \text{ by i.i.h.}$$

Therefore $\alpha \leq \Delta^2 \omega(w)$. 
\[
\text{If } \lim (\alpha), \quad \Omega^2_\alpha(\omega) = \lim_{n \to \omega} \Omega^2_{\Sigma_\alpha(n)}(\omega)
\]

\[
\geq \lim_{n \to \omega} \Sigma_\alpha(n) \quad \text{by r. i. h.}
\]

\[
\Rightarrow \alpha
\]

Thus, the case for index \( \alpha \) is proved by transfinite induction, for \( p = 2 \).

Suppose for 1.H. \((\forall \alpha < \omega_1)(\forall p < \beta)\), for some \( p < \omega_1 \), such that \( p > 2 \), that \( \alpha \leq \Omega^p_\beta(\omega) \).

Then consider the expression \( \Omega^p_\alpha(\omega) \). By definition 3.1, (iii), (iv), there are six cases:

For \( \alpha = 3 \):

\[
\Omega^3_\omega(\omega) = \left\{ \begin{array}{ll}
\Omega^3_2(\omega) \Omega^3_0(\omega) & \text{if } \text{Succ}(p) \\
\lim_{n \to \omega} \Omega^3_{\Sigma_2(n)}(\omega) & \text{if } \lim(n) 
\end{array} \right.
\]

\[
\Omega^3_2(\omega) \Omega^3_0(\omega) = \Omega^3_2(\omega), \omega
\]

\[
\geq \Omega^3_3(\omega)
\]

\[
\Rightarrow \alpha = 3 \quad \text{by 1.H.}
\]

Therefore, \( \lim_{n \to \omega} \Omega^3_{\Sigma_2(n)}(\omega) \geq \alpha = 3 \) by 1.H.
For \( \alpha > 3 \):

\[
\mathcal{I}_\alpha^p(\omega) = \begin{cases} \\
\mathcal{I}_{\alpha}^{p_x}(\omega) + 1(\omega) & \text{if } \text{Suc}(\alpha, p) \land \alpha = 1 \\
\mathcal{I}_{p_x}^{\omega+1}(\omega) & \text{if } \text{Suc}(\alpha, p) \land \alpha < 1 \\
\lim \mathcal{I}_\alpha^{p_x}(\omega) & \text{if } \text{Lin}(p) \\
\lim \mathcal{I}_\alpha^{p_x}(\omega) + 1(\omega) & \text{if } \text{Lin}(\omega) \land \text{Suc}(p)
\end{cases}
\]

by definition 3.1. By theorem 4.1, the first two of these expansions are each strictly greater than \( \mathcal{I}_{\alpha}^{p_x}(\omega) > \alpha \) by 1.H. Furthermore:

\( \forall n < \omega \), \( \mathcal{I}_\alpha^{p_x}(\omega) \geq \alpha \) by 1.H., therefore

\[
\lim \mathcal{I}_\alpha^{p_x}(\omega) \geq \alpha \text{ by 1.H.}
\]

And finally, \( \forall n < \omega \), \( \mathcal{I}_\alpha^{p_x}(\omega) + 1(\omega) \geq \mathcal{I}_{p_x}^{\omega+1}(\omega) \) by theorem 4.1

therefore \( \lim \mathcal{I}_\alpha^{p_x}(\omega) + 1(\omega) \geq \alpha \) by 1.H.
Thus it is shown that \((V\alpha < \omega), \Pi_a^B(\omega) \geq \alpha\),
and so by transfinite induction and the result for the values of the index \(p\), it follows:

\[
(V\alpha, p < \omega), \alpha > 0, p > 1, \max(\alpha, p) \leq \Pi_a^B(\omega).
\]

The case for values \(> \omega\) of the index \(\xi\) remains.

**Lemma 4.1d** \((\forall p, r)\) \(p \geq 0, r \geq \omega \land \xi. \Pi_{\xi}^B(\xi)\)
is non-decreasing (n. d.).

**Proof.** \(\Pi_{\xi}^B(\xi) = 0 \leq \Pi_1^B(\xi) = \xi + \beta \leq \Pi_2^B(\xi) = \xi^\beta\)

Suppose for I. H. that for some \(\alpha > 3\), the functions of Lemma 4.1d have been shown
to be n. d. for all \(\xi \leq \alpha\). Then there are the four cases of Definition 3.1(iv) to be
considered. For \(\beta = \omega p + x, x < \omega,\)

if \(\text{Succ}(\alpha, \beta) \& x = 1\),

\[
\Pi_{\alpha, \omega p}^B(\xi) + 1(\xi) = \Pi_{\beta, \xi}^B(\xi), \text{ by}
\]

\[
\Pi_{\alpha, \omega p}^B(\xi) + 1 = \beta, \text{ and by theorem 4.1.}
\]

If \(\text{Succ}(\alpha, \beta) \& x > 1,\)
\[
\text{Lim} \left\{ \prod_{\omega}^{\alpha'_n+\omega-1}(\beta) \right\} \text{ by corollary 4.1.4}
\]
and
\[
\prod_{\omega}^{\alpha'_n+\omega-1}(\beta) \geq \omega \beta^{+} \alpha'_n \text{ by theorem 4.1}
\]
= \beta \dot{+} \phi, \text{ which is a successor number,}

therefore
\[
\prod_{\omega}^{\omega \beta^{+} \alpha'_n}(\beta) > \beta \dot{+} \phi \text{ and } \prod_{\omega}^{\alpha'_n+\omega-1}(\beta) \geq \beta
\]
So by theorem 4.1,
\[
\prod_{\omega}^{\omega \beta^{+} \alpha'_n}(\beta) \geq \prod_{\omega}^{\beta \dot{+} \phi}(\beta).
\]
If \( \text{Lim}(\alpha) \) and \( \text{Succ}(\beta) \), \( (\forall \eta < \omega) \Rightarrow \)
\[
\prod_{\omega}^{\omega \beta^{+} \alpha'_n}(\eta) + 1(\eta) \geq \prod_{\omega}^{\beta \dot{+} \phi}(\eta) \Rightarrow \prod_{\omega}^{\beta \dot{+} \phi}(\eta) + 1
\]
\( \geq \beta \) by theorem 4.1, therefore by 1.H.
\[
\lim_{n<\omega} \prod_{\omega}^{\alpha'_n}(\beta) \Rightarrow \prod_{\omega}^{\beta \dot{+} \phi}(\beta), \text{ for all } \alpha' < \alpha, \text{ since } \forall \alpha' < \alpha \exists n < \omega \exists n < \omega.
\]
Finally, if \( \text{Lim}(\beta) \), then \( \prod_{\omega}^{\beta}(\beta) = \)
\[
\lim_{n<\omega} \prod_{\omega}^{\beta}(\eta), \text{ and } (\forall \eta' < \alpha)
\]
\[
\prod_{\omega}^{\beta}(\beta) = \lim_{n<\omega} \prod_{\omega}^{\beta}(\eta).
\]
Thus, after a finite number of substitutions in this case, if definition 3.1 (iv), these two
1. h. s. will be expressed in terms of the three remaining cases of definition 3.1(ii), all of which have already been dealt with.

Thus it is proved that \((\forall \alpha' < \alpha)(\forall \beta, \gamma < \omega), \gamma > \omega \\Rightarrow \mathbb{I}^\beta_i(\alpha') \leq \mathbb{I}^\beta_i(\gamma)\), and so by transfinite induction Lemma 4.1d is proved.

To return to the proof of theorem 4.1c:

\((\forall \gamma < \omega) \\Rightarrow \mathbb{I}^2(\gamma) \geq \gamma, i = 1, 2, 3\). By lemma 4.1d, \(\forall \epsilon, \mathbb{I}^2(\gamma)\) is non-decreasing, therefore \((\forall \gamma < \omega) \Rightarrow \mathbb{I}^2_{\alpha}(\gamma) \geq \gamma\). And by theorem 4.1 the exponent function of \(\mathbb{I}^2\) is strictly increasing and hence non-decreasing, therefore \((\forall \alpha, \beta, \gamma < \omega), \alpha > 0, \beta > 1, \gamma \geq \omega \Rightarrow \gamma \leq \mathbb{I}^\beta_{\alpha}(\gamma)\), and so theorem 4.1c is proved.

**COROLLARY 4.2** \((\forall \beta, \gamma < \omega), \beta > 1, \gamma \geq \omega \\Rightarrow \mathbb{I}^\beta_{\alpha}(\gamma)\) is ultimately increasing \((\forall \alpha)\). **PROOF.** By lemma 4.1d, the functions of this corollary are non-decreasing, and by theorem 4.1c, \((\forall \beta, \gamma < \omega), \beta > 1, \gamma \geq \omega \Rightarrow \alpha \leq \mathbb{I}^\beta_{\alpha}(\gamma), \alpha < \alpha\), so \(\mathbb{I}^\beta_{\alpha}(\gamma)\) is ultimately unbounded. \(\square\) 4.2
COROLLARY 4.3 \((Vx, y < w, x \neq 0) \iff (y \leq \mathcal{I} \mathcal{B}(y))\) in n.d.

PROOF Definition 3.1 gives the explicit definitions:

\((Vx < w) \quad \mathcal{I} \mathcal{B}(\alpha) = x + \beta\)

\((Vx < w) \quad \mathcal{I} \mathcal{B}(\alpha) = x + 1\)

\((Vx < w) \quad \mathcal{I} \mathcal{B}(\alpha) = x\)

These functions of \(\gamma\) are n.d., and all the functions \(\mathcal{I} \mathcal{B}\), \(x, \beta > 1\) are obtained from these functions by the application of superposition of non-decreasing functions, or by taking limits.

Since neither of these operations leads out of the class of non-decreasing functions, \((Vx, y < w, x \neq 0) \iff (y \leq \mathcal{I} \mathcal{B}(y))\) in n.d. \(\square 4.3\)

COROLLARY 4.3a \((Vx, y < w, x \neq 0, \beta > 1) \iff (x \leq \mathcal{I} \mathcal{B}(x))\) in n.i. for \(\omega \leq \gamma\).

PROOF By theorem 4.1c, \(\gamma \geq \gamma \Rightarrow (Vx, y < w, x \neq 0, \beta > 1) \iff (x \leq \mathcal{I} \mathcal{B}(x)) \square 4.3a\)

Now suppose for \(f : \omega^{[2]} \to \omega, \omega_{n_2} = \omega_{n_3}\):
107

(i) \((\forall \alpha \leq \omega_1) \Rightarrow \exists \beta. f(\alpha, \beta) \text{ is } \text{u.i.}\)

(ii) \((\forall \beta \leq \omega_1) \Rightarrow \exists \alpha. f(\alpha, \beta) \text{ is } \text{u.i.}\)

Thus for each \(\alpha \leq \omega_1\), if \(\alpha' < \alpha\), then

\[ g(\alpha) = \inf f(\alpha, \alpha) \geq f(\alpha', \alpha) \text{ by (ii)} \]
\[ \geq f(\alpha', \alpha') \text{ by (i)} \]
\[ = g(\alpha') \text{, therefore} \]

\[ g(\alpha') \leq g(\alpha) \text{, and so } g \text{ is non-decreasing}. \]

Now, for each \(\alpha\), let \(\alpha^{(n)}\), \(\alpha^{(\omega)}\) denote the least ordinals such that \(f(\alpha, \alpha) = f(\alpha^{(n)}, \alpha)\);

\[ f(\alpha, \alpha) < f(\alpha, \alpha^{(\omega)}). \text{ Now let } \alpha^* = \max(\alpha^{(n)}, \alpha^{(\omega)}). \text{ Then } \alpha < \alpha^*, \text{ and} \]

\[ g(\alpha) = f(\alpha, \alpha) < f(\alpha^{(n)}, \alpha) \]
\[ \leq f(\alpha^*, \alpha) \]
\[ \leq f(\alpha^*, \alpha^*) \]
\[ = g(\alpha^*). \text{ Thus } g(\alpha) \leq g(\alpha^*), \]

and therefore \(g\) is \(\text{u.i.}\).

In general, there is the following theorem, which can readily be proved by induction:

**THEOREM 4.4.** Let \(f: \omega^{[n]} \to \omega\), be such that for any \(\alpha \in \omega^{[n]}, \beta \in \omega^{[\omega^{(n-r)}]}\) for natural numbers
n, r such that \( r < n \) and \( \lambda \exists . f(\bar{x}, \bar{y}, \bar{z}) \in \mu . i \). Then if the function \( g: \omega \to \omega \) is obtained from \( f \) by choosing fixed values for a subsequence of the parameters of \( f \), and identifying the remaining parameters of \( f \), then \( g \in \mu . i \). \( \square 4.4 \)

Thus

COROLLARY 4.4a. The following functions are \( \mu . i \): \((\forall \alpha, \beta, \delta < \omega_1). \alpha > 0, \beta \geq 1, \exists \gamma \in \lambda. \delta \leftarrow \Xi_\delta^\alpha (\gamma), \lambda. \Xi_\delta^\alpha (\gamma), \lambda. \Xi_\delta^\beta (\gamma), \lambda. \Xi_\delta^\gamma (\gamma). \square \)

COROLLARY 4.5. Let \( \alpha, \beta, \gamma \) be the arguments of the function \( \Xi \) of definition 3.1. If \( \Xi \) is written as \( \forall \alpha_1, \alpha_2, \alpha_3 \in (\alpha, \beta, \gamma) \), then whatever substitution is made for the arguments \( \alpha_1, \alpha_2, \alpha_3 \), \((\forall \alpha_1, \alpha_2, \alpha_3 < \omega_1) \Rightarrow \omega \alpha_2 < \alpha \) \( \forall \alpha_2 \in (\alpha, \beta, \gamma) \) \( \leq_m \gamma \alpha_2 \in (\alpha, \beta, \gamma) \)

(\text{In corollary 4.5, '} \leq_m \text{' denotes partial majorization.})

PROOF by theorem 4.1 and corollaries 4.2, 4.3 \( \square \)
Definitions 3.1, 3.2 were chosen in order that the definitions of functions \( D^\alpha \) and operators \( U^\alpha \) may be derived directly from them. It is now shown, on the basis of theorem 4.1, that definition 3.1 can be Simplified. The only change to this definition is that limits of sequences whose indices form fundamental sequences are replaced by limits of sequences whose indices form Segments. Thus, in order to avoid rewriting much of the definition, the function symbols \( \Lambda, \Lambda' \) are used to denote certain of its functions.

**Definition 4.6.** Let \( \Lambda, \Lambda' \) be such that

\[
\begin{align*}
\Xi^B_\alpha (\delta) &= \lim_{n \to \omega} \Lambda (\alpha, \beta, \Omega_\alpha(n), \delta) \quad \ldots \quad (1a) \\
\Xi^B'_\alpha (\delta') &= \lim_{n \to \omega} \Lambda' (\alpha, \beta', \Omega'_{\alpha'}(n), \delta') \quad \ldots \quad (1b)
\end{align*}
\]

are equations within the system of transfinite recursion of definition 3.1 whose right hand sides are defined by limiting operators in the cases \( \lim (\alpha) \), \( \lim (\beta) \) respectively. Then the class \( \Xi^B_\alpha \) of functions \( \Xi^B_\alpha \) is defined exactly as for the class \( \Xi^B_\omega \) of 3.1, excepting that every occurrence of an equation of form (1a) is replaced by

\[
\begin{align*}
\Xi^B_* \alpha (\delta) &= \lim_{\mu < \alpha} \Lambda (\alpha, \beta, \mu, \delta)
\end{align*}
\]

and of form (1b) replaced by

\[
\begin{align*}
\Xi^B'_* \alpha (\delta') &= \lim_{\xi < \beta'} \Lambda' (\alpha, \beta', \xi, \delta')
\end{align*}
\]
Thus, definition 4.6 makes no reference to fundamental sequences for limit ordinals which occur as parameters in the definitions.

**THEOREM 4.8** For each $f:s.a. \Omega$,

$$D_{\alpha}^{(2)} = D_{\beta}^{(2)}$$

**PROOF.** Suppose that element $D_{\alpha}^{(*)}$ of $D_{\beta}^{(2)}$ is defined by the clause

$$D_{\alpha}^{(*)}(\delta) = \lim_{\mu < \alpha} \Lambda(\alpha, \beta, \mu, \delta)$$

By definition 4.6 and lemma 4.1d, the function $\Lambda$ is non-decreasing in the $3^{rd}$ argument. Thus

$$\{\Lambda(\alpha, \beta, \mu, \delta)\}_{\mu < \alpha}$$

and

$$\{\Lambda(\alpha, \beta, \Omega_\alpha(n), \delta)\}_{n < \omega}$$

are non-decreasing, and so

$$\forall \mu < \alpha \exists n < \omega \forall \Lambda(\alpha, \beta, \mu, \delta) \leq \Lambda(\alpha, \beta, \Omega_\alpha(n), \delta)$$

and

$$\forall n < \omega \exists \mu < \alpha \forall \Lambda(\alpha, \beta, \Omega_\alpha(n), \delta) \leq \Lambda(\alpha, \beta, \mu, \delta)$$

Thus the sequences (1) are congruent.

Similarly, if $D_{\alpha}^{(*)}$ is defined in $D_{\beta}^{(2)}$ by

$$D_{\alpha}^{(*)}(\delta) = \lim_{\mu \leq \beta} \Lambda(\alpha, \beta, \mu, \delta)$$

the sequences corresponding to (1) to limit $\beta$ are congruent.
The remaining clauses of definitions 3.1, 4.6 coincide, so they are identical. \[ \Box 4.8 \]

It was shown on p. 46 that distinct f.s.a.'s can be exhibited, however the following is now clear:

**Corollary 4.10** For all f.s.a.'s \( \Omega, \Omega' \),
\[ \mathcal{D}^{(a)}_\Omega = \mathcal{D}^{(a)}_{\Omega'} \]

**Corollary 4.11** \((\forall \alpha, \beta, \gamma < \omega_1)\) for all f.s.a.'s \( \Omega \),
\[ \Gamma^\alpha_{\Omega} \cap \mathcal{U}_\alpha (\beta, \gamma) = \begin{cases} \cap_{\alpha} \mathcal{U}_\alpha (\gamma), \alpha < \omega_1 & \mathcal{U}_\alpha (\beta, \gamma), \alpha < \omega_1, \end{cases} \]
where \( \mathcal{U}_\alpha \) is the projection function, satisfying \((\forall \beta, \gamma < \omega_1)\)
\[ \mathcal{U}_\alpha (\beta, \gamma) = \alpha. \]

**Proof.** Since \( \mathcal{U}_\alpha (\beta, \gamma) = \gamma \), for \( \alpha = 1 \) it follows
\[ \cap_{\alpha} \mathcal{U}_\alpha (\beta, \gamma) = \prod_{\beta} \mathcal{U}_\alpha (\beta, \gamma) \text{ by definition 3.2,} \]
\[ = \mathcal{U}_\alpha (\beta, \gamma) \]

checking through the remaining clauses of definitions 3.2, 4.6 proves corollary 4.11 for \( \prod_{\beta} \mathcal{U}_\alpha (\beta, \gamma) \) as defined in 4.6, and so the corollary is proved by theorem 4.8. \[ \Box 4.11 \]
In particular, from this corollary,

\[ \alpha \circ (\beta + 1) = \alpha \circ \left( \bigcup_{\beta' = \beta}^\infty \mathcal{U}_2(\beta', \alpha) \right) \]

\[ \alpha \circ (\beta + 1) = \left( \bigcup_{\beta' = \beta}^\infty \mathcal{U}_2(\beta', \alpha) \right) \circ \alpha \]

**THE LEADING TERM**

In the operational notation, for each countable \( \sigma \), \( \beta \geq 2 \), \( \alpha \geq 3 \)

\[ \mathcal{D}^{\beta}(\sigma) = \sigma \circ \beta \]

\[ = \underbrace{\sigma \circ \sigma \circ \cdots \circ \sigma}_{\omega \text{ terms}} \circ \left( \underbrace{\sigma \circ \sigma \circ \cdots \circ \sigma}_{\beta \text{ terms}} + 1 \right) \]

Now, the operations \( \Gamma_{\alpha} \) of definition 3.2 give

\[ \Gamma_{\alpha} \circ_{\beta} = \begin{cases} 
\left( \underbrace{\sigma \circ \sigma \circ \cdots \circ \sigma}_{\omega \text{ terms}} + 1 \right) \circ_{\omega} \cdots \circ_{\beta} \\
\left( \underbrace{\sigma \circ \sigma \circ \cdots \circ \sigma}_{\omega \text{ terms}} + 1 \right) \circ_{\omega} \cdots \circ_{\beta} \\
\left( \underbrace{\sigma \circ \sigma \circ \cdots \circ \sigma}_{\omega \text{ terms}} + 1 \right) \circ_{\omega} \cdots \circ_{\beta} \\
\end{cases} \]

\[ \text{for } \alpha \leq 2 \]

\[ \text{for } \beta \leq \beta \]

\[ \text{for } 3 \leq \alpha \]
In the case for \( \beta < \gamma \), \( \gamma_0 \) is called the leading term.

There is a special case, which can arise quite often, in which each term in the expression takes the same value \( \delta \), except the leading term, \( \delta' \).

Thus, for \( 0 \leq \beta' \leq \beta \), \( \delta_{\beta'} = \delta' \), for each such countable index, and for \( \gamma_0 = \gamma \), the following notation is adopted:

\[
\Gamma_{\alpha} \delta_{\beta'} = \begin{cases} 
\bigoplus_{\alpha + 1} (\delta; \delta') & \text{for } \alpha < \omega \\
\bigoplus_{\beta'} (\delta; \delta') & \text{for } \omega \leq \alpha 
\end{cases}
\]

**Some Properties of the Leading Term**

In the first instance, attention is restricted to the finite ordinals.

\((\forall x < \omega) \), let \( h(0) = 0 \); \( g(0) = 1 \); \( f(0) = 2 \)

\( h(1) = 2 \); \( g(1) = 4 \); \( f(1) = 2 \)

\( h(x+1) = 3^{h(x)} \); \( g(x+1) = 2^{g(x)} \); \( f(x+1) = (x+2)^{f(x)} \)

Thus, for example

\( h(5) = 3^{3^{3^2}} \); \( g(5) = 2^{2^{2^4}} \); \( f(5) = 6^{5^{4^{3^2}}} \) then
THEOREM 4.12 \( h \leq m g \leq f \).

PROOF by induction.

\begin{align*}
  h(0) &= 0 < 1 = g(0) \\
  h(1) &= 2 < 4 = g(1) \\
  h(2) &= 3^2 = 9 < 16 = 2^4 = g(2) \\
  h(3) &= 3^9 = (27)^3 < (32)^3 = 2^{16} = g(3)
\end{align*}

Now in fact \( h(3) = 19,683 < g(3) = 32,768 \times 2 \)

Therefore \( \frac{g(3)}{h(3)} = 1.5 \times 2 = 3 \)

Also, \( h(4) = 3^{h(3)} \) \( g(4) = 2^{g(3)} = (2^{\log_2(3)})^{h(3)} > 2^{3 \cdot h(3)} = 2^{\log_2(3) \cdot h(3)} \)

Therefore
\[
  \frac{g(4)}{h(4)} > \frac{2^{3 \cdot h(3)}}{2^{\log_2(3) \cdot h(3)}} = 2^{h(3)(3-\log_2(3))} > 2^{h(3)} > 4 \text{ since } 2 < h(3).
\]

Now, for induction hypothesis, suppose for some \( n > 4 \) that \( \frac{g(n)}{h(n)} > 4 \). Then

\[
  h(n+1) = 3^{h(n)} = (2^{\log_2(3)})^{h(n)}
\]
\[
  g(n+1) = 2^{g(n)} > 2^{4 \cdot h(n)} \text{ by 1.4, therefore}
\]
\[
g(n+1) > \frac{2^4 \cdot h(n)}{2^{\log_2(3)} \cdot h(n)} = 2^{\log_2(3)} \cdot h(n) > 2 \cdot h(n) > 4, \text{ since } 2 < h(n).
\]
Thus it is proved by induction that for each \( n \geq 3 \)
4. \( h(n) < g(n) \).
On the other hand, it can readily be seen that
\[
\begin{align*}
g(0) &= 1; f(0) = 2 \\
g(1) &= 4; f(1) = 2 \\
g(2) &= 2^4 = 16; f(2) = 3^2 = 9 \\
g(3) &= 2^{16}; f(3) = 4^9 = (2^2)^9 = 2^{18}
\end{align*}
\]
Thus \( g(3) < f(3) \). Since for \( n \geq 3 \), \( f(n) \) is
defined by the exponentiation of a larger base
than that for \( g(n) \), it follows \( g(n) < f(n) \), and
\( g < m f \)
\[\boxed{4.12}\]
The leading term in the expression of \( g(n) \), \( n \geq 0 \), is \( f \).
A less sharp but more general result follows. It
shows that despite the \( r \) \( h \) relation of theorem 4.12,
if the leading term is large enough, then in many
cases it is decisive.
THEOREM 9.13. Suppose \( \forall x . m_x \) is majorized by \( \forall x . 2^x \). Then there is an integer \( M \) such that
\[
\forall x . \prod_{i < x+1} m_i \leq m \forall x . \prod_{i \leq x} (2^i M)
\]

PROOF Make the definitions \( g(1) = 2 \), \( f(1) = 8 \), and for \( n > 0 \), \( g(n+1) = (2^{n+1})^g(n) \)
\[
f(n+1) = 2f(n)
\]
Then \( g(2) = 4^2 < 2^5 = \frac{f(2)}{2^3} \). For induction hypothesis, suppose for some \( n > 2 \), \( g(n) . 2^{n+1} < f(n) \). Then
\[
f(n+1) = 2f(n) > 2^{(n+1)}g(n) \quad \text{by IH}, \quad \text{also} \ g(n+1) = 2^{(n+1)}g(n)
\]
therefore
\[
\frac{f(n+1)}{g(n+1)} = \frac{2^{(n+1)}g(n)}{2^{(n+1)}g(n)} = \frac{g(n)(2^{n+1})}{2^{n+1}}
\]
\[
> 2 \cdot 2^{n+1}
\]
induction \( (\forall n < \omega) g(n) . 2^n < f(n) \).
Now suppose \( \forall x \leq \omega m_x \) is majorized by \( \forall x . 2^x \) at \( x = \omega \). Then put \( M = \prod_{i < \omega} m_i \). Then since
\[
\forall m_1, m_2, \ldots, m_x . \prod_{i < x+1} m_i \quad \text{is strictly increasing in each argument}, \ (\forall x < \omega) \prod_{i < x+1} m_i \leq \prod_{i \leq x} (2^i M)
\]
It seems likely that 4.13 can be improved by replacing the majorizing function \( \forall x . 2^x \) for \( \forall x \) by the function of \( x \) that is \( 2^x \), for fixed integers \( s, t \).
LEADING TERMS IN EXPRESSIONS
CONTAINING TRANSFINITE ORDINALS

Since, according to definition 3.2, transfinite sum and product expressions are built up by additive and multiplicative right hand concatenation respectively, these expressions do not have right hand leading terms when the bounds of the operations are limit ordinals. Thus a definition of the left hand Hausdorff transfinite product for countable ordinals, for each \( f : \omega_1^{(\omega)} \rightarrow \omega_1 \), is now given

\[
(\forall \beta \in \omega_1^{(\omega-2)})(\forall \gamma \in \omega_1) \quad \beta \leq \omega \Rightarrow \\
\prod_{\beta < \gamma} f(\beta, \gamma) = \prod_{\beta < \gamma} f(\beta, \gamma), \text{ and } \gamma > \omega \Rightarrow \\
\prod_{\beta < \gamma} f(\beta, \gamma) \text{ for some positive } n < \omega.
\]

\[
\prod_{\beta < \gamma} f(\beta, \gamma) = (\mu \alpha < \omega_1) \left( (\forall \gamma < \mu) f(\beta, \gamma) \prod_{\beta < \gamma} f(\beta, \gamma) \right)
\]

Clearly, this product differs from that of definition 3.2, as ordinal multiplication is not commutative.

The leading term of \( f(\beta, \gamma) \prod_{\beta < \gamma} \) is \( f(0, \gamma) \).

THEOREM 4.14

\[
(\forall \beta \in \omega_1) \omega_\beta = \bigcup_{\beta < \gamma} f(\beta, \gamma) \prod_{\beta < \gamma} \leq \prod_{\beta < \gamma} \bigcup_{\beta < \gamma} f(\beta, \gamma) = \omega^\beta
\]
PROOF by i.e. First note that

\[ U_2 (\beta', \omega)^\prod_{p \leq \omega} = \prod_{p \leq \omega} U_2 (\beta', \omega) \]

\[ = \lim_{n<\omega} \prod_{p \leq n} U_2 (\beta', \omega) \text{ by definition 3.2} \]

\[ = \omega^\omega. \]

It is readily shown by induction that \((\forall \beta < \omega) \omega^\beta \omega.

\[ \prod_{p < \beta} U_2 (\beta', \omega) = \omega^\beta. \text{ But} \]

\[ U_2 (\beta', \omega)^\prod_{p < \omega + 1} = (\forall \beta < \omega + 1) \left[ U_2 (\beta', \omega)^\prod_{p < \beta', \omega + 1} \leq \alpha \right] \]

\[ = \omega. U_2 (\beta', \omega)^\prod_{p < \omega + 1} \]

\[ = \omega^\omega = \omega^{\omega + 1} = \omega^\omega. \]

Suppose for i.H. that theorem 9.14 has been proved for all \( \beta' \) such that \( \omega < \beta' < \beta \), for some \( \beta > \omega + 1 \).

Then if \( \text{Snc}(\beta) \), \( U_2 (\beta', \omega)^\prod_{p < \beta} \)

\[ = (\forall \beta < \omega) \left[ (\forall \beta' < \beta) \left[ U_2 (\beta', \omega)^\prod_{p < \beta} \leq \alpha \right] \right] \]

\[ = \omega. U_2 (\beta', \omega)^\prod_{p < \beta} \]

\[ = \omega^\omega \text{ by induction hypothesis} \]

\[ = \omega^\omega. \]

If \( \text{Lim}(\beta) \), then

\[ U_2 (\beta', \omega)^\prod_{p < \beta} = \lim_{p^* < \beta} \left\{ \omega. U_2 (\beta', \omega)^\prod_{p < \beta, \omega} \right\} \]

\[ = \lim_{p^* < \beta} (\omega^\omega)_{p^*} \text{ by induction hypothesis} \]

\[ = \omega^\omega. \]
Thus, by transfinite induction, and since \((\forall \alpha < \omega_1 \omega^\alpha < \omega^\beta)\), theorem 4.14 follows.

The distinction between finite and infinite ordinals with respect to leading terms is witnessed by the inequality \(\omega^{\omega^2} > \varepsilon_0 = \omega^{\varepsilon_0}\).

Whereas \((\forall n, m, x < \omega) \geq 1, m < n < x \Rightarrow x^m < m^x < m^x \leq y^x\), where \(y = \max(m, n)\).

**SOME FUNCTIONS OF A FINITE OPERAND**

It follows from definitions 3.1, 3.7c that for all numbers \(x, y, n < \omega\), \(D_n^x(y) = D_n^y(x)\).

Theorem 4.1a2 to corollary 4.4a inclusive deal only with transfinite values of the operand \(x\).

**THEOREM 4.15**

(i) \((\forall x < \omega)_\omega D_\omega^x(2) = 4\)

(ii) \((\forall n < \omega)_\omega D_\omega^2(n; D_\omega^2(2)) = \varepsilon_0\)

**PROOF** (i) by t. i. For \(x = 1\), \(D_\omega^2(2) = 4\).

Suppose 4.15 has been proved for all \(x < \alpha\), for some \(\alpha > 1\). Then if \(\text{Succ}(\alpha)\)

\[D_\omega^2(2) = D_\omega^{\text{Succ}(\alpha)}(2), \text{ by } 3.1(ii)\]

\[= 4, \text{ by } 1.11.\]

If \(\text{Lim}(\alpha)\), \(D_\omega^2(2) = \lim_{n<\omega} D_\omega^2(2)\)

\[= \lim_{n<\omega} (4), \text{ by } 1.11.\]

\[= 4.\]
(ii) $\prod^\omega (n; \prod^2 (\omega)) = \lim_{n^\omega} \left[ \omega^\omega, n^\omega, n^\omega, \cdots \right]$

Now, $n^\omega = n^{\omega(1+\omega)} = n^{\omega \cdot \omega} = (n^\omega)^\omega = \omega^\omega$,

since $(\forall n < \omega) \lim_{k \geq n} n^k = \omega$.

Suppose for some $i < \omega$, $i > 2$, that

$\prod^2 (\omega) = \prod^{i-2} (n; \prod^2 (\omega))$. Then

$\prod^{i-1} (n; \prod^2 (\omega)) = \prod^{i-2} (n; \prod^2 (\omega))$

$= \prod^{i-3} (n; \prod^2 (\omega))$

$= \prod^{i-4} (n; \prod^2 (\omega))$

$= \cdots$

$= \prod^{i-(i-2)} (n; \prod^2 (\omega))$

$= \prod^{i-(i-3)} (n; \prod^2 (\omega))$

$= \cdots$

$= \prod^{i-(i-\omega)} (n; \prod^2 (\omega))$

$= \omega^\omega$

$= \prod^{i+1} (\omega)$ by 1.11.

Therefore $\prod^\omega (n; \prod^2 (\omega))$

$= \lim_{i \leq \omega} \prod^{i} (n; \prod^2 (\omega))$

$= \lim_{i \leq \omega} \prod^{i} (\omega) = \varepsilon_0 \quad \square$ 9.15

But note that $\prod^{i+1} (n; \prod^2 (\omega)) < \prod^{i+1} (\omega)$,

so $\varepsilon_0 = n^{\varepsilon_1} = n^{\varepsilon_0} = n^{\varepsilon_0} = n^{\varepsilon_0} = \varepsilon_0$ as $\varepsilon_0$ is a nonceal number of multiplication

$= (n^\omega) \varepsilon_0$

$= \omega \varepsilon_0$

$= \varepsilon_0 \cdot \omega$

$\leq \varepsilon_0 \cdot \omega = \omega \cdot \omega = \omega^\omega = \text{r.b.s.}$

Still, it can readily be shown that

$\omega^{\varepsilon_0} = \omega^\omega = \varepsilon_0$.
SOME ARITHMETIC PROPERTIES OF THE OPERATIONS

It is well-known that the operations of ordinal addition and multiplication are not commutative, but are associative:

\[(\forall \alpha, \beta, \gamma < \omega_1) \alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma ; \]
\[\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma \]

and that ordinal multiplication is left distributive over ordinal addition:

\[(\forall \alpha, \beta, \gamma < \omega_1) \alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma \]

The operation of exponentiation is not even associative; however, the following laws hold, expressed using the notation \(\Delta \alpha \cdot \beta(\gamma) = \gamma \otimes \alpha \cdot \beta:

\[\alpha \otimes (\beta \otimes \gamma) = (\alpha \otimes \beta) \otimes \gamma \]
\[\alpha \otimes (\beta \circ \gamma) = (\alpha \otimes \beta) \circ (\alpha \otimes \gamma) \]

It may be asked whether these and perhaps other laws apply to operations belonging to the class \(\Delta^\omega(\alpha)\) and which are more complex than exponentiation.

It appears that this is not the case, and there is a curious distinction between the operations \(i^, \]
\[i \leq 3\], and the operations \(\otimes, \]
\[q \leq \alpha \]. In the first instance, only the restriction of these operations to the natural numbers need be considered.
By virtue of the inductive definition of operations \(\otimes\) from operations \(\otimes'\), \(\alpha' \leq \alpha\), for \(4 \leq \alpha\) none of the operations \(\otimes\) is either commutative or associative; and it can be proved by induction that \((\forall x, y, z < \omega)\):

\[
x \otimes (y \otimes z)
\]

\[
x \not\leq x \otimes (y \otimes z)
\]

\[
\not\leq (x \otimes y) \otimes z
\]

\[
\not\leq x \otimes (y \otimes z)
\]

\[
\not\leq (x \otimes y) \otimes (x \otimes z)
\]

\[
\not\leq (x \otimes y) \otimes (x \otimes z)
\]

Similar inequalities hold for operations \((k)\), \(4 < k < \omega\).

Now suppose \(\text{Sne}(p, Pp)\), and \(3 \leq \alpha\). Then for each \(\gamma \leq \omega\),

\[
\gamma \otimes (\beta + 1) = \gamma \otimes (\gamma \otimes (\beta + 1)) \tag{1}
\]

For example, \(\beta = \omega + 1\), \(\alpha = 3\), so that

\[
\gamma \otimes (\omega + 2) = \gamma \otimes (\gamma \otimes (\omega + 1)) = \gamma \otimes (\omega + 1)
\]

Now suppose \(\text{Sne}(p)\), and \(\alpha \leq 2\). Then

\[
\gamma \otimes (p + 1) = (\gamma \otimes (p))^\otimes \gamma \tag{2}
\]
For example $p = \omega + 1$, $\alpha = 2$ so that

$$\gamma \circ (\omega + 2) = (\gamma \circ (\omega + 1)) \circ \gamma = \gamma \wedge \gamma \wedge \gamma$$

$$= \gamma^{\omega + 1 + 1}$$

$$= \gamma^{\omega + 2}$$

Now, had the expression $\gamma \circ (\omega + 2)$ been evaluated on the basis of formula (1) in place of formula (2), a different value would have been obtained:

$$\gamma \circ (\omega + 2) = \gamma \circ (\gamma \circ (\omega + 1)) = \gamma \circ \gamma^{\omega + 1}$$

$$= \gamma^{\omega + 1 + 1}$$

$$= \gamma^{\omega + 2}$$

Thus, definition 3.1, in generating the operations $\circ$, has to include formulas (1), (2) in accordance with the value of $\alpha$. 
§ 5. GENERALIZED EPSILON NUMBERS.

NORMAL FUNCTIONS. Strictly increasing continuous functions (normal functions) defined on the countable ordinals, together with some of their elementary properties, were introduced in § 1, p. 13.

Veblen [20] has studied a number of properties of normal functions, and an operation which leads to a new normal function from a given normal function. Throughout the remainder of § 5, each citation of this author is a reference to [20]. The operation is that of obtaining a first derived function $f'$ from a normal function $f$, according to the following definition:

\[ f'(0) = (\mu \beta) \left[ f(\beta) = \beta \right] \]

\[ f'(\alpha + 1) = (\mu \beta) \left[ f'(\beta) = \beta \text{ and } f(\beta) = \beta \right] \]

For \( \text{Lim} (\alpha) \) : \[ f'(\alpha) = \text{Lim}_{\alpha' < \alpha} f'(\alpha') \]

Thus by lemma 1.4, $f'$ enumerates the fixed points of the function $f$. Furthermore, in theorem 1.3 it was proved that for $f$ a normal function, the smallest
fixed point of \( f \), when \( f(0) = 0 \), is \( f^\omega(0) \), and the smallest fixed point greater than some \( \beta < \omega \) is the number \( f^\omega(\beta+1) \). Thus, to evaluate \( f' \), write

\[
\begin{align*}
  f(0) &= f^\omega(0) \\
  f'(x+1) &= f^\omega(f'(x) + 1), \text{ and for } \operatorname{Lim}(\alpha), \\
  f'(x) &= \lim_{\alpha' \to \alpha} f'(\alpha').
\end{align*}
\]

Thus by definition 1.5 it follows that for each normal \( f \), \( f^\omega = f' \). Now let \( f \) be the normal function \( \chi \cdot 1 + \alpha \). Then \( \forall \alpha < \omega \), \( f'(\alpha) = \omega + \alpha \). This is readily proved by induction, for \( f'(0) = \omega \).

Suppose \( f'(\alpha) = \omega + \alpha \). Then \( f'(\alpha+1) = \lim_{n+\omega+\alpha+1} \) and \( \forall \alpha < \omega \), \( f'(\alpha) = \omega + \alpha \). Then \( f'(\alpha) = \lim_{\alpha' < \alpha} f'(\alpha') = \omega + \alpha' \). Then \( f'(\alpha) = \lim_{\alpha' \to \alpha} f'(\alpha') = \omega + \alpha \).

As was observed in §1, it follows from the strictly increasing property of \( f \), and from Lemma 1.4, that \( f \) normal \( \Rightarrow f^\omega \) normal.
Tabulated below are some normal functions, and their corresponding first derived functions. For proofs of the exponential case, see theorem 5.25.

\[
\begin{align*}
\omega & \rightarrow \omega + \beta & \text{and} & \omega \cdot \beta + \gamma & \rightarrow \omega \cdot \beta + \gamma \\
\alpha & \rightarrow \beta + \beta & \text{and} & \alpha \cdot \beta + \beta & \rightarrow \alpha \cdot \beta + \beta \\
\lambda & \rightarrow \beta & \text{and} & \lambda \cdot \beta + \beta & \rightarrow \lambda \cdot \beta + \beta \\
\lambda & \rightarrow \beta & \text{and} & \lambda \cdot \beta + \gamma & \rightarrow \lambda \cdot \beta + \gamma \\
\lambda & \rightarrow \beta & \text{and} & \lambda \cdot \beta + \beta & \rightarrow \lambda \cdot \beta + \beta \\
\lambda & \rightarrow \beta & \text{and} & \lambda \cdot \beta + \beta & \rightarrow \lambda \cdot \beta + \beta \\
\lambda & \rightarrow \beta & \text{and} & \lambda \cdot \beta + \gamma & \rightarrow \lambda \cdot \beta + \gamma \\
\lambda & \rightarrow \beta & \text{and} & \lambda \cdot \beta + \beta & \rightarrow \lambda \cdot \beta + \beta \\
\lambda & \rightarrow \beta & \text{and} & \lambda \cdot \beta + \beta & \rightarrow \lambda \cdot \beta + \beta \\
\end{align*}
\]

for \( \omega \leq \alpha' + \beta \), where \( \alpha' = [\mu \eta] [\gamma \leq \varepsilon \gamma] \).

The notion of 1st derived function is developed by Veblen to that of \( \alpha \)-th derived function \( f_\alpha \) of a normal function \( f = f_0 \).

**DEFINITION 5.1.** For each non-zero \( \alpha < \omega \), the \( \alpha \)-th derived function \( f_\alpha \) of a normal function \( f \) is defined as:

\[
(\forall \beta = \omega) \, f_\alpha (\beta) = f^F (\beta) \quad \text{and} \quad \alpha > 1 \,, \, f_\alpha (\beta) = (\mu \eta) \left( (\forall \beta' < \beta) \, f_\alpha (\beta') = \eta \wedge \alpha \right)
\]

\[
\eta \in \bigcap_{1 \leq \alpha'} \left\{ \xi \mid f_{\alpha'} (\xi) = \xi \right\}
\]

\[
\alpha' < \alpha
\]
SOME VEBLEN FUNCTIONS $\phi$. The notion of derived function of a normal function $\phi$ is extended still further in [20] to that of functions $\phi$ defined for a positive countable number of arguments such that

(i) each argument is non-zero

(ii) only finitely many arguments simultaneously take values exceeding unity

(iii) the number of arguments is a successor number.

In fact the definition given in [20] allows for the condition of countability to be replaced by restriction to ordinals $X = \omega_\tau$, for some arbitrary ordinal $\tau$. Thus, in the definition given here, $\tau = 1$:

Suppose $\phi$ is a normal function. Then $(\forall \beta, \beta < \omega_1) > 1$

1. $(\forall \beta, \beta < \omega_1) \phi ((\beta), \beta) = \phi (\beta)$

2. $\phi ((\beta), 1, 2, 3, \ldots, \beta) = \phi$

$$(\forall \beta, \beta < \omega_1) \left[ (\forall \beta', \beta' < \beta) \phi ((\beta'), 1, 2, 3, \ldots, \beta) \right] \implies \eta$$

& $\eta \in \bigcap_{\alpha < \beta} \{ \alpha \mid \phi ((1), 1, 2, \ldots, (\alpha)) = \alpha \}$

3. $(\forall \beta, \beta < \omega_1) \phi ((\beta), 1, 2, 3, \ldots, \beta) = \omega_1$
\[ \phi((\rho_1), (1)_2, (1)_3, \ldots, (\rho_2), \ldots, (\rho_p)) = \eta \]
\[ (\mu \eta) < \nu \left[ \left( \forall \rho'_1 < \rho_1 \right) \phi((\rho'_1), (1)_2, (1)_3, \ldots, (1)_p) < \eta \right] \]
\[ \land \eta \in \bigcap \bigcap \{ \alpha \mid \phi((1), (1)_2, \ldots, (\alpha), (1)_2, (1)_3, \ldots, (\rho'_1), \ldots, (\rho_p)) = \alpha \} \bigg\} \]

Veldkamp goes on to show that \( \forall p \phi((1), (1)_2, \ldots, (1)_p) \)
in fact normal, from which it follows that there exists a bound \( \exists E(\sigma) < \omega \), of solutions to the equation \( \phi((1), \ldots, (1)_E(\sigma)) = E(\sigma) \). Each number \( E(\sigma), \sigma < \omega \), is inexpressible in terms of ordinals smaller than \( E(\sigma) \), and by means of a function symbol \( \phi \) involving subscripts smaller than \( E(\sigma) \).

Some of the properties of the \( \alpha \)-th derived function of a normal function are examined below.

Suppose \( f_\alpha : \omega \rightarrow \omega \), is a normal function.
Then the sequence of functions \( \{ f_\alpha \}_{\alpha < \omega} \), obtained in definition 5.1 is also obtained by the scheme of definition 5.1 a to follow. The identity is proved in theorem 5.7.
DEFINITION 5.1a.

\((\forall \alpha \leq \omega_1) \left[ f_{\alpha+1}(0) =_{df} f^\omega_{\alpha}(0) \right. \&

\left. (\forall \eta \leq \omega_1) \left[ f_{\alpha+1}(\eta+1) =_{df} f^\omega_{\alpha} \{ f_{\alpha+1}(\eta) + 1 \} \right. \&

\left. (\forall \lambda \leq \omega_1) \lim(\lambda) \left[ f_{\alpha}(\lambda) =_{df} \lim_{\gamma \to \lambda} f_{\alpha}(\gamma) \right. \&

\left. f_{\eta}(0) =_{df} \lim_{\gamma \to \eta} f_{\eta}(0) \&

\left. f_{\eta}(\eta+1) =_{df} \lim_{\gamma \to \eta} f_{\eta} \{ f_{\eta}(\gamma) + 1 \} \right] \right] \right]

where, \((\forall \alpha, \eta \leq \omega_1)(\forall n \leq \omega) = 0 \]

\[ f_{\alpha}(\eta) =_{df} f_{\alpha}(\eta) \&

\[ f_{\alpha}^{n+1}(\eta) =_{df} f_{\alpha} \{ f_{\alpha}^n(\eta) \} \&

\[ f_{\alpha}(\eta) =_{df} \lim_{n \to \omega} f_{\alpha}(\eta) \]

\[ \square 5.1a \]

THEOREM 5.2 \((\forall \alpha \leq \omega_1) f_{\alpha} is normal .\)

PROOF by nested transfinite induction .

By hypothesis, \(f_0\) is normal .

Suppose for some \(\alpha < \omega_1\), it has been proved \(\forall \alpha' \leq \alpha\) \(f_{\alpha'}\) is normal . Then suppose \((\exists \eta \leq \omega_1)(\forall \eta', \eta'' \leq \eta) \]

\[ \eta'' < \eta' \Rightarrow f_{\alpha}(\eta'') < f_{\alpha}(\eta') \& \lim (\eta') \Rightarrow 

\[ f_{\alpha}(\eta') = \lim_{\eta^* < \eta'} f_{\alpha}(\eta^*) . \]
Now suppose \( \text{Succ}(\alpha, \eta) \).

Then \( f_\alpha(\eta) = \lim_{\gamma' < \eta} f_\alpha(\gamma') \) by definition 5.1a, and

\[
\geq f_\alpha(\eta) + 1, \quad \text{since } f_\alpha \text{ is normal, by 1.H. on } \alpha
\]

\[
> f_\alpha(\eta) \), \text{ also since } f_\alpha \text{ is normal, such that } (\forall p \in \omega_1) f_\alpha(p) \geq p.
\]

Thus \( f_\alpha \) increases at the successor ordinal \( \eta \).

Now suppose \( \text{Succ}(\alpha), \text{ Lim}(\eta) \).

Then \( f_\alpha(\eta) = \lim_{\gamma' < \eta} f_\alpha(\gamma') \) by definition 5.1a, and

\[
\leq f_\alpha(\eta), \text{ for each } \gamma' < \eta \text{ by 1.H. on } \eta.
\]

Thus \( f_\alpha \) increases at the limit ordinal \( \eta \).

Now suppose \( \text{Lim}(\alpha) \). Then for \( \text{Succ}(\eta) \),

\[ f_\alpha(\eta) = \lim_{\alpha' < \alpha} \{ f_\alpha(\eta) + 1 \} \text{ by definition 5.1a,} \]

\[
\geq f_0 \{ f_\alpha(\eta) + 1 \} \geq f_\alpha(\eta) + 1 > f_\alpha(\eta),
\]

thus \( f_\alpha \) increases at successor ordinal \( \alpha \).

Furthermore, if \( \text{Lim}(\gamma) \), then \( f_\alpha(\eta) = \lim_{\gamma' < \eta} f_\alpha(\gamma') \),

thus \( f_\alpha \) is continuous at the limit ordinal \( \eta \).

Also \( f_\alpha(\eta) > f_\alpha(\eta') \), for each \( \gamma' < \eta \) by 1.H. on \( \gamma \),
and thus $f_\alpha$ increases at the limit ordinal $\eta$. Thus it follows by transfinite induction that the function $f_\alpha$ is normal, and from this it follows again by transfinite induction that $(\forall \alpha < \omega_1) f_\alpha$ is normal. □ 5.2

THEOREM 5.3 $(\forall \alpha < \omega_1) f_{\alpha+1}$ enumerates in increasing order (i.e., in order of the fixed points of $f_\alpha$)

(b) $(\forall \alpha < \omega_1) \text{Lim} (\alpha) \Rightarrow f_\alpha$ c.e. in $\text{Lim} (\alpha)$, the set of ordinal numbers

\[ \bigcap_{\alpha' < \alpha} \text{FP}(f_{\alpha'}) \]

where $\text{FP}(f_{\alpha'})$ is the set of fixed points of the function $f_{\alpha'}$, and $f_0$ is a rel. fn. satisfying $f_0(0) > 0$.

PROOF (a) For every countable $\alpha$, the smallest fixed point of $f_\alpha$ is given by $f_\alpha(0) = f_{\alpha+1}(0)$. Suppose $f_{\alpha+1}(\eta)$ is the $\eta$-th fixed point of $f_\alpha$. Then the least fixed point of $f_\alpha$ greater than this number is given by $f_\alpha(f_{\alpha+1}(\eta)+1) = f_{\alpha+1}(\eta+1)$

Now suppose $\{f_{\alpha+1}(\eta)\}_{\eta \in \omega}$ for some $\chi < \omega_1$. 
such that Lin (x), is a segment of the sequence of fixed points of \( f_\alpha \). Then, the smallest ordinal greater than any in the segment is \( \lim_{x \to \infty} f_\alpha(x') \), but by theorem 5.2, \( f_\alpha \) is normal, therefore

\[
\lim_{x \to \infty} \left[ \lim_{x' \to \infty} f_{\alpha + 1}(x') \right] = \lim_{x \to \infty} f_\alpha \left( \lim_{x' \to \infty} f_{\alpha + 1}(x') \right) \text{ by congruence } (*)
\]

\[
= \lim_{x \to \infty} f_{\alpha + 1}(x'), \text{ as the numbers } f_{\alpha + 1}(x'), \quad x \leq x' \text{ are fixed points of } f_\alpha
\]

\[
= f_{\alpha + 1}(x), \text{ thus } f_{\alpha + 1}(x) \text{ is the smallest fixed point of } f_\alpha \text{ greater than any in the sequence } \{f_{\alpha + 1}(x')\}_{x' \to \infty}. \quad \Box 5.3(\alpha)
\]

\[(*) \text{ Suppose } f \text{ is a normal function. Then for Lin (x), } f(x) = \lim_{x' \to \infty} f(x'). \text{ Suppose }
\]

\[
\eta = \lim_{\mu' < \mu} \eta_{\mu'}, \text{ where Lin (}\mu\text{) and } \{\eta_{\mu'}\}_{\mu' < \mu} \text{ is an increasing sequence. Then the sequences } \{f(\eta')\}_{\eta' \leq \eta}
\]

\[
\text{and } \{f(\eta_{\mu'})\}_{\mu' < \mu} \text{ are congruent, as } f \text{ is strictly increasing, thus } f(\eta) = \lim_{\mu' < \mu} f(\eta_{\mu'}).\]
PROOF (b). Suppose \( \alpha < \omega \) is a limit ordinal. Then the smallest point belonging to \( \bigcap \alpha' \subseteq \alpha FP(\alpha) \) is the number \( \lim_{\alpha' \subseteq \alpha} f(0) = f(0) \).

Now suppose the \( \eta \)-th point belonging to \( \bigcap \alpha' \subseteq \alpha FP(\alpha) \) is \( f_\alpha(\eta) \). Then the next largest ordinal number belonging to this set is the least which is a fixed point, greater than \( f_\alpha(\eta) \), of each of the functions \( f_\alpha' \), \( \alpha' \subseteq \alpha \). Thus, it is the limit of the sequence \( f_\alpha( [ f_\alpha(\eta) + 1 ] ) \), \( \alpha' \subseteq \alpha \), and by definition 5.1(a), this limit is \( f_\alpha(\eta + 1) \).

Now suppose \( \{ f_\alpha(\eta') \} \eta' < \eta \) for some \( \eta < \omega \), such that \( \lim_{\eta' < \eta} f_\alpha(\eta') \) is a segment of the sequence of points in \( \bigcap \alpha' \subseteq \alpha FP(\alpha) \). Then the smallest ordinal greater than any in the segment is \( \lim_{\eta' < \eta} f_\alpha(\eta') \), since by theorem 5.2, \( f_\alpha \) is normal, thus strictly increasing. Furthermore,

\[
(\forall \alpha' \subseteq \alpha)(f_\alpha' \{ \lim_{\eta' < \eta} f_\alpha(\eta') \}) = \lim_{\eta' < \eta} f_\alpha \{ f_\alpha(\eta') \}
\]

\[
= \lim_{\eta' < \eta} f_\alpha(\eta'), \text{ as } \alpha' \subseteq \alpha.
\]
THEOREM 5.4 \( \forall \alpha. f_\alpha(0) \) is normal, if 
\( 0 < f_\alpha(0) \).

PROOF. \( f_\alpha(0) = k > 0 \). As \( f_\alpha \) is normal, \( f_\alpha(1) > f_\alpha(0) \).
\( k = f_\alpha(0) \), therefore \( f_\alpha(k) > f_\alpha(0) = k \). Thus, as 
\( f_\alpha(1) = \lim \{ f_\alpha(0), f_\alpha(1), \ldots \}, f_\alpha(0) = k \),
\( k = f_\alpha(0) \), therefore \( f_\alpha(0) > f_\alpha(0) = k \).

The proof proceeds by transfinite induction:

Suppose for I.H. that \( f_\alpha(0) \) has been proved that
for some non-zero \( \alpha < \omega \), \( (\forall \beta < \alpha) \beta > 0 \).

\( \mbox{Suc}(\beta) \Rightarrow f_\beta(0) = \sup f_\beta(0) \) &
\( \mbox{Lim}(\beta) \Rightarrow f_\beta(0) = \sup f_\beta(0), (\forall \beta' < \beta) \).

Then if \( \mbox{Suc}(\alpha) \),
\( f_\alpha(0) = k^* < k \), for some \( k^* < \omega \), by I.H.,
and as \( f_\alpha \) is normal, \( f_\alpha(1) = f_\alpha(0) = k^* \),
\( f_\alpha k^* > f_\alpha(0) = k^* \). Thus, \( \alpha \).
\[ f_\alpha (0) = \lim \{ f_\nu (0), f_\xi (0), f_\mu (0), \ldots \} \]
\[ f_\alpha (0) > k^*, \text{ therefore } f_\alpha (0) > f_\nu (0). \text{ Thus } \]
\[ \forall \alpha. \ f_\alpha (0) \text{ is strictly increasing at } \alpha. \]
\[ \text{If } \lim (\alpha), \text{ by definition 5.1a, } f_\alpha (0) = \]
\[ \lim \{ f_\nu (0), \ldots \}, \text{ thus } \forall \alpha. \ f_\alpha (0) \text{ is continuous at } \alpha. \]

But by 1.1.1, \[ \{ f_\alpha (0) \}_{\alpha' < \alpha} \text{ is strictly increasing,} \]
\[ \text{thus its limit, } f_\alpha (0), \text{ is greater than any of its terms, thus } \forall \alpha. \ f_\alpha (0) \text{ is s.i. at } \alpha. \]
\[ \text{Thus it follows by transfinite induction that } \]
\[ \forall \alpha. \ f_\alpha (0) \text{ is normal.} \quad \square 5.4 \]

**COROLLARY 5.5** \((\forall \beta, \gamma < \omega_1)(\forall \alpha < \beta)\)

\[ (i) \lim (\gamma) \Rightarrow \lim \{ f_\alpha (\gamma) \} \in \text{Rge}(f_\alpha) \]
\[ (ii) \text{Rge}(f_\alpha) \subseteq \omega_1 \]
\[ (iii) \text{Rge}(f_\omega) = \omega_1 \]

\[ (iv) (\forall \alpha' < \alpha) \quad \text{Rge}(f_\alpha) \subseteq \text{Rge}(f_{\alpha'}) \]
\[ (v) \text{Rge}(f_\beta) \subseteq \bigcap_{\alpha < \beta} \text{Rge}(f_\alpha) \]
\[ (vi) \lim (\gamma) \Rightarrow \lim \{ f_\beta (\gamma') \} \in \text{Rge}(f_\beta) \]
\[ (vii) \text{Rge}(f_\beta) = \omega_1 \]
where \( f_0 \) is a normal function.

**Proof.** (i) from definition 5.1a, (ii) by definition 5.1a, as \( f_0 \) is only defined by the two operations of taking successor and constant limit; (iii) \((\forall \eta, \eta' \in \omega). \eta \neq \eta' \Rightarrow f_\alpha(\eta) \neq f_\alpha(\eta')\), by theorem 5.2, (iv), (v) from theorem 5.3, (vi) from definition 5.1a, (vii) as for (iii). \( \square \)

**Corollary 5.6**

\[ \text{FP}(\forall \alpha. f_\alpha(0)) \subseteq \text{FP}(f_0), \text{ if } f_0(0) > 0. \]

**Proof.** By corollary 5.5, (iv),

\( (\forall \alpha \in \omega). f_\alpha(0) \in \text{Rge}(f_\alpha). \)

Therefore \((\forall \alpha \in \omega). f_\alpha(0) \) is a fixed point of \( f_0 \). Furthermore, \( f_0(0) = k > 0 \), and by theorem 5.4 \( \forall \alpha. f_\alpha(0) \) is normal, therefore \( f_1(0) = f_0(0) = k \). Therefore, the smallest fixed point \( \mu \) of \( \forall \alpha. f_\alpha(0) \) which satisfies \( f_\mu(0) = \mu \), is therefore itself a fixed point of \( f_0 \), since \( \mu > 0 \).

Also, since \( \text{FP}(\forall \alpha. f_\alpha(0)) \subseteq \text{Rge}(\forall \alpha. f_\alpha(0)) \), corollary 5.6 follows. \( \square \)

Now let \((\forall \beta = \omega). \forall \xi. \varphi((\omega_1), (\omega_2), \ldots (\omega_\beta)) \) be the Veblen functions. Let \((\forall \beta = \omega_1) \varphi((\omega_1)) = f_0(\omega_1)\).
Then:
\[ \phi(((\alpha_1), (1)_1)) = \alpha \] is the number \( \alpha \) such that
\[ \phi((\alpha), (1)) = \alpha \]
\[ \phi(((\alpha_1), (2)_2)) = \alpha_2 \] is the number \( \alpha_2 \) such that
\[ \phi((\alpha), (1)_1) = \alpha_2 \]
\[ \phi(((\alpha_1), (\alpha_2)_2)) = \alpha_1 \] is the number belonging to
\[ \bigcap \{ \alpha \mid \phi((\alpha), (\alpha)_2) = \alpha \} \]

Thus:
**THEOREM 5.7** \((\forall \omega_1, \alpha_2 < \omega_1)\)
\[ \phi(((\alpha_1),)) = f_0(\alpha_1) \] & \( \phi(((\alpha_1), (\alpha_2)_2)) = f_{\alpha_2}(\alpha_1) \)

**COROLLARY 5.8** \((\forall \beta < \omega_1)\) the sequence of sets \( \{ \text{Rge } f_\beta \}_\beta \) and the set \( \text{Rge } f_\beta \)
have the properties of the set \( \{ \text{S}_\alpha \} \) and of the set \( S \) of theorem 5 of [20], p. 284.

**PROOF.** Immediate from corollary 5.5 \( \square \).

Thus it follows that \((\forall \beta < \omega_1)\)
\[ f_\beta(\beta) = f^F(\beta) \]
\[ f_{\alpha+1}(\beta) = f^F_\alpha(\beta) \]
and for \( \lim (\alpha) \)

\[
f_{\alpha} (\beta) = \begin{cases} 
\lim_{\alpha' < \alpha} f_{\alpha'} (\beta) & \text{if } \beta = 0 \\
\lim_{\beta' = \beta} f_{\alpha} (\beta') & \text{if } \lim (\beta) \\
\lim_{\alpha' < \alpha} \{ f_{\alpha'} (\beta) + 1 \} & \text{if } \text{Succ} (\beta)
\end{cases}
\]

For some values of \( \beta \), \( f_{\alpha} \) can be computed inductively, without introducing the higher arithmetic operations. For example, if \( f = \alpha \cdot 1 + \beta \)

then \( f^{\alpha} = \alpha \cdot \omega + \beta \). Thus \( f_{1} (\beta) = \omega + \beta \); if \( f_{2} (\beta) = \omega \cdot \beta \), for some \( \omega \) and all \( \beta < \omega \), then

\[
f_{\alpha+1} (\beta) = \begin{cases} 
\lim_{\beta' = \beta} \omega + \beta' & \text{if } \lim (\beta) \\
\lim_{n < \omega} \left[ \frac{n \omega + \omega \cdot \cdots + \omega \cdot \beta}{n} \right] & \text{if } \text{Succ} (\beta)
\end{cases}
\]

and if \( \lim (\alpha) \), and for \( \alpha' < \alpha \), \( f_{\alpha'} (\beta) = \omega^{\alpha'+1} \beta \),

then:
\[ f_\alpha (\beta) = \begin{cases} \lim_{\alpha' \to \alpha} \omega^{\alpha'} + \beta = \omega^{\alpha} \alpha' = \alpha \\ \lim_{\beta' = \beta} \omega^{\alpha} + \beta' = \omega^{\alpha} + \beta \end{cases} \]

Therefore, it follows by transfinite induction that 
\((\forall \alpha, \beta < \omega_1) \alpha > 0 \ f_\alpha (\beta) = \omega^{\alpha} + \beta \).

There are, of course, functions definable from the function \( \supr \) which are strictly increasing but not continuous and therefore non-normal. For example, let 
\((\forall \alpha < \omega_1) (\forall x < \omega) \ g (\omega \alpha + x) = \supr \ \omega^{\alpha + x} \),

for \( \beta = \omega \alpha + x \). Then
\[
\begin{align*}
g (x) &= \omega \cdot x, \ x < \omega \\
g (\omega + x) &= \omega (\omega + x) \\
g (\omega \cdot x) &= \frac{\omega \cdot x}{\omega} = \epsilon_1 \\
g (\omega \cdot 2 + x) &= \frac{x}{\omega} \epsilon_1 + 1 \\
g (\omega \cdot 3) &= \supr \ \omega^{\cdot 3} (\omega)
\end{align*}
\]

**Theorem.** The function \( g \) is strictly increasing, but is discontinuous.
PROOF. Suppose $p < q < \omega_1$. Then for
$p^* = \omega \alpha_1 + \alpha_1$, $p = \omega \alpha_2 + \alpha_2$, $g\alpha_1 = \alpha_2 = \alpha$
$g(p^*) = \square_{2+\alpha_1} (\omega) \supset \square_{2+\alpha_2} (\omega)$ by theorem 4.1
$= g(p^*)$

Thus without loss of generality, let $\alpha_1 \equiv \alpha_2$. Then
$g(p) = \square_{2+\alpha_2} (\omega)$
$\supset \square_{2+\alpha_1} (\omega)$ by corollary 4.2
$\supset \square_{2+\alpha_2} (\omega)$ by theorem 4.1
$= g(p^*)$.

Therefore $g$ is strictly increasing, but from the
examples given above, can be seen to be discontinuous:

\begin{itemize}
  \item e.g. $p = \omega \cdot 2$
  \item $g(\omega \cdot 2) = \square_{\omega \cdot 2} (\omega)$
  \item $= \omega_1$
  \item $\supset \omega_1$
  \item $= \lim_\omega \omega_1^\alpha$
  \item $x = \omega$
  \item $= \lim_\omega \square_{2+\alpha} (\omega) = \lim_\omega \supset \omega_2 \supset g(p^*)$.
\end{itemize}

\begin{itemize}
  \item e.g. $p = \omega^2$
  \item $g(\omega^2) = \square_{2+\omega^2} (\omega)$.
\end{itemize}
\[(\forall x \leq \omega)\]
\[g(\omega, x) = \mathbb{II}_{2+\omega}^x(\omega) \leq \mathbb{II}_{2+\omega}^{2x+x+1}(\omega) \leq \mathbb{II}_{2+\omega}^{3x+1}(\omega) \leq \mathbb{II}_{2+\omega}^{2x+1}(\omega) \leq \mathbb{II}_{2+\omega}^{x+1}(\omega) \leq \mathbb{II}_{2+\omega}^{x+4}(\omega)\]

Therefore,
\[\lim_{x \to \omega} g(\omega, x) \leq \lim_{x \to \omega} \mathbb{II}_{2+\omega}^{x+4}(\omega) = \mathbb{II}_{2+\omega}^{x+4}(\omega) \leq \mathbb{II}_{2+\omega}^{x+3}(\omega) \leq \mathbb{II}_{2+\omega}^{x+2}(\omega) \leq g(\omega^+) \]

\[\square\]
EPSILON NUMBERS DERIVED FROM GENERALIZED ARITHMETIC OPERATIONS

In the first instance, the sequence of countable epsilon numbers \( \{ \varepsilon_p \} \) is examined. These numbers can be completely characterized in terms of the operations \( \Pi \).

**THEOREM 5.11.** Each of the countable \( \varepsilon \)-numbers is of the form \( \varepsilon_p = \Pi_q^{\omega(p)}(\omega) \).

**PROOF.** by transfinite induction. Firstly, suppose \( p = 0 \). Then

\[
\Pi_0^\omega(\omega) = \lim_{n < \omega} \Pi_0^n(\omega)
\]

\[
= \lim \{ \omega, \omega^\omega, \omega^{\omega^\omega}, \ldots \} = \varepsilon_0
\]

Now suppose that \( p \) is such that \( 0 < p < \omega \), and that \((\forall p < p) \varepsilon_p = \Pi_q^{\omega(p)}(\omega)\). Then if \( \varepsilon_{\omega(p)} \), the smallest \( \varepsilon \)-number greater than \( \varepsilon_p \) is the limit of the increasing sequence

\[
\left\{ \frac{n}{\omega} (\varepsilon_{p+1}) \right\}_{n < \omega}
\]

Thus \( \varepsilon_p = \lim_{n < \omega} \left\{ \frac{n}{\omega} (\varepsilon_{p+1}) \right\}_{n < \omega} \)

\[
= \lim_{n < \omega} \left\{ \Pi_q^{\omega(p+1)}(\omega) \right\}_{n < \omega}
\]

\[
= \Pi_q^{\omega(p+1)}(\omega) = \Pi_q^{\omega(p)+\omega}(\omega)
\]
If $\text{Lim}(p)$, then the smallest $\varepsilon$-number greater than any of the form $\varepsilon_p$, $p < p$, is

$$
\lim_{p < p} \varepsilon_p = \lim_{p < p} \mathcal{D}_q(\omega^{(1)})(\omega) \quad \text{by 1.11.}
$$

$$
= \lim_{n < \omega} \mathcal{D}_q(\omega^{(1)+N}p(n))(\omega) \quad \text{by theorem 4.1}
$$

$$
= \mathcal{D}_q(\omega^{(1)+p})(\omega)
$$

Thus theorem 5.11 is proved by transfinite induction. □

Since for transfinite $p$, $1 + p = p$, theorem 5.11 can be profounded as:

$$(\forall p < \omega) \quad \varepsilon_p = \begin{cases}
\mathcal{D}_q(\omega^{(1)})p(n) & \text{if } p < \omega \\
\mathcal{D}_q(\omega^{(p)}) & \text{if } \omega \leq p
\end{cases}
$$

**Theorem 5.12** $(\forall p < \omega)$ \( p \notin \{ \varepsilon_p \} \quad \Rightarrow \quad \varepsilon_0 \notin \{ \varepsilon_p \} \quad \forall p < \omega, \quad \Rightarrow \quad \varepsilon_0 > \varepsilon_n \)

**Proof.** Suppose $p = n < \omega$. Then for $n > 1$,

\[\omega^{\varepsilon_0^n} = \omega^{\varepsilon_0}^{\varepsilon_0^n}
= (\omega^{\varepsilon_0})^{\varepsilon_0^n} \quad \text{by multiplication law for powers of an ordinal}
= \varepsilon_0^{\varepsilon_0^n}
\]

$\geq \varepsilon_0^{\varepsilon_0}$

$> \varepsilon_0^n$, therefore $\varepsilon_0^n$ is not an $\varepsilon$-number.
Now suppose \( p \geq \omega \). Then \( 1 + p = p \), therefore
\[
\omega^p_0 = \omega^{\omega^p_0 + p} = \omega^{\omega^p_0} = (\omega^{\omega^p_0})^p = (\omega^p_0)^p \text{ by multiplication law}
\]
\[
= \varepsilon_0^p
\]
\[
\geq \varepsilon_0^p,
\]
so \( p \leq \omega^p_0 \) by assumption.

Therefore \( \varepsilon_0^p \) is not an \( \varepsilon \)-number. \( \square \) THEOREM 5.12

G. Hessenberg introduces the terminology \( \langle \varepsilon \rangle \)-numbers' for ordinals \( \alpha > 0 \) satisfying \( \omega \cdot \alpha = \alpha \). These numbers, as non-zero fixed points of \( \omega \cdot \alpha \cdot \omega \), are the numbers \( \omega^\alpha \), \( \alpha > 0 \), so that \( \varepsilon_0 = \omega^\omega \); every \( \varepsilon \)-number is a delta number, but not vice versa.

Since \( (\forall \mu \leq \omega) \quad (\forall \alpha \leq \omega) \quad \Pi_\alpha \mathcal{E}_\mu(\omega) = \mathcal{E}_\mu \), for each \( \mu \leq \omega \), the notation \( \mathcal{E}_\mu^{(4)} \) is adopted for the number \( \mathcal{E}_\mu \).

None of the operations \( \Pi_\alpha \), \( \omega < \omega \), when applied to ordinals \( \rho, \varepsilon \leq \varepsilon_0 \), are sufficient to generate any ordinal \( \geq \varepsilon_0 \). Thus theorems 5.11, 5.12 emphasize that the definition of \( \varepsilon \)-numbers
requires a new arithmetical operation, \( \aleph_4 \).

For \( \beta = \omega \), \( \varepsilon_0 = \aleph_4^\omega (\omega) = \aleph_5^\omega (\omega) \). This corresponds with the standard notation for certain smaller ordinals:

\[
\begin{align*}
\omega + 2 &= \aleph_1^2 (\omega) \\
\omega \cdot 2 &= \aleph_2^2 (\omega) \\
\omega^2 &= \aleph_3^2 (\omega) \\
\omega^\omega &= \aleph_4^\omega (\omega) = \varepsilon_0 \\
\varepsilon_0 &= \aleph_5^\omega (\omega)
\end{align*}
\]

**Definition 5.12a** \((\forall \alpha < \omega_1, \alpha \geq 4 \setminus \varepsilon \alpha (\omega) \downarrow \varepsilon_0, \)

denotes the sequence of numbers \( \varepsilon_0 \) which are solutions to the equation \( \aleph_5^\alpha (\omega) = \varepsilon_0 \), if \( \varepsilon \prec (\omega) \), or \((\forall \alpha \leq \alpha) \) to the equations \( \aleph_\alpha^\omega (\omega) = \varepsilon_0 \), if \( \varepsilon \prec (\omega) \).

The alternative notation already adopted for \( \varepsilon \)-numbers thus satisfies definition 5.12a.

Note that every \( \varepsilon (\alpha) \) number, for \( \alpha \) such that \( 4 < \alpha < \omega_1 \), is a limit number. For suppose otherwise, and let \( \alpha^+ \) be such an \( \varepsilon (\alpha) \) number.

Then if \( \varepsilon \prec (\omega), (\forall \alpha < \omega_1) \), \( \aleph_\omega^\alpha (\omega) > \alpha^+ \) by theorem 4.1, but \( \varepsilon \prec (\omega) \) and \( \varepsilon \prec (\omega) \) by theorem 4.1a 2, therefore \( \alpha^+ \leq \aleph_\omega^\alpha (\omega) \).

* ordered according to magnitude
If \( \lim (\alpha) \), then \( (\forall \beta < \omega) \)

\[
\Pi_{\alpha}^{\beta+1}(\omega) \geq \beta + 1 \quad \text{by theorem 4.1, but}
\]

\[
\Pi_{\alpha}^{\beta+1}(\omega) = \lim_{\alpha' \to \alpha} \Pi_{\alpha'}^{\beta}(\omega) + 1(\omega),
\]

therefore \( \lim (\text{l.h.s.}) \) by theorem 4.1a2 and corollary 4.2, and so \( \beta + 1 \leq \Pi_{\alpha}^{\beta+1}(\omega) \).

Thus every \( \varepsilon^{(\alpha)} \)-number, for \( q < \alpha < \omega \), is a limit number.

**THEOREM 5.13** \( (\forall \alpha < \omega) (n > q) \) \( \text{Succ}(\omega) \Rightarrow \)

a) \( \varepsilon^{(n)}_{\omega} = \lim_{n < \omega} \Pi_{\alpha}^{\beta}(\omega) \)

b) \( (\forall \beta < \omega) \) \( \text{Succ}(\beta) \Rightarrow \) the least \( \varepsilon^{(\alpha)} \)-number greater than \( \varepsilon^{(\alpha)}_{\beta} \) is the limit of the sequence

\[
\left\{ \varepsilon^{(\alpha)}_{\beta}, \Pi_{\beta}^{\varepsilon^{(\alpha)}_{\beta}+1}(\omega), \Pi_{\beta}^{\varepsilon^{(\alpha)}_{\beta}+1}(\omega), \ldots \right\}
\]

c) \( (\forall \beta < \omega) \) \( \lim (\beta) \Rightarrow \varepsilon^{(\alpha)}_{\beta} = \lim_{\beta < \varepsilon^{(\alpha)}_{\beta}} \varepsilon^{(\alpha)}_{\beta} \).

**PROOF.** In the case \( \alpha = q \), the cases a), b), c) are standard properties of the epsilon numbers which are identical with the \( \varepsilon^{(\alpha)} \)-numbers.

Suppose \( \alpha \) is a successor number satisfying...
$4 < \alpha < \omega_1$. By theorem 4.1, $\nabla p : \mathcal{P}_\alpha^\beta(\omega)$ is strictly increasing. Thus, in the various cases:

(a) Consider the sequence
\[
\{ \omega, \mathcal{P}_\alpha^\omega(\omega), \mathcal{P}_\alpha^\omega(\omega)(\omega), \mathcal{P}_\alpha^\omega(\omega)(\omega)(\omega), \ldots \}
\]
\[
= \{ \mathcal{P}_\alpha^n(\omega) \}_{n=0}^\infty , \text{ since by definition 3.1,}
\]
\[
\mathcal{P}_\alpha^\omega(\omega) = \mathcal{P}_\alpha^\omega(\omega), \text{ and } (\forall n < \omega) \mathcal{P}_\alpha^n(\omega)
\]
\[
\mathcal{P}_\alpha^{n+1}(\omega) = \mathcal{P}_\alpha^{n+1}(\omega).
\]
Since $\omega = \mathcal{P}_\alpha^\omega(\omega)$, the sequence is strictly increasing, and its limit is denoted by
\[
\varepsilon = \lim_{n<\omega} \mathcal{P}_\alpha^n(\omega) . \quad \text{Now } \mathcal{P}_\alpha^\varepsilon(\omega)
\]
\[
= \lim_{n<\omega} \mathcal{P}_\alpha^{n+1}(\omega)
\]
\[
= \lim_{n<\omega} \mathcal{P}_\alpha^\omega(\omega)
\]
\[
= \mathcal{P}_\alpha^\omega(\omega) = \varepsilon .
\]
Thus $\varepsilon$ is a fixed point of $\nabla p : \mathcal{P}_\alpha^\beta(\omega)$. It is the smallest such number, for if $\mathcal{P}_\alpha^\beta(\omega) = \beta$, then $\beta > 1$, so that $\omega = \mathcal{P}_\alpha^\beta(\omega)$.

Now suppose $\mathcal{P}_\alpha^n(\omega) = \beta$, for some $n$. Then
\[ \overline{\mathcal{I}}_{E_\alpha}^\alpha(w) = \overline{\mathcal{I}}_{E_\alpha} P(w), \text{ therefore} \]

\[ \overline{\mathcal{I}}_{E_\alpha}^\alpha(w) \leq \overline{\mathcal{I}}_{E_\alpha} P(w) = \beta, \text{ by theorem 4.1, thus} \]

by induction \( \forall n < \omega \) \( \overline{\mathcal{I}}_{E_\alpha}^n(w) = \beta \), so

\[ \lim_{n \leq \omega} \overline{\mathcal{I}}_{E_\alpha}^n(w) = \overline{\mathcal{I}}_{E_\alpha}^\alpha(w) \leq \beta. \text{ Thus } \mathcal{E}_0^\alpha = \alpha \varepsilon. \]

Now let \( \beta \) be a successor number, and suppose that \( (\forall \beta' < \beta) \) numbers \( \mathcal{E}_{\beta'}^{(a)} \) have been defined according to 5.13 a), b), c). Then for case b) consider the number \( \mathcal{E}_{\beta}^{(a)} \). Let \( \gamma_i = \mathcal{E}_{\beta}^{(a)} + 1 \), and for each \( n \) such that \( 0 < n < \omega \), let

\[ \gamma_{n+1} = \gamma_n \cdot \overline{\mathcal{I}}_{E_{\beta}}^{\gamma_n}(w). \]

Now \( \mathcal{E}_{\beta}^{(a)} + 1 \leq \overline{\mathcal{I}}_{E_{\beta}}^{\mathcal{E}_{\beta}^{(a)} + 1}(w) \); but since \( \text{Sec}(\text{r.h.s}) \)

and \( \text{Lin}(\text{r.h.s}) \) by theorem 4.1 a, 2,

\[ \mathcal{E}_{\beta}^{(a)} + 1 \leq \overline{\mathcal{I}}_{E_{\beta}}^{\mathcal{E}_{\beta}^{(a)} + 1}(w), \text{ and so by induction} \]

\( \forall n < \omega \) \( \gamma_n \leq \gamma_{n+1} \). Thus, as in case a) it follows:

\[ \overline{\mathcal{I}}_{E_{\beta}}^{\gamma_n}(w) = \lim_{n \leq \omega} \overline{\mathcal{I}}_{E_{\beta}}^{\gamma_n}(w) \]

\[ = \lim_{n \leq \omega} \gamma_{n+1} \]

\[ = \lim_{n \leq \omega} \gamma_n. \text{ This is the smallest} \]
\( \varepsilon^{(a)} \) is number greater than \( \varepsilon_{EP} \), for if for some \( S > \), \( \Pi_{EP} S (w) = S \) and \( \varepsilon_{EP} < S \) \( \leq \lim_{n \to \infty} S_n \), then since \( \lim_{n \to \infty} S_n \), \( \varepsilon^{(a)}_{EP} + 1 \neq S \), and \( S_n < S \Rightarrow \Pi_{EP} S_n (w) < \Pi_{EP} S (w) \), therefore \( S_{n+1} < S \), thus \( \lim_{n \to \infty} S_n = S \), and so \( \varepsilon^{(a)}_{EP} = \lim_{n \to \infty} S_n \).

Now let \( p \) be a limit number, and suppose that \( (\forall p' < p) \), numbers \( \varepsilon^{(a)}_{EP} \) have been defined according to 9.13(b), c). Then for case c).

c) Let \( \varepsilon = \lim_{p' < p} \varepsilon^{(a)}_{EP} \).

Then \( \Pi_{EP} \varepsilon (w) = \lim_{p' < p} \Pi_{EP} \varepsilon^{(a)}_{EP} (w) \) by corollary 4.11.1

\[ = \lim_{p' < p} \varepsilon^{(a)}_{EP} \]

\[ = \varepsilon; \text{ thus } \varepsilon \text{ is an } \varepsilon^{(a)} \text{ number. Since } \varepsilon = \lim_{p' < p} \varepsilon^{(a)}_{EP}, \text{ it is the smallest } \varepsilon^{(a)} \text{ number greater than any of the numbers } \varepsilon^{(a)}_{EP}, p' < p, \text{ so } \varepsilon = \varepsilon^{(b)} p, \text{ thus } \varepsilon^{(a)}_{EP} = \lim_{p' < p} \varepsilon^{(a)}_{EP}. \]

\[ \square \text{ THEOREM 5.13.} \]
COROLLARY 5.14 \((\forall \alpha < \omega_1) \text{ Succ}(\alpha) \Rightarrow \text{ the least } \varepsilon(\alpha) \text{ number is } \Delta^2_\omega(\omega)\).

PROOF. From theorem 5.13(a),

\[ \varepsilon_0(\alpha) = \lim_{n \to \infty} \Delta^2_\alpha(\omega) = \Delta^2_\alpha(\omega) \text{ by definition 3.1 = } \Delta^2_{\alpha+1}(\omega) \text{ by definition 3.1 } \]

COROLLARY 5.15 \((\forall \alpha, \beta < \omega_1) \beta < \alpha \Rightarrow \text{ Succ}(\alpha) \)

\[ \varepsilon_\beta(\alpha) = \Delta^\omega_{\alpha + \beta}(\omega) \]

PROOF by transfinite induction on index \(\beta\).

For \(\beta = 0\), \(\Delta^\omega_\alpha(\omega) = \varepsilon_0(\alpha) \text{ by theorem 5.13 a).}\)

Suppose the cases for \(\beta < \beta\) have already been proved, for some \(\beta\) such that \(0 < \beta < \omega_1\). Then if \(\text{ Succ}(\beta), \text{ by theorem 5.13 b) }\)

\[ \varepsilon_\beta(\alpha) = \lim \{ \Delta^\omega_{\alpha + \beta}(\omega) + 1, \Delta^\omega_{\alpha + \beta + 1}(\omega), \ldots \} \]

\[ = \lim \{ \Delta^\omega_{\alpha + \beta + 1}(\omega) + 1, \Delta^\omega_{\alpha + \beta + 2}(\omega), \Delta^\omega_{\alpha + \beta + 3}(\omega), \ldots \} \]

\[ = \Delta^\omega_{\alpha + \beta + 1}(\omega) \]

\[ = \Delta^\omega_{\alpha + \beta}(\omega) \]

If \(\text{ Lim}(\beta)\), then \(\varepsilon_\beta(\alpha) = \lim_{\beta < \alpha} \varepsilon_\beta(\alpha) \text{ by theorem 5.13 c).}\)
Therefore, \( E^p = \lim_{p \to \infty} II_\alpha w(\alpha + p) \) by 1.H.

\[ = II_\alpha w(\alpha + p) \]

Thus, corollary 5.15 is proved.

Sierpinski [18], p. 328, has shown that numbers \( p > \omega \) satisfying \( 2^p = \omega \) are each \( \epsilon \)-numbers.

Since it is not in general true that \( II_\alpha w(\alpha + p) \)

\[ = II_\alpha w(\alpha + p) \]

Sierpinski's method of proof cannot be directly generalized to the \( \epsilon_\omega \)-numbers.

But the theorem itself can be used to obtain a

similar result for \( \epsilon_{15} \)-numbers. Thus

**Lemma 5.16** \( (\forall p \leq \omega), II_3^p(2) \leq II_4^p(2) \).

**Proof** by T.1. on \( p \).

It is true that \( II_4(2) \geq II_3(2) = 2^2 \).

Suppose for some \( p \) such that \( 1 \leq p \leq \omega \), that \( (\forall p < \omega), II_4^p(2) \geq II_3^p(2) \).

Then, if \( (\text{Suc}(p)) \), \( II_4^p(2) \geq 2 \cdot II_4^p(2) \)

\[ \geq 2^p \cdot 2 \text{ by 1.H.} \]

\[ = 2^p = II_3^p(2) \]

If \( \text{Suc}(p) \) and \( \text{Lim}(p) \)

\[ II_4^p(2) = 2 \cdot II_4^p(2) + 1 \]
So \( \mathcal{D}^{p^*}(2) \geq 2^{\mathcal{D}^{p_1}(2)} + 1 \) by 1. H.
\[
= 2^{p^*} + 1
= 2^{p^*} - 2
\geq 2 \cdot 2 = 2^p = \mathcal{D}^{p}(2).
\]

And if \( \text{Lim}(p) \),
\[
\mathcal{D}^{p^*}(2) = \lim_{p \to p^*} \mathcal{D}^{p}(2)
\geq \lim_{p \to p^*} 2^p \quad \text{by 1. H.}
\]

\[
= 2^p = \mathcal{D}^{p}(2)
\]

therefore \( (\forall \beta < \omega). \mathcal{D}^{p^*}(2) \geq \mathcal{D}^{p}(2) \quad \square \ 5.16
\]

**Lemma 5.17**

\( (\forall \beta < \omega). \mathcal{D}^{p^*}(2) \geq \mathcal{D}^{p}(\omega) \).

**Proof.** The relation is true for \( p = 1 \). Suppose each case true for \( p^* < p \), for some \( p \) such that \( 1 \leq p < \omega \). Then if \( \text{Suc}(p) \):

\[
\mathcal{D}^{p^*}(2) - \mathcal{D}^{\omega p^* + \omega}(2) = \lim_{n \to \omega} \mathcal{D}^{p^* + \omega n}(2)
\]

\[
\lim \left\{ \mathcal{D}^{p^*}(\omega) + 1, 2^{\mathcal{D}^{p^*}(\omega) + 1}, 2^{2^{\mathcal{D}^{p^*}(\omega) + 1}}, \ldots \right\} \text{ by 1. H.}
\]

Let \( \gamma + 1 = \mathcal{D}^{p^*}(\omega) + 1 \), and consider

\[
\gamma = 2^{2^{2^{2^\gamma + 1}}}
\]

Now \( 2^\gamma = 2 \cdot 2 \geq \gamma + 2 \)
\[ 2^{\gamma + 2} = 2^{\gamma + 1} \cdot 2 \]
\[ 2^{2 \cdot \gamma} = \left( 2^{2 \cdot \gamma + 1} \right)^2 = 2^{2 \cdot \gamma + 1} \cdot 2^{2 \cdot \gamma + 1} \]
\[ \geq \omega \cdot 2^{2 \cdot \gamma + 1} \]
\[ 2^{\omega \cdot 2^{\gamma + 1}} = (2^\omega)^{2 \cdot \gamma + 1} = \omega^{2 \cdot \gamma + 1} \geq \omega^{\gamma + 1} \]

Therefore \( S \geq \omega^{\gamma + 1} \); but \( \omega \omega^\beta(2) \geq S \), therefore
\( \omega \omega^\beta(2) \geq \omega \omega^\beta(\omega) = \omega \omega^\beta(\omega) \).

If \( \lim (p) \), then \( \omega \omega^\beta(2) = \lim_{p < p} \omega \omega^\beta(2) \)
\[ \geq \lim_{p < p} \omega \omega^\beta(\omega) \text{ by 1.11.} \]
\[ = \omega \omega^\beta(\omega) \]

Thus the lemma is proved. \( \square \ 5.17 \)

Now suppose \( p > \omega \) and \( p = \omega \omega^\beta(2) \).
Therefore by lemma 5.16, \( p \geq \omega \omega^\beta(2) \), thus by Sierpinski's theorem cited above, \( p \) is an epsilon number, so \( \omega \cdot p = p \). Therefore \( p = \omega \omega^\beta(2) \)
and therefore $p \geq \mathcal{D}^0_\ast (\omega)$ by lemma 5.15. But
by theorem 4.1, strict inequality cannot hold,
therefore $p = \mathcal{D}^0_\ast (\omega)$, and so $p$ is an $\varepsilon^{(3)}$ number.

DEFINITION Let $\varepsilon^{(3)}_p, p = \omega$, $\varepsilon^{(3)}_\ast p = \omega$, denote the $\varepsilon$- and $\ast$-numbers respectively.

THEOREM 5.18. Suppose $\alpha = \varepsilon^{(3)}_p$ for
some $p, \mu = \omega$, such that $2 < \mu$. Then $(\forall \mu < \mu)_2
(\exists \mu') \alpha = \varepsilon^{(3)}_{p, \mu'}$

PROOF. In the cases $\varepsilon^{(3)}_p$, by definition $\varepsilon^{(3)}_p$
is a solution $\xi$ to every equation $\mathcal{D}^5_{\mu'}(\omega) = \xi$, for $\mu':\mu$, therefore in these cases there exist the
required indices $\mu'$.

Thus, suppose $\varepsilon^{(3)}_\ast (\mu)$. Then by hypothesis,
$\mathcal{D}^{\alpha}_p (\omega) = \alpha$. Let $\mu'$ be any number such that
$2 < \mu' = \mu$. Then

$\mathcal{D}^{\alpha}_{\mu'} (\omega) \geq \alpha$ by theorem 4.1.

Also $\mathcal{D}^{\alpha}_{\mu'} (\omega) \leq \mathcal{D}^{\alpha}_p (\omega)$ by corollary 4.2,
therefore $\mathcal{D}^{\alpha}_{\mu'} (\omega) = \alpha$, thus for some $\mu'$,

$\alpha = \varepsilon^{(3)}_{p, \mu'}$. □ 5.18

Thus:
COROLLARY 5.19 \((\forall \alpha, \beta < \omega_1) \Rightarrow \rho < \alpha \Rightarrow\)

Every \(\epsilon^{(\alpha)}\) number is an \(\epsilon^{(\beta)}\) number. \(\square\)

Thus, for example, it is proved that every epsilon number is also a delta and a gamma number.

Corollary 5.15 characterizes the \(\epsilon^{(\alpha)}\) numbers, for \(\alpha\) a countable successor number. Now the case for \(\alpha\), a countable limit, is dealt with.

THEOREM 5.20 \((\forall \mu, \alpha < \omega_1) \lim(\mu), \gamma \leq \alpha \) let

\(\{ \beta\}_{\beta < \mu} \) be a strictly increasing sequence,

and let \(\{ \epsilon^{(\alpha)}_{\beta}\}_{\beta < \mu} \) be a \(\mu\)-sequence of \(\epsilon^{(\alpha)}\) numbers. Then the number \(\epsilon = \lim_{\beta < \mu} \epsilon^{(\alpha)}_{\beta}\) is an \(\epsilon^{(\alpha)}\) number, of index \(\lim_{\beta < \mu} \beta\).

PROOF. Since the sequence of \(\epsilon^{(\alpha)}\) numbers is strictly increasing, and so \(\{ \epsilon^{(\alpha)}_{\beta}\}_{\beta < \mu} \) and \(\{ \epsilon^{(\alpha)}_{\beta}\}_{\beta < \mu} \) are convergent, in the case \(\text{Suc}(\alpha)\) it follows from theorem 5.13 c)

that \(\lim_{\beta < \mu} \epsilon^{(\alpha)}_{\beta} = \lim_{\beta < \mu} \epsilon^{(\alpha)}_{\beta} = \epsilon^{(\alpha)}_{\lim_{\beta < \mu} \beta}\).
Suffix now \( \lambda \alpha \). Then \( (\forall \mu < \alpha)(\forall \nu < \alpha) \)
\[
\Delta \omega^\alpha (\omega) = \epsilon_p^{(\alpha)}(\omega).
\]

Therefore \( (\forall \mu < \alpha) \)
\[
\Delta \omega \mu^\alpha (\omega) = \lim_{\mu' \searrow \mu} \Delta \omega^\alpha (\omega) \text{ by theorem 4.1a}.
\]
\[
= \lim_{\mu' \searrow \mu} \epsilon_p^{(\alpha)}(\omega).
\]

Furthermore, by the congruence just indicated, this expression denotes the least \( \epsilon^{(\alpha)} \) number greater than any \( \epsilon_p^{(\alpha)} \), \( \mu < \lim_{\mu' \searrow \mu} \nu' \), therefore
\[
\lim_{\mu' \searrow \mu} \epsilon_p^{(\alpha)}(\omega) = \epsilon_p^{(\alpha)}(\omega).
\]

\( \square \) THEOREM 5.20

THEOREM 5.21. \( (\forall \alpha \in \omega) \lim_{\alpha} (\alpha) \Rightarrow \epsilon_0^{(\omega)} = \Delta_\omega^{\omega^2}(\omega) \).

PROOF, by transfinite induction on the limit ordinals smaller than \( \omega \).

Let \( \omega = \omega \). Then \( \epsilon_0^{(\omega)} \) is the smallest ordinal \( \xi < \omega \), such that \( (\forall n < \omega) \Delta_n^\xi (\omega) = \xi \).

Thus \( \epsilon_0^{(\omega)} = \lim_{n < \omega} \Delta_n^\omega (\omega) \) by corollary 5.14
\[
= \Delta_\omega^{\omega^2}(\omega) \text{ by definition 3.1}
\]

But \( (\forall n \in \omega) \Delta_n^\omega (\omega) = \lim_{n < \omega} \Delta_n^{\omega^*}(\omega) \)
by Lemma 4.1d, therefore \( (\forall \alpha^* < \alpha) \)

\[
\mathcal{D}_{\alpha^*} (\omega) = \mathcal{D}_{\alpha^*} \lim_{\alpha' \rightarrow \alpha^*} \mathcal{D}_{\alpha' + T_1 (\omega)} (\omega)
\]

\[= \mathcal{D}_{\alpha} (\omega) \text{ by corollary 5.14, corollary 5.19 and theorem 5.20.}
\]

Therefore \( E_0 (\omega) = \mathcal{D}_{\alpha} (\omega) \).

Now suppose for induction hypothesis that, for some \( \alpha \) such that \( \omega = \alpha < \omega \), and \( \lim (\alpha) \),

\( (\forall \alpha' < \alpha) \lim (\alpha') \Rightarrow E_0 (\omega) = \mathcal{D}_{\alpha^*} (\omega) \).

Let \( T (\beta) \) be the characteristic function of the predicate \( \text{Succ} \), so that

\[
T (\beta) = \begin{cases} 1 & \text{if } \text{Succ} (\beta) \\ 0 & \text{o.w.} \end{cases}
\]

Then \( E_0 (\omega) \geq \lim_{\alpha' < \alpha} \mathcal{D}_{\alpha' + T (\alpha')} (\omega) \text{ by corollary 5.14 and by induction hypothesis} \)

\[= \lim_{\alpha' < \alpha} \mathcal{D}_{\alpha' + 1} (\omega) \text{ by Lemma 4.1d} \]

\[= \mathcal{D}_{\alpha} (\omega) . \]

But \( (\forall \alpha^* < \alpha) \mathcal{D}_{\alpha} (\omega) = \lim_{\alpha' < \alpha} \mathcal{D}_{\alpha' + \alpha^*} (\omega) \text{ by lemma 4.1d, therefore } \( (\forall \alpha^* < \alpha) \)

\[
\mathcal{D}_{\alpha^*} (\omega) = \mathcal{D}_{\alpha^*} \lim_{\alpha' \rightarrow \alpha^*} \mathcal{D}_{\alpha' + T_1 (\omega)} (\omega)
\]
Therefore $\mathcal{D}_{\alpha^*}^\infty(\omega) = \lim_{\alpha' \leq \alpha^*} \mathcal{D}_{\alpha'}^2(\omega)$ by corollaries 5.14, 5.19, and by theorem 5.20, and by the inductive hypothesis.

Thus $\mathcal{D}_{\alpha^*}^\infty(\omega) = \mathcal{D}_{\alpha^*}^2(\omega) = \varepsilon_0(\omega)$. 

[\Box \text{THEOREM 5.21}]

\textbf{LEMMA 5.22} \quad (\forall \alpha \leq \omega_1) \Rightarrow (\forall \alpha' \leq \omega_1) \Rightarrow (\forall \beta \in \varepsilon_{\omega}) \Rightarrow (\forall \alpha' \leq \alpha) \Rightarrow (\forall \alpha' \leq \alpha) \Rightarrow \mathcal{D}_{\alpha'}^{\varepsilon + 1}(\omega) = \varepsilon^{\alpha'}.

\text{PROOF, by transfinite induction. Let } \alpha \text{ be an ordinal in the given range. For } \alpha' = 3, \text{ the equality is true by definition. Suppose it has been proved to hold for all } \alpha' \text{ such that } 3 \leq \alpha' \leq \alpha^* < \alpha, \text{ for some } \alpha^*. \text{ Then if } \text{Suc}(\alpha^*)

\mathcal{D}_{\alpha^*}^{\varepsilon + 1}(\omega) = \mathcal{D}_{\text{Suc}(\alpha^*)}^{\varepsilon + 1}(\omega) \text{ by definition 3.1}

= \mathcal{D}_{\alpha^*}^{\varepsilon + 1}(\omega) \text{ by corollary 5.19, as } \varepsilon \in \varepsilon^{(\alpha^*)} \text{ number}

= \varepsilon^{\alpha'} \text{ by inductive hypothesis.}

If \text{Lim}(\alpha^*), then

\mathcal{D}_{\alpha^*}^{\varepsilon + 1}(\omega) = \lim_{\alpha' \leq \alpha^*} \mathcal{D}_{\alpha'}^{\varepsilon + 1}(\omega)

= \lim_{\alpha' \leq \alpha^*} \mathcal{D}_{\alpha'}^{\varepsilon + 1}(\omega) \text{ by corollary 5.19,}
$\Rightarrow \exists \varepsilon$ is an $\varepsilon$ (a) number, therefore

$$\lim_{\alpha' < \alpha^*} \varepsilon^{\varepsilon_1} (\omega) \Rightarrow \lim_{\alpha' < \alpha^*} (\omega^{\varepsilon_1})_{\alpha'}$$

by induction hypothesis

$$= \omega^{\varepsilon_1},$$

therefore it follows by transfinite induction that $(\forall \alpha' < \alpha)$

$$\lim_{\alpha' < \alpha} \varepsilon^{\varepsilon_1} (\omega) = \omega^{\varepsilon_1}$$

$\Box$ Lemma 5.22

Lemma 5.23 $(\forall \alpha, \varepsilon < \omega_1) (\forall n < \omega) \varepsilon_1$

$\lim_{\alpha} (\alpha)$ $\Rightarrow (\exists \alpha' < \alpha)$ such that:

$$\lim_{\alpha' < \alpha} \varepsilon^{\varepsilon_1} (\omega) > \left\{ \begin{array}{c}
\frac{\alpha' + 1}{\omega} \\
\frac{\omega^{\alpha'} (\omega)}{\omega^{\alpha'} (\omega)} \\
\frac{\omega^{\alpha'} (\omega)}{\omega^{\alpha'} (\omega)} \\
\frac{\omega^{\alpha'} (\omega)}{\omega^{\alpha'} (\omega)} \\
\frac{\omega^{\alpha'} (\omega)}{\omega^{\alpha'} (\omega)}
\end{array} \right\}

where $\Omega$ is an f.s.a.

Proof. Suppose $n \geq 3$, and choose $\alpha' = \omega^{\alpha'} (\omega) + 1$.

Then l.h.s. $> \lim_{\alpha'} \varepsilon^{\alpha' + 1} (\omega)$

$$= \lim_{\alpha'} \varepsilon^{\alpha' + 2} (\omega)$$

$$= \lim_{\alpha'} \varepsilon^{n + 1} (\omega)$$

$$= \lim_{\alpha'} \varepsilon^{n + 1} (\omega)$$

Now suppose again that $\Omega$ is an f.s.a.
LEMMA 5.23a \((\forall \alpha, \varepsilon < \omega_1) (\forall n < \omega)_{n \geq 3} \)
\[ \lim(n) \Rightarrow \]

\[
\begin{array}{c}
\frac{\varepsilon + 1}{k} \\
\frac{\varepsilon + 1}{\Omega^{\omega}_2(n)} \\
\frac{\varepsilon + 1}{\Omega^{\omega}_2(n)} \end{array}
\]

\[
\begin{array}{c}
\frac{\varepsilon + 1}{k + 1} \\
\frac{\varepsilon + 1}{\Omega^{\omega}_2(n)} \\
\frac{\varepsilon + 1}{\Omega^{\omega}_2(n)} \end{array}
\]

PROOF by induction. For \(k = 0\), \(\varepsilon + 1 = \)

\[ \Omega^{\varepsilon + 1}_2(n)(\omega) \], as by theorem 4.12p. \(\Omega^{\varepsilon + 1}_2(n)(\omega)\)

is strictly increasing, and \(\text{Suc}(\text{l.h.s.})\) and \(\text{Lim}(\text{r.h.s.})\) by corollary 4.1a4. Suppose the

inequality of lemma 5.23a has been proved

for all cases of continued exponents of index \(\leq k\),

for some \(k > 0\). Then

\[
\begin{array}{c}
\frac{\varepsilon + 1}{k - 1} \\
\frac{\varepsilon + 1}{\Omega^{\omega}_2(n)} \\
\frac{\varepsilon + 1}{\Omega^{\omega}_2(n)} \end{array}
\]

\[
\begin{array}{c}
\frac{\varepsilon + 1}{k} \\
\frac{\varepsilon + 1}{\Omega^{\omega}_2(n)} \\
\frac{\varepsilon + 1}{\Omega^{\omega}_2(n)} \end{array}
\]

\[ \leq \frac{\varepsilon + 1}{\Omega^{\omega}_2(n)} \]

\[ \leq \frac{\varepsilon + 1}{\Omega^{\omega}_2(n)} \]

by 1.H.
But since (l.h.s), and \( \text{lim}(r.h.s.) \) by theorem 4.1a2, therefore strict inequality holds, and the lemma is proved by induction. \( \square \)

Theorems 5.20, 5.21 and Lemmas 5.22, 5.23, 5.23a enable a characterization of the \( E^{(\omega)} \) numbers for limit \( \alpha \) to be given:

**THEOREM 5.24.** Suppose \( \mu \) is a countable ordinal uniquely represented as \( \mu = \omega \beta + n \), \( n < \omega \). Then

\[
\text{lim}(\alpha) \Rightarrow E_\mu(\alpha) = \begin{cases} 
\Pi_{\alpha}^{(n+1)}(\omega) & \text{when } \mu < \omega \\
\Pi_{\omega}^{\omega \beta + n}(\omega) & \text{when } \omega \leq \mu < \omega,
\end{cases}
\]

**PROOF.** By transfinite induction on \( \mu \). The cases for \( \mu = \omega \) are identical with the general cases given below, except that \( E_\omega^{(\omega)} = \Pi_\omega^{(2)}(\omega) \), and therefore in the cases \( \mu < \omega \), there is an increment of 2 in the exponent of the exponent defining \( E_\mu^{(\omega)} \).

For each \( \alpha \) such that \( \text{lim}(\alpha) \) and \( \alpha < \omega \), suppose the exponent for \( E_\mu^{(\omega)} \) has been proved to hold for all \( \mu' < \mu \), for some non-zero \( \mu < \omega \). Then if \( \text{Suc}(\mu) \) such that \( \mu = \omega \beta + n \):

(1) By theorem 5.21.
\[ \varepsilon_{\alpha'} = \lim_{\alpha' \to \alpha} \left[ \lim_{\alpha' \to \alpha} \left\{ \frac{1}{(\omega^+ + (\omega^+)^2) + 1}, \frac{1}{(\omega^+ + (\omega^+)^2) + 1} \omega, \frac{1}{(\omega^+ + (\omega^+)^2) + 1} \omega, \ldots \right\} \right] \]

\[ = \lim_{\alpha' \to \alpha} \left[ \lim_{\alpha' \to \alpha} \left\{ \frac{1}{(\omega^+ + (\omega^+)^2) + 1}, \frac{1}{(\omega^+ + (\omega^+)^2) + 1} \omega, \frac{1}{(\omega^+ + (\omega^+)^2) + 1} \omega, \ldots \right\} \right] \]

by

Theorem 5.13(b), and by theorem 5.20.

This expansion has the property that \((\forall k \in \omega)(\forall \alpha^* \in \alpha)\)

\[ \frac{k}{(\omega^+ + (\omega^+)^2) + 1} (\omega) \leq \frac{k+1}{(\omega^+ + (\omega^+)^2) + 1} (\omega) \]

\[ \alpha^* \leq \alpha' \Rightarrow \frac{1}{(\omega^+ + (\omega^+)^2) + 1} (\omega) \leq \frac{1}{(\omega^+ + (\omega^+)^2) + 1} (\omega) \]

Thus the double limit expansion can be evaluated by choosing any \(\alpha \) s.t. \(\alpha' \in \alpha^* \) and taking the limit of a diagonal \(\omega\)-sequence (i.e., a sequence whose coordinates \(\omega, \alpha' (\omega)\) decrease ultimately, increasing function of \(\omega, \gamma\) ).
Thus
\[ E_\mu(\omega) = \lim_{x \to \omega} \left\{ \frac{\mu^2}{x^2 - 1} \right\} \]
by Lemma 5.22. Furthermore, by Lemma 5.23, the converse to Lemma 5.23 holds, i.e. that
\((\forall \alpha' < \alpha)\) there exists an \(n < \omega\) such that the
direction of the inequality of 5.23 may be
reversed; therefore
\[ E_\mu(\omega) = \lim_{\alpha' < \alpha} \left\{ \frac{E_{\mu^{(n+1)}}(\omega)}{\mu^{n+1}(\omega)} + 1 \right\} \]
by induction hypothesis, and by Lemma 5.22, therefore
\[ E_\mu(\omega) = \lim_{\alpha' < \alpha} \frac{\mu^2}{\alpha^{n+1}(\omega)} \]
by definition 3.1.
In the case \(\lim(\mu)\), \(E_\mu(\omega) = \lim_{\mu' \to \mu} E_{\mu'}(\omega)\)
\[ = \lim_{\mu' \to \mu} \frac{\mu^2}{\alpha^{n+1}(\omega)} \]
\[ = \alpha^\omega(\omega), \text{ thus theorem} \]
5.24 is proved by transfinite induction. □

The sequence of $\mathbb{E}^\omega$ numbers enumerates the fixed points common to all functions $\gamma_p. \mathbb{E}_\alpha^\omega (\omega)$, $\omega < \alpha$, by definition and by Corollary 5.19.

Thus, taken with Corollary 5.15, theorem 5.24 gives:

$$(\forall \alpha < \omega_1)(\exists \text{ suc } (\alpha)) \Rightarrow$$

$$[ f = \gamma_p. \mathbb{E}_\alpha^\omega (\omega) \Rightarrow f^f = \gamma_p. \mathbb{E}_\alpha^{\omega+1} (\omega)] \wedge$$

$$\alpha > \omega \wedge \text{ lein } (\alpha) \Rightarrow$$

$$[ f = \gamma_p. \lim_{p < \alpha} \mathbb{E}_\alpha^\omega (\omega) \Rightarrow \mathbb{E}_\alpha^{\omega+p+n} (\omega), \text{ for } p = \omega_\beta + n,$$

in the $p$-th fixed point of the function $f$ ]

The function defined here in the limit case is continuous and ultimately increasing, but it is not strictly increasing.

Theorem 5.24 indicates the effect of a slight alteration in definition 3.1. Suppose that in part $(iv)$ of this definition, the fourth segment is replaced by

$$(\forall \alpha < \omega_1)(\forall \beta < \omega_1)_{\beta > \omega} \lim (\alpha) \wedge \text{ suc } (\beta) \Rightarrow \text{ suc } (\beta) \Rightarrow$$

$$\mathbb{E}_\alpha^{\omega+1} (\omega) = \lim_{n < \omega} \mathbb{E}_\alpha^n (\omega),$$

that is, in the cases indicated, the unit increment is omitted from
the exponents of the r.h.s. Call this definition 3.1'.

Now consider the number \( \overline{\omega}^o \omega (v) = \lim_{m \to \omega} \overline{\omega}^o \omega (m) \).

By definition 3.1'. Then

\[
\overline{\omega}^o \omega (v) = \lim_{n \to \omega} \left[ \lim_{m_n \to \omega} \left[ \lim_{m_{n-1} \to \omega} \left[ \lim_{m_{n-2} \to \omega} \left[ \cdots \lim_{m_3 \to \omega} \left[ \lim_{m_2 \to \omega} \left[ \lim_{m_1 \to \omega} \left[ \lim_{m_0 \to \omega} \overline{\omega}^o \omega (m_v) \right] \right] \right] \cdots \right] \right] \right] \right]
\]

Now suppose it has been proved that \((\forall n < \omega) \geq 0\)

\(\overline{\omega}^{n+2} (\omega) = \lim_{m \to \omega} \left[ \overline{\omega}^{n+2} (\omega) \right] \by \text{definition 3.1'} \)

Then

\[
\overline{\omega}^{n+3} (\omega) = \lim_{m_{n+3} \to \omega} \overline{\omega}^{n+2} (\omega) \by \text{definition 3.1'}
\]

\[
= \lim_{m_{n+3} \to \omega} \left[ \lim_{m_{n+2} \to \omega} \left[ \lim_{m_{n+1} \to \omega} \left[ \lim_{m_n \to \omega} \left[ \lim_{m_{n-1} \to \omega} \left[ \lim_{m_{n-2} \to \omega} \left[ \cdots \lim_{m_3 \to \omega} \left[ \lim_{m_2 \to \omega} \left[ \lim_{m_1 \to \omega} \left[ \lim_{m_0 \to \omega} \overline{\omega}^{n+3} (\omega) \right] \right] \right] \cdots \right] \right] \right] \right] \right] \right]
\]

by choosing a diagonal from the double limit. Thus it is proved by induction that:
\[ \tilde{\mathcal{D}}_\omega(w) = \lim_{n \to \omega} \left\{ \frac{\mathcal{D}_n}{\mathcal{D}_{n+1}} (w) \right\} \]

by definition 3.1,

\[ = \mathcal{E}_i(w), \]

by substitution into the expression derived for the \( \mathcal{E}_i \) numbers or \( p \).

\[ = \tilde{\mathcal{D}}_\omega(w), \]

in the notation of definition 3.1,

by theorem 5.24.

A relation is now established between the \( \alpha \)-th derived function of an exponential function, and the \( \mathcal{E}_\alpha \) numbers, as a corollary to the following

THEOREM 5.25. Suppose \( f = \lambda p \cdot \omega^p \). Then

\[ (\forall \alpha, \beta < \omega) \quad f_\alpha (p) = \begin{cases} \mathcal{D}_\beta^{\omega^p} (w) & \text{if} \; \mathcal{E}_\alpha (w) \\ \mathcal{D}_\beta^{\omega^p} (w) & \text{if} \; \mathcal{E}_\alpha (w) \end{cases} \]

PROOF by nested transfinite induction. Let \( \alpha = 1 \).

Then \( (\forall \beta < \omega) \; f_1 (p) = f_\beta (p) = \mathcal{D}_\beta^{\omega^p} (w) \), by theorem 5.11. Suppose for outer inductive hypothesis the relation of (0.1H) that 5.25 is true for all \( p < \omega_1 \), and for all \( \alpha' < \alpha \), for some \( \alpha < \omega_1 \). Suppose for inner inductive hypothesis (i.e., 4.) that \( (\forall p < \beta') \)
for some $p < \omega$,

$$f_\alpha(p) = \begin{cases} \Gamma^{\omega(1+p)}(\omega) \\ \Gamma^{\omega(1+p)}(\omega) \text{ if } \text{Succ}(\alpha) \end{cases}$$

Now, by theorem 5.7, for every normal function $f = f_\alpha$, $(\forall \alpha < \omega)$, $f_\alpha$ is the $\alpha$-th derived function of $f$, according to the formulation of $f_\alpha$ given in definition 5.1a. For $p = 0$:

$$f_\alpha(0) = \begin{cases} \lim_{\alpha' \downarrow \alpha} f_{\alpha'}(0) \text{ if } \text{Succ}(\alpha) \\ \lim_{\alpha' \downarrow \alpha} \Gamma^{\omega^2}(\omega) \text{ by O.I.H.} \\ \Gamma^{\omega^2}(\omega) \end{cases}$$

OR

$$f_{\alpha}^{\omega_0}(0) \text{ if } \text{Succ}(\alpha)$$

$$= \lim_{n \leq \omega} \Gamma^{\omega^2}(\omega), \Gamma^{\omega^2}(\omega), \Gamma^{\omega^2}(\omega), \ldots$$

Now suppose that Succ($\alpha, p$). Then

$$f_\alpha(p) = f_{\omega_0} \{ f_\alpha(P\beta) + 1 \} \text{ by definition 5.1a}$$

$$= f_{\omega_0} \{ \Gamma^{\omega(1+p\beta)}(\omega) + 1 \} \text{ by O.I.H.}$$
Therefore \( f_\alpha(p) \)

\[
= \lim \left\{ \prod_{3+\alpha} \omega_i(1+p_\alpha) \omega \right\}, \quad \prod_{3+p_\alpha} \omega_i(1+p_\alpha) \omega_1, \ldots \}
\]

\[
= \lim \left\{ \prod_{3+\alpha} \omega_i(1+p_\alpha) \omega_1, \prod_{3+p_\alpha} \omega_i(1+p_\alpha) \omega \right\} \ldots \ldots (1)
\]

but

\[
\prod_{3+\alpha} \omega_i(1+p_\alpha) \omega_1 \leq \prod_{3+p_\alpha} \omega_i(1+p_\alpha) \omega
\]

and the suitable induction hypothesis on the

number \( n \omega \) of applications of operation \( \prod_{3+p_\alpha} \)

gives the \( n \)-th term in limit (1) as:

\[
\leq \prod_{3+\alpha} \omega_i(1+p_\alpha) \omega_1 \ldots (\omega)
\]

therefore

\[
f_\alpha(p) \leq \lim \left\{ \prod_{3+\alpha} \omega_i(1+p_\alpha) \omega_1, \prod_{3+p_\alpha} \omega_i(1+p_\alpha) \omega \right\} \ldots \ldots (2)
\]

but by direct comparison of the corresponding terms
of the two non-decreasing sequences of limits (1),
(2), strict inequality cannot hold, thus

\[
f_\alpha(p) = \lim (2) = \lim_{n \leq \omega} \prod_{3+\alpha} \omega_i(1+p_\alpha) + n(\omega)
\]

\[
= \prod_{3+\alpha} \omega_i(1+p_\alpha) + \omega(\omega) = \prod_{3+\alpha} \omega_i(1+p_\alpha)(\omega)
\]
Now suppose $\text{Suc}(\alpha)$ and $\text{Lim}(\beta)$. Then

$$f_\alpha(p) = \lim_{p' \leq p} f_\alpha(p') \text{ by definition 5.1a}$$

$$= \lim_{p' = p} D_{3+\alpha} \omega(1+p') (\omega) \text{ by i, ii, b.}$$

$$= D_{3+\alpha} \omega_{\omega(p')} (\omega) \text{ by corollary 4.1a1}$$

$$= D_{3+\alpha} \omega_{\omega(p')} (\omega), \text{ since } (\forall p' \leq p') 1+p' \leq p' + 1 \leq p$$

$$= D_{3+\alpha} \omega_{\beta} (\omega) \text{ as } \beta = \omega_{\omega(p')} \text{ is normal}$$

$$= D_{3+\alpha} \omega(1+p_{\beta}) (\omega) \text{ as } \beta \text{ is transfinite}.$$ 

Now suppose $\text{Lim}(\alpha, \beta)$. Then

$$f_\alpha(p) = \lim_{p' = p} f_\alpha(p') \text{ by definition 5.1a}$$

$$= \lim_{p' \leq p} D_{3+\alpha} \omega_{2+p'} (\omega) \text{ by i, ii, b.}$$

$$= D_{3+\alpha} \omega_{\beta} (\omega)$$

$$= D_{3+\alpha} \omega_{2+p} (\omega) \text{ as } \beta \text{ is transfinite}.$$ 

Only one case remains: Suppose $\text{Lim}(\alpha)$ and $\text{Suc}(\beta)$. Then

$$f_\alpha(p) = \lim_{d \leq \alpha} f_\alpha, \{ f_\alpha(p_{d}), 1 \} \text{ by definition 5.1a}$$
therefore \( f_\alpha (\beta) \)

\[
= \lim_{\alpha' < \alpha} f_{\alpha'} \left\{ \prod_{3^{\alpha+1}}^{2+\beta} (\omega) + 1 \right\} \text{ by i. i. h.}
\]

\[
= \lim_{\alpha' < \alpha} \prod_{3^{\alpha+1}}^{2+\beta} (\omega) + 1 \text{ by O.1. H.}
\]

\[
= \prod_{2^{3^{\alpha+1}(\omega)}}^{2+\beta} (\omega), \text{ as } \prod_{3^{\alpha+1}+1}^{2+\beta} (\omega) + 1 \text{ is transfinite}
\]

\[
= \prod_{2^{\alpha+1}}^{2+\beta} (\omega).
\]

Thus the case for \( \alpha \) is proved for every \( \beta < \omega \), so the theorem is proved by transfinite induction. \( \square \) 5.25.

**COROLLARY 5.26** \((\forall \alpha, \beta < \omega) \) \( \text{Succ}(\alpha) \) & \( f = \Psi \text{ at } \omega^\beta \Rightarrow f_\alpha (\beta) = \varepsilon_{\beta^{(\alpha+1)}}. \)

**PROOF** follows from Corollary 5.15 \( \square \)
§6. NORMAL FORM
THEOREMS.

The object here of defining normal form expressions is to assign to certain ordinals \( \alpha \geq \varepsilon_0 \) indices smaller than \( \alpha \) such that \( \alpha \) can be expressed arithmetically in terms of these indices.

**Theorem 6.8.** \(( \forall \alpha \in \Pi^2_\varepsilon (\omega) ) ( \exists ! \eta < \omega) \)

\(( \forall i \leq n ) ( \exists ! \alpha_i < \omega_i ) ( \exists ! \alpha < \omega ) ( \exists ! \beta < \alpha )

\[\alpha = \begin{cases} \\
\sum_{i \leq n} \omega^{\varepsilon_i} \cdot \alpha_i & \text{if } ( \exists ! \mu < \omega ) \; \varepsilon_\mu < \alpha < \varepsilon_{\mu+1} \\
\Pi^2_{\varepsilon+\alpha+1} (\omega) & \text{if } \omega^\alpha = \alpha \; \text{(in this case, } \varepsilon = \varepsilon_1, \alpha = \varepsilon_1) \end{cases}\]

**Proof.** The first case holds as \( \alpha \) is simply the Cantor normal form applied to ordinals \( \leq \Pi^2_\varepsilon (\omega) \) which are not \( \varepsilon \)-numbers, as in the example below. The relation in the second case is proved from the fact that if \( \alpha = \Pi^2_{\varepsilon+1} (\omega) \), and \( \beta = \alpha \), then by theorem 4.1 \( \beta \leq \Pi^2_\varepsilon (\omega) \leq \Pi^2_{\varepsilon+1} (\omega) \leq \beta \), therefore \( \Pi^2_{\varepsilon} (\omega) = \beta \), and thus \( \beta = \alpha \) would be an \( \varepsilon \)-number, and the least such number is, by corollary 5.14,

\( \Pi^2_\varepsilon (\omega) \), which contradicts the assumption that \( \alpha \leq \Pi^2_\varepsilon (\omega) \).

\[\Box \quad 6.8\]

Thus, for example if \( \alpha = \varepsilon_0 + \omega^\omega + \omega^5 \), then \( \alpha = \omega^5 + \omega + \omega^5 \)

\( = \omega^\alpha \cdot \alpha_1 + \omega^\alpha \cdot \alpha_2 + \omega^\alpha \cdot \alpha_3 + \alpha_4 \), such that

\( \alpha_1, \alpha_2, \alpha_3 < \alpha \).

Now suppose \( \alpha < \Pi^2_\varepsilon (\omega) \) and \( \lim (\alpha) \). Then a fundamental sequence \( \{ \delta^\alpha (x) \}_{x=\omega} \) for \( \alpha \) can be inductively defined by:

---

\[\delta^\alpha (x) = \begin{cases} \\
\varepsilon \text{ if } x = \omega \\
\delta^\alpha (x-1) + \omega^\alpha & \text{if } x < \omega \\
\end{cases}\]

---
\[ \Omega_{\alpha}(x) = \begin{cases} 
\sum_{i=1}^{n-1} \omega^a_i a_i + \omega^a_\infty (a_n-1) & \text{if } \alpha < \omega, \\
\omega^a_\infty \text{ if } \text{Suc}(\alpha), \\
\Omega_{\alpha_n}(x) \text{ if } \text{Lim}(\alpha_n) & \text{otherwise}.
\end{cases} \]

If \( \alpha \) is of the form \( \alpha = \Omega^{\omega^a + \beta}(\omega) \). 

THE NUMBER \( \Theta_0 \).

In expressing an ordinal \( \alpha \) arithmetically, in terms of smaller indices, certain operations have been used, each of which belongs to the class of functions \( \Omega^{\omega^a}(\omega) = \Omega^{\omega^a} \) of definition 3.1, and is thus indexed by an ordinal. 

E.g. \( \omega^{\alpha_1} = \Omega^{\omega^{\alpha_1} \omega}(\omega) = \omega \ominus \alpha_1 \). Arbitrary operations \( \oplus \) might be applied to obtain a normal form, but only those whose index \( \alpha \) is smaller than a certain fixed upper bound \( \Theta_0 \) will ensure.
a proper decomposition of a segment of the ordinals \(<\omega_1\),
in fact the ordinals smaller than \(\theta_\omega\).
\(\theta_\omega\) is the least fixed point of \(\varphi\). \(\varphi^\omega_\omega(\omega)\).
Let \(\tilde{\theta}_1 = \omega\)
\(\tilde{\theta}_{n+1} = \varphi^\omega_\omega(\omega)\)
\(\theta_\omega = \lim_{n<\omega} \tilde{\theta}_n\).

By theorem 4.16, \(\varphi\) is strictly increasing, therefore since \(\tilde{\theta}_1 = \omega < \varphi^\omega_\omega(\omega) = \tilde{\theta}_2\), and
\((\forall n < \omega) \tilde{\theta}_n < \tilde{\theta}_{n+1} \Rightarrow \tilde{\theta}_{n+1} < \varphi^\omega_\omega(\omega) = \tilde{\theta}_{n+2}\), it follows
by induction that the sequence \(\{\tilde{\theta}_n\}_{n<\omega}\) is strictly increasing. Furthermore,

\(\varphi^\omega_\omega(\omega) = \lim_{n<\omega} \tilde{\theta}_n\)
\(= \lim_{n<\omega} \varphi^\omega_\omega(\omega)\) by definition 3.1
\(= \lim_{n<\omega} \tilde{\theta}_{n+1}\)
\(= \tilde{\theta}_\omega\).

\(\theta_\omega\) is the least such fixed point; for suppose
otherwise, such that \((\exists \theta' < \Theta_0) \; I_{\theta'}^2(\omega) = \theta'\).
Then \(\theta' \leq \omega\) or \((\exists n < \omega) \; \bar{\theta}_n < \theta'.\) Clearly \(\omega < \theta'\), so suppose for some \(n\) such that \(1 < n < \omega\), \(\bar{\theta}_n < \theta'.\) Then

\[ I_{\bar{\theta}_n}^2(\omega) = \bar{\theta}_{n+1} \]

\[ < I_{\bar{\theta}_n}^2(\omega), \text{ by theorem } 4.1b \]

\[ = \theta', \text{ so by induction } (\forall n < \omega). \]

\(\bar{\theta}_n < \theta'\), therefore \(\Theta_0 < \theta'\), a contradiction .

The notation \(\{\theta_{\mu}\}_{\mu < \omega}\) is adopted for the sequence of countable solutions, ordered according to magnitude, of the equation

\[ I_{\theta}^2(\omega) = \theta. \]

By theorem 4.1b, \(\Theta_0\) is greater than any \(I_{\alpha}^2(\omega)\), \(\alpha < \Theta_0\), and is, thus, by corollary 5.14, greater than any \(\varepsilon_{0}^{\omega}\)-number, for \(\alpha < \Theta_0\).

The process of applying \(\varepsilon_{\alpha}\)-numbers to the determination of normal form decompositions of ordinals \(\alpha\), for \(p = \alpha < \Theta_0\) is now considered .
DEFINITION 6.8a. \((\forall \alpha < \omega_1) \alpha \text{ is an exact } \xi^p - \text{number iff } \alpha \text{ is an } \xi^p - \text{number and } \forall \mu < \omega_1 \exists \beta \alpha \leq \Sigma^\beta_\mu (\omega). A \text{ number which is an exact } \xi^{(\omega)} - \text{number is called an exact epsilon number.} \Box 6.8a

In general, for each \(p\) such that \(1 < p < \omega\), there are many exact \(\xi^p - \text{numbers, and by theorem 4.1b, the smallest } \xi^p - \text{number is always an exact } \xi^p - \text{number. Thus in the case } p = 4, the numbers } \xi_0, \xi_1 = \lim \{\xi_0 + 1, \omega^{\xi_0 + 1}, \ldots\} \text{ are exact epsilon numbers. The smallest epsilon number which is not exact is}

\[
\xi_0^{(5)} = \Sigma^2_4 (\omega) = \Sigma^2_4 \omega^3 (\omega)
= \Sigma^2_4 \omega^2 (\omega)
= \xi_0^{(4)}.
\]

THEOREM \((\forall \mu < \omega) \xi_\mu \text{ is exact iff } \mu \text{ is not an } \xi^{(\omega)} - \text{number, for some } \xi \text{ such that } 4 < \xi < \omega. \)

PROOF. Suppose \(\mu = \xi_\alpha (\omega)\), for some \(\alpha, \beta < \omega\), such that \(4 < \alpha\). Then
\[ e_\mu = I_\mu^{\omega + e_\mu^\omega}(\omega) = I_\mu \varepsilon_\mu^\omega(\omega) = \varepsilon_\mu^\omega, \text{ as } 4 < \alpha \text{ and by theorem 5.18. Therefore } e_\mu \text{ is not exact.} \\
\]
Now suppose \( e_\mu \) is not exact. Then (3) \( \exists \alpha, \mu < \omega_1, \delta > 4 \)
\[ e_\mu = \varepsilon_\mu^\omega = I_\mu \varepsilon_\mu^\omega(\omega), \text{ as } 4 < \alpha \]
\[ = I_\mu^{\omega(1 + e_\mu^\omega)}(\omega) \]
\[ = \varepsilon_\mu_{\mu}^\omega, \text{ therefore } \mu = \varepsilon_\mu^\omega \quad \square \]

Note that \((\forall \varepsilon < \omega, \exists \text{ an exact } \varepsilon^\omega \text{ number } \Rightarrow \alpha \text{ is unique.} \) For suppose otherwise, such that \( \varepsilon \) is an exact \( \varepsilon^\omega \), \( \varepsilon^{\alpha'} \) number, where \( \alpha \neq \alpha' \).

Then without loss of generality suppose \( \alpha' < \alpha \). Then by definition \( \varepsilon \) cannot be an exact \( \varepsilon^{\alpha'} \) number.

The following lemmas are used:

**Lemma 6.9** \((\forall \alpha < \theta_0)(\exists! \mu < \omega_1, \delta > 4) \varepsilon_\mu < \alpha [\alpha < \omega^\alpha \text{ or } \alpha \text{ is an exact } \varepsilon^\mu \text{ number}]\)

**Proof.** Suppose \( \alpha < \theta_0 \), and that \( \alpha \) is an \( \varepsilon \)-number.

Let \( \{ \eta \mid \eta < \eta \} \) be the strictly increasing sequence of countable ordinals such that \( I_\alpha \eta^\omega(\omega) = \alpha \).
Then by theorem 5.18, each ordinal \( \alpha \) greater than 3\( \nu_0 \) and less than some \( \lambda \), is such that 

\[
\exists \alpha \in \lambda. \quad \varpi^\alpha(\omega) = \alpha,
\]

and so \( \{ \lambda \eta \}_{1 < \eta < \gamma} = \{ \lambda \} \subseteq \lambda \), where \( \lambda = \text{lin} \lambda \eta \). Suppose \( \gamma > 2 \) is a successor number. Then by definition of \( \lambda \), \( \lambda \) is an \( \varepsilon(\gamma + 1) \)-number, and by theorem 4.1c, \( \gamma \leq \alpha \).

Suppose \( \gamma \) is a limit number. Then, since for all \( \lambda < \gamma \), \( \lambda \) is an \( \varepsilon(\lambda + 2) \)-number by definition 5.12a, it is true that \( (\forall \lambda' \leq \lambda_\gamma) \lambda_\gamma \subseteq \lambda \) is an \( \varepsilon(\lambda + 2) \)-number, and therefore by theorem 5.20, \( \lambda \) is an \( \varepsilon(\lambda + 1) \)-number. But by definition of \( \gamma \), \( (\forall \lambda' \leq \lambda_\gamma) \lambda_\gamma \subseteq \lambda \) is an \( \varepsilon(\lambda + 1) \)-number, and by theorem 4.1c, \( \gamma \leq \alpha \).

**Lemma 6.10.** The least solution \( \gamma \) to the equation 

\[
\varepsilon_0(\alpha) = \gamma
\]

is the number \( \theta_\alpha \).

**Proof.** Clearly \( \omega < \gamma \). Firstly, suppose \( \text{Suc} (\gamma) \). Then \( \varepsilon_0(\alpha) = \varpi^\nu \omega (\omega) = \varpi^\nu (\omega) \geq \gamma + 1 \) by theorem 4.1c,
therefore \( \gamma < \varepsilon_0(\lambda) \), and so \( \gamma \) cannot be a successor number.

But for \( \lim (\lambda) \), by definition \( 5.12a \), \( \gamma = \varepsilon_0(\lambda) \Rightarrow \gamma = (\mu^\lambda) \left[ (\forall \lambda < \gamma) \Rightarrow \varepsilon_0(\lambda) = \mu \right]. \) Therefore

\( \gamma = \varepsilon_0(\lambda) \), by theorem 5.21

\( = \varepsilon_0(\lambda) \). Thus \( \gamma = \theta_0. \) \( \square \) \textbf{LEMMA 6.10}

Returning now to the proof of Lemma 6.9, suppose \( \alpha \) is an exact \( \varepsilon^\langle \mu \rangle \)-number, such that \( \alpha = \varepsilon^\langle \mu \rangle_\beta \), for some \( \beta < \omega_1 \), and also suppose \( \mu = \alpha \).

Then \( \beta > 0 \Rightarrow \varepsilon^\langle \mu \rangle_\beta > \varepsilon^\langle \mu \rangle_0 \geq \mu = \alpha \), therefore \( \beta = 0 \).

Thus, since \( \alpha = \varepsilon^\langle \mu \rangle_\beta \), and \( \alpha < \theta_0 \) by assumption, then by Lemma 6.10, \( \mu < \alpha \). Furthermore, by the remark on \( \beta \), \( (\forall \alpha < \theta_0) \alpha \) is an exact \( \varepsilon^\langle \mu \rangle \)-number \( \Rightarrow \mu \) is unique.

\( \square \) \textbf{LEMMA 6.9}

\textbf{LEMMA 6.10} a. Suppose \( \alpha < \omega_1 \), is an exact \( \varepsilon^\langle \mu \rangle \)-number, such that \( \alpha = \varepsilon^\langle \mu \rangle_\beta \), for some \( \mu, \beta < \omega_1, \mu \geq 2 \). Then \( \beta \leq \alpha \).
PROOF. Suppose otherwise, so that $\alpha = \varepsilon_\omega^{(n)}$. Then by corollary 5.15 and by theorem 5.24

$$\alpha = \begin{cases} 
\omega \omega^{(n)}(\omega) & \text{if } \text{Suc}(\mu) \\
\omega^\alpha(\omega) & \text{if } \text{Lim}(\mu), \text{ so } \alpha \text{ is a limit number.}
\end{cases}$$

But as $\omega^\alpha = \alpha$, $\omega^{(n)}(\omega) = \omega^\alpha(\omega)$, therefore in each case $\alpha = \omega^\alpha(\omega) \Rightarrow \alpha$ is an $\varepsilon^{(n+1)}$-number, contrary to weakness assumption.

Therefore $p < \alpha$. □

**LEMMA 6.10a**

Lemmas 6.9, 6.10a enable a normal form expression to be assigned to each ordinal number smaller than $\theta_0$.

**THEOREM 6.11**

\[
\begin{align*}
& (\forall \alpha < \omega_1) 
& (\exists ! n < \omega) 
& \left( \forall i \leq n \right) 
& \left( \exists ! \alpha_i < \omega \right)_{1 \leq i < n} \\
& (\exists ! \alpha_n < \omega) 
& (\exists ! p, \mu < \omega_1)_{0 \leq n}
\end{align*}
\]
\[ \alpha = \begin{cases} 
\sum_{i=1}^{\omega} \omega^{a_i} \alpha_i & \text{if } \alpha < \omega \\
\text{otherwise} & \\
\Xi (\mu, \beta, n) & \text{if } \mu \in L(\alpha) \\
\Xi (\mu, \beta, n) & \text{if } \mu \in L(\alpha) \\
\Xi (\mu, \beta, n) & \text{if } \mu \in L(\alpha) \\
\end{cases} \]

where

\[ \Xi (\mu, \beta, n) = \begin{cases} 
\Pi (\mu, \beta + n + 1) & \text{if } \mu \leq \beta + n + 1 \\
\Pi (\mu, \beta + n + 1) & \text{if } \mu \leq \beta + n + 1 \\
\Pi (\mu, \beta + n + 1) & \text{if } \mu \leq \beta + n + 1 \\
\end{cases} \]

and where \( \mu \) is the unique ordinal smaller than \( \alpha \) such that \( \alpha \) is an exact \( \varepsilon(\mu) \)-number.

**Proof.** The first clause follows immediately from the Cantor normal form and the definition of the \( \varepsilon \)-numbers. The relations of the remaining clauses follow from corollary 5.15, and from theorem 5.24. The strict inequalities quantifying the indices follow from lemmas 6.9, 6.10a.

**Theorem 6.11.**

No \( \omega \)-sequence \( \alpha \) is definable.
the following scheme, where \( \alpha = \varepsilon_1^\beta \), such that \( \alpha \) is an exact \( \varepsilon_\beta \)-number, if \( \alpha \) is an epsilon number:

\[
\Omega_\alpha(x) = \begin{cases} \\
\sum_{i=1}^{\gamma_1} \omega x_i + \omega x_n (\alpha_n - 1) & \text{if } \alpha < \omega_\beta \\
\begin{cases} \\
\Pi_{\beta+1}^\omega(x) & \text{if } \beta < \omega \\
\Pi_{\beta+1}^{\omega+1}(x) & \text{if } \text{Succ}(\beta), \omega < \beta \\
\Pi_\beta^{\omega \Omega_\beta(x)}(x) & \text{if } \text{Lim}(\beta) \\
\end{cases} & \text{if } \text{Succ}(\beta), \text{otherwise} \\
\end{cases}
\]

Where

\[
\Xi(p, \beta', n) = \begin{cases} \\
\Xi_{\alpha_1}^\omega(x) & \text{if } n = 0 \\
\Xi_{\alpha_1}^{\omega+1}(x) & \text{if } 0 < n < \omega \\
\Xi_{\alpha_1}(x) & \text{when } p = n < \omega \\
\Xi_{\alpha_1}^{\omega+\omega+p+1}(x) & \text{when } p = \omega + n, 0 < n, \beta' \\
\Xi_{\alpha_1}(x) & \text{when } \text{Lim}(p) \\
\end{cases}
\]

and where \( p = \omega \beta' + n, n < \omega \).
CONCLUSION

With regard to the unsolved problem stated in § 2, it is clear that the class \( \mathcal{D}^{(a)} = \mathcal{D}^{(a)} \), or indeed any class containing only countably many normal functions, will not provide a means of expressing an arbitrary limit number \( \gamma \in \mathcal{Z}(5) \) as an arithmetical expression containing ordinal indices \( \alpha \) all of which satisfy \( \alpha < \gamma \).

Consider however, the class \( \mathcal{D}^{(0)} \), for some fundamental sequence assigning \( \mathcal{S} \). This class is extensive by comparison with \( \mathcal{D}^{(0)} \), the subclass of its members which are primitive recursive, since \( \mathcal{S} \) enables new functions to be defined within \( \mathcal{D}^{(0)} \) by diagonalization processes.

The class \( \mathcal{D}^{(a)} \) may be viewed as providing a framework of primitive recursive functions \( \mathcal{D}^{(b)} : \omega \rightarrow \omega \), and this view suggests the possibility of defining a class \( \mathcal{D}^{(b)} \) which contains the members of \( \mathcal{D}^{(a)} \) and which also contains functions defined on \( W \), which are defined by diagonalization processes of type \( \omega \), these processes being enabled by a fundamental sequence assigning \( \mathcal{S}^{*} \) defined for limit numbers of \( \mathcal{Z}(5) \). (c.f. Boehm [3]).

Such a notion perhaps suggests that the problem as it applies to the second number class is unlikely to become resolved in isolation.
APPENDIX I

THE ACKERMANN FUNCTION.

\[(\forall n, x, y < \omega)\]
\[g_0(x, y) = y + 1\]
\[g_1(x, y) = x + y\]
\[g_2(x, y) = x \cdot y\]
\[g_{n+3}(0, y) = 1\]
\[g_{n+3}(1, y) = y\]
\[g_{n+3}(2, y) = g_{n+2}(y, y)\]
\[g_{n+3}(x+1, y) = g_{n+2}(g_{n+3}(x, y), y)\]

More than the minimum number of equations are given here, in order to show the initial behaviour of the function.

\[(\forall n < \omega) \Rightarrow x, y \cdot g_n(x, y) \text{ is primitive recursive,}\]

but \(\forall n, x, y \cdot g_n(x, y) \text{ is not. cf. Fefer [15].}\)
APPENDIX II

EXPONENTIAL NOTATION.

It is sometimes convenient to write out the expression \( \prod_{d}^{b} (\varphi) \), when \( b \) is a successor number, in repeated exponential form. For example

\[
\prod_{d}^{b} (\varphi) = \prod_{d}^{b-1} (\varphi) (\varphi) = \prod_{d}^{(\varphi_{d})(\varphi_{d})} (\varphi) (\varphi) (\varphi) \]

where \( n \) is such that \( 1 \leq n \leq \omega \), and \( b \) of the form \( b = b' + n + 2 \).

The indices \( (\varphi_{i}) \), \( 1 \leq i \leq n \), are applied only in this instance, to indicate which operands \( d \) one to be paired with which indices \( d \).
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INDEX

Canonical sets 8
Cantor normal form 32
class $\mathcal{D}$ 52
Class $\mathcal{E}$ 35
Class $\mathcal{N}$ 109
Diagonalization 45
Delta numbers 144
Epsilon numbers 32
Exponent 37
Exponential notation 183
Exact $\varepsilon^{(p)}$-numbers 175
Fixed points 13
Function 13
Ackermann 22, 182
normal 13, 124
$\varepsilon^{(\alpha)}$-derived 126
elementary 17
Fundamental sequence 30
assignment 31
Grzegorczyk hierarchy 44
Index 37
Inverses 69
Leading term 112
Majorization 65
Normal form expression 171, 179, 180
$\varepsilon^{(\omega)}$-numbers 145
Perfect f.s.a. 53
Operand 37
Ordinal number 7
Schemata R
Second number class 10
Segment 9
Theta numbers 174
Ultimately increasing sequence 11
Unique form 36, 37