THE FARADAY EFFECT

IN SEMICONDUCTORS.

by

JANET WEBSTER, B. Sc.

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Equations for the Faraday rotation and ellipticity in an anisotropic material are derived in terms of quantities related to the components of the high frequency magnetoconductivity tensor. The theory is valid for all orientations of $\mathbf{H}$, (the magnetic field), and $\mathbf{E}$, (the electric vector of the incident radiation), with respect to the crystal axes. The magnetoconductivity tensor is calculated for the [111] and [100] ellipsoid band models of a cubic semiconductor, assuming the scattering to be isotropic. The treatment is semiclassical, being based on the solution of the Boltzmann equation for a single valley which is an ellipsoid of revolution. It is convenient to consider axes such that $\mathbf{H}$ is along one axis. The tensors are therefore transformed to systems in which $\mathbf{H}$ lies in a (100) or (110) plane, and $\mathbf{E}$ takes any orientation in the plane perpendicular to the field. The tensor for the isotropic model having spherical surfaces of constant energy is obtained as a special case. The series expansion for the current density in powers of $\mathbf{H}$ is also developed to $O(H^2)$, and the associated tensor coefficients evaluated for $\mathbf{H}$ in the (100) and (110) planes, as for the closed solution. Detailed calculations are made of the frequency, field and temperature dependence of the rotation and ellipticity in two specimens of n-type germanium, assuming the isotropic model. One is near intrinsic and acoustic phonon scattering is considered, while the other is moderately doped and ionized impurity scattering is assumed. Calculations are also made for the lattice scattering specimen only, using the [111] ellipsoid band model. An analysis is made of the anisotropy of the rotation and ellipticity as a function of the orientation of $\mathbf{H}$ and $\mathbf{E}$, in addition to a consideration of the field and frequency dependence. Approximate expressions are derived and discussed in certain regions. Finally some calculations are presented for the same specimen, using the series expansion, and a comparison of the two treatments is made.
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CHAPTER 1

Introduction

The Faraday effect is the rotation of the electric vector which is observed when a plane polarized electromagnetic wave is passed through a material medium, (which may be solid, liquid or gaseous), in the presence of a static longitudinal magnetic field. It was first observed in glass by Faraday in 1845 and subsequently in many more substances by other workers. Excluding optically active materials, the rotation $\mathbf{\hat{R}}$ was found to be approximately proportional to the magnetic field $\mathbf{H}$ and the path length $d$. The Verdet constant $V$, a characteristic of a particular substance, was defined in the following way

$$\mathbf{\hat{R}} = VHd.$$ (1.1)

The rotations were of the order of $10^{-1}$ to $10^{-2}$ degrees cm$^{-1}$ oersted$^{-1}$, except in ferromagnetics where very large effects of the order of $10^{5}$ degrees cm$^{-1}$ were observed under conditions of saturation magnetization.

The earliest theoretical analysis was based on the Lorentz treatment of the particles, (generally electrons), as a system of classical oscillators which are set into forced vibration by the radiation of frequency $\omega$. Classical electrodynamics then gives an expression for the complex refractive index (Moss, 1959, chapter 2). The effect of a magnetic field is easily included in the single particle equation of motion by adding the magnetic Lorentz force term, and the solutions show that the linear component of the oscillatory motion in the plane perpendicular to $\mathbf{H}$ is split magnetically into two counter-rotating circularly polarised components. At low fields these have apparent frequencies $\omega \pm eH/2mc$ respectively (for particles of mass $m$), i.e. they are separated by twice the Larmor precession frequency. Thus two modes having slightly different refractive indices are propagated. Large absorption occurs when the external frequency is equal to a natural frequency of the system and in general,
away from such regions, a rotation of the plane of polarization is observed in the emitted radiation when the two modes are recombined. A full account of this approach was given by Moss (1959, chapter 5), except that the damping term was not included. The free electron case can be considered as the limit in which the natural frequency of the system becomes zero.

Most of the work on solids prior to about 1950, whether theoretical or experimental, was concerned either with the Faraday effect due to bound electrons in insulators and ionic crystals analyzed in terms of the Lorentz theory, or with ferromagnetics where the internal field concept had to be invoked to explain the large rotations. The free electron contribution in non-ferromagnetic metals was not of much interest because of the very high metallic absorption. This necessitates the use of very thin samples which produce correspondingly small rotations. The magneto-Kerr effect (polar reflection Faraday effect), the analogue in reflection of the Faraday effect in transmission, is also so small as to be unmeasurable with good accuracy until recently (Stern, McGroddy and Harte, 1964) and, since it is only present because of the differing reflectivities of the two modes, is given by a more complex formula (Donovan and Medcalf, 1963).

The attention given to semiconductors has begun to compare favourably with that accorded to metals only in comparatively recent times, and thus the possible importance of Faraday rotation measurements of the free electron effect in these materials was not considered until the mid nineteen fifties. Following the first measurement of the resonant absorption of one of the circularly polarized components when the radiation frequency is equal to the cyclotron frequency of the electrons (or holes) spiralling round the magnetic field direction (Dresselhaus, Kip and Kittel, 1953), it was natural that other effects associated with the dispersion around the cyclotron frequency, of which the Faraday effect is one, (others are the magneto-reflection and magnetoabsorption), should receive some attention. The suggestion that effective masses could be measured directly in the high frequency
when the radiation frequency greatly exceeds the relaxation frequency of the free carrier system, was first made by Mitchell (1955). He obtained the well known formula showing that $\beta \propto \lambda^2$ from the dispersion relation. The first measurements were reported at the Rochester Semiconductor Conference by Smith, Moss and Taylor (1958) on indium antimonide. Later (1959) they gave a fuller account of the work. The energy wave vector relation was deduced from the dependence of $m^*$ on the electron doping level. Work on other materials soon followed; for example Moss and Walton in three separate papers in 1959 reported measurements on gallium arsenide, indium phosphide and n-type germanium, and Austin (1960) reported results for the hexagonal crystal bismuth telluride. Where a comparison with values obtained by independent methods was possible, as for n germanium, the agreement was usually good. More recently workers have sometimes been concerned with other aspects, such as the dependence of $m^*$ on both temperature and carrier concentration for indium antimonide and gallium arsenide (Cardona, 1961), and combined Faraday and Voigt measurements in indium arsenide and indium antimonide (Palik, Teitler and Wallis, 1961 (two papers)). The method is now accepted as a useful way of obtaining information on $m^*$, particularly if cyclotron resonance measurements are difficult or impossible. The microwave effect has been less exploited, probably because of the greater technical difficulties and less direct information obtainable. Rau and Caspari (1955) reported the first measurement on n-type germanium. This was not followed up until the work of Furdyna and Broersma (1960) and Bouwknecht and Volger (1964), both groups again using n germanium.

The theoretical treatment of the effect was placed on a more satisfactory footing by Stephen and Lidiard in 1958. These authors derived the field components for the two independent modes in the presence of a magnetic field, together with the associated refractive indices, by solving the appropriate Maxwell equations for the medium, substituting the complex field dependent current density $J$. The unrealistic model of Lorentz oscillators producing a nett electric
polarization of the material was therefore replaced by a calculation of the electric field components in an electromagnetic wave. The time of relaxation was assumed to be a constant independent of the electron energy. It was shown that the current density could either be obtained from the equation of motion for a classical quasi-free electron (Hau and Caspari, 1955) or, more generally, from the electron distribution function satisfying the high frequency Boltzmann equation. For this latter method the analysis of Stephen and Lidiard was limited to low magnetic fields such that the current density can be written (Abeles and Meiboom, 1954),

\[
J_i = \sum_{ij} E_j + \sum_{ijk} E_j \mathbf{H}^k
\]

where \(\sum_{ij}\) and \(\sum_{ijk}\) are frequency dependent tensor components and \(E_j\) electric field components. Also, only the special case of cubic symmetry for which \(\sum_{ij}\) and \(\sum_{ijk}\) can be shown to be independent of the orientation of the magnetic field with respect to the crystal axes was considered. The exact conductivity tensor has the special form for which their analysis was made only when \(\mathbf{H}\) is along [100] or [111] directions in cubic crystals or along the c axis in hexagonal crystals. The Boltzmann equation approach is an improvement on the equation of motion treatment in that a non isotropic band structure may be considered, leading to the replacement of the single effective mass associated with spherical surfaces of constant energy which was introduced into the equation of motion, by an appropriate function of the energy band parameters. For spherical surfaces the two methods give the same result, though the Boltzmann equation is clearly more general, and the method is capable of refinement to include for example non isotropic crystal symmetries with non spherical surfaces of constant energy, and an energy dependent relaxation time. These features cannot be covered by the equation of motion method. Stephen and Lidiard presented calculations for spherical energy surfaces, and used the [111] ellipsoid model of n-type germanium only to derive the high and low frequency limits for \(H\).
In October 1959 when the work reported here was begun, the earliest microwave and infra-red measurements had been made and the Stephen and Lidiard theory published. Two possible lines of progress were considered. One was the incorporation into the Stephen and Lidiard theory of a more realistic relaxation mechanism in which \( T \), the relaxation time, was a function of the electron energy, the current density being derived from the distribution function in order to make this extension possible. Systematic calculations of the rotation, and also of the Faraday ellipticity, as a function of the scattering mechanism, frequency, field and temperature were carried out. An account of this work is given in chapter five and Donovan and Webster (1961). The second development was the setting up of a theory of the Faraday effect in an anisotropic material, assumed to be described by a magnetoconductivity tensor not simplified by symmetry requirements. The independent modes are found to be elliptically polarized and the equations for the field components, refractive indices and rotation and ellipticity are considerably more complicated than in the isotropic situation analysed by Stephen and Lidiard, which is a special case of the new theory when the tensor has a particular symmetry. In particular cubic symmetry was considered, with the magnetic field orientated in various crystallographic planes. This work is set out in chapter two and in Donovan and Webster (1962 (two papers, the second correcting minor algebraic errors)). As an application of the theory, the high frequency magnetoconductivity tensor was derived for the [111] ellipsoid model of germanium by extending the work of Abeles and Meiboom (1954), and many numerical results calculated. This aspect is treated in chapters three and six and in Donovan and Webster (1963). Also included (chapters four and seven) is the generalisation to high frequencies of the series expansion for \( J \) including the \( R^2 \) term, to which order the current density is no longer isotropic in cubic materials, together with numerical results for this approximation.

Finally, in chapter eight, the limitations of the work are discussed, and some indication of developments which are either
based on the present work, or may be made in the future on the basis of the new theories, is given.
CHAPTER 2

Theoretical Treatment of Faraday Rotation and Ellipticity

The expressions derived in the present chapter are valid for semiconductors at all frequencies and fields, subject to the restrictions discussed in chapter eight. The conditions under which they may be applied to other materials, e.g. ferromagnetics, are also discussed briefly in chapter eight.

2(a) The Current Density

We define a system of cartesian axes \((x,y,z)\) such that the incident plane polarised radiation is propagated along the \(z\) direction, and the electric vector of this radiation is initially parallel to the \(x\) direction. The external magnetic field \(\mathbf{H}\) then has components \((0,0,\mathbf{H}_z)\) and the electric vector \(\mathbf{E} (\mathbf{E}_x, 0, 0)\).

It will be assumed in this chapter that the second order high frequency magnetoconductivity tensor \(S_{jk}(\mathbf{H}, \omega)\), referred to the \((x,y,z)\) axes, is known. This calculation can be carried out in a number of ways, and is discussed in detail in chapters three and four. When the \(S_{jk}\) are calculated for an arbitrary orientation of \(\mathbf{H}\) with respect to the principal axes of the crystal, these being denoted by \((1,2,3)\), the nine components are, in general, all different. Considerable simplifications in the theory result from the relations between the tensor components which arise when particular lattice symmetries, and field directions with respect to the lattice, are considered. The number of independent components in the simplest case is three, and intermediate situations are also of considerable interest. These special cases are dealt with in 2(e), and in the earlier sections it is assumed that the nine components of \(S_{jk}(\mathbf{H}, \omega)\) are independent. The displacement or polarisation current can be represented by a polarisation tensor, \(\mathbf{E}_{jk}(\omega)\), whose symmetry properties are determined by the lattice symmetry.
The conduction current density $J$ and the displacement current density $D$ can always be written as

$$
\begin{align*}
J_j &= S_{jk} E_k \\
D_j &= \varepsilon_{jk} E_k
\end{align*}
$$

The equations are referred to the $(x,y,z)$ system, and $(j,k)$ are summed over $(x,y,z)$. Terms of higher order in $\xi$ are negligible for the electric fields normally associated with electromagnetic radiation.

2(b) Solution of Maxwell's Equations

Wave propagation in the material is governed by Maxwell's equations. Using Gaussian units these are

$$
\begin{align*}
\nabla \times \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} \\
\nabla \times \mathbf{H} &= \frac{4\pi}{c} \mathbf{J} = \frac{1}{c} \left[ 4\pi \mathbf{J} + \frac{\partial D}{\partial t} \right]
\end{align*}
$$

$\mathbf{H}$ is the magnetic field of the electromagnetic wave and $\mathbf{J}$ the total current density. The magnetic permeability is set equal to unity for a semiconductor. On eliminating $\mathbf{H}$ (2.2) becomes

$$
\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c^2} \left[ 4\pi \frac{\partial \mathbf{J}}{\partial t} + \frac{\partial^2 D}{\partial t^2} \right]
$$

The useful solutions of this equation are complex plane waves, depending on the $z$ co-ordinate alone for propagation in the $z$ direction. They can be written

$$
\mathbf{E}(\omega, z) = \mathbf{E}_0(\omega, z) \exp(i\omega t - \mu z),
$$

where $\mathbf{E}_0(\omega, z)$ is an amplitude term which may be complex, and $\mu = \alpha + i\beta$ is the complex propagation constant, whose real part determines the attenuation, and imaginary part the phase of the wave. (Some authors use $K$ where $\mu = iK$.) $z$ is measured from the front surface of the specimen. Since derivatives with respect to $x$ and $y$ are zero, (2.3) yields the following three equations
in component form on substituting the solution (2.4)

\[ \frac{\partial^2 \mathbf{E}_x}{\partial t^2} = \frac{1}{c^2} \left[ \frac{4\pi \partial \mathbf{J}_x}{\partial t} + \frac{\partial^2 \mathbf{D}_x}{\partial t^2} \right] \]

\[ \frac{\partial^2 \mathbf{E}_y}{\partial t^2} = \frac{1}{c^2} \left[ \frac{4\pi \partial \mathbf{J}_y}{\partial t} + \frac{\partial^2 \mathbf{D}_y}{\partial t^2} \right] \]

\[ 0 = \frac{1}{c^2} \left[ \frac{4\pi \partial \mathbf{J}_z}{\partial t} + \frac{\partial^2 \mathbf{D}_z}{\partial t^2} \right] \]  

(2.5)

The components \( \mathbf{D}_j \) and \( \mathbf{J}_j \) can be substituted in (2.5) from (2.1), and the time derivatives evaluated for solutions of the type (2.4). The right hand side of equation (2.5) may then be written formally as \( \mathbf{F}_{jk} \mathbf{\varepsilon}_{k} \), where the component \( \mathbf{F}_{jk} \) is defined by

\[ \mathbf{F}_{jk} = \frac{-\omega^2 \mathbf{E}_j}{c^2} + \frac{4\pi \partial \omega}{c^2} \mathbf{S}_{jk} \]  

(2.6)

The third equation determines \( \mathbf{E}_z \), and elimination of the \( \mathbf{E}_z \) terms from the first two equations is thus possible.

The two fundamental equations for the electric field components in the x and y directions then reduce to the form

\[ \frac{\partial^2 \mathbf{E}_x}{\partial t^2} = \left[ \mathbf{F}_{xx} - \frac{\mathbf{F}_{x2} \mathbf{F}_{2x}}{\mathbf{F}_{22}} \right] \mathbf{E}_x + \left[ \mathbf{F}_{xy} - \frac{\mathbf{F}_{x2} \mathbf{F}_{2y}}{\mathbf{F}_{22}} \right] \mathbf{E}_y \]

\[ \frac{\partial^2 \mathbf{E}_y}{\partial t^2} = \left[ \mathbf{F}_{yx} - \frac{\mathbf{F}_{y2} \mathbf{F}_{2x}}{\mathbf{F}_{22}} \right] \mathbf{E}_x + \left[ \mathbf{F}_{yy} - \frac{\mathbf{F}_{y2} \mathbf{F}_{2y}}{\mathbf{F}_{22}} \right] \mathbf{E}_y \]  

(2.7)

They may be conveniently abbreviated to

\[ \begin{align*}
\frac{\partial^2 \mathbf{E}_x}{\partial t^2} &= \mathbf{A} \mathbf{E}_x + \mathbf{B} \mathbf{E}_y \\
\frac{\partial^2 \mathbf{E}_y}{\partial t^2} &= \mathbf{C} \mathbf{E}_x + \mathbf{D} \mathbf{E}_y 
\end{align*} \]  

(2.8)

\( \mathbf{A}, \mathbf{B}, \mathbf{C} \) and \( \mathbf{D} \) are therefore complex scalar functions of \( \mathbf{S}_{jk}(\omega, \Xi), \mathbf{E}_{jk}(\omega) \) and \( \omega \), the magnetic field dependence being contained in the \( \mathbf{S}_{jk} \) tensor.

Solutions for the components \( \mathbf{E}_x \) and \( \mathbf{E}_y \) may be written,
following (2.4), as
\[
\begin{align*}
\mathcal{E}_x &= a \exp(\imath \omega t - \mu z), \\
\mathcal{E}_y &= b \exp(\imath \omega t - \mu z),
\end{align*}
\]
where \(a\) and \(b\) are complex functions of \(\alpha, \beta, \varepsilon\) and \(\delta\) to be determined by the boundary conditions of the problem.

Substitution of (2.9) in (2.8) yields the equations
\[
\begin{align*}
(\mu^2 - \alpha) a - \beta b &= 0 \\
-\varepsilon a + (\mu^2 - \delta)b &= 0
\end{align*}
\]
which have non-trivial solutions only if the determinant of the coefficients is zero. The two roots of (2.10) for \(\mu^2\) are found to be
\[
2\mu^2 = \alpha + \delta \pm \left[ (\alpha - \delta)^2 + 4\beta\varepsilon \right]^{\frac{1}{2}}.
\]
Hence two independent modes, specified by \(\mu^\pm\), \(a^\pm\) and \(b^\pm\) to correspond with the sign of the square root in (2.11), can be propagated. For general values of \(a^\pm\) and \(b^\pm\) the modes will be elliptically polarized in the xy plane.

2(c) Boundary Conditions

At the boundary of the medium a fraction of the incident wave is reflected and the remainder transmitted, the proportions being determined by continuity conditions on the \(\mathcal{E}\) and \(\mathcal{H}\) fields of the wave. The emergent wave is thus the sum of an infinite number of waves of varying phase and diminishing amplitude, set up as a result of multiple internal reflections at the front and back faces of the specimen. If the specimen absorption is low enough, as may be the case for example with high magnetic fields, low conductivity samples or small thicknesses, the effect of this reflected wave system on the numerical predictions of the no reflection theory has been shown to be considerable, (Donovan and Medcalf 1963). The necessary modifications to the present theory can, however, be expressed by introducing complex transmission coefficients \(k^\pm\) associated with the two modes into the equations.
to be derived, in a manner which does not affect their overall form very significantly. We consider here the simple case in which the reflected fraction is neglected. Hence we take the boundary conditions to be

$$\begin{aligned}
[E_x]_{z=0} &= \mathcal{E}_0 \\
[E_y]_{z=0} &= 0
\end{aligned} \quad (2.12a)$$

This requires, using (2.9),

$$\begin{aligned}
a_+ + a_- &= \mathcal{E}_0 \\
b_+ + b_- &= 0
\end{aligned} \quad (2.12b)$$

However, since the final equations do not contain $\mathcal{E}_0$, the theory is in fact valid for the case where the reflection coefficients of the two modes are assumed to be equal, i.e. differential rather than overall absorption is neglected. If $\pm$ subscripts are introduced in (2.10) to correspond with the two solutions $\mu_\pm = \alpha_\pm + i \beta_\pm$, the equations may be rearranged as follows

$$\begin{aligned}
\mathbb{A} a_+ + \mathbb{B} b_+ &= a_+ \mu_+^2 \\
\mathbb{C} a_+ + \mathbb{D} b_+ &= b_+ \mu_+^2
\end{aligned} \quad (2.13)$$

and, using the boundary conditions to solve these equations, we find

$$\begin{aligned}
a_\pm &= \pm \frac{\mathcal{E}_0 (A - \mu_\pm^2)}{\mu_+^2 - \mu_-^2} = \pm \frac{\mathcal{E}_0}{2} \left[ 1 \mp \frac{A - \mathbb{D}}{\mu_+^2 - \mu_-^2} \right] \\
b_\pm &= \pm \frac{\mathcal{E}_0 (\mu_+^2 - \mathbb{A})(\mu_-^2 - \mathbb{A})}{\beta (\mu_+^2 - \mu_-^2)} = \mp \frac{\mathcal{E}_0 \mathbb{C}}{\mu_+^2 - \mu_-^2}
\end{aligned} \quad (2.14)$$

where

$$\mu_+^2 - \mu_-^2 = \left[ (\mathbb{A} - \mathbb{D})^2 + 4 \beta \mathcal{E} \right]^{\frac{1}{2}}.$$ 

The coefficients $\mathbb{A}$, $\beta$, $\mathcal{E}$ and $\mathbb{D}$ are complex, and a notation in which they are replaced by four real quantities $P$, $Q$, $R$ and $T$ is therefore introduced. We let
\[
\frac{A - D}{C} = P + iQ \quad \text{and} \quad \frac{B}{C} = R + iT.
\] (2.15)

Hence
\[
\frac{\mu^2 - \mu^2}{C^2} = \left[(P + iQ)^2 + 4(R + iT)^2\right]^{\frac{1}{2}}.
\] (2.16)

This can be written as \( G + iH \).

The ambiguity of sign which always occurs in the evaluation of a complex square root such as (2.16) is resolved by taking the positive sign for \( H \). This convention is shown in section e(i) to give the correct sign in the isotropic limit where the need not appear. \( a_\pm \) and \( b_\pm \) can now be expressed in terms of \( P, Q, R \) and \( T \) as follows
\[
a_\pm = \pm \frac{E_0}{2} \left[ \frac{G \pm P + i(H \pm Q)}{G + iH} \right]
\] (2.17)
\[
b_\pm = \frac{E_0}{G + iH}
\]

Hence the four real quantities defined as
\[
\begin{align*}
2L_\pm &= G \pm P \\
2M_\pm &= H \pm Q
\end{align*}
\] (2.18)

enable the amplitudes of all the transmitted components to be expressed in a convenient notation. \( L_+ \) and \( M_+ \) will be shown to specify the form of one of the independent modes, while \( L_- \) and \( M_- \) describe the second mode in a similar manner. Using the above definitions, it is straightforward to show that the following relations, which are of use in subsequent sections, are true
\[
\begin{align*}
g^2 - H^2 &= P^2 - Q^2 + 4R \\
GH &= PQ + 2T \\
L_+M_- + L_-M_+ &= \frac{1}{2}(GH - PQ) \\
L_+(L_+^2 + M_+^2) &= RL_+ + TM_+ \\
M_+(L_+^2 + M_+^2) &= TL_+ - RM_+ \\
(L_+^2 + M_+^2)(L_-^2 + M_-^2) &= R^2 + T^2 \\
L_+L_- - M_+M_- &= R
\end{align*}
\] (2.19)
The first two relations are obtained by taking the real and imaginary parts of (2.16), and are useful in deriving the remaining results.

The emitted wave is now described completely as follows,

\[ \mathcal{E}_x = \frac{\mathcal{E}_0}{G + iH} \left( (L_+ + iM_+) \exp(i\omega t - \mu_+ d) + (L_- + iM_-) \exp(i\omega t - \mu_- d) \right) \]

\[ \mathcal{E}_y = \frac{\mathcal{E}_0}{G + iH} \left[ \exp(i\omega t - \mu_- d) - \exp(i\omega t - \mu_+ d) \right] \]

where \( d \) is the specimen thickness.

2(d) Equations for the Rotation and Ellipticity

The transmitted wave (2.20) is an ellipse and, by definition, its major axis is inclined at the Faraday angle \( \Phi \) to \( \mathcal{E}_0 \), and the ratio of its minor to its major axis is the Faraday ellipticity \( \Delta \).

\( \Phi \) and \( \Delta \) are found by eliminating the time dependent terms between the above equations. Summarizing \( \mathcal{E}_x \) and \( \mathcal{E}_y \) as

\[
\begin{align*}
\mathcal{E}_x &= J \cos(\omega t - \gamma) \\
\mathcal{E}_y &= K \cos(\omega t - \chi)
\end{align*}
\]

the equation to the resultant ellipse is

\[
K^2 \mathcal{E}_x^2 + J^2 \mathcal{E}_y^2 - 2JK \cos(\gamma - \chi) \mathcal{E}_x \mathcal{E}_y = J^2 K^2 \sin^2(\gamma - \chi) .
\]

The ellipse is transformed to its major axes by a rotation about the \( z \) axis in the sense \( x \rightarrow y \) through an angle which makes the coefficient of the \( \mathcal{E}_x \mathcal{E}_y \) term zero. The rotation is thus positive when clockwise to an observer looking along the direction of propagation. The required angle is \( \Phi \) and we find

\[
\tan 2\Phi = \frac{2JK \cos(\gamma - \chi)}{J^2 - K^2} ,
\]

and hence
The sign of $\Delta$ is at present arbitrary, since only $\Delta^2$ can be found from (2.22). $2\pi$ is assigned to the quadrant fixed by the signs of the numerator and denominator of $\tan 2\pi$ in the usual manner, $(0 \leq 2\pi \leq 2\pi)$. This is consistent with the above convention as a rotation such that $\pi > \pi > \pi/2$ may be interpreted by the observer as a negative or anticlockwise rotation.

We are concerned with a rotation, and a ratio of the minimum and maximum amplitude components of an elliptically polarized wave. An absolute measure of the amplitude is not required in either case, so that there is no loss of generality if the common factor $E_0/G + iH$ is henceforth omitted from (2.20) to simplify the algebra.

This equation must now be reduced to the form (2.21). On taking either the real or the imaginary part of (2.20), and comparing the coefficients of the time dependent terms in the two equations, we obtain

$$J \begin{pmatrix} \cos \gamma \\ \sin \gamma \end{pmatrix} = \begin{cases} L+ \begin{pmatrix} \cos \beta \cdot d \\ \sin \beta \cdot d \end{pmatrix} & + M+ \begin{pmatrix} \sin \beta \cdot d \\ \cos \beta \cdot d \end{pmatrix} e^{-A d} \\ L- \begin{pmatrix} \cos \beta \cdot d \\ \sin \beta \cdot d \end{pmatrix} & - M- \begin{pmatrix} \sin \beta \cdot d \\ \cos \beta \cdot d \end{pmatrix} e^{-A d} \end{cases}.$$  \hspace{1cm} (2.25)

$$K \begin{pmatrix} \cos \chi \\ \sin \chi \end{pmatrix} = \begin{cases} \cos \beta \cdot d \cdot e^{-A d} + \cos \beta \cdot d \cdot e^{-A d} \\ \sin \beta \cdot d \cdot e^{-A d} - \sin \beta \cdot d \cdot e^{-A d} \end{cases}.$$  \hspace{1cm} (2.25)

The equations (2.23) and (2.24) can now be evaluated by forming suitable combinations of the four equations constituting (2.25). We find, letting $\beta = (\beta_+ - \beta_-)$ and $\alpha = (\alpha_+ - \alpha_-)$,

$$J^2 + K^2 = (L^2 + M^2 + 1) e^{-2A d} + (L^2 + M^2 + 1) e^{-2A d} + 2 e^{-2A d} \begin{pmatrix} L+ M- (M+ M- 1) \cos \beta \cdot d + (M+ L- L+ M-) \sin \beta \cdot d \end{pmatrix}.$$
\[ 2JK \cos(\beta - \chi) = 2 \left[ L_+ e^{-2 \alpha d} - L_- e^{-2 \alpha d} \right] + e^{-2(\alpha + \chi) d} \left( (L_- - L_+) \cos \beta d - (M_+ + M_-) \sin \beta d \right) \]

\[ 2JK \sin(\beta - \chi) = 2 \left[ -M_+ e^{-2 \alpha d} + M_- e^{-2 \alpha d} \right] + e^{-2(\alpha + \chi) d} \left( (L_+ + L_-) \cos \beta d + (M_+ - M_-) \sin \beta d \right) \]

(The amplitude term omitted from (2.26) is \( \frac{\epsilon_0}{\sqrt{\sigma + M^2}} \))

Hence, factorising \( e^{-2(\alpha + \chi) d} \), we find

\[ \tan 2\beta = \frac{2 \left[ L_+ e^{-2 \alpha d} - L_- e^{-2 \alpha d} \right]}{(L_+^2 + M_+^2) e + (L_-^2 + M_-^2) e + 2(L_+ + M_+ + i) \cos \beta d + 2(M_+ + M_-) \sin \beta d} \]

and

\[ \Delta = \frac{\pm 2 \left[ -M_+ e^{-2 \alpha d} + M_- e^{-2 \alpha d} \right]}{(L_+^2 + M_+^2 + 1) e + (L_-^2 + M_-^2 + 1) e + 2(L_+ + M_+ + i) \cos \beta d + 2(M_+ + M_-) \sin \beta d + x \frac{1}{4} \left[ J^2 + K^2 \right]} \]

Using the equations of sections 2(a) to 2(d), \( \sigma \) and \( \Delta \) can be determined for an arbitrary direction of \( \mathbf{H} \) in the crystal, provided the \( F_{jk} \) are known.

2(e) Special Cases

The formulation given above may be simplified for some configurations of \( \mathbf{H} \) and \( \epsilon_0 \). Three cases of interest are discussed in this section.

(i) Isotropic Case (\( \alpha = \sigma \) and \( \beta = -\epsilon \))

It is shown in section I of appendix A that, when \( \mathbf{H} \) is parallel to \([100]\) or \([111]\) type directions in a cubic crystal, the magnetoconductivity tensor may be written in the form...
Furthermore, the components of $S$ may be expanded as power series in ascending powers of $H$ and, if terms of $O(H^2)$ and higher are neglected, the tensor reduces to the form (2.29), independently of the orientation of $H$ and $\xi_0$. The validity of this approximation is dependent on the fields and frequencies employed, and is discussed in chapter 7, where the expansion approach is used to calculate $\mathbf{X}$. In passing it may be mentioned that (2.29) also applies when $H$ is parallel to the c axis in a hexagonal crystal, (Austin, 1959).

When $S$ has the symmetry of (2.29) it follows from (2.6), (2.7) and (2.8) that

$$A_b = \Xi \text{ and } \beta = -\mathcal{E}$$

(2.30)

and hence, using the appropriate definitions, that

$$P = Q = T = G = 0, \quad R = -1, \quad H = 2$$

$$L^\pm = 0, \quad M^\pm = 1$$

(2.31)

Applying these results to (2.26) we obtain

$$J^2 \pm K^2 = (1 \pm 1) \left[ e^{-2\phi d} + e^{-2\phi d} \right] + 2e^{-(\phi + \phi)}d(1 \mp 1) \cos \beta d$$

$$2JK \cos(\phi - \chi) = -4e^{-(\phi + \phi)}d \sin \beta d$$

$$2JK \sin(\phi - \chi) = -2 \left[ e^{-2\phi d} - e^{-2\phi d} \right]$$

and finally, from (2.23) and (2.24),

$$\mathbf{X} = -\frac{\beta d}{2}$$

(2.33)

and

$$\Delta = \pm \frac{-e^{-2\phi d} + e^{-2\phi d}}{e^{-2\phi d} + e^{-2\phi d} + 2e^{-(\phi + \phi)}d}$$

$$= \pm \tanh \frac{\phi d}{2}$$

(2.34)

Equations (2.33) and (2.34) are well known and have been used by several authors, (Mitchell 1955, Rau and Gaspari 1955, etc.).
A straightforward derivation is possible when $S$ is given by (2.29). Equations (2.8) can then be written

$$\begin{align*}
\mu^2 \varepsilon_x &= \mathcal{A} \varepsilon_x + \beta \varepsilon_y \\
\mu^2 \varepsilon_y &= -\beta \varepsilon_x + \mathcal{A} \varepsilon_y
\end{align*}
$$

(2.35)

We introduce $i$ to represent a phase lag of $\pi / 2$ between the $x$ and $y$ components of the electric field and form the combinations

$$
\mu^2 (\varepsilon_x \pm i \varepsilon_y) = (\mathcal{A} \mp i \beta)(\varepsilon_x \pm i \varepsilon_y),
$$

(2.36)

showing that the propagation constants are given by

$$
\mu^2 = \mathcal{A} \mp i \beta
$$

(2.37)

and that the independent modes, which may be denoted by

$$
\varepsilon_+ = \varepsilon_x \pm i \varepsilon_y,
$$

are contrarotating and circularly polarized. $\varepsilon_+$ represents a clockwise and $\varepsilon_-$ an anticlockwise sense of rotation of the electric vector, to an observer looking along the direction of propagation. This argument was used by Stephen and Lidiard (1958), except that these authors substituted explicit expressions for the components of $S$. It is shown in many textbooks dealing with optical activity (see for example Jenkins and White, Optics) that, when two contrarotating circularly polarized vibrations are combined, $\mathcal{A}$ and $\Delta$ are given by (2.33) and (2.34).

Thus the anisotropic theory developed in this chapter reduces to the correct form in the isotropic case. We can also see that the choice of the positive sign for $H$ in section 2(c) is correct in the isotropic case by substituting (2.37) in (2.16). Recalling that $\beta = -\epsilon$, (2.16) reduces to the identity $i = i$ for $H = +2$. All previous experimental work has been limited to isotropic situations, (Rau and Caspari, 1955; Smith and Moss, 1958; etc.).

(ii) $\mathcal{A} = \mathcal{D}$ ($\beta$ and $\epsilon$ unrestricted)

It is shown in section II(c) of appendix A that, letting $\phi$ denote the angle between $\mathcal{H}$ and the 3 axis; when $\mathcal{H}$ lies in the (110) plane and the $x$ and $y$ axes are equally inclined at an angle
Cos\(\frac{1 + \cos \phi}{2}\) to the 1 and 2 axes respectively; or when \(H\) lies in the (100) plane and the \(x\) and \(y\) axes are equally inclined to the 2 axis at an angle Cos \((\cos \phi \sqrt{2})\), \(S\) reduces to the form

\[
\begin{pmatrix}
S_{xx} & S_{xy} & S_{xz} \\
S_{yx} & S_{xx} & S_{yz} \\
S_{yz} & S_{xz} & S_{zz}
\end{pmatrix}
\]

(2.38)

It can be seen from (2.7) and (2.8) that \(A = \mathcal{D}\) and \(\beta \neq -C\) in this case and hence, from (2.15) and (2.18),

\[
P = Q = 0 \quad \text{and} \quad L_+ = L_ - = L = G/2
\]

(2.39)

Substitution of these simplifications in (2.27) and (2.28) finally leads to

\[
\tan 2 \Omega = \frac{-2\left[L \sinh \xi d + M \sin \beta d\right]}{(L^2 + M^2 - 1) \cosh \xi d + (L^2 + M^2 + 1) \cos \beta d} = \frac{\mathcal{J}}{\mathcal{G}}
\]

(2.40)

and

\[
\Delta = \frac{\pm 2\left[M \sinh \xi d - L \sin \beta d\right]}{(L^2 + M^2 + 1) \cosh \xi d + (L^2 + M^2 - 1) \cos \beta d + \left[J_0 + K_0 \right]^{1/2}}
\]

(2.41)

(iii) \(\beta = -C\) (\(A\) and \(D\) unrestricted)

It is shown in sections II(a) and (b) of appendix A that, defining \(\phi\) as above; when \(H\) lies in the (100) plane with the \(x\) axis along \([100]\); or when \(H\) lies in the (110) plane with the \(x\) axis along \([110]\); \(S\) reduces to the form

\[
\begin{pmatrix}
S_{xx} & S_{xy} & S_{xz} \\
-S_{xy} & S_{yy} & S_{yz} \\
-S_{xz} & S_{yz} & S_{zz}
\end{pmatrix}
\]

(2.42)

Using this symmetry, it can be seen from (2.7) and (2.8) that \(A \neq \mathcal{D}\) and \(\beta = -C\), and hence, from (2.15) and (2.19), we find
\[
R = -1, \quad T = 0
\]
\[
g^2 - H^2 = P^2 - Q^2 - 4
\]
\[
\Omega = PQ
\]

It follows easily from (2.19) that
\[
L_+ M_- = -L_- M_+
\]
\[
1 + L_+ L_- = M_+ M_-
\]
and, using (2.44), we find from (2.27)

\[
\tan 2(\Phi) = \frac{2\left[ L_+ e^{-z} d - L_- e^{z} d + (L_+ - L_-)\cos \beta d - (M_+ + M_-)\sin \beta d\right]}{(L_+^2 + L_-^2 - 1)e^{-z} d + (L_+^2 + L_-^2 - 1)e^{z} d + 4M_+ M_- \cos d + 4M_- L_+ \sin \beta d}
\]

with a similar reduction for \( \Delta \).

2(f) Analysis of Individual Modes

It has been shown in the preceding sections that, in the material, the initially plane polarized radiation splits into two independent contrarotating modes, which are elliptically polarized in the \( xy \) plane, and propagate with different phase velocities \( \beta_\pm \) and attenuations \( \alpha_\pm \). To complete the theory we discuss the electric field configuration in the component waves for an arbitrary value of \( z \). If \( \mathbf{E} \) is used now to denote the electric field in the individual modes, these are specified by

\[
\mathbf{E}_x = \pm a \exp(i\omega t - \mu_\pm z) = \frac{\mathbf{E}_0}{\delta \pm (L \pm iM_\pm) \exp(i\omega t - \mu_\pm z)}
\]

\[
\mathbf{E}_y = \pm b \exp(i\omega t - \mu_\pm z)
\]

with the convention that, wherever a sign option is shown, the upper sign refers to the positive wave. The time dependent terms
can be eliminated using the method of section (d), and the
equations of the ellipses referred to the xy axes are found to be
\[
\begin{align*}
\varepsilon_x^2 + (L_+^2 + M_+^2) \varepsilon_y^2 &= 2L_x \varepsilon_x \varepsilon_y = \frac{\varepsilon_o^2 M_+^2 e^{-2A_z}}{0^2 + h^2}.
\end{align*}
\]
(2.47)

We now define \( \theta_\pm \) to be the angles between the x axis and a
principal axis of the appropriate ellipse, measured with a right
hand screw relation with the positive z direction, and chosen so
that \( \theta_\pm < \frac{\pi}{2} \). From (2.22) and (2.23) they are given by
\[
\tan 2 \theta_\pm = \frac{\pm 2L_\pm}{L_+^2 + M_+^2 - 1}.
\]
(2.48)

\( 2 \theta_\pm \) are assigned to the quadrant fixed by the sign of \( \tan 2 \theta_\pm \)
with \( 0 \leq 2 \theta_\pm \leq \pi \). The ellipses may now be imagined to be
rotated back through \( \theta_\pm \), so that the principal axes are
coincident with the x and y axes. The individual ellipticities
\( \xi_\pm \) are then defined as the ratio of the axis parallel to y to the
axis parallel to x. For the resultant ellipse, \( \Delta \) was defined to
be \(< 1\) by ensuring that \( \theta \) was always measured from the major axis.
In the present case, however, \( \xi_\pm \) \(< 1\) or \( > 1\) depending on whether
\( \theta_\pm \) are measured from the major or the minor axis. It follows from
the sign convention for \( \theta_\pm \) that \( \xi_\pm \) \(< 1\) if \( \pm L_\pm \) \( > 0 \), and \( \xi_\pm \) \( > 1\) if
\( \pm L_\pm \) \( < 0 \). If the former is the case, \( \xi_\pm \) are given by (2.24) as it
stands, while, if the latter, this equation must be inverted. Hence
we find, evaluating the terms in equation (2.24),
\[
\xi_\pm = \pm \left[ \frac{\bar{T} 2M_\pm}{1 + L_\pm^2 + M_\pm^2 + \left\{ (L_\pm^2 + M_\pm^2 - 1)^2 + 4L_\pm^2 \right\}^{1/2}} \right]
\]
(2.49)

if \( \pm L_\pm \) \( > 0 \), with the inverse expression if \( \pm L_\pm \) \( < 0 \). The choice
of sign outside the square bracket arises from (2.24), and is not
part of the sign convention of this section.
We now consider the special cases of section (e). For the isotropic case \( \tan 2 \Theta \) becomes indeterminate and \( \delta \) reduces to \( \mp 1 \). If the sign of the ellipticity is defined to be the same as the sense of rotation of the \( E \) vector, (i.e. positive for a rotation which is clockwise to an observer looking along the direction of propagation), we require \( \delta = \pm 1 \), so that the negative sign outside the square bracket must be taken in (2.49). Since the ellipticity is unity the modes are circularly polarized and \( \Theta \) are not defined. Thus the complete theory gives the same result as the simple argument used in section e(i).

For the case considered in e(ii) where \( A = B \) it follows immediately that

\[
\begin{align*}
\tan 2 \Theta_+ &= -\tan 2 \Theta_- \\
\Theta_+ - \Theta_- &= \frac{\pi}{2}
\end{align*}
\]

Also, since the numerators of \( \tan 2 \Theta_+ \) and \( \tan 2 \Theta_- \) have opposite signs,

\[
\delta_+ = -\frac{1}{\delta_-} \quad .
\]

The field configuration is drawn schematically in Fig. (2.1), which shows that the two ellipses are symmetrically disposed with respect to the x axis.

For the case considered in e(iii) where \( \beta = -C \), the following relations are obtained as special cases of some of the equations given in (2.19)

\[
\begin{align*}
L_+^2 (L_+^2 + M_+^2) &= -L_+ \\
M_+^2 (L_+^2 + M_+^2) &= L_+ \\
(L_+^2 + M_+^2)(L_-^2 + M_-^2) &= 1
\end{align*}
\]

Using these, the field components for the negative ellipse may be put in the form

\[
\begin{align*}
\varepsilon_x &= \frac{E_0}{G + iH} (L_-^2 + M_-^2)(-L_+ + iM_+) \exp(i\omega t - \mu z) \\
\varepsilon_y &= \frac{E_0}{G + iH} \exp(i\omega t - \mu z)
\end{align*}
\]
Fig. (2.1). Schematic diagram showing electric field configuration in component waves for the special case of section 2e(ii), drawn assuming $L_+ > 0$ and $(L_+^2 + M_+^2 - 1) > 0$. The ellipses are shown for simplicity as identical, whereas the ratio of the major axis of the positive ellipse to that of the negative ellipse is $\exp(-2d)$, owing to the unequal attenuations.
Hence, again following the method of section 2(d), the equation to the ellipse is
\[ \mathcal{E}_x^2(L_+^2 + M_+^2) + \mathcal{E}_y^2 - 2L_+ \mathcal{E}_x \mathcal{E}_y = \frac{\mathcal{E}_0^2 M_+^2 (L_+^2 + M_+^2)^2}{c^2 + H^2} e^{-2\kappa z} \tag{2.54} \]
and the rotation and the ellipticity are given by
\[ \tan 2 \Theta_\pm = \frac{2L_+ (L_-^2 + M_-^2)}{(L_-^2 + M_-^2)^2 (L_+^2 + M_+^2) - 1} \tag{2.55} \]

Since \((L_+^2 + M_+^2)\) is positive, the numerator and denominator of \(\tan \Theta_\pm\) and \(\delta_\pm\) can be multiplied by this factor without affecting the sign convention. Then, using the last relation of (2.52), we find
\[ \tan 2 \Theta_\pm = \frac{2L_+}{1 - L_-^2 - M_-^2} \tag{2.56} \]
\[ \delta_\pm = -\left( \frac{2M_+}{L_+^2 + M_+^2 + 1 + \left\{ (1 - L_-^2 - M_-^2)^2 + 4L_-^2 \right\}^{\frac{1}{2}}} \right) \]
Therefore
\[ \Theta_+ + \Theta_- = \pi/2 \]
\[ \delta_+ = -\delta_- \tag{2.57} \]
\(\delta_\pm\) must be inverted if \(L_+\) is negative. The field configuration is drawn in Fig. (2.2). In this case the ellipses are symmetrically disposed with respect to the line \(x = y\).
Fig. (2.2). The component waves for the special case of section 2a(iii). Details as for Fig. (2.1).
CHAPTER 3

The Magnetoconductivity Tensor (Closed Solution)

3(a) Solution of the Boltzmann Equation for Ellipsoidal Energy Surfaces

The equilibrium distribution of electrons in momentum or p space, in the presence of external fields, is governed by a distribution function $f(p)$, which is the solution of the Boltzmann equation. This integro-differential equation may be written in its most general form as

$$f(p) = 0, \quad (3.1)$$

and it expresses the balance which must occur in the steady state between the two effects, namely scattering and applied fields, which tend to alter the distribution. It is assumed that the perturbation produced by such fields is small, so that we may write

$$f = f_0 - \phi \frac{\partial f_0}{\partial E}, \quad (3.2)$$

where $f_0$ is the distribution function in the absence of fields, i.e. the Fermi-Dirac function, and $\phi$ is a scalar function of $p$. Provided a time of relaxation $\tau$ can be defined, which we shall assume in this work, the collision term can be put in the following form

$$\left(\frac{df}{dt}\right)_{\text{collisions}} = -\frac{f - f_0}{\tau} = \phi \frac{\partial f_0}{\partial E}, \quad (3.3)$$

Electron-electron scattering is excluded in deriving (3.3).

If the material is homogeneous, and at a uniform temperature throughout, there are no field terms arising from either internal strains or thermal gradients, and we need to consider only the effect of the $H$ and $E$ fields on $f$. The magnetic field quantizes
the electronic energy levels in the plane perpendicular to the direction of the field. However, provided the field is low enough to satisfy $R \ll 1$, where $\lambda$ is the cyclotron frequency of an electron, calculated using the appropriate effective mass, it is permissible to treat $E$ as a continuous variable and neglect the discrete nature of the levels. Satisfactory agreement between theory and experiment is in fact often obtained when the condition is not satisfied, provided the fields are not too high, and the method has the advantage over a quantum mechanical approach of relative simplicity. (See for example Pippard, 1963.) Within the framework of this classical approach we have

$$-\left(\frac{df}{dt}\right) = -e \left[ \frac{1}{c} \nabla_p E \cdot \mathbf{H} \right] \cdot \nabla_p f + \frac{\partial f}{\partial t}, \quad (3.4)$$

where $e$ is the modulus of the electronic charge. The last term is necessary only if the applied fields themselves are time dependent. In the present case $H$ is not time dependent, (we neglect the interaction of the $H$ field of the radiation with the electrons as it is of the order of $v/c$ smaller than the $E$ interaction, $v$ being the electron velocity), and $E$ varies harmonically with time so that

$$E = E_0 \exp(i\omega t). \quad (3.5)$$

Since $J$ is assumed to be a linear function of the electric field, (equation 2.1), the same must be true of $\Phi$. Substituting (3.3), (3.4) and (3.5) in the Boltzmann equation and grouping the terms $\Phi$ together we have

$$\left[ \frac{1+i\omega c}{c} \right] \nabla E \frac{\partial f_0}{\partial E} + e \left[ \frac{1}{c} \nabla_p E \cdot \mathbf{H} \right] \cdot \nabla_p f = 0. \quad (3.6)$$

Considering the field terms

$$\nabla P f = \nabla P f_0 - \nabla \left( P \frac{\partial f_0}{\partial E} \right) \quad (3.7)$$

and

$$(\nabla_p E \cdot \mathbf{H}) \cdot \nabla P f = \frac{\partial f_0}{\partial E} \left( \nabla_p E \cdot \mathbf{H} \right) \cdot \nabla P E - \frac{\partial f_0}{\partial E} \left( \nabla_p E \cdot \mathbf{H} \right) \cdot \nabla P \Phi$$

$$- \Phi \frac{\partial^2 f_0}{\partial E^2} \left( \nabla_p E \cdot \mathbf{H} \right) \cdot \nabla P E \quad (3.8)$$
We neglect the terms containing products of $\varepsilon$ and $\Phi$, which are of second order in $\varepsilon$, and also, in (3.8), the term containing the second derivative of $f^\prime_0$, so the Boltzmann equation finally becomes, (see for example Wilson, 1954),

$$\left[ \frac{1 + i\omega\tau}{\tau} \right] \Phi + e\varepsilon \cdot \nabla_p E - \frac{c}{2} \left( \nabla_p E \cdot H \right) \cdot \nabla_p \Phi = 0 \quad . \quad (3.9)$$

For positive holes, rather than electrons, the sign of the terms containing $e$ must be changed.

This differential equation was solved exactly for an ellipsoidal surface of constant energy in momentum space, and an isotropic time of relaxation $\tau(E)$, which is constant over a constant energy surface by Blochinzev and Nordheim, (1933). Little use appears to have been made of this solution, however, until a multi-ellipsoid band model was found to be applicable to many semiconductors. Abeles and Meiboom, (1954), first gave the components of $s$, the magnetoconductivity tensor for a single ellipsoid, for the D.C. case in the course of their work on the interpretation of magnetoresistance data for germanium. The equation to a constant energy surface is

$$E = \frac{1}{2} \sum_j \sum_k \frac{p_j p_k}{m_{jk}} = \frac{1}{2} \sum_j \sum_k f_{jk} p_j p_k \quad , \quad (3.10)$$

where $m_{jk}$ are components of the second rank symmetric effective mass tensor, and $f_{jk} = 1/m_{jk}$ are the components of a tensor we call $\mathcal{F}$. Then the relation $\mathcal{F}' \mathcal{F} = 1$ defines a tensor which is the inverse of $\mathcal{F}$. With this notation it may be verified by substitution in (3.9) and some manipulation of terms that

$$\Phi = -T_c \nabla_p E \left[ \frac{\varepsilon - T \left( \mathcal{F}' \varepsilon \right) \cdot H + T^2 \left( \mathcal{F}' H \right) \cdot \left( \varepsilon \cdot H \right) / \varepsilon' \varepsilon' \right] \right] \quad , \quad (3.11)$$

where $|\mathcal{F}'| = $ determinant of $\mathcal{F}'$

and $T = \frac{e \tau}{c(1 + i\omega\tau)}$. 
It is convenient to choose a set of cartesian axes which coincide with the principal axes of the energy ellipsoid. We label these (1, 2, 3) for convenience, but they are not to be confused with the cubic axes of the crystal. In this representation the mass tensor is diagonal and we have

\[ E = \frac{1}{2} \sum_j \frac{p_j^2}{m_{jj}} = \frac{1}{2} \sum_j f_{jj} p_j^2, \tag{3.12} \]

while the solution becomes

\[ \Phi = -\frac{T_c}{m_1} \left[ \frac{p_1 - T(p_2 H_3/m_2 - p_3 H_2/m_3) + T^2 H_1/m_2 m_3(p_1 H_1 + p_2 H_2 + p_3 H_3)}{1 + T^2 (H_1^2/m_2 m_3 + H_2^2/m_3 m_1 + H_3^2/m_1 m_2)} \right] \epsilon (3.13) \]

+ 2 terms obtained by cyclic permutation of the subscripts (123) in the orders (231) and (312). The solution in this form was given by Abeles and Meiboom, (1954), except that \( e \tau / c \) rather than \( T \) appeared where appropriate.

3(b) The Magnetoconductivity Tensor for a Single Ellipsoid referred to the Principal Axes of the Ellipsoid

In this section we follow the general procedure of Abeles and Meiboom, (1954, Section IV), making certain modifications as required for the high frequency case. We return to equation (2.1) for the components of \( s \),

\[ J_j = s_{jk} \epsilon_k. \tag{3.14} \]

The conduction current density \( J \) for a single ellipsoid arising from the perturbation \( \Phi \) is given by, (see for example Wilson, 1954),

\[ J = \frac{e}{4 \pi^3 \hbar^3} \int \int \int \frac{\partial F_0}{\partial E} \nabla_\perp E \, d\mathbf{p}, \tag{3.15} \]

where \( d\mathbf{p} \) is the volume element in momentum space. This integral
can be evaluated by integrating first over a constant energy surface, and then performing the energy integration, if use is made of the relation

\[
\frac{d\varphi}{dE} = \frac{dS}{\nabla_{p} E} = \left[ \left( \frac{p_1}{m_1} \right)^2 + \left( \frac{p_2}{m_2} \right)^2 + \left( \frac{p_3}{m_3} \right)^2 \right]^{1/2} \frac{dS}{dE}.
\]  

(3.16)

dS is an element of the constant energy surface which is given by

\[
dS = \left[ 1 + \left( \frac{\partial p_3}{\partial p_1} \right)^2 + \left( \frac{\partial p_3}{\partial p_2} \right)^2 \right]^{1/2} dp_1 dp_2
\]

\[
= \frac{m_3}{p_3} \left[ \left( \frac{p_3}{m_3} \right)^2 + \left( \frac{p_2}{m_2} \right)^2 + \left( \frac{p_1}{m_1} \right)^2 \right]^{1/2} dp_1 dp_2.
\]

(3.17)

\[ \frac{\partial F_0}{\partial E} \]

is given by

\[
- \frac{n h^3}{2(2\pi kT)^{3/2}} \left( m_1 m_2 m_3 \right)^{1/2} \exp \left( - \frac{E}{kT} \right)
\]

(3.18)

\( n \) being the density of electrons occupying the valley. The Fermi-Dirac distribution has been approximated by the Maxwell-Boltzmann distribution, as the system is non-degenerate at normal semiconductor densities. The integral can therefore be taken over the range 0 to infinity. We can now substitute (3.13), (3.16) and (3.18) in the expression for \( \mathbf{J} \) given above, equate the result to (3.14), and deduce the nine components of \( s \) by comparing coefficients of \( \mathbf{E}_k \) in each of the three components of \( \mathbf{J} \). The component \( s_{11} \), which is the coefficient of \( \mathbf{E}_1 \) in the \( j = 1 \) component of \( \mathbf{J} \), will be treated in some detail as an example. We find, using (3.17),
First consider the momentum integrals. Since $\tau$ is $\tau(E)$ only, they are clearly all of the form

$$ I_{jk} = \int_{0}^{\infty} \int_{0}^{\infty} \frac{p_{j} p_{k}}{p_{3}} \, dp_{1} \, dp_{2} \quad (3.20) $$

The ellipsoids are required by cubic symmetry to possess an axis of revolution for the commonly occurring arrangements in the Brillouin zone. We choose the 3 axis to be the axis of revolution, hence $m_{1} = m_{2}$. For this case the integrals are easily evaluated for constant $E$ by transforming to cylindrical polar coordinates. The general result is

$$ I_{jk} = \frac{\eta}{3} \prod \left( \frac{m_{1} m_{2}}{m_{3}} \right)^{1/3} m_{j}^{3/2} \delta_{jk} \quad (3.21) $$

and, substituting in $s_{11}$, we now have

$$ s_{11} = \frac{4 \pi e^{2}}{3 \pi^{4} m_{1} (kT)^{3/2}} \left[ \frac{1 + \frac{\eta}{3} \prod \left( \frac{m_{1} m_{2}}{m_{3}} \right)^{1/3} m_{j}^{3/2} \delta_{jk}}{1 + \frac{\eta}{3} \prod \left( \frac{m_{1} m_{2}}{m_{3}} \right)^{1/3} m_{j}^{3/2} \delta_{jk}} \right] \exp \left( \frac{-E}{kT} \right) \, dE \quad (3.22) $$

Changing the variable to $y = E/kT$ and setting

$$ K = \frac{m_{3}}{m_{1}}, \quad u = \frac{4 \pi e^{2}}{3 \pi^{4} m_{3}}, \quad \Omega^{2} = \frac{e^{2} K}{m_{3} c^{2}} \left[ H_{1}^{2} + H_{2}^{2} + KH_{3}^{2} \right] \quad (3.23) $$
(3.22) becomes

\[ s_{11} = uK \int_0^\infty y^{3n} \frac{1 + \frac{1}{T H_1} \frac{1}{m_2 m_3}}{1 + \rho 2 \tau^2 / (1 + i\omega \tau)^2} \exp(-y) \, dy \]  

(3.24)

\( \rho \) is the cyclotron frequency of an ellipsoid when \( H \) is referred to its principal axes.

The remaining components may be calculated by a similar method, and the tensor is conveniently summarized by the two expressions

\[ s_{jj} = \frac{um_j}{m_j} \int_0^\infty y^{3n} \frac{1 + \frac{1}{T H_j} \frac{1}{m_k m_j}}{1 + \rho 2 \tau^2 / (1 + i\omega \tau)^2} \exp(-y) \, dy \]  

(3.25)

and

\[ s_{jk} = \frac{um_j}{m_j} \int_0^\infty y^{3n} \frac{-\frac{1}{T H_k} \frac{1}{m_k} + \frac{1}{T H_j \rho} \frac{1}{m_k m_j}}{1 + \rho 2 \tau^2 / (1 + i\omega \tau)^2} \exp(-y) \, dy \]  

(3.26)

\[ = s_{kj}(-\rho) \]  

The components are obtained by setting \((jkl)\) equal to \((123)\), \((231)\) and \((312)\) in turn.

It can be seen from (3.25) and (3.26) that the energy integrals are of three kinds. They may be expressed in the form

\[ I(\tau^s) = \int_0^\infty y^{3n} \left( \frac{\tau}{1 + i\omega \tau} \right)^s \frac{1 + \rho 2 \tau^2 / (1 + i\omega \tau)^2}{1 + \rho 2 \tau^2 / (1 + i\omega \tau)^2} \exp(-y) \, dy \]  

(3.27)

where \( s = 1, 2 \) or \( 3 \). It is convenient to call these complex integrals \( \alpha (\rho) \), \( \beta (\rho) \) and \( \gamma (\rho) \), corresponding to the cases \( s = 1, 2 \) and \( 3 \) respectively. They are of course different from the D.C. integrals obtained by Abeles and Meiboom, and therefore require detailed consideration. On rationalizing we find
\[ \alpha (\omega) = \int_{0}^{\infty} \frac{e^{-y}}{1 + \tau^2(\omega^2 - \omega^2)} \left[ 1 + \tau^2(\omega^2 - \omega^2) \right] \ dy, \quad (3.28) \]

\[ \beta (\omega) = \int_{0}^{\infty} \frac{\tau^2 e^{-y}}{1 + \tau^2(\omega^2 - \omega^2)} \left[ 1 + \tau^2(\omega^2 - \omega^2) \right] \ dy, \quad (3.29) \]

and

\[ \gamma (\omega) = \int_{0}^{\infty} \frac{\tau^2 e^{-y}}{1 + \tau^2(\omega^2 - \omega^2)} \left[ 1 + \tau^2(\omega^2 - \omega^2) \right] \ dy. \quad (3.30) \]

The denominator can be factorized since

\[ \left[ 1 + \tau^2(\omega^2 - \omega^2) \right]^2 + 4\omega^2 \tau^2 \]

\[ = \left[ 1 + \tau^2(\omega + \omega^2) \right] \left[ 1 + \tau^2(\omega - \omega^2) \right] \] \quad (3.31)

This suggests that \( \alpha (\omega) \), \( \beta (\omega) \) and \( \gamma (\omega) \) may be simplified by the method of partial fractions. After some manipulation all six integrals are found to reduce to sums and differences of integrals of the type

\[ I_s(\tau) = \int_{0}^{\infty} \frac{\tau^2 e^{-y}}{1 + \tau^2 \tau^2} \ dy \quad (s = 1 \text{ or } 2) \] \quad (3.32)

where \( \tau \) may be \( \omega, (\omega + \omega) \) or \( (\omega - \omega) \). The expressions are

\[ \alpha (\omega) = \frac{1}{2} \left[ I_1(\omega + \omega) + I_1(\omega - \omega) \right] - \frac{1}{2} \left[ (\omega + \omega)I_2(\omega + \omega) + (\omega - \omega)I_2(\omega - \omega) \right], \quad (3.33) \]

\[ \beta (\omega) = \frac{1}{2\omega} \left[ (\omega + \omega)I_2(\omega + \omega) - (\omega - \omega)I_2(\omega - \omega) \right] + \]

\[ \frac{1}{2\omega} \left[ I_1(\omega + \omega) - I_1(\omega - \omega) \right] \quad (3.34) \]

and

\[ \gamma (\omega) = \frac{1}{2\omega} \left[ 2I_1(\omega) - I_1(\omega + \omega) - I_1(\omega - \omega) \right] - \]

\[ \frac{1}{2\omega} \left[ 2\omega I_2(\omega) - (\omega + \omega)I_2(\omega + \omega) - (\omega - \omega)I_2(\omega - \omega) \right]. \quad (3.35) \]
I_1 and I_2 are in fact the same as \( \alpha(\omega) \) and \( \beta(\omega) \) defined in (4.3) of Abeles and Meiboom, as may be seen by setting \( \omega = 0 \) and \( t \propto e^{-\frac{t}{\tau}}, \) (lattice scattering). Clearly integrals of this form will still be required if the surfaces of constant energy are spherical, (section 3(i)), so that an ellipsoidal energy surface does not necessitate the evaluation of more complicated integrals; it only introduces more complicated functions of the same integrals.

In terms of \( \alpha(\mathbf{x}) \), \( \beta(\mathbf{x}) \) and \( \gamma(\mathbf{x}) \), the tensor components for a single ellipsoid are

\[
\begin{align*}
\sigma_{11} &= uK\alpha(\mathbf{x}) + u\nu^2k^2H_1^2\gamma(\mathbf{x}) \\
\sigma_{22} &= uK\alpha(\mathbf{x}) + u\nu^2k^2H_2^2\gamma(\mathbf{x}) \\
\sigma_{33} &= u\alpha(\mathbf{x}) + u\nu^2k^2H_3^2\gamma(\mathbf{x}) \\
\sigma_{12} &= -u\nu k^2H_3^2\beta(\mathbf{x}) + u\nu^2k^2H_1^2H_2\gamma(\mathbf{x}) = \sigma_{21}(-\mathbf{H}) \\
\sigma_{23} &= -u\nu k H_1 H_2 \beta(\mathbf{x}) + u\nu^2k^2H_2^2H_3\gamma(\mathbf{x}) = \sigma_{32}(-\mathbf{H}) \\
\sigma_{31} &= -u\nu k H_2 H_3 \beta(\mathbf{x}) + u\nu^2k^2H_3^2H_1\gamma(\mathbf{x}) = \sigma_{13}(-\mathbf{H})
\end{align*}
\]

(3.36)

where \( \nu = e/m_3c \).

3(c) The Magnetoconductivity Tensor referred to the Crystal Axes

It is well known that the conduction band, (and in some cases the valence bands as well), of many cubic semiconductors can be represented, close to an extremum where the electrons fill the levels (or leave vacant holes), by a quadratic dependence of \( E \) upon \( k \), the wave vector. The extrema are arranged in the first Brillouin zone in a manner consistent with cubic symmetry. The most common arrangements, which are treated in the present section, are

(i) 4 or 8 extrema which are ellipsoids of revolution, having their main axis directed along a \([111]\) type direction in the Brillouin zone.

(ii) 3 or 6 extrema which are ellipsoids of revolution,
having their main axis directed along a [100] type direction in the Brillouin zone.

The total magnetoconductivity tensor $S$ is found by summing the contributions from all the valleys, which are assumed to contribute independently to the conductivity. Herring (1955), first discussed the problem of intervalley, as opposed to intravalley scattering, showing that it was important at room temperature, and could be described by an isotropic time of relaxation. This means that it can be included in calculations of the present type if a $\tau$ which includes both types of scattering is used, in a similar manner to the calculation of a resultant $\tau$ for intravalley scattering, when several competing mechanisms are important. The position of the extrema in the Brillouin zone is then important in specifying the wave vectors of the phonons through which intervalley scattering takes place. These, in turn, will affect $\tau$, but apart from the possibility of this indirect dependence, the theory does not distinguish between 8 or 6 extrema within the zone, or 4 or 3 extrema at the zone centre or boundary. Hence we need to consider only a summation over 4 or 3 valleys in cases (i) and (ii) respectively, and the assumption of independent conductivities is justifiable, within the framework of the Boltzmann equation.

(i) $[111]$ orientated ellipsoids

We first transform $s$ from the principal axes of the ellipsoid to the right handed set of cubic axes to which the 3 axis of the ellipsoid under consideration forms a body diagonal. This procedure is of course the same for all ellipsoids, and is illustrated by Fig. A II(c) if $\phi$ is set equal to $\cos^{-1}(1/3^{1/3})$. However, the cubic axes concerned are different in the four cases, and these are shown in Fig. (3.1) in four different colours. The matrix for this transformation is, (Abeles and Meiboom, 1954, equation 3.25),

$$
\begin{pmatrix}
1 & 1+3^{1/3}, 1-3^{1/3}, 2 \\
2 & 3^{1/3}, 1-3^{1/3}, 2 \\
-2 & 3^{1/3}, 1+3^{1/3}, 2 \\
-2 & -2, -2, 2 \\
\end{pmatrix}
$$

(3.37)
Relative orientations of four sets of cubic axes to which the magnetoconductivity tensors for the four [111] ellipsoids are initially referred.
The back transformation to be applied to $H_j$, (see appendix A II), is of the form
\[ H_j' = \frac{1}{k_j} H_k' \quad \text{(3.38)} \]

For convenience we use primes to distinguish quantities referred to the cubic axes of the crystal from those referred to the principal axes of the ellipsoid, until it is stated otherwise. We therefore replace $H_1$, $H_2$ and $H_3$ by the following expressions
\[
\begin{align*}
H_1 &= \frac{1}{(2,3^\frac{1}{2})} \left[ (1+3^\frac{1}{2})H_1' + (1-3^\frac{1}{2})H_2' - 2H_3' \right] \\
H_2 &= \frac{1}{(2,3^\frac{1}{2})} \left[ (1-3^\frac{1}{2})H_1' + (1+3^\frac{1}{2})H_2' - 2H_3' \right] \\
H_3 &= \frac{1}{(2,3^\frac{1}{2})} \left[ 2H_1' + 2H_2' + 2H_3' \right]
\end{align*}
\]

Then, transforming we have
\[ s = \frac{1}{12} \begin{pmatrix}
1+3^\frac{1}{2}, 1-3^\frac{1}{2}, 2 \\
1-3^\frac{1}{2}, 1+3^\frac{1}{2}, 2 \\
-2, -2, 2
\end{pmatrix}
\begin{pmatrix}
s_{11} & s_{12} & s_{13} \\
s_{21} & s_{22} & s_{23} \\
s_{31} & s_{32} & s_{33}
\end{pmatrix}
\begin{pmatrix}
1+3^\frac{1}{2}, 1-3^\frac{1}{2}, -2 \\
1-3^\frac{1}{2}, 1+3^\frac{1}{2}, -2 \\
2, 2, 2
\end{pmatrix} \quad \text{(3.40)}
\]

Working out each component in turn from (3.40), and substituting for $s_{jk}$ and $H_j$ from (3.36) and (3.39) respectively, we find, after considerable reduction, that the components become
\[
\begin{align*}
s'_{11} &= \frac{\nu}{3}(2K+1)\alpha(\mathbf{n}') + uv^2k^2H_1'\gamma(\mathbf{n}') \\
s'_{22} &= \frac{\nu}{3}(2K+1)\alpha(\mathbf{n}') + uv^2k^2H_2'\gamma(\mathbf{n}') \\
s'_{33} &= \frac{\nu}{3}(2K+1)\alpha(\mathbf{n}') + uv^2k^2H_3'\gamma(\mathbf{n}') \\
s'_{12} &= -\frac{\nu}{3}(K-1)\alpha(\mathbf{n}') - \frac{\nu}{3}vq^{(12)(\mathbf{H})}\beta(\mathbf{n}') + uv^2k^2H_1'H_2'\gamma(\mathbf{n}') = s'_{21}(-H') \\
s'_{23} &= -\frac{\nu}{3}(K-1)\alpha(\mathbf{n}') - \frac{\nu}{3}vq^{(23)(\mathbf{H})}\beta(\mathbf{n}') + uv^2k^2H_2'H_3'\gamma(\mathbf{n}') = s'_{32}(-H') \\
s'_{31} &= -\frac{\nu}{3}(K-1)\alpha(\mathbf{n}') - \frac{\nu}{3}vq^{(31)(\mathbf{H})}\beta(\mathbf{n}') + uv^2k^2H_3'H_1'\gamma(\mathbf{n}') = s'_{13}(-H')
\end{align*}
\]
\( \mathcal{R}' \) is obtained by applying the back transformation of the field to \( \mathcal{R} \) and is given by

\[
\mathcal{R}' = v^{2K/3} \left[ (K+2)(H_1'^2 + H_2'^2 + H_3'^2) + 2(K-1)(H_1'H_2' + H_2'H_3' + H_3'H_1') \right].
\] (3.42)

\( q(jk) \) is a convenient abbreviation for the expression

\[
q(jk)(H) = (K-1)H_j' + (K-1)H_k' + (K+2)H_1'.
\] (3.43)

\( (jk1) \) is again one of the cyclic permutations (123), (231) or (312), whichever is appropriate to the component \( s_{jk} \).

The total tensor is now obtained by applying to \( S' \) a rotation about the 3 axis of Fig (3.1), through angles of 0, \( \pi/2 \), \( \pi \), and \( 3\pi/2 \) in turn. This has the effect of expressing \( S' \) for all the ellipsoids with reference to a common set of cubic axes, which we shall henceforth refer to as (123). The appropriate components can then be added directly. The transformation matrices and back field transformations are summarized below

\[
\begin{array}{cccc}
0 & \pi/2 & \pi & 3\pi/2 \\
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} & \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} & \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} & \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\end{array}
\]

\[
\begin{array}{c}
H_1' \rightarrow H_1 \\
H_2' \rightarrow H_2 \\
H_3' \rightarrow H_3 \\
\end{array} \quad \begin{array}{c}
H_1' \rightarrow -H_1 \\
H_2' \rightarrow -H_2 \\
H_3' \rightarrow -H_3 \\
\end{array} \quad \begin{array}{c}
H_1' \rightarrow -H_2 \\
H_2' \rightarrow -H_1 \\
H_3' \rightarrow -H_3 \\
\end{array} \quad \begin{array}{c}
H_1' \rightarrow H_2 \\
H_2' \rightarrow H_1 \\
H_3' \rightarrow H_3 \\
\end{array}
\]

(3.44)

The final result for \( S \) referred to the cubic axes of the crystal is, (dropping the primes),

\[
S_{11} = u/3(2K+1) \sum_{i=1}^{4} \Psi (\mathcal{R}_i) + vu^2k^2H_1^2 \sum_{i=1}^{4} \Psi (\mathcal{R}_i)
\]

\[
S_{22} = u/3(2K+1) \sum_{i=1}^{4} \Psi (\mathcal{R}_i) + vu^2k^2H_2^2 \sum_{i=1}^{4} \Psi (\mathcal{R}_i)
\]

\[
S_{33} = u/3(2K+1) \sum_{i=1}^{4} \Psi (\mathcal{R}_i) + vu^2k^2H_3^2 \sum_{i=1}^{4} \Psi (\mathcal{R}_i)
\]

\[
S_{12} = S_{21} = S_{13} = S_{31} = S_{23} = S_{32} = 0.
\]
\[ S_{33} = u/3(2K+1) \sum_{i=1}^{4} \alpha (\mathcal{J}_i) + u v^2 K^2 H_3 \sum_{i=1}^{4} \delta (\mathcal{J}_i) \]  
\[ S_{12} = u/3(K-1) \sum_{i=1}^{4} (-1)^i \alpha (\mathcal{J}_i^{(12)}) - 1/3 u v K \sum_{i=1}^{4} q_{i}^{(12)} \beta (\mathcal{J}_i^{(12)}) + \]  
\[ u v^2 K^2 H_1 H_2 \sum_{i=1}^{4} \varsigma (\mathcal{J}_i) = S_{21}(-H) \]  
\[ S_{23} = u/3(K-1) \sum_{i=1}^{4} (-1)^i \alpha (\mathcal{J}_i^{(23)}) - 1/3 u v K \sum_{i=1}^{4} q_{i}^{(23)} \beta (\mathcal{J}_i^{(23)}) + \]  
\[ u v^2 K^2 H_2 H_3 \sum_{i=1}^{4} \varsigma (\mathcal{J}_i) = S_{32}(-H) \]  
\[ S_{31} = u/3(K-1) \sum_{i=1}^{4} (-1)^i \alpha (\mathcal{J}_i^{(31)}) - 1/3 u v K \sum_{i=1}^{4} q_{i}^{(31)} \beta (\mathcal{J}_i^{(31)}) + \]  
\[ u v^2 K^2 H_3 H_1 \sum_{i=1}^{4} \varsigma (\mathcal{J}_i) = S_{13}(-H) \]  

The summations are over the four cyclotron frequencies, and four values of \( q^{(jk)} \), obtained by back transforming \( J_{i}^{'} \) and \( q^{(jk)}(\mathcal{H}^{*}) \) respectively, according to (3.44). In general each ellipsoid has a different cyclotron frequency for an arbitrary orientation of the field. They are, taking the back transformations of (3.44) in the order given,

\[ \mathcal{J}_1^2 = v^2 K/3 \left[ (K+2)(H_1^2 + H_2^2 + H_3^2) + 2(K-1)(H_j H_k + H_k H_l + H_l H_j) \right] \]  
\[ \mathcal{J}_2^2 = v^2 K/3 \left[ (K+2)(H_1^2 + H_2^2 + H_3^2) + 2(K-1)(-H_j H_k + H_k H_l - H_l H_j) \right] \]  
\[ \mathcal{J}_3^2 = v^2 K/3 \left[ (K+2)(H_1^2 + H_2^2 + H_3^2) + 2(K-1)(H_j H_k - H_k H_l - H_l H_j) \right] \]  
\[ \mathcal{J}_4^2 = v^2 K/3 \left[ (K+2)(H_1^2 + H_2^2 + H_3^2) + 2(K-1)(-H_j H_k - H_k H_l + H_l H_j) \right] \]  

The \( q^{(jk)} \) are given by

\[ q_1^{(jk)} = (K-1)H_j + (K-1)H_k + (K+2)H_1 \]  
\[ q_2^{(jk)} = - (K-1)H_j + (K-1)H_k + (K+2)H_1 \]  
\[ q_3^{(jk)} = - (K-1)H_j - (K-1)H_k + (K+2)H_1 \]  
\[ q_4^{(jk)} = - (K-1)H_j - (K-1)H_k + (K+2)H_1 \]  

\[ (3.47) \]
(jk1) are again to be interpreted as (123), (231) or (312), whichever is appropriate to the off diagonal component $S_{jk}$. The superscript $(jk)$ on the cyclotron frequency is necessary in connection with the first two summations in the off diagonal components because the sign factor, $(-1)^i$, and the coefficient $q^{(jk)}$ respectively, make the ordering of the frequencies important. It is omitted from other terms where the ordering is immaterial, because they all have positive signs and a common coefficient. These components are identical with those given in equations (4.11), (4.12) and (4.13) of Abeles and Meiboom, (1954), provided $\alpha (J)$, $\beta (J)$ and $\gamma (J)$ are interpreted using (3.33) to (3.35).

(ii) $[100]$ orientated ellipsoids

In this case the principal axes of an ellipsoid coincide with a set of cubic axes. The relative orientations of these three sets of cubic axes are illustrated in Fig (3.2), and the matrices for the transformations needed to refer all three ellipsoids to a common set of cubic axes, together with the associated back transformations, are as follows

$$
\begin{align*}
&\begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix},
&\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix},
&\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \\
&H_1' \rightarrow H_2, \\
&H_2' \rightarrow H_3, \\
&H_3' \rightarrow H_1 \\
&H_1' \rightarrow H_3, \\
&H_2' \rightarrow H_1, \\
&H_3' \rightarrow H_2 \\
&H_1' \rightarrow H_1, \\
&H_2' \rightarrow H_2, \\
&H_3' \rightarrow H_3
\end{align*}
$$

(3.48)

Transforming (3.36) we find for $S_{11}$ on summing the contributions from the three ellipsoids

$$
S_{11} = u \left[ K \sum_{i=1}^{3} \alpha (J_i) - (K-1) \alpha (J_1) + v^2 K^2 H_1^2 \sum_{i=1}^{3} \gamma (J_i) \right]
$$

$$
S_{22} = u \left[ K \sum_{i=1}^{3} \alpha (J_i) - (K-1) \alpha (J_2) + v^2 K^2 H_2^2 \sum_{i=1}^{3} \gamma (J_i) \right]
$$
Relative orientations of three sets of cubic axes to which the magnetoconductivity tensors for the three [100] ellipsoids are initially referred. Ellipsoid associated with each set of axes also shown schematically.
$$S_{33} = \left[ \sum_{i=1}^{3} \alpha (\mathcal{R}_i) - (K-1)\alpha (\mathcal{R}_3) + \nu^2 k^2 H_2 \frac{1}{3} \sum_{i=1}^{3} \gamma (\mathcal{R}_i) \right]$$

$$S_{12} = -uvKH_3 \left[ \sum_{i=1}^{3} \beta (\mathcal{R}_i) + (K-1)\beta (\mathcal{R}_3) \right] + uv^2 k^2 H_2 H_3 \frac{1}{3} \sum_{i=1}^{3} \gamma (\mathcal{R}_i)$$

$$S_{23} = -uvKH_1 \left[ \sum_{i=1}^{3} \beta (\mathcal{R}_i) + (K-1)\beta (\mathcal{R}_1) \right] + uv^2 k^2 H_2 H_3 \frac{1}{3} \sum_{i=1}^{3} \gamma (\mathcal{R}_i)$$

$$S_{31} = -uvKH_2 \left[ \sum_{i=1}^{3} \beta (\mathcal{R}_i) + (K-1)\beta (\mathcal{R}_2) \right] + uv^2 k^2 H_2 H_3 \frac{1}{3} \sum_{i=1}^{3} \gamma (\mathcal{R}_i)$$

The three cyclotron frequencies are, from (3.23) and (3.48),

$$\mathcal{L}_1^2 = v^2 k \left[ \frac{H_2^2 + H_3^2 + KH_3^2}{3} \right]$$

$$\mathcal{L}_2^2 = v^2 k \left[ \frac{H_3^2 + H_1^2 + KH_1^2}{3} \right]$$

$$\mathcal{L}_3^2 = v^2 k \left[ \frac{H_1^2 + H_2^2 + KH_2^2}{3} \right]$$

3(d) The Magnetoconductivity Tensor for $H$ parallel to a cubic axis

This may be deduced directly from the equations of the preceding section. The magnetic field may be taken to have components $(0,0,H_z) \equiv (0,0,H_3)$. For [111] ellipsoids all the cyclotron frequencies become equal, because all the ellipsoids are symmetrically placed with respect to the field. It can be seen from (3.46) that this frequency, (written $\mathcal{L}_1$), is

$$\mathcal{L}_1^2 = v^2 k H_z^2 (K+2)/3$$

Using (3.47), the $q^{(jk)}$ reduce to
\[ q^{(12)} = (K+2)H_z \quad \text{for all ellipsoids} \]

\[
\begin{align*}
q^{(23)}_1 &= q^{(23)}_2 = (K-1)H_z, \quad q^{(23)}_3 = q^{(23)}_4 = -(K-1)H_z \quad (3.52) \\
q^{(31)}_1 &= q^{(31)}_4 = (K-1)H_z, \quad q^{(31)}_2 = q^{(31)}_3 = -(K-1)H_z
\end{align*}
\]

Applying these results to (3.45) we find

\[
\begin{align*}
S_{11} &= S_{22} = 4u/3(2K+1)\alpha(J) \\
S_{33} &= 4u/3(2K+1)\alpha(J) + 4uv^2H_z^2\gamma(J) \\
S_{12} &= -S_{21} = -4/3uvK(K+2)H_z\beta(J) \\
S_{23} &= S_{32} = S_{31} = S_{13} = 0
\end{align*}
\]

(3.53)

For \([100]\) ellipsoids there are two cyclotron frequencies, as only the first and second ellipsoids are symmetrically placed with respect to \(H\). They are

\[
\begin{align*}
\omega_1^2 &= v^2KH_z^2 = \omega_2^2 \\
\omega_3^2 &= v^2KH_z^2
\end{align*}
\]

(3.54)

and the magnetoconductivity tensor reduces to

\[
\begin{align*}
S_{11} &= S_{22} = u\left[(K+1)\alpha(J_1) + K\alpha(J_3)\right] \\
S_{33} &= u\left[2K\alpha(J_1) + \alpha(J_3) + v^2KH_z^2\sum_{i=1}^{3}\gamma(J_1)\right] \\
S_{12} &= -S_{21} = -uvKH_z\left[2\beta(J_1) + K\beta(J_3)\right] \\
S_{23} &= S_{32} = S_{31} = S_{13} = 0
\end{align*}
\]

(3.55)

Both (3.53) and (3.55) have the correct symmetry for the field along a cubic axis, as a comparison with equation (A.5) will show.
In the following three sections we calculate $S$ for the three choices of $(x,y,z)$ axes discussed from the point of view of symmetry in sections (a), (b) and (c) of appendix A II.

3(e) $\mathbf{H}$ in (100) plane

In this case $x$ is parallel to the [100] direction, and the $(1,2,3)$ and $(x,y,z)$ systems of axes are illustrated in Fig A II(a). The matrix for a transformation from the former to the latter system can be written down immediately from this diagram. It is

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \phi & -\sin \phi \\
0 & \sin \phi & \cos \phi
\end{pmatrix}
$$

In the $(x,y,z)$ system $\mathbf{H}$ is, by definition, $(0,0,H_z)$. Hence the associated back transformation of the field is

$$
\begin{align*}
H_1 &= 0 \\
H_2 &= H_z \sin \phi \\
H_3 &= H_z \cos \phi
\end{align*}
$$

It is shown in appendix A II(a) that the number of independent components to be calculated reduces to six in the present case, the exact form of the tensor being given by equation (A.16).

(i) [111] orientated ellipsoids

There are only two different cyclotron frequencies because the ellipsoids divide into two pairs, with each member of a pair being symmetrically placed with respect to the field. The frequencies and equal pairs for the cyclic permutation of $(jkl)$ relevant to the component $S_{jk}$, and given in brackets after the frequency, are, from (3.46),
\[ \mathcal{N}_1^{2(jkl)} = v^2 K H_z^2 / 3 \left[ K + 2 + (K-1) \sin 2\phi \right] \]
\[ = \mathcal{N}_2^{2(123)} = \mathcal{N}_3^{2(231)} = \mathcal{N}_4^{2(312)} \]  
and
\[ \mathcal{N}_3^{2(123)} = v^2 K H_z^2 / 3 \left[ K + 2 - (K-1) \sin 2\phi \right] = \mathcal{N}_4^{2(123)} \]
\[ = \mathcal{N}_2^{2(231)} = \mathcal{N}_3^{2(231)} \]
\[ = \mathcal{N}_2^{2(312)} = \mathcal{N}_3^{2(312)} \]

(3.58)

In deriving the transformed tensor, substitution of the cyclotron frequencies appropriate to the orderings of \((jkl)\) will be carried out using (3.58) and (3.59), and the results will be written in terms of the two frequencies given by

\[ \mathcal{N}_1^2 = v^2 K H_z^2 / 3 \left[ K + 2 \pm (K-1) \sin 2\phi \right] \]  
\[ = q_{(jkl)} \]

Similarly, from (3.47), the \(q_{(jk)}\) for the various permutations are

\[ q_1^{(12)} = \left[ (K-1) \sin \phi + (K+2) \cos \phi \right] H_z = q_2^{(12)} \]
\[ q_3^{(12)} = \left[ -(K-1) \sin \phi + (K+2) \cos \phi \right] H_z = q_4^{(12)} \]  
\[ q_1^{(23)} = \left[ \sin \phi + \cos \phi \right] (K-1) H_z = -q_3^{(23)} \]
\[ q_2^{(23)} = \left[ -\sin \phi + \cos \phi \right] (K-1) H_z = -q_4^{(23)} \]  
\[ and \]
\[ q_1^{(31)} = \left[ (K-1) \cos \phi + (K+2) \sin \phi \right] H_z = q_4^{(31)} \]
\[ q_2^{(31)} = \left[ -(K-1) \cos \phi + (K+2) \sin \phi \right] H_z = q_3^{(31)} \]  

(3.61)

Henceforth we shall abbreviate \(\alpha (\mathcal{N}_i)\), \(\beta (\mathcal{N}_i)\) and \(\gamma (\mathcal{N}_i)\) to \(\alpha_i\), \(\beta_i\) and \(\gamma_i\) respectively. Using this notation, the transformed tensor is
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \phi & -\sin \phi \\
0 & \sin \phi & \cos \phi
\end{pmatrix}
\begin{pmatrix}
S_{11} & S_{12} & S_{13} \\
S_{21} & S_{22} & S_{23} \\
S_{31} & S_{32} & S_{33}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{pmatrix}
\]

(3.64)

and we find, substituting for \( S_{jk} \) from (3.45), \( H \) from (3.57) and using equations (3.58) to (3.63)

\[
S_{xx} = S_{11} = 2u/3(2K+1)(\alpha_1+\alpha_2)
\]

\[
S_{xy} = S_{12}\cos \phi - S_{13}\sin \phi
\]

(3.65)

\[
= \frac{u}{3}(K-1)\left[ \cos \phi \left( -\alpha_1+\alpha_3+\alpha_1 \right) - \sin \phi \left( -\alpha_1+\alpha_3+\alpha_1 \right) \right] - \\
\frac{uvKH}{3}\left[ \left\{ (K-1)\sin \phi +(K+2)\cos \phi \right\} 2\beta_1 \cos \phi +\left\{ -(K-1)\sin \phi +(K+2)\cos \phi \right\} 2\beta_3 \cos \phi \\
+ \left\{ (K-1)\cos \phi +(K+2)\sin \phi \right\} 2\beta_1 \sin \phi +\left\{ -(K-1)\cos \phi +(K+2)\sin \phi \right\} 2\beta_3 \sin \phi \right] \\
= \frac{-2}{3}uvKH\left[ \beta_1 \left\{ K+2-(K-1)\sin 2\phi \right\} + \beta_3 \left\{ K+2-(K-1)\sin 2\phi \right\} \right].
\]

Other components are calculated in a similar manner. The complete tensor is given in appendix B I.

(ii) \([100]\) orientated ellipsoids

The three cyclotron frequencies are all different, except when \( H \) is parallel to a \([100]\) or \([110]\) type direction, \( \phi = 0, \pi/4, \pi/2 \).

In this case two of the frequencies become equal. The three frequencies are obtained by substituting (3.57) in (3.50) and are

\[
\mathcal{\Omega}_1^2 = v^2KH_z^2
\]

\[
\mathcal{\Omega}_2^2 = v^2KH_z^2\left[ \cos^2 \phi + K \sin^2 \phi \right]
\]

\[
\mathcal{\Omega}_3^2 = v^2KH_z^2\left[ \sin^2 \phi + K \cos^2 \phi \right]
\]

(3.66)
The transformed tensor is again given by (3.64), provided $S_{jk}$ are substituted from (3.49) and $H_j$ from (3.57). We find for example

$$S_{xy} = S_{12} \cos \phi - S_{13} \sin \phi$$

$$= -uvKH_z \left[ \cos^2 \phi \left( \beta_1 + \beta_2 + K \beta_3 \right) + \sin^2 \phi \left( \beta_1 + \beta_2 + K \beta_3 \right) \right]$$

$$= -uvKH_z \left[ \sum_1^3 \beta_i + (K-1)(\beta_2 \sin^2 \phi + \beta_3 \cos^2 \phi) \right]. \quad (3.67)$$

The complete tensor is given in appendix B II.

3(f) $H$ in $(1\bar{1}0)$ plane ($x$ parallel to $[\bar{1}10]$)

The $(1,2,3)$ and $(x,y,z)$ sets of axes are illustrated in Fig A II(b). The transformation between the two systems can be conveniently visualized in two stages. First we perform a rotation of $\pi/4$ about the 3 axis in the sense $2 \rightarrow 1$. The new 1 axis is now the $[\bar{1}10]$ direction, and the transformation (3.56) applied about this axis makes the resultant set of axes coincide with $(x,y,z)$. Hence, multiplying the two matrices together, we obtain for the transformation matrix required

$$\begin{pmatrix}
1/2^\frac{1}{3} & -1/2^\frac{1}{3} & 0 \\
\cos \phi/2^\frac{1}{3} & \cos \phi/2^\frac{1}{3} & -\sin \phi \\
\sin \phi/2^\frac{1}{3} & \sin \phi/2^\frac{1}{3} & \cos \phi
\end{pmatrix}. \quad (3.68)$$

The back transformation for $H$ is

$$H_1 = H_z \sin \phi/2^\frac{1}{3}$$

$$H_2 = H_z \sin \phi/2^\frac{1}{3}$$

$$H_3 = H_z \cos \phi$$

It is shown in appendix A II(b) that symmetry arguments again require the tensor to have the form (A.16), with only six independent components.
(i) \([111]\) orientated ellipsoids

There are three cyclotron frequencies, as the second and fourth ellipsoids are symmetrically placed with respect to the field. This is reduced to two when \(H\) is along the \([110]\) direction, \((\phi = \pi/2)\). With the convention of \(3(e)(i)\), the frequencies and equal pairs for the various permutations of \((jkl)\) are

\[
\mathcal{A}_1^2(jkl) = \frac{v^2KH_z^2}{3} \left[ k + 2 + (k-1)(\sin^2\phi + \frac{3}{2}\sin 2\phi) \right]
\]

\[
\mathcal{A}_2^2(123) = \frac{v^2KH_z^2}{3} \left[ k + 2 - (k-1)\sin^2\phi \right] = \mathcal{A}_4^2(123)
\]

\[
\mathcal{A}_3^2(123) = \frac{v^2KH_z^2}{3} \left[ k + 2 + (k-1)(\sin^2\phi - \frac{3}{2}\sin 2\phi) \right]
\]

In the transformed tensor, all cyclotron frequencies are to be understood in terms of these three frequencies, which will be called \(\mathcal{A}_1, \mathcal{A}_2 = \mathcal{A}_4, \text{ and } \mathcal{A}_3\), in the order of equations (3.70) to (3.72) respectively. The \(q(jk)\) are, using (3.47),

\[
q_1^{(12)} = \left[ (k-1)\frac{3}{2}\sin\phi + (k+2)\cos\phi \right] H_z
\]

\[
q_2^{(12)} = \left[ k + 2 \right] \cos\phi H_z = q_4^{(12)}
\]

\[
q_3^{(12)} = \left[ -(k-1)\frac{3}{2}\sin\phi + (k+2)\cos\phi \right] H_z
\]

\[
q_1^{(23)} = \left[ (k-1)\sin\phi/\frac{3}{2} + (k-1)\cos\phi + (k+2)\sin\phi/\frac{3}{2} \right] H_z
\]

\[
q_2^{(23)} = \left[ -(k-1)\sin\phi/\frac{3}{2} + (k-1)\cos\phi + (k+2)\sin\phi/\frac{3}{2} \right] H_z
\]

\[
q_3^{(23)} = \left[ -(k-1)\sin\phi/\frac{3}{2} - (k-1)\cos\phi + (k+2)\sin\phi/\frac{3}{2} \right] H_z
\]

\[
q_4^{(23)} = \left[ (k-1)\sin\phi/\frac{3}{2} - (k-1)\cos\phi + (k+2)\sin\phi/\frac{3}{2} \right] H_z
\]
The transformed tensor is

\[
\begin{pmatrix}
\frac{1}{2}, -\frac{1}{2}, 0 \\
\cos\phi/2,\cos\phi/2, -\sin\phi \\
\sin\phi/2, \sin\phi/2, \cos\phi
\end{pmatrix}
\begin{pmatrix}
S_{11} & S_{12} & S_{13} \\
S_{21} & S_{22} & S_{23} \\
S_{31} & S_{32} & S_{33}
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2}, \cos\phi/2, \sin\phi/2 \\
-\frac{1}{2}, \cos\phi/2, \sin\phi/2 \\
0, -\sin\phi, \cos\phi
\end{pmatrix}
\]

and we find for example, substituting for \( S_{jk} \) from (3.45), \( H_j \) from (3.69), and using equations (3.70) to (3.75)

\[
S_{xx} = \frac{1}{3}(S_{11} + S_{22}) - \frac{1}{6}(S_{12} + S_{21})
\]

\[
= \frac{u}{3}(2K+1) \sum_{i=1}^{4} \alpha_i + uv^2K^2H_r^2\sin^2\phi \sum_{i=1}^{4} \gamma_i - \frac{u}{3}(K-1)(-\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4) - uv^2K^2H_r^2\sin^2\phi \sum_{i=1}^{4} \gamma_i
\]

\[
= \frac{u}{3}(2K+1) \sum_{i=1}^{4} \alpha_i - \frac{u}{3}(K-1) \sum_{i=1}^{4} (-1)^i \alpha_i
\]

The complete tensor is given in appendix B III.

(ii) \([100]\) orientated ellipsoids

There are two cyclotron frequencies as the first and second ellipsoids are symmetrically placed with respect to the field. For the special case of \( \phi = \cos^{-1}(1/3^{1/2}) \), when \( H \) is parallel to the \([111]\) direction, these two frequencies become equal. The frequencies are
\[ \mathbf{J}_1^2 = v_{KH}^2 \mathbf{J}_2 \mathbf{J}_3 \left[ \sin^2 \phi \left( \frac{k+1}{2} \right) + \cos^2 \phi \right] = \mathbf{J}_2^2 \]

\[ \mathbf{J}_3^2 = v_{KH}^2 \mathbf{J}_2 \mathbf{J}_3 \left[ \sin^2 \phi + K \cos^2 \phi \right] \]

and, using (3.76), (3.69) and (3.49), we have for example

\[ S_{xx} = \frac{1}{2} (S_{11} + S_{22}) - \frac{1}{2} (S_{12} + S_{21}) \]

\[ = uK \sum_{i=1}^{3} \lambda_i - (K-1) \lambda_1 \]  

The complete tensor is given in appendix B 17.

3(g) \text{ H in (110) plane (x and y inclined equally to 1 and 2 axes respectively)}

The transformation is illustrated in Fig A II (c). The operation consists of a rotation through \( \phi \) about the \([110]\) axis and, since the matrix must be orthogonal and we require

\[ l_{11} = l_{22}, \quad l_{12} = l_{21}, \quad l_{13} = l_{32}, \quad l_{13} = l_{23} \]

by symmetry, (the \( l_{jk} \) are direction cosines), it is possible to deduce all the \( l_{jk} \). They cannot be written down in this case by inspection of the diagram.

The matrix is found to be

\[ \begin{pmatrix}
\frac{1+\cos \phi}{2} & \frac{-1+\cos \phi}{2} & -\sin \phi / 2^\frac{1}{2} \\
-\frac{1+\cos \phi}{2} & \frac{1+\cos \phi}{2} & -\sin \phi / 2^\frac{1}{2} \\
\sin \phi / 2^\frac{1}{2} & \sin \phi / 2^\frac{1}{2} & \cos \phi
\end{pmatrix} \]

It can easily be verified from (3.80) that the back transformation for the field is again given by (3.69), as would be expected, since the direction of the field is the same in both cases. The symmetry properties of \( S \) are discussed in appendix A II (c), where it is shown that there are again six independent components, with the form of the tensor given by (A.20).
(i) \( [\text{III}] \) orientated ellipsoids

The cyclotron frequencies and \( q^{(jk)} \) are naturally again given by equations (3.70) to (3.75). Hence, substituting for \( S_{jk} \) from (3.45) and \( H_j \) from (3.69) into the transformed tensor which is

\[
\begin{pmatrix}
\frac{(1+\cos \phi)}{2}, \frac{(-1+\cos \phi)}{2}, \frac{-\sin \phi}{2} \\
\frac{(-1+\cos \phi)}{2}, \frac{(1+\cos \phi)}{2}, \frac{-\sin \phi}{2} \\
\frac{\sin \phi}{2}, \frac{-\sin \phi}{2}, \cos \phi
\end{pmatrix}
\begin{pmatrix}
S_{11} & S_{12} & S_{13} \\
S_{21} & S_{22} & S_{23} \\
S_{31} & S_{32} & S_{33}
\end{pmatrix}
\begin{pmatrix}
\frac{(1+\cos \phi)}{2}, \frac{(-1+\cos \phi)}{2}, \frac{\sin \phi}{2} \\
\frac{(-1+\cos \phi)}{2}, \frac{(1+\cos \phi)}{2}, \frac{\sin \phi}{2} \\
\frac{-\sin \phi}{2}, \frac{-\sin \phi}{2}, \cos \phi
\end{pmatrix},
\]

we find for example, after some reduction,

\[
S_{xx} = \left( \frac{1+\cos \phi}{2} \right) \left\{ \frac{S_{11}}{2} + \frac{(1+\cos \phi)}{2} - \frac{S_{13}}{2\sin \phi} \right\} + \\
\left( \frac{1+\cos \phi}{2} \right) \left\{ \frac{S_{21}}{2} + \frac{(1+\cos \phi)}{2} - \frac{S_{23}}{2\sin \phi} \right\} - \\
\left( \frac{1+\cos \phi}{2} \right) \left\{ \frac{S_{31}}{2} + \frac{(1+\cos \phi)}{2} - \frac{S_{33}}{2\sin \phi} \right\}
\]

\[
S_{xx} = \frac{u}{3}(2K+1) \sum_{i=1}^{4} \alpha_i - \frac{u}{3}(k-1) \left[ \frac{\sin^2 \phi}{2} \sum_{i=1}^{4} (-1)^i \alpha_i + \frac{\sin 2\phi}{2\sin \phi} (\alpha_3 - \alpha_1) \right].
\]

The complete tensor is given in appendix B V.

(ii) \( [\text{I00}] \) orientated ellipsoids

The cyclotron frequencies are given by (3.78) and the transformed tensor by (3.81), provided the \( S_{jk} \) are substituted from (3.49). Then, using (3.69), we find

\[
S_{xx} = \frac{u}{2} \left[ (2K+1 + K \sin^2 \phi + \cos^2 \phi) \alpha_1 + \\
(K + K \cos^2 \phi + \sin^2 \phi) \alpha_3 \right].
\]

The complete tensor is given in appendix B VI.
3(h) The Magnetoconductivity Tensor for \( H \) parallel to \([111]\) direction

This is easily deduced from the tensors derived in either 3(f) or 3(g) by setting \( \phi = \text{Cos}^{-1}(1/\sqrt{3}) \). For \([111]\) ellipsoids the second, third and fourth ellipsoids are equivalently placed with respect to the field, so there are two cyclotron frequencies. From (3.70), (3.71) and (3.72) these are

\[
\mathfrak{L}_1^2 = \frac{v^2 K H_z^2}{3} \left[ 1 + 2 - \frac{2}{3}(K-1) \right] = \mathfrak{L}_3^2 = \mathfrak{L}_4^2 \tag{3.85}
\]

Then we find for \( S \), from appendix B III or B V,

\[
S_{xx} = S_{yy} = \frac{u}{3}(2K+1) \sum_{i=1}^{4} \alpha_i - \frac{u}{3}(K-1)(\alpha_2 - \alpha_1)
\]

\[
S_{zz} = \frac{u}{3}(2K+1) \sum_{i=1}^{4} \alpha_i + \frac{2u}{3}(K-1)(\alpha_2 - \alpha_1) + uv^2 K H_z^2 \sum_{i=1}^{4} \gamma_i \tag{3.86}
\]

\[
S_{xy} = -S_{yx} = -\frac{uv K H_z}{3} \left[ (K+2) \sum_{i=1}^{4} \beta_i + 2(K-1)(\beta_1 - \beta_2) \right]
\]

\[
S_{yz} = S_{zy} = S_{zx} = S_{xz} = 0
\]

The symmetry of \( S \) is consistent with that of equation (A.5), which is shown in appendix A I(b) to apply to this case.

For the case of \([100]\) ellipsoids, which are all symmetrically placed with respect to the field, there is only one cyclotron frequency, namely

\[
\mathfrak{L}_1^2 = \frac{v^2}{3} K(K+2) H_z^2 \tag{3.87}
\]

The tensor components are, from appendix B IV or B VI,
\[
\begin{align*}
S_{xx} &= S_{yy} = u(2k+1)\alpha_1 \\
S_{zz} &= u(2k+1)\alpha_1 + 3uv^2K^2H_z^2\gamma_1 \\
S_{xy} &= -S_{yx} = -uvK(K+2)H_z\beta_1 \\
S_{yz} &= S_{zy} = S_{zx} = S_{xz} = 0 \\
\end{align*}
\]

which are consistent with (A.5)

3(i) **Isotropic Effective Mass**

If the energy surfaces are spherical, the effective mass is isotropic and the magnetoconductivity tensor is independent of the orientation of \( H \) with respect to the cubic axes. We have therefore

\[
m_1 = m_3 = m^*, \quad k = 1
\]

and, in the \((x,y,z)\) co-ordinate system, a single cyclotron frequency which, from (3.46) or (3.50), is

\[
\Omega = \frac{eH_z}{m^*c}
\]

If \( N \) is the total carrier density, then \( N = 4n \) and \( N = 3n \) for the [111] and [100] ellipsoid models respectively. Hence we find for \( S \), from (3.45) or (3.49)

\[
\begin{align*}
S_{xx} &= S_{yy} = \frac{4Ne^2}{3\pi^2m^*} \alpha(\Omega) \\
S_{zz} &= \frac{4Ne^2}{3\pi^2m^*} \alpha(\Omega) + \frac{4Ne^2\gamma H_z^2}{3\pi^2m^*} \gamma(\Omega) \\
S_{xy} &= -S_{yx} = -\frac{4Ne^2\nu H_z}{3\pi^2m^*} \beta(\Omega) = -\frac{4Ne^2\nu}{3\pi^2m^*} \beta(\Omega) \\
S_{yz} &= S_{zy} = S_{zx} = S_{xz} = 0. \\
\end{align*}
\]
CHAPTER 4

The Magnetoconductivity Tensor (Series Solution)

In section (a) of chapter 3 it was shown that \( \mathbf{\Phi} \), the perturbation term in the distribution function which arose from the external fields present, could be obtained by solving the differential equation

\[
(1 + i\omega t/c)\mathbf{\Phi} + \mathbf{\xi} \cdot \nabla \mathbf{E} - \frac{\mathbf{\varepsilon}}{c} (\nabla \times \mathbf{E} \times \mathbf{H}) \cdot \nabla \mathbf{\varepsilon} \mathbf{\Phi} = 0 \quad (4.1)
\]

A closed solution was shown to exist for an ellipsoidal surface of constant energy and an isotropic time of relaxation. However, (4.1) may also be solved by expressing \( \mathbf{\Phi} \) as an infinite series in ascending powers of \( \mathbf{H} \). The general solution was first given by Jones and Zener, (1934). It has been widely used to first order in \( \mathbf{H} \) to interpret Faraday rotation measurements at infrared frequencies, (see for example Stephen and Lidiard, 1958), and in the present chapter the theory is extended to enable terms of order \( \mathbf{H}^2 \) to be included in the calculation.

Some numerical work was carried out using this theory, prior to the application of the closed solution of chapter 3. The method was found, however, to be of limited use in practice, because inclusion of the term in \( \mathbf{H}^2 \) only gives a satisfactory agreement with the exact result for a restricted range of values of \( \omega \) and \( \mathbf{H} \). At very low fields the term is unnecessary, and at higher fields, when it is no longer negligible, the increasingly slow convergence of the series with increasing field rapidly renders it insufficient without even higher order terms. Ultimately the series diverges. The validity of the approach is discussed fully in chapter 7, following the account of the numerical work. The expansion method will probably prove most useful in cases where the energy-momentum...
relation is more complex than (3.10), and a simple closed solution to the Boltzmann equation is no longer available. The $H^2$ term is also needed to calculate effects such as the Voigt ellipticity, which is of second order in $H_z$ (Cardona, 1961; Donovan and Webster, 1962). In the present case all the results of this chapter can of course also be obtained by expanding the appropriate magnetoconductivity tensor of chapter 3 to order $H^2$. The coefficients of this series expansion, which are themselves field independent tensors, are calculated from first principles in the following sections. They specify the magnetoconductivity tensor in its expanded form completely.

4(a) The Magnetoconductivity Tensor for a Single Ellipsoid

Once again we use the D.C. theory of Abeles and Meiboom (1954, section III) as a basis for the discussion. The series solution to (4.1) is easily obtained by an iterative procedure. We set $\Phi = 0$ in the right hand term, solve for $\Phi$ to give a first approximation, and substitute the result in the right hand term. We can then solve again for $\Phi$, and use this second approximation to substitute in the right hand term. Repetition of this process gives the series directly. Rearranging the scalar triple products, the result is, to terms of order $H^2$,

$$\Phi = -\frac{e\tau}{1+i\omega\tau} \left\{ \varepsilon \cdot \nabla \frac{E}{p} - \frac{e}{c} \frac{H}{H} \left\{ \nabla \cdot \left( \varepsilon \cdot \nabla \frac{E}{p} \right) \right\} + \frac{e^2}{c^2} \frac{H}{H} \left\{ \nabla \cdot \left( \varepsilon \cdot \nabla \frac{E}{p} \right) \right\} \right\} \right\} ... (4.2)$$

Unlike (3.11), however, (4.2) is not restricted by any assumptions about either $\tau$ or the shape of a constant energy surface. We make such assumptions in the following work as they introduce simplifications, but (4.2) can be applied without them. If $\tau$ is isotropic, as discussed in 3(a), a term of the form
where \( f \) and \( g \) are scalar functions of \( C \) and \( P \) respectively, becomes

\[
\nabla_P E \cdot \nabla_P \left\{ f(C) \nabla_P s(P) + s(P) \nabla_P f(C) \right\} = \nabla_P E \cdot \left\{ f(C) \nabla_P s(P) + s(P) \frac{\partial f(C)}{\partial E} \nabla_P E \right\} \quad (4.3)
\]

The right-hand term is identically zero, and it follows therefore that terms involving \( C \) only can be taken outside the vector differential operators in (4.2). Also, choosing the co-ordinate system to coincide with the principal axes of the energy ellipsoid, as in 3(a), we see from (3.12) that \( \partial^2 E/\partial P_k \partial P_j \) is zero unless \( j = k \).

It is convenient to use the convention of summing over repeated subscripts adopted by Abeles and Meiboom, (1954). In this notation we find

\[
(\nabla_P E \cdot \nabla_P)_j = \varepsilon_{jkl} \frac{\partial E}{\partial P_k} \frac{\partial}{\partial P_l} \quad (4.4)
\]

where \( \varepsilon_{jkl} \) is the epsilon tensor. The current density for a single ellipsoid may be written to order \( H^2 \) as

\[
J_j = \sigma_{jk} \varepsilon_k \varepsilon_k + \sigma_{jkl} \varepsilon_k H_1 + \sigma_{jklm} \varepsilon_k H_1 H_m \quad (4.5)
\]

where \( \sigma_{jk}, \sigma_{jkl} \) and \( \sigma_{jklm} \) are second, third and fourth rank tensors respectively. They are not functions of \( H \), and the back transformation of the field considered in chapter 3 and appendix A is no longer necessary when they are transformed to new axes.

Substituting (4.2) in (3.15), and using the above results where appropriate to expand the vector operators, we can reduce the current density to a form which may be compared directly with (4.5), namely
\[ J_j = -\frac{e^2}{4\pi^3 \hbar^3} \int_0^\infty \int \int \frac{\partial f_0}{\partial E} \frac{\partial E}{\partial p_j} \left( \frac{\tau}{1+i\omega \tau} \right) \left\{ \epsilon \frac{\partial E}{\partial k \partial p_k} - \frac{e}{c(1+i\omega \tau)} \left[ \epsilon \frac{\partial E}{\partial p_j} \frac{\partial E}{\partial p_k} \right] \right\} \]
\[ + \frac{e^2 \tau^2}{c^2(1+i\omega \tau)^2} \left[ \frac{\hbar}{m} \frac{\partial E}{\partial p} \frac{\partial E}{\partial p} \left( \frac{\hbar \epsilon}{l z u k} \frac{\partial E}{\partial p} \frac{\partial E}{\partial p} \right) \right] \cdots \right) \frac{dp}{p} \quad (4.6) \]

Hence we find

\[ \sigma_{jk} = -\frac{e^2}{4\pi^3 \hbar^3} \int_0^\infty \int \int \frac{\partial f_0}{\partial E} \frac{\partial E}{\partial p_j} \left( \frac{\tau}{1+i\omega \tau} \right) \frac{\partial E}{\partial p_k} \frac{dp}{p} \quad (4.7) \]
\[ \sigma_{jkl} = \frac{e^3}{4\pi^3 \hbar^3} \int_0^\infty \int \int \frac{\partial f_0}{\partial E} \frac{\partial E}{\partial p_j} \left( \frac{\tau}{1+i\omega \tau} \right)^2 \frac{\partial E}{\partial p_k} \frac{\partial E}{\partial p_k} \frac{dp}{p} \quad (4.8) \]

and

\[ \sigma_{jklm} = -\frac{e^4}{4\pi^3 \hbar^3} \int_0^\infty \int \int \frac{\partial f_0}{\partial E} \frac{\partial E}{\partial p_j} \left( \frac{\tau}{1+i\omega \tau} \right)^3 \frac{\partial E}{\partial p_k} \frac{\partial E}{\partial p_k} \left( \frac{\partial E}{\partial p_k} \right) \frac{dp}{p} \quad (4.9) \]

Using the equations of 3(b), the integration over momentum space can again be transformed to an integration over a constant energy surface, followed by an energy integration. It can then be seen from (3.12) that the momentum integrals are of the form

\[ \int_0^\infty \int \frac{1}{p_3} \frac{\partial E}{\partial p_j} \frac{\partial E}{\partial p_k} dp_1 dp_2 \quad (4.10) \]

and, using (3.20) and (3.21), we see that the integrals are zero unless \( j = k \). This shows immediately that

\[ \sigma_{jk} = 0 \quad \text{unless} \quad j = k \quad (4.11) \]
\[ \sigma_{jkl} = 0 \quad \text{unless} \quad j \neq k \neq l \quad (4.12) \]

and, from (4.9), because
\[\epsilon_{\text{lj}} \epsilon_{\text{mju}} = \delta_{\text{jl}} \delta_{\text{km}} - \delta_{\text{jk}} \delta_{\text{lm}} \quad , \quad (4.12)\]
\[\sigma_{\text{jklm}} = 0 \quad \text{unless the subscripts are equal in pairs} \quad . \quad (4.13)\]

On substituting for \(\partial f_0 / \partial E\) from (3.18), and using (3.21) to evaluate the momentum integrals, we find

\[\sigma_{\text{jk}} = \frac{4ne^2}{3\pi^2m_3(kT)^2} \frac{m_3}{m_j} \int_0^\infty \left(\frac{\mathcal{C}}{1+i\omega T}\right)^{3/2} e^{-E/kT} dE \quad , \quad (4.14)\]

\[\sigma_{\text{jk1}} = -\frac{4ne^2}{3\pi^2m_3(kT)^2} \frac{e^2}{m_3^2} \left(\frac{m_3}{m_jm_k}\right) \epsilon_{\text{mjuk}} \int_0^\infty \left(\frac{\mathcal{C}}{1+i\omega T}\right)^{2} e^{-E/kT} dE \quad (4.15)\]

and

\[\sigma_{\text{jk1m}} = \frac{4ne^2}{3\pi^2m_3(kT)^2} \frac{e^2}{m_3^2} \left(\frac{m_3}{m_jm_km_u}\right) \epsilon_{\text{mjuk}} \epsilon_{\text{lkj}} \int_0^\infty \left(\frac{\mathcal{C}}{1+i\omega T}\right)^{3/2} e^{-E/kT} dE \quad , \quad (4.16)\]

where \(u\) is the number from the set \((1,2,3)\) which is not included in the set \((j,k,l,m)\). Hence, if once again

\[y = \frac{E}{kT} \quad , \quad u = \frac{4ne^2}{3\pi^2m_3} \quad , \quad v = \frac{e}{m_3c} \quad , \quad \]

we have

\[\sigma_{\text{jk}} = \frac{um_3}{m_j} \int_0^\infty \left(\frac{\mathcal{C}}{1+i\omega T}\right)^{3/2} e^{-y} dy \quad , \quad (4.17)\]

\[\sigma_{\text{jk1}} = -\frac{uvm_3^2}{m_jm_k} \int_0^\infty \left(\frac{\mathcal{C}}{1+i\omega T}\right)^{2} e^{-y} dy \quad (4.18)\]

and

\[\sigma_{\text{jk1m}} = \frac{uvm_3^3}{m_jm_km_u} \epsilon_{\text{mjuk}} \epsilon_{\text{ljk}} \int_0^\infty \left(\frac{\mathcal{C}}{1+i\omega T}\right)^{3/2} e^{-y} dy \quad . \quad (4.19)\]
Finally, rationalizing the complex integrals, the tensor coefficients for a single ellipsoid become

\[ \sigma_{jk} = \frac{u m_j}{m_j} \delta_{jk} \left[ \int_{0}^{\infty} \frac{c y^{3/2} e^{-y}}{1 + \omega^2 c^2} dy - i \omega \int_{0}^{\infty} \frac{c y^{3/2} e^{-y}}{1 + \omega^2 c^2} dy \right] \], \quad (4.20)

\[ \sigma_{jkl} = \frac{-uv m^2 j}{m_j m_k} \delta_{jkl} \left[ \int_{0}^{\infty} \frac{c^2(1 - \omega^2 c^2) y^{3/2} e^{-y}}{(1 + \omega^2 c^2)^2} dy - 2i \omega \int_{0}^{\infty} \frac{c^2 y^{3/2} e^{-y}}{(1 + \omega^2 c^2)^2} dy \right] \], \quad (4.21)

and

\[ \sigma_{jklm} = \frac{uv^2 m^3}{m_j m_k m_l} \delta_{jkl} \delta_{mju} \left[ \int_{0}^{\infty} \frac{c^3(1 - \omega^2 c^2) y^{3/2} e^{-y}}{(1 + \omega^2 c^2)^3} dy - i \omega \int_{0}^{\infty} \frac{c^4(1 - \omega^2 c^2) y^{3/2} e^{-y}}{(1 + \omega^2 c^2)^3} dy \right] \]. \quad (4.22)

4(b) The Magnetoconductivity Tensor referred to the Crystal Axes

We again consider the two band structures discussed in 3(c), namely [111] and [100] ellipsoids. The total current density referred to the cubic axes may be written

\[ J_j = \sum \delta_{jk} \mathcal{E}_k + \sum \delta_{jkl} \mathcal{E}_k \mathcal{H}_1 + \sum \delta_{jklm} \mathcal{E}_k \mathcal{H}_1 \mathcal{H}_m \], \quad (4.23)

where \( \sum \delta_{jk} \), \( \sum \delta_{jkl} \) and \( \sum \delta_{jklm} \) are respectively the second, third and fourth rank tensors which are obtained by transforming the tensors for each individual ellipsoid to a common set of cubic axes and summing over all the valleys. The number of independent components to be evaluated may be considerably reduced by symmetry arguments because, as \( \sum \delta_{jk} \), \( \sum \delta_{jkl} \) and \( \sum \delta_{jklm} \) are referred to the cubic axes, they must be invariant under any transformation which belongs to
the cubic point group. Thus, for example, $\Sigma_{jk}$ must be invariant under the two transformations which consist of rotations of $\pi/2$ applied in opposite senses about the 3 axis. The matrices are

$$
\begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} \quad \text{(4.24)}
$$

and

$$
\begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} \quad \text{(4.25)}
$$

From these, applying them in the order in which they are quoted, we have

$$
\begin{pmatrix}
\Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\
\Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\
\Sigma_{31} & \Sigma_{32} & \Sigma_{33}
\end{pmatrix} \equiv \begin{pmatrix}
\Sigma_{22} & \Sigma_{21} & -\Sigma_{23} \\
\Sigma_{12} & \Sigma_{11} & \Sigma_{13} \\
\Sigma_{32} & \Sigma_{31} & \Sigma_{33}
\end{pmatrix} \equiv \begin{pmatrix}
\Sigma_{22} & -\Sigma_{21} & \Sigma_{23} \\
-\Sigma_{12} & \Sigma_{11} & -\Sigma_{13} \\
\Sigma_{32} & -\Sigma_{31} & \Sigma_{33}
\end{pmatrix} \quad \text{(4.26)}
$$

It can be seen from a comparison of the equivalent components in (4.26), together with corresponding results obtained from other point group transformations, that the second rank tensor must be of the form

$$
\Sigma_{jk} = \Sigma^{(2)} \delta_{jk} \quad \text{(4.27)}
$$

where $\Sigma^{(2)}$ is a scalar coefficient common to all the components.

Analogous arguments can be used to show that the third rank tensor must have the form

$$
\Sigma_{jkl} = \Sigma^{(3)} \epsilon_{jkl} \quad \text{(4.28)}
$$

The transformation of $\Sigma_{jkl}$ may be written

$$
\Sigma_{jkl} = l^1_{j} l^1_{k} l^1_{l} \Sigma_{uvw} \quad \text{(4.29)}
$$
and, applying the transformations (4.24) and (4.25) in that order, we have
\[
\begin{align*}
\varepsilon_{123} &\rightarrow -\varepsilon_{213} \rightarrow -\varepsilon_{213} \\
\varepsilon_{112} &\rightarrow \varepsilon_{221} \rightarrow -\varepsilon_{221} \\
\varepsilon_{233} &\rightarrow -\varepsilon_{133} \rightarrow -\varepsilon_{133} \\
\varepsilon_{111} &\rightarrow -\varepsilon_{222} \rightarrow -\varepsilon_{222}
\end{align*}
\] 
\]
(4.30)

etc.

Hence
\[
\varepsilon_{112} = \varepsilon_{221} = \varepsilon_{233} = \varepsilon_{133} = \varepsilon_{111} = \varepsilon_{222}
\]
(4.31)

Suitable point group transformations are easily found to show that all components with at least two subscripts equal are zero, and, if all three subscripts are different, the six components are numerically equal, with the sign depending on whether the subscripts are arranged in a cyclic permutation of 1, 2 and 3 or otherwise. Hence (4.28) is established. The fourth rank tensor \(\varepsilon_{ijklm}\), which has 81 components and transforms in a manner similar to (4.29) but involving products of four direction cosines, can also be simplified.

We can choose suitable pairs of point group transformations such as (4.24) and (4.25) to show that every component which does not have even numbers of equal subscripts, i.e. two pairs, or all four equal, must vanish, because we obtain the same component in the two cases but with opposite signs. For example, it can be shown using (4.24) and (4.25) that
\[
\varepsilon_{1333} = \varepsilon_{2333} = \varepsilon_{1223} = \varepsilon_{2113}
\]
(4.32)

The same transformations also show that the three components with four equal subscripts must be identical, and, similarly, that components such as \(\varepsilon_{1122}\), \(\varepsilon_{1212}\), and \(\varepsilon_{1221}\), with two independent non equal pairs of subscripts, must be invariant under all permutations of 1, 2 and 3. Thus we can have only four
independent non zero components in the fourth rank tensor. These may be written as

\[ \varepsilon_{(4)}^{(j=k=l=m)}, \varepsilon_{(4)}^{(j=k\neq l=m)}, \varepsilon_{(4)}^{(j=l\neq k=m)}, \varepsilon_{(4)}^{(j=m\neq k=l)} \]  (4.33)

It does not follow from the argument outlined above that components of the type \( \sigma_{12}^\prime, \sigma_{133}^\prime, \sigma_{1233}^\prime \), etc. are necessarily zero when the tensors for a single ellipsoid are transformed from the principal axes of the ellipsoid to the cubic axes of the crystal. (As in (c) the primes are again used to denote single ellipsoid/tensors referred to the latter axes.) For \([111]\) ellipsoids, in fact, none of these components vanish. However, the transformations (3.44) and (3.48), which are applied to express the tensors for each ellipsoid with reference to a common set of cubic axes, have only one non zero component in each row and column, and it therefore follows that they must transform any particular component into a single component having the same number of different subscripts. The pattern of equal groups of numbers must also be maintained, i.e. \( \sigma_{133}^\prime \) could be transformed into \( \sigma_{233}^\prime \) but never into \( \sigma_{223}^\prime \). Thus, when the summation is carried out, \( \sigma_{12}, \sigma_{133}, \sigma_{1233} \), etc. are only required to evaluate \( \varepsilon_{12}, \varepsilon_{133}, \varepsilon_{1233} \), etc. which are known by symmetry arguments to vanish. The individual ellipsoid contributions to each component which is one of the kinds indicated above must therefore sum identically to zero, and we do not need to calculate such components, either for a single ellipsoid or in the total tensor. It must be remembered, however, that the equalities between the non zero components expressed in (4.27), (4.28) and (4.33) are not valid for the corresponding components of the tensors, as a single ellipsoid does not have cubic symmetry. It is therefore necessary to calculate individually all the components of \( \sigma_{jk}^\prime, \sigma_{jkl}^\prime, \) and \( \sigma_{jklm}^\prime \) which are required for the summations giving the non zero \( \varepsilon_{jk}, \varepsilon_{jkl} \) and \( \varepsilon_{jklm} \) components.

(i) \([111]\) orientated ellipsoids

It is only necessary to evaluate one example of each of the six
different components defined in (4.27), (4.28) and (4.33). We consider $\Sigma_{11}$, $\Sigma_{123}$, $\Sigma_{1111}$, $\Sigma_{1122}$, $\Sigma_{1212}$ and $\Sigma_{1221}$ below.

First we find the components of the $\sigma'$ tensors required for the summation, by using the transformations of (3.44) to express all the individual ellipsoid tensors with reference to a common set of cubic axes. We obtain, in terms of the components of $\sigma'_{jk}$, $\sigma'_{jkl}$ and $\sigma'_{jklm}$,

\[
\begin{align*}
\Sigma_{11} &= \sigma'_{11} + \sigma'_{22} + \sigma'_{11} + \sigma'_{22} \\
\Sigma_{123} &= \sigma'_{123} + \sigma'_{213} + \sigma'_{123} + \sigma'_{213} \\
\Sigma_{1111} &= \sigma'_{1111} + \sigma'_{2222} + \sigma'_{1111} + \sigma'_{2222} \\
\Sigma_{1122} &= \sigma'_{1122} + \sigma'_{2111} + \sigma'_{1122} + \sigma'_{2111} \\
\Sigma_{1212} &= \sigma'_{1212} + \sigma'_{2121} + \sigma'_{1212} + \sigma'_{2121} \\
\Sigma_{1221} &= \sigma'_{1221} + \sigma'_{2112} + \sigma'_{1221} + \sigma'_{2112}
\end{align*}
\]

(4.34)

For the particular case of [111] ellipsoids the 1, 2 and 3 axes are equivalently placed with respect to an ellipsoid, and the components of all the $\sigma'$ tensors must therefore be invariant under permutations of the subscripts 1, 2 and 3, despite the absence of cubic symmetry. (The cyclic or non cyclic ordering must be maintained for $\sigma'_{jkl}$ as the sign alters on interchanging 2 subscripts only.) Applying these conditions to (4.34), we see that the summation reduces to multiplying the first component on the right hand side of the equations by four. For these components, using (3.37) to transform to the cubic axes as in 3(c)(i), we find

\[
\begin{align*}
\sigma'_{11} &= \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \\
\sigma'_{123} &= \frac{1}{3}(\sigma_{123} + \sigma_{231} + \sigma_{312}) = -\sigma'_{213} \\
\sigma'_{1111} &= \frac{2}{9} \left( \frac{1}{4} \sigma_{1122} + \sigma_{233} + \sigma_{331} - \frac{2}{4} \sigma_{1212} \right) (j=k=1=m)
\end{align*}
\]
\[
\sigma'_{1122} = \frac{1}{18}(7\sigma_{1122} + 4\sigma_{2233} + 4\sigma_{3311} - 3\sigma_{1212}) \quad (j=k, l=m)
\]
\[
\sigma'_{1212} = \frac{1}{18}(\sigma_{1122} - 2\sigma_{2233} - 2\sigma_{3311} + 15\sigma_{1212}) \quad (j=1, k=m)
\]
\[
\sigma'_{1221} = \frac{1}{18}(\sigma_{1122} - 2\sigma_{2233} - 2\sigma_{3311} - 3\sigma_{1212}) \quad (j=m, k=1)
\]

Finally we substitute (4.20), (4.21) and (4.22) in (4.35) and, on multiplying by four, find for the required components of the \( \Sigma \) tensors
\[
\begin{align*}
\Sigma_{jk} &= \frac{4}{3}u(2K+1)\varepsilon_{jk} A(\omega) \\
\Sigma_{jkl} &= -\frac{4}{3}uvK(2K+2)\varepsilon_{jkl} B(\omega) \\
\Sigma_{jklm} &= -\frac{8}{9}uv^2(K-1)^2 C(\omega) \quad (j=k=l=m) \\
\Sigma_{jklm} &= -\frac{4}{9}uv^2(K+2) C(\omega) \quad (j=k,l=m) \\
\Sigma_{jklm} &= \frac{4}{9}uv^2(K+7K+1) C(\omega) \quad (j=1,k=m) \\
\Sigma_{jklm} &= \frac{4}{9}uv^2(K+1)^2 C(\omega) \quad (j=m,k=1)
\end{align*}
\]

where \( A(\omega) \), \( B(\omega) \) and \( C(\omega) \) represent the contents of the square brackets in (4.20), (4.21) and (4.22) respectively. If we now introduce the reciprocity of the magnetic field as an additional symmetry property of the general field configuration, we see from (4.23) that interchange of the 1 and \( m \) subscripts in the fourth rank tensor must leave the conductivity invariant and hence, while components such as \( \Sigma_{1111} \) and \( \Sigma_{1122} \) are not affected, we must have \( \Sigma_{1212} = \Sigma_{1221} \), etc. The last two components of (4.36) may therefore be replaced by a single averaged component of the form
\[
\Sigma_{jklm} = \frac{2}{9}uv^2(K+2) C(\omega) \quad (j\neq k \text{ and } l\neq m) \quad (4.37)
\]

All other components are zero.
(ii) [100] orientated ellipsoids

The transformations in (3.48) are used, as in 3(c)(ii), to refer the tensors for the three ellipsoids to a common set of cubic axes, prior to summing the contributions. The six non equal components of the $\Sigma$ tensors are found to be

\[
\begin{align*}
\Sigma_{11} &= \sigma_{33} + \sigma_{22} + \sigma_{11} \\
\Sigma_{123} &= \sigma_{312} + \sigma_{231} + \sigma_{123} \\
\Sigma_{1111} &= \sigma_{3333} + \sigma_{2222} + \sigma_{1111} = 0 \\
\Sigma_{1122} &= \sigma_{3311} + \sigma_{2233} + \sigma_{1122} \\
\Sigma_{1212} &= \sigma_{3131} + \sigma_{2323} + \sigma_{1212} \\
\Sigma_{1221} &= \sigma_{3113} + \sigma_{2332} + \sigma_{1221} = 0 \\
\end{align*}
\]

(4.38)

We substitute the components of the $\sigma$ tensors from (4.20), (4.21) and (4.22) and, introducing the reciprocity of the magnetic field as in (i), we find for the general form of the required components of the $\Sigma$ tensors

\[
\begin{align*}
\Sigma_{jkc} &= u(2K+1) \delta_{jk} A(\omega) \\
\Sigma_{jkl} &= -uvK(K+2) \epsilon_{jkl} B(\omega) \\
\Sigma_{jklm} &= -uv^2K(K^2+K+1) \alpha(\omega) & (j=k\neq l=m) \\
\Sigma_{jklm} &= \frac{3}{2}uvK^2 \alpha(\omega) & (j\neq k \text{ and } l\neq m) \\
\end{align*}
\]

(4.39)

All other components are zero.
4(c) The Magnetoconductivity Tensor for $\mathbf{H}$ contained in certain Crystallographic Planes

(i) $\mathbf{H}$ parallel to cubic axis

The $x$, $y$ and $z$ axes are coincident with the 1, 2 and 3 axes respectively, and the components of the current density, which are given by (4.23), therefore assume the form

$$
\begin{align*}
J_1 &= \mathcal{E}_{11} \mathbf{E}_1 + \mathcal{E}_{123} \mathbf{E}_2 H_3 + \mathcal{E}_{1133} \mathbf{E}_1 H_3^2 + \ldots, \\
J_2 &= \mathcal{E}_{22} \mathbf{E}_2 + \mathcal{E}_{213} \mathbf{E}_1 H_3 + \mathcal{E}_{2233} \mathbf{E}_2 H_3^2 + \ldots, \\
J_3 &= \mathcal{E}_{33} \mathbf{E}_3 + \mathcal{E}_{3333} \mathbf{E}_3 H_3^2 = 0.
\end{align*}
$$

(4.40)

It can be seen that only the three independent components $\mathcal{E}_{11}$, $\mathcal{E}_{123}$ and $\mathcal{E}_{1133}$, which are given in (4.36) and (4.39) for [111] and [100] ellipsoids respectively, are required to evaluate the Faraday effect. Using (3.14) and (4.40), the components of $S$, the closed magnetoconductivity tensor, can be related to the $\mathcal{E}$ tensors as follows

$$
\begin{align*}
S_{11} &= S_{22} \rightarrow \mathcal{E}_{11} + \mathcal{E}_{1133} H_3^2 + o(H^4) \\
S_{12} &= -S_{21} \rightarrow \mathcal{E}_{123} H_3 + o(H^3) \\
S_{33} &= \mathcal{E}_{33} + \mathcal{E}_{3333} H_3^2 + o(H^4)
\end{align*}
$$

(4.41)

In the following sub-sections the $\mathcal{E}$ tensors are transformed to the $(x, y, z)$ co-ordinate system defined in the similarly entitled section of chapter 3. We note first however that, apart from a scalar coefficient, the second and third rank tensors are respectively the kronecker delta and the epsilon tensor, both of which are invariant under all orthogonal co-ordinate transformations, (see for example Jeffreys and Jeffreys, 1962). Thus $\mathcal{E}_{jk}$ and $\mathcal{E}_{jkl}$ are given by the first two equations of (4.36) or (4.39), independently
of the relative orientations of the (1,2,3) and (x,y,z) axes. In the latter system the most general form for the components of the current density is, from (4.23),

\[
J_x = \varepsilon^{(2)}_x + \varepsilon^{(3)}_y H_z + (\varepsilon_{xxzz} \varepsilon_x + \varepsilon_{yyzz} \varepsilon_y + \varepsilon_{zzzz} \varepsilon_z) H_z^2
\]

\[
J_y = \varepsilon^{(2)}_y - \varepsilon^{(3)}_x H_z + (\varepsilon_{yxzz} \varepsilon_x + \varepsilon_{yyzz} \varepsilon_y + \varepsilon_{zzzz} \varepsilon_z) H_z^2
\]

\[
J_z = \varepsilon^{(2)}_z + (\varepsilon_{xxzz} \varepsilon_x + \varepsilon_{yyzz} \varepsilon_y + \varepsilon_{zzzz} \varepsilon_z) H_z^2
\]

(4.42)

\[\Sigma_{jk}\] is abbreviated to \[\Sigma^{(2)}\] and \[\Sigma_{jkl}\] to \[\Sigma^{(3)}\], the positive sign being required for a cyclic permutation of (x,y,z) and the negative sign otherwise. The symmetry relations derived for the (1,2,3) system are no longer valid, and the nine fourth rank components in (4.42) must all be considered in the calculation. The transformation is of the form

\[
\Sigma_{jklm} = \delta_{js} \delta_{kt} \delta_{lu} \delta_{mv} \Sigma_{stuv}
\]

(4.43)

and the connection between the tensor coefficients of the series solution and \(S\) can be obtained from a comparison of (3.14) and (4.42). We find

\[
S_{xx} = \Sigma^{(2)} + \Sigma_{xxzz} H_z^2 + \ldots
\]

\[
S_{yy} = \Sigma^{(2)} + \Sigma_{yyzz} H_z^2 + \ldots
\]

\[
S_{zz} = \Sigma^{(2)} + \Sigma_{zzzz} H_z^2 + \ldots
\]

\[
S_{xy} = \Sigma^{(3)} H_z + \Sigma_{xyzz} H_z^2 + \ldots
\]

\[
S_{yx} = -\Sigma^{(3)} H_z + \Sigma_{yxzz} H_z^2 + \ldots
\]

(4.44)
\[ S_{yz} = \sum yzzz H_z^2 + \ldots \]
\[ S_{zy} = \sum yzze H_z^2 + \ldots \]
\[ S_{zx} = \sum zzzz H_z^2 + \ldots \]
\[ S_{xz} = \sum xzzz H_z^2 + \ldots \]

\( A, \beta, \epsilon, \text{ and } \Omega, \) and hence \( J^2, \) may be evaluated for the series solution using (4.44) in conjunction with (2.6), (2.7) and (2.8).

(ii) \( H \) in (100) plane

Some reduction in the calculation is brought about on comparing (4.44) with the appropriate form for \( S, \) as given in (A.16). This shows that, for consistency, we require

\[
\begin{align*}
\sum yzzz &= -\sum yxzz \\
\sum zzzz &= -\sum xzzz \\
\sum yzzz &= \sum yz zz
\end{align*}
\]

Using (3.56) to transform the six independent components which remain we find, expressing the results in terms of \( \Sigma_{1111}, \Sigma_{1122} \) and \( \Sigma_{1212}, \)

\[
\begin{align*}
\Sigma_{zzzz} &= \Sigma_{1111} + 2 \sin^2 \phi \cos^2 \phi (\Sigma_{1122} + 2 \Sigma_{1212} - \Sigma_{1111}) \\
\Sigma_{zzzz} &= \Sigma_{1122} \\
\Sigma_{yyzz} &= \Sigma_{1122} + 2 \sin^2 \phi \cos^2 \phi (\Sigma_{1111} - 2 \Sigma_{1212} - \Sigma_{1122}) \\
\Sigma_{xyzz} &= 0 = \Sigma_{yxzz} \\
\Sigma_{xzzz} &= 0 = \Sigma_{zxzz}
\end{align*}
\]
\[ \Sigma_{yzzz} = \frac{1}{2} \sin 2\phi \cos 2\phi (\Sigma_{1122} + 2 \Sigma_{1212} - \Sigma_{1111}) \]

\[ = \Sigma_{zyzz} \]

The final forms of these components for [111] and [100] ellipsoids, obtained by substituting for \( \Sigma_{1111}, \Sigma_{1122} \) and \( \Sigma_{1212} \) from (4.36), (4.37) and (4.39) respectively, are given in sections I and II of appendix C.

(iii) \( H \) in (110) plane (x parallel to [110])

The relations given in (4.45) are also applicable to the present case, as \( S \) again has the symmetry of \( (4.16) \). The transformation (3.68) may be applied to find the six independent components which assume the form

\[ \Sigma_{zzzz} = \Sigma_{1111} + \frac{\sin^2 \phi}{2}(3 \cos^2 \phi + 1)(\Sigma_{1122} + 2 \Sigma_{1212} - \Sigma_{1111}) \]

\[ \Sigma_{xxzz} = \Sigma_{1122} + \frac{\sin^2 \phi}{2}(\Sigma_{1111} - \Sigma_{1222} - 2 \Sigma_{1212}) \]

\[ \Sigma_{yyzz} = \Sigma_{1122} + \frac{3 \sin^2 \phi}{2} \cos^2 \phi (\Sigma_{1111} - \Sigma_{1122} - 2 \Sigma_{1212}) \]

\[ \Sigma_{xyzz} = 0 = \Sigma_{yxxx} \]

\[ \Sigma_{xzzz} = 0 = \Sigma_{zxzz} \]

\[ \Sigma_{yxzz} = \frac{1}{2} \sin \phi \cos \phi (3 \cos^2 \phi - 1)(\Sigma_{1122} + 2 \Sigma_{1212} - \Sigma_{1111}) \]

\[ = \Sigma_{zyzz} \]

The components for [111] and [100] ellipsoids are given in sections III and IV respectively of appendix C.
(iv) \textit{H in (110) plane}

(x and y equally inclined to 1 and 2 axes respectively)

The symmetry of S is given by (4.20) and, by comparison with (4.44), the following equalities may be deduced

\[
\begin{align*}
\Sigma_{xxzz} &= \Sigma_{yyzz} \\
\Sigma_{zyzz} &= \Sigma_{zzzz} \\
\Sigma_{yzzz} &= \Sigma_{zzzz}
\end{align*}
\]

\begin{align}
(4.48)
\end{align}

Transforming the six independent components according to (3.80) we find

\[
\begin{align*}
\Sigma_{zzzz} &= \Sigma_{1111} + \frac{\sin^2 \phi (3 \cos^2 \phi + 1)}{2} (\Sigma_{1122} + 2 \Sigma_{1212} - \Sigma_{1111}) \\
\Sigma_{xxzz} &= \Sigma_{1122} + \frac{\sin^2 \phi (3 \cos^2 \phi + 1)}{4} (\Sigma_{1111} - \Sigma_{1122} - 2 \Sigma_{1212}) \\
&= \Sigma_{yyzz} \\
\Sigma_{zyzz} &= \frac{\sin \phi}{2,2^2} (3 \cos^2 \phi - 1) (\Sigma_{1122} + 2 \Sigma_{1212} - \Sigma_{1111}) \\
&= \Sigma_{xzzz} = \Sigma_{yzzz} = \Sigma_{zzzz} \\
\Sigma_{xycz} &= \frac{\sin^2 \phi}{4} (3 \cos^2 \phi - 1) (\Sigma_{1111} - 2 \Sigma_{1212} - \Sigma_{1122}) \\
&= \Sigma_{yxzz}
\end{align*}
\]

(4.49)

The components for \([111]\) and \([100]\) ellipsoids are given in sections V and VI respectively of appendix C.
CHAPTER 5

The Faraday Effect in an Isotropic Semiconductor

Calculations of the Faraday rotation and ellipticity for a single band model of a hypothetical semiconductor with spherical surfaces of constant energy are presented in this chapter. The model is referred to as 'isotropic'. The two cases of lattice scattering and ionized impurity scattering are dealt with separately, and the dependence of $\rho$ and $\Delta$ on the external parameters $\omega$, $H$, and $T$ is considered in some detail.

5(a) General Formulation

The closed form of the magnetoconductivity tensor for $K = 1$ was derived in 3(i). Substituting the components from (3.91) in (2.6), (2.7) and (2.8), we find that the so-called 'plasma terms' $F_{xx}, F_{zx}, F_{xz}$, etc. arising from $E_z$ are all zero and

$$\begin{align*}
\mathcal{A} &= \mathcal{D} = F_{xx} = -\frac{3e}{c^2} + \frac{4\pi i\omega}{c^2} u \alpha(\mathcal{R}) \\
\mathcal{B} &= -\mathcal{C} = F_{xy} = -\frac{4\pi i\omega}{c^2} u \beta(\mathcal{R})
\end{align*}$$

where

$$u = \frac{4ne^2}{3\pi^2 m^*} .$$

Hence the isotropic case discussed in 2(e)(i) applies to this model, independently of the orientation of $H$ with respect to the crystal axes, and the propagation constants are therefore given by (2.37),

$$\mathcal{R}^2 = (\alpha_\perp + i\beta_\perp)^2 = \mathcal{A} + i\mathcal{B} .$$

Expressing them directly as functions of the integrals $I_s(\gamma)$, by substituting (5.2), (3.33) and (3.34) in (5.3), we find
\[ \mu_\pm = \left[ -\frac{\omega \epsilon}{c^2} + \frac{4\pi i\omega u}{c^2} \left\{ I_1(\omega \mp \lambda) - i(\omega \mp \lambda) I_2(\omega \mp \lambda) \right\} \right]^{\frac{1}{2}} \tag{5.4} \]

\( \Theta \) and \( \Delta \) are then given by (2.33) and (2.34). The numerical work therefore consists of the tabulation of \( I_1(\gamma) \) and \( I_2(\gamma) \) for the appropriate values of \( \gamma, \omega \) and \( u \), evaluation of the real and imaginary parts of the complex square root (5.4), and direct substitution in the simple equations (2.33) and (2.34).

5(b) **Scattering Mechanisms**

Since the integrals \( I_\alpha(\gamma) \) depend on \( \gamma \), assumptions must now be made about the nature of the scattering occurring in the semiconductor. If \( \gamma \) is taken to be a constant, independent of the electron energy, the integrals can be evaluated analytically, the general result being

\[ I_\alpha(\gamma) = \frac{\gamma^2}{1+\gamma^2} \left\{ \frac{3\pi^{\frac{3}{2}}}{4} \right\} \tag{5.5} \]

The Faraday rotation has been calculated in this approximation by Stephen and Lidiard, (1958), and we shall not consider it any further, except for purposes of comparison, as it is not theoretically justified for a non-degenerate system of electrons obeying Maxwell-Boltzmann statistics.

We may assume alternatively that scattering arises from the interaction of electrons and lattice vibrations, (see for example Bardeen and Shockley, 1950). The scattering is isotropic, with a mean free path independent of the velocity and inversely proportional to the temperature, provided the edge of the band occurs at \( k = 0 \), as for a spherical energy surface. The approximations leading to this result involve neglecting the temperature dependence of quantities such as the effective mass and the longitudinal elastic constants of the material. Such effects are generally considered to be small, and \( \gamma_L \) can thus be written as
\[ T_L = \frac{p_l}{V} = \frac{p_L'}{kT} y^{-\frac{3}{2}} = \frac{p_L'}{(kT)^{3/2}} y^{-\frac{3}{2}} = ay^{-\frac{3}{2}}, \quad (5.6) \]

where \( a \) is proportional only to \( T^{-3/2} \) for a given specimen.

The integrals then become

\[
I_1(\omega \mp \lambda) = a \int_0^\infty \frac{y e^{-y} dy}{y + a^2(\omega \mp \lambda)^2} = a I_1^L (\omega \mp \lambda),
\]

\[
I_2(\omega \mp \lambda) = a^2 \int_0^\infty \frac{y e^{-y} dy}{y + a^2(\omega \mp \lambda)^2} = a^2 I_2^L (\omega \mp \lambda),
\]

and the propagation constants may be written

\[
\mu_\pm = \left[ -\frac{\omega^2 e}{c^2} + \frac{4 \pi i \omega e}{c^2} \sigma_o^L \left\{ I_1^L (\omega \mp \lambda) - ia(\omega \mp \lambda) I_2^L (\omega \mp \lambda) \right\} \right]^{1/2}, \quad (5.8)
\]

where \( \sigma_o^L \) is the D.C. conductivity for lattice scattering,

\[
\sigma_o^L = \frac{4Ne^2a}{3\pi^{3/2}m^*}. \quad (5.9)
\]

Lattice scattering is always present, but in addition, in extrinsic semiconductors, scattering from ionized impurities may be important. An approximate classical treatment by Conwell and Weisskopf, (1950), considered Rutherford scattering from a system of ions distributed uniformly throughout the lattice. This led to the following form for \( \tau_T \)

\[
\tau_T = \frac{\varepsilon^2m^*V^3}{2\pi N_i e^4} \left[ \log \left\{ 1 + \frac{\varepsilon^2m^*V^4}{4e^4N_i^{2/3}} \right\} \right]^{-1}, \quad (5.10)
\]

\( N_i \) being the density of ionized impurities which is not temperature dependent at temperatures sufficiently high to produce virtually complete ionization. It is usual to neglect also the velocity dependence of the slowly varying logarithmic term. It can be replaced by a suitable mean value so that we may write for \( \tau_T \),
where \( g \) depends only on \( T \) for a given specimen.

The integrals become

\[
I_1(\omega \mp \lambda) = \int_0^{\infty} \frac{y^3 e^{-y}}{1 + \frac{y^2}{3} (\omega \mp \lambda)^2} \, dy = \int_1 \left( \frac{y}{y_1} \right) \left( \frac{y}{y_2} \right) \, dy
\]

and the propagation constants may be written

\[
\sqrt{\frac{\omega}{c^2} + \frac{4\pi e \omega \sigma_o}{c^2}} \left\{ \frac{I_1(\omega \mp \lambda)}{I_2(\omega \mp \lambda)} - i g(\omega \mp \lambda) \right\} = \frac{I}{2},
\]

where \( \sigma_o \) is the D.C. conductivity for ionized impurity scattering, given by

\[
\sigma_o = \frac{8\pi e^2 n}{N \lambda_{\text{m*}}^3}.
\]

Scattering from dislocations in the lattice was discussed by Dexter and Seitz, (1952), but several authors, (see for example Debye and Conwell, 1954), have deduced from experimentally determined mobilities that it is not important in good material, except at temperatures of only a few °K. Scattering from neutral impurities is also not very important for many specimens, except at low temperatures, as many impurity atoms are ionized unless the samples are very highly doped.

In the present work calculations are made for the first two mechanisms only, using the approximate equations (5.11) and (5.12) for \( I_1 \) and \( I_2 \) respectively. It seems reasonable to expect that the use of more sophisticated expressions for \( I \) would not alter the main features of the calculated Faraday rotation, but would merely change the numerical values by a few percent. The temperature dependence graphs will be most susceptible to modification in this respect. It is possible to take both lattice
and impurity scattering into account if they are regarded as independent mechanisms, so that the resultant $\Sigma$ may be calculated by the reciprocal sum rule:

$$\frac{1}{\Sigma} = \frac{1}{\Sigma_L} + \frac{1}{\Sigma_\text{I}}$$

from which we find

$$\Sigma = \frac{\frac{\delta \gamma}{a}}{1 + \frac{\delta \gamma}{a} \gamma^2}$$

Impurity scattering increases in importance with increasing carrier concentration, and also with decreasing temperature as the lattice vibrations become less energetic. However, the lattice contribution only becomes negligible at very low temperatures, so that (5.16) should be used for high conductivity specimens to fit experimental data. A mixed scattering calculation was not undertaken at present because the integrals are complicated, and it is also difficult to estimate $a$ and $\delta$ separately, so that the ratio $\delta / a$ must normally be regarded as an adjustable parameter.

The data used was obtained from the D.C. conductivity and Hall effect measurements of Debye and Conwell, (1954), on N-type germanium specimens, which were assumed to satisfy the conditions of the present chapter. For lattice scattering we use sample 51. The Hall coefficient $R_H$ is given by

$$R_H = -\frac{3 \pi}{\text{Ohm} \cdot \text{cm}}$$

Thus, taking approximate values for $\sigma_0^L$ and $R_H$ from figures 2 and 3 respectively, we can find $N$ and the relaxation parameter $a$, if a value is assumed for the effective mass. We use $m^* = m/4$ and this gives at 300°K

$$\sigma_0^L = 1.0 \times 10^{11} \text{ e.s.u.}$$

$$N = 1.3 \times 10^{14} \text{ cm}^{-3}$$

$$a = 1.0 \times 10^{-12} \text{ sec}$$

$$m^* = 2.28 \times 10^{-28} \text{ gm}$$

(5.18)
Reference to figure 4 of the same paper shows that, over the range 100-300 °K, the temperature dependence of the Hall mobility is in good agreement with the theoretical $T^{-3/2}$ power law for lattice scattering, so that impurity scattering can reasonably be neglected over this range for such a lightly doped specimen. The values of $a$ used to calculate the temperature dependence were computed from the experimental data at 300 °K, by the use of the $T^{-3/2}$ law, this procedure being justified to a good approximation by the experimental data of Debye and Conwell (1954). $N$ was constant over the range due to complete ionization of the impurities.

The impurity scattering calculations are based on data from sample 61 over the range 15-100 °K. The Hall coefficient is given by

$$R_H = -\frac{9/2! \cdot 3/2!}{3! \cdot 3! \cdot \text{Nec} \cdot \text{Nec}} = -1.23,$$

(5.19)

and, using (5.14) and (5.19) in conjunction with the experimental measurements, we find at 100 °K

$$\sigma_0 = 3.0 \times 10^{13} \text{ e.s.u.}$$
$$N = 4.0 \times 10^{16} \text{ cm}^{-3}$$
$$\sigma = 1.6 \times 10^{-13} \text{ sec}$$
$$m^* = 2.28 \times 10^{-28} \text{ gm}$$

(5.20)

The theoretical $T^{-3/2}$ power law for the temperature dependence of the Hall mobility, (neglecting the logarithmic term), is not found experimentally to be valid, as can be seen from figure 4 of Debye and Conwell's paper. However, the mobility at least decreases with decreasing temperature over the range, and it seems probable that impurity scattering is dominant, though the data (5.20) calculated on this basis must be regarded as approximate only, since the scattering is not adequately described by this simple picture.
5(c) Approximate Expressions for the Rotation and Ellipticity

In two limiting cases simple approximate formulae for \( \Theta \) and \( \Delta \) can be derived. Both formulae are valid for low magnetic fields, i.e. \((\lambda \tau)^2 \ll 1\), the more useful being the high frequency expressions, \((\omega \tau \gg 1)\). The region appropriate to the low frequency expressions, \((\omega \tau \ll 1)\), is not generally accessible experimentally. We consider this case first.

(i) Low frequency, lattice scattering

Putting in the two conditions \((\lambda \alpha)^2 \ll 1\) and \((\omega \alpha)^2 \ll 1\), and evaluating the integrals \((5.7)\) to a first approximation we find

\[
\begin{align*}
I_1^L (\omega + \lambda) & \to 1 \\
I_2^L (\omega - \lambda) & \to \frac{\pi}{2}
\end{align*}
\]

Hence, substituting in \((5.8)\),

\[
\mu_\perp^2 \to -\frac{\omega^2 \epsilon}{c^2} + 4 \frac{\pi \sigma_0^L}{c^2} \omega a(\omega + \lambda) \frac{\pi}{2} + \frac{4 \pi \omega \sigma_0^L}{c^2}
\]

This equation can be solved for \( \delta_\perp \) and \( \beta_\perp \) in the usual way. Then, if the displacement current is much smaller than the conduction current, the real part of \((5.22)\) is small compared to the imaginary part, and the square roots in the standard forms for \( \delta_\perp \) and \( \beta_\perp \) can be expanded appropriately. Retaining only first order terms in the ratio of the real to imaginary parts we find

\[
\delta_\perp = \frac{1}{2^{a}} \left[ \frac{4 \pi \omega \sigma_0^L}{c^2} \right]^{1/2} \left[ 1 + \frac{1}{2} \left( -\frac{\omega^2 \epsilon}{c^2} + 4 \frac{\pi \sigma_0^L}{c^2} \omega a(\omega + \lambda) \frac{\pi}{2} + \frac{4 \pi \omega \sigma_0^L}{c^2} \right) \right]^{-1}
\]

\[
\beta_\perp = \frac{1}{2^{a}} \left[ \frac{4 \pi \omega \sigma_0^L}{c^2} \right]^{1/2} \left[ 1 - \frac{1}{2} \left( -\frac{\omega^2 \epsilon}{c^2} + 4 \frac{\pi \sigma_0^L}{c^2} \omega a(\omega + \lambda) \frac{\pi}{2} + \frac{4 \pi \omega \sigma_0^L}{c^2} \right) \right]^{-1}
\]

and hence

\[
\frac{J}{\beta} = \mp \frac{1}{2^{a}} \left[ \frac{4 \pi \omega \sigma_0^L}{c^2} \right]^{1/2} \frac{eaH}{m^*c} \frac{\pi}{2}
\]
But, from (5.9) and (5.17),

$$\sigma_L^L = -\frac{\frac{3}{2}e^2a}{2m^*c}, \quad (5.25)$$

where $R_H^L$ is the Hall coefficient for electrons for lattice scattering. Substituting this result and using (2.33) and (2.34), we finally obtain

$$\Theta = \text{Tanh}^{-1}\frac{\Delta}{Hd} = \frac{B_H^L}{c} \left[ \frac{\pi \omega (\sigma_L^L)}{2} \right]^{\frac{1}{2}}, \quad (5.26)$$

choosing the positive sign for $\Delta$ in (2.34).

(ii) Low frequency, impurity scattering

In this case the integrals can be approximated as follows

$$I_1 (\omega \mp \lambda) \rightarrow 3!, \quad I_2 (\omega \mp \lambda) \rightarrow \frac{9!}{2^3}, \quad (5.27)$$

and we find

$$\mu^2 \rightarrow -\frac{\omega^2}{c^2} + \frac{4 \pi \sigma_o^I}{c^2} \omega g(\omega \mp \lambda) \frac{9!}{2^3} + i \frac{4 \pi \omega \sigma_o^I}{c^2}. \quad (5.28)$$

Expanding the real and imaginary parts of (5.28) as in (i) gives

$$\alpha^* = \frac{1}{2^{\frac{3}{2}}} \left[ \frac{4 \pi \omega \sigma_o^I}{c^2} \right] \frac{1}{2} \left[ 1 + \frac{1}{2} \left\{ -\frac{\omega^2}{c^2} + \frac{4 \pi \sigma_o^I}{c^2} \omega g(\omega \mp \lambda) \frac{9!}{2^3} \right\} \left( \frac{4 \pi \omega \sigma_o^I}{c^2} \right) \right]. \quad (5.29)$$

$$\beta^* = \frac{1}{2^{\frac{3}{2}}} \left[ \frac{4 \pi \omega \sigma_o^I}{c^2} \right] \frac{1}{2} \left[ 1 - \frac{1}{2} \left\{ -\frac{\omega^2}{c^2} + \frac{4 \pi \sigma_o^I}{c^2} \omega g(\omega \mp \lambda) \frac{9!}{2^3} \right\} \left( \frac{4 \pi \omega \sigma_o^I}{c^2} \right) \right]$$

and these equations lead to

$$\frac{\alpha^*}{\beta^*} = \pm \frac{1}{2^{\frac{3}{2}}} \left[ \frac{4 \pi \omega \sigma_o^I}{c^2} \right] \frac{1}{2} \frac{e^2}{m^*c} \frac{2^3}{2^3} \frac{1}{3}. \quad (5.30)$$

However, from (5.14) and (5.19),

$$\sigma_o^I \frac{R_H^I}{R_H^T} = \frac{9/2!}{3/2!} \frac{3/2!}{3!}. \quad (5.31)$$
so that (5.30) may be reduced to the form

\[
\Theta = \frac{\text{Tanh}^{-1} \Delta}{\text{Hd}} = \frac{B_H}{\text{Hd}} \left[ \frac{\pi \omega (\sigma_e)^3}{2} \right]^{1/2}.
\]

(5.32)

It can be seen that the two kinds of scattering give the same expressions for \( \Theta \) and \( \Delta \) in this limit, provided the appropriate forms for the D.C. conductivity and Hall coefficient are used. This result is general, and may be derived directly from (5.4) without introducing a specific mechanism, by writing \( \sigma_e \) and \( B_H \) in terms of formal Maxwellian averages throughout. Since \( \text{Tanh}^{-1} \Delta = \Delta \) for small ellipticities, \( \Theta \) and \( \Delta \) are also equal in magnitude in this limit. The rotation is anticlockwise to an observer looking along the direction of propagation, and the resultant ellipse is rotating in a clockwise sense to the same observer.

These formulae have not so far been verified experimentally, as the specimens used in microwave experiments to date have had fairly low carrier concentrations, e.g. Purdyana and Broersma, (1960), had \( N = 3.6 \times 10^{13} \text{ cm}^{-3} \), and the necessary condition that the conduction current be greater than the displacement current has therefore only been satisfied at frequencies lower than the normal microwave range. It should, however, be possible to use thinner specimens with higher carrier concentrations so that the overall absorption is unchanged, thus making the \( \omega^{1/2} \) region of (5.26) and (5.32) experimentally accessible.

(iii) High frequency, lattice scattering

The appropriate conditions are \( (\omega a)^2 \gg 1 \) and \( \omega^2 \gg \sigma_e a^{2} \). These are easily satisfied at infrared frequencies and room temperatures for most semiconductors. For example, fields of ten thousand oersted can generally be used without invalidating the second condition. The integrals (5.7) are approximated by

\[
\begin{align*}
I_1 (\omega \mp \lambda) & \to \frac{2}{a^2} (\omega \mp \lambda)^2 \\
I_2 (\omega \mp \lambda) & \to \frac{3 \pi^{1/2}}{4a^2} (\omega \mp \lambda)^2
\end{align*}
\]

(5.33)
The imaginary part of (5.8) can be neglected in evaluating $\beta_+$, as $\omega a \gg 1$ and the displacement current terms are much greater than the conduction current terms. Hence, expanding the square root to first order and retaining only the term in $\sqrt{\omega}$, i.e. the term linear in $H$, we obtain

$$\beta_+ \to \frac{\omega e_0}{c} \left[1 - \frac{\omega p}{\omega^2}\right]^{1/2} \left[1 \frac{1}{2} \frac{4 \pi \omega \sigma_0}{c^2} \cdot \frac{3 \pi}{4} \frac{1}{\omega^2} \right] \frac{\omega^2}{c^2} \left[1 - \frac{\omega p}{\omega^2}\right]^{-1/2}$$

which leads to

$$\frac{\omega}{Hd} = \frac{2 \pi N e^3}{c^2 \omega^3 \varepsilon_0} \left[1 - \frac{\omega p}{\omega^2}\right]^{1/2}$$

(5.35)

The plasma frequency $\omega_p$ is defined by

$$\omega_p^2 = \frac{4 \pi N e^2}{m^*}$$

(5.36)

The imaginary part of (5.8) must be retained to evaluate $\Delta$, which is zero in the approximation used to calculate $\Theta$. We find

$$\alpha_+ \to \frac{1}{2} \frac{4 \pi \omega \sigma_0}{c^2} \frac{2}{a^2(\omega \mp \lambda)^2} \left[\frac{\omega^2}{c^2} - \frac{4 \pi \omega \sigma_0}{c^2} \cdot \frac{3 \pi}{4} \frac{1}{\omega^2(\omega \mp \lambda)}\right]^{-1/2}$$

(5.37)

from which, keeping only the term linear in $H$ and the first order plasma frequency correction which may be important for high conductivity specimens, the ellipticity is given by

$$\frac{\text{Tanh}^{-1} \Delta}{Hd} = \frac{32 \pi^3 N e^3}{3c^2 m^* \omega^3 \varepsilon_0} \left[1 + \frac{3 \omega p}{4 \omega^2}\right]$$

(5.38)

(iv) High frequency, impurity scattering

The integrals (5.12) are approximated by

$$I_1 \to \frac{1}{g^2} (\omega \mp \lambda)^2$$

$$I_2 \to \frac{3 \pi^3}{4 g^2} (\omega \mp \lambda)^2$$

(5.39)
Since only the term involving $I_2$ is retained to calculate the rotation, $\Theta$ is again given by \((5.35)\). However, for the ellipticity we have, using \((5.13)\) and \((5.39)\),

$$\alpha_\perp \rightarrow \frac{1}{4} \frac{4 \pi \omega \sigma_0}{6c^2} \cdot \frac{1}{g^2(\omega + \lambda)^2} \left[ \frac{\omega^2 \varepsilon}{c^2} - \frac{4 \pi \omega \sigma_0}{6c^2} \cdot \frac{3 \pi \frac{1}{2}}{4g(\omega + \lambda)} \right]^{-\frac{1}{2}} \quad (5.40)$$

and hence, to the same order as the case of lattice scattering,

$$\frac{Tanh^{-1} \Lambda}{Hd} = \frac{16 \pi \frac{1}{2} N \varepsilon^3}{3c^2 m^* \omega^3 g \varepsilon^2} \left[ 1 + \frac{3\omega_p^2}{4\omega^2} \right] \quad (5.41)$$

The plasma terms may be regarded as a correction to the effective dielectric constant, which is significant at frequencies close to the plasma edge, $(\omega = \omega_p)$. As an example, inserting the N-germanium data of \((5.18)\) in \((5.36)\) gives $\omega_p = 3.2 \times 10^{11} \text{ sec}^{-1}$. Thus $N$ must be of the order of $10^{17} \text{ cm}^{-3}$ for plasma effects to be significant. These conditions have, however, been attained in recent thin film Faraday rotation measurements, (Palik et al., 1965). Also, for the data of \((5.18)\) and \((5.20)\), $\omega a(\omega g) \gg 1$ is well satisfied for $\omega \sim 10^{14} \text{ sec}^{-1}$, and similarly for the magnetic field condition as $\mathcal{J} \sim 7 \times 10^7 \text{ H sec}^{-1}$. The approximation \((5.35)\) to the rotation has been used by a number of investigators to measure Faraday effective masses since the original work of Smith and Moss, (1958), on indium antimonide. It is particularly useful in that it is independent of the relaxation time, and hence of the details of the scattering mechanism which are always rather uncertain. Also, the magnetic fields required are much lower than for a cyclotron resonance experiment. There are, however, two disadvantages. Firstly, $N$ must be known, and the usual method is to find it from the Hall coefficient, whose numerical factor is in fact scattering dependent. Furthermore, for band structures which do not have spherical symmetry, the Faraday mass is a complicated function of the components of the effective mass tensor, and these cannot be determined individually from such an experiment.
The ellipticity is smaller by a factor of the order of \((\omega a)^{-1}\) or \((\omega g)^{-1}\), and also more difficult to measure accurately. However, the ratio \(\Theta/\Delta\) should give information on the scattering parameter, \(a\) or \(g\).

5(d) Evaluation of the Integrals

Four integrals are required, two for the lattice scattering case given in (5.7), and two for the impurity scattering case given in (5.12). Initially the lattice scattering integrals were calculated either from tables drawn up by Dingle, Arndt and Roy, (1957), or from asymptotic series also given in their paper. They tabulated the integral

\[
I(\gamma) = \frac{1}{p!} \int_0^\infty \frac{x^p e^{-x}}{x + \gamma} \, dx
\]

for integer and half integer values of \(p\) over the range \(0 \leq \gamma \leq 20\). In general it was possible to carry out satisfactory interpolation using the Besselian formula

\[
f(a_o + hm) = f(a_o) + (m) \delta f_{1\frac{1}{2}} + (m) \delta^2 f_{1\frac{1}{2}} + \frac{1}{3} (m-\frac{1}{2})(m) \delta^3 f_{1\frac{1}{2}} + \cdots, (5.43)
\]

where \(h\) is the interval length and \(m\) the fraction of the interval between \(f(a_o)\) and \(f(a_1)\). \(\delta^{2n+1} f_{1\frac{1}{2}}\) denotes an odd order central difference, e.g. \(\delta f_{1\frac{1}{2}} = f(a_1) - f(a_0)\), and \(\delta^{2n} f_{1\frac{1}{2}}\) the mean value of the two adjacent even order central differences, e.g.

\[
\delta^2 f_{1\frac{1}{2}} = \frac{1}{2} (\delta^2 f_0 + \delta^2 f_1).
\]

\((m)\) etc. are the usual binomial coefficients. The tables give four significant figures, and it was estimated that errors in the interpolated values did not exceed two units in the last decimal place. This method could not be used for \(\gamma < 0.1\), the interval in this range being only 0.1, or for \(\gamma > 20\). The \(p = 3/2\) case was dealt with for small \(\gamma\) by using the ascending series given in equation (17) of Dingle et al., namely
\[
\int_0^{\infty} \frac{p \cdot e^{-x}}{x + \frac{p}{p-1}} \, dx = (p-1)! \left[ 1 - \frac{T}{p-1} + \frac{T^2}{(p-1)(p-2)} - \cdots - \frac{p \cdot e^{-x}}{\sin \pi \frac{p}{p-1}} \right], \quad (5.44)
\]

and the \( p = 2 \) case by use of the series, (equation (18), Dingle et al.),
\[
\int_0^{\infty} \frac{p \cdot e^{-x}}{x + \frac{p}{p-1}} \, dx = (p-1)! \left[ 1 - \frac{T}{p-1} + \frac{T^2}{(p-1)(p-2)} - \cdots + \frac{(-1)^{p-1}}{p-1!} \right] + \left\{ (-1)^p \sum_{t=0}^{\infty} \frac{T^t}{t!} \left[ \frac{\Psi(t) - \ln T}{t} \right] \right\}, \quad (5.45)
\]

\( \Psi(t) \) being \( \frac{d}{dt} \ln(t!) \), which is tabulated in Jahnke and Emde, (1945), page 18. Finally, for \( T > 20 \), the descending series, (Dingle et al., equation (20)),
\[
\int_0^{\infty} \frac{p-1 \cdot e^{-x}}{x + \frac{p}{p-1}} \, dx = (p-1)! \left[ 1 + \frac{p}{(p+T)^2} + \frac{p(p-2T)}{(p+T)^4} + \frac{p(p^2-8pT+6T^2)}{(p+T)^6} + \cdots \right], \quad (5.46)
\]

was suitable for both integrals. Accuracy was generally rather better than for the interpolated values.

These methods were applied to some only of the lattice scattering calculations, as it was possible to use the University of London Mercury Computer to complete this work, and also to carry out the impurity scattering and all subsequent calculations. A library sub-programme issued by the Computer Unit for integrals of the type
\[
I = \int_C f(x) \, dx, \quad (5.47)
\]

with no singularities over the range, was incorporated into the main programmes. The lower limit of integration \( c \), and a desired accuracy chosen to be one part in \( 10^5 \) were specified, and a subroutine to evaluate the appropriate integrands was added. The method involved dividing the interval into several finite ranges, (neglecting the tail at infinity), and using five and six point Gaussian quadrature formula over each range. The contributions
were treated as a convergent series and numerical methods used to estimate the limit, i.e. the integral. If the five and six point formulae did not agree to within the specified accuracy, the intervals were sub-divided and the process repeated with more terms until the results were satisfactory. This method was accurate for all four integrals. It was, however, complicated for the well behaved integrals required, and much faster routines, described in the following chapter, were developed for the anisotropic work.

5(e) Calculation of the Propagation Constants

In the present section the evaluation of the real and imaginary parts of a propagation constant with the form (5.8) or (5.13) is discussed. For convenience we use the notation

\[
\frac{\mu}{c} = \alpha + i \beta
\]

\[
\mu^2 = -D + J_2 + i J_1
\]

where

\[
D = \frac{\omega \epsilon}{c^2}
\]

\[
J_1 = \frac{4 \pi \omega \sigma \rho L}{c^2} I_1(\omega + \mathcal{L})
\]

\[
J_2 = \frac{4 \pi \omega \sigma \rho L}{c^2} a(\omega \mp \mathcal{L}) I_2(\omega \mp \mathcal{L})
\]

(or the equivalent expressions for impurity scattering).

Hence

\[
\alpha = \frac{1}{2^{3/4}} \left\{ (-D+J_2) + \left\{ (-D-J_2)^2 + J_1^2 \right\}^{1/2} \right\}^{1/2}
\]  

\[
\beta = \frac{1}{2^{3/4}} \left\{ (D-J_2) + \left\{ (-D-J_2)^2 + J_1^2 \right\}^{1/2} \right\}^{1/2}
\]

As long as \( D < |J_1| \) and \( |J_2| \), (5.49) may be used as it stands to evaluate \( \alpha \) and \( \beta \), sufficient figures being carried to utilize all of the four or five figures to which the integrals are known, and to minimize rounding errors. The magnetic field terms, which
occur only in $J_1$ and $J_2$, affect the first, or at least the second significant figure of $\alpha_\pm$ and $\beta_\pm$ at fields which produce experimentally realistic rotations and ellipticities, so that, at most, only two out of the four or five figures will be lost in forming the differences $\Delta$ and $\overline{\beta}$.

The final accuracy of $\Delta$ and $\overline{\Delta}$ is thus better than one part in a hundred, which is adequate.

However, as $\omega$ increases, $D$ increases and $|J_1|$ and $|J_2|$ decrease. This has the effect of increasing $\beta_\pm$ rapidly, while $\overline{\beta}$ remains of the same order of magnitude. The dominant term in $\beta_\pm$ is the field independent term $D^2$, which increases linearly with $\omega$, but is essentially lost by subtraction in evaluating $\overline{\beta}$.

Thus, if the overall accuracy is to remain approximately the same, the number of figures carried in the $\beta_\pm$ calculation must be gradually increased as more are lost by the final subtraction. When more than seven or eight figures are required in $\beta_\pm$ for a final two-figure accuracy, the direct method becomes impractical. The alternative adopted was to expand the square roots and subtract out the dominating term $D^2$, thus tabulating essentially $(\beta - D^2)$. After expanding the inner root of (5.49b) we can factorize $(D - J_2)^{\frac{1}{2}}$, and express the result formally as

$$\beta \to \left[ D - J_2 \right]^{\frac{1}{2}} \left[ 1 + D' \right]^{\frac{1}{2}}, \text{ where } |D'| \ll 1 \quad (5.50)$$

Then, factorizing $D^2$ and expanding both square roots, the general form becomes

$$\beta \to D^2(1 + D')(1 + D'') \text{, where } |D''| \text{ and } |D'''| \ll 1 \quad (5.51)$$

The leading term for both $\beta_+$ and $\beta_-$ is $D^2$, which subtracts out of $\overline{\beta}$. For $\beta_+$, $D'$, $D''$ and $D'''$ may be tabulated from the known values of $D$, $J_1$ and $J_2$. Six significant figures were carried, and sufficient terms were included from the square root expansions to give convergence to this number of figures. Then, if we write

$$\beta' = D^2(D'' + D''' + D''') \quad (5.52)$$

and calculate $\beta'_1$, $\overline{\beta}$ is given by the difference of $\beta'_1$ and $\beta'_1$, with the advantage that fewer significant figures are lost in the subtraction, and it is therefore again possible to obtain finally
at least two figure accuracy.

With the same values of $D$, $J_1$ and $J_2$, it can be seen from (5.49a) that the situation when calculating $\chi$ differs in that the significant figures are lost when the term $(-D+J_2)$ is subtracted from the inner square root, which is only slightly greater in magnitude. This root can be expanded, however, and the large terms subtracted out, giving

$$\chi \sim \frac{1}{2\pi} \left[ \frac{D-J_2}{-\frac{1}{2}\left(\frac{J_1}{D-J_2}\right)^2 - \frac{1}{8}\left(\frac{J_1}{D-J_2}\right)^4 + \ldots} \right]^{\frac{1}{2}}. \quad (5.53)$$

On expanding the first root, as above, this can be put in the form

$$\chi \sim \frac{D^{\frac{1}{2}}}{2\pi} (1+D)D'' \quad (5.54)$$

Thus $\chi_\mp$ may be found directly if $D''$ and $D'''$ are tabulated. It should be noted, however, that the expansion of the inner root becomes necessary at lower frequencies than the expansion of $(D-J_2)^{\frac{1}{2}}$, because the leading term involves the second rather than the first power of the ratio of the conductivity to the displacement current.

In hand calculations the optimum point for the changeover in methods could be determined by inspection. It was in the region of $\omega = 5 \times 10^{11}$ to $10^{12}$ sec$^{-1}$ for both $\chi_\mp$ and $\beta_\pm$, when using the lattice scattering parameters given in (5.16) and taking $\lambda = 10^{11}$ sec$^{-1}$. (This corresponds to a field of 1400 oersteds with the effective mass quoted.) The changeover moved to higher frequencies as $H$ was increased and, for $\lambda = 10^{13}$ sec$^{-1}$, was about $5 \times 10^{12}$ to $10^{13}$ sec$^{-1}$. When using the computer for the impurity scattering work, the two methods were written into the programme as alternatives. The change from the direct to the expansion method was made to occur when either $D/|J_1|$ or $D/|J_2|$ exceeded twenty, provided the other ratio was greater than five. Five terms were programmed in all the square root expansions, and checks using both methods for a few trial values showed that the programme always gave at least two, and generally three figure accuracy in $\chi$ and $\beta$. 

5(f) Lattice Scattering – Results

(i) Rotation – Frequency dependence

Using the lattice scattering parameters, which were given in (5.18) and are appropriate to a typical specimen of n-type germanium at room temperature, the Faraday rotation \( \varrho \) was calculated from \( \beta_z \) over the angular frequency range \( 10^9 \) to \( 10^{14} \) sec\(^{-1} \). Four cyclotron frequencies were taken, namely \( \omega_c = 10^{10}, 10^{11}, 10^{12} \) and \( 10^{13} \) sec\(^{-1} \), and the results for the final three are shown in Fig. (5.1) for a thickness of one centimetre. The curve for \( \omega_c = 10^{10} \) was not included as it is very similar to that for \( 10^{11} \), apart from a factor of ten in the overall magnitude. In any case the rotation is unrealistically small at this field, (142 oersteds). It should be noted that the lowest field curve, (a), has been multiplied by ten to give a scale comparable with those for \( 10^{12} \) and \( 10^{13} \).

The general behaviour of \( \varrho \) is similar to that described by Stephen and Lidiard, (1958). They showed that the magnitudes of \( \omega_c \) and \( \omega \) were the main factors governing the behaviour in the degenerate case. It turns out that their conclusions are essentially valid for the case of lattice scattering if \( \omega \) is replaced by \( a \). Thus, for weak fields, i.e., \( (\omega a)^2 \ll 1 \), (curve a), the rotation is negative and proportional to \( \omega^{1/2} \) at frequencies \( \ll 5 \times 10^9 \), being given very closely by (5.26). It then decreases less rapidly and passes through a minimum at about \( 10^{11} \), increases again, and changes sign in the region of \( a^{-1} \), the natural frequency of the system of free carriers. In the degenerate case the system has only the single frequency \( \omega^{-1} \) and, at low fields, the sign change occurs almost exactly at this value, (Stephen and Lidiard, (1958), Fig (1), curve (i)). In the present instance \( a^{-1} \) is only approximately correct, due to the distribution of relaxation times depending on the energy of the electron. In fact the sign change always occurs at a frequency slightly higher than \( a^{-1} \); about \( 1.1 \times 10^{12} \) for curve (a). As \( \omega \) increases further the rotation passes through a small maximum, and finally begins to decrease, eventually reaching the \( \omega^{-2} \) region of (5.35) at about \( 10^{13} \). At this frequency we have \( \omega a = 10 \),
Fig. (5.1). Variation of Faraday rotation ($10^p \theta$ rad) with frequency for lattice scattering $n$-type germanium specimen with $a = 10^{-12}$ sec, $N = 1.3 \times 10^{14}$ cm$^{-3}$ at $300^o$K, $m^* = m/4$, $e = 16$: a, $\mathcal{L} = 10^{11}$ (p = 1); b, $\mathcal{L} = 10^{12}$ (p = 0); c, $\mathcal{L} = 10^{13}$ (p = 0); $d = 1$ cm and $T = 300^o$K for all curves.
\[ \frac{\omega}{J} = 10^2 \] and, since \( J \) is easily calculated to be \( 3.2 \times 10^{11} \), the plasma term is negligible. The exact value is 10\% lower than the approximate value calculated from (5.35) at \( 10^{13} \). The most important correction term to the high frequency expression is of order \( (\omega a)^{-2} \), and it is straightforward to show that (5.35) should be multiplied by \( (1 - 15/2a^2 \omega^2) \). Inclusion of this first order correction reduces the discrepancy to about 2\%. Thus the value of \( \omega a \) is generally the chief factor which determines the low frequency limit to the applicability of (5.35). A value of \( \omega \) 10 would seem to be the minimum acceptable, in view of the factor 15/2. For the specimen under consideration the limit is not important, as \( 10^{13} \) is below the experimentally accessible infra-red frequency range. However, with shorter relaxation time specimens, care should be taken when using (5.35) in the near infra-red. The cyclotron frequency in the low field case is not therefore important and, provided \( (\omega a)^2 \ll 1 \), the whole curve (a) is merely altered proportionately if \( \omega \) is changed.

The next curve, (b), is the transition curve, as \( \omega a = 1 \). The low frequency approximation is no longer valid, though the high frequency approximation can still be used, as \( \omega/J \) remains very large. The most noticeable changes from (a) are that \( \Re \) has increased overall by a factor less than ten, although \( \Re \) has been increased tenfold, while the minimum has flattened out and the sign change moved to a slightly higher frequency.

The rotation in fact is a maximum in general when \( \omega a = 1 \) and, as the field is further increased to satisfy the high field condition \( (\omega a)^2 \gg 1 \), \( \Re \) decreases, (curve (c), \( J = 10^{13} \)). This conclusion is complicated, however, by changes in the overall shape of the curve. A small minimum occurs at \( \omega \simeq 10^{10} \) and, thereafter, the magnitude of \( \Re \) remains essentially constant and independent of \( \omega \), until a frequency quite close to the cyclotron frequency, when a resonance curve somewhat similar to that found, for example, in the refractive index of an insulator at a natural frequency of the system, occurs. It consists of a sharp minimum, a sign change at the resonant frequency \( \omega \), and a sharp maximum on the high frequency side, and is due to the vanishing of the term \( (\omega - \omega) \) associated with the positive
mode. This causes \( I_1^L(\omega, -\lambda) \) and \( I_2^L(\omega, -\lambda) \) to pass through very sharp maxima at the resonance, with consequent sudden changes in \( \beta^+ \), and hence in \( \beta \) and \( \Delta \). The negative mode is not absorbed selectively, so \( \beta^- \) varies quite smoothly through the cyclotron frequency. Well beyond the positive peak the high frequency approximation is again valid, when \( \omega \gg \lambda \) is satisfied. At even higher fields the general pattern is similar, but the magnitude of \( \Delta \) decreases approximately as \( \lambda^{-1} \), and the dispersion curve around \( \omega = \lambda \) becomes even narrower and sharper, while moving to the appropriate higher frequency, with the consequent extension of the flat region. However, since \( \lambda = 10^{13} \) corresponds to a field of 140,000 oersteds, higher fields would seem to be unrealistic for the present specimen.

The low frequency minimum is interesting as it occurs at the so-called 'magnetoplasma edge'. In the negatively rotating wave the signs of the conductivity term involving \( I_2^L(\omega, -\lambda) \) and the dielectric term, \( \omega^2 \epsilon / c^2 \), are different and, provided \( \lambda > \omega \) and \( (\lambda a)^2 > 1 \), they are easily shown to cancel one another when \( \omega \lambda = \omega_p^2 \).

Then, since the \( I_1^L(\omega + \lambda) \) term is small under the same conditions, the absorption coefficient \( \alpha^- \) of the negative wave is large at lower frequencies as the real part of \( \mu^- \) is positive, and falls sharply when the frequency \( \omega_p^2 / \lambda \) is reached, at this point, the real part of \( \mu^- \) becomes negative. On the other hand the real part of \( \mu^+ \) is always negative, and hence \( \alpha^+ \) is low at all the frequencies under consideration and shows no sudden transition at the magnetoplasma edge. The effect on \( \Delta \) will be discussed in \( f(iv) \). The minimum in \( \Delta \) arises because \( \beta^- \) is increasing steadily with increasing frequency, while \( \beta^- \) remains very low until the real part of \( \mu^- \) changes sign and becomes negative, and then increases more sharply than \( \beta^- \). This causes \( \beta^- \) and \( \beta^- \) to diverge from one another until the frequency \( \omega_p^2 / \lambda \), and converge thereafter. Figure (5.2), which shows \( \alpha^- \) and \( \beta^- \) in this frequency region, illustrates these points.

The conditions considered above are very similar to those delineating the helicon region in a semiconductor. If additionally \( (\omega_p a)^2 > 1 \), the zero field absorption \( \alpha_0 \) is high because the
Fig. (5.2). Variation of propagation constants \((\alpha_+ + i\beta_+)\) with frequency in the vicinity of the magnetoplasma edge for lattice scattering n-type germanium specimen: \(J = 10^{13}\) and \(T = 300^\circ\text{K}\). The left hand ordinate scale is to be used for \(\alpha_+\) and the right hand for \(\beta_+\).
conductivity term is greater than the dielectric term. Also, as discussed above, $\alpha$ is high below the magnetoplasma edge. However, $\alpha_+$ is very much lower, so that, although at normal thicknesses the incident wave may be completely attenuated in the absence of an external field, and similarly for the negative wave when a field is applied, the positive wave may be easily observable. This is the circularly polarized helicon mode, whose phase velocity is generally of the same order as the velocity of sound, so that coupling with acoustic waves can be observed, (Agrain, 1960). This picture is correct so long as $\omega p^2/\omega_\lambda \gg 1$. However, when $\omega_\lambda$ becomes equal to $\omega p^2$, either by increasing $\omega$ or $H$ sufficiently, $\alpha$ drops sharply to a value only slightly above $\alpha_+$ (Fig. 5.2), and, because both modes are transmitted, the helicon mode will be elliptically polarized. The true helicon conditions are thus $\omega_\lambda \gg \omega$, $(\alpha a)^2 \gg 1$, $\omega p^2 \gg \omega_\lambda$ and $(\omega a)^2 \gg 1$. The present specimen has too low a conductivity to satisfy the last requirement, so that the difference between $\alpha$ and $\alpha_+$ is not as great as it would be in a true helicon region.

A simple expression for $\rho$ in the region beyond the 'kink' is easily obtained by putting the conditions $(\alpha a)^2 \gg 1$, $\omega_\lambda \gg \omega$ and $\omega p^2/\omega_\lambda \ll 1$ into (5.3). The integrals are approximately given by (5.33) and, expanding $\beta_+$ in a manner similar to that used for the high frequency formula in 5(c)(iii), but in powers of $\omega/\alpha$ rather than $\omega/\kappa$, we obtain

$$\rho = \frac{\omega p^2 \varepsilon \beta_+ \omega_a}{2 \kappa c}.$$  \hspace{1cm} (5.55)

This expression agrees to better than 1% with the exactly calculated rotation over the frequency range $2 \times 10^{10}$ to $2 \times 10^{12}$, which covers the normal microwave region.

(ii) Rotation - Field dependence

The dependence of the Faraday rotation on field strength is illustrated at three frequencies in Fig. (5.3). It is desirable to use a log scale for both axes, in order to span a wide range of fields between the limits $\omega a \ll 1$ and $\omega a \gg 1$. This means,
Fig. (5.3). Variation of Faraday rotation with field strength for lattice scattering n-type germanium specimen at various frequencies: a, $\omega = 10^9$; b, $\omega = 10^{11}$; c, $\omega = 10^{12}$; d = 1 cm and $T = 300^\circ K$ for all curves. (Note: all values of $\Theta$ are negative).
however, that sign changes cannot be adequately represented. Therefore only frequencies for which $\mathfrak{R}$ is always negative are shown, and $\log_{10}|\mathfrak{R}|$ is plotted, thereby avoiding the negative infinity at the value $\mathfrak{R} = 0$. The highest frequency for which $\mathfrak{R}$ does not change sign as $H$ varies is given approximately by $a^{-1}$. At any higher frequency $\mathfrak{R}$ will change from positive to negative when the field is high enough to satisfy $\mathfrak{R} = \omega$.

We consider first curves (a) and (b), which are together typical of all frequencies such that $\omega a < 1$, excluding only frequencies close to $10^{12}$ when $\lambda a$ is also $< 1$. At low fields it can be seen that the gradient of (a) and (b) is almost unity until a field equivalent to $\lambda a \sim 0.2$ is reached, so that $\mathfrak{R}$ is proportional to $H$ at these frequencies. This is consistent with the low frequency approximation, (5.26), for $(\omega = 10^9$, and in fact remains valid to within $2\%$ up to frequencies as high as $\omega \sim 5 \times 10^{11} (\omega a = \frac{1}{3})$, although (5.26) itself breaks down when the displacement current term can no longer be neglected, at about $10^{10}$. It does not seem to be possible to obtain a simple expression for $\mathfrak{R}$ with the field dependence of the form $H^{-1}$ for the region beyond $10^{10}$. As the field is increased so that $\lambda a > 0.2$, the deviation from the $H^{-1}$ rule becomes noticeable, and $\mathfrak{R}$ increases less rapidly, eventually passing through a maximum when $\lambda a$ is slightly greater than one. Again this slight displacement is due to the energy dependent relaxation time, the maximum being exactly at $\mathfrak{R} = 1$ for the degenerate case.

When the high field condition $(\lambda a)^2 \gg 1$ is satisfied, two limiting cases are important. For both we have $(\lambda a)^2 \gg 1$ and $\lambda^2 \gg \omega^2$, so that the integrals are given by (5.33). For frequencies well below the magnetoplasma edge, i.e. $\omega_p^2 / \omega \lambda \gg 1$, (curve (a)), we can neglect one in comparison to $\omega_p^2 / (\omega + \lambda)$ in the expressions for $\beta_\pm$ obtained by substituting (5.33) in (5.8) and extracting the imaginary parts. If the resultant is expanded appropriately, having regard to the opposite signs of $(\omega - \lambda)$ and $(\omega + \lambda)$, we find
\[ \beta_+ \rightarrow \frac{\omega_p}{c} \left[ \frac{\omega e}{\nu l} \right]^{\frac{1}{2}} \]  

and

\[ \beta_+ \rightarrow \beta_+ \left[ \frac{4}{3 \pi^{\frac{3}{2}} a} \right] \]  

Hence \( \beta_+ \ll \beta_+ \) and, as a rough approximation, \( \mathbb{R} \) is equal to \( -\frac{\beta_+ d}{2} \), (see Fig. 5.2). Then (5.56) predicts that \( \mathbb{R} \) be proportional to \( H^{-\frac{1}{2}} \). This is approximately borne out by (a), the main discrepancy arising from the neglect of \( \beta_- \) as, with \( \omega = 10^9 \) and \( \mathcal{L} = 10^{13} \), (5.56) gives a value for \( \beta_+ \) which is only about 4% too high. (5.57) is rather poorer, giving a value about 20% too high.

For frequencies well above the magnetoplasma edge, i.e. \( \omega_p^2/\omega \mathcal{L} \ll 1 \), (curve (b)), the appropriate expression was derived in the previous section and is given in (5.55). \( \mathbb{R} \) should be proportional to \( H^{-1} \), in very good agreement with (b). At intermediate frequencies a gradual transition between the two limiting cases \( H^{-\frac{1}{2}} \) and \( H^{-1} \) is found.

Curve (c) is plotted for \( \omega = 10^{12} \) which, at low fields, is very close to the sign change. It does not conform to the pattern of (a) and (b) in this range, as \( \mathbb{R} \) increases at a rate greater than the first power of \( H \), and the maximum occurs at a somewhat higher field. It was found in general that consistent behaviour was not obtained when the values of \( \omega \) and \( \mathcal{L} \) were such that \( \mathbb{R} \) and \( \Delta \) were about to change sign. The field dependence can change quite markedly for a relatively small change in the frequency, so that (c) cannot be regarded as typical. A simple description in terms of approximate expressions analogous to (5.56) and (5.57) is never possible, as none of the various limiting conditions are applicable. Thus \( \mathbb{R} \) should always be calculated exactly close to a sign change.

At higher fields (c) and (b) merge, because the sign change moves away to the higher frequency \( \mathcal{L} \), and hence \( 10^{12} \) lies within the range of the approximations leading to (5.55), as for curve (b).

At frequencies above \( 10^{12} \) and fields for which the sign change is at \( a^{-1} \), i.e. for fields up to and including \( \mathcal{L} a = 1 \), \( \mathbb{R} \) is positive and approximately proportional to \( H \). Deviations from this
rule are generally less than 10%. At higher fields the proportionality will continue as long as \( \lambda \) is less than \( \omega \). However, since we have \( \lambda > 1 \), a resonance similar to that found in the frequency dependence, (Fig. 5.1), occurs as \( \lambda \) passes through the region in which \( \lambda \sim \omega \). There is a sharp peak in the rotation, followed by a rapid fall, a change of sign at \( \lambda = \omega \), and then a sharp minimum. Finally, the rate of change of \( \Theta \) with \( \lambda \) decreases, until the \( \lambda^{-1} \) region described by (5.55) is reached when \( \lambda^2 \gg \omega^2 \). For this specimen there are no complications arising from plasma terms, as both the plasma and magnetoplasma edges occur at much lower frequencies.

(iii) Rotation - Temperature dependence

The variation of \( \Theta \) with temperature at a number of fixed frequencies and field strengths was evaluated between 100 \( ^\circ \)K and 300 \( ^\circ \)K. As discussed in 5(b), it is a reasonable approximation for the specimen under consideration to assume that \( N \) is constant over the range, while \( a \) varies as \( T^{-3/2} \). Using the data of (5.18), \( a \) was calculated at seven temperatures equally spaced throughout the range. It varied between the limits \( 10^{-12} \) and \( 5 \times 10^{-12} \). For the first time, use was made of the computer, so that it was possible to calculate at a large number of frequencies in the range \( 10^{10} - 10^{13} \), with both \( \lambda = 10^{11} \), (low field case), and \( \lambda = 10^{13} \), (high field case).

In Fig. (5.4) the results for \( \lambda = 10^{11} \) are shown at four of these frequencies, (solid curves). In general it is clear that \( |\Theta| \) increases quite rapidly as the temperature is lowered, except in the region of the sign change. We consider the general effect on the frequency dependence as the temperature falls, noting that \( (\lambda a)^2 \) remains less than one over the entire range, though at 100 \( ^\circ \)K it is \( \sim 0.25 \), so that the low field approximation becomes rather poor. The low frequency expression (5.26) remains valid only up to rather lower frequencies, as the most restrictive approximation, which involved the assumption that the real part of \( \mu_z^2 \) was much less than the imaginary part, required \( 4 \omega_p^2 a / 3 \pi^{3/2} \omega \gg 1 \) and

\[ |\Pi^{3/2} a(\omega = \lambda)/2| \ll 1, \text{ and though the first condition can therefore be extended from an upper limit } \sim 5 \times 10^9 \text{ at } 300^\circ \text{K to about } 10^{11} \text{ at} \]
Fig. (5.4). Variation of Faraday rotation with temperature for lattice scattering n-type germanium specimen at various frequencies and fields: a, $\omega = 10^{10}$, $\lambda = 10^{11}$; b, $\omega = 5 \times 10^{-10}$, $\lambda = 10^{11}$; c, $\omega = 10^{11}$, $\lambda = 10^{11}$; d, $\omega = 10^{12}$, $\lambda = 10^{11}$; e, $5 \times 10^{10}$ $< \omega < 10^{12}$, $\lambda = 10^{13}$; d = 1 cm for all curves.
100^0K, the second condition becomes a very poor approximation, as \( \mathcal{N}a \) is \( \sim 1/2 \) at 100^0K. In fact, despite important higher order terms in \( a(\omega + \mathcal{N}) \) at the low temperatures, and the poorness of the displacement current approximation at higher temperatures, (first condition), a plot of \( \log_{10}|\theta| \) against \( \log_{10}T \) shows that the temperature dependence is \( \sim T^{-2.1} \) for curves (a), (b) and (c), in quite good agreement with (5.26), which predicts a variation of \( T^{-2.25} \) because of the factor \( (\sigma_0^L)^{3/4} \). This indicates that a dependence \( \sim T^{-2} \) applies generally in low magnetic fields until frequencies quite close to the sign change are reached, even though (5.26) is quite incorrect over most of the range. It can be concluded that, on a frequency plot, the low frequency minimum becomes narrower as well as larger in magnitude as the temperature falls, due to the movement of the sign change to a lower frequency as \( a \) increases. Curve (d) represents a case where the sign change lies within the range covered by \( a^{-1} \). It happens fairly close to the high temperature limit, and the feature of \( \theta \) increasing as \( T \) falls can still be seen at the lower temperatures. At higher frequencies \( \theta \) is positive at all the temperatures considered. In the high frequency limit, (equation (5.36)), \( \theta \) is independent of \( a \) and hence of \( T \) and, since this approximation requires \( (\omega a)^2 \gg 1 \), it is reached first at the lowest temperature. Thus the curves eventually begin to flatten out close to 100^0K and finally, at about 10^13, become constant over the whole range. The low field temperature dependence is clearly complicated by the fact that, unless extremely small values of \( H \) are considered, \( (\mathcal{N} < 10^{11} \) or \( H < 1,400 \) oersted), the approximation \( (\mathcal{N} a)^2 \ll 1 \) will be very poor at the lowest temperatures.

Conversely, the high field condition, \( (\mathcal{N} a)^2 \gg 1 \), improves as the temperature drops and, for \( \mathcal{N} = 10^{13} \), is excellent over the whole range. However, on substituting the high field approximations to the integrals as given in (5.33) into equation (5.8) for \( \gamma^2 \), it can be seen that only the imaginary part depends on \( a \), and that this is smaller than the real part by a factor \( a(\omega + \mathcal{N}) \). Excepting only the case of \( \beta \) at frequencies below the magnetoplasma edge,
(section (ii)), the real part of $\mu_+^2$ is negative, besides being large so that, in evaluating $\beta_\pm$, this imaginary term is always of second order, and therefore not very important. Thus, as $\beta_\pm \ll \beta_+$ below the magnetoplasma edge, $\Re$ is, to a good approximation, independent of the temperature at all frequencies, except in the vicinity of cyclotron resonance, where the resonance peaks resulting from the vanishing of $(\omega - \kappa)$ become sharper and larger in magnitude as $a$ decreases. Thus, in this very small region only, $|\Re|$ would increase as $T$ decreased. This exception apart, the lack of dependence of $\Re$ on $T$ is confirmed by expressions (5.55) and (5.56), which do not depend on $a$. The dotted curve (e) on Fig. (5.4) represents the exactly calculated results for $\Re$ at any frequency between $5 \times 10^{10}$ and $10^{12}$.

(iv) Ellipticity - Frequency dependence

Calculations of the ellipticity $\Delta$ were made over the same range of frequencies and at the same fields as for $\Re$. The results for $\kappa = 10^{11}, 10^{12}$ and $10^{13}$ are presented in Fig. (5.5). The ordinate plotted is $\Delta/2$ rather than $\Delta = \tanh \kappa d/2$, because, when $\Delta d/2$ becomes large, $\Delta$ tends to one, corresponding to the total absorption of one of the circularly polarized modes, and in general therefore the shape of the curve is dependent on the value of $d$ taken, and $\Delta$ is not directly proportional to $d$ as for the rotation. Curves for $\Delta$ have the peaks flattened perceptibly if $\Delta d/2 < 0.5$. This is illustrated by the dashed curves (d) and (e), which represent $\Delta$ for $d = 1 \text{ cm}$ and $\kappa = 10^{12}$ and $10^{13}$ respectively, over the regions where $\Delta$ and $\Delta d/2$ can be separated.

In practice the overall absorption must be low enough to permit measurements on the emergent radiation, and this condition is not generally satisfied if $d$ is so large that $\Delta \to 1$. Also, neither $\Re$ nor $\Delta$ can be measured accurately because of the difficulty in determining the major and minor axes of an ellipse which is almost circularly polarized. The thicknesses used experimentally are therefore usually low enough for the approximation $\Delta \approx \Delta d/2$ to be valid. However, $\Delta$ should not be too small, as leakages, etc. can produce a small ellipticity, which could be of the same order as the
Fig. (5.5). Variation of $\mathcal{Z}/2$ (= $\tanh^{-1} \Delta/d$) with frequency for lattice scattering $n$-type germanium specimen ($10^p \mathcal{Z}/2$ plotted): a, $\mathcal{Z} = 10^{11}$ ($p = 1$); b, $\mathcal{Z} = 10^{12}$ ($p = 0$); c, $\mathcal{Z} = 10^{13}$ ($p = 0$); d and e, variation of $\Delta$ with frequency for $d = 1$ cm, other details as for b and c respectively; $T = 300^\circ K$ for all curves.
Faraday ellipticity in extreme cases.

From Figs. (5.1) and (5.5) it can be seen that, in general, the rotation is approximately a maximum or minimum when the ellipticity is zero, and vice versa. At low fields, i.e. curve (a), $|\alpha/2|$ is equal to $|\beta/2|$ in the low frequency limit, and varies as $\omega^{3/2}$, (equation (5.26)). The divergence between the two is 10% or less for $\omega \leq 5 \times 10^9$, $|\alpha/2|$ is less than $|\beta/2|$, and the separation increases as the frequency rises, so that $|\alpha/2|$ passes through its turning point at a lower frequency than $\Theta$. It then changes sign at the frequency corresponding to the minimum in $\Theta$, and rises quite steeply to a maximum value at $\omega^{-1}$, the natural frequency of the system. The value $a^{-1}$ should be regarded as approximate only, as the actual turning point is at a rather lower frequency. However, the position is independent of the field so long as $(\lambda a)^2 \ll 1$ is satisfied, so the displacement probably results from the use of an energy dependent relaxation time, (cf. (H) section f(i)). Finally $\alpha/2$ decreases quite rapidly and, when $\omega = 6 \times 10^{12}$, the high frequency approximation (5.38), in which $\Delta$ is proportional to $\omega^{-3}$ (neglecting the plasma term), is only 15% too high. The agreement improves rapidly thereafter so that, by $10^{13}$, (5.38) may be used with high accuracy. The ellipticity is smaller than the rotation by the factor $16\pi^{3/2}/3\omega$ in the high frequency limit, and, as a very small ellipticity is in any case more difficult to measure than a very small rotation, it would be undetectable at normal thicknesses $\sim 1$ cm for the present specimen. Much higher carrier concentrations would be necessary if the ellipticity were to be measurable, (i.e. $\sim 0.1$), at infra-red frequencies, and so far no measurements have been reported. The ellipticity plotted for $d = 1$ cm is virtually indistinguishable from (a). Also, the curve for $\lambda = 10^{10}$ differs only by a factor of ten in magnitude, deviations from this proportionality being only about 1 or 2% over the whole range of $\omega$.

The effect of increasing $\lambda$ to $10^{12}$ so that $\lambda a = 1$ is illustrated by (b). The magnitude has increased by a factor of about 6 overall, an effect similar to that observed in the rotation. This shows that the linear dependence on $H$ breaks down as the condition $(\lambda a)^2 \ll 1$
becomes less satisfactory. The general shape of the curve is, however, more similar to the low field curve (a) than to the high field curve (c). The low frequency minimum still occurs at $3 \times 10^{10}$, and the sign change has moved to a frequency only marginally higher than for (a). The maximum has also moved to a somewhat higher frequency although the shape of the peak is similar to (a), and the high frequency approximation can be used at the same frequencies.

The deviation between $\bar{z}/2$ and $\Delta$ for a thickness of 1 cm is now obvious at both peaks.

Considerable changes are evident, however, when the high field condition is satisfied, as for curve (c). The maximum and minimum have moved further apart, to lower and higher frequencies respectively. The minimum now occurs at about $5 \times 10^9$, and, up to this value, the magnitude of $\bar{z}/2$ has only decreased slightly in comparison with the corresponding results for $\mathcal{A} = 10^{12}$. The rotation showed similar behaviour, in that (b) and (c) of Fig. (5.1) were not resolved below the magnetoplasma edge at $10^{10}$. Just beyond the turning point $\bar{z}/2$ rises very sharply at first, and then levels off rather abruptly when close to zero, so that it changes slowly with $\omega$, and actually crosses the axis at about the same frequency as the lower field curves. It continues to increase only slowly with frequency until $\omega$ approaches $\mathcal{A}$, when it rises suddenly and passes through a very high and narrow maximum at $\omega = \mathcal{A}$, before reaching the $\omega^{-3}$ region of (5.38) at about $6 \times 10^{13}$. As the field is increased still further, the two peaks become narrower, while the minimum moves to a lower frequency as the magnetoplasma edge occurs at a lower frequency, and the maximum moves to the appropriate higher cyclotron frequency. The curve generally approximates more closely to two sharp peaks separated by a region of zero ellipticity.

As in the case of $\mathcal{H}$, approximate expressions may be obtained for $\bar{z}/2$ at frequencies well below and well above the magnetoplasma edge at $\omega = \omega_p^2/\mathcal{A}$. In the former case we have $\omega_p^2/\omega \mathcal{A} \gg 1$, and it can be seen from Fig. (5.2) that $\lambda_- \gg \lambda_+$ as discussed in f(i).

We may also assume $\lambda_-^2 \gg \beta_-^2$ so that, using the standard high field
approximations, (5.33), for the integrals, we find

\[ \alpha_+^2 \to \frac{\omega e^2}{c^2} \left[ \frac{\omega_p^2}{\omega (\omega + \omega)} - 1 \right] . \quad (5.58) \]

Then, taking \( \omega \ll \omega \) and neglecting \( \lambda \) in comparison with \( \omega_p^2/\omega \)

\[ \alpha_+ \to \frac{\omega_p}{c} \left[ \frac{\omega e}{\omega \lambda} \right] \alpha - \frac{\lambda}{\omega} . \quad (5.59) \]

At \( 10^9 \) (5.58) gives 0.42 for \( \alpha_+ \). This is in good agreement with
the exactly computed value of 0.41, and a reasonable approximation

to \( -\lambda \) which is 0.38. It can be seen on comparing (5.56) and (5.59)
that, in this limit, \( \alpha_+ \ll \beta_+ \) or, since it may generally be assumed
that \( \lambda \gg \alpha \) and \( \beta_+ \gg \beta_-, \) \( \beta_+ \ll \alpha_+ \). The same result holds for
the low field case, (equation (5.26)). \( |J/\omega| \) is therefore
proportional to \( \omega \) at very low frequencies. Approximate expressions
for \( \alpha_+ \) cannot be found in the vicinity of the magnetoplasma edge, as
\( \alpha_+ \) is changing rapidly with frequency and \( \alpha_+ \) can no longer be neglected.

However, when \( \omega_p^2/\omega \lambda < 1 \) and \( \beta_+^2 \gg \alpha_+^2 \), we find

\[ \beta_+ \to \frac{\omega e^2}{c^2} \left[ 1 - \frac{\omega_p^2}{2\omega (\omega + \lambda)} \right] . \quad (5.60) \]

and

\[ \alpha_+ \beta_+ \to \frac{4 \pi \omega \sigma_0 \lambda}{\omega^2 c^2 (\omega + \lambda)^2} . \quad (5.61) \]

On substituting (5.60) in (5.61), and retaining the first term in \( \omega_p^2 \),

\[ \alpha_+ \to \frac{4 \pi \omega \sigma_0 \lambda}{\omega^2 c^2 (\omega + \lambda)^2} \left\{ \frac{1}{(\omega + \lambda)^2} \left\{ 1 + \frac{\omega_p^2}{2\omega (\omega + \lambda)} \right\} \right\} , \quad (5.62) \]

which leads to

\[ \alpha \to \frac{16 \pi \omega \sigma_0 \lambda}{\omega^2 c^2 \omega^3 \lambda^3} \left[ 1 - \frac{\omega_p^2}{4\omega^2} \right] . \quad (5.63) \]

This equation gives a good representation of that portion of (c)
for which \( J/\omega \) is small, i.e., between the magnetoplasma edge and
the cyclotron frequency. It predicts a sign change at \( \omega_p/2 \) or
1.22 \times 10^{11} \text{ for this specimen, and the change is actually at } 1.8 \times 10^{11}. \text{ Taking specific examples, we find for } \frac{\alpha}{2} \text{ at } 5 \times 10^{10} \text{ a value of } 9.5 \times 10^{-3}, \text{ in comparison with } 9.2 \times 10^{-3} \text{ from the exact formula, while at } 10^{12} \text{ the results are } 2.05 \times 10^{-2} \text{ from (5.63) and } 2.07 \times 10^{-2}, (\text{exact}). \text{ In general (5.63) is satisfactory to within } 10\% \text{ over the range } 3 \times 10^{10} \rightarrow 5 \times 10^{12}. \text{ Equation (5.63) has also been given by Purdya and Brodwin, (1961), equation (10), in terms of Maxwellian averages over } \Upsilon. \text{ However, their suggestion that it might be used to determine } \omega_p \text{ by measuring the node is not very realistic, because experiments in this region must be done with a fixed frequency microwave cavity, and the zero is not dependent on } H \text{ which is the only parameter that can be adjusted. Also, } \Delta \text{ is small over the region and tends rapidly to zero as the field is increased.}

The maximum at the cyclotron frequency, where } \Delta \text{ is one, is due to the almost complete resonant absorption of the positive wave, which rotated with the same sense as the charge carriers. The approximation (5.33) to the integrals breaks down because, as } \omega \rightarrow \lambda, a^2(\omega - \lambda)^2 \text{ becomes very small over a narrow frequency band. Hence } I_1^L \rightarrow 1 \text{ and } I_2^L \rightarrow \frac{\pi^2}{2}. \text{ It is easy to show that}

\[ \delta_+ \rightarrow \frac{2\pi\sigma_{\perp}^L}{c} \times \frac{\lambda}{\delta} \text{ as } \delta_+ > \delta_- \quad . \quad (5.64) \]

The peak value of } \frac{\alpha}{2} \text{ for curve (c) is } 2.62, \text{ from both (5.64) and the exact formula.}

The sign of the ellipticity has been arbitrarily chosen to give the same sign for } \delta \text{ and } \Delta \text{ in the low frequency limit. In fact the sign has a physical significance, in that it is linked to the sense of rotation of the emergent ellipse, which is the same as the sense of rotation of the mode which is attenuated least. If the ellipse rotates clockwise to an observer looking along the direction of propagation, we have } \delta_- > \delta_+, \text{ i.e. } \frac{\lambda}{\delta} \text{ negative, and the ellipticity is negative with the above convention, as at low frequencies. In the high frequency region the sense of rotation is anti-clockwise and } \Delta \text{ is conventionally positive.
(v) Ellipticity - Field Dependence

The sign change is not strongly field dependent for the specimen under consideration. Hence \(|\mathcal{E}/2|\) can be conveniently shown as a function of the field at frequencies above and below the sign change, using log scales for both axes. The results for four frequencies are presented in Fig. (5.6). In all cases the gradient is one for \(\mathcal{R}a \lesssim 0.25\), showing that \(|\mathcal{E}/2|\), like \(\mathcal{R}\), is closely proportional to \(H\) at low fields. This linear behaviour is predicted in the high and low frequency limits by equations (5.38) and (5.26) respectively, but appears in fact to be maintained over the entire frequency range. As \(H\) is increased further, all the curves pass through a maximum value when \(\mathcal{R}a\) is close to unity.

The high field behaviour is more complicated. For frequencies well below the magnetoplasma edge, \(|\mathcal{E}/2|\) should vary as \(H^{-3}\) in accordance with (5.59), while well above an \(H^{-3}\) dependence is predicted by (5.63). At frequencies in the vicinity of the edge neither limit is applicable, and intermediate behaviour is found. However, the edge itself is field dependent, as it occurs at \(\omega_p^2/\mathcal{R}\), and it therefore moves to a lower frequency as \(H\) increases, so that frequencies which at first belong to the region below the edge must move through it and ultimately follow the \(H^{-3}\) law if \(H\) is increased sufficiently. Thus all the curves in Fig. (5.6) eventually have a gradient of minus three, but the transitional behaviour can extend over an order of magnitude of the field for frequencies which are below or at the edge when the high field condition \((\mathcal{R}a)^2 \gg 1\) is first satisfied, (e.g. curve (a)). This argument also applies to the rotation under the same conditions, but the transition is not so marked as the change is only from \(H^{-3}\) to \(H^{-1}\). Clearly the plasma frequency is a very important parameter in determining the high field behaviour of \(\mathcal{R}\) and \(\Delta\). For the present specimen it is comparatively low, and therefore the behaviour appropriate to \(\omega_p^2/\mathcal{R} \ll 1\) occurs in the experimentally accessible microwave region. However, if \(\omega_p^2\) was increased by an order of magnitude to \(10^{24}\), both types of behaviour would be important.
Fig. (5.6): Variation of $\tau/2$ with field strength for lattice scattering n-type germanium specimen at various frequencies: a, $\omega = 10^{10}$; b, $\omega = 3.16 \times 10^{10}$; c, $\omega = 10^{11}$; d, $\omega = 10^{12}$; $T = 300^\circ K$ for all curves. (Note: $\tau/2$ is negative for a, b and c and positive for d).
When the radiation frequency is high enough to cause the condition $\mathcal{R}^2 \gg \omega^2$ to break down in the high field region, $\mathcal{R}a^2 \gg 1$, (5.63) will not be appropriate at all high fields. The high frequency limit (5.38) shows that $\mathcal{A}/2$ will be proportional to $H^{-1}$ so long as $\omega^2 \gg \mathcal{R}^2$. Then, in the vicinity of $\omega = \mathcal{R}$, the resonance peak will occur in the field dependence also, and (5.63) will only become applicable when we have $\mathcal{R}^2 \gg \omega^2$. For high frequencies, i.e., $\omega \gg 10^{13}$, for the specimen under consideration, this condition is more restrictive than $\mathcal{R}a^2 \gg 1$.

(vi) Ellipticity - Temperature dependence

The calculation covered the range 100-300 °K, as for the rotation. The results for a number of frequencies are shown in Fig. (5.7), with $\mathcal{R} = 10^{11}$ for the solid curves and $10^{13}$ for the dashed curves. As in (iii), we discuss the behaviour as the temperature is lowered and $a$ increases.

In the low field case, $|\mathcal{A}/2|$ increases as $a$ increases for frequencies below the sign change. This is to be expected when the approximation (5.26) is valid, as this equation shows that $|\mathcal{A}/2|$, like (H), should vary as $T^{-2.25}$. The actual dependence was found from a log plot to be about $T^{-2.2}$ for (a), while for (b) it varied between about $T^{-2}$ at low temperatures and $T^{-2.5}$ at the higher temperatures. The agreement with $T^{-2.25}$ is thus quite good and, as in the case of the rotation, appears to extend up to frequencies quite close to the sign change, where (5.26) itself is no longer valid. However, the sign change is at a lower frequency than for (H), so that curve (c), for which $\omega = 10^{11}$, does not conform to the pattern set by (a) and (b). At the high temperature end of the range the dependence for (c) is about $T^{-3}$ but, as $a$ increases, the sign change moves to a lower frequency as it is linked to the position of the maximum at $^{-1}$, so that the curve turns round and would eventually pass through zero and increase in the positive sense if $a$ was further increased. All the low frequency graphs would ultimately show this effect if $a$ were made large enough, while $\mathcal{R}a$ remained small. Curve (b) in fact is beginning to flatten close to 100 °K,
Fig. (5.7). Variation of $\zeta/2$ with temperature for lattice scattering $n$-type germanium specimen at various frequencies and fields: a, $\omega = 10^{10}, \mathcal{L} = 10^{11}$; b, $\omega = 5 \times 10^{10}, \mathcal{L} = 10^{11}$; c, $\omega = 10^{11}, \mathcal{L} = 10^{11}$; d, $\omega = 10^{12}, \mathcal{L} = 10^{11}$; e, $\omega = 10^{10}, \mathcal{L} = 10^{13}$; f, $\omega = 5 \times 10^{10}, \mathcal{L} = 10^{13}$. 
due to the close approach of the sign change at this temperature.

Curve (d) is for $\omega = 10^{12}$, a frequency well above the sign change for the 100–300 °K range. $|\vec{z}/z|^2$ increases as $T$ decreases, so long as the value of $\omega$ remains on the low frequency side of the positive low field peak. However, this maximum moves to a lower frequency as $a$ increases, so that a point at a fixed frequency effectively travels across the peak. Thus $|\vec{z}/z|^2$ should begin to decrease, once $\omega$ is greater than $a^{-1}$. For (d), $\omega = a^{-1}$ at 300 °K, but the maximum in the temperature dependence occurs at a lower temperature, i.e. when $\omega a > 1$, as the above effect is partially offset by the increase in the overall magnitude of the peak as $a$ is increased. These two competing factors result in a rather broad flat maximum, with $\vec{z}/z$ changing quite slowly with $T$ over the region where $\omega a$ is just greater than one. At higher frequencies, when $(\omega a)^2 > 1$, the high frequency approximation (5.38) can be used. Unlike (5.38), $\vec{z}/z$ is not independent of $a$ in this limit, and should vary as $T^{3n}$. Curve (d) is tending to the $T^{3n}$ region at the low temperature end, but somewhat larger values of $a$ are required to satisfy $(\omega a)^2 > 1$. Thus, so long as $(\omega a)^2 < 1$, the magnitude of $(\omega a)$ mainly governs the temperature dependence of the ellipticity.

At high fields (5.59), which is appropriate to frequencies well below the magnetoplasma edge, shows that $|\vec{z}/z|^2$ is independent of $\omega$ and hence of $T$, in this limit. The position of the edge, which is at $10^{10}$ for $\omega = 10^{13}$, is not temperature dependent either. However, the approximation (5.63) for frequencies above the edge predicts that $|\vec{z}/z|^2$ should vary as $T^{3n}$. Log plots for $\omega = 5 \times 10^{10}$ and $10^{12}$ have slopes of 1.46 and 1.58 respectively, confirming the validity of (5.63) between the edge and the cyclotron frequency $\omega$. This range itself is independent of temperature. Curve (e) for $\omega = 10^{10}$ is at the edge, so that behaviour intermediate between the two limits represented by (5.59) and (5.63) is found. $|\vec{z}/z|^2$ increases with temperature, but the rate is considerably less rapid than $T^{3n}$. The $T^{3n}$ behaviour is illustrated by (f), for which $\omega = 5 \times 10^{10}$. For frequencies in the vicinity of $\omega = \omega$ the behaviour is complicated,
as the peak becomes narrower and higher as the temperature decreases. Just on either side of the peaks \( \bar{\varphi} / 2 \) will decrease with decreasing temperature, but very close to \( \omega = \lambda \), (5.64) shows that \( \bar{\varphi} / 2 \) must increase as the temperature falls. Finally, when \( \omega^2 > \lambda^2 \), the high frequency approximation becomes valid again, and \( \bar{\varphi} / 2 \) varies as \( \frac{1}{n} \), as on the low frequency side of the resonance.

5(g) Impurity Scattering — Results

The details of the impurity scattering specimen are given in (5.20). It is clear from the discussion of (f) that, for lattice scattering, the main features of the behaviour of \( \bar{\varphi} \) and \( \Delta \) are determined by \( \omega_p^2 \), and the values of \( \lambda a \) and \( \omega a \) in relation to unity. If we take account of the change in \( \omega_p^2 \) occasioned by the increased carrier density, and consider the magnitudes of \( \lambda g \) and \( \omega g \), rather than \( \lambda a \) and \( \omega a \), much of the discussion of the preceding section can be applied in the present case to predict the general features of the rotation and ellipticity for given ranges of values of any of the external parameters. There are some modifications to numerical factors in the approximate expressions which depended on \( a \), but the effects of the type of scattering are basically confined to detail changes in the curves.

1) Rotation — Frequency dependence

The frequency dependence is shown in Fig. (5.9) for the three fields corresponding to \( \lambda = 10^{11}, 10^{12} \) and \( 10^{13} \). High field results comparable to curve (c) of Fig. (5.1) would entail \( \lambda > 10^{14} \) for this specimen, as \( g \) is shorter than \( a \). The calculation was not performed, as the corresponding field is over \( 10^6 \) oersted and is unrealistic experimentally. It turns out, however, that (c), with \( \lambda = 10^{13} \), is a good approximation to the high field limit, although \( (\omega g)^2 \) is only 2.6.

It is easily shown that, so long as \( (\omega g)^2 << 1 \) and \( \omega_p^2 g / \omega >> 1 \), the zero field absorption \( \lambda_0 \) is given by

\[
\lambda_0 = \frac{2 \omega_p^2}{c} \left[ \frac{\epsilon \omega}{\pi} \right]^{\frac{1}{3}}
\]

(5.65)
Fig. (5.8). Variation of Faraday rotation \(10^p \theta\) (rad) with frequency for impurity scattering n-type germanium specimen with \(g = 1.6 \times 10^{-13}\) sec, \(N = 4.0 \times 10^{16}\) cm\(^{-3}\) at 100 K, \(m^* = m/4\), \(e = 16\): a, \(\lambda = 10^{11}\) \((p = 0)\); b, \(\lambda = 10^{12}\) \((p = -1)\); c, \(\lambda = 10^{13}\) \((p = -1)\); d = 1 cm and \(T = 100^0 K\) for all curves.
Since $\alpha_{\pm}$ do not deviate greatly from $\alpha_0$ under these conditions, which hold up to $10^{12}$, the overall absorption for low fields can be estimated from (5.65). For $\omega = 10^{11}$, $\alpha_0 \approx 10^2$, and therefore only thicknesses $< 0.02$ cm could be used for a microwave transmission experiment, without attenuating the wave completely. The absorption is less for high fields, (see Fig. (5.2)), and thicker specimens could be used in this limit. However, the thickness is always a critical parameter at microwave frequencies for such a high conductivity specimen, unlike the lattice scattering sample where $N$, and hence $\alpha_0$, were lower by a factor $\approx 250$. The graph shows the rotation per cm, as this is the most useful quantity for comparison purposes.

The low frequency approximation (5.32), which requires the real part of $\mu^2$ in equation (5.28) to be much smaller than the imaginary part, is valid up to about $10^{11}$ for curve (a). The displacement current term satisfies this condition up to $\omega = 10^{12}$, the criterion being $\delta \omega_p^2 \omega / \omega_0^3 \omega \gg 1$. However, for the conductivity term to be small, we must have $|g(\omega \mp \omega)/(3^2)| < 1$, and this is only valid for $\omega < 10^{11}$ with $\omega = 10^{11}$. The formula (5.32) and the exact equations give values differing by 2% at $10^{10}$, rising to 12% at $10^{11}$. For the lattice scattering specimen the displacement current condition restricted the frequency beyond which (5.32) was not satisfactory, but for this sample, because $\omega^2_p$ is higher, the conductivity term becomes the limiting factor. However, the $\omega_p^{1/2}$ region extends into the microwave range, and is more important than for the low conductivity specimen. The minimum is at $3.3 x 10^{11}$, and $|\Theta|$ then decreases gradually and becomes zero at $5 x 10^{12}$, a little below the natural frequency $g^{-1} = 6.3 x 10^{12}$. This may be contrasted with the lattice scattering case, where the energy dependence of $\Phi$ caused the sign change to occur at a frequency somewhat higher than $a^{-1}$. The small positive peak is actually closer than the sign change to $g^{-1}$. Thus the exact position of the zero is a function of the energy dependence of $\Phi$. Beyond the maximum $\Phi$ tends to the high frequency $\omega^{-2}$ approximation, (5.35), which gives a value only 4% too high at $4 x 10^{13}$. 
For (b), with $J_L = 10^{12}$, the low frequency approximation (5.32) is never satisfactory, because $g\lambda(9/2)^!3! = 1.4$. The large numerical factor $(9/2)^!3! = 8.7$ arises from the realisation time averages in $I_1^L$ and $I_2^L$. For the degenerate case and lattice scattering the corresponding numbers are 1 and $\Pi^{3/2}$ respectively. This illustrates the kind of way in which the type of scattering can affect $\Theta$ as, for impurity scattering, the low frequency approximation breaks down at much lower fields than it does for lattice scattering, taking the values of $a$ and $g$ to be equal. For (b) $\Theta$ decreases less rapidly, and the minimum is at a higher frequency than for curve (a) though, corresponding to the change in $H$, the maximum value of $|\Theta|$ is increased very closely by a factor of ten. Beyond about $3 \times 10^{12}$ (a) and (b) become indistinguishable. Since $\Theta$ is multiplied by $10^{-1}$ for (a), the rotation varies linearly with $H$ in this region.

In the case of (c), $J_L g$ is slightly greater than one. In view of the similarity to Fig. (5.1), curve (c), it can most usefully be considered as a high field curve. Since the maximum in the field dependence occurs for $J_L g < 1$, (section (ii)), typical high field behaviour may be expected at rather lower fields than for lattice scattering with the same $a$. The magneto-plasma edge is at $3.2 \times 10^{12}$ for $J_L = 10^{13}$, as the plasma frequency is $5.66 \times 10^{12}$. The region between the edge and the cyclotron frequency, over which $\Theta$ is independent of $\omega$, is therefore very short. However, the edge appears clearly, beginning at $2.5 \times 10^{12}$ in quite good agreement with the theoretical value, and terminating at about $4 \times 10^{12}$. Beyond this frequency, the resonance, which is very sharply peaked despite the low value of $J_L g$, occurs, while at still higher frequencies, when $\omega^2 \gg \lambda^2$, $\Theta$ decreases rapidly. The high frequency approximation is very accurate at $6 \times 10^{13}$. At higher fields the edge would move to a lower frequency and the resonance to a higher frequency, thus extending the 'flat' portion of the curve.

As for lattice scattering, approximate expressions for $\Theta$ at frequencies above and below the edge can be found. The edge itself is independent of $\lambda$, and remains at $\omega_p^2/\omega \lambda^1$ so long as $\lambda^2 \gg \omega^2$.
and \( (\mathcal{L} g)^2 \gg 1 \). At lower frequencies, where \( \omega_p^2/\omega \mathcal{L} \gg 1 \), \( \beta_+ \) is still given by (5.56) as this equation is independent of \( a \), and hence also of \( g \) for impurity scattering. Equation (5.57) for \( \beta_- \) is modified to

\[
\beta_- \rightarrow \beta_+ \left[ \frac{2}{3\pi \mathcal{L} g \mathcal{L}} \right] .
\]

Exactly calculated values of \( \beta_+ \) agree with (5.56) to within 4% up to \( 10^{17} \) but, as in the case of lattice scattering, the agreement is rather poor for \( \beta_- \). Since \( \mathcal{L} g \) is relatively small, \( \beta_- \) is not negligible compared to \( \beta_+ \), and this contributes to the 10% deviation between the exact \( \mathcal{H} \) and the value calculated from (5.56) and (5.66) at \( 10^{11} \). It is hardly possible to satisfy \( \omega_p^2/\omega \mathcal{L} \ll 1 \) adequately for \( \mathcal{L} = 10^{13} \), as the flat region is so short, but (5.55), which is valid for all types of scattering as it does not depend on \( a \), gives \(-2.13 \times 10^2 \) at \( \omega = 4 \times 10^{12} \), while the exact value is \(-2.01 \times 10^2 \). Rather surprisingly it seems that \( (\mathcal{L} g)^2 \) only needs to be \(< 3 \) for the high field results to be valid. This may be due to fortuitous near cancellation of higher order terms in the expansions used.

(ii) Rotation - Field dependence

A graph of \( \log_{10} \mathcal{H} \) against \( \log_{10} \mathcal{L} \) does not differ greatly from the lattice scattering case shown in Fig. (5.3), provided the lower horizontal scale is interpreted as \( \log_{10} \mathcal{L} g \), rather than \( \log_{10} \mathcal{L} a \), and has its origin at \( \log_{10} \mathcal{L} = 12.8 \), corresponding to \( \mathcal{L} g = 1 \). For \( \mathcal{L} < 3 \times 10^{11} \), (10) is closely proportional to \( H \) at all frequencies. It may be deduced from (a) and (b) of Fig. (5.8) that, as the field is raised further, the curves will increase less steeply for frequencies below about \( 10^{12} \). This is due to the onset of the transition from low field to high field behaviour, during which the minimum shifts towards the cyclotron frequency. The maximum occurs when \( \mathcal{L} g \) is rather less than one and, for high fields, i.e. \( (\mathcal{L} g)^2 \gg 3 \), (10) initially decreases as \( H^{-\frac{3}{2}} \), following (5.56). However, eventually the field is such that the magnetoplasma edge is at a value only slightly higher than the frequency under consideration. In this region (10) begins to decrease more rapidly with increasing \( H \), and
finally varies as $H^{-1}$ when the edge lies well below the given frequency. It should be noted that the maximum is flatter, and spreads over a wider range of field before tending to the $H$ or $H^{1/2}$ limits, than for lattice scattering. The low field $H$ dependence breaks down when $\lambda g < 0.05$, while previously it was valid for $\lambda a < 0.2$. Only partial compensation is achieved by the shift of the maximum to $\lambda g < 1$, rather than $\lambda a > 1$. This latter effect is clearly a consequence of the type of scattering assumed. If $\Upsilon$ is $\Upsilon(E)$ only, it may be written generally as

$$\Upsilon = \ell \left( \frac{E}{kT} \right)^p$$

(5.67)

The maximum in the field dependence is at $\lambda L < 1$ if $p$ is positive, ($p = 3/2$ for impurity scattering), and the position varies smoothly with $p$, passing through $L \lambda = 1$ for $p = 0$, and occurring for $L \lambda > 1$ when $p$ is negative, ($p = -1/2$ for lattice scattering).

For $O > 10^12$, when (a) and (b) are indistinguishable in Fig. (5.8), $H$ remains proportional to $H$ until $\lambda g < 0.2$, as for lattice scattering. Up to the frequency at which $H$ changes sign at low fields, i.e. $5 \times 10^{12}$, the field dependence for higher values of $\lambda$ still resembles the case described above but, like (c) of Fig. (5.3), the gradient of the log plot becomes > 1 close to $\lambda g = 1$, before the maximum is reached. Also, the maximum is sharper, and its position may be shifted slightly. Close to the sign change the behaviour is rather unpredictable in the region of the maximum, as for the previous specimen. Well above $5 \times 10^{12}$ the high field behaviour exactly parallels that described in f(ii), since the magnetoplasma edge is still below the cyclotron frequency for all fields such that $(\lambda g)^2 > 1$. If $\omega_p^2$ is increased so that the edge at $\omega_p^2 / \lambda$ is above the cyclotron frequency, i.e. $\omega_p^2 > \lambda^2$, as might be the case in highly doped materials, equation (5.56) remains valid until the resonance region is reached, and the 'flat' portion of the frequency dependence is eliminated. The plasma edge effects are then as described by Palik et al. (1962), and the edge is at $\omega = \omega_p \pm \lambda / 2$, the shift from the zero field edge at $\omega_p$ being small. At high enough fields, however, $\omega_p^2$ becomes less than $\lambda^2$. 
and the field dependence reverts to the $H^{-1}$ region of (5.55).

(iii) Rotation - Temperature dependence

The temperature dependence of $\delta$ was computed over the range $15 - 100^\circ K$, using values of $N$ and $g$ taken from the experimental results of Debye and Conwell, (1954), for sample 61. The density of ionized donors decreases as $T$ decreases in this range, so that $N$, which was found from the Hall effect data, decreased from $4.0 \times 10^{16}$ at $100^\circ K$ to $6.8 \times 10^{14}$ at $15^\circ K$. The relaxation parameter $g$ was calculated from $N$ and the measured D.C. conductivity, using equation (5.14). It varied between $1.6 \times 10^{-13}$ at $100^\circ K$ and $1.1 \times 10^{-13}$ at $15^\circ K$. Alternatively, $g$ could have been found by using the theoretical $T^{3/2}$ law to extrapolate from the $100^\circ K$ data. However, as the temperature dependence of the experimentally derived $g$ values was very different from $T^{3/2}$, it seemed more consistent to use the experimental measurements as a basis throughout. As discussed in (b), impurity scattering alone is not adequate for this specimen, though it appears to be the dominant mechanism. For this reason the results described should only be regarded as showing the general trends expected in $\delta$ as $T$ is varied.

In Fig. (5.9) $\delta$ is shown at various frequencies, with $\omega = 10^{11}$ or $10^{13}$ as indicated. $|\delta|$ always decreases as $T$ decreases, in contrast to the case of lattice scattering. This effect is due much more to the decrease in $N$ by a factor of 59 between 100 and $15^\circ K$, than to the relatively small decrease of about 50% in $g$. The present results are not therefore very dependent on the scattering mechanism. In general it is to be expected that the decreasing carrier density will be the most important single factor governing the variation of $\delta$ with $T$ at low temperatures, for any specimen. However, for a $\tau$ such that $p$ in (5.67) is positive, the rate of decrease of $\delta$ with $T$ is generally increased by relaxation effects, as $\propto T^p$, and vice versa if $p$ is negative, (assuming that the mean free path is not dependent on the temperature). The dependence on $N$ usually lies between the limits $N_0^2$ and $N$. For the lattice scattering sample, by contrast, the relaxation effects alone were responsible for the variation of $\delta$ with $T$, as $N$ remained constant over the range considered.
Fig. (5.9). Variation of Faraday rotation (10^P \theta \text{ rad}) with temperature for impurity scattering n-type germanium specimen at various frequencies and fields: 

- a, \( \omega = 10^{10}, \lambda = 10^{11} (p = 0) \)
- b, \( \omega = 5 \times 10^{10}, \lambda = 10^{11} (p = 0) \)
- c, \( \omega = 10^{12}, \lambda = 10^{11} (p = 0) \)
- d, \( \omega = 10^{13}, \lambda = 10^{11} (p = 0) \)
- e, \( \omega = 5 \times 10^{10}, \lambda = 10^{13} (p = -1) \)
- f, \( \omega = 3 \times 10^{11}, \lambda = 10^{13} (p = -1) \)
- g, \( \omega = 10^{14}, \lambda = 10^{13} (p = 0) \)
- h, \( \omega = 10^{14}, \lambda = 10^{13} (p = 0) \)

d = 1 cm for all curves.
The low field curves are (a) to (d). Only (a) and (b) are within the scope of the low frequency approximation (5.32) at all temperatures, as \( \omega_p^2 \) decreases to \( 5.4 \times 10^{23} \) at 15 °K, and the displacement current condition considered in (i) is only valid up to \( \omega = 5 \times 10^{10} \) for this value. According to (5.32), \( \mathcal{H} \propto N^{3/2} \), so that \( \mathcal{H} \) should decrease by a factor of about 14 between 100 and 15 °K. At \( 10^{10} \) and \( 5 \times 10^{10} \) the ratios are 13.7 and 13.3 respectively, indicating good agreement. For \( 10^{11} \) the value is 13.1, which suggests that the dependence predicted by (5.32) remains a good approximation up to the frequency of the minimum. Beyond the minimum \( \mathcal{H} \) begins to decrease more rapidly with \( T \), and for \( \omega \gg 10^{12} \) up to frequencies beyond the region of the sign change, the dependence is approximately \( N^2 \). This is illustrated by the large change, (by a factor \( \approx 80 \)), in (c) and (d), for which \( \omega = 10^{12} \) and \( 10^{13} \) respectively. The sign change varies between \( 5 \times 10^{12} \) and \( 8 \times 10^{12} \) over the range considered. At higher frequencies (5.35) shows that \( \mathcal{H} \) is proportional to \( N \) only. The shape of the curves reflects the way in which \( N \) changes, fairly slowly at high temperatures, and increasingly rapidly as \( T \to 0 \).

At high fields \( \mathcal{H} \) is independent of \( g \) to a good approximation at all frequencies, except in the vicinity of the resonance at \( \omega = \lambda \). Thus, for \( \omega \ll 10^{10} \), \( \mathcal{H} \) varies as \( N^{3/2} \) from (5.56), as \( \omega \) is well below the magnetoplasma edge at all temperatures. At higher frequencies \( \mathcal{H} \) remains proportional to \( N^{1/2} \) as long as \( \omega_p / \lambda \gg \omega \), but, as the temperature and \( \omega_p^2 \) decrease, the dependence will alter gradually until, when \( \omega_p^2 / \omega \lambda \ll 1 \), (5.55) will become valid and \( \mathcal{H} \) will be proportional to \( N \). For curve (f'), with \( \omega = 3 \times 10^{11} \), \( \omega_p^2 = \omega \lambda \) at 23 °K and the rate of decrease of \( \mathcal{H} \) with \( T \) at low temperatures is more marked than for (e), which has \( \omega = 5 \times 10^{10} \). The linear dependence on \( N \) is also appropriate in the high frequency limit, e.g. curve (h), though it is not correct in the region of the resonance, where \( \mathcal{H} \) decreases much more rapidly with \( T \), (curve (g)).

(iv) Ellipticity - Frequency dependence

The dependence of \( \mathcal{E} / \lambda \) on frequency at the fields corresponding to \( \lambda = 10^{11}, 10^{12} \) and \( 10^{13} \) is shown in Fig, (5.10). The values are
Fig. (5.10). Variation of $10^3 \tilde{\mathcal{V}}/2$ with frequency for impurity scattering n-type germanium specimen: a, $\mathcal{R} = 10^{11}$ (p = 0); b, $\mathcal{R} = 10^{12}$ (p = -1); c, $\mathcal{R} = 10^{13}$ (p = -1); $T = 300^\circ K$ for all curves.
so large that, over most of the range, barring the high frequency limit, \( \Delta = 1 \) when \( d = 1 \) cm. A thickness of 0.01 cm would give reasonable values for \( \Delta \) at microwave frequencies. Again the rotation is a maximum or a minimum when \( \bar{\alpha} / 2 \) is zero, and vice versa.

For the low field curve (a) the low frequency approximation, in which \( \bar{\alpha} = \bar{\alpha} d / 2 \), is valid up to about \( 10^{10} \). The agreement between \( \bar{\alpha} \) and \( \bar{\alpha} d / 2 \) is better than 1% at \( 10^9 \), but \( |\bar{\alpha} / 2| \) increases less rapidly with \( \omega \) until, at \( 10^{10} \), the discrepancy is about 7%. Thereafter \( |\bar{\alpha} / 2| \) falls well below the value given by the approximate expression, and passes through the turning point at \( 10^{11} \). Thus (5.32) breaks down at lower frequencies than it did for the rotation. The sign change is at \( 2.8 \times 10^{11} \), almost exactly in coincidence with the minimum in the rotation. The broad positive double humped peak may be explained as follows. It is not possible to simplify the integrals in the frequency range where it occurs, as \( \omega \) \( \gg \lambda \), and thus approximate expressions for \( \bar{\alpha} / 2 \) cannot be developed. However, if the steps of the calculation are followed through, it becomes apparent that \( \bar{\alpha} / 2 \) starts to decrease when the real part of \( \mu^2 \), which up till then has been positive, goes through zero and becomes negative. When this happens, (at nearly equal frequencies for \( \mu^2 \) and \( \nu^2 \)), \( \alpha \) fall rapidly as discussed in section f(i). For the lattice scattering specimen the corresponding effect happened at a much lower frequency \( \approx 10^{10} \), and in fact would only be present in \( \alpha_4 \) for \( \omega \gg \lambda \), i.e. at very low fields. It did not result in an anomaly in the frequency dependence. In the present case the effect seems to be to cause an extension of the normal positive peak, which is the lower frequency hump, to the frequency where \( \alpha^+ \) exhibit an absorption edge. It is to be expected that all specimens for which the real part of \( \mu^2 \) only becomes zero in the region of, or above the positive peak, will show this broadening of the maximum up to the edge, as the high frequency approximation (5.41) does not become valid until the real part is negative, with the displacement current contribution greatly exceeding the conductivity term. This happens at about \( 6 \times 10^{13} \). The rotation also shows a slight 'kink' at the edge frequency, where \( \beta \) will rise suddenly as \( \alpha \) fall.
For curve (b), (5.32) is not valid, as explained in (i). However, the general result that $\mathbb{A} \propto \mathcal{L}d/2$ as $\omega \to 0$ still holds. At $10^9$ the discrepancy is much less than $1\%$, and this has only risen to $7\%$ by the time $\omega = 4 \times 10^{10}$. Thus the agreement holds to higher frequencies than for (a), due to the minimum occurring at a higher frequency. Thereafter the zero again coincides with the minimum in $\mathbb{H}$, though the appropriate frequency is higher than for (a). As found for the rotation, (a) and (b) become almost indistinguishable in the region of the positive peak. The double hump arises in the same way as for (a), since $\mathcal{L} \ll \omega$, and the increased field therefore causes only a small alteration in the real part of $\mu^2$. In fact the real part becomes zero at a slightly lower frequency than for $\mathcal{L} = 10^{11}$, so that $\mathcal{L}/2$ also starts to decrease at a correspondingly lower frequency, but the displacement is too small to be resolved in Fig. (5.10).

Curve (c) represents the high field limit. Equation (5.59) is independent of the scattering, and is therefore valid for frequencies well below the magnetoplasma edge at $3.2 \times 10^{12}$. It is about $20\%$ too high at $10^{11}$, but this is mainly due to the poorness of the high field approximation, $(\mathcal{L}g)^2 \gg 1$, as the agreement is very little better at $10^9$. For frequencies well above the edge (5.63) must be modified, as it depends on the type of scattering. We find for impurity scattering

$$\mathcal{L} \to \frac{4\pi \omega \sigma_0 I}{3\sigma_0^2 e^2} \left[ 1 - \frac{\omega_p^2}{4\omega^2} \right]$$

(5.68)

However, the high field approximation $(\mathcal{L}g)^2 \gg 1$ seems to be more critical in deriving (5.68) than it was for the corresponding approximation (5.55) to the rotation. The region between the magnetoplasma edge and the cyclotron frequency, which was quite extensive for the lattice scattering specimen, has almost vanished, being represented only by a slight 'kink' at about $3 \times 10^{12}$, and (5.68) does not give a good approximation to the exactly calculated values at, or in the vicinity of, this frequency, even though (5.55) could be used for $\mathbb{H}$, (see (i)). The effect of the resonant
absorption of the positive wave at $\omega \approx \lambda$ extends on the low frequency side so close to the edge that the intermediate region, governed by (5.68), is lost. The resonance maximum is very high. The expression corresponding to (5.64), which gave $\alpha_+$ at $\omega = \lambda$, is now modified to

$$\alpha_+ \to \frac{2\pi \sigma_0^1}{c \epsilon^2}, \propto \lambda^2 \quad \text{as} \quad \lambda_+ \gg \lambda_-. \quad (5.69)$$

However, this involved the assumption that, at $\omega = \lambda$, the real part of $\mu_+^2$, (the dielectric term only), was much greater than the imaginary part, and this is not true for the present high conductivity specimen when the cyclotron frequency is only $10^{13}$. Thus (5.69) cannot be applied to predict the peak value of $\lambda/2$, though both (5.68) and (5.69) would clearly become valid at high enough fields. Eventually, at about $10^{14}$, the high frequency formula (5.41) becomes accurate to within 2%.

(v) Ellipticity - Field dependence

This is very similar to the field dependence of the rotation. For frequencies below the low field sign change the details concerning the shape of the maximum in the vicinity of $\arg g = 1$ are unchanged from section (ii), and $|\lambda/2|$ is linear with $H$ at low fields, and proportional to $H^{-1/2}$ at high fields. However, at high enough fields, when the magnetoplasma edge lies well below the radiation frequency, (5.68) shows that $|\lambda/2|$ will ultimately vary as $H^{-1}$, rather than $H^{-1/2}$ as for the rotation. The behaviour at frequencies close to the low field sign change is again rather unpredictable, and dependent on the scattering mechanism assumed. While, well above $g^{-1}$, $\lambda/2$ is proportional to $H$ as long as $\omega^2 \gg \lambda^2$, passes through a resonance similar to that observed in the frequency dependence at $\omega \approx \lambda$, and reverts to the high field $H^{-3}$ dependence when $\lambda^2 \gg \omega^2$. If the edge is very close to the cyclotron frequency over a certain field range, when $(\arg g)^2 \gg 1$, the initial variation will be as $H^{-1/2}$, tending to $H^{-3}$ when $\omega_p^2/\lambda$ becomes less than the applied frequency. This behaviour
is essentially identical with that described for the lattice scattering specimen in $f(\nu)$, since only the magnitude of $\mathcal{X}/2$ is affected by the scattering mechanism, the limits delineating the various regions being independent of $a$ or $g$.

**(vi) Ellipticity - Temperature dependence**

The calculation is again based on the data given in section (iii). As for the rotation, $|\mathcal{X}/2|$ decreases in general as $T$ decreases, mainly due to the decreasing concentration of free carriers. We consider first the low field case, e.g. $\mathcal{L} = 10^{11}$. The low frequency approximation (5.32), in which $|\mathcal{X}/2|$ varies as $N^g^{3/2}$, is only valid over the whole of the temperature range up to about $3 \times 10^9$, as it breaks down at a lower frequency when applied to $|\mathcal{X}/2|$ than to $\mathcal{H}$. At higher frequencies it will be valid for part of the range at the high temperature end, and will finally break down, even at $100^\circ K$, at about $10^{10}$. As in the case of the rotation, however, the low frequency dependence on $N$ and $g$ continues to be a good approximation up to frequencies approaching the sign change at $2.3 \times 10^{11}$. At $10^{10}$, for example, the ratio of the values at $100^\circ K$ and $15^\circ K$ is $13.9$, in very good agreement with the value of $14.1$ predicted by a variation of $N^g^{3/2}$. In the vicinity of the sign change and positive maximum the dependence tends more closely to $N^g$. An exact analysis is not possible, due to complicating factors such as the temperature dependence of the position of the normal positive maximum, (lower frequency peak), which is at $\omega_p g^{-1}$. This causes $|\mathcal{X}/2|$ to decrease even more rapidly with decreasing $T$ above the sign change, and vice versa below it. Furthermore, the 'absorption edge' peak discussed in (iv) moves to a lower frequency as $T$, and hence $\omega_p^2$, decrease, and eventually vanishes when the two peaks become coincident. Thus a very rapid decrease of $\mathcal{X}/2$ with $T$ is found over certain temperature ranges, if $\omega$ lies on the higher frequency hump in Fig. (5.10). Beyond this absorption edge, however, the behaviour tends rapidly to the $N^g^{-1}$ dependence predicted by the high frequency approximation, (5.41).

For $\mathcal{L} = 10^{12}$ the low frequency behaviour is only modified insomuch as (5.32) is never valid. However, the temperature
dependence remains very closely $N^{-1/2}$ almost up to the sign change, which is of course at a rather higher frequency than for $\omega = 10^{11}$ at all temperatures. Beyond this point the behaviour is essentially similar in the two cases, since $\tilde{\sigma}/2$ is proportional to $H$ at all frequencies.

For the high field case (5.59) shows that $|\tilde{\sigma}/2|$ varies as $N^{3/2}$ at frequencies which are well below the magnetoplasma edge at all the temperatures considered. With $\omega = 10^{13}$ this is true for $\omega \ll 5.4 \times 10^{10}$. In the opposite limit well above the edge, $|\tilde{\sigma}/2|$ is approximately proportional to $N^2g^{-2}$ below the sign change, as shown by (5.60). Thus, under these conditions, $|\tilde{\sigma}/2|$ decreases considerably more rapidly with $T$ than it does for frequencies below the edge. This limit is not reached at any frequency for $T = 100^\circ K$, (see (iv)), but is attained over an increasingly wide frequency range between $\omega_p^2/\omega$ and $\omega$ as the temperature falls. If $\omega_p^2/\omega \ll \omega$ the behaviour is transitional between $N^{3/2}$ and $N^2g^{-2}$. Thus, over this range, $|\tilde{\sigma}/2|$ decreases relatively slowly with $T$ at high temperatures, and at a much more rapid rate as the temperature falls. Above the sign change, when $\omega_p^2/4\omega^2 \ll 1$, $\tilde{\sigma}/2$ will vary approximately as $Ng^{-2}$.

The position of the resonance is of course independent of the temperature, but the height of the peak is very sensitive to $N$, and $\tilde{\sigma}/2$ decreases very rapidly with $T$ for $\omega \gg \omega_p$. Beyond this, when $\omega^2 \gg \omega_p^2$, the high frequency $Ng^{-1}$ dependence becomes valid.
Calculations of the rotation and ellipticity in N-type germanium are presented in this chapter, assuming the [111] ellipsoidal model for the conduction band. The most interesting aspect of this work is the anisotropy of the Faraday effect for such a band structure. The model discussed in chapter 5 had spherical symmetry as the surfaces of constant energy were spherical, and \( \Theta \) and \( \Delta \) were therefore isotropic, i.e., independent of the orientation of the magnetic field with respect to the crystal axes. For the correct ellipsoidal model, which has cubic symmetry only, this is no longer true. As discussed in chapter 3 and appendix A, it is first necessary to evaluate the components of the magnetoconductivity tensor for any given choice of the \((x,y,z)\) system of axes. In this system \( H \) is parallel to the \( z \) axis and the initial electric vector \( E_0 \) is along the \( x \) axis. The general equations derived in chapter 2 may then be used to find \( \Theta \) and \( \Delta \) for this configuration of \( H \) and \( E_0 \), and the orientation dependence can thus be investigated. The work of the present chapter establishes the necessary conditions for the degree of anisotropy to be experimentally significant. It is found that, at room temperature and for microwave frequencies, the magnetic fields required are quite moderate for most semiconductors, and the effects should be easily detectable in experiments.

The general features of the dependence of \( \Theta \) and \( \Delta \) on temperature and on the type of scattering assumed are not, however, likely to be significantly different for different orientations of \( H \) with respect to the crystal. Calculations have therefore been performed only for the lattice scattering specimen discussed in 5(f), and the temperature dependence was not considered, the values chosen for \( N \) and \( a \) being those appropriate to room temperature. The frequency and field dependence, and the dependence of \( \Theta \) and \( \Delta \) on the orientation of the magnetic field in \((100)\) and \((1\bar{1}0)\) planes at
fixed frequency and field, are calculated and discussed. We consider also the effect of rotating the initial direction of polarization of the electric vector in the plane perpendicular to the field for a fixed direction of $\mathbf{H}$.

6(a) General Formulation

It is convenient to discuss separately the two cases mentioned above.

(i) $\mathbf{H}$ in (100) plane

The $(x,y,z)$ system of axes is shown in Fig. A II(a) and the six independent components of the magnetoconductivity tensor are given in appendix B I. The angle $\phi$ measures the inclination of $\mathbf{H}$ to the [001] axis in the (100) plane. Using the notation of chapter 2 we have $\beta = -\phi$, and the resulting simplifications to the general expressions given in 2(d) have been discussed in 2(e)(iii).

Although the components $S_{yz}$, $S_{zy}$, $S_{zx}$, and $S_{xz}$ change sign under the transformation $\phi \rightarrow (\pi/2 - \phi)$, they occur only in pair products in $\mathbf{A}$, etc. and it is easily seen that $\mathbf{A} (\phi) = \mathbf{A} (\pi/2 - \phi)$, so that the rotation (and ellipticity also) is symmetrical about the [011] direction. Thus the orientation dependence need only be calculated for the range $0 \leq \phi \leq \pi/4$.

In contrast to the isotropic case, the plasma terms $F_{xz} F_{zx}/F_{zz}$, etc. vanish only for $\phi = 0$, $\pi/4$ and $\pi/2$. This result follows from the forms of the tensors for these directions which were derived in sections (a) and (c) of appendix A I. Approximate numerical estimates showed, however, that the plasma terms would generally be small, i.e. $\ll F_{xx}$, $F_{xy}$, etc., and they were not therefore included in the calculations reported here. Later calculations by Medcalf (1963 and 1964), in connection with the multiple reflection effect did include the terms and, as expected, it was found that they only displaced the orientation curves slightly, the effect being a maximum around $\pi/3$ and tending to zero as $\phi$ approached 0 or $\pi/4$. 
Using this approximation we have therefore

\[
\begin{align*}
\mathcal{A} &= -\frac{\omega c}{c^2} + \frac{4\pi i\omega}{c^2} S_{xx} \\
\beta &= \frac{4\pi i\omega}{c^2} S_{xy} = -C \\
\mathcal{D} &= -\frac{\omega c}{c^2} + \frac{4\pi i\omega}{c^2} S_{yy}
\end{align*}
\]  

(6.1)

with \( S_{xx}, \) etc. defined in equation (B.2) of appendix B. They are functions of \( \alpha (\lambda_j) \) and \( \beta (\lambda_j) \) which are given in (3.33) and (3.34). From (2.11) the propagation constants are

\[
2 \mu^2 = \mathcal{A} + \mathcal{D} \pm \left[ (\mathcal{A} - \mathcal{D})^2 - 4\beta^2 \right]^{\frac{1}{2}}. \quad (6.2)
\]

The numerical work therefore requires first the tabulation of \( I_1(\omega \pm \lambda_j) \) and \( I_2(\omega \pm \lambda_j) \), followed by the calculation of the real and imaginary parts of \( \alpha (\lambda_j) \) and \( \beta (\lambda_j) \) and hence of the necessary tensor components \( S_{xx}, S_{yy} \) and \( S_{xy} \). These lead to the complex quantities \( \mathcal{A}, \mathcal{B} \) and \( \mathcal{D} \), using (6.1), and from these the propagation constants can be obtained directly according to (6.2). Finally, to find \( \mathcal{H} \) and \( \Delta \) from equations (2.45) and (2.28) (with the appropriate simplifications), it is necessary to calculate \( P \) and \( Q \) and \( L_+ \) and \( M_+ \) which were defined in (2.15) and (2.18) respectively. The calculation is considerably more complicated than that required for the isotropic case, mainly because the integrals in that case were functions only of \( \omega \) and not of \( H \) and \( \phi \), whereas now they depend on all three variables. Furthermore there are two values of \( \lambda \) to be considered for each value of \( H \). Thus eight integrals in all must be evaluated for every different set of the variables \( \omega, \ H \) and \( \phi \). Also, the steps of the subsequent calculation are more involved. The University of London Mercury Computer was used for all the anisotropic calculations, and it was therefore possible to cover wide ranges of the parameters \( \omega, \ H \) and \( \phi \) for the lattice scattering specimen of N-type germanium.

(ii) \( H \) in \((1\overline{1}0)\) plane

The \((x,y,z)\) system of axes is shown in Fig. A II(c). This is
the case for which \( \epsilon_0 \) is inclined at a variable angle \( \cos^{-1}\left(\frac{1+\cos\phi}{2}\right) \) to the 1 crystallographic axis, \( \phi \) being measured now in the (110) plane. Calculations were not carried out for the configuration illustrated in Fig. A II(b) since it has the same symmetry as the (100) plane case, though the numerical magnitudes of the tensor components are of course different. The situation denoted by Fig. A II(c) will be subsequently referred to simply as the (110) case.

The Faraday effect is now symmetrical about \( \phi = \pi/2 \), the [110] direction. The components \( S_{yz} \), etc. change sign under the transformation \( \phi \rightarrow (\pi - \phi) \) in a similar manner to the (100) plane case, but, as they occur in pairs in the plasma terms, the overall symmetry is maintained. The orientation dependence is therefore calculated for the range \( 0 \leq \phi \leq \pi/2 \). The high symmetry directions contained in the plane are \( \phi = 0 \), [001], \( \phi = \cos^{-1}(1/\sqrt{3}) \), [111] and \( \phi = \pi/2 \), [110]. The plasma terms vanish for these directions because the tensors \( S_{yz} \), etc. are zero, as proved in appendix A I, sections (a), (b) and (c) and, as for the (100) plane, their effect is small for intermediate values of \( \phi \), so they are omitted from the calculation.

In the notation of chapter 2, \( \mathcal{A} = \mathcal{D} \) for this case. The propagation constants are therefore given by

\[
\mathcal{A}^2 = \mathcal{A} \pm i\left[\beta \varepsilon\right]^{1/2},
\]

with

\[
\begin{align*}
\mathcal{A} & = -\frac{\omega^2\epsilon}{c^2} + \frac{4\pi i\omega}{c^2} S_{xx} = \mathcal{D} \\
\beta & = \frac{4\pi i\omega}{c^2} S_{xy} \\
\varepsilon & = \frac{4\pi i\omega}{c^2} S_{yx}
\end{align*}
\]

\( S_{xx}, \) etc. are defined in equation (B.5) of appendix B. The rotation and ellipticity are given by (2.40) and (2.41) with \( L_+ = L_- = L \) and \( M_+ = M_- = M \) as defined in (2.39). The calculation is similar at most stages to that for the (100) plane case, except that 12 integrals must now be evaluated for each set of the variables \( \omega, H \) and \( \phi \), as there are in general three different cyclotron frequencies.
(iii) Isotropic Directions

To conclude this section we consider the extent to which the isotropic model of chapter 5, and the isotropic directions for the ellipsoid model, are equivalent. The isotropic theory is valid when the magnetoconductivity tensor has the form (2.29). This is only true for [111] ellipsoid band structure when \( \mathbf{H} \) is parallel to a [100] or [111] type direction, as proved in sections (a) and (b) of appendix A I. The actual tensor components were derived in chapter 3, and are given by equations (3.53) and (3.86) for [100] and [111] directions respectively. They are of course not the same for the two cases. Faraday experiments which are interpreted according to the isotropic theory should therefore only be carried out with \( \mathbf{H} \) along a [100] or [111] direction. Even then the use of a single isotropic effective mass is not correct as, for the ellipsoid model, it is not possible to define a single effective mass, a function of \( m_\| \) and \( K \), which makes the tensor components equivalent to those in the isotropic case. For example, for the [001] direction, the equivalent cyclotron masses obtained by comparing \( \lambda \) for the ellipsoid and isotropic models are

\[
m^* = m_\| \left[ \frac{3}{K(K+2)} \right]^{\frac{1}{2}} \quad (6.5)
\]

whereas the equivalent masses obtained by comparing \( S_{xx} \) are

\[
m^* = \frac{3m_\|}{2K+1} \quad (6.6)
\]

Comparing \( S_{xy} \) again gives (6.5). For the [111] direction the situation is even more involved as there are two cyclotron frequencies for the ellipsoid model. Clearly the approximation of an isotropic effective mass should not be used to interpret experimental data, unless an approximate expression for \( R \) (or \( \Delta \) ) which enables a single 'Faraday' mass to be defined is valid. A case where this is possible is the high frequency approximation in which \( \mathbf{H} \) is proportional to the real part of \( S_{xy} \), so that a Faraday effective mass may be defined by (6.5), thus making the isotropic model valid for the appropriate ranges of frequencies and fields.
6(b) Evaluation of the Integrals

The integrals required are those appropriate to the lattice scattering case. They are, from (5.7)

\[
I_1^L (\omega \pm \lambda) = \int_0^\infty \frac{y^2 e^{-y} dy}{y + a^2 (\omega \pm \lambda)^2} \quad \text{and} \quad I_2^L (\omega \pm \lambda) = \int_0^\infty \frac{3/2 y e^{-y} dy}{y + a^2 (\omega \pm \lambda)^2}
\]

(6.7)

However, the methods adopted in connection with the isotropic model, interpolation in tables using a desk calculating machine and use of a library programme which took, on average, 10 seconds of Mercury time to perform one integration, were not considered to be satisfactory for the very large number of integrals required for the present calculation. Alternative procedures were developed therefore, which enabled about 4 integrals to be performed in one second by the computer.

Gaussian quadratures were considered first. Any integral \(I\) with at least one infinite limit can be expressed as follows

\[
I = \int_a^\infty W f(y) dy = \sum_{j=1}^n f(a_j) H_j + \text{Remainder},
\]

(6.8)

where \(W\) is a weighting factor ensuring convergence, \(H_j\) are weight coefficients and \(a_j\) are abscissae lying within the interval of integration. The 2n disposable values of \(H_j\) and \(a_j\) are chosen to make (6.8) exact, without the remainder term, when \(f(y)\) is a polynomial of degree \((2n-1)\). In the present case where \(W = e^{-y}\) and \(a = 0\), the \(a_j\) are the roots of the Laguerre polynomial of order \(n\). These, together with the weights, \(H_j\), are tabulated for \(n \leq 15\) (see for example Kopal (1961)). The procedure adopted was to calculate the Gauss Laguerre approximation to \(I_1^L\) for \(n = 13, 14\) and 15, treat these three values as a sequence \((a, b, c)\) and estimate the limit using the formula

\[
L_n \to \infty (a, b, c) = c + \frac{(b+c)^2}{2b-(a+c)}.
\]

(6.9)
This process was employed as it was found to improve the result by about one order of magnitude.

A final error \( \leq 1 \) part in \( 10^6 \) is desirable if possible. For \( I_1(\eta) \) this was approximately achieved by using the Gauss Laguerre method if \( \eta \geq 1 \), as the following example will show. With \( \eta = 1 \), the limit found from the 12, 13 and 14 point values is

\[ I_1^L(1) = 0.5963468 \]

while the corresponding limit obtained from the 13, 14 and 15 point values is

\[ I_1^L(1) = 0.5963473 \]

a difference of 5 parts in \( 10^7 \). It seems that the error in the second value is not likely to exceed 1 part in \( 10^6 \). This is confirmed since the value obtained by another method discussed below, which is accurate to 7 significant figures, is 0.59634738. Therefore the Gauss Laguerre limit process was used for \( I_1^L(\eta) \) when \( \eta \) was greater than one.

Most of the contribution to the integrals arises from \( \eta \leq 1 \), and the polynomial approximation to \( f(\eta) \) is naturally worst therefore when \( \eta \leq 1 \). A better approach may be made using the identity

\[ I_1^L(\eta) = 1 - \eta^2 + \eta^4 \exp(\eta^2) \int_{\eta}^{\infty} \frac{e^{-t}}{t} \, dt \quad (6.10) \]

The integral in the last term is, apart from the sign, the logarithmic integral defined by

\[ Ei(-x) = -\int_{-\infty}^{x} \frac{e^{-t}}{t} \, dt \quad (\infty > x > 0) \quad (6.11) \]

The series expansion

\[ Ei(-x) = \ln \gamma x - x + \frac{x^2}{2!2} - \frac{x^3}{3!3} + \frac{x^4}{4!4} - \frac{x^5}{5!5} + \ldots, (6.12) \]

with

\[ \ln \gamma = \int_{0}^{1} \frac{1-e^{-t}}{t} \, dt - \int_{1}^{\infty} \frac{e^{-t}}{t} \, dt \]

\[ = 0.577215665 \]

was used to evaluate \( Ei(-x) \) for \( \eta < 1 \). The series converges quite
rapidly over this range, requiring only terms up to $x^{11}$ for an accuracy of ten significant figures when $\gamma = 1$. The exact value mentioned above for $\gamma = 1$ was obtained by this method. For $\gamma > 1$ the procedure quickly becomes less satisfactory as $(1 - \gamma^2)$ and $-\gamma^4 \exp(\gamma^2) \text{Li}(-x)$ increase rapidly, so that $I_{1L}^{L}(\gamma)$ is the small difference of two large terms and significant figures are lost.

Since the computer worked only to an accuracy of about nine decimal digits, the method was not suitable for $\gamma > 1$ and, for $I_{1L}^{L}$, the change-over to a Gauss Laguerre quadrature was therefore made at this value.

$I_{2L}^{L}$ presents more difficulty as the integrand does not tend to a polynomial for any value of $\gamma$. A Gauss Laguerre quadrature, as for $I_{1L}^{L}$, is only satisfactory for $\gamma \geq 3$. The result of the (13,14,15) point limit for $\gamma = 3$ is

$$I_{2L}(3) = 0.1175534 \quad (6.13)$$

An expression similar to (6.10) may also be obtained. It is

$$I_{2L}(\gamma) = \prod^{1} \left[ 1/2 - \gamma^2 + \prod \gamma^3 \exp(\gamma) F(\gamma) \right], \quad (6.14)$$

where $F(\gamma)$ is related to the Gaussian error function Erf($\gamma$) and is defined by

$$F(\gamma) = \frac{2}{\pi^{3/2}} \int_{\gamma}^{\infty} e^{-u^2} du = 1 - \frac{2}{\pi^{1/2}} \int_{0}^{\gamma} e^{-u^2} du = 1 - \text{Erf}(\gamma). \quad (6.15)$$

For $\gamma \leq 1$ the integral $\int_{0}^{\gamma} e^{-u^2} du$ may be evaluated to the desired accuracy of nine to ten significant figures by expanding the integrand and integrating term by term. This gives

$$F(\gamma) = 1 - \frac{2}{\pi^{3/2}} \left[ \gamma - \frac{\gamma^3}{1!3} + \frac{\gamma^5}{2!5} - \frac{\gamma^7}{3!7} + \frac{\gamma^9}{4!9} - \cdots \right] \quad (6.16)$$

For $\gamma = 1$ it is sufficient to include terms of the order $\gamma^{11}$ to utilize the working capacity of the computer to the full. For $\gamma > 1$ the integral $2/\pi^{1/2} \int_{0}^{\gamma} e^{-u^2} du$ tends rapidly to 1 as $F(\gamma)$ is very small, and significant figures are lost in computing $F(\gamma)$. However, the last term of (6.14) is still large because $\exp(\gamma^2)$ increases rapidly and, as in the case of $I_{1L}^{L}$, further significant figures
are lost on subtracting \((\gamma^2 - \frac{1}{2})\). Thus the method is not satisfactory beyond \(\gamma = 1.5\), at which value an error of \(\approx 1\) part in \(10^7\) in \(\int_0^\gamma e^{-u^2} du\) gives rise to an error \(\approx 1\) part in \(10^7\) in \(I_2^L(\gamma)\). It was actually used only for \(\gamma \leq 1\), as an alternative method is available for \(1 < \gamma < 3\). We have, setting

\[
2^{3/2} \gamma = \eta \quad \text{and} \quad 2^{3/2} u = t,
\]

\[
P(\eta) = e^{-\eta^2/2} \int_{-\infty}^{\infty} e^{-t^2/2} dt = 2^{3/2} \gamma^2 \int_{-\infty}^{\infty} e^{-u^2} du = P(2^{3/2} \gamma) = (\pi/2)^{3/4} \exp(\gamma^2) P(\gamma).
\]

Thus \((6.14)\) may be written as

\[
I_2^L(\gamma) = \pi^{3/4} \left[ 1/2 - \gamma^2 + 2^{3/2} \gamma^3 P(2^{3/2} \gamma) \right]. \tag{6.18}
\]

\(P(\eta)\) is called the probability integral, and it may be calculated from the continued fraction

\[
P(\eta) = \frac{1}{\eta} + \frac{1}{\eta} + \frac{2}{\eta} + \frac{3}{\eta} + \cdots \tag{6.19}
\]

It was found that, for \(\eta = 2^{3/2}\), i.e. \(\gamma = 1\), and \(n = 80\), this series agreed to nine significant figures with the corresponding value derived from \((6.16)\). It also agreed, to the same accuracy, with Shepherd's Tables of the Probability Integral (1939). As \(\gamma\) was increased, with \(n = 80\), the agreement improved, so that this method was satisfactory until \(\gamma\) was large enough for too many significant figures to be lost when taking the difference of the first two and last terms of \((6.18)\). At the changeover to a Gauss Laguerre quadrature, made when \(\gamma = 3\), two significant figures were lost, so that a final accuracy of at least 1 part in \(10^6\) should be achieved for \(I_2^L\) when using the computer. The value obtained from the probability integral Tables for \(\gamma = 3\) gave for \(I_2^L\), using \((6.18)\),

\[
I_2^L(3) = 0.11755783.
\]
The accuracy is better than 1 part in $10^6$. Thus the Gauss Laguerre value given in (6.13) is too low by $\sim 4$ parts in $10^6$ which is just acceptable.

Thus two methods were employed for $I^L_1$ and three for $I^L_2$, the one selected depending on the value of $\gamma$. All the integrals were accurate to at least 5 parts in $10^6$ and most were considerably better than this.

6(c) Details of the Computer Programme

Separate programmes incorporating the simplifications arising from the particular symmetry of the field configuration with respect to the cubic axes were written for the (100) and (110) planes. In the former case $\mathcal{B}$ and $\Delta$ were calculated, for a given frequency and field, at $5^\circ$ intervals between 0 and $\pi/4$, and in the latter case at $10^\circ$ intervals between 0 and $\pi/2$. In both cases the entire calculation, including the routines to evaluate the integrals, was contained in one programme, the data required being only the specimen parameters and the frequency, field and orientation angle $\phi$.

The integrals were calculated first as described in the previous section. From these the real and imaginary parts of the conductivity tensor components and hence of $\Phi$, etc. were obtained. The propagation constant calculation as described in 5(e) had to be modified somewhat as complex square roots of expressions of the type (6.2) and (6.3), rather than (5.3), are required. Difficulties arise because of the optional sign for the root in the square brackets. If we denote this by the form

$$U + iV = \left[W + iX\right]^{1/2}, \quad (6.20)$$

the known sign of $X$ only determines whether or not the signs of $U$ and $V$ should be the same or different. It is possible to appeal to the isotropic limit, in which the root becomes simply $\pi\beta$, to decide the overall sign. However, it is difficult to specify the sign of the real and imaginary parts of $\beta$ in this limit with complete certainty. The imaginary part particularly can have different signs at different frequencies and fields. In a hand calculation it is not difficult to decide the correct signs for $U$ and $V$ by inspection, but a more precise criterion is necessary for a computer programme. It is essential that
no mistake be made as the wrong choice has the effect of interchanging \( \mu^+ \) and \( \mu^- \). For isotropic directions this merely changes the overall sign of \( \Theta \) and \( \Delta \) but, for other directions, as \( L_+ \) and \( L_- \) and \( M_+ \) and \( M_- \) are not also changed appropriately, a wrong answer is obtained in general. (The (110) plane is an exception as \( L_+ = L_- \) and \( M_+ = M_- \) so that only the sign is wrong in this case.)

An alternative form for \( \mu^+ \) may be used to eliminate any uncertainty over the correct identification of \( \mu^+ \) and \( \mu^- \). From (2.13) we have, since \( a_+ \) and \( b_\pm \) are never zero,

\[
\mu^2 = \mathcal{A} + \mathcal{B} b_\pm / a_\pm = \mathcal{D} + \mathcal{C} a_\pm / b_\pm.
\]  
(6.21)

The second form is the more convenient in practice as, on substituting (2.18) into (2.17), we find

\[
a_\pm = \frac{\mathcal{C} \pm \sqrt{\frac{L_\pm + iM_\pm}{G + iH}}}{2 - \frac{\mathcal{C}}{G + iH}}
\]
and hence

\[
\mu^2 = \mathcal{D} \pm \mathcal{C} (L_\pm + iM_\pm). 
\]  
(6.23)

The square root, with its associated sign uncertainty, is therefore eliminated, and \( \mu^2 \) has the same form as the isotropic expression (5.3). Thus, after evaluating \( L_\pm \) and \( M_\pm \), the real and imaginary parts of (6.23) may be found directly, and the procedure described in 5(e) applied from that point onwards.

The square root problem now arises only in connection with \( G \) and \( H \), which are contained in \( L_\pm \) and \( M_\pm \) (see section 2(c)). It was shown in 2(e) that \( H \) is always +2 in the isotropic limit. As far as could be ascertained, the quantity remained positive for all \( \omega \), \( H \) and \( \phi \), and this property was therefore suitable to fix the sign convention in the programme. Furthermore, as inspection of the equations (2.18), (6.23), (2.27) and (2.28) will show, if for any reason \( G \) and \( H \) are given the wrong signs, though \( \mu^+ \) and \( \mu^- \) are interchanged and \( L_\pm \to -L_\mp \) and \( M_\pm \to -M_\mp \), the rotation and ellipticity
are unaffected. This method is therefore foolproof. The reason is that the overall sign of \((G+iH)\) is a matter of convention when the full theory of the Faraday effect, as set out in chapter 2, is used. Thus, provided the same convention is used for both \(\mathcal{M}_\pm\) and \(L_\pm\) and \(M_\pm\), as the above method ensures, the final result is always right. The only error possible is that the independent modes will be wrongly labelled if \((G+iH)\) is given the wrong sign.

Knowing \(\psi^+, L_\pm\) and \(M_\pm\) unambiguously, the rotation and ellipticity are obtained by straightforward substitution in the appropriate equations of chapter 2, taking care to ensure that the sign convention for the rotation is given correctly by the programme. Finally, the rotation and ellipticity of the independent modes which were discussed in 2(f) were calculated.

Provision was also made for the calculation of the rotation and ellipticity as a function of the orientation of \(\xi\) in the plane perpendicular to \(H\). The tensor has to be transformed to new axes such that \(x\) is rotated about the \(z\) axis or field direction through an angle \(\psi\), in the sense \(x \rightarrow y\). The symmetry of the tensor under such a transformation is discussed in appendix A II(d), where it is shown that, for all orientations in the planes considered, \(\mathcal{R}(\psi) = \mathcal{R}(\pi + \psi)\) and, at intervals of \(\pi/4\), the two symmetries \(\mathcal{S} = \mathcal{O}\) and \(\beta = -\xi\) occur alternately. The \(\psi\) dependence was evaluated at \(10^0\) intervals in the range \(0 \leq \psi \leq \pi\). The programme was written in such a manner that this calculation could be performed, if desired, for not more than 3 orientations of \(H\) in the plane, i.e. 3 values of \(\phi\), for a given frequency and field. Control was exercised by extra data parameters. In this calculation neither of the two symmetries are applicable for an arbitrary \(\psi\), and \(P, Q, R\) and \(T\) must all be found and the general expressions (2.27) and (2.28) used to calculate \(\mathcal{R}\) and \(\Delta\) respectively.

The computer needed, on average, about one minute to perform a complete orientation dependence calculation, excluding the plane of polarization dependence. Thus it was possible to obtain fairly comprehensive results as a function of \(\omega, H, \phi\) and \(\psi\).
6(d) Approximate Expressions for the Rotation and Ellipticity

The two limiting cases discussed in 5(c) in connection with
the isotropic model are considered. They are both low field
approximations, i.e. $(\mathcal{J}a)^2 \ll 1$ and one, $(\omega a)^2 \ll 1$, represents the
low frequency limit, while the other case is the high frequency limit
$(\omega a)^2 \gg 1$. The approximations derived are linear in $H$, as only terms
of order $H$ in the expansion in powers of the magnetic field are
retained. The formulae are therefore isotropic, (see section 4(b)),
and are essentially identical with those obtained in 5(c), except that
$m^*$ is replaced by an effective mass appropriate to the ellipsoid model.

(i) Low Frequency

The [001] direction is considered first. Substituting the
appropriate tensor components from appendix B into (5.3) and using the
approximation (5.21) to the integrals, as it is still valid for
$(\mathcal{J}a)^2 \ll 1$ and $(\omega a)^2 \ll 1$, we find

$$\frac{\omega}{c}^{2} \to -\frac{\omega^2}{c^2} + \frac{4\pi i \omega}{c^2} \frac{4u(2K+1)a}{3} \left[1 - i \frac{T^2}{2}\right]$$

$$\frac{\omega}{c}^{2} \to \frac{4\pi \omega}{c^2} \frac{4u}{3} \frac{vK(K+2)H}{2} a^2$$

The D.C. conductivity and Hall coefficient for lattice scattering are

$$\sigma_{\omega}^{L} = \frac{4u}{3}(2K+1)a$$

$$R_{H}^{L} = \frac{9\pi K(K+2)}{8Ne\zeta (2K+1)^2}$$

If, for the reasons given in 5(c)(i), the real part of (6.24) is much
less than the imaginary part, we find, on following through the steps
of the corresponding isotropic calculation

$$\alpha_{\omega}^{L} \to \frac{1}{2^2} \left[\frac{4\pi \omega \sigma_{L}^{L}}{c^2}\right]^{1/2} \times$$

$$\left[1 + \frac{1}{2} \left\{ -\frac{\omega^2}{c^2} + \frac{4\pi \omega a \sigma_{L}^{L}}{2} \left(\omega + \frac{eH a K(K+2)}{n^2 \zeta (2K+1)}\right) \left(\frac{4\pi \omega \sigma_{L}^{L}}{c}\right)^{-1}\right\}\right].$$

(6.26)
The bracketed sign alternative applies to $\beta_\perp$. Hence

$$\left[ \frac{\alpha}{\beta} \right] = \frac{1}{2} \left[ \frac{4 \pi \omega \sigma_0^L}{c^2} \right]^{1/2} \frac{a e H K(K+2)}{m_3^c (2K+1) 2^{1/2}}$$ \hspace{1cm} (6.27)

But, as (6.25) gives

$$R_H^L \sigma_0^L = \frac{\pi^{1/2} e a H K(K+2)}{2 m_3^c (2K+1)}$$ \hspace{1cm} (6.28)

we obtain finally the same form of equation as (5.26), namely

$$\frac{H}{H_d} = \frac{\text{Tanh}^{-1} \Delta}{\text{Tanh}^{-1} \Delta} = \frac{R_H^L}{c} \left[ \frac{\pi \omega (\sigma_0^L)^3}{2} \right]^{1/2}$$ \hspace{1cm} (6.29)

The equivalent Faraday mass $m^*_F$ is thus given by

$$m^*_F \frac{3^{1/2}}{2} \equiv \left[ \frac{3m_3^c (2K+1)}{K(K+2)} \right]^{1/2}$$ \hspace{1cm} (6.30)

Equation (6.27) is easily seen to be valid for any direction of the field, because all the anisotropic terms are zero in the approximation considered above, so that equation (6.24) is still correct. However, for non-isotropic directions, the reasoning used only establishes that $-\frac{P}{H_d}$ and $-\frac{Q}{H_d}$ are given by (6.29). It is necessary to show further that $P$ and $Q$, or $R$ and $T$, are equal to their isotropic values to the same approximation. It turns out that, for the (100) plane case, $P$ and $Q$ contain a term which is linear in $H$. It is anisotropic, however, as it arises from terms $O(H^2)$ in $(\mathbf{A} - D)$. If the integrals are approximated by (5.21), $P$ and $Q$ are zero. It is straightforward to show that the term is

$$P + iQ \rightarrow \frac{(K-1)^2 \sin^2 \phi \left( e a / m_3^c \right) H (1 - i 3 \pi^{3/2} a \omega)}{3(K+2)(1 - 14 a \omega)}$$ \hspace{1cm} (6.31)

with a similar type of expression for the (110) plane. The cyclotron resonance effective mass values are, (Dresselhaus et al. (1955),
Levinger and Frankl (1961))

\[ m_1 = 0.08 m_0, \quad m_3 = 1.6 m_0, \quad K = 20 \quad (6.32) \]

The carrier concentration used for the isotropic calculation is retained for convenience. The Hall coefficient for the ellipsoidal model with \( K = 20 \) is in fact numerically quite close to the value for the isotropic model. With the D.C. conductivity equal to \( 9 \times 10^{10} \) e.s.u. (Debye and Conwell (1954), sample 51) we have

\[
\begin{align*}
\alpha &= 4.3 \times 10^{-13} \text{ sec.} \\
N &= 1.3 \times 10^{14} \text{ cm}^{-3}
\end{align*}
\quad (6.33)
\]

If terms in \((\omega a)\), which are of second order, are neglected, the maximum value of \( P = 2.6 \times 10^{-5} H \) \((Q = 0)\) when \( H \) is along the \([011]\) direction. The condition \((\lambda a)^2 \ll 1\) must be satisfied for the larger value of \( \lambda \), and is therefore more restrictive for the \([011]\) than for the \([001]\) direction. The values for the germanium band parameters are \(1.83 \times 10^8 H\) and \(1.34 \times 10^8 H\) respectively. For the \([011]\) direction \((\lambda a)^2 = 0.25\) with \( H = 10^4 \) oersted. Thus the whole approximation breaks down long before \( P \) and \( Q \) become large enough to make the isotropic formula for \( \Theta \) and \( \Delta \) no longer satisfactory, and the Faraday effect is therefore completely isotropic at low fields as \( P \) and \( Q \) are negligibly small. However, for the ellipsoidal model, the field up to which \((6.29)\) is valid is lower by up to 50\% for an arbitrary direction of \( H \) than for the \([001]\) direction. Equation \((6.29)\) is also valid for impurity scattering if the numerical factor is adjusted from \(3\pi/8\) to 1.93.

(ii) High Frequency

For \( \omega^2 \gg \lambda^2 \) and \((\omega a)^2 \gg 1\) the approximations to the integrals are given by \((5.33)\). For the \([001]\) direction \((5.3)\) becomes
\[
\alpha_+^2 \rightarrow -\frac{\omega^2 \epsilon}{c^2} + \frac{4\pi iwa}{c^2} \frac{4u(2K+1)}{3} \left[ \frac{2}{a^2 \omega^2} - i \frac{3\pi^{\frac{1}{2}}}{4aw} \right]
\]
(6.34)

\[
\beta_+ \rightarrow \frac{\omega \epsilon}{c} \left[ 1 - \frac{\omega^2}{\omega^2} \right] \left[ 1 - \frac{\omega^2 e^{\frac{e^{K}(2K+2)H}}}{2m_3 c \omega^2 (2K+1)} \left\{ 1 - \frac{\omega^2}{\omega^2} \right\}^{-1} \right].
\]
(6.38)

These equations lead to

\[
\frac{\Delta}{2Hd} = \frac{\text{Tanh}^{-1}\Delta}{Hd} = \frac{32\pi^{\frac{1}{2}}Ne^{K}(2K+2)}{9m_3 c^2 \omega^3 a \epsilon^2} \left[ 1 + \frac{3\omega^2}{4\omega^2} \right].
\]
(6.39)

The Faraday effective mass for this approximation is thus given by (6.5).

The anisotropic terms are all zero in the present approximation, so that (6.35) and (6.39) are in fact valid for any orientation of \(H\) so long as \(P,\) etc. tend to the isotropic values. As for the low frequency case, \(P,\) etc. contain a term linear in \(H\) though, taking the (100) plane as an example, it arises from terms \(O(H^2)\) in \((A - D)\) and is therefore anisotropic. It is given by
Calculation shows that $P$ and $Q$ are negligible at all fields and frequencies to which the high frequency approximation may be applied. Thus, as long as $\omega^2 \gg \lambda^2$ is satisfied for the largest cyclotron frequency, the isotropic expressions are valid for all directions of the field. Formulae corresponding to (6.35) and (6.39) may be obtained for impurity scattering.

6(e) Rotation - Results

(i) Frequency dependence

The room temperature parameters for the specimen of N-type germanium are given in (6.32) and (6.33). The calculations covered the frequency range $10^{-10} - 10^{14}$ for fields of $10^4$, $2 \times 10^4$, and $3 \times 10^4$ oersted, together with additional fields at a few frequencies only. The cyclotron frequencies for the high symmetry directions and the values of $\lambda$ for $H = 10^4$, $2 \times 10^4$ and $3 \times 10^4$ are

<table>
<thead>
<tr>
<th>Direction</th>
<th>$\mathcal{R}$</th>
<th>$\lambda_a (10^4)$</th>
<th>$\lambda_a (2 \times 10^4)$</th>
<th>$\lambda_a (3 \times 10^4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>[100]</td>
<td>$1.33 \times 10^8$</td>
<td>0.57</td>
<td>1.14</td>
<td>1.71</td>
</tr>
<tr>
<td>[111]</td>
<td>$2.20 \times 10^8$</td>
<td>0.95</td>
<td>1.89</td>
<td>2.84</td>
</tr>
<tr>
<td></td>
<td>$8.67 \times 10^7$</td>
<td>0.37</td>
<td>0.75</td>
<td>1.12</td>
</tr>
<tr>
<td>[110]</td>
<td>$1.82 \times 10^8$</td>
<td>0.78</td>
<td>1.56</td>
<td>2.35</td>
</tr>
<tr>
<td></td>
<td>$4.92 \times 10^7$</td>
<td>0.21</td>
<td>0.42</td>
<td>0.63</td>
</tr>
</tbody>
</table>

This table shows that the fields chosen fall in the region $\lambda_a \ll 1$ for N-type germanium. The chief emphasis in this case is laid on the results for experimentally attainable fields, because they can be compared with actual measured values as the correct band structure has been used. The high and low field limits are discussed, however, as for the isotropic case. It will be clear from the table...
that, except for the [100] direction, the conditions are more restrictive than for the isotropic case as they must be satisfied for all the cyclotron frequencies, and these can differ by a factor as great as three. However, tendencies towards the expected low and high field behaviour are clearly shown for the values $10^4$ and $3 \times 10^4$ respectively.

Fig. (6.1) shows the frequency dependence of the rotation for $H$ parallel to the [001] direction with $H = 10^4$, $2 \times 10^4$ and $3 \times 10^4$ for curves (a), (b) and (c) respectively. The thickness is 5 mm. At very low fields the low frequency approximation (6.29), in which $\Theta$ varies as $\omega^{1/3}$, is valid. With $H = 5 \times 10^3$ for example, the value obtained from (6.29) is only 5% too high at $10^9$ and 1% too high at $10^{10}$. This equation is, in general, a reasonable approximation for $\omega < 3 \times 10^{10}$ and $H < 5 \times 10^3$ ($\Re a = \frac{1}{4}$). The upper frequency limit is slightly higher than for the isotropic calculation because the natural frequency $a^{-1}$ is higher. Thus the low field pattern shown in curve (a) of Fig. (5.1) is shifted to a higher frequency for the present specimen. For (a) of Fig. (6.1) the field is too high for (6.29) to be very accurate, the value being about 20% too high for $\omega = 10^9$. However, the typical isotropic low field pattern is reproduced quite closely, except for a slight flattening of the negative peak. The sign change at $\omega = 3 \times 10^{12}$ is slightly higher than the natural frequency which is $2.3 \times 10^{12}$. This effect was observed in the isotropic case (section 5(f)(i)), and is due to the energy dependence of $\Upsilon$. The high frequency positive peak and $\omega^{-2}$ region are also at correspondingly higher frequencies. At $4 \times 10^{13}$ (6.35) is only 3% too high, $1.10 \times 10^{-2}$ as compared with the exact value of $1.07 \times 10^{-2}$ for a thickness of $1 \text{cm}$. The discrepancy is due to the higher order term in $(\omega a)^{-2}$ which is exactly the same as for the isotropic case. The plasma term is negligible.

A detailed comparison between the isotropic and anisotropic results is difficult as $a$, as well as the effective masses, differs in the two cases. However, if the ratio of the two results is evaluated at frequencies below the sign change for the same values of $\Re a$, it remains fairly constant at about $1.1 \rightarrow 1.2$ for all fields such that $\Re a \leq 1$. Close to the sign change and in the region of
Fig. (6.1). Variation of Faraday rotation with frequency for n-type germanium specimen with $a = 4.3 \times 10^{-13}$ sec, $N = 1.3 \times 10^{14}$ cm$^{-3}$, $m = 1.6m_0$, $K = 20$, $\epsilon = 16$: a, $H = 1.0 \times 10^4$ Oe; b, $H = 2.0 \times 10^4$ Oe; c, $H = 3.0 \times 10^4$ Oe; $H$ along [001] and $d = 5$ mm for all curves.
the positive peak it varies considerably, while in the high frequency region it finally tends to 0.23. The non constancy round the sign change is due to the different natural frequencies in a region where ℭ is changing very rapidly with ω. It is largely eliminated if the ratio of corresponding points on the two curves, rather than the ratio at a given frequency, is considered. The high frequency ratio for a given value of \( \lambda a \) is, from (5.35) and (6.35),

\[ \frac{3m_s^2}{m K(K+2)} \begin{bmatrix} H_{iso} \\ H_{anis} \end{bmatrix} = (6.42) \]

where \( H_{iso} \) and \( H_{anis} \) represent the magnetic fields required to make \( \lambda a \) equal in the two theories. The value of (6.42) is 0.23 for the present parameters. It seems likely therefore, that, if \( \lambda a \) were made equal in the two cases, a single isotropic effective mass could be defined which would give results quite close to those obtained using the correct band structure over the entire frequency range apart from the high frequency region. Here a different mass defined by (6.5) would be required. This conclusion, which may be true for any band structure, is rather surprising as an approximate formula for the rotation in terms of a single effective mass can only be derived for very low frequencies and fields. The isotropic model should therefore be useful in interpreting the results of microwave experiments, as long as the field direction remains fixed with respect to the crystal axes.

Curve (b), for which \( \lambda a = 1 \), shows further flattening of the negative maximum. The rotation has only increased by a factor of 25-50%, due to the breakdown of the low field linear variation with \( H \).

Curve (c) has \((\lambda a)^2 \approx 3\). The negative peak is considerably broader and flatter than for (b), and the magnitude of \( \lambda \) has decreased in some frequency regions, indicating that the maximum in the field dependence generally lies between 2 and \( 3 \times 10^4 \) oersted. It was noted in the isotropic case also that the maximum occurred for \( \lambda a \) just greater than unity. This field is not sufficiently high for either the magnetoplasma edge 'kink', or the resonance at the cyclotron frequency \( \omega = \lambda a \times 4 \times 10^{12} \) to be apparent.

As for the isotropic case, approximate formulae may be derived
in the high field limit. If the appropriate tensors for the [001] direction are substituted into (6.2) and the usual approximations that \((d_l)^2 \gg 1\) and \(\mathcal{K}^2 \gg \omega^2\) made, we obtain after some straightforward substitutions

\[
\mathcal{M}^2 \rightarrow -\frac{\omega e^2}{c^2} \left[ 1 + \frac{\omega^2}{\mathcal{K}^2} + \frac{\omega^2}{\omega \mathcal{K} (2K+1)} \right] - \frac{3 \omega^2}{3 \pi^2 \omega a \mathcal{K}^2} + i \frac{\omega^2 16 m^3 e}{(2K+1) \pi \frac{3}{2} \alpha \mathcal{K}^2}
\]

(6.43)

Exactly as in the isotropic case, the first and third terms of (6.43) cancel for the negative wave when the product of \(\omega\) and \(\mathcal{K}\) has a certain value. This is the magnetoplasma edge which is now defined by the condition

\[
\frac{\omega_p^2 [3K(K+2)]^{1/2}}{\omega \mathcal{K} (2K+1)} = 1
\]

(6.44)

The multiplying function of \(K\) cannot exceed 1 and varies only between 1 for \(K = 1\) and \(3^{1/2}\) as \(K \to \infty\), so that the condition is in practice very close to the isotropic equivalent \(\omega_p^2/\omega \mathcal{K} = 1\). \(\alpha\) falls sharply and \(\beta\) rises correspondingly when the frequency defined by (6.44) is reached, in a similar manner to that illustrated in Fig. (5.2) for the isotropic model.

Well above the edge, i.e. when \(\omega_p^2/\omega \mathcal{K} \ll 1\), we can neglect \(\alpha^2 \ll \beta^2\) when taking the real part of (6.43) and, on expanding the square root appropriately, find

\[
\Theta = -\frac{2\pi N e d}{\mathcal{K}^{1/2}}
\]

(6.45)

This is exactly the result of (5.55) when the isotropic values of \(\omega_p^2\) and \(\mathcal{K}\) are substituted in that equation. Similarly, the conditions defining the magnetoplasma edge are in fact the same in the two cases. (6.45) is interesting as it is independent of the effective mass, relaxation time and frequency and might therefore be used to determine \(N\) as an alternative to Hall measurements in
a material such as indium antimonide where the necessary conditions could be satisfied at reasonable fields and carrier concentrations. The disadvantage lies in the multiple reflection correction $T$, (Medcalf (1963), equation 3), which is not negligible under the same conditions because the absorption is low. The measurements of Mansfield and Borst (1964) on indium antimonide show the tendency to a saturation value independent of the field in the high field limit; $(\omega_p^2/\omega\kappa \sim 0.5$ for their highest field). However, this value is about 27% higher than (6.45) predicts, 63.3°, as compared with 53.6°. However, further investigation of the multiple reflection correction might be of interest as the Mansfield and Borst experiment shows that it must also tend to a saturation value, and a simple formula might be obtainable.

The result for the other limiting case for which $\omega_p^2/\omega\kappa >> 1$ also follows from (6.43). As in the isotropic case attention must be paid to signs when expanding the inner square root occurring in $\beta_\pm$. We find

$$\beta_+ \to \frac{\omega_p}{c} \left[ \frac{3m_ee\kappa}{e\hbar(2K+1)} \right]^{\frac{1}{2}} = \left[ \frac{4\pi N\kappa e\omega}{e\hbar} \right]^{\frac{1}{2}}, \quad (6.46)$$

and

$$\beta_- \to \beta_+ \frac{4}{3} \frac{\omega_p}{3\pi^{\frac{3}{2}} a \kappa} \left[ \frac{2K+1}{3K(K+2)} \right]^{\frac{1}{2}}. \quad (6.47)$$

The comparable isotropic formulae are equations (5.56) and (5.57). On substituting for $\omega_p^2$ it can be seen that (5.56) and (6.46) are identical. Since the multiplying function of $K$ is the reciprocal of the one discussed in connection with (6.44), it is never much greater than one, so that $\Theta$ is still given approximately by $-\beta_+ d/2$.

The edge, as defined by (6.44), is at $5 \times 10^{10}$ for curve (c), quite close to the onset of the flat region of the negative maximum. However, (6.45), (6.46) and (6.47) represent very poor approximations to the exactly calculated values at this field for frequencies well above and below the edge, the discrepancies being as great as 50%. This is not surprising in view of the low value of $\kappa a$, only 1.7.
Furthermore, as the maximum in the field dependence occurs for \( J_\alpha > 1 \) and not for \( J_\alpha = 1 \), correspondingly higher fields are necessary to attain the high field behaviour. The frequency independent maximum, though not given accurately by (6.45), is nevertheless clearly evident and spans the right frequency range, and an increase in the field by a factor of about 2-3 should bring good agreement between the exact and the approximate formulae in both limits.

The usual \( \omega^{-2} \) high frequency approximation is satisfactory for (b) and (c) for \( \omega \gtrsim 4 \times 10^{13} \), or above an appropriately increased frequency such that \( \omega^2 \gg J^2 \) as \( H \) is increased. Therefore, in the high field limit, the rotation is independent of the relaxation mechanism over the entire frequency range, except close to the resonant frequency \( \omega = J \) where the detailed form of the resonance is a function of \( \gamma \). Since, in addition, the rotation in the region above the magnetoplasma edge is not a function of the effective mass, the isotropic theory using the Faraday effective mass defined by (6.36), which is in fact the usual conductivity mass, \( \rho_m/(2k+1) \), should give a very good representation of the rotation below cyclotron resonance, while the high frequency mass given in equation (6.42) is again required above the resonant frequency.

Figure (6.2) shows the rotation for \( H = 2 \times 10^4 \) oersted, with \( H \) parallel to certain crystallographic directions selected to show the extent of the variation in \( \phi \) due to anisotropic effects. Curve (a) is identical with (b) of Fig. (6.1), but it is included as the rotation is a maximum for this orientation, except close to the sign change. \( H \) is along the [011] direction for curve (b) with the electric vector of the incident radiation along the [100] axis, so that (a) and (b) correspond to the values \( \phi = 0 \) and \( \pi/4 \) for the (100) plane case discussed in appendix A II(a). \( \gamma \) defines the direction of \( \xi_\alpha \) and the convention chosen for this chapter is explained in (iv). For curve (c) \( H \) is along the [111] direction, with \( \xi_\alpha \) inclined at \( \pi/4 \) to the [1\overline{1}0] axis measured in the (111) plane with a right handed sense of rotation to an observer looking along the [111] direction. Thus (a) and (c) show typical curves for the (1\overline{1}0) plane case discussed in appendix A II(c), though not the maximum anisotropy as \( |\overline{\kappa}| \) is generally a
Fig. (6.2). Variation of Faraday rotation with frequency for n-type germanium specimen:

- a, $\mathbf{H}$ along [001];
- b, $\mathbf{H}$ along [011] in (100) plane with $\psi = 0$;
- c, $\mathbf{H}$ along [111];
- d, $\mathbf{H}$ along [011] in (100) plane with $\psi = \pi/4$; $\mathbf{H} = 2.0 \times 10^4$ Oe and $d = 5$ mm for all curves.
minimum for $\phi \sim 35^\circ$. Curve (d) illustrates the rotation for a configuration of axes for which $|\Theta|$ is close to its minimum value over most of the range. $\mathbf{H}$ is along the [011] direction, and the electric vector is at $\pi/4$ to the [100] direction measured in the (011) plane with a right handed sense of rotation to an observer looking along the field direction.

All the curves have a common crossing point at $\sim 2.5 \times 10^{12}$ where the anisotropy is negligible. This corresponds very closely to the natural frequency of the carriers, $a^{-1}$. The low field, low frequency approximation (6.25) is no longer applicable if the anisotropy is significant. It was proved in a(i) that $\mathbf{H}$ is isotropic to this order. Similarly, the high frequency $\omega^{-2}$ approximation is isotropic, and all the curves coincide in this region as the condition $\omega^2 \gg \lambda^2$ is satisfied for all the cyclotron frequencies. Below this region the anisotropy is quite marked at $2 \times 10^4$ oersted. It is a maximum in the neighbourhood of $\lambda a = 1$ and decreases as $\lambda a \to 0$ or $\infty$. The general features of the frequency dependence described above clearly persist for all orientations of $\mathbf{H}$, but anisotropic effects modify the sign change frequency slightly, while the negative maximum is more rounded for the non-isotropic directions.

With $\lambda a \sim 1$, as for Fig. (6.2), there are no approximate formulae below the high frequency region, but in the high field limit the two cases of $\omega_p^2/\omega \lambda << 1$ and $\gg 1$ again arise. It is assumed that the conditions $(\lambda a)^2 \gg 1$ and $\lambda^2 \gg \omega^2$ are satisfied for all the cyclotron frequencies. Then, considering the (100) plane case as an example, and using these approximations to find first $\alpha$, $\beta$, $\epsilon$ and $\phi$, and hence $P$ and $Q$, $G$ and $H$, and $L_\perp$ and $M_\perp$ using equations (2.15), (2.16) and (2.18) respectively, we obtain on substituting in (6.23) for $1/\tau^2$ in terms of $C$, $D$, $L_\perp$ and $M_\perp$, and simplifying where appropriate,
Care must be exercised in deriving (6.48), because of the difficulty over deciding which are the most significant terms. The real part includes terms up to $O(H^{-2})$ only, since the last term, which is $O(H^{-1})$, gives rise to the leading term in expressions for the rotation. Thus terms of $O(H^{-3})$ are omitted, though these have been derived to ensure that they are all small. The term of $O(H^{-3})$ must be retained in the imaginary part, however, as it turns out to be the leading term in approximate formulae for the ellipticity. The magnetoplasma edge condition is now given by the cancellation of the first and last terms of the real part of (6.48). It is the same as (6.44), and independent of the band structure, though it cannot now be conveniently written in terms of $\mathcal{A}$ and $\omega_p^2$. Thus the rotation well above the edge is again given by (6.45), and is independent of the orientation of $\mathbf{H}$ in this limit. Similarly, well below the edge the isotropic formula (6.46) for $\beta_+$ is again valid. $\beta_-$, which derives from higher order field terms, is anisotropic. We find

$$\beta_- \to \beta_+ - \frac{4}{3\pi a} \left[ \frac{1}{\mathcal{A}_1^2} + \frac{1}{\mathcal{A}_2^2} + \frac{(k-1)\sin 2\phi}{2(2k+1)} \left( \frac{1}{\mathcal{A}_1^2} - \frac{1}{\mathcal{A}_2^2} \right) \right] \left( \frac{2k+1}{6\mathcal{A}_3} \right) \frac{\omega \mathbf{H}}{a m_c}.$$ (6.49)

Like (5.57) and (6.47), (6.49) is not a very good approximation. This can be traced in all three cases to higher order terms, which are more important than in the case of $\beta_+$. However, $\beta_-$ is still $\beta_+/a\mathcal{A}$, so that the rotation is given approximately by $-\beta_+/a\mathcal{A}$. 
The rotation is thus independent of the orientation of $\mathbf{H}$ with respect to the crystal axes in both the high field approximations. Together with the high frequency $\omega^{-2}$ approximation, these cover most of the frequency range. However, when $\mathbf{H}$ is along non-isotropic directions, higher fields are required for (6.45) and (6.46) to be valid, because the high field conditions must be satisfied for the smaller cyclotron frequency, which is always less than the isotropic direction frequencies. Approximate expressions cannot be found for the rotation at fields which give rise to significant anisotropic effects, since these require $\lambda a > 1$.

For the (110) plane case only equation (6.49) for $\phi_0$ needs to be modified appropriately.

(ii) Field Dependence

The dependence of the rotation on magnetic field strength is illustrated at four frequencies in Fig. (6.3). Logarithmic scales are used for both axes, to emphasize the linear dependence on $H$. All the frequencies are below the low field sign change, so that $\log_{10}(\phi)$ is plotted. For (a), (b) and (c), which are well below the sign change, $|\phi|$ varies linearly with $H$ to within about 5% for $H < 5 \times 10^3$; ($\lambda a \leq 0.3$). This is in accordance with the low frequency approximation (6.29). As in the isotropic case, the linear dependence persists in fact to frequencies where (6.29) is no longer valid. At higher fields the rotation increases less rapidly with field. However, at $10^9$ and $10^{10}$, (curves (a) and (b) respectively), $|\phi|$ is still increasing at $3 \times 10^4$ ($\lambda a = 1.7$), suggesting that a value of $\lambda a = 2$ is necessary in order to reach the turning point. This helps to explain why numerical values taken from curve (c) of Fig. (6.1) did not give good agreement with the high field approximations. It is clearly necessary to be well beyond the maximum before the field dependence can tend to $H^{-1}$ or $H^{-\frac{3}{2}}$, as predicted by (6.45) and (6.46) respectively. For curve (c), with $\omega = 10^{11}$, the maximum occurs at $2.4 \times 10^4$; ($\lambda a = 1.4$). Thus it moves towards the value $\lambda a = 1$ as the frequency increases. Curves (a) and (b) of Fig. (5.3) show exactly the same feature for the isotropic case, confirming that the effect arises from the energy.
Fig. (6.3). Variation of Faraday rotation with field strength for n-type germanium specimen at various frequencies:

- a, \( \omega = 10^9 \)
- b, \( \omega = 10^{10} \)
- c, \( \omega = 10^{11} \)
- d, \( \omega = 10^{12} \)

\( H \) along [001] and \( d = 5 \text{ mm} \) for all curves.
dependence of $\mathcal{C}$, rather than from the nature of the energy surfaces. Curve (d), for which $\omega = 10^{12}$, is not very far from the low field sign change frequency and, like (c) of Fig. (5.3), $|\mathbf{a}|$ increases at a rate greater than the first power of $H$ at low fields, before passing through the turning point at about the same field as curve (c), with which it merges as $H$ is further increased.

At higher fields the slopes of (a) and (b) would tend first to $-\frac{1}{2}$, as (6.46) became valid. However, as the field was further increased, first (a) and (b) would tend finally to the $H^{-1}$ dependence of (6.45) as the magnetoplasma edge frequency, $\omega = 4\pi N e c / \mathcal{E} H$, became much less than $10^{10}$ and $10^9$ respectively. The gradient of (c) and (d) would tend directly to $-1$, because $10^{11}$ and $10^{12}$ are above the edge frequency by the time the high field condition is adequately satisfied.

At frequencies in the vicinity of the sign change the field dependence is unpredictable, and the rotation can change from a positive to a negative value as the field passes from the region of $\mathcal{R} \mathbf{a} < 1$ to $\mathcal{R} \mathbf{a} > 1$. Well above the sign change $\mathcal{R}$ is positive and proportional to $H$ as long as $\omega^2 \gg \mathcal{R}^2$, since the high frequency approximation (6.35) is valid. The rotation will then show a sharp resonance and change of sign in the vicinity of $\omega = \mathcal{R}$, and finally tend to the $H^{-1}$ variation when $\mathcal{R}^2 \gg \omega^2$. The behaviour is similar to the isotropic case, but the detailed shape of the resonance in particular is affected by the use of the ellipsoid band model.

Fig. (6.4) shows the rotation as a function of field at a fixed frequency, $10^{11}$, for the four orientations of the $(x,y,z)$ axes described in (i) in connection with Fig. (6.2). It should be noted that logarithmic scales have not been used for this graph. The anisotropy tends to zero at $H \approx 5 \times 10^3$, about the field at which the approximation (6.29) breaks down. As $H$ is increased the four curves diverge uniformly. The maximum separation is reached when $H$ is about $2 \times 10^4$ oersted, and maintained fairly constant up to $3 \times 10^4$. The positions of the maxima vary slightly for the different orientations. The tendency is for the turning point to move to a lower field, and become rather broader and flatter, when $H$ is along the more anisotropic
Fig. (6.4). Variation of Faraday rotation with field strength for n-type germanium specimen with various orientations of field: a, $\mathbf{H}$ along [001]; b, $\mathbf{H}$ along [011] in (100) plane with $\psi = 0$; c, $\mathbf{H}$ along [111]; d, $\mathbf{H}$ along [011] in (100) plane with $\psi = \pi/4$; $\omega = 10$ and $d = 5$ mm for all curves.
directions for which $|\mathbf{a}|$ is smallest. The maximum anisotropy expressed as a percentage of the value for the $[001]$ direction, (curve (a)), is about 35%, and should therefore be easily detectable experimentally. At higher fields the separation of the curves will gradually decrease as $|\mathbf{a}|$ decreases. When the anisotropy is negligible equation (6.45) will apply, so that the rotation will tend to zero as $H^{-1}$.

Corresponding curves for other frequencies well below the low field sign change are very similar to Fig. (6.4), except for the slight variations in the positions of the turning points noted in (i). The maximum percentage anisotropy remains approximately constant and independent of the frequency.

Just below the sign change frequency the anisotropy tends to zero when $\mathbf{a} \leq 1$, (see Fig. (6.2)), and the curves therefore merge at this frequency and then occur with the positions interchanged, $|\mathbf{a}|$ being smallest for (a) and increasing for (b), (c) and (d) in that order, as $\omega$ is increased. They invert again and resume the ordering of Fig. (6.4) when the rotation becomes positive at a slightly higher frequency. The anisotropy then diminishes to zero and (a), (b), (c) and (d) finally merge at a frequency a little above the positive peak.

In the region of the sign change, i.e., $\omega \gtrsim 10^{12}$, the anisotropic effects, like the field dependence, are rather complicated when $\mathbf{a} > 1$, because of the field dependence of the sign change. Between the change and the natural frequency, the overall rotation increases at first and the anisotropy reduces to zero and reverts to the pattern of (6.4) as the negative maximum flattens and extends to higher frequencies with the increasing field. Then ultimately, when $(\mathbf{a})^2 \gg 1$, both $|\mathbf{a}|$ and the anisotropy decrease and the latter actually tends to zero as (6.45) becomes valid. For frequencies which are above the sign change at low fields, $\mathbf{a}$ will remain positive, showing anisotropic effects similar to Fig. (6.4), which increase with $H$ until $\omega = \omega^\text{res}$. The resonance and sign change will occur in this region, the peak heights increasing as the value of $\mathbf{a}$ at which resonance occurs increases. The anisotropy is large at the resonance, due to the differing cyclotron frequencies and, for directions with more than one cyclotron frequency, the peaks are broadened. Finally the flat negative maximum is reached when
\( \lambda^2 >> \omega^2 \). The exact details of the curves for various orientations of \( H \) can only be derived by exact calculations in this region.

As in the isotropic case, the magnetoplasma and plasma edges are always at much lower frequencies than the cyclotron resonance, so that the complications described in 5(e)(ii) are avoided.

(iii) Dependence on the Orientation of the Magnetic Field

We consider the (100) and (110) plane cases separately, taking the former first. For all frequencies \( \leq 6 \times 10^{11} \), i.e. appreciably below the low field sign change, the anisotropy shown when the rotation is plotted as a function of \( \phi \) has a characteristic pattern. This is valid up to the highest field considered in the present calculations, \( 3.0 \times 10^4 \) oersted, and seems likely to persist to all fields, though this has not been verified. Below about \( 5.0 \times 10^3 \) oersted the degree of anisotropy becomes very small, but the pattern remains the same. Similar behaviour would be expected in the high field limit where the anisotropy also tends to zero.

Fig. (6.5) shows the dependence of \( \mathcal{R} \) on the angle \( \phi \) at a typical microwave frequency, \( (\lambda = 3 \text{ cm}) \), with \( H = 1.0 \times 10^4, 1.5 \times 10^4, 2.0 \times 10^4 \) and \( 3.0 \times 10^4 \) for curves (a), (b), (c) and (d) respectively. The rotation is symmetrical with respect to the [011] direction, \( (\phi = \pi/4) \). The general shape suggests a sinusoidal dependence. However, dividing \( \sin 2\phi \) by \( [\mathcal{R} (\phi = 0) - \mathcal{R} (\phi)] \) for typical curves always gives a result which varies considerably with \( \phi \), so this simplest approximation is not suitable, but repeating the ratio test using \( \sin^2 2\phi \) gives much better constancy, and this function appears to be the best relatively simple functional form for the orientation dependence in the (100) plane. The ratio

\[
\frac{\sin^2 2\phi}{[\mathcal{R} (\phi = 0) - \mathcal{R} (\phi)]}
\]

falls in fact by about 30% as \( \phi \) goes from \( 5^\circ \) to \( \pi/4 \). This is true at all fields for frequencies \( \leq 6 \times 10^{11} \), except that a slight improvement in the fit to \( \sin^2 2\phi \) is noticeable as either \( \omega \) or \( H \) are decreased.
Fig. (6.5). Variation of Faraday rotation with orientation of $\mathbf{H}$ in (100) plane for $n$-type germanium specimen at a fixed frequency: a, $H = 1.0 \times 10^4$ Oe; b, $H = 1.5 \times 10^4$ Oe; c, $H = 2.0 \times 10^4$ Oe; d, $H = 3.0 \times 10^4$ Oe; $\omega = 6.28 \times 10^{10}$, $\psi = 0$ and $d = 5$ mm for all these curves; e, variation of $-\beta d/2$ with other details as for c.

There are two contributions to the anisotropy. The first arises from the anisotropy of the penetration constant and the second from
The magnitude of the anisotropy increases with field in the range covered, as was noted earlier in connection with Figs (6.2) and (6.4). In Fig. (6.5) the progressive increase from $1.0 \times 10^4$, through $1.5 \times 10^4$ to $2.0 \times 10^4$ is clearly evident. For $3.0 \times 10^4$ the magnitude of the rotation has not changed much from curve (c), but the anisotropy has still increased slightly, (see the discussion of Fig. (6.4) in (ii)). Quantitatively it is convenient to calculate the maximum variation of $H$, (or $\Delta$), in the particular crystal plane, and express it as a percentage of the value for the [001] direction. The value is denoted by $\rho_R$ for the rotation and $\rho_\Delta$ for the ellipticity. Thus, for example,

$$\rho_R = \left[ \frac{H(\phi=0) - H(\phi)}{H(\phi=0)} \right] \times 100\% \quad (6.50)$$

Table 6(a) shows the values of $\rho_R$ for frequencies up to and including $3.16 \times 10^{12}$. Dashes appear where the values were not calculated for certain fields. The upper number of each pair is appropriate to a thickness of 5 mm, and the lower is for 1 mm. Certain trends emerge. At all fields $\rho_R$ tends to a constant value independent of the frequency at low frequencies for both values of $d$. In this region the values are also quite closely proportional to $H$ for $H \leq 2.0 \times 10^4$ oersted, and are levelling off gradually at the highest fields where the linear dependence of $H$ on $H$ also breaks down. At even higher fields they should begin to decrease as $H$ decreases. As the frequency is increased beyond $10^{10}$, $\rho_R$ begins to decrease significantly for $d = 5$ mm and falls gradually to zero a little before the sign change. This point is just under $10^{12}$ at the lowest field, and rises to about $3 \times 10^{12}$ at $3.0 \times 10^4$ oersted. At the higher fields $\rho_R$ rises slightly around $5 \times 10^{11}$ before falling to zero. This is due to the rapid decrease in the overall rotation in this region which, over a small frequency range, outstrips the decrease in the anisotropy. For 1 mm the behaviour is similar, except that $\rho_R$ does not begin to decrease much until $\omega$ is about $10^{11}$.

There are two contributions to the anisotropy. The first arises from the anisotropy of the propagation constants and the second from
the complex formulae (2.27) and (2.28) for $\rho$ and $\Delta$. $\rho$ values, for $\bar{\beta}$, called $\rho(\bar{\beta})$, are higher than the corresponding $\rho_R$ values. This effect is small at low frequencies, $\lesssim 10^6$, but increases as $\omega$ increases, since the percentage anisotropy in $\bar{\beta}$ stays virtually constant up to frequencies very close to the frequency at which $\rho_R \to 0$. ($\rho_R$ and $\rho(\bar{\beta})$ become zero at almost exactly the same point.) Thus the effect of the formula (2.45) for the (100) plane case is to decrease the anisotropy in general. This is illustrated by curves (c) and (e) of Fig. (6.5), which show $\mathbf{R}$ and $-\bar{\beta}d/2$ for the same parameters. The effect of (6.45) in the negative rotation region is least when $\bar{\beta}d$ and $\bar{\beta}d$ are small, so that $\sin \bar{\beta}d \sim \bar{\beta}d$, $\cos \bar{\beta}d \sim 1$, and $\exp(\pm \bar{\beta}d) \sim 1$. Under these conditions $\rho_R$ is not very dependent on the thickness, as hand calculations show, the anisotropic terms in (2.45) are small, so that $\mathbf{R} = -\bar{\beta}d/2$ and $\rho_R \to \rho(\bar{\beta})$. At very low frequencies $< 10^9$ this is true for both 5 mm and 1 mm with the present specimen parameters; (the values for $\bar{\beta}d$ are 0.73 and 0.15 respectively at $10^{10}$ with $H = 2.0 \times 10^4$). For 1 mm at $10^{11}$, with $H$ again $2.0 \times 10^4$, $\bar{\beta}d = 0.38$.

Thus the criterion for (2.45) to be approximately equal to the isotropic formula also depends on $\omega$ and $H$, and cannot be stated simply as an upper bound to $|\bar{\beta}d|$. However, $|\bar{\beta}d|$ should certainly be $< \pi/4$.

More generally, in the microwave region when $H$ is along non-isotropic directions, the rotation should show a small fluctuation periodic in $d$ superimposed on the basic value if the thickness is varied sufficiently. This is because $\bar{\beta}d \sim \pi$ and (2.27) contains sine and cosine terms. This effect was not investigated systematically, but the decreasing $\rho_R$ values as $\bar{\beta}d$ increases may well be a consequence of this periodicity, and for values of $d > 5$ mm $\rho_R$ may in fact increase again. This conclusion could be modified by the non-periodic exponential terms if $\bar{\beta}d$ is also not small. The necessary thicknesses would, however, correspond to unrealistically high values for the absorption, and the effect would also be masked anyway by the multiple reflection correction which is approximately periodic in $2\bar{\beta}d$, and therefore has a much shorter period, (Bouwknecht and Volger (1964)).

Fig. (6.6) shows the $\phi$ dependence at a number of frequencies and fields in the vicinity of the natural frequency, where the behaviour is
Fig (6.6). Variation of Faraday rotation with orientation of \( \mathbf{H} \) in (100) plane for n-type germanium specimen at various frequencies and fields: a, \( H = 1.0 \times 10^4 \) Oe, \( \omega = 10^{12} \); b, \( H = 2.0 \times 10^4 \) Oe, \( \omega = 2.24 \times 10^{12} \); c, \( H = 2.0 \times 10^4 \) Oe, \( \omega = 3.16 \times 10^{12} \); d, \( H = 2.0 \times 10^4 \) Oe, \( \omega = 6.31 \times 10^{12} \); \( \psi = 0 \) and \( d = 5 \) mm for all curves. The left hand ordinate scale is to be used for a and b, the lower right hand for c and the upper right hand for d.
less systematic. Curve (a) shows that the low frequency pattern is maintained almost up to the frequency at which the anisotropy becomes zero, about $1.3 \times 10^{12}$ for this field as Table 6(a) shows. Curve (b) is even closer to the zero point at $\omega = 2.4 \times 10^{12}$, for the higher field of $2.0 \times 10^4$ oersted, and shows some deviation from the characteristic pattern in that the peak is considerably flattened. With the same field, but an increased frequency of $3.2 \times 10^{12}$, (curve (c)), the pattern is inverted, but otherwise unchanged. Thus the region of near zero anisotropy, over which the $\phi$ dependence is changing in form and must therefore be found by calculation, is very small. It always moves to higher frequencies as $H$ is increased, though this effect is small at low fields as Table 6(a) shows. As $\omega$ is increased there is a small frequency region where the rotation changes sign as $\phi$ varies though, if $\Theta$ is plotted with a vertical scale changing from positive to negative, the form of the variation is unchanged. For (d) $\Theta$ is positive for all $\phi$, and the basic pattern is again obtained.

Fig. (6.7) shows a set of typical curves of various fields for the $(110)$ plane case, again with $\lambda = 3$ cm. The rotation is symmetrical with respect to the $[011]$ direction, ($\phi = \pi/2$). As before the patterns illustrated are typical of all frequencies $\leq 6 \times 10^{11}$, though they are more complicated than for the $(100)$ plane, and vary somewhat with field. For all the curves $|\Theta|$ rises steeply between $\phi = 0$ and the first turning point at $35-45^\circ$. This turning point occurs at the largest value of $\phi$ for the smallest fields and higher frequencies, and moves towards $\phi = \cos^{-1}(2/3)$, ($\approx 35^\circ$), as $H$ increases or $\omega$ decreases. At very low fields, $5.0 \times 10^{-3}$ for example, $|\Theta|$ is actually just greater for $\phi = \pi/2$ than for $\phi = 0$, though this is never true of $B$ itself. This is the only exception found in the present work to the general rule that, in this region, $|\Theta|$ is always a maximum for $H$ along the $[100]$ type directions with respect to which all the ellipsoids are symmetrically placed. For a $(100)$ ellipsoid band structure on the other hand, the rotation is generally a maximum when $H$ is along the $[111]$ directions. As the field increases the level of the curve at $\phi = \pi/2$ relative to $\phi = 0$ rises gradually. This can be seen in Fig. (6.7), taking (a), (b), (c) and (d).
Fig. (6.7). Variation of Faraday rotation with orientation of $\mathbf{H}$ in (110) plane for n-type germanium specimen at a fixed frequency: a, $H = 1.0 \times 10^4$ Oe; b, $H = 1.5 \times 10^4$ Oe; c, $H = 2.0 \times 10^4$ Oe; d, $H = 3.0 \times 10^4$ Oe; $\omega = 6.28 \times 10^{10}$, $\psi = 0$ and $d = 5$ mm for all these curves; e, variation of $-\beta d/2$ with other details as for c.
in that order. For (c) and (d), with the highest fields, a second turning point appears at about $70^\circ$, and $|\Theta|$ then rises again. This is barely perceptible for (c), but quite marked for (d). There is thus a lengthy region over which $\Theta$ is almost independent of $\phi$. At higher fields than were considered, the second turning point will probably become even more prominent before the anisotropy tends to zero in the high field limit. It does not seem to be possible to find a simple function of $\phi$ which will represent the anisotropy of $\Theta$ in the $(1\bar{1}0)$ plane.

$\rho_R$ is defined in a similar way to equation (6.50). However, at low fields, and in the vicinity of the anisotropy zero, the values sometimes range from a positive to a negative limit since, as noted above, the base value from which the anisotropy is measured, $\Theta(0)$, is not always the maximum value in the $(1\bar{1}0)$ plane. Once again $\rho_R$ tends to a constant value independent of frequency at low frequencies, and the linearity with $H$ is valid up to about $1.5 \times 10^4$ oersted before the values begin to level off. The anisotropy in this region is greater than for the $(100)$ plane at all fields by about 25%. Also, there is almost no difference between the values for 5 mm and 1 mm at low frequencies and, although $\rho_R$ again starts to decrease first for the larger thickness, this does not happen until about $10^{11}$ for 5 mm and about $6 \times 10^{11}$ for 1 mm. However, the anisotropy still tends to zero at the same frequency as for the $(100)$ plane.

On analyzing the two contributions to the anisotropy, the behaviour of $\overline{\beta}$ is found to be somewhat different to that of the rotation itself. However, $\Theta$ and $-\overline{\beta}d/2$ must intersect at $\phi = \cos^{-1}(1/3)^{1/3}$, as the isotropic formula applies to the $[11\bar{1}]$ direction. The curve rises initially, though less steeply than the rotation, and in the region of $40-45^\circ$ passes through a turning point or flattens. At the lowest fields, such as $5.0 \times 10^3$ oersted, it turns over and slopes downwards again, but much less than the corresponding rotation curve. As $H$ is increased, $|\overline{\beta}|$ first tends to flatten out at about $45^\circ$ and stay almost constant up to $\pi/2$. This flat region then changes gradually into a maximum at about $40^\circ$, followed by a small minimum close to the $[11\bar{1}]$ direction, before $|\overline{\beta}|$ finally increases again as $\phi$ approaches $\pi/2$. Thus, in
contrast to the rotation, the extreme values of $\theta$ occur for $0$ and $\pi/2$, except at low fields. Curves (c) and (e) of Fig. (6.7) show $\theta$ and $-\theta - d/2$ respectively for the same parameters, illustrate the two turning points.

In the present case the rotation is given by (2.40), whose form suggests that the rotation should be closely proportional to $d$, thus making $\rho_\kappa$ independent of thickness, as long as $\theta d$ and $\theta d$ are small. The near independence of $d$ shown by $\rho_\kappa$ over such a wide frequency range is somewhat surprising, however, as these conditions do not hold at the higher frequencies. In addition to the earlier conclusion that $\rho_\kappa$ must depend on $\omega$ and $H$, as well as $d$, it is now clear that there are considerable differences between various orientations of $H$ as, in the present case, the effect on $\rho_\kappa$ of a change in the thickness by a factor of 5 is negligible. It may be that this type of symmetry, $(\kappa = \phi)$, associated with equation (2.40) for the rotation, will always lead to a more extensive low frequency region over which $\rho_\kappa$ is essentially independent of both $\omega$ and $d$ than the (100) plane symmetry, $(\theta = -\epsilon)$, but considerable checking would be necessary to substantiate this. It seems clear, however, that at microwave frequencies, varying $d$ will have much less effect on the anisotropy for the (110) plane case than for the (100) plane case, probably because the anisotropic terms periodic in $d$ are zero, as a comparison of (2.27) and (2.40) will show.

Fig. (6.8) shows the $\phi$ dependence in the vicinity of the anisotropy zero. In fact, the anisotropy does not go almost to zero at one frequency, but the transition takes place rather by a continuous change of the shape which always retains considerable anisotropy. The patterns of Fig. (6.7) persist until very close to this region, then modify rapidly, and the inverted patterns are soon established. Curve (a), for which $\omega = 2.24 \times 10^{12}$ and $H = 1.0 \times 10^4$, is very similar, apart from being inverted, to (b) of Fig. (6.7). At the higher field represented by (b) the frequency is closer to the zero and the pattern is transitional, while for (c), with $H = 2.0 \times 10^4$, the sign change has not quite been reached, and the low frequency pattern is basically present but shows some distortion in details. These three curves illustrate the narrowness of the transition region. Finally curve (d), for which the rotation has
Fig. (6.3). Variation of Faraday rotation with orientation of $H$ in (110) plane for $n$-type germanium specimen at various frequencies and fields: a, $H = 1.0 \times 10^4$ Oe, $\omega = 2.24 \times 10^{12}$; b, $H = 1.5 \times 10^4$ Oe, $\omega = 2.24 \times 10^{12}$; c, $H = 2.0 \times 10^4$ Oe, $\omega = 2.24 \times 10^{12}$; d, $H = 2.0 \times 10^4$ Oe, $\omega = 6.31 \times 10^{12}$; $\psi = 0$ and $d = 5$ mm for all curves. The left hand ordinate scale is to be used for b and c, the lower right hand for a and the upper right hand for d.
changed sign, shows typical behaviour in this region, similar, but not quite the same as the low frequency patterns.

We conclude that, if the anisotropic changes are to be at least several percent, it is necessary, for n-type germanium, to have

\[ \lambda a > 1/4 \quad \text{and} \quad \Omega / \omega > 1/4 \]

In the microwave region the first condition is generally the more restrictive but, at infrared frequencies, it is the second condition which is not satisfied, as fields \( \approx 10^5 \) oersted at least are required.

(iv) Dependence on the Orientation of the Electric Vector

The effect of rotating the electric vector of the incident radiation round the direction of the magnetic field, which remains fixed with respect to the crystal, is discussed from the point of view of symmetry in appendix AII(d). Details of the programming were given in (c). In A(23) the tensor components are expressed in terms of those referred to an initial configuration of the axes. This may be any convenient arrangement for which the components are known. The azimuthal angle \( \psi \), which is measured from the initial direction of the electric vector in the sense \( x \to y \), then specifies the direction of \( \varepsilon \) in the plane perpendicular to \( H \). For the (100) plane case \( \psi = 0 \) is taken as \( \varepsilon \) along the \( l \) axis of the crystal. This direction is convenient because, being perpendicular to the (100) plane, it is the only direction which lies in the plane perpendicular to \( H \) for all values of \( \phi \). The \( y \) axis lies in the (100) plane and makes an angle \( \phi \) with the \( 2 \) axis. For the same reason \( \psi = 0 \) is chosen to be the \([110]\) direction for the (110) plane case. This is the configuration discussed in appendix AII(b), and represents a rotation of \( 7\pi/4 \) from the usual (110) plane case. Since the rotation has the symmetry \( \Theta(\psi) = \Theta(\pi + \psi) \), however, this is equivalent to a rotation of \( 3\pi/4 \).

The \( \psi \) dependence was evaluated at all the frequencies considered.
for \( H = 1.0 \times 10^4 \) and \( 2.0 \times 10^4 \) oersted. The values of \( \phi \) chosen were 15°, 30° and \( \pi/4 \) in the (100) plane, and 35° (\( \approx \cos^{-1}(2/3)\frac{1}{2} \)), and 70° in the (110) plane. As for the \( \phi \) dependence, the general form of the variation is largely independent of frequency for \( \omega \leq 6.0 \times 10^{11} \). Fig. (6.9) shows typical behaviour at a wavelength of 3cm, with \( H = 2.0 \times 10^4 \) oersted. Curves (a), (b) and (c) correspond to \( \phi = 15°, 30° \) and \( \pi/4 \) respectively in the (100) plane, for a thickness of 5mm. They are all similar in form, but the amount of anisotropy increases fairly uniformly with \( \phi \), becoming a maximum for \( \phi = \pi/4 \). The variation is closely sinusoidal in character, with the maximum and minimum always separated by about \( \pi/2 \). The positions of the turning points, however, depend on \( \omega, H \) and \( \phi \). The maximum occurs at about \( \pi/4 \) and the minimum at about \( 3 \pi/4 \) for \( \omega = 10^9, H = 1.0 \times 10^4 \) and \( \phi = 15° \). As \( H \) and \( \omega \) increase, the turning points move to slightly higher values of \( \psi \) and, as \( \phi \) increases, to slightly lower values. This latter trend can just be seen in Fig. (6.9). The \( \phi \) and \( H \) variation is only a few degrees for the widest ranges covered, but the frequency dependence is more marked and, for \( 6.28 \times 10^{10} \) as in Fig. (6.9), the turning points have moved to about 60 - 65° and 150 - 155°. They move in fact beyond 65° and 155° at about \( 2 \times 10^{11} \) with \( H = 1.0 \times 10^4 \). This trend then reverses quite rapidly and, just prior to the zero in the anisotropy, the turning points are close to 0° and \( \pi/2 \). As the field increases and the anisotropy zero moves to higher frequencies, the reversal in the movement of the turning points is correspondingly delayed to a higher frequency.

\( \alpha \) and \( \beta \) are of course not dependent on \( \psi \), so anisotropy arises solely from the complex formula for \( \Theta \). Table 6(c) in which \( \rho_\Theta(\psi) \), defined as the difference between the maximum and minimum values of \( \Theta \) divided by the maximum value and expressed as a percentage, is tabulated for the [110] direction, shows the extent of the anisotropy. Like \( \rho_\Theta \), the values are approximately linear with field up to \( 1.5 \times 10^4 \) oersted, and almost independent of frequency until quite close to the anisotropy zero, when they decrease sharply. They increase again beyond this region, and eventually the pattern is inverted, i.e. the phase
Fig. (6.9). Variation of Faraday rotation with orientation of $\xi^0$ for various orientations of $H$ in (100) plane for $n$-type germanium specimen: a, $\phi = 15^\circ$, $d = 5$ mm; b, $\phi = 30^\circ$, $d = 5$ mm; c, $\phi = \pi/4$, $d = 5$ mm; d, $\phi = \pi/4$, $d = 1$ mm (times five): $\omega = 6.28 \times 10^{10}$ and $H = 2.0 \times 10^4$ Oe for all curves.
is altered by $\pi/2$. This behaviour persists until, at high frequencies, the anisotropy becomes too small to be significant. At some point the rotation will of course change sign as $\psi$ varies, but with the ordinate chosen appropriately, the behaviour of $\mathcal{H}$ is unaffected.

The dotted curve (d) is for 1mm, multiplied by five to make the comparison with (c) straightforward. Clearly the turning point positions are also a function of the thickness. At low frequencies such as $10^7$, when $\mathcal{A}d$ and $\mathcal{B}d$ are small, the rotation as a function of $\psi$ is almost exactly proportional to the thickness but, as $\omega$ increases, this breaks down. The effect is mainly to change the phase, so the dependence of the turning points on $\omega$, $H$ and $\phi$ is probably almost entirely a consequence of the increasing values of $\mathcal{A}d$ and $\mathcal{B}d$ at constant thickness. For 1mm the turning points hardly move beyond $\pi/4$ and $3\pi/4$, as illustrated by curve (d). In support of this conclusion, $\mathcal{A}d$ and $\mathcal{B}d$ always increase in the parameter ranges considered as $\phi$ is decreased and $\omega$ and $H$ are increased, except at frequencies approaching the anisotropy zero, where the turning point movement reverses as would be expected since $\mathcal{B}$ decreases. Close to the anisotropy zero the curves are considerably out of phase for the two thicknesses. However, this region is small, and the overall behaviour changes so rapidly with $\omega$ that consistency cannot be expected.

Fig. (6.10) shows the $\psi$ dependence in the (110) plane, again at 3cm and $2.0 \times 10^4$ oersted. The $\psi$ values chosen are $35^\circ$ and $70^\circ$. The former represents approximately the maximum anisotropy, and the latter illustrates the phase change in the pattern as $\phi$ passes through the [111] direction. The patterns are again sinusoidal, and the positions of the turning points vary in exactly the way described for the (100) plane. Again the effect may be ascribed to the magnitude of $\mathcal{B}d$ and $\mathcal{A}d$, as it is much less marked for 1mm than for 5mm. This is shown by curves (a) and (c). The percentage anisotropy also behaves in a similar manner to that shown in Table 6(c), though the values are always smaller as the effect is a maximum for the [110] direction, which in this respect is the most anisotropic direction for a [111]
Fig. (6.10). Variation of Faraday rotation with orientation of $\psi$ for various orientations of $H$ in (110) plane for n-type germanium specimen: a, $\phi = \cos^{-1} (2/3)^{1/2}$, $d = 5$ mm; b, $\phi = 70^\circ$, $d = 5$ mm; c, $\phi = \cos^{-1} (2/3)^{1/2}$, $d = 1$ mm (times five); $\omega = 6.28 \times 10^{10}$ and $H = 2.0 \times 10^{4}$ Oe for all curves.
ellipsoid band structure. For \( \phi < \cos^{-1}(1/3)^{\frac{1}{3}} \), curve (a) is typical. The anisotropy is a maximum for the direction shown, and decreases as \( \phi \) approaches the [001] and [111] directions. For \( \phi > \cos^{-1}(1/3)^{\frac{1}{3}} \), the anisotropy increases again to a maximum for \( \phi = \pi/2 \), the [110] direction, but is almost \( \pi/2 \) out of phase with (a), as curve (b) demonstrates. This would be expected as, when \( \phi = \pi/2 \), the value of \( \Psi \) which corresponds to \( \Psi = 0 \) for the (100) plane is \( \pi/2 \) in this case.

Anisotropic behaviour resulting from changes in the orientation of the electric vector is thus as great as that arising from variations in the orientation of \( \mathbf{H} \), with the practical advantage that experimentally it is much easier to produce the former effect. For [110] type directions in fact, the anisotropy is large, even at \( 1.0 \times 10^4 \) oersted, and exceeds the \( \rho_{\text{H}} \) values for both the (100) and (110) planes by a significant amount.

6(f) Ellipticity - Results

(i) Frequency dependence

Fig. (6.11) shows the frequency dependence of the ellipticity for \( \mathbf{H} \) along the [001] direction, and equal to \( 1.0 \times 10^4 \), \( 2.0 \times 10^4 \) and \( 3.0 \times 10^4 \) for (a), (b) and (c) respectively. \( \Delta \) is plotted in preference to \( Jd/2 \), since the values are always < 1, and the flattening of the peaks is not therefore very pronounced. Also, for non-isotropic directions, the effect of the formula for \( \Delta \) is important in connection with the anisotropy. The overall behaviour is reminiscent of that illustrated in Fig. (3.5) for the isotropic case. The negative maximum in the rotation occurs at the same frequency as the ellipticity zero, and the rotation zero is close to the positive peak in the ellipticity, as noted in connection with Figs. (5.1) and (5.5). At very low fields the rotation and ellipticity tend to the same value in the low frequency limit, given by equation (6.29). Agreement between this equation and the exactly calculated values is rather poorer than for \( |\mathbf{H}| \), as \( |\Delta| \) begins to decrease sooner than \( |\mathbf{H}| \), so that (6.29) soon overestimates the ellipticity. At \( 10^3 \), with \( \mathbf{H} = 5.0 \times 10^3 \), the approximate value is about 15% too high,
Fig. (6.11). Variation of Faraday ellipticity with frequency for n-type germanium specimen:

- a, $H = 1.0 \times 10^4 \text{ Oe}$
- b, $H = 2.0 \times 10^4 \text{ Oe}$
- c, $H = 3.0 \times 10^4 \text{ Oe}$
- d, $H = 0$ along [001] and $d = 5 \text{ mm}$

for all these curves, $d$, variation of $2d/2$ with other details as for a.
and the $\omega^{\frac{1}{3}}$ approximation can only really be used for $\omega \leq 5 \times 10^9$ and $H \leq 5.0 \times 10^3$.

For (a) the field is too high for (6.29) to be accurate. The zero and positive peak occur at slightly higher frequencies than for the isotropic case because of the shorter relaxation time. The peak is once again slightly below the natural frequency $a^{-1} = 2.3 \times 10^{12}$. The ellipticity then decreases rapidly, and the high frequency approximation (6.39) is only 1% too high at $3 \times 10^{13}$. It becomes valid at lower frequencies than the corresponding rotation approximation, but its practical usefulness is limited by the very small values of $\Delta$ in this region.

For curve (b), with the field doubled, the ellipticity has only increased by about 50%, indicating the approach of the maximum in the field dependence. In the high field limit there is a region of near zero ellipticity around the sign change, which was analysed for the isotropic case in 5f(iv). The onset of this effect is just visible as a slight 'kink' in curve (b). Also, the positive peak moves from $a^{-1}$ towards the higher frequency $\Lambda$ as the influence of the cyclotron resonance becomes stronger. Curve (c) is similar, but the ellipticity has decreased with increasing field in some regions, and the flattening close to the zero is more noticeable. The high frequency approximation cannot be used until a slightly higher frequency than in the low field case, because of the movement of the positive peak. The dotted curve, which represents $\Delta d/2$ at $3.0 \times 10^4$ oersted, shows the relatively small degree of peak flattening produced by the $\tanh$ function in the present case.

If the ratio of the ellipticities for the isotropic and anisotropic cases is evaluated for the same values of $\Delta a$, as for the rotation, the result is nearly independent of frequency for $\Delta a \leq 1$ well below the sign change. Between the sign change and the region of the high frequency approximation, i.e. around the positive peak, the ratio remains almost constant if evaluated at corresponding points on the curves to allow for the different values of $\Delta$ affecting the peak positions. In the high frequency limit the ratio is easily derived from (5.38) and (6.39), and it depends on $a$ as well as the effective masses. Thus, if $a$ is the same in both cases, the isotropic
theory using the effective mass defined in (6.5) for the high frequency region, and a single suitable isotropic mass at all other frequencies, will give an excellent approximation to the results using the correct band structure. This means that the above conclusion is true for the Faraday effect in general when the magnetic field lies along an isotropic direction, though it will not be such a good approximation for other directions.

High field formulae similar to (5.59) and (5.63) can be derived for the regions well below and above the magnetoplasma edge respectively. In the former case the reasoning of 5f(iv), which showed that \( \alpha \approx \beta \gg \alpha \), is still valid, so that \( \bar{x} d/2 \approx H \). The ellipticity thus follows directly from (6.46) in this limit. Well above the edge we have, from (6.43),

\[
\alpha_{\pm} \beta_{\pm} \to \frac{4 \omega_p \omega_e}{3 \pi^2 a_0 \lambda^2} \left[ 1 \pm \frac{6 m_0 \omega}{(2K+1)eH} \right] \tag{6.51}
\]

and

\[
\beta_{\pm} \to \frac{\omega e^{3/2}}{\omega_p} \left[ 1 - \frac{\omega_p^2}{2 \lambda^2} \pm \frac{\omega_p^2 \{3K(K+2)\}^{1/2}}{2 \omega \lambda (2K+1)} \right] \tag{6.52}
\]

These equations lead to

\[
\bar{x} \to \frac{16 \pi \omega \alpha \beta}{\omega_p^2 a^2 \varepsilon \lambda^3 (2K+1)} \left[ 1 - \frac{\omega_p^2}{4 \omega} \right] \tag{6.53}
\]

The condition for zero ellipticity, \( \omega^2 = \omega_p^2/4 \), is thus the same as in the isotropic case, (cf. equation (5.63)), but the multiplying function depends on the band structure. The variation with \( a, \omega \) and \( H \) is of course unaltered. Numerically (6.53) is a poor approximation to the exactly calculated ellipticity for \( H = 3.0 \times 10^4 \) oersted. This was found in the corresponding situation for \( \hat{\sigma} \), the cause being the relatively small value of \( (\lambda a) \). However, the zero is nearly correct. Equation (6.53) predicts \( 2.3 \times 10^{11} \), and the value taken from Fig. (6.11) is \( 2.4 \times 10^{11} \). In fact Figs. (5.5) and (6.11) show that the zero changes very little with field in either theory, even at
low fields.

In Fig. (6.12) the ellipticity is plotted for $H = 2.0 \times 10^4$ oersted, with the field along the same directions as for the corresponding curves of Fig. (6.2). Clearly $|\Delta|$, like $|\Theta|$, is a maximum in general for $H$ along the [001] axis, and the ordering of the curves, except very close to the zero, is the same as for the rotation. This suggests that the ellipticity will generally behave somewhat similarly to the rotation as a function of $\phi$. There is no one frequency at which the anisotropy goes to zero, as the various curves intersect over a small range of frequencies. As for the rotation, however, this region is just below the zero.

In the low frequency approximation $|\Theta|$ and $\Delta$ are equal and the formula is isotropic, so the curves would superimpose at fields low enough for (6.29) to be valid. Similarly the ellipticity is isotropic in the high frequency approximation, so all the curves merge in the region where (6.39) may be used. Between these two limits the anisotropy is rather less for the negative maximum, and considerably greater for the positive peak, than that shown by the rotation. The sign change frequency also varies slightly with the orientation of $H$.

At very high fields the two approximations are easily obtained. Since $\alpha - \alpha \beta$ below the magnetoplasma edge, the ellipticity is derived directly from (6.46), and is therefore isotropic in the high field limit. Above the edge $\mathcal{J}$ can be deduced from (6.48) for the (100) plane case. It turns out to contain two terms, one of which is similar to the isotropic formula except that both the cyclotron frequencies are involved, and the other which contains the multiplying factor $(K-1)\sin 2\phi$ and thus vanishes for $K=1$ and $\phi=0$. The approximation is

$$
\mathcal{J} \to \frac{32 \pi \frac{1}{3} \text{New}}{3 \alpha \hbar \epsilon} \left[ \left( \frac{1}{\mathcal{J}_1^2} + \frac{1}{\mathcal{J}_2^2} \right) \left( 1 - \frac{\omega_p^2}{4 \omega^2} \right) \right] - (6.54)
$$

$$
\frac{3(K-1)\sin 2\phi}{8(2K+1)} \left( \frac{1}{\mathcal{J}_1^2} - \frac{1}{\mathcal{J}_2^2} \right) \left( \frac{(2K+1)(K-1)^2 \sin^2 2\phi}{9K[(K+2)^2 - (K-1)^2 \sin^2 2\phi]} + \frac{\omega_p^2}{\omega^2} \right)
$$
Fig. (6.12). Variation of Faraday ellipticity with frequency for n-type germanium specimen:
a, $H$ along [001]; b, $H$ along [011] in (100) plane with $\Psi = 0$; c, $H$ along [111]; d, $H$
along [011] in (100) plane with $\Psi = \pi/4$; $H = 2.0 \times 10^4$ Oe and $d = 5$ mm for all curves.
The frequency at which the zero occurs varies with the orientation of \( H \), but it is still independent of the magnitude of the field and relaxation parameter in this limit. It should be noted, however, that the second term vanishes when \( H \) is along the [011] direction, and is in any case always small, so the zero will not be much displaced as the orientation of \( H \) is varied. The greatest effect of the anisotropy arises from the change in the multiplying factor \( \left( \frac{1}{\beta_1^2} + \frac{1}{\beta_2^2} \right) \) as \( \phi \) varies. Therefore the ellipticity, unlike the rotation, is not isotropic in the high field limit above the magnetoplasma edge. An equation similar to (6.54) may be derived for the (1\( \overline{1} \)0) plane.

(ii) Field Dependence

The modulus of the ellipticity is plotted as a function of the field at four frequencies in Fig. (6.13). The magnetic field is along the [001] direction and logarithmic scales are used for both axes. At frequencies well below the sign change, as for curve (a), the gradient tends to one as \( \Delta \) is proportional to \( H \). At 3.0x10^4 the ellipticity is still increasing so the maximum must occur, as for the rotation, at \( \lambda a \approx 2 \). At higher fields \( \Delta \) will decrease at first as \( H^{-2} \) as long as 10^{10} lies below the magnetoplasma edge, and then finally as \( H^{-3} \), governed by equation (6.53). This pattern persisted up to frequencies close to a \(-1\) for the rotation, but it begins to vary only a little above 10^{10} in the present case as (b), for which \( \omega = 6.278\times10^{10} \), shows. The linearity with \( H \) breaks down at lower fields, below the lowest value considered which is 5.0x10^3, while the maximum in the field dependence also moves gradually to a lower field. For (b) it occurs at about 1.5x10^4, and it moves to even lower fields closer to the zero. The high field \( H^{-3} \) limit, to which (b) must tend because 6.278x10^{10} is above the magnetoplasma edge by the time \( (\lambda a)^2 \gg 1 \) is satisfied, will be reached at correspondingly lower fields, though 3.0x10^4 is still too low. For curve (c) \( \omega = 4.467\times10^{11} \), which is a little above the sign change. However, the pattern is similar to (b), except that the movement of the point with frequency has reversed in the region of the zero, and it is now returning to higher fields.
Fig. (6.13). Variation of Faraday ellipticity with field strength for n-type germanium specimen at various frequencies: a, $\omega = 10^{10}$; b, $\omega = 6.28 \times 10^{10}$; c, $\omega = 4.47 \times 10^{11}$; d, $\omega = 2.24 \times 10^{12}$; $H$ along [001] and $d = 5$ mm for all curves.
as $\omega$ increases. This trend is continued for (d), by which time the
turning point is back at $\Delta a = 1$.

At higher frequencies such that $(\omega a)^2 >> 1$, the ellipticity
will remain linear with $\mathbf{H}$, according to the high frequency approx-
imation (6.39), as long as $\omega^2 >> \mathbf{H}$ and then, as $\mathbf{H}$ is increased,
the cyclotron resonance will show up at $\omega = \mathbf{H}_0$, the $\mathbf{H}^{-3}$ region being
reached only when $\mathbf{H}^2 >> \omega^2$. For non isotropic directions the transition
region $\omega = \mathbf{H}$ is extensive, due to the differing cyclotron frequencies.

It seems that the turning point position varies quite a lot with
field frequency for $\Delta$, more so than for the rotation, so that the condition
$\Delta a = 1$ for the maximum is approximate only over much of the microwave
region for the present specimen. This effect is undoubtedly present
also in the isotropic case, but, owing to the wider range of $\mathbf{H}$ covered,
sufficient data was not available close to the turning point to predict
its position with enough accuracy to detect the shifts, as these would
be small on the scale of Fig. (5.6).

Fig. (6.14) shows the ellipticity as a function of $\mathbf{H}$ at a fixed
frequency for various orientations of $\mathbf{H}$ and $\mathbf{E}_0$, the details being as
for Fig. (6.2). It cannot be regarded as typical of all frequencies,
as the pattern changes considerably over the range. For $\omega < 10^{10}$,
behaviour is similar to Fig. (6.4) for the rotation as regards the
ordering of the curves, relative positions of the turning points
and region of maximum anisotropy. At higher frequencies the turning
points move to lower fields as noted above. They no longer occur to
successively lower fields for (a), (b), (c) and (d) in that order in
every case, and the shape of the peaks alters with frequency, as can
be seen by comparing Figs. (6.4) and (6.14). Also, the usual ordering
is not maintained in some cases such as (b) and (c) of Fig. (6.14).

In general the field dependence curves must be computed between
about $3 \times 10^{10}$ and $10^{12}$ to find the variation, since no standard pattern
exists as for the rotation. This lack of systematic behaviour by the
ellipticity in this region seems to arise from the onset of the high
field pattern, namely the 'kinks' and generally small and decreasing
values of $\Delta$ around the zero, at fields such that $\Delta a < 1$. It covers
the frequency range over which the negative rotation maximum is flat
Fig. (6.14). Variation of Faraday ellipticity with field strength for n-type germanium specimen with various orientations of field: a, $\mathbf{H}$ along [001]; b, $\mathbf{H}$ along [011] in (100) plane with $\Psi = 0$; c, $\mathbf{H}$ along [111]; d, $\mathbf{H}$ along [011] in (100) plane with $\Psi = \pi/4$: $\omega = 4.47 \times 10^{11}$ and $d = 5 \text{ mm}$ for all curves.
but, unlike (ii), which does not actually start to decrease until \( \varphi a > 1 \) although the high field form is approached at lower fields, the ellipticity does decrease as well as tend to the high field pattern for \( \varphi a < 1 \). Fig. (6.1) shows this clearly.

(iii) Dependence on the Orientation of the Magnetic Field

The form of the \( \varphi \) dependence at various frequencies is much more variable than for the rotation. We consider the (100) plane case first. Only at frequencies below the negative peak, in the present instance \( \leq 4 \times 10^{10} \), is the behaviour similar to that of Fig. (6.5). This is the region in which \( \Theta \) and \( \Delta \) tend to the same values in the high and low field limits. The variation is again given quite closely by \( \sin^2 2 \varphi \), and is illustrated by (a) of Fig. (6.15), which shows the \( \varphi \) dependence at two frequencies for \( 2.0 \times 10^4 \) oersted. Above \( 4 \times 10^4 \) the turning point at \( \pi/4 \) begins to broaden and flatten at the highest fields. It eventually moves to a lower value of \( \varphi \), giving a small minimum at \( \pi/4 \) and, as \( \omega \) increases, this becomes more pronounced until the curve has inverted completely. (c) of Fig. (6.15), for which \( \omega = 1.585 \times 10^{11} \), is a typical example. For higher fields the same transition occurs at a somewhat higher frequency. Beyond this, in the region of \( 2-3 \times 10^{11} \), the ellipticity passes through zero. The inverted patterns are not affected by this, and the behaviour does not alter much until frequencies just beyond the positive peak. Fig. (6.16) shows a set of curves at various fields for a frequency in this region, \( 4.467 \times 10^{11} \). The anisotropy is a maximum for \( H = 1.5 \times 10^4 \) oersted, but the curves are otherwise similar to (c) of Fig. (6.15), except for a tendency to flatten more noticeably in the vicinity of \( \varphi = 0 \). In fact, at the highest field \( 3.0 \times 10^4 \), there is a barely perceptible minimum at \( 0^\circ \) and a maximum at about \( 20^\circ \). This effect occurs in corresponding curves at other frequencies in this region, and because of it \( \sin^2 2 \varphi \) is not a good representation of the variation. Beyond the peak the \( \varphi \) dependence inverts again. This happens first at low fields, since the positive peak is at a lower frequency in this case. There is a small frequency band over
Fig. (6.15). Variation of Faraday ellipticity with orientation of $\mathbf{H}$ in (100) plane for n-type germanium specimen at various frequencies: a, $H = 2.0 \times 10^4$ Oe, $\omega = 10^{10}$, $d = 5$ mm; b, $H = 2.0 \times 10^4$ Oe, $\omega = 10^{10}$, $d = 1$ mm (times five); c, $H = 2.0 \times 10^4$ Oe, $\omega = 1.58 \times 10^{11}$, $d = 5$ mm; $\psi = 0$ for all these curves; d, variation of $\frac{\mathcal{F}d}{2}$ with other details as for c. The left hand ordinate scale is to be used for a and b and the right hand for c and d.
Fig. (6.16). Variation of Faraday ellipticity with orientation of $H$ in (100) plane for $n$-type germanium specimen at a fixed frequency: $a, H = 1.0 \times 10^4 \text{ Oe}; b, H = 1.5 \times 10^4 \text{ Oe};
\hspace{1em} c, H = 2.0 \times 10^4 \text{ Oe}; d, H = 2.5 \times 10^4 \text{ Oe}; e, H = 3.0 \times 10^4 \text{ Oe};$
\hspace{1em} $\omega = 4.47 \times 10^{-11}, \psi = 0$ and $d = 5 \text{ mm}$ for all curves.
The right hand ordinate scale is to be used for $a$ and $b$ and the left hand for $c$, $d$ and $e$. 
which the curves change in a similar way to that described for the  
other inversion region, and then the behaviour returns to the low  
frequency pattern, (a) of Fig. (6.15). Finally the anisotropy tends  
to zero as the high frequency approximation (6.39) becomes valid.

The percentage anisotropy $\rho_A$, defined similarly to equation  
(6.50), is given in table 6(d) for 5mm,(upper value),and 1mm,(lower  
value). At low frequencies $\rho_A$ is less than $\rho_R$ and, for 5mm, it  
immediately begins to decrease appreciably as $\omega$ increases. However,  
the values remain approximately linear in $H$ up to $2.0\times10^4$ oersted  
for $\omega \leq 10^{10}$ before levelling off. They become negative when the  
curves have inverted, the first instance in the present calculations  
occurring at $6.278\times10^{10}$ with $H = 3.0\times10^4$ oersted. In the inversion  
range some of the curves show additional maxima and minima, and have  
both positive and negative values. However, these are always small.  
Because of the definition of $\rho_A$ the values become very large close  
to the zero where $\Delta$ is small, and in fact sometimes changes sign as  
$\phi$ varies. These values are omitted from the table as they are not  
meaningful over this small range. Beyond the sign change $\rho_A$ is  
positive and decreases steadily at the lower fields, before passing  
through zero and becoming negative just after the positive peak.  
It then increases for a while, and finally tends to zero in the high  
frequency $\omega^3$ limit. At higher fields, where the inversion is delayed  
to higher frequencies, the positive values begin to increase again  
in the region of the positive peak before decreasing as the inversion  
region approaches. In this region $\rho_A$ is only linear with $H$ at very  
low fields, $\leq 1.0\times10^4$ oersted, and the values tend to be fairly  
constant over the range $1.5\times10^4 - 2.0\times10^4$, i.e. when $\Delta a$ for the  
isotropic direction is just less than one. For the smaller thickness  
the anisotropy is generally greater, as shown for example by curves  
(a) and (b) of Fig. (6.15). At the lowest frequency, $10^9$, the dif-  
fERENCE is several percent, and this increases with frequency, since  
for 1mm $\rho_A$ does not begin to decrease much until $\omega \approx 3\times10^{10}$. The  
inversion occurs to slightly higher frequencies than for 5mm, so that  
$\rho_A$ is only negative over a very short region. When $\Delta$ is positive
the anisotropy is still greater by a factor as big as 1.5 - 2.0, and
does not tend to zero as rapidly. The flattening around $\phi = 0$ at
$3.0 \times 10^{-4}$ is still observable, but it is less pronounced and the second
turning point is not present.

As for the rotation, the behaviour may be analyzed into anisotropy
in the differential absorption $\mathcal{I}$, and further anisotropy resulting
from the complex formula for $\Delta$. $\rho(\mathcal{I})$ is always greater than the
consequent $\rho_\Delta$, though the overall shape of the $\phi$ dependence is
similar except in the inversion region where the curves are changing
rapidly with $\omega$ and $H$. Curve (d) of Fig. (6.15), which shows $\mathcal{I}d/2$
for the same parameters as (c) for the ellipticity, illustrates this
diversity. The inversion occurs at a slightly lower frequency for
$\mathcal{I}d/2$ than for $\Delta$. However, the field dependence, and in particular
the value of $H$ for which the anisotropy is a maximum at any given
frequency, is generally similar to that for the ellipticity. Beyond
the sign change, which for any given field and $\phi$ value is at a
slightly lower frequency than for the ellipticity, the behaviour is
fairly similar to $\Delta$, though the anisotropy is significantly greater
than for 1mm and about twice the 5mm values. The high field flattening
effect around $\phi = 0$ is still evident, but it is less noticeable than
for $\Delta$ with a thickness of 1mm. The final inversion takes place at
a somewhat lower frequency than for the ellipticity. In general,
therefore, the anisotropy seems to decrease fairly uniformly with
increasing thickness, with $\rho_\Delta$ tending to $\rho(\mathcal{I})$ for very small
thicknesses such that $\tanh \mathcal{I}d/2 \approx \mathcal{I}d/2$, $\sin \beta d \approx \beta d$ etc. As
noted in connection with the rotation, it is possible that the decrease
is a periodic effect and $\rho_\Delta$ may increase again for larger thicknesses.

For the (110) plane the inversion regions are similarly situated,
but the intermediate patterns are more complicated. There are gradual
modifications as $\omega$ and $H$ change over much of the frequency and field
range. These are best explained by following the behaviour at a partic-
ular field as the frequency varies. Then the behaviour at other
fields is similar in the main, but the various modifications may occur
at somewhat different frequencies. Fig. (6.17) illustrates the
Fig. (6.17). Variation of Faraday ellipticity with orientation of $\mathbf{H}$ in (110) plane for n-type germanium specimen at various frequencies and fields: a, $H = 1.0 \times 10^4$ Oe, $\omega = 10^{10}$; b, $H = 3.0 \times 10^4$ Oe, $\omega = 10^{10}$; c, $H = 2.0 \times 10^4$ Oe, $\omega = 3.93 \times 10^{10}$; d, $H = 2.0 \times 10^4$ Oe, $\omega = 6.28 \times 10^{10}$; e, $H = 2.0 \times 10^4$ Oe, $\omega = 1.59 \times 10^{11}$ (times two); f, $H = 2.0 \times 10^4$ Oe, $\omega = 1.59 \times 10^{11}$ (times two); $\psi = 0$ and $d = 5$ mm for all curves.
behaviour below the first inversion region for a thickness of 5mm.
At the lowest frequencies the behaviour is similar to that shown by \( \mathbf{H} \). Curves (a) and (b) are typical of this limit. There is a sharp rise followed by a turning point at \( \phi \approx 40^\circ \), and \( \Delta \) then decreases by an amount depending on the field. At low fields \( \lesssim 5\times10^3 \), the curve drops steeply to below the starting value, giving the small negative \( \rho_\Delta \) values in Table 6(e), but, as \( H \) rises, the rate of descent relative to the \( \phi = 0 \) value decreases. For (a), which has \( H = 1.0\times10^4 \) oersted, \( \Delta (\pi/2) \) lies only slightly above \( \Delta (0) \) while, at the highest fields, the difference is considerable, and an intermediate minimum appears in the vicinity of \( 70 - 80^\circ \) with a small maximum at \( \pi/2 \), (c.f. (d) of Fig. (6.7)). These features are shown in (b). This behaviour pattern persists with relatively little change up to \( \omega \lesssim 4\times10^{10} \). There is, however, a slight movement of the extra minimum when it occurs towards a lower value of \( \phi \), \((60 - 70^\circ)\), and it also appears at large \( \phi \) for smaller fields between \( 10^{10} \) and \( 4\times10^{10} \). At about \( 4\times10^{10} \) the \( 40^\circ \) maximum begins to move perceptibly to a higher value of \( \phi \), and the intermediate minimum continues to move to a lower \( \phi \) value, or appears for the first time close to \( \pi/2 \), as appropriate. Curve (c), with \( \omega = 3.981\times10^{10} \) and \( H = 2.0\times10^4 \), is typical. This trend continues and, at the same time, the maximum concerned flattens out giving a pattern such as (d) at \( \omega = 6.278\times10^{10} \) and \( H = 2.0\times10^4 \). Meanwhile the percentage anisotropy is beginning to decrease. By \( 10^{11} \) the intermediate turning points have disappeared, as for (e), and the curve rises slightly with increasing \( \phi \) over the whole range. The actual inversion region, over which the anisotropy is small and not very systematic in its behaviour, follows. Curve (f), with \( \omega = 1.585\times10^{11} \), lies in this region. \( \rho_\Delta \) is again small and, by \( 2.239\times10^{11} \), the patterns are inverted at all fields and are closely similar to the set of curves for the positive region which are shown in Fig. (6.18). The sign of \( \Delta \) changes as \( \phi \) varies at most fields for this frequency but, as in the case of the rotation, this does not affect the orientation dependence. Curves (c), (d) and (e) are fairly representative of all fields just below the inversion
frequency, except for the absence of the intermediate minimum at low fields, in which case the maximum simply moves to higher $\phi$ values until it vanishes at $\pi/2$. By $10^{11}$ and $1.585 \times 10^{11}$ the behaviour has become similar at all fields.

Fig. (6.18) shows the progressive change in the $\phi$ dependence at the fixed field of $2.0 \times 10^4$ oersted, using two vertical scales as indicated. At first the pattern is typified by (a), with minima at $\sim 35^\circ$ and $\pi/2$ which are approximately equal in value, and maxima at $60^\circ$ and $0^\circ$, the latter being much the higher of the two. As $\omega$ increases the minimum at $\pi/2$ gradually becomes shallower, as shown in curve (b), and eventually almost vanishes for (c) with $\omega = 2.239 \times 10^{12}$, the proportions of the rest of the curve remaining unchanged. The trend continues for (d) as the inversion region approaches, with the main minimum beginning to move to a larger value of $\phi$, around $50^\circ$ in the case illustrated, and the final vanishing of the second minimum, so that there is now a maximum at $\pi/2$. This rises gradually in relation to the value at $\phi = 0$ as $\omega$ increases. Curve (e) is typical of the inversion region. The percentage anisotropy is small and the curve just rises at first, showing a shallow maximum and minimum at $25^\circ$ and $50^\circ$ respectively, and, as $\phi \to \pi/2$, a final relatively steep rise well above the value for $\phi = 0$. (f) and (g) are inverted, the latter being the pattern finally established as the anisotropy tends to zero, and the former a logical transitional stage between (e) and (g). At other fields the sequence of changes is almost identical for $\omega > 2 \times 10^{11}$ but, at lower fields, every change takes place at a correspondingly lower frequency, and vice versa at higher fields. The inversion region is always just above the positive peak which is field dependent. Clearly the frequency region appropriate to (a) is very short at low fields. To give examples, at $5.0 \times 10^3$ and $1.0 \times 10^{12}$ the $\phi$ dependence looks very like (d) of Fig. (6.18), which is for $3.162 \times 10^{12}$, and at $3.0 \times 10^4$ and $2.239 \times 10^{12}$ it is on the other hand more like (b) than (c).

The percentage anisotropy is given in Table 6(f). As before some of the values range between positive and negative limits. As noted for $\Theta$, they are in general greater than the corresponding
Fig. (6.18). Variation of Faraday ellipticity with orientation of $\mathbf{H}$ in (110) plane for n-type germanium specimen at various frequencies: $a$, $\omega = 4.47 \times 10^{11}$; $b$, $\omega = 10^{12}$; $c$, $\omega = 2.24 \times 10^{12}$; $d$, $\omega = 3.16 \times 10^{12}$; $e$, $\omega = 4.47 \times 10^{12}$; $f$, $\omega = 6.31 \times 10^{12}$; $g$, $\omega = 1.59 \times 10^{13}$; $H = 2.0 \times 10^4$ Oe, $\psi = 0$ and $d = 5$ mm for all curves. The right hand ordinate scale is to be used for $a$ and $g$ and the left hand for the remaining curves.
(100) plane values by ~ 50%. The linearity with field for \( H \leq 1.5 \times 10^4 \) is again found at low frequencies. Also, the values remain essentially constant over a much greater range, up to ~ \( 2 \times 10^{10} \), and fall off much less sharply as the inversion approaches than for the (100) plane case. It occurs at a slightly higher frequency in the (1\( \overline{1} \)0) plane, so the region of negative values is very small. Above the sign change the gradual inversion can be seen, as the negative values occur first at \( 5.0 \times 10^3 \) and, for the higher fields, only when \( \omega \) is increased. The ellipticity is like the rotation in that the formula for \( \Delta \) does not contribute much to the anisotropy, which is mainly in \( \mathcal{J} \). At 1mm the \( \rho_\phi \) values are slightly higher than \( \rho(\mathcal{J}) \) in general, though this effect is small. Increasing the thickness to 5mm hardly decreases \( \rho_\phi \) at all below the first inversion and not a great deal above it, though the frequency for the second inversion is slightly thickness dependent. The \( \phi \) dependence for 1mm is not, however, always exactly like that for 5mm, as the positions of the intermediate turning points often vary somewhat between the two thicknesses. The initial modifications preceding the inversion regions are delayed in both cases to noticeably higher frequencies for 1mm than for 5mm, at a given field. The behaviour becomes identical only in the actual inversion regions. In the positive ellipticity region the maximum anisotropy occurs between \( 2.0 \times 10^4 \) and \( 3.0 \times 10^4 \) oersted, over which range \( \rho_\phi \) varies very little. This is in contrast to the (100) plane, where the values are a maximum between \( 1.5 \times 10^4 \) and \( 2.0 \times 10^4 \) and decrease significantly for \( 3.0 \times 10^4 \).

Although the percentage anisotropies for \( \mathcal{J} \) and \( \Delta \) are of the same order, the actual orientation dependence of \( \mathcal{J} \) is generally somewhat different from that of the ellipticity. This effect was also found for the rotation in the (1\( \overline{1} \)0) plane. Excepting only the inversion region, \( \rho(\mathcal{J}) \) is never negative, so that \( \mathcal{J} \) is a maximum for \( H \) along the [100] axis. At low frequencies the curve for \( \mathcal{J} \) rises initially and passes through a turning point at 55° for the lowest field, moving to 40° at the highest field. This behaviour is similar to \( \Delta \). Between the [111] and [110] directions, however, the curves for
always rise considerably above those for \( \Delta \). Except at \( 5.0 \times 10^3 \) there is a small minimum around \( 55 - 60^\circ \), followed by a rise which becomes proportionately greater as \( H \) is increased. The inversion region is small, there being no change in the form of any of the graphs until \( \omega = 10^{11} \) while, by \( 2.239 \times 10^{11} \), the inverted patterns are established. Between these limits the behaviour is changing so rapidly that \( I \) and \( \Delta \) are quite different. In the region of the positive peak \( I \) undergoes a gradual transformation as a function of frequency at fixed field, exactly as \( \Delta \) does. By comparison with Fig. (6.18), after following the same pattern up to \( 55^\circ \), \( I \) falls much more steeply than (a) between \( 55^\circ \) and \( \pi/2 \), finishing up well below the value at the \( 35^\circ \) minimum. The same feature can be seen in the graphs for \( I \) corresponding to (b) and (c) of Fig. (6.18). For \( 3.162 \times 10^{12} \) the maximum at \( 55^\circ \) is no longer present, and the \( 35^\circ \) minimum has moved to \( 55^\circ \), beyond which the curve rises again, but much less than (d). By the frequency corresponding to (e), \( I \) and \( \Delta \) are similar in form and remain so thereafter. The behaviour of \( I \) and \( \Delta \) are linked in this way at all fields, the pattern merely shifting to a different frequency as \( H \) changes.

(iv) Dependence on the Orientation of the Electric Vector

The \( \psi \) dependence was evaluated using the same parameters as for the rotation. The variation is again closely sinusoidal in character under all conditions. Fig. (6.19) shows typical results for \( H \) in the (100) plane at \( 3.981 \times 10^{10} \). (a) and (b) are for \( \phi = 30^\circ \) and \( \pi/4 \) respectively, and show that the maxima and minima for these orientations are closely in phase. This is true at all frequencies, except in the inversion regions and regions where \( I \) is large, where disparities of a few degrees are observed. The larger \( \phi \) value corresponds to the smaller \( \psi \) value for the turning points, as in the case of the rotation. At very low frequencies the separation of the maximum and minimum is exactly \( \pi/2 \), the extreme values being at \( 50^\circ \) and \( 140^\circ \) respectively at all fields. These immediately begin to move to higher \( \psi \) values as \( \omega \) is increased, the movement being slowest.
Fig. (6.19). Variation of Faraday ellipticity with orientation of $\xi_0$ for various orientations of $\mathbf{H}$ in (100) plane for $n$-type germanium specimens: a, $\phi = 30^\circ$, $H = 2.0 \times 10^4$ Oe, $d = 5$ mm; b, $\phi = \pi/4$, $H = 2.0 \times 10^4$ Oe, $d = 5$ mm; c, $\phi = \pi/4$, $H = 2.0 \times 10^4$ Oe, $d = 1$ mm (times five); d, $\phi = \pi/4$, $H = 1.0 \times 10^4$ Oe, $d = 5$ mm; $\omega = 3.98 \times 10^{10}$ for all curves.
Initially, by $3.981 \times 10^{10}$, the frequency of Fig. (6.19), the turning points are at $69^\circ$ and $162^\circ$ for $1.0 \times 10^4$ and $74^\circ$ and $174^\circ$ for $2.0 \times 10^4$. The movement is more rapid for the higher field as long as $|\omega|$ is greater at the higher field. This is consistent with the $\phi$ variation, as $|\omega|$ is greatest when $\phi$ is least. The trend continues until, at $1.585 \times 10^{11}$, the corresponding values have advanced to $97^\circ$ and $11^\circ$ for $1.0 \times 10^4$ and $94^\circ$ and $10^\circ$ for $2.0 \times 10^4$. The separation of the maxima and minima also deviates somewhat from $\pi/2$. In the vicinity of the inversion the movement is rapid, and beyond this a more regular pattern is established, with maxima and minima at approximately $165^\circ$ and $70^\circ$ for $1.0 \times 10^4$ and $175^\circ$ and $73^\circ$ for $2.0 \times 10^4$. The patterns are $\sim \pi/2$ out of phase with the low frequency values, i.e. they have inverted, like the $\phi$ dependence. The turning points begin to move again about $10^{12}$, as the second inversion region approaches. They change quite rapidly over the range $10^{12} - 10^{13}$ and finally, by $10^{13}$, are almost in phase with the low frequency curves, having a maximum at $\sim 50^\circ$ and a minimum at $\sim 140^\circ$. The exact inversion region of course moves to higher frequencies as the field increases. Mainly because the anisotropy of the ellipticity passes through two inversions, with associated phase changes of $\sim \pi/2$, the turning point positions are more variable than for the rotation, which showed only one inversion at about the same frequency as the second ellipticity inversion. This region apart, the maximum shift of the turning points from the low frequency positions was only about $15^\circ$ over most of the frequency range.

The values of $\rho_2(\omega)$ as a function of frequency are given in Table 6(1) for $H = 1.0 \times 10^4$ and $2.0 \times 10^4$ oersted at 5mm and 1mm. At low frequencies they tend to the same limit as the corresponding $\rho_2(\omega)$. They are approximately linear with field up to about $1.5 \times 10^4$ oersted, and do not decrease much until $\omega > 10^{11}$, which is very close to the inversion region. They become large almost immediately close to the ellipticity zero, and beyond it decrease at first but then show either a small increase or, in the case of 1mm, a region of almost stationary values before falling off as the second inversion approaches. Here they go through zero and increase again to higher frequencies,
before tending finally to zero in the high frequency limit. The
anisotropy is not great beyond the inversion, the maximum values cal-
culated being about 15%, but it is significantly larger than for $\hat{H}$
at corresponding fields and frequencies, and persists to a higher fre­
quency. The effect on the percentage anisotropy of changing the thick­
ness is clearly very small. The 5mm and 1mm values only differ by
appreciable amounts in the inversion regions where they are changing
very rapidly. The turning point positions are, however, more dependent
on thickness, as for the rotation. In the low frequency limit they
are the same for 1mm and 5mm to within a few degrees but, as $\omega$
increases, the 1mm turning points are less displaced and, by $4 \times 10^{-10}$ for
example, occur about 20° earlier than for 5mm. This increases to as
much as 40° in the inversion region, but decreases again to 10 - 20°
somewhat before the positive peak, and to zero close to the inversion
frequency. Beyond this the 5mm values move to lower $\Psi$ for a short
period but, by $\sim 10^{-13}$, the separation has reduced almost to zero again.
For 1mm in fact the turning points do not move much except very close
to the inversions, so that the $\pi/2$ phase changes are confined to much
smaller frequency regions than for 5mm. As noted in connection with
$\hat{H}$, this is almost certainly because $\mathcal{J} \cdot \mathbf{d}$ always remains small for 1mm.
The separation between the two thicknesses is least at low and high
frequencies, where $\mathcal{J} \cdot \mathbf{d}$ is still fairly small for 5mm. This agrees with
the trend observed when the $\mathcal{H}$ and $\phi$ variation was considered. Generally
speaking, therefore, if $\mathcal{J} \cdot \mathbf{d} \ll 1$, the turning points are close to $\pi/4$
and $3 \pi/4$ with the separation $\sim \pi/2$, $\pi/4$ being a maximum at low
and high frequencies and a minimum between the two inversions, while
the inversion regions are confined to very narrow frequency ranges.
When $\mathcal{J} \cdot \mathbf{d}$ is not small the turning points move to proportionately
higher $\Psi$ values. The effect is exactly as for the rotation, except
that the shifts begin at lower values of $\mathcal{J} \cdot \mathbf{d}$ than of $\mathbf{p} \cdot \mathbf{d}$ for the rotation.

Fig. (6.20) shows the $\Psi$ dependence in the $(\mathbf{110})$ plane. Curves
(a) and (c), for which $\phi = 35^\circ$, are typical of the behaviour for
$\phi < \cos^{-1}(1/3)^{1/3}$, the $[111]$ direction. The inversion regions occur
at the same frequencies as for the $(100)$ plane. As for the rotation,
the anisotropy is generally a maximum when $\phi \sim 35^\circ$. When $\mathcal{J} \cdot \mathbf{d} \ll 1$
Fig. (6.20). Variation of Faraday ellipticity with orientation of \( \mathbf{E} \) for various orientations of \( \mathbf{H} \) in (110) plane for n-type germanium specimen: a, \( \phi = \cos^{-1}(2/3) \), \( H = 2.0 \times 10^4 \) Oe, \( d = 5 \) mm; b, \( \phi = 70^\circ \), \( H = 2.0 \times 10^4 \) Oe, \( d = 5 \) mm; c, \( \phi = \cos^{-1}(2/3) \), \( H = 1.0 \times 10^4 \), \( d = 5 \) mm; d, \( \phi = 70^\circ \), \( H = 2.0 \times 10^4 \) Oe, \( d = 1 \) mm (times five); \( \omega = 3.98 \times 10^{10} \) for all curves.
the maximum and minimum are at $\pi/4$ and $3\pi/4$ respectively at frequencies below the first inversion and above the second inversion, and vice versa in the intermediate region. When $\xi d$ is larger the turning points move in exactly the way described for the $(100)$ plane case when $\omega$, $H$ and $\phi$ are varied. With $H$ along the [111] direction the anisotropy is of course zero, and beyond this value it increases again to a maximum when $H$ is along the [110] direction with $\phi = \pi/2$. At this value the curves are as illustrated in Fig. (6.19), except that the position for $\psi = 0$ differs by $\pi/2$. The phase in this region is always exactly $\pi/2$ different from the $0 - 55^\circ$ range in the limit of $\xi d \ll 1$, so that (b) and (d) are inverted with respect to (a) and (c) in Fig. (6.20). This is not exactly true for the present parameters, as the turning points are displaced. For (d) the thickness is only $1\text{mm}$, and the maximum and minimum are close to the expected values of $135^\circ$ and $45^\circ$ respectively. The percentage anisotropy behaves similarly to Table 6(f) at both $35^\circ$ and $70^\circ$, but is always less than for the [110] direction.

6(g) Individual Modes - Results

The individual rotations $\Theta_+$ and ellipticities $\delta_+$ were also calculated. Because of the relations between $\Theta_+$ and $\Theta_-$ and $\delta_+$ and $\delta_-$ for the two symmetries $A = B$ and $B = -C$, (see 2(f)), it is only necessary to consider 1 mode. The results presented here are for the positive wave. For the $(100)$ plane case $P$ and $Q$ were also tabulated. For $(\omega a)^2 \ll 1$, i.e. $\leq 10^{-12}$ in the present case, $P$ is positive and independent of $\omega$ while $Q$, which is negative and much smaller, is approximately proportional to $\omega$. It is straightforward to show that, provided $Q^2$ and $P^2 \ll 1$, in this limit

$$\tan 2\Theta_+ \rightarrow \frac{P}{Q}$$

and

$$\delta_+ \rightarrow \frac{1}{(1 + P/2)}$$

$\tan 2\Theta_+$ is very large and, when defined as in 2(f), $\Theta_+$ is therefore just greater than $\pi/4$ by an amount which is proportional to $Q$ and
hence to $\omega$. However, the values are so small, only $1.8 \times 10^{-2}$ radians for $\omega = 10^{11}$ and $H = 1.0 \times 10^4$ for example, that $\Theta_+$ are both practically $\pi/4$ over most of this range. By $10^{12}$ and $1.0 \times 10^4$ $\Theta_+$ is $1.9 \times 10^{-1}$, so that it is still closely proportional to $\omega$, but no longer negligibly small. Since $P$ increases with $H$ approximately like $H$, being a maximum for $\omega$ just $> 1$, and $|Q|$ decreases with increasing field for $H \geq 1.0 \times 10^4$ oersted, $\Theta_+$ measured with respect to $\pi/4$ actually decreases with increasing field over the range considered, which is $5.0 \times 10^3$ to $3.0 \times 10^4$. At very low fields of course ($\Theta_+ - \pi/4$) must tend to zero. In the region $\omega a > 1$ $P$ and $Q$ become of the same order of magnitude, but $P$ begins decreasing at about $10^{12}$ at low fields, while $|Q|$ is still increasing, though it is no longer proportional to $\omega$. $|Q|$ does not start to decrease until about $3 \times 10^{12}$, and always remains numerically greater than $P$ while both are tending to zero at high frequencies. It is easily shown that the rotation is in fact still given by (6.55) whatever the relative values of $P$ and $Q$, provided $P^2$ and $Q^2$ are $<< 1$. Thus, above $10^{12}$, $\Theta_+$ continues to increase since $|\tan 2 \Theta_+|$ decreases, but the rate of increase is less rapid. Finally it tends to $\pi/2$ at very high frequencies, as $P/Q$ approaches zero from the negative side. At the highest frequency considered, $3.981 \times 10^{13}$, $\Theta_+$ is $1.527$ radians or $87.5^\circ$, so that the axes of the two modes are almost perpendicular. For high fields such that $\omega a \geq 1$, the transitional region where $P \sim Q$ is delayed to the frequency $\omega = \omega_0$.

The ellipticity on the other hand is almost independent of frequency for $(\omega a)^2 << 1$. With $H = 1.0 \times 10^4$ and $\omega = 10^{11}$, (6.56) gives $0.929$ in comparison with the exact value of $0.927$. At $10^{12}$ the discrepancy is only $1 - 2\%$, and if (6.56) is modified to allow for $P \sim Q$, we find

$$\Theta_+ \Rightarrow \frac{1}{1 + \frac{1}{2} \left[ P^2 + Q^2 \right]^{1/2}} \quad (6.57)$$

This equation represents the ellipticity very accurately. At high frequencies $\Theta_+$ tends first to $(1 + Q/2)^{-1}$, and ultimately to $1$ as $Q$ approaches zero. The modes thus become circularly polarized and $\Theta_+$ loses its significance.
In the (1\bar{1}0) plane similar results are found to hold, but the approximate expressions for $\Theta_+$ and $\delta_+$ are obtained in terms of $R$ and $T$. We find, since $|1 - |R|| >> |T|$ for $(\omega a)^2 << 1$,

$$\Theta_+ \rightarrow T/(|R| - 1)$$

and

$$\delta_+ \rightarrow \begin{cases} |R|^{1/2} & \text{for } 1 > |R| \\ |R|^{1/2} & \text{for } 1 < |R| \end{cases}$$

$|R|$ is independent of frequency and $T$ is proportional to frequency. Between the [001] and [111] directions $T$ is positive with a maximum at about 35°, and $|R|$ is > 1. On the other hand, between the [111] and [1\bar{1}0] directions $T$ is negative and continuously decreasing, and $|R|$ is < 1. However, both combinations lead to the conclusion that $\Theta_+$ is > 0, but small and proportional to $\omega$, for $(\omega a)^2 << 1$. At $1.0 \times 10^4$ and $10^{11}$ for example $\Theta_+$ is $3.43 \times 10^{-2}$ radians at $\phi = 35^\circ$, rather greater than the corresponding maximum rotation $(\Theta_+ = \pi/4)$ in the (100) plane. Since $|T|$ is a maximum when $H = 1.0 \times 10^4$, while $|T| = 1$ increases with $H$ for all the fields considered, $\Theta_+$ generally decreases as $H$ increases, as in the (100) plane. As long as $T^2 << |R| - 1$, which is always true for the present calculation, the rotation is given by (6.58) at all frequencies. Beyond $10^{12}$ $|R| - 1$ begins to decrease, but $|T|$ increases less rapidly, so that $\Theta_+$ in fact also increases less rapidly. At about $2 \times 10^{12}$, $(\omega a - 1)$, $|T|$ starts to decrease but always remains greater than $(|R| - 1)$, so that $\Theta_+ \rightarrow \pi/4$ in the high frequency limit. At $3.93 \times 10^{13}$ and $1.0 \times 10^4$ the maximum value of $\Theta_+$ is about 42°9'. As previously, transitional behaviour dependent on the value of $\omega a$ at low fields is a function of $\omega / R$ at high fields when $(R a)^2 >> 1$. The ellipticity, as calculated from (6.59), is less than unity for all $\phi$, and is almost independent of frequency for $(\omega a)^2 << 1$. It increases as $H$ increases and may be quite large, e.g. 0.519 for $\omega = 10^{11}$, $H = 3.0 \times 10^4$ and $\phi = \pi/2$. This is closely equal to the corresponding maximum value in the (100) plane, which suggests that $\delta_+$ are not very dependent on $\psi$ in the low frequency region. At high frequencies $\delta_+$ tend to zero, as in the (100) plane.
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Table 6(b) \( \rho_R \) - \( \bar{H} \) in \((1\bar{1}0)\) plane

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CHAPTER 7

The Faraday Effect in N Germanium (Series Solution)

7(a) General Formulation

The components of the magnetoconductivity tensor were derived in the form of a series expansion in powers of the magnetic field in chapter four. The field independent term $\Sigma^{(2)}$, and the coefficients of $H$ and $H^2$, $\Sigma^{(3)}$ and $\Sigma_{jklm}$ respectively, are given in chapter four and appendix G for the (100) and (110) plane cases. The components $\delta_{xx}$ etc. are expressed in terms of the $\Sigma$ tensors in equation (4.44), and thus $\delta$, $\delta$, $C$, and $D$ and $\mu_+^2$ may be expressed as functions of the $\Sigma$ components. Using the University of London Mercury Computer the rotation and ellipticity were evaluated for the (100) and (110) plane cases. The lattice scattering specimen parameters given in (6.32) and (6.33) were used, thus making possible a direct comparison between the results for the closed and series solutions.

The integrals needed for the tensor components, which are given in (4.20), (4.21) and (4.22), are functions of $\omega t$ only. Since only a small number were required, the relatively slow library programme for infinite interval integrations was used in preference to developing special methods for the six complicated integrands of (4.20) to (4.22). After evaluating $\mu_+^2$ the remainder of the programme was essentially as described for the high field case, except that dependence on the rotation of the plane of polarization of the electric vector was not considered. Detailed results were obtained for fields $\leq 1.0 \times 10^4$ oersted over the frequency range $10^9 - 10^{11}$. This work in fact preceded the high field calculation described in chapter six, but is reported here only briefly as, for reasons to be discussed in section (c), the method does not give a satisfactory representation of the anisotropy.
7(b) Results

All the curves plotted in Figs. (7.1) to (7.7) are chosen to illustrate the divergence between the results for the closed and series solutions. The two methods are subsequently referred to as the high field and weak field cases respectively. Fig. (7.1) shows that the overall form of the frequency dependence of the rotation is initially similar in the two cases. At $5.0 \times 10^3$ the weak field curve is slightly displaced, $|\Theta|$ being greater than the exact value. The effect is $\sim 10\%$ at low frequencies. This tends to zero somewhat before the sign change, and the two curves then change positions, but the discrepancy is small in this region and tends to zero in the high frequency limit. At $1.0 \times 10^4$ the weak field pattern is the same but the numerical discrepancy is considerably greater, $\sim 40\%$ at low frequencies. Also, while the sign change frequency remains essentially unchanged for the weak field case, it has moved to a higher frequency in the high field case. The contrasts increase as $H$ is further increased, because the weak field rotation simply increases in proportion to the change in $H$ to within a few percent.

The major features such as turning points and sign changes are hardly displaced as $\Omega$ approaches and exceeds unity, whereas the high field curves modify considerably, both in magnitude and shape, under the same conditions. Thus, at fields much greater than $1.0 \times 10^4$, the high and weak field curves become completely different in many respects. Curve (e) shows that, at $1.0 \times 10^4$, the anisotropy for the weak field case, represented by the difference between (c) and (e) which are the limiting values in the (110) plane, is much less than the difference between the corresponding weak and high field curves (c) and (d).

The pattern for the ellipticity, which is shown in Fig. (7.2), is fairly similar. The rotation and ellipticity still tend to the same values at low frequencies as would be expected, since (6.29) is a low field approximation. Differences between the high and weak field curves are again about $10\%$ and $40\%$ at $5.0 \times 10^3$ and $1.0 \times 10^4$ respectively for low frequencies. They tend to zero just before the
Fig. (7.1). Variation of Faraday rotation with frequency for n-type germanium specimen with $a = 4.3 \times 10^{-13}$ sec, $N = 1.3 \times 10^{14}$ cm$^{-3}$, $m_3 = 1.6m_e$, $k = 20$, $\epsilon = 16$: $a$, $H = 5.0 \times 10^3$ Oe and along [001]; $c$, $H = 1.0 \times 10^4$ Oe and along [001]; $e$, $H = 1.0 \times 10^4$ Oe and along [110] in (110) plane with $\psi = \pi/4$; all for weak field case: $b$ and $d$, $H$ as for $a$ and $c$ respectively for high field case: $d = 5$ mm throughout.
Fig. (7.2). Variation of Faraday ellipticity with frequency for n-type germanium specimen: a, $H = 5.0 \times 10^3$ Oe; c, $H = 1.0 \times 10^4$ Oe; both for weak field case; b and d, as for a and c respectively for high field case; $H$ along $[001]$ and $d = 5$ mm for all curves.
ellipticity changes sign. Thus $|\Delta|$ is less for the high field case than for the weak field case, except in the small region between the inversion and the ellipticity zero. Beyond the zero and through the positive peak region the percentage separation is about the same as at low frequencies until just beyond the peak, where it tends to zero rather suddenly, so that the two curves again change positions. The separation thereafter is, however, very small, even at $1.0 \times 10^4$. It is noteworthy that, for both $\mathbf{H}$ and $\Delta$, the inversions between the high and weak field curves occur at similar frequencies to those in the anisotropic behaviour for the high field case, (provided $H$ is small if the inversion itself is field dependent).

It seems that $(\mathbf{a})^2$ must be $\leq 1/4$ for the general pattern of the frequency dependence of $\mathbf{H}$ and $\Delta$ to be correct, but for the overall magnitude to be correct to within $10^\circ$, it is necessary to have $(\mathbf{a})^2 \leq 1/16$. This is valid for $(\omega a)^2 \ll 1$. As the frequency increases and $\omega^2$ approaches $\mathbf{a}^2$ and $(\omega a)^2$ approaches one, (at $\omega \approx 10^{12}$ for $H = 1.0 \times 10^4$), the curves tend together while, when $\omega^2 \gg \mathbf{a}^2$ and $(\omega a)^2 \gg 1$, the series expansion gives the same value as the closed solution to a good approximation, whatever the value of $(\mathbf{a})^2$. Thus, in the high frequency region, the series solution is adequate except at extremely high fields, whereas in the microwave region its range of validity is very limited.

The way in which the separation of the curves increases as the field increases is illustrated for a frequency of $10^{11}$ in Fig. (7.3). This curve may be regarded as typical, barring inversion regions, for both $\mathbf{H}$ and $\Delta$, at frequencies such that $(\omega a)^2 \ll 1$. The weak field curves are closely proportional to $H$. This continues to even higher fields as a few calculations for fields up to $2.0 \times 10^4$ oersted showed. In the case of the ellipticity the proportionality remains for $\mathbf{d}$, though $\Delta$ will tend to one when $\mathbf{d}$ is no longer small. The high field curves begin to deviate slightly from linearity at about $2.0 \times 10^3$, though not markedly below $\approx 5.0 \times 10^3$. It may be noted again that typical anisotropic effects, illustrated for the high field case by the group (b), (d) and (f) and at weak fields by the group (a), (c).
Fig. (7.3). Variation of Faraday rotation with field strength for n-type germanium specimen: a, $\phi = 0$; c, $\phi = \pi/4$ in (100) plane and $\psi = 0$; e, $\phi = \pi/2$ in (110) plane and $\psi = \pi/4$; all for weak field case: b, d and f, $\phi$ and $\psi$ as for a, c and e respectively for high field case: $\omega = 10^{-11}$ and $d = 5$ mm for all curves.
and (e), are much less than the weak field - high field separation for corresponding parameters at all fields such that the anisotropy is great enough to be significant, i.e., for $H \gg 5.0 \times 10^3$ oersted. Also, the ordering within the groups is not the same, e.g., curve (c) lies above (a), but the corresponding high field pair, (b) and (d), are interchanged. These two features suggest that the anisotropy calculated for the weak field case is unlikely to be satisfactory.

The last group of Figs. (7.4) to (7.7), confirm this very clearly. They show the rotation and ellipticity as a function of $\phi$ with $H = 1.0 \times 10^4$ oersted in the (100) and (110) planes, $\omega$ being $10^{11}$ for $\Theta$ and $4.467 \times 10^{11}$ for $\Delta$. The vertical scale is correct for the weak field curves, but the high field results, plotted to show the contrast in behaviour, have been displaced because of the 40% discrepancy in the overall magnitude. In the (100) plane the rotation shows a variation similar to $-\sin^2 \phi$. This is true for all fields, and for frequencies below the inversion region at about $8 \times 10^{11}$. Beyond this the pattern is inverted to $\sin^2 \phi$, and it remains so until about $5 \times 10^{12}$ where it inverts back again to the original low frequency form. However, the anisotropy is very small in this region. This variation is the mirror image of that found for the high field case, illustrated by (b) of Fig. (7.4) and Fig. (6.5). In the (110) plane, the weak field curve falls initially, as in the (100) plane, passes through a minimum at $\sim 35^\circ$, and then rises steeply above the [100] direction value. The minimum moves to slightly larger $\phi$ values, $\sim 40^\circ$, for larger values of $\Theta$, but this is probably a thickness effect. Otherwise the pattern is constant at all fields and for frequencies up to about $2 \times 10^{12}$, where it inverts rapidly and remains unaltered thereafter in general form up to the highest frequency considered. The inversion is therefore at a somewhat higher frequency than for the (100) plane case. The rotation is always nearly equal for $H$ along the isotropic [001] and [111] directions, which is not usually true in the high field case. This is due to the very small anisotropy shown by $X$ and $\bar{X}$, discussed later in this section. The behaviour described is again the opposite of that found in the high field case, being a mirror image of the $\phi$ dependence calculated.
Fig. (7.4). Variation of Faraday rotation with orientation of $\mathbf{H}$ in (100) plane for n-type germanium specimen: a, weak field case; b, high field case; both for $d = 5$ mm; c, weak field case for $d = 1$ mm (times five); $\omega = 10^{11}$, $H = 1.0 \times 10^4$ Oe and $\psi = 0$ for all curves: (b, ordinate scale shifted).
Fig. (7.5). Variation of Faraday rotation with orientation of $\mathbf{H}$ in (110) plane for n-type germanium specimen: a, weak field case; b, high field case: $\omega = 10^{11}, \mathbf{H} = 1.0 \times 10^4$ Oe, $\psi = \frac{\pi}{4}$ and $d = 5$ mm for both curves: (b, ordinate scale shifted).
Fig. (7.6): Variation of Faraday ellipticity with orientation of $\mathbf{H}$ in (100) plane for n-type germanium specimen: a, weak field case; b, high field case: $\omega = 4.47 \times 10^{11}$, $H = 1.0 \times 10^4$ Oe, $\Psi = 0$ and $d = 5mm$ for all curves: (b, ordinate scale shifted).
Fig. (7.7). Variation of Faraday ellipticity with orientation of \( \mathbf{H} \) in (1\(\bar{1}\)0) plane for n-type germanium specimen: a, weak field case; b, high field case: \( \omega = 4.47 \times 10^{11}, \mathbf{H} = 1.0 \times 10^4 \) Oe, \( \psi = \frac{\pi}{4} \) and \( d = 5 \) mm for all curves; (b, ordinate scale shifted).
from the closed solution at very small fields, \((< 5 \times 10^3 \text{ oersted})\).

Even in the region \(\omega^2 > \mathcal{R}^2\), where the results for the two methods should be tending together, the form of the anisotropy never becomes the same.

The \(\phi\) dependence of the ellipticity is similar to (a) of Fig. (7.4) at low frequencies. The curves invert around \(10^{-11}\), just before the zero at about \(2 \times 10^{-11}\). Curve (a) of Fig. (7.6) shows the variation at \(4.467 \times 10^{-11}\), and is typical of the intermediate frequency region.

Just beyond the positive peak at \(\omega = 1^{-1}\) the anisotropy decreases sharply to zero, and inverts again back to the form of (a) of Fig. (7.4) though, as for the rotation, the anisotropic effects are very small.

The behaviour is again exactly the opposite of that described for the high field case in all regions, as a comparison of (a) with (b) of Fig. (7.6) or with Fig. (6.16) shows. The situation in the (110) plane, illustrated by Fig. (7.7), is similar, but the inversions are at \(\omega = 2 \times 10^{-10}\) and \(6 \times 10^{-12}\).

The percentage anisotropies \(\rho_R\) and \(\rho_A\) for the weak field case are given in Tables 7(a) and 7(b) respectively for \(5.0 \times 10^3\) and \(1.0 \times 10^4\), with thicknesses of \(5\text{mm}\) and \(1\text{mm}\) (lower value). For the (100) plane the anisotropy is always small, and is generally less than for the corresponding high field calculation. Also, it is more nearly proportional to \(H^2\) than to \(H\) for both \(\theta\) and \(\Delta\). For both methods \(\rho_R\) is approximately independent of \(\omega\) at low frequencies, but it then increases for the weak field case before decreasing and inverting at about \(8 \times 10^{-11}\), the same frequency as found for the closed solution at low fields. In the region \(\omega a < 1\), \(\rho_R\) tends to zero, inverts again, increases slightly though remaining very small, and soon tends very rapidly to zero for the last time when \((\omega a)^2 > 1\). The second inversion was also observed in the high field calculation, though at a higher frequency because no fields lower than \(2.0 \times 10^4\) were considered. For \(\omega < 10^{-10}\) \(\rho_R\) is independent of \(d\), while beyond this value it always decreases when the thickness is decreased from \(5\text{mm}\) to \(1\text{mm}\), (see curve (c) of Fig. (7.4)). Above \(10^{-11}\) this decrease is very marked, with differences as great as a multiplying factor of five in many cases. Thus for \(1\text{mm}\) the anisotropy is very small indeed at all frequencies, even for \(H = 1.0 \times 10^4\), and
is always much less than for the high field case.

In the $(110)$ plane $\rho_\phi$ is much greater, and often exceeds the high field values of Table 6(b). The total anisotropy (sum of both contributions regardless of sign) is, however, approximately linear in $H$ in this case, except close to inversions and sign changes. It is independent of frequency for $\omega \lesssim 10^{11}$, and then decreases gradually, inverts at the same frequency as the high field case, and increases again to values comparable with the low frequency results before tending to zero. There is no second inversion below $10^{13}$, and $\rho_\alpha$ always remains much greater than the corresponding $(100)$ plane value. The magnitude of $\rho_\phi$ is more nearly comparable for the high and weak field calculations than in the $(100)$ plane, though the angular variation is always in the opposite sense. Decreasing $d$ to $1\text{mm}$ has no effect on $\rho_\phi$ at low frequencies where $|\Theta|$ is small. At intermediate frequencies $|\Theta|$ is mainly a little larger for the smaller thickness, as found in the high field case. $\rho_\alpha$ is similar to $\rho_\phi$, except for the differing inversion regions. It is small and approximately proportional to $H^2$ in the $(100)$ plane, and considerably larger and approximately proportional to $H$ in the $(110)$ plane. The effect of varying $d$ is also similar, and the anisotropy undergoes the second inversion at a higher frequency for the $(110)$ plane than for the $(100)$ and tends to zero less rapidly.

The propagation constants were also calculated. In the $(100)$ plane at low frequencies the $\phi$ dependence of $\beta$ is very similar to that of $\Theta$, though the anisotropy is rather less. This decreases with increasing $\omega$ until $\beta$ becomes inverted at about $10^{11}$. This inversion extends over a larger frequency range than those observed in $\Theta$ and $\Delta$. The percentage anisotropy, $\rho(\beta)$, is very small at all frequencies higher than $10^{11}$, the highest values recorded, even for $H = 1.0 \times 10^4$, being $\approx 1\%$. There are further inversions, again rather protracted, in the vicinity of $10^{12}$ and $10^{13}$, and all the time $\rho(\beta)$ is decreasing. For practical purposes it is zero above $10^{11}$. In the $(110)$ plane $\beta$ again shows most anisotropy at low frequencies, and, as previously, $\rho(\beta)$ is again less than $\rho_\phi$. The curves are...
rather different. The minimum occurs when \( \overline{\mathbf{H}} \) is along the \([111]\) direction, and the curve does not rise very much thereafter. Beyond \( 10^{11} \) the variation changes as the anisotropy slowly inverts, and tends to a form with a small maximum and minimum at \( 40^\circ \) and \( 55^\circ \) respectively, superimposed on a curve which rises gradually with increasing \( \phi \). \( \rho(\overline{\beta}) \) is the same as for the (100) plane in this region, as the \([100]\) and \([110]\) values are the extrema in both cases. It is therefore very small. The curves invert again about \( 10^{12} \) and assume a form which is the mirror image of the behaviour just described, before finally inverting again at about \( 10^{13} \). The anisotropy of \( \Delta \) is more complex, though still negligibly small at all save very low frequencies where it behaves like the corresponding \( \overline{\beta} \) in both planes, with \( \rho(\overline{\alpha}) \) slightly less than \( \rho(\overline{\beta}) \). Deviations from this pattern are evident at \( 2 \times 10^{10} \) in the (100) plane and \( 4 \times 10^{10} \) in the (110) plane. The inversion is fairly rapid in this case. Once \( \Delta \) has changed sign, \( \rho(\overline{\alpha}) \) is always less than \( 1\% \). The second inversions occur at about \( 3 \) and \( 6 \times 10^{11} \) respectively in the two planes, and are followed in both cases by a third at about \( 3 \times 10^{12} \), the natural frequency of the system. Clearly the anisotropy of the propagation constants as given by the weak field calculation bears no relation at all to the exactly calculated behaviour obtained from the closed solution, neither in respect of orders of magnitude, form of angular variation, or inversion frequencies. The propagation constants appear to behave less systematically than the rotation and ellipticity in the weak field approximation.

At higher fields the \( \phi \) dependence was found to alter rapidly with field at a given frequency, so that no form of behaviour can be regarded as typical. The anisotropy is, however, so small in general that the anisotropy in \( \mathcal{H} \) and \( \Delta \) essentially arises solely from the complex formulae, the \( \phi \) dependence of \( \overline{\beta} \) and \( \overline{\alpha} \) being negligible by comparison. The behaviour of \( \mathcal{H} \) and \( \Delta \) is more systematic than that of \( \overline{\beta} \) and \( \overline{\alpha} \), but the only features in common between the \( \phi \) dependence as calculated from the high and weak field methods are that the patterns invert at similar frequencies, (provided the low field limit is considered in the former case), and are approximately mirror images of one another. The numerical magnitudes are always wrong, apart from
the incorrect signs for \( \rho \) values, and in particular the field, frequency and thickness dependence of \( \rho_2 \) and \( \rho_3 \) are generally given wrongly.

7(c) Discussion

The results presented in the preceding section suggest that the frequency dependence is represented fairly well by the series expansion to \( O(\mathbf{H}^2) \) provided \( \mathcal{J} \alpha \leq 1/4 \), (\( \mathcal{J} \) being the cyclotron frequency for the [100] direction). This is the region \( \omega a \leq 1 \). When \( \omega a > 1 \) the criterion is less restrictive and will probably require \( \mathcal{J}/\omega \leq 1/4 \), though this has not been fully verified by calculation.

The anisotropy, which may be regarded as a relatively small variation superimposed on the isotropic rotation is, on the other hand, apparently never given correctly, even for fields such that \( \mathcal{J} \alpha < 1/4 \). The lowest field considered corresponds to \( \mathcal{J} \alpha \approx 0.1 \), and still does not give agreement. At such a field the anisotropy is in any case negligibly small. The minimum value giving significant anisotropy was found in section 6(e) to be about \( \mathcal{J} \alpha = 1/4 \) for the high field case when \( (\omega a)^2 \leq 1 \). Thus there is no range of \( H \) over which agreement in this connection might be expected, when the degree of anisotropy is experimentally measurable. The same conclusion is likely to hold in the region where \( \omega a > 1 \), since we require \( \mathcal{J}/\omega \geq 1/4 \) for significant anisotropy. In fact there is never agreement, as noted above. As the highest field considered in the present calculations is \( 1.0 \times 10^4 \), the high and weak field methods should agree closely as regards magnitude beyond \( 10^{13} \). In practice the agreement is good beyond about \( 6 \times 10^{12} \) for \( \mathcal{J} \) and \( 3 \times 10^{12} \) for \( \Delta \).

In the weak field approximation anisotropy arises solely from the term of \( O(\mathbf{H}^2) \). This term is small compared to the leading isotropic term \( O(\mathbf{H}) \) as long as \( \mathcal{J} \alpha < 1/4 \), (for \( \omega a < 1 \)), and the resulting anisotropy is therefore also small. The reason for the wrong representation of the anisotropy must clearly be sought in higher order field terms which become important at even lower fields, i.e. the suitability of the criterion \( (\mathcal{J} \alpha)^2 \ll 1 \) to imply satisfactory convergence of the series, particularly for non isotropic directions,
must be investigated. To this end the high field integrals for lattice scattering, given in (6.7), were expanded to find the first field term omitted in the weak field approximation. This is the term $O(H^4)$ for $I_1$ and the term $O(H^2)$ for $I_2$. The tensors required are $S_{xx}$, $S_{yy}$, $S_{xy}$ and $S_{yx}$, and, considering the (100) plane as an example, it is clear from appendix B1 that the significant quantities are $(\alpha_1 + \alpha_2)$ and $(\beta_1 + \beta_2)$. These are easily derived from $I_1$ and $I_2$ to the required order in $H$. For the real parts we find, on expanding in terms of $\mathcal{R}$ and substituting for the two cyclotron frequencies,

$$\alpha_1 + \alpha_2 \rightarrow 2A'(\omega) + 2V^2(K+2)C'(\omega) + 2V^4\left[(K+2)^2 + (K-1)^2\sin^22\phi\right]B'(\omega), (7.1)$$

$$\alpha_1 - \alpha_2 \rightarrow 2V^2(K-1)\sin2\phi \left[C'(\omega) + 2V^2(K+2)B'(\omega)\right], (7.2)$$

$$\beta_1 + \beta_2 \rightarrow 2B'(\omega) + 2V^2(K+2)D'(\omega) (7.3)$$

and

$$\beta_1 - \beta_2 \rightarrow 2V^2(K-1)\sin2\phi D'(\omega), (7.4)$$

where $A'(\omega)$, $B'(\omega)$ and $C'(\omega)$ are the real parts of the integrals $A(\omega)$, $B(\omega)$ and $C(\omega)$ defined in (4.20), (4.21) and (4.22) respectively, (apart from factors $a^2$ and $a^3$ arising from substituting $\mathcal{C} = a\sqrt{2}$). $D'(\omega)$ and $B'(\omega)$ are the corresponding integrals for the higher order terms and are given by

$$D'(\omega) = \int_0^\infty \frac{y^{3/2}e^{-y}}{(y + a^2\omega^2)^4} \left[-y^2 + 6a^2\omega^2y - a^4\omega^4\right] dy \quad (7.5)$$

and

$$B'(\omega) = \int_0^\infty \frac{y^2e^{-y}}{(y + a^2\omega^2)^5} \left[y^2 - 10a^2\omega^2y + 5a^4\omega^4\right] dy \quad (7.6)$$

$V^2$ is $(aeH/m_\omega)^2k/3$. The neglect of the imaginary parts is justified at low frequencies, as the real parts of the tensors are always much greater in this region.

The terms in $A'(\omega)$ and $B'(\omega)$ represent the isotropic contributions to $S_{xx}$ etc. We now consider the relative values of the terms
in \( C'(\omega) \) and \( E'(\omega) \) as a function of \( H \), since these affect \( S_{xx} \) and \( S_{yy} \), and similarly for \( B'(\omega) \) and \( D'(\omega) \) in connection with \( S_{xy} \). We take the low frequency limiting values for \( A'(\omega) \), \( B'(\omega) \), \( C'(\omega) \) and \( D'(\omega) \), as these are approximately correct up to \( \omega \sim 10^{12} \). They are 1, \( \pi^2/2 \), \(-\pi^2/2\) and \( \pi^2 \) respectively. \( B'(\omega) \) tends to \(+\infty\) as \( \omega \to 0 \). A few values for \( E'(\omega) \) were therefore calculated numerically, (to within \( \pm 5\% \)), at \( 2.3 \times 10^{10} \), \( (\omega a = 0.01) \), \( E'(\omega) \sim 2.3 \) while, at \( 2.3 \times 10^{11} \), \( (\omega a = 0.1) \), \( E'(\omega) \sim 1.3 \). At higher frequencies it eventually changes sign. We consider the value at \( 2.3 \times 10^{10} \) as being typical of the low frequency region. Then, substitution in (7.2) shows that \( (\alpha_1 - \alpha_2) \) is positive in the weak field approximation when the term in \( E'(\omega) \) is not included, but when this term is considered \( (\alpha_1 - \alpha_2) \) becomes negative for \( H \sim 8 \times 10^3 \) oersted. This field decreases as \( \omega \) decreases because \( E'(\omega) \) increases. It is interesting to note, however, that the \( \phi \) dependence is unaffected, being still \( \sin 2\phi \). Thus, as \( H \) increases the contribution made to the anisotropy by this term decreases and eventually changes sign, but the angular dependence is unchanged. In (7.1) the term in \( E'(\omega) \) is again of opposite sign to the \( C'(\omega) \) term and, for \( \phi = \pi/4 \), cancellation again occurs when \( H \sim 8 \times 10^3 \) oersted. In this case, however, the cancellation field is a function of \( \phi \), so the error incurred through neglecting the term in \( E'(\omega) \) increases with \( \phi \). On evaluating \( S_{xy} \) from (7.3) and (7.4) we find that it tends to

\[
\frac{-2uvK}{3} \left[ \left( K+2 \right) - 2v^2 \left\{ \left( K+2 \right)^2 + (K-1)^2 \sin^2 2\phi \right\} \right], \quad (7.7)
\]

from which it may be deduced that \( S_{xy} \) itself changes sign at about \( 9 \times 10^3 \) oersted when the \( H^3 \) term is included. Thus the additional term considered here becomes equal to the preceding term, which was considered in the weak field expansion, at a field below \( 10^4 \) oersted in all cases. This will hold up to about \( 10^{12} \). It seems probable, moreover, that the succeeding terms will also be of the same order at the same fields, or possibly at even lower fields. Since all terms do not have the same sign the anisotropy must be considerably affected
by including the higher order terms one by one. It seems likely that
the terms will in fact diverge and have alternating signs, giving an
oscillating series of increasing amplitude whose sum to infinity,
however, remains finite. This divergence will probably set in at some
term in the series for any field, but at very low fields there may
be convergence for a few terms before this effect becomes evident.
As \( H \) is increased the divergence starts in gradually earlier terms
until it reaches the first. If this is the case, the \( S_{xy} \) series will
diverge in all terms above \( \sim 9 \times 10^3 \), and the \( S_{xx} \) series will diverge
in all terms bar the first at a similar field, and in the first above
\( \sim 1.3 \times 10^4 \) for the present example.

As a test \( \beta \) was evaluated at \( H = 1.0 \times 10^4 \) and \( \omega = 2.3 \times 10^{10} \)
for the orientations \( \phi = 0 \) and \( \pi/4 \). These represented the extreme
values in the high field calculations. This calculation was approx­
iminate in the sense that the imaginary parts of \( S_{xx} \) etc. were neglected.
In the weak field approximation \( \beta \) is 2.36 for \( \phi = \pi/4 \), the exact
value established by interpolation being about 2.3, so that neglect
of the imaginary terms does not appear to introduce an error greater
than 1%. However, when the higher terms are included we find
\( \beta \sim -0.22 \). Errors present because of the terms omitted can hardly
account for more than a small part of this difference. It is mainly
due to the fact that \( S_{xy} \) has changed sign, the effect of the changes
in \( S_{xx} \) and \( S_{yx} \) being small by comparison. For \( \phi = 0 \) the weak field
approximation is little altered from the value for \( \pi/4 \). This is
because the only different term is \( S_{yx} \), which is \( \phi \) dependent because
of the small \( \alpha_1 - \alpha_2 \) term. In the higher approximation the'
corresponding difference is much greater, because \( S_{xy} \) is considerably
altered in magnitude and in fact changes sign between \( \phi = 0 \) and
\( \pi/4 \). We find in this case \( \beta \sim +0.69 \). The Table below summarizes
the results for \( \beta \) using the various approximations:

<table>
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<tr>
<th>( \phi )</th>
<th>Weak Field</th>
<th>Exp. to ( O(H^4) )</th>
<th>High Field ( (1.995 \times 10^{10}) )</th>
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<tr>
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<td>Approx.</td>
<td>Exact ( (1.995 \times 10^{10}) )</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>2.27</td>
<td>2.13</td>
<td>0.69</td>
</tr>
<tr>
<td>( \pi/4 )</td>
<td>2.36</td>
<td>2.21</td>
<td>-0.22</td>
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</tbody>
</table>
The weak field approximation overestimates the overall magnitude, and gives an anisotropy which is in the wrong sense. On the other hand including the next term produces exactly the opposite effect, underestimating the magnitude of $\mathcal{B}$ by a similar amount, (the high field values for both cases are scarcely different from the mean values of the two approximate methods, which are $1.48$ and $1.29$ for $\phi = 0$ and $\pi/4$ respectively), and overestimating the anisotropy to such a degree that $\mathcal{B}$ actually changes sign between $0$ and $\pi/4$. While it would not be wise to conclude too much from this sample consideration of the higher order terms, the results do reinforce the supposition that the series expansions in ascending powers of $H$ for either $S_{xx}$ and $S_{xy}$, or $S_{xy}$ and $S_{yx}$, consist of a series of terms which oscillate about the exact value with an amplitude which, for $1.0 \times 10^4$ oersted, is roughly constant for the earliest terms, though it will probably increase for higher terms. It is not surprising that the $\phi$ dependence is not greatly affected by including the extra terms, apart from the sense of the variation, as (7.1) etc. show that all the anisotropic terms still vary as $\sin^2(\phi)$ or $\sin^2(2\phi)$. At higher fields the deviations on either side of the high field values are likely to increase in both approximations. For lower fields, while the discrepancies in the overall magnitudes will clearly decrease, it seems probable that the sense of the anisotropic variation will not change in either case. As the frequency increases and $\omega a$ approaches unity there will be modifications to this behaviour. This follows qualitatively from the fact that the integrals appropriate to the higher order anisotropic terms fall off more rapidly in this region, and even undergo a change of sign, so that these terms will decrease in comparison to the $\Sigma (2)$ and $\Sigma (3)$ terms, and the weak field approximation will progressively improve as a consequence. This is subject to the additional requirement that $\kappa / \omega$ remains small, otherwise the divergence due to the higher powers of $H$ will more than outweigh the decreasing integrals. It seems likely that considerations of the sort outlined above will apply when $(\omega a)^2 \gg 1$ and $\kappa$ approaches $\omega$. Thus, for the present model, it is necessary to set a limit
considerably lower than \( \lambda a = 1 \) to ensure convergence to two or three terms, and obtain a reasonable value for the overall magnitude, though it seems to be impossible to have a description of the anisotropy which is even approximately correct to \( O(H^2) \). To \( O(H^4) \) the situation is hardly more satisfactory as the anisotropy, though in the correct sense, is far too great at useful fields. It should, however, be better than the weak field approximation below \( 5.0 \times 10^3 \). Only in the case of a phenomenon whose leading term depends on \( H^2 \) can the anisotropy be predicted correctly by the series expansion. Even in this case we must have \( (\lambda a)^2 \leq 1/16 \) for \( (\omega a)^2 \leq 1 \), and \( (\omega / \omega)^2 \leq 1/16 \) for \( (\omega a)^2 > 1 \). (The infrared Voigt ellipticity, (Cardona 1961), is an example of such an effect.) \( \mathfrak{B} \) is isotropic in such regions, and if we can reasonably neglect the \( H^3 \) and \( H^4 \) terms in the series expansions the \( H^2 \) terms can probably be omitted also, as they are of the same order down to very low fields, but much smaller than the linear term. To illustrate this, at \( 10^9 \) the low frequency approximation (6.29) gives 0.178 with \( H = 5.0 \times 10^3 \), while the weak field approximation gives 0.187, showing that the effect of the \( H^2 \) term is small in comparison with the linear term.

The discussion has been confined to the imaginary parts of the propagation constants. The effect of the complex formula for \( \mathfrak{B} \) for non-isotropic directions can easily be shown to be similar in the high field and weak field methods because \( \mathfrak{P} \) (\( \mathfrak{Q} \) is negligible at low frequencies), is given approximately correctly by the weak field expressions. This could be a coincidence, and the discrepancy might be more apparent if the \( H^3 \) and \( H^4 \) terms were included, since the denominator \( \mathcal{C} \) would then change sign as \( \phi \) varied while the numerator would retain the same sign for all \( \phi \).

A similar analysis can probably be made for the \( (1\overline{1}0) \) plane and for the ellipticity.
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*Note: The values in the table represent changes in some physical parameter, where the change is denoted by an arrow (\( \rightarrow \)). The sign of the change is indicated by the sign column.*
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In semiconductor science, the mean free path of carriers is a function of \( \omega \) and the distribution function is determined by the interaction between the spin moment and the magnetic field of the crystal. The distribution function is not important at these frequencies. In calculating the conductivity, the Fermi liquid model could therefore be used. The mean free path is given by the Fermi surface in semiconductors.
Summary and Conclusions

In the preceding chapters a theory of the Faraday effect in an isotropic material has been given, and its application to the case of the semiconductor n-type germanium developed in some detail. The expressions for $\mathbf{H}$ and $\Delta$ derived in chapter two are general, and are thus applicable also to other Faraday phenomena, whatever the origin of the static field and the nature of its interaction, in the presence of an incident plane polarized electromagnetic wave, with the electrons (or holes) in the material. It is only necessary to assume that the interaction can be described by a conductivity tensor which is a function of $\omega$ and $\mathbf{H}$. This assumption has been justified for a wide range of experimental conditions and is certainly a very good approximation for both bound and conduction electrons up to and including infra-red frequencies. Even in metals which have a small skin depth the dependence on the wave vector of the electron is not important at these frequencies. Equations (2.27) and (2.28) could therefore be used for example in calculations of the Faraday effect in ferromagnetics.

In semiconductors the most important contribution to the conductivity tensor usually arises from the interaction between the orbital motion of the charge carriers induced by an external magnetic field and the electromagnetic radiation. The intraband contribution to this effect can be found by semiclassical arguments, based on the calculation of a field modified distribution function for the carriers from the Boltzmann equation. Spin-orbit effects have generally been regarded as small though the spin-orbit term has been included in the Hamiltonian in quantum calculations by various authors, e.g., Boswarva, Howard and Lidiard (1962); Roth (1964); Halpern, Lax and Nishina (1964) and Bennett and Stern (1965). More recently Stern (1965), using a semiclassical method involving
the Boltzmann equation, has argued that, in non ferromagnetic materials, the spin-orbit term may be regarded as giving rise to a nett electric polarization which also contributes to the current. It becomes significant in magnitude in the infra-red, but it is negligible at lower frequencies. The valence to conduction band transitions begin to make an important contribution when the radiation frequency approaches the frequency corresponding to the energy gap. This frequency usually lies in the far infra-red. Some form of quantum calculation, (see Boswarva etc. above), is necessary to describe the various allowed interband transitions which contribute to the conductivity, so that this effect is outside the scope of the distribution function method based on the Boltzmann equation.

The solution of the Boltzmann equation is generally much simpler than attempting the corresponding problem formulated quantum mechanically, and less assumptions are needed in order to obtain numerical results with reasonable ease. To date the quantum calculations have been concerned with the limit in which the displacement current greatly exceeds the conductivity current, and attention has been concentrated on obtaining either $S_{xy}$, the off diagonal component which determines the Faraday rotation in this limit, or $S_{xx}$ and $S_{zz}$ which are required for the Voigt effect. Relaxation effects are introduced where necessary by replacing $\omega$ by $(\omega - i/\tau)$ in the resultant components obtained from a first order perturbation calculation, (Halpern etc., (1964)).

We are chiefly concerned with microwave frequencies, as the anisotropic effects are usually most evident in this region. It is clearly unnecessary to use such a complicated method at microwave frequencies, where the spin-orbit effects and the interband orbital contribution must be negligible. The results of the quantum calculation would merely reduce to the free electron terms when reasonable approximations were made. The calculation of all the tensor components to all orders in the field, including relaxation effects and considering the correct energy surfaces as required to discuss anisotropy, would be prohibitively lengthy and of academic
interest only.

In the present work therefore, the Boltzmann equation approach has been used and interband and spin-orbit effects excluded. This method, which treats the electron energy as a continuous variable, is quite adequate provided the magnetic field satisfies $\hbar J << kT$, so that the discrete nature of the Landau levels is unimportant. This is questionable at the highest fields and lowest temperatures used in the isotropic calculation, but satisfactory for the more realistic parameters used in the anisotropic case.

The specific calculations which are detailed in chapters three to seven are concerned with solving the Boltzmann equation for the distribution function, and hence obtaining the magnetococonductivity tensor and Faraday rotation and ellipticity in a semiconductor having either spherical surfaces of constant energy, or ellipsoidal surfaces arranged along $[111]$ or $[100]$ axes, i.e. having cubic symmetry. The latter structures are appropriate to the conduction band of many cubic semiconductors such as germanium and silicon. The isotropic model is a convenient approximation as far as the Faraday effect, whose leading term in either the high or low field limits is isotropic, is concerned. Certain major trends not involving directional effects, e.g. the frequency, field and temperature dependence, can be deduced.

It was assumed that a time of relaxation existed, since without this simplification the Boltzmann equation is not exactly soluble for these models. Furthermore the scattering was assumed to be isotropic, the two possible mechanisms being acoustic mode scattering from the lattice vibrations and scattering from ionized impurities. Intervallen scattering was therefore excluded, along with other less important mechanisms such as scattering from neutral impurities and dislocations. These restrictive assumptions on the scattering are likely to be more serious at microwave frequencies than the use of the Boltzmann equation itself as opposed to a quantum calculation, but could hardly be avoided if detailed numerical results were to be obtained with reasonable ease.

No satisfactory theory of intervalley scattering exists which
would have made its inclusion a straightforward matter, even though it is isotropic to a good approximation and would therefore only affect the form of the energy integrals. Moreover, the objection made in chapter six to the combined lattice and impurity scattering calculation, namely the absence of reliable information about the relative weights to be assigned to the intra and inter-valley contributions, must again be considered. It seems likely that the additional effort involved would be worth while only if attempting to fit experimental results in a specimen where the inter-valley contribution was known to be appreciable. The phonon contribution is believed to be small, though the impurity scattering contribution may be significant. For example, Fritzsche and Cuevas (1962) estimated from piezoresistance measurements that about 35% of the scattering in germanium doped with $\sim 3 \times 10^{13}$ arsenic donors cm$^{-3}$ was due to inter-valley transitions. The effect is a function of the type of impurity, the arsenic value being high, and decreases with decreasing doping. For most of our calculations therefore, the neglect of this effect should not matter. In any case the two mechanisms considered in chapter five produced only detail differences in the shape of the various curves and small differences in the overall magnitudes, if the parameters $a$ and $g$ were suitably chosen, and similarly $\Omega$ and $\Delta$ are not likely to be sensitive functions of the type of scattering, as long as it is isotropic.

The anisotropy of $\gamma$ for lattice and ionized impurity scattering is a more significant problem in the present work, and again rather difficult to treat adequately. The relaxation tensor approach, first initiated by Herring and Vogt, (1956), has been invoked recently by a number of authors, ( see for example Neuringer 1964 ; Tsidilkovski, Sokolov and Kharus 1964, and others ). Herring and Vogt considered the case when the relaxation tensor has the same symmetry as the effective mass tensor and the same energy dependence as the isotropic case. They showed that, since the D.C. Boltzmann equation is a function of the ratio ($m/\gamma$) only, in order to take account of anisotropy in the scattering it is only necessary to replace $m_3/\gamma$. 


and \( m_1/\tau \) by \( m_3/\tau_3^0 \) and \( m_1/\tau_1^0 \). \( \tau_3^0 \) and \( \tau_1^0 \) being the two independent coefficients of the energy in the diagonal relaxation tensor referred to the cubic axes of the crystal. This theory has been used with some success by the above authors and others in the interpretation of D.C. Hall effect and magnetoresistance data. However, the high frequency generalization is much more complicated and has not been worked out in any detail. The effect could in principle be included, given sufficient data and assuming that the amount of anisotropy in \( \tau \) is small enough to render Herring and Vogt’s approximate treatment valid, but the theory was not so developed as at present when the work reported in this thesis was carried out, and its inclusion at that time was considered to be too difficult. However, we may assess the probable effect of anisotropic scattering in general terms as follows. Herring and Vogt showed that in the D.C. case the theory can be developed in terms of a modified mass ratio parameter defined by

\[
K' = \frac{m_3 \tau_3^0}{m_1 \tau_1^0} = K K_{\tau}
\]

with \( m_3/\tau \) also replaced by \( m_3/\tau_3^0 \). \( K_{\tau} \) for lattice scattering is quite close to one for germanium at least, according to the deductions of the authors listed above from the D.C. data, though it may be larger for impurity scattering. At low frequencies, i.e. \( (\omega \tau)^2 \ll 1 \), the simple replacement of \( K \) by \( K' \) and introduction of a suitable average \( \tau \) where necessary elsewhere in the equations would be a reasonable approximation. This covers the major part of the microwave range in the present germanium calculations. At high frequencies, i.e. \( (\omega \tau)^2 \gg 1 \), \( \Theta \) is independent of the scattering. In the region \( \omega \tau \sim 1 \) anisotropic scattering may affect the detailed structure of the dependence on various external parameters to a greater extent. However, since the relaxation tensor has the same symmetry as the effective mass tensor, the orientation dependence is likely to be affected more through changed \( \rho \) values than an altered angular dependence. On balance it seems likely that
corrections additional to the $K \to K'$ approximation will be small in specimens where lattice scattering predominates. This should be particularly good for the rotation where terms involving $\mathcal{T}$ are usually of second order. Relaxation effects are more critical for the ellipticity, and the influence of anisotropic scattering could be important in the region $\omega \tau \gtrsim 1$. For highly doped samples where impurity scattering predominates, the situation is still obscure, as it is theoretically possible for $\tau_{ij}^* / \tau_{ii}$ to be as large as ten. At the present time, the incorporation of anisotropic lattice scattering into the germanium calculation seems to be feasible and would be of some interest. However, it is considered that the isotropic scattering approximation does not introduce serious errors into the field, frequency and orientation dependence, and the general conclusions which have been drawn should be valid.

One other effect known to be important at microwave frequencies has been omitted. This is the multiple reflection correction (Champlin 1962; Donovan and Medcalf 1963 and 1964). The correction is mainly to the magnitude and is very large when the wavelength in the sample is $\gtrsim d$, (i.e., $\beta \lesssim \pi$), and oscillatory about the no reflection curve with decreasing amplitude at shorter wavelengths, (Donovan and Medcalf 1964). The $\phi$ and $\psi$ dependence should not be affected as much. It would be straightforward to repeat the present calculations including the multiple reflection correction as all the relevant expressions have been derived. They are quoted in Medcalf (1965) and a few numerical results presented. The equivalent problem for the Voigt configuration has been analyzed by Donovan and Webster (1965).

Using the approximations discussed above, the exact conductivity tensor in the high frequency case has been derived in chapter three for the $[111]$ and $[100]$ ellipsoid band models, and transformed appropriately to enable the dependence on the orientation of $\mathcal{H}$ to be computed in $(100)$ and $(1\overline{1}0)$ planes, these being the obvious cases to consider for cubic symmetry. Attention has also been directed for the first time to the dependence on the orientation of the $\mathcal{E}$.
vector when $H$ is not along an isotropic direction. The necessary tensor transformations to enable this effect to be calculated have been carried out. The conductivity tensor for spherical surfaces of constant energy was also obtained by letting the effective mass ratio $K$ tend to unity. In chapter four the equivalent results for the current expanded as a series in ascending powers of $H$ were calculated. This method was considered because it is useful to compare the predictions of the series with the exact results in a case where a closed solution of the Boltzmann equation is possible, with a view to assessing the probable usefulness of the series expansion in cases where a closed solution cannot be found but the series expansion method can be developed with reasonable ease. The warped spheres appropriate to the valence bands of germanium and silicon are a case in point.

Chapters five, six and seven present detailed numerical calculations for n-type germanium. In chapter five the isotropic model was assumed and attention focussed on the frequency, field and temperature dependence for lattice and impurity scattering. Previously only the frequency dependence of the rotation had been investigated in any detail, (Stephen and Lidiard 1958), and the ellipticity had received virtually no attention. The significance of microwave experiments in the measurement of the field and temperature dependence was pointed out. The only measurement for which a useful comparison between theory and experiment could be made was that of Rau and Gaspari, (1955), on a specimen of n-type germanium having a resistivity of 16 ohm cm and a mobility of $3.9 \times 10^3 \text{ cm}^2 \text{ v}^{-1} \text{ sec}^{-1}$. The carrier density was $1.0 \times 10^{14} \text{ cm}^{-3}$ and, assuming lattice scattering as the conductivity is close to the intrinsic value, the parameter $a$ was $7.4 \times 10^{-13} \text{ sec}$ for an effective mass $m^* = m/4$. The measured value was $3.5^\circ$, the quoted field and specimen thickness being $1.43 \times 10^3$ oersted and $0.46 \text{ cm}$ respectively, compared with a theoretical magnitude of $4.37^\circ$ calculated from the above parameters. The low frequency limit (5.26) is not a particularly good approximation, (it gives $3.9^\circ$), as the
frequency is rather too high even though \( \lambda a \approx 0.07 \) is very small.

It can be seen from (5.26) that, for given mobility and conductivity values, a variation of \( m^* \) will not affect the result in this approximation as only the ratio \( m^*/\sigma \) appears in the equation, and this is proportional to the mobility and therefore constant, while \( N \) also must be fixed, so that better agreement could only result from changes in \( \sigma \) or the mobility. Without information on the possible experimental uncertainties in these quantities therefore, the comparison is not very meaningful. Without calculation also it is not possible to decide with certainty whether or not the multiple reflection correction would increase or decrease the rotation. A decrease seems more likely on the basis of Fig. 1 of Donovan and Medcalf, (1964), as they find oscillatory behaviour beginning at \( \omega = 1.3 \times 10^{11} \) for \( d = 0.5\text{mm} \) and an increase in \( d \) by a factor \( \approx 10 \) should move this point closer to \( 10^{10} \), and place the Rau and Caspari frequency, \( \omega = 5.5 \times 10^{10} \), between this point and the next crossing of the no reflection and reflections included graphs. The specimen parameters are not dissimilar so this deduction seems reasonable. However, the absorption is fairly high for this thickness and the correction should not be very great on this account.

The isotropic calculations are chiefly useful in giving an approximate general picture of the behaviour of \( \Theta \) and \( \Delta \) over a very wide range of external conditions and sample parameters.

An exact comparison with experiment clearly requires the inclusion of the correct band structure and multiple reflection correction at microwave frequencies. The calculation presented in chapter six fulfils the first of these conditions for \( n \)-type germanium, and a detailed analysis of the dependence on \( \omega \), \( H \), \( \phi \), \( \psi \) and \( d \) is given for a typical lattice scattering specimen. The range of field for which anisotropic effects are significant was deduced.

Recently the experiments of Bouwknecht and Volger, (1964), have provided confirmation of the anisotropy of the Faraday effect in \( n \)-germanium. Their measurement of the dependence of \( \Theta \) on the orientation of \( \mathbf{E}_0 \) for \( \mathbf{H} \) along the [011] axis shows the correct symmetry,
and the peak positions and $\rho_{R}(\psi)$ values appear reasonably consistent with the behaviour discussed in §6(e)(iv). Bouwknegt and Volger's own calculations based on the theory developed in this thesis, but including multiple reflections, seem to agree well.

Further measurements of $\Theta$ and $\Delta$ on n germanium would be of interest. The field dependence has not yet been plotted for various orientations, cf. Figs. (6.4) and (6.14), and the $\psi$ dependence of the ellipticity also remains to be investigated. Measurements on other multivalley structures, e.g. n-type silicon, would be useful. The high field asymptotic behaviour also merits consideration. Only Mansfield and Borst (1965) have approached this region, the material being indium antimonide. In germanium these effects should be attainable without great difficulty at lower temperatures. Furthermore the experimental precision attained by Bouwknegt and Volger ($\sim 2\%$) suggests that the relatively small variations arising from the different scattering mechanisms, or mixed scattering, may be profitably analyzed in future experiments. In fact Bouwknegt and Volger considered both the lattice scattering and constant 'C approximations separately, and concluded that the latter gave a better fit, probably because of the admixture of a small amount of ionized impurity scattering.

In addition to the particular numerical results which have been evaluated, the theory of chapter two is also useful in other contexts as indicated previously. Also the complete magnetoconductivity tensors and appropriate general symmetry properties, given in the appendices for both the [111] and [100] ellipsoid configurations in various high symmetry planes in the crystal, are a convenient basis for the calculation of any magneto-optical phenomenon such as the Voigt effect or the magnetoabsorption.

The series calculation considered in chapter seven is rather disappointing in that agreement with the closed solution is only adequate for $\omega a \leq 1/4$ in the microwave range where $\omega a \leq 1$, while the anisotropic variation is predicted quite inaccurately at all fields. This suggests that the expansion method should be used only with caution in this region.
APPENDIX A

Symmetry Properties of the Magnetoconductivity Tensor in Cubic Crystals

I. \( \mathbf{H} \) parallel to certain crystallographic directions

Let the matrix \( \mathbf{R} \) denote a transformation applied to the \((x,y,z)\) axes, which consists of a rotation about the magnetic field direction, \((z\) axis). If the crystal axes are sent into an equivalent position by \( \mathbf{R} \), i.e. if the transformation applied to the crystal axes is an operation of the \(O_h\) cubic point group, then, by symmetry, the magnetoconductivity tensor, which in matrix notation may be denoted by \( \mathbf{S} \), must be invariant under the transformation \( \mathbf{R} \).

Hence

\[
\mathbf{R} \mathbf{S} \mathbf{R}^{-1} = \mathbf{S} \quad . \tag{A.1}
\]

Three cases are of interest:

(a) \( \mathbf{H} \) parallel to [001]

The \(z\) axis in this case is the \(3\) axis of the crystal, so that a rotation of \( \pi /2 \) about the \(z\) axis in the direction \(x \rightarrow y\) applies the same transformation to the crystal axes. The matrix is given by

\[
\mathbf{R} = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} \quad . \tag{A.2}
\]

which belongs to the \(O_h\) point group as required. Hence, writing (A.1) out in full, we find

\[
\begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
S_{xx} & S_{xy} & S_{xz} \\
S_{yx} & S_{yy} & S_{yz} \\
S_{zx} & S_{zy} & S_{zz}
\end{pmatrix}
\begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
=
\begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
S_{xx} & S_{xy} & S_{xz} \\
S_{yx} & S_{yy} & S_{yz} \\
S_{zx} & S_{zy} & S_{zz}
\end{pmatrix}
\]
Comparing equivalent terms gives the following relations between the components of $S$

\[
\begin{align*}
S_{xx} &= S_{yy}, \\
S_{xy} &= -S_{yx}, \\
S_{xz} &= S_{yz}, \\
S_{zx} &= S_{zy}, \\
S_{zz} &= \text{unrestricted}.
\end{align*}
\]  

The last four equations can be satisfied only if $S_{xz} = S_{zx} = S_{yz} = S_{zy} = 0$.

The conductivity tensor therefore reduces to the form

\[
\begin{pmatrix}
S_{xx} & S_{xy} & 0 \\
-S_{xy} & S_{xx} & 0 \\
0 & 0 & S_{zz}
\end{pmatrix}
\]  

The argument given above is independent of the orientation of the $x$ and $y$ axes in the $(001)$ plane, and hence of the direction of $\mathbf{E}_0$ which is parallel to the $x$ axis.

(b) $\mathbf{H}$ parallel to $[111]$.

If $\mathbf{R}$ is a rotation of $2\pi/3$ about the $z$ axis in the direction $x \rightarrow y$, the transformation applied to the crystal axes is easily shown to be

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\]

which belongs to the $O_h$ point group. Also the matrix $\mathbf{R}$ is given by
Applying the condition \((A.1)\) and equating components of the equivalent tensors to zero gives the following equations:

\[
\begin{align*}
3^{3/2}(S_{xx} - S_{yy}) &= -(S_{yx} + S_{xy}) \\
3^{3/2}(S_{xy} + S_{yx}) &= (S_{xx} - S_{yy}) \\
S_{zz} &= S_{zx} :: S_{zz} \text{ unrestricted}
\end{align*}
\]

\[(A.8)\]

Further simplification in orientations of the (111) plane.

The first two equations are consistent only if

\[S_{xx} = S_{yy} \text{ and } S_{xy} = -S_{yx},\]

and the last four if

\[S_{xz} = S_{zx} = S_{yz} = S_{zy} = 0\]

so that \(S\) again reduces to the form given in \((A.5)\), independently of the orientation of \(\mathbf{E}_0\) in the (111) plane.

(c) \(\mathbf{H}\) parallel to [011]

The transformation of the crystal axes corresponding to a rotation of \(\pi\) about the \(z\) axis also belongs to the \(O_h\) point group, and is given by

\[
\begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]

\[(A.9)\]
In this case \( \mathbf{R} \) is
\[
\begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
(A.10)
and, applying (A.1) and equating equivalent components to zero, it follows that

\[
S_{xx}, S_{yy}, S_{zz}, S_{xy} \text{ and } S_{yx} \text{ are unrestricted and}
\]

\[
S_{xz} = -S_{zx}, \quad S_{zx} = -S_{xz}, \quad S_{yz} = -S_{zy}, \quad S_{zy} = -S_{yz}.
\]

The only restriction on \( S \) is therefore
\[
S_{xz} = S_{zx} = S_{yz} = S_{zy} = 0.
\]
Further simplifications will be shown to arise for particular orientations of the \( x \) and \( y \) axes with respect to the crystal axes.

II \( \mathbf{H} \) contained in certain crystallographic planes

The magnetoconductivity tensor can be simplified by the method of A I only when \( \mathbf{H} \) is parallel to \([001], [111] \) and \([110] \) type directions, these being four, three and two fold axes respectively of a cubic crystal. However, an extension of this method can be used to determine the symmetry properties of \( S \) when \( \mathbf{H} \) lies in certain planes.

A transformation \( \mathbf{R} \) is used which rotates the \( z \) axis by \( \pi \) in the plane considered, about an axis perpendicular to the plane. The magnetic field is rotated along with the \( z \) axis and, since \( S \) will generally be a function of \( \mathbf{H} \), \( \mathbf{H} \) must be transformed back to its original direction in the resultant expressions for the components \( S_{jk} \). In the present case the back transformation required is just a reversal of the field direction, and it can be accomplished by using the Onsager relation
\[
S_{jk}(\omega, \mathbf{H}) = S_{kj}(\omega, -\mathbf{H}).
\]
(A.12)
Three cases are considered which, by the argument of A I, leave $S$ invariant. For the first two the $x$ axis is the axis of rotation, so that $R$ is given by

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (A.13)$$

(a) $H$ in (100) plane, making an angle $\phi$ with the 3 axis; $x$ parallel to [100]; (Fig A II(a)).

Clearly the transformation applied to the crystal axes by $R$ is given by (A.13). Hence

$$R S R^{-1} = \begin{pmatrix} S_{xx} & -S_{xy} & -S_{xz} \\ -S_{yx} & S_{yy} & S_{yz} \\ -S_{zx} & S_{zy} & S_{zz} \end{pmatrix} \quad (A.14)$$

and (A.13) belongs to the $0_h$ point group, so that, provided (A.12) is used to back transform $H$, the condition (A.1) may be applied and we find

$$S = \begin{pmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{pmatrix} \equiv \begin{pmatrix} S_{xx} & -S_{yx} & -S_{zx} \\ -S_{xy} & S_{yy} & S_{zy} \\ -S_{xz} & S_{zy} & S_{zz} \end{pmatrix} \quad (A.15)$$

It follows that

$$S_{xx}, S_{yy} \quad \text{and} \quad S_{zz} \quad \text{are unrestricted},$$

$$S_{xy} = -S_{yx}$$

and

$$S_{xz} = -S_{zx}, \quad S_{yz} = S_{zy}$$

The magnetococonductivity tensor thus has six independent components and reduces to the form
Since $K$ is the same as for the previous case, $Q$ again satisfies the form (4.10) for the same reasons.  The validity place in comparison to the apparently similar arrangement in II(a) for which (4.17) was derived, and the faster arguments therefore used not be realistic for the two situations (one component in fact turns out to be different).  The two cases are therefore physically different, and give rise to different Faraday rotations, etc.  This non-equivalence is proved by symmetry arguments alone in II(b).
\[
\begin{pmatrix}
S_{xx} & S_{xy} & S_{xz} \\
-S_{xy} & S_{yy} & S_{yz} \\
-S_{xz} & S_{yz} & S_{zz}
\end{pmatrix}
\]  \quad (A.16)

For the special case when \( \mathbf{H} \) is parallel to the [011] direction, \( (\phi = \pi/4) \), and \( y \) along the [011] direction, both (A.11) and (A.16) must be valid, and \( S \) reduces to the four component form

\[
\begin{pmatrix}
S_{xx} & S_{xy} & 0 \\
-S_{xy} & S_{yy} & 0 \\
0 & 0 & S_{zz}
\end{pmatrix}
\]  \quad (A.17)

When \( \phi = 0 \), \( (\mathbf{H} \parallel [001]), \) (A.16) does not contain any restriction not included in (A.5).

(b) \( \mathbf{H} \) in (110) plane, making an angle \( \phi \) with the 3 axis; \( x \) parallel to (110); (Fig A II(b)).

The \( 0_{h} \) point group transformation of the crystal axes effected by \( \mathcal{R} \) is

\[
\begin{pmatrix}
0 & -1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}
\]  \quad (A.18)

Since \( \mathcal{R} \) is the same as for the previous case, \( S \) again assumes the form (A.16) for arbitrary \( \phi \). Also, when \( \mathbf{H} \) is parallel to the [110] direction (\( \phi = \pi/2 \)), and \( y \) along the [001] direction, (A.17) is valid. However, it can be seen that the \( x \) and \( y \) axes have changed places in comparison to the apparently similar arrangement in II(a) for which (A.17) was derived, and the tensor components therefore need not be identical for the two situations; (one component in fact turns out to be different). The two cases are therefore physically different, and give rise to different Faraday rotations, etc. This non-equivalence is proved by symmetry arguments alone in A II(d).
Finally we consider the case in which

(3) \[ H \] makes an angle to the \((110)\) plane, making an angle \(\phi\) with
the \(z\) axis, but where the \(x\) and \(y\) axes are inclined at an angle
\[
\sin \left( \frac{\pi}{2} \right)
\]
by the \(1\) and \(2\) axes respectively. (Fig A II(b)).

![Diagram](image)

**Fig. A II(b).** Orientation of field and crystallographic
axes in case II(b). The solid lines lie in the \((110)\) plane.
Finally we consider the case in which

(c) \( H \) again lies in the (1\overline{1}0) plane, making an angle \( \phi \) with the 3 axis; but where the x and y axes are inclined at an angle \( \cos^{-1} \left( \frac{1+\cos \phi}{2} \right) \) to the 1 and 2 axes respectively; (Fig A II(c)).

This arrangement preserves a symmetrical arrangement of the x and y axes with respect to the cubic axes. As in case (b), \( \mathcal{R} \) is a rotation of \( \Pi \) about the [1\overline{1}0] direction. The transformation which this operation applies to the \((x,y,z)\) system is obtained as follows. The transformation from the cubic axes to the \((x,y,z)\) system is a rotation of \( \phi \) about the [1\overline{1}0] direction. Because the axis of rotation for \( \mathcal{R} \) is also the [1\overline{1}0] direction, the same matrix is appropriate to both \( \mathcal{R} \), and the equivalent transformation applied to the crystal axes, (c.f. case A II(a)). The matrix required is (A.18) and, applying (A.1) and (A.12), we find

\[
\mathcal{R} \mathbf{S} \mathcal{R}^{-1} = \begin{pmatrix}
S_{yy} & S_{xy} & S_{zy} \\
S_{yx} & S_{xx} & S_{zx} \\
S_{yz} & S_{xz} & S_{zz}
\end{pmatrix} = \mathbf{S} \tag{A.19}
\]

and, by equating equivalent terms, the magnetoconductivity tensor may be reduced to

\[
\begin{pmatrix}
S_{xx} & S_{xy} & S_{xz} \\
S_{yx} & S_{xx} & S_{yz} \\
S_{yz} & S_{xz} & S_{zz}
\end{pmatrix} \tag{A.20}
\]

There are six independent components as for the previous case, but the symmetry properties are quite different from (A.16). A similar argument also leads to the form (A.20) when \( H \) is in the (100) plane inclined at an angle \( \phi \) to the 3 axis, and the x and y axes are equally inclined to the 2 axis at an angle \( \cos^{-1} \left( \cos \frac{\phi}{2} \right) \).
Fig. A II(c). Orientation of field and crystallographic axes in case II(c). The solid lines lie in the (110) plane and the two angles marked similarly are equal.
When $\phi = \pi / 2^\circ$, with $\mathbf{H}$ in the (110) plane, $\mathbf{H}$ is along the [110] direction and, using (A.11), $S$ may be further simplified to the four component form

$$
\begin{pmatrix}
S_{xx} & S_{xy} & 0 \\
S_{yx} & S_{xx} & 0 \\
0 & 0 & S_{zz}
\end{pmatrix}, \quad (A.21)
$$

which may be contrasted with (A.17).

(d) Further consideration of [110] direction

Equations (A.11), (A.17) and (A.21) taken together show that the symmetry properties of $S$ are dependent on the orientation of the $x$ and $y$ axes in the plane perpendicular to the magnetic field direction. To clarify this situation we consider a rotation of the axes, through an angle $\psi$, about the $z$ direction, in the sense $x \to y$. The transformation matrix is

$$
\begin{pmatrix}
\cos \psi & \sin \psi & 0 \\
-\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad (A.22)
$$

The transformed components, which are denoted by primes, are

$$
\begin{align*}
S'_{xx} &= S_{xx} \cos^2 \psi + S_{yy} \sin^2 \psi + (S_{xy} + S_{yx}) \sin \psi \cos \psi \\
S'_{yy} &= S_{xx} \sin^2 \psi + S_{yy} \cos^2 \psi - (S_{xy} + S_{yx}) \sin \psi \cos \psi \\
S'_{xy} &= S_{xy} \cos^2 \psi - S_{yx} \sin^2 \psi + (S_{yy} - S_{xx}) \sin \psi \cos \psi \\
S'_{yx} &= S_{xy} \sin^2 \psi - S_{yx} \cos^2 \psi - (S_{yy} - S_{xx}) \sin \psi \cos \psi \\
S'_{zz} &= S_{zz}
\end{align*}, \quad (A.23)
$$
Other components are zero because of (A.11). If the initial state of the axes is taken to be -H along [110], x along [110], y along [001], (case A II(b) with $\phi = \pi/2$), (A.17) may be used to simplify (A.23) and the following special cases derived.

(i) $\psi = 0$; $S'_{xx} = S_{xx} \neq S'_{yy} = S_{yy}$, $S'_{xy} = S_{xy}$, $S'_{yx} = -S_{xy}$

(ii) $\psi = \pi/4$; 

$$\begin{align*} 
S'_{xx} &= \frac{1}{2}(S_{xx} + S_{yy}) = S'_{yy} \\
S'_{xy} &= S_{xy} + \frac{1}{2}(S_{yy} - S_{xx}) \\
S'_{yx} &= -S_{xy} + \frac{1}{2}(S_{yy} - S_{xx}) 
\end{align*}$$  \quad (A.24)

The x and y axes are now symmetrically placed with respect to the crystal axes, and the arrangement of the (x,y,z) axes which lead to (A.21) has been obtained.

(iii) $\psi = \pi/2$; 

$$\begin{align*} 
S'_{xx} &= S_{yy} \neq S'_{yy} = S_{xx} \\
S'_{xy} &= S_{xy}, \quad S'_{yx} = -S_{xy} 
\end{align*}$$  \quad (A.25)

We now have the situation, x along [001] and y along [110], and the same symmetry as for $\psi = 0$. However, the axes are equivalent to the situation described by A II(a) with $\phi = \pi/4$, rather than to (i), and $S'_{xx}$ and $S'_{yy}$ have been changed over by the rotation of $\pi/2$. This proves that interchange of the x and y axes alters S even though the symmetry of S is unchanged, thus making the two cases physically different for a Faraday experiment, as mentioned at the end of A II(b).

(iv) $\psi = 3\pi/2$; 

$$\begin{align*} 
S'_{xx} &= \frac{1}{2}(S_{xx} + S_{yy}) = S'_{yy} \\
S'_{xy} &= S_{xy} - \frac{1}{2}(S_{yy} - S_{xx}) \\
S'_{yx} &= -S_{xy} - \frac{1}{2}(S_{yy} - S_{xx}) 
\end{align*}$$  \quad (A.26)

The symmetry of case (ii) re-occurs, but $S'_{xy}$ and $S'_{yx}$ are different for the two cases, and hence the situations corresponding to (ii) and (iv) are, like (i) and (iii), physically different.
(v) From (A.23) it is easily seen that $S(\psi) = S(\pi + \psi)$, so that the Faraday effect has a symmetry of $\pi$ for a rotation of $\varepsilon_0$ round the magnetic field direction.

To sum up, (A.11) is always valid for $\mathbf{H}$ along $[110]$ type directions and, in addition, the special forms (A.17) and (A.21) apply when the $x$ and $y$ axes also lie along certain directions which occur at intervals of $\pi/4$.

In conclusion two further points may be made briefly. Firstly, the transformation (A.22) leaves $S$ unaltered when $\mathbf{H}$ is parallel to $[100]$ or $[111]$ type directions, verifying that the Faraday effect itself, in addition to the symmetry of $S_0$, (see A.I), is independent of the orientation of $\varepsilon_0$ in these two cases.

Secondly, for a general orientation of $\mathbf{H}$ when $S_{xz}$ etc. are not zero, $S'_{xz}$ etc. do not obey $S'_{jk}(\psi) = S_{jk}(\pi + \psi)$, though this relation is still valid for $S'_{xx}$, $S'_{yy}$, $S'_{zz}$, $S'_{xy}$ and $S'_{yx}$. However, it is easy to show that the combinations of tensor components occurring in $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are such as to leave these quantities, and hence the Faraday effect, invariant under the transformation $\psi \rightarrow (\pi + \psi)$. Furthermore, for the situations defined in A II(a), (b) and (c), with $\phi$ arbitrary, either $S_{xy} = -S_{yx}$ or $S_{xx} = S_{yy}$ and, by applying (A.23), $S_{xx}$ etc. can be shown to transform in a manner similar to cases (i) to (iv) above. Continued rotations of $\pi/4$ alter the symmetry successively from $S'_{xx} = S'_{yy}$ to $S'_{xy} = -S_{yx}$ and back again, or including also the terms in $S'_{xz}$ etc. and considering $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$, from $\mathbf{A} = \mathbf{D}$ to $\mathbf{B} = -\mathbf{C}$ and back again.
Appendix B

Components of the Magnetooconductivity Tensor (Closed Solution)

I. \( \mathbf{H} \) in (100) plane, [111] ellipsoids

The two cyclotron frequencies are

\[
\mathcal{L}_1^2 = \frac{v^2KH_z^2}{3} \sqrt{K^2 + (K-1)\sin 2\phi} = \mathcal{L}_3^2. \quad (B.1)
\]

The tensor components are

\[
S_{xx} = \frac{2}{3}u(2K+1)(\alpha_1 + \alpha_2)
\]

\[
S_{yy} = \frac{2}{3}u(2K+1)(\alpha_1 + \alpha_2) + \frac{2}{3}u(K-1)\sin 2\phi (\alpha_1 - \alpha_2)
\]

\[
S_{zz} = \frac{2}{3}u(2K+1)(\alpha_1 + \alpha_2) + \frac{2}{3}u(K-1)\sin 2\phi (\alpha_1 - \alpha_2)
\]

\[
S_{xy} = \frac{2}{3}uvKH_z \left[ (K+2)(\beta_1 + \beta_2) + (K-1)\sin 2\phi (\beta_1 - \beta_2) \right] = -S_{yx}
\]

\[
S_{yz} = \frac{2}{3}u(K-1)\cos 2\phi (\alpha_2 - \alpha_1) = S_{zy}
\]

\[
S_{zx} = \frac{2}{3}uvKH_z(K-1)\cos 2\phi (\beta_2 - \beta_1) = -S_{xz}
\]

II. \( \mathbf{H} \) in (100) plane, [100] ellipsoids

The three cyclotron frequencies are

\[
\begin{aligned}
\mathcal{L}_1^2 &= \frac{v^2KH_z^2}{3} \\
\mathcal{L}_2^2 &= \frac{v^2KH_z^2}{3} \left[ \cos^2\phi + K \sin^2\phi \right] \\
\mathcal{L}_3^2 &= \frac{v^2KH_z^2}{3} \left[ \sin^2\phi + K \cos^2\phi \right]
\end{aligned}
\quad (B.3)
\]
The tensor components are
\[ S_{xx} = uK \sum_{i=1}^{3} \alpha_i - u(K-1)\alpha_1 \]
\[ S_{yy} = uK \sum_{i=1}^{3} \alpha_i - u(K-1)(\alpha_2 \cos^2 \phi + \alpha_3 \sin^2 \phi) \]
\[ S_{zz} = uK \sum_{i=1}^{3} \alpha_i - u(K-1)(\alpha_2 \sin^2 \phi + \alpha_3 \cos^2 \phi) + \]
\[ \sum_{i=1}^{3} \gamma_i \]
\[ S_{xy} = -uvKH \left[ \sum_{i=1}^{3} \beta_i + (K-1)(\beta_2 \sin^2 \phi + \beta_3 \cos^2 \phi) \right] = -S_{yx} \]
\[ S_{yz} = u \sin \phi \cos \phi (K-1)(\alpha_3 - \alpha_2) = S_{zy} \]
\[ S_{zx} = uvKH \sin \phi \cos \phi (K-1)(\beta_3 - \beta_2) = -S_{xz} \]

III \( H \) in \( [110] \) plane, \( [111] \) ellipsoids
(x along \( [110] \) direction)

The three cyclotron frequencies are
\[ \mathcal{E}_1^2 = v^2KH_z^2/3 \left[ K + 2 + (K-1)(\sin^2 \phi + 2\frac{1}{3} \sin 2\phi) \right] \]
\[ \mathcal{E}_2^2 = v^2KH_z^2/3 \left[ K + 2 - (K-1) \sin^2 \phi \right] = \mathcal{E}_4^2 \]
\[ \mathcal{E}_3^2 = v^2KH_z^2/3 \left[ K + 2 + (K-1)(\sin^2 \phi - 2\frac{1}{3} \sin 2\phi) \right] \]

(B.5)
The tensor components are

\[ S_{xx} = \frac{u}{3} (2K+1) \sum_{i=1}^{4} \alpha_i - \frac{u}{3} (K-1) \sum_{i=1}^{4} (-1)^i \alpha_i \]

\[ S_{yy} = \frac{u}{3} (2K+1) \sum_{i=1}^{4} \alpha_i + \frac{u}{3} (K-1) \cos^2 \phi \sum_{i=1}^{4} (-1)^i \alpha_i - \frac{u}{3} (K-1)^2 \sin 2\phi (\alpha_1 - \alpha_2) \]

\[ S_{zz} = \frac{u}{3} (2K+1) \sum_{i=1}^{4} \alpha_i + \frac{u}{3} (K-1) \sin^2 \phi \sum_{i=1}^{4} (-1)^i \alpha_i + \]

\[ \frac{u}{3} (K-1) \frac{2}{3} \sin 2\phi (\alpha_1 - \alpha_2) + \frac{u^2}{6} K^2 H_z^2 \sum_{i=1}^{4} \beta_i \]

\[ S_{xy} = -\frac{u v H_z}{K} \left[ (K+2) \sum_{i=1}^{4} \beta_i - (K-1) \sin^2 \phi \sum_{i=1}^{4} (-1)^i \beta_i + (K-1)^2 \sin 2\phi (\beta_1 - \beta_2) \right] = -S_{yx} \]

\[ S_{yz} = \frac{u}{3} (K-1) \left[ \sin \phi \cos \phi \sum_{i=1}^{4} (-1)^i \alpha_i + 2 \sin \phi \cos 2\phi (\alpha_3 - \alpha_1) \right] = -S_{zy} \]

\[ S_{zx} = \frac{u v H_z}{K} \left[ \sin \phi \cos \phi \sum_{i=1}^{4} (-1)^i \beta_i + 2 \sin \phi \cos 2\phi (\beta_3 - \beta_1) \right] = -S_{xz} \]
IV. In (110) plane, [100] ellipsoids

The two cyclotron frequencies are

\[ \mathcal{I}_1^2 = \nu^2 K H z \left[ \frac{(K+1)}{2} \sin^2 \phi + \cos^2 \phi \right] = \mathcal{I}_2^2 \]

\[ \mathcal{I}_3^2 = \nu^2 K H z \left[ \sin^2 \phi + K \cos^2 \phi \right] \]

The tensor components are

\[ S_{xx} = u K \sum_{i=1}^{3} \alpha_i - u(K-1) \alpha_1 \]

\[ S_{yy} = u K \sum_{i=1}^{3} \alpha_i - u(K-1)(\alpha_1 \cos^2 \phi + \alpha_3 \sin^2 \phi) \]

\[ S_{zz} = u K \sum_{i=1}^{3} \alpha_i - u(K-1)(\alpha_1 \sin^2 \phi + \alpha_3 \cos^2 \phi) + \]

\[ u v^2 K H z \sum_{i=1}^{3} \gamma_i \]

\[ S_{xy} = -u v K H z \left[ \sum_{i=1}^{3} \beta_i + (K-1)(\beta_1 \sin^2 \phi + \beta_3 \cos^2 \phi) \right] \]

\[ S_{yz} = u \sin \phi \cos \phi (K-1)(\alpha_3 - \alpha_1) = S_{zy} \]

\[ S_{zx} = u v K H \sin \phi \cos \phi (K-1)(\beta_3 - \beta_1) = -S_{xz} \]

V. In (111) plane, [111] ellipsoids

(x and y equally inclined to 1 and 2 axes respectively)

The cyclotron frequencies are again given by (B.5).

The tensor components are
\[ S_{xx} = \frac{u}{3}(2K+1) \sum_{i=1}^{4} \alpha_i - \frac{u}{6}(K-1)\sin^2 \phi \sum_{i=1}^{4} (-1)^i \alpha_i - \frac{u}{3}(K-1)\sin 2 \phi (\alpha_3 - \alpha_1) = S_{yy} \]

\[ S_{zz} = \frac{u}{3}(2K+1) \sum_{i=1}^{4} \alpha_i + \frac{u}{3}(K-1)\sin^2 \phi \sum_{i=1}^{4} (-1)^i \alpha_i + \frac{u}{3}(K-1)\frac{1}{2} \sin 2 \phi (\alpha_3 - \alpha_1) + uvz^2 (K-2) \sum_{i=1}^{4} \gamma_i \]

\[ S_{xy} = \frac{u}{3}(K-1) \left( \frac{1+\cos^2 \phi}{2} \right) \sum_{i=1}^{4} (-1)^i \alpha_i - \frac{u}{3}(K-1)\sin 2 \phi (\alpha_3 - \alpha_1) - \frac{nvz^2}{3}(K+2) \sum_{i=1}^{4} \beta_i - (K-1)\frac{1}{2} \sin 2 \phi (\beta_3 - \beta_1) \]

\[ S_{yx} = \frac{u}{3}(K-1) \left( \frac{1+\cos^2 \phi}{2} \right) \sum_{i=1}^{4} (-1)^i \alpha_i - \frac{u}{3}(K-1)\sin 2 \phi (\alpha_3 - \alpha_1) + \frac{nvz^2}{3}(K+2) \sum_{i=1}^{4} \beta_i - (K-1)\frac{1}{2} \sin 2 \phi (\beta_3 - \beta_1) \]

\[ S_{yz} = \frac{u}{3}(K-1) \left[ \sin \phi \cos \phi \frac{1}{2^\frac{1}{2}} \sum_{i=1}^{4} (-1)^i \alpha_i + \cos 2 \phi (\alpha_3 - \alpha_1) \right] + \frac{nvz(K-1)H^2}{3}(K+2) \left[ \sin \phi \cos \phi \frac{1}{2^\frac{1}{2}} \sum_{i=1}^{4} (-1)^i \beta_i + \cos 2 \phi (\beta_3 - \beta_1) \right] \]

\[ = S_{zx} \]
The cyclotron frequencies are again given by \((B.7)\).

The tensor components are

\[
S_{xx} = \frac{u}{2} \left( \sum_{i=1}^{3} \alpha_i \right) + (K-1)(\alpha_1 \sin^2 \phi + \alpha_3 \cos^2 \phi) \]

\[
S_{yy} = \frac{u}{2} \left( \sum_{i=1}^{3} \alpha_i \right) - (K-1)(\alpha_1 \sin^2 \phi + \alpha_3 \cos^2 \phi) \]

\[
S_{zz} = uK \sum_{i=1}^{3} \alpha_i - u(K-1)(\alpha_1 \sin^2 \phi + \alpha_3 \cos^2 \phi) + uvK^2H_z \sum_{i=1}^{3} \beta_i \]

\[
S_{xy} = -\frac{u}{2}(K-1)\sin^2 \phi(\alpha_3 - \alpha_1) \]

\[
u v K H_z \left[ \sum_{i=1}^{3} \beta_i + (K-1)(\beta_1 \sin^2 \phi + \beta_3 \cos^2 \phi) \right] \]

\[
S_{yx} = -\frac{u}{2}(K-1)\sin^2 \phi(\alpha_3 - \alpha_1) + \]

\[
u v K H_z \left[ \sum_{i=1}^{3} \beta_i + (K-1)(\beta_1 \sin^2 \phi + \beta_3 \cos^2 \phi) \right] \]
\[ S_{yz} = u(K-1) \sin \phi \cos \phi / 2^b \left[ (\alpha_3 - \alpha_1) + vKH_z (\beta_3 - \beta_1) \right] \]

\[ = S_{zx} \]

\[ S_{zy} = u(K-1) \sin \phi \cos \phi / 2^b \left[ (\alpha_3 - \alpha_1) - vKH_z (\beta_3 - \beta_1) \right] \]

\[ = S_{xz} \]
Appendix C

Anisotropic Components of the Magnetoconductivity Tensor

(Series Solution)

The nine fourth rank tensor components required to evaluate the Faraday effect are given for the orientations of $\mathbf{H}$ discussed in chapter 4.

I \hspace{1em} \mathbf{H} \text{ in (100) plane, } [111] \text{ ellipsoids}

\[ \sum zzzz = -\frac{3}{9} uv^2 K(K-1)^2(1 - 2 \sin^2 \phi \cos^2 \phi) C(\omega) \]
\[ \sum xzzz = -\frac{4}{9} uv^2 K(2K+1)(K+2) C(\omega) \]
\[ \sum yzzz = -\frac{4}{9} uv^2 (2K+1)(K+2) C(\omega) \]
\[ \sum yzzz = \frac{4}{9} uv^2 (K-1)^2 \sin 2\phi \cos 2\phi C(\omega) = \sum_{zyzz} \]
\[ \sum xzzz = \sum_{zzzz} = \sum_{xzzz} = \sum_{yzzz} = 0 \]

II \hspace{1em} \mathbf{H} \text{ in (100) plane, } [100] \text{ ellipsoids}

\[ \sum zzzz = -\frac{u}{2} v^2 K(K-1)^2 \sin^2 2\phi C(\omega) \]
\[ \sum xzzz = -uv^2 K(K^2+K+1) C(\omega) \]
\[ \sum yzzz = -uv^2 \left[ K^2+K+1 - \frac{(K-1)^2}{2} \sin^2 2\phi \right] C(\omega) \]
\[ \sum yzzz = -\frac{u}{2} v^2 (K-1)^2 \sin 2\phi \cos 2\phi C(\omega) = \sum_{zyzz} \]
\[ \sum xzzz = \sum_{zzzz} = \sum_{xzzz} = \sum_{yzzz} = 0 \]
III \( \mathbb{H} \) in (1\{1\}) plane, [111] ellipsoids
(x along [11\{1\}] direction)

\[ \Sigma_{zzzz} = - \frac{8}{9} uv^2 K(K-1)^2 \left[ 1 - \frac{\sin^2 \phi}{2} (3 \cos^2 \phi + 1) \right] \cos(\omega) \]

\[ \Sigma_{xxzz} = - \frac{4}{9} uv^2 K \left[ (2K+1)(K+2) + (K-1)^2 \sin^2 \phi \right] \cos(\omega) \]

\[ \Sigma_{yyzz} = - \frac{4}{9} uv^2 K \left[ (2K+1)(K+2) + 3(K-1)^2 \sin^2 \phi \cos^2 \phi \right] \cos(\omega) \] . (C.3)

\[ \Sigma_{yzzz} = \frac{4}{9} uv^2 K(K-1)^2 \sin \phi \cos \phi (3 \cos^2 \phi - 1) \cos(\omega) = \Sigma_{zyzz} \]

\[ \Sigma_{xxzz} = \Sigma_{xzzz} = \Sigma_{xyzz} = \Sigma_{yxzz} = 0 \]

IV \( \mathbb{H} \) in (1\{1\}) plane, [\{1\}00] ellipsoids
(x along [11\{1\}] direction)

\[ \Sigma_{zzzz} = - \frac{u}{2} v^2 K(K-1)^2 \sin^2 \phi (3 \cos^2 \phi + 1) \cos(\omega) \]

\[ \Sigma_{xxzz} = - uv^2 K \left[ K^2 + K + 1 - \frac{(K-1)^2}{2} \sin^2 \phi \right] \cos(\omega) \]

\[ \Sigma_{yyzz} = - uv^2 K \left[ K^2 + K + 1 - 3 \frac{(K-1)^2}{2} \sin^2 \phi \cos^2 \phi \right] \cos(\omega) \] . (C.4)

\[ \Sigma_{yzzz} = - \frac{u}{2} v^2 K(K-1)^2 \sin \phi \cos \phi (3 \cos^2 \phi - 1) \cos(\omega) = \Sigma_{zyzz} \]

\[ \Sigma_{xxzz} = \Sigma_{xzzz} = \Sigma_{xyzz} = \Sigma_{yxzz} = 0 \]

V \( \mathbb{H} \) in (1\{1\}) plane, [111] ellipsoids
(x and y equally inclined to 1 and 2 axes respectively)

\[ \Sigma_{zzzz} = - \frac{8}{9} uv^2 K(K-1)^2 \left[ 1 - \frac{\sin^2 \phi}{2} (3 \cos^2 \phi + 1) \right] \cos(\omega) \]
\[
\sum_{xxzz} = -\frac{4uv^2}{9}K \left[ (K+2)(2K+1) + \frac{(K-1)^2}{4} \sin^2 \phi (3\cos^2 \phi + 1) \right] C(\omega) \\
= \sum_{yyzz}
\]

\[
\sum_{zyzz} = \frac{4uv^2}{9}K \left[ \frac{(K-1)^2}{2} \sin \phi \cos \phi (3\cos^2 \phi - 1) \right] C(\omega) \\
= \sum_{yzzz} = \sum_{zzzz} = \sum_{xzzz}
\]

\[
\sum_{xysz} = -\frac{2}{9}uv^2K(K-1)^2 \sin^2 \phi (3\cos^2 \phi - 1) C(\omega) \\
= \sum_{yxzz}
\]

VI. \( \{110\} \) plane, \([100]\) ellipsoids
(x and y equally inclined to 1 and 2 axes respectively)

\[
\sum_{zzzz} = -\frac{u}{2}v^2K(K-1)^2 \sin^2 \phi (3\cos^2 \phi + 1) C(\omega) \\
\sum_{xxzz} = -uv^2K \left[ K^2 + K + 1 - \frac{(K-1)^2}{4} \sin^2 \phi (3\cos^2 \phi + 1) \right] C(\omega) \\
= \sum_{yyzz}
\]

\[
\sum_{yzzz} = uv^2K \left[ \frac{(K-1)^2}{2} \sin \phi \cos \phi (3\cos^2 \phi - 1) \right] C(\omega) \\
= \sum_{zyzz} = \sum_{zzzzz} = \sum_{xzzzz}
\]

\[
\sum_{xyzz} = uv^2K \left[ \frac{(K-1)^2}{4} \sin^2 \phi (3\cos^2 \phi - 1) \right] C(\omega) \\
= \sum_{yxzz}
\]
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The Faraday Effect in Non-degenerate Semiconductors

BY B. DONOVAN AND JANET WEBSTER
Department of Physics, Bedford College, London

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Abstract. The theory of the Faraday effect, due to free charge carriers, is developed for non-degenerate semiconductors with spherical energy surfaces. General expressions for the Faraday rotation $\theta$ and the ellipticity $\delta$ are given, which are valid for all frequencies from the microwave region to the infra-red, and for all magnetic field strengths within the limitations of the Boltzmann equation.

The cases of lattice scattering and impurity scattering are considered separately and the numerical calculations are based on experimental data for n-type germanium. The rotation $\theta$ changes sign at the cyclotron resonance frequency in strong fields and at a lower frequency (determined by the relaxation time) in weak fields. The ellipticity $\delta$ passes through zero at a frequency corresponding to maximum $\theta$, and reaches a maximum when $\theta$ is zero. In the limit of high frequencies and weak fields $\theta$ (but not $\delta$) is independent of the relaxation mechanism. Both $\theta$ and $\delta$ vary linearly with field strength in weak fields but subsequently pass through a maximum.

The temperature dependence of $\theta$ and $\delta$ is examined in the range 100–300 °K for lattice scattering and 15–100 °K for impurity scattering. As the temperature is lowered, in general $\theta$ and $\delta$ both increase in the case of lattice scattering, and both decrease in the case of impurity scattering.

§ 1. Introduction

When a plane polarized electromagnetic wave passes through a medium in the presence of a constant magnetic field parallel to the direction of propagation, a rotation of the plane of polarization takes place. This effect is known as the Faraday rotation and its exploitation in connection with semiconductors is a comparatively recent development. The rotation was observed at microwave frequencies in germanium by Rau and Caspari (1955) and, more recently, it has been investigated in the infra-red region in indium antimonide (Smith, Moss and Taylor 1959), in germanium (Walton and Moss 1959) and in bismuth telluride (Austin 1960). In these experiments the rotation is due to the free charge carriers and the infra-red Faraday effect has provided a valuable additional method for obtaining information on the effective mass parameters (for a detailed account see Moss 1959).

For a degenerate system of free electrons the theory of the Faraday effect follows straightforwardly from the high frequency modification of the Boltzmann equation, and the complex propagation constants, which determine the rotation, have been given for this case (but excluding the displacement current) by Donovan (1954). The free-electron model has also been used by Stephen and Lidiard (1959, to be referred to as SL) to discuss the frequency dependence of
the Faraday rotation and they have also derived particular results applicable to non-spherical energy surfaces.

The purpose of the present paper is to put forward a theory more appropriate to semiconductors, in which the relaxation time of the carriers is given the proper dependence upon energy in the cases of lattice scattering and impurity scattering. The behaviour of the Faraday rotation as a function of frequency, magnetic field strength and temperature is discussed, and a similar study is made of the ellipticity of the radiation, which has so far received little attention. For the purposes of numerical calculation, experimental data relevant to n-type germanium have been incorporated where necessary. The present calculations are based on the simple quasi-free electron model, for which the surfaces of constant energy are spherical and the density of states has the normal parabolic form. The additional complications arising from anisotropic energy surfaces will be dealt with in a subsequent paper.

§ 2. General Formulation

A system of axes is chosen such that the incident radiation is propagated along the positive z direction. The electric field in the medium is then specified by \((\varepsilon_x, \varepsilon_y, 0) \exp (i\omega t)\), where \(\omega\) is the angular frequency, and the constant magnetic field has components \((0, 0, H)\). Inside the medium the initially plane polarized radiation may be regarded as split into two circularly polarized waves, rotating in opposite senses and represented by \(\varepsilon_x = \varepsilon_x \pm i\varepsilon_y\), where the positive sign denotes clockwise rotation to an observer looking along the positive z direction.

These two circularly polarized waves travel with different velocities and undergo different attenuations so that, in general, the resultant wave is elliptically polarized. We define the ellipticity \(\delta\) as the ratio of the minor to the major axis of the ellipse, and the rotation \(\theta\) as the angle between the major axis of the ellipse and the initial direction of the electric vector.

The essential problem is the evaluation of the complex propagation constants for the two component waves and we now give a general formulation for a system of carriers with relaxation time \(\tau\) and effective mass \(m^*\), described by the Fermi-Dirac function \(f_\nu(E)\), where the energy \(E\) is a quadratic function of momentum. The calculation of the current density from the Boltzmann equation follows standard lines and the analysis may be rendered more concise if, as first pointed out by Dingle (1956), the symbol \(i\) is reserved for phase displacement and the operator \(j\) is introduced for spatial rotation round the magnetic field. Thus, if \(a\) and \(b\) are two harmonically varying quantities confined to the \((x, y)\) plane, the combination \(a + jb\) signifies that \(a\) and \(b\) differ in phase by \(\frac{1}{2}\pi\), whereas \(a + j b\) signifies an angular displacement of \(\frac{1}{2}\pi\) between the directions of \(a\) and \(b\). Since \(i^2 = j^2 = -1\), but \(ij \neq -1\), the process of rationalization must be carried out separately for \(i\) and \(j\).

Thus, by introducing \(\varepsilon_x = \varepsilon_x + j\varepsilon_y\) and \(\mathbf{J} = J_x + jJ_y\), we have (cf. Moore 1958)

\[
\mathbf{J} = -\left[\frac{16}{3\hbar^2} \int_0^\infty \frac{\tau E_{\nu}^2}{1 + (i\omega + j\Omega)\tau} \frac{\partial f_\nu}{\partial E} dE\right] \mathbf{E}, \quad \ldots \ldots \quad (1)
\]

where \(\Omega = eH/m^*c\), i.e. the cyclotron resonance frequency (Gaussian units are used throughout).

It is thus possible to express \(\mathbf{J}\) in the form

\[
\mathbf{J} = (M + jN) \mathbf{E} \quad \ldots \ldots \quad (2)
\]
where $2M = I_1(\omega + \Omega) + I_1(\omega - \Omega) - i[(\omega + \Omega)I_2(\omega + \Omega) + (\omega - \Omega)I_2(\omega - \Omega)]$  

$2N = -[(\omega + \Omega)I_2(\omega + \Omega) - (\omega - \Omega)I_2(\omega - \Omega)] - i[I_1(\omega + \Omega) - I_1(\omega - \Omega)]$  

and 

$I_\pm(\xi) = -\frac{16\sqrt{2}\pi^2\mu_0}{3\hbar^2} \int_0^\infty \frac{\tau^3E_3^{\text{sh}^2} \partial_\tau^2 E}{1 + \xi^2 \tau^2} \text{d}E$.  

If we now combine these results with Maxwell's equations, we obtain 

$$\frac{d^2 E}{dz^2} = (A + jB)E$$  

where 

$$A = \frac{4\pi \omega i}{c^2} \left(M + i\omega \epsilon\right), \quad B = \frac{4\pi \omega i}{c^2} N,$$

and $\epsilon$ is the dielectric constant.

At this stage we may conveniently obtain the two circularly polarized waves by setting $j = \pm i$, and if we write 

$$\mathcal{E}_\pm(z) = \mathcal{E}_\pm(0) \exp(-\mu_\pm z),$$

we may derive the complex propagation constants from (6) in the form

$$\mu_\pm = \alpha_\pm + j\beta_\pm = (A \pm iB)^{1/2}.$$  

Substitution from (3) and (4) leads finally to the expressions

$$\mu^2 = -\frac{\omega^2 \epsilon}{c^2} + \frac{4\pi \omega}{c^2} (\omega + \Omega)I_2(\omega + \Omega) + \frac{4\pi \omega i}{c^2} I_1(\omega + \Omega),$$

and the problem reduces to the extraction of the real and imaginary parts of the $\mu_{\pm}$, following evaluation of the integrals in (5). For a highly degenerate electron gas, we may note that, from (5):

$$I_1(\xi) = \frac{\sigma_0}{1 + \xi^2 z}, \quad I_2(\xi) = \frac{\sigma_0}{{\xi^2 + z^2}},$$

where $\sigma_0$ is the d.c. conductivity, and in this case (9) is identical with Eqn (2.6) of SL.

Having evaluated $\alpha_\pm$ and $\beta_\pm$ (all of which must be positive), the parameters of the Faraday effect may be obtained for a given distance $z$ travelled in the specimen. The rotation is given by

$$\theta = \frac{1}{2}(\beta_+ - \beta_-)z,$$  

and is positive when it appears clockwise to an observer looking along the positive $z$ direction (i.e. in sense of rotation of $\mathcal{E}_\pm$). The ellipticity is given by

$$\delta = \frac{\exp(-\alpha_+ z) - \exp(-\alpha_- z)}{\exp(-\alpha_+ z) + \exp(-\alpha_- z)} = \tanh \left(\frac{1}{2}(\alpha_+ - \alpha_-)z\right),$$  

and the choice of sign is entirely arbitrary. With the convention of (11), $\theta$ and $\delta$ tend to the same expression at very low frequencies and in this region both are negative for electrons. If the attenuations are nearly equal a plane polarized wave will be obtained ($\delta \rightarrow 0$), but if they are widely different $\delta$ will approach unity, corresponding to the suppression of one of the circularly polarized components. The expressions (10) and (11) are both exact, but the ellipticity may be written in the approximate form

$$\delta = \frac{1}{2}(\alpha_+ - \alpha_-)z$$  

provided that $|\alpha_+ - \alpha_-|z| < 1$.

$^\dagger$ The sign convention is stated incorrectly in SL.
§ 3. Single Band Model with Lattice Scattering

We now apply the results of the previous section to a semiconductor in which conduction is due predominantly to a single type of carrier and the system is described by Boltzmann statistics. Accordingly, for the distribution function we write

\[ f_0(E) = \frac{1}{\pi n h^3} (2\pi m^* kT)^{-3/2} \exp \left( -\frac{E}{kT} \right), \quad \ldots \ldots (13) \]

where \( n \) is the carrier density.

For the case of scattering by the lattice vibrations we assume the mean free path \( l \) to be independent of energy and the relaxation time may be expressed as

\[ \tau = ax^{-1/2}, \quad \ldots \ldots (14) \]

where \( x = E/kT \) and \( a = l(m^*/2kT)^{1/2} \). Substituting (13) and (14) in (5) leads to the essential integrals, in terms of \( \xi = \omega \pm \Omega \),

\[ I_1(\xi) = a_0 \int_0^\infty \frac{x^3 e^x}{x + a^2 x^2} \, dx \quad \ldots \ldots (15) \]

and

\[ I_2(\xi) = a_0 \int_0^\infty \frac{x^3 e^x}{x + a^2 x^2} \, dx \quad \ldots \ldots (16) \]

where we have introduced the d.c. conductivity

\[ \sigma_0 = \frac{4ne^2}{3(2\pi mk^* T)^{1/2}} = \frac{4ne^2a}{3\pi^{1/2}m^{1/2}}. \quad \ldots \ldots (17) \]

The above integrals have been discussed previously in connection with high frequency conductivity in semiconductors (Donovan and March 1956). They have since been tabulated by Dingle, Arndt and Roy (1957), but at some stages of the present calculation, particularly for small values of \( \xi \), the tables were found to be insufficiently detailed and additional numerical integration was required.

In general the expressions (10) and (11) must be evaluated numerically but approximate forms may be derived in certain limiting cases. For very low frequencies (such that the displacement current is negligible) and weak fields, we have

\[ \frac{\theta}{2H} = \frac{\delta}{2H} = \frac{R_0}{c} \left( \frac{\pi \omega \sigma_0^2}{2} \right)^{1/2}, \quad \ldots \ldots (18) \]

where \( R_0 \) is the Hall constant (\( = -3\pi/8ne \)). A similar expression for \( \theta \) was derived for the degenerate case by SL (Eqn (2.16)). For high frequencies and weak fields (\( \omega a \gg 1, \omega \gg \Omega \)), we find

\[ \frac{\theta}{2H} = \frac{2\pi ne}{c^2 e^{1/2} m^* \omega}, \quad \ldots \ldots (19) \]

and

\[ \frac{\delta}{2H} = \frac{32\pi ne}{3c^2 e^{1/2} m^* \omega a}. \quad \ldots \ldots (20) \]

It should be noted that in these circumstances the rotation is independent of \( a \) whereas the ellipticity is not.

3.1. Faraday Rotation

For specific numerical calculation we take initially \( \sigma_0 = 10^{11} \) c.s.u., \( a = 10^{-12} \) sec, \( c = 16 \), and we shall consider these values as applying at \( T = 300^{\circ}K \) to a typical specimen of n-type germanium satisfying the conditions of the present section.
Thus the room temperature parameters are:

resistivity $= 9 \text{ ohm cm}$, $n \approx 1.3 \times 10^{14} \text{ cm}^{-3}$, $m^* = \frac{1}{2} m$

(Debye and Conwell 1954; our data correspond closely to sample 51).

Figure 1 shows the Faraday rotation plotted as a function of frequency for various values of the field strength (with the effective mass quoted above, $\Omega = 10^{11}$ corresponds to $H \approx 1400 \text{ Oe}$). The general behaviour of $\theta$ is similar to that described by SL; in particular, a change in sign occurs at $\omega \sim a^{-1}$ for weak fields ($a\Omega \ll 1$) and at the cyclotron resonance frequency, $\omega \sim \Omega$, for strong fields ($a\Omega \gg 1$). An additional feature exhibited in strong fields is the small minimum at low frequencies ($\omega \sim 10^{10}$ in curve $c$). This is a consequence of the cancellation of the first two terms on the right-hand side of expression (9) and hence, in the present instance, it occurs (provided $\omega \ll \Omega$) at $\omega \sim 3\pi a^2 \sigma_0/\epsilon \Omega a$. Above this frequency $\theta$ is independent of $\omega$ almost up to the cyclotron resonance frequency.

![Graph showing Faraday rotation vs. frequency](image)

**Fig. 1.** Variation of Faraday rotation ($10^6 \theta \text{ rad cm}^{-1}$) with frequency for n-type germanium specimen with $\sigma_0 = 10^{11}$ e.s.u., $a = 10^{-11} \text{ sec}$, $m^* = \frac{1}{2} m$, $\epsilon = 16$ (lattice scattering only): $a$, $\Omega = 10^{11} (p = 1)$; $b$, $\Omega = 10^{12} (p = 0)$; $c$, $\Omega = 10^{13} (p = 0)$.

The dependence of the Faraday rotation upon field strength is complicated by the change in sign illustrated in Fig. 1. For small fields ($\Omega \leq 10^{11}$) $\theta$ is proportional to $H$ over the entire frequency range shown in Fig. 1. For $\omega < a^{-1}$ the rotation is always negative and the variation with field strength is shown in Fig. 2. The rotation reaches a maximum value in the neighbourhood of $\Omega a = 1$ and subsequently decreases to zero. For $\omega > a^{-1}$ the rotation is positive for small fields and proportional to $H$, in accordance with (19); as the field is increased $\theta$ will eventually change sign abruptly at $\Omega \sim a$, pass through a minimum and ultimately decrease to zero. For very strong fields, such that $\Omega a \gg 1$, conclusions based on the Boltzmann equation must, however, be treated with reserve.

It is also of interest to consider the variation of $\theta$ with temperature, at a fixed frequency and field strength. For this purpose we have taken the mean free path to be inversely proportional to $T$, so that $a \propto T^{-3/2}$; furthermore, over the temperature range considered, the carrier density may be regarded as sensibly constant (Debye and Conwell 1954, Fig. 10). Some typical results are illustrated in Fig. 3 (the data of Fig. 1 corresponding to $T = 300^\circ \text{K}$) and it will be seen that, in
The Faraday Effect in Non-degenerate Semiconductors

In general, the magnitude of \( \theta \) increases steadily as the temperature decreases. In the microwave region, with \( \Omega = 10^{11} \), \( \theta \) increases by a factor of 10 in cooling from room temperature to 100°K. For the same field strength and \( \omega = 10^{12} \), \( \theta \) is negative at 300°K but passes through zero and becomes appreciably larger and positive with decreasing temperature. In strong fields the variation with \( T \) is

Fig. 2. Variation of Faraday rotation with field strength for n-type germanium specimen with details as for Fig. 1 (lattice scattering only): \( a, \omega = 10^8; \ b, \omega = 10^{11}; \ c, \omega = 10^{12} \)
(Note: all values of \( \theta \) are negative.)

Fig. 3. Variation of Faraday rotation with temperature for n-type germanium specimen with \( c_p = 10^{11} \) e.s.u. and \( \sigma = 10^{-13} \) sec at \( T = 300^\circ K \); \( m^* = \frac{1}{2} m; \ e = 16 \) (lattice scattering only): \( a, \omega = 10^9; \ b, \omega = 10^{11}; \ c, \omega = 10^{12}; \ d, 5 \times 10^{10} < \omega < 10^{12} \).
extremely small (except in the vicinity of cyclotron resonance) as shown by the broken line, which represents all frequencies on the flat portion of Fig. 1, curve c.

In considering the application of the present theory to experimental results, we may note that the infra-red measurements may be interpreted in terms of expression (19), which is independent of the state of degeneracy. A significant comparison may, however, be made with the microwave experiments of Rau and Caspari (1955); in particular we consider their sample of n-type germanium, with a resistivity of 16 ohm cm. Taking the experimental value of the mobility to be 3900 cm\(^2\) V\(^{-1}\) sec\(^{-1}\) and using the previous effective mass \(m^* = \frac{1}{3}m\), we find \(\alpha = 7.44 \times 10^{-13}\) sec. This effective mass gives \(\Omega = 1.01 \times 10^{11}\) for the quoted field strength of 1430 Oe. At the frequency of the measurement (\(\omega = 5.49 \times 10^{10}\)) the present theory then gives \(\theta = 4.37^\circ\) for the quoted path length of 0.46 cm, as compared with the value 3.5\(^\circ\) obtained by Rau and Caspari. If the effective mass is changed (thus altering \(\alpha\) and \(\Omega\)), keeping the mobility constant, the rotation does not vary appreciably. Thus with \(m^* = \frac{1}{3}m\) we find \(\theta = 4.30^\circ\) and with \(m^* = \frac{1}{2}m\), \(\theta = 4.43^\circ\). The fact that the observed value is smaller than this estimate may possibly be due to the effect of positive holes, thus indicating that a single-band model is not adequate for this particular specimen.

3.2. Ellipticity

The essential function which governs the ellipticity is \(\frac{1}{2}(\alpha_+ - \alpha_-)\) and from this \(\delta\) may be obtained, using (11), for any desired value of the specimen thickness. Using the same values for \(c_0\), \(\alpha\), \(m^*\) and \(\epsilon\) as in § 3.1, the expression \(\frac{1}{2}(\alpha_+ - \alpha_-)\) has been evaluated as a function of frequency and the results are shown in Fig. 4.

![Fig. 4. Variation of \(\frac{1}{2}(\alpha_+ - \alpha_-)\) with frequency for n-type germanium specimen with details as for Fig. 1 (lattice scattering only): \(a, \Omega = 10^{11}; b, \Omega = 10^{12}; c, \Omega = 10^{13}\). The broken lines represent ellipticities calculated for 1 cm path length.](image)

Curves for the ellipticity would have the same general characteristics, but with the peaks flattened; as an illustration the broken lines in Fig. 4 show \(\delta\) calculated for a path length \(z\) of 1 cm. In moderate fields the ellipticity passes through zero at a frequency corresponding to the maximum negative value of \(\theta\) and rises to a
maximum in the neighbourhood of zero \( \theta \). In the optical region the radiation is effectively plane polarized.

In strong fields \( \delta \) is very small over a wide band of frequencies, corresponding roughly to the flat portion of Fig. 1, curve \( c \), and exhibits a sharp peak at cyclotron resonance. Here \( \delta \) attains the value unity, which is to be expected in view of the absorption of one of the circularly polarized components. It should be noted that the polarization may be almost circular also at \( \omega \tau \sim 1 \), for appropriate values of the field, as shown in curve \( b \). At constant frequency \( \delta \) varies linearly with field strength for small \( \Omega \) and subsequently passes through a maximum and decreases to zero after the manner of Fig. 2.

![Diagram](image)

Fig. 5. Variation of ellipticity with temperature for 1 cm path length in n-type germanium specimen with \( \sigma_0 = 10^{12} \) e.s.u. and \( a = 10^{-12} \) sec at \( T = 300^\circ \text{K} \); \( m^* = \frac{1}{2} m_0 \), \( \epsilon = 16 \) (lattice scattering only): \( a, \omega = 10^{19} \); \( b, \omega = 5 \times 10^{19} \); \( c, \omega = 10^{11} \); \( d, \omega = 10^{12} \); \( e, \omega = 10^{10} \).

As a typical illustration of the temperature dependence of the ellipticity, the expression (11) has been evaluated with \( z = 1 \) cm and the results are shown in Fig. 5 (the data of Fig. 4 corresponding to \( T = 300^\circ \text{K} \)). In moderate fields the magnitude of the ellipticity increases with decreasing \( T \) and ultimately passes through a maximum at a temperature which depends upon the frequency. For \( \omega = 10^{11} \) (curve \( c \)) the turning point occurs at \( T \sim 130^\circ \text{K} \); for lower frequencies (curves \( a \) and \( b \)) \( |\delta| \) is still increasing at \( T \sim 100^\circ \text{K} \), and in this region ellipticities approaching unity should be obtainable at wavelengths around 3 cm. In strong fields the variation with \( T \) is greatly reduced, as in the case of \( \theta \), and as shown by the broken line \( |\delta| \) diminishes slowly as the temperature decreases.

§ 4. SINGLE BAND MODEL WITH IMPURITY SCATTERING

We now take up the case of an impurity semiconductor, in which the predominant scattering mechanism is that due to the impurity ions. The calculations in the present section are based on the treatment of Conwell and Weisskopf (1950), which leads to a mean free path proportional to the square of the electronic energy. Thus for the relaxation time we write

\[
\tau = g \lambda^{3/2},
\]

\( .... (21) \)
with \( x = E/kT \) as before and

\[
g = \frac{(2m^*)^{1/2} (kT)^{m^*} n^2}{\pi Ne^4 \log (1 + 36e^2 dr^2 T^4/e^4)},
\]

where \( N \) is the number of impurity ions per unit volume and \( 2d \) their average separation.

The essential integrals for this case are obtained by substituting (21) in (5), together with the distribution function (13); in this way we find

\[
I_1(\xi) = \frac{\sigma_0}{6} \int_0^\infty \frac{x^3 \exp (-x)}{1 + e^x \xi^2 x^3} \, dx \quad \ldots \ldots \quad (22)
\]

and

\[
I_2(\xi) = \frac{g\sigma_0}{6} \int_0^\infty \frac{x^{3/2} \exp (-x)}{1 + e^{2\xi^2 x^3}} \, dx \quad \ldots \ldots \quad (23)
\]

where the d.c. conductivity is now given by

\[
\sigma_0 = \frac{8ne^2 g}{\pi^{1/2} m^*} \quad \ldots \ldots \quad (24)
\]

The above integrals have been discussed previously (Donovan and March 1956, Dingle, Arndt and Roy 1957) but the existing tabulations were found to be unsatisfactory for the present purpose and it was decided to obtain all the required values numerically \textit{ab initio}.

As in §3, asymptotic forms for the Faraday rotation and ellipticity may be obtained from (10) and (11). At low frequencies (18) is again valid, with the appropriate expression for the Hall constant; for high frequencies \( \theta \) is again given by (19), but the expression corresponding to (20) for the ellipticity is now

\[
\frac{\delta}{zH} = \frac{32\pi^{1/2} ne^3}{3e^{1/2} m^* g \omega g} \quad \ldots \ldots \quad (25)
\]

4.1. Faraday Rotation

For numerical presentation of the results we again choose parameters relevant to \( n \)-type germanium and the following calculations are based on sample 61 studied by Debye and Conwell (1954). In particular, we consider the temperature range 15–100°K where the observed behaviour is representative of impurity scattering only. Thus at \( T = 100°K \) we take

\[
\sigma_0 = 3 \times 10^{13} \text{e.s.u.}, \quad n = 4 \times 10^{16} \text{cm}^{-3}, \quad g = 1.6 \times 10^{-12} \text{sec},
\]

together with the previous values \( \epsilon = 16 \) and \( m^* = \frac{1}{4} m \).

The Faraday rotation at \( T = 100°K \) is shown in Fig. 6 as a function of frequency, for various values of the field strength, and the behaviour differs markedly in several respects from that shown in Fig. 1. Firstly, as a consequence of the higher values for \( \sigma_0 \) and \( n \), the magnitudes of \( \theta \) are considerably larger; for example, in the microwave region (\( \lambda \sim 1 \text{cm} \)) rotations of the order of 50 rad cm\(^{-1} \) should be obtainable with a field of 15000 Oe. For the strongest field represented (curve c), \( \theta \) varies extremely rapidly between large negative and positive values in the neighbourhood of cyclotron resonance, and only a vestigial flat region appears on the low frequency side. In smaller fields, such that \( \Omega g < 1 \), \( \theta \) changes sign at a frequency slightly lower than \( g^{-1} \).

The variation of \( \theta \) with field strength is similar to the case of lattice scattering (\( \theta \propto H \) up to \( \Omega \sim 10^{11} \) for all frequencies) and need not be discussed in detail. The temperature dependence of \( \theta \) is, however, entirely different, as is apparent from
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Fig. 6. Variation of Faraday rotation \((10^6 \theta \text{ rad cm}^{-1})\) with frequency for n-type germanium specimen with \(\sigma_0 = 3 \times 10^{12} \text{ e.s.u., } g = 1.6 \times 10^{-13} \text{ sec, } m^* = \frac{1}{2} m, \epsilon = 16\) (impurity scattering only): \(a, \Omega = 10^{11} (p = 0)\); \(b, \Omega = 10^{13} (p = -1)\); \(c, \Omega = 10^{13}\) \((p = -1)\).

Fig. 7. Variation of Faraday rotation \((10^6 \theta \text{ rad cm}^{-1})\) with temperature for n-type germanium specimen with \(\sigma_0 = 3 \times 10^{12} \text{ e.s.u. and } g = 1.6 \times 10^{-13} \text{ sec at } T = 100^\circ\text{K}; m^* = \frac{1}{2} m, \epsilon = 16\) (impurity scattering only):
\[
\begin{align*}
\alpha, \omega &= 10^{10} (p = 0); & \quad d, \omega &= 10^{13} (p = 0); & \quad g, \omega &= 10^{13} (p = -1); \\
b, \omega &= 5 \times 10^{10} (p = 0); & \quad e, \omega &= 5 \times 10^{10} (p = -1); & \quad h, \omega &= 10^{14} (p = 0); \\
c, \omega &= 10^{11} (p = 0); & \quad f, \omega &= 3 \times 10^{11} (p = -1); & \quad \eta, \omega &= 10^{11} (p = -1).
\end{align*}
\]

Fig. 7. For the purposes of calculation \(\sigma_0\) and \(n\) were obtained as functions of \(T\) from the data of Debye and Conwell; the values of \(g\) subsequently given by (24) were found to be in reasonable agreement with those calculated directly. Since \(g\)
is an increasing function of $T$ (in contrast to the parameter $a$ in §3) the magnitude of $\theta$ decreases with decreasing temperature when impurity scattering is predominant. As shown in Fig. 7, the values of $\theta$ are greatly reduced at $T \sim 15^\circ K$ and the behaviour in strong fields (illustrated by the broken lines) is similar in general. The only exception is provided by the cyclotron resonance region (curve $g$) in which the variation in $\theta$ is, as might be expected, much more rapid.

\[ \begin{align*}
\text{Fig. 8. Variation of } 10^p \times \frac{1}{2}(\alpha_+ - \alpha_-) \text{ with frequency for n-type germanium specimen with details as for Fig. 6 (impurity scattering only):} \\
a, \Omega = 10^{11} (p = 0); \\
b, \Omega = 10^{13} (p = -1); \\
c, \Omega = 10^{13} (p = -1).
\end{align*} \]

\[ \begin{align*}
\text{Fig. 9. Variation of ellipticity with temperature for n-type germanium specimen with} \\
\alpha_0 = 3 \times 10^{13} \text{ e.s.u. and } g = 1.6 \times 10^{-13} \text{ sec at } T = 100^\circ K; \\
m^* = \frac{1}{2}m, \epsilon = 16 \text{ (impurity} \\
\text{scattering only):} \\
a, \omega = 10^{10}; \\
b, \omega = 10^{11}; \\
c, \omega = 10^{12}; \\
d, \omega = 10^{13}; \\
e, \omega = 3 \times 10^{11}; \\
f, \omega = 10^{12}; \\
g, \omega = 10^{13}; \\
h, \omega = 10^{14}.
\end{align*} \]

4.2. Ellipticity

Using the same data at $T = 100^\circ K$ the expression $\frac{1}{2}(\alpha_+ - \alpha_-)$ has been evaluated as a function of frequency and the results are shown in Fig. 8. The same general features are exhibited as in Fig. 4, but the orders of magnitude are very much larger.
as a consequence of the higher values for $\sigma$ and $n$. The function in Fig. 8 passes through zero at a frequency corresponding to the maximum negative value of $\theta$ in Fig. 6 and rises to a maximum in the neighbourhood of zero $\theta$. In particular, an extremely sharp peak occurs with strong fields (Fig. 8, curve c). It will be seen that, over most of the frequency range displayed, very small path lengths would be required to avoid one or other of the circularly polarized components being totally attenuated.

As an illustration of the temperature dependence of the ellipticity, the expression (11) has been evaluated with $z = 1$ cm for $\Omega = 10^{11}$, and with $z = 1$ mm for $\Omega = 10^{13}$. The results are shown in Fig. 9 (with the data of Fig. 8 corresponding to $T = 100^\circ$K). It will be seen that, at sufficiently low temperatures, $|\delta|$ tends to zero in all cases, but in strong fields, even with $z$ as small as 1 mm, the ellipticity remains at an extreme value over a substantial temperature range for all frequencies up to the infra-red.

§ 5. Conclusion

In this paper the theory of the Faraday effect in semiconductors has been developed on the basis of a model with spherical energy surfaces, in which the free carriers constitute a non-degenerate system. The object has been to consider an energy-dependent relaxation time, for the cases of lattice scattering and impurity scattering, and to investigate the behaviour of the rotation and ellipticity as functions of frequency, field strength and temperature. The numerical calculations were carried out on the University of London mercury computer, which was programmed to perform the numerical integrations and subsequent calculations in one continuous operation. It would be straightforward, in principle, to extend these calculations to a two-band model or to mixed conduction (cf. Donovan and March 1956) but this does not seem particularly profitable at present.

The essentially new information which has emerged concerns the ellipticity, and in this connection experimental evidence would be of considerable interest. Within the limits of the approximate expression (12) $\delta$ is not strictly independent of $\theta$ and one might anticipate that a relation of the Kramers-Kronig type would apply. However, expression (12) is invalid for values of $\delta$ exceeding roughly 0.2 and may therefore be used only in restricted circumstances. Finally, it may be remarked that, for $\theta$ as well as $\delta$, further experimental results at various temperatures would be valuable at microwave frequencies and, if possible, in the far infra-red.

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References

The Theory of the Faraday Effect in Anisotropic Semiconductors

By B. DONOVAN† and JANET WEBSTER†

Department of Physics, Bedford College, London

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Abstract. A formal treatment of the Faraday effect, due to free charge carriers, is developed for arbitrary directions of propagation in anisotropic semiconductors. General expressions for the Faraday rotation and the ellipticity are derived, in terms of four quantities which involve components of the tensors representing the dielectric constant and the high frequency conductivity in the presence of a magnetic field.

It is found that, in contrast to the isotropic case, both the rotation and ellipticity depend upon the real and also the imaginary parts of the complex propagation constants. An examination is also made of the two elliptically polarized component waves into which the initially plane polarized radiation is resolved. Two special cases are discussed, of particular relevance to crystals with cubic symmetry, in which a simplification of the general expressions may be effected.

§ 1. INTRODUCTION

The use of magneto-optical experiments in connection with semiconductors is now well established, and in recent years an increasing amount of attention has been directed towards the Faraday effect, i.e. the rotation of the plane of polarization of an electromagnetic wave in the presence of a static longitudinal magnetic field. Following the original microwave experiments by Rau and Caspari (1955) on germanium, the Faraday rotation due to free charge carriers has been investigated for a number of semiconductors in the infra-red region, where it provides an additional method for obtaining effective mass parameters (see, for example, Moss 1959). The ellipticity of the radiation has also been measured at microwave frequencies in silicon and germanium by Furdyna and Broersma (1960), who interpreted their results in terms of the classical Drude-Zener theory. A more detailed theoretical treatment of the free carrier Faraday effect in non-degenerate semiconductors has been given recently by Donovan and Webster (1961).

A matter of considerable interest, in the light of the experimental work, concerns the influence of the energy band structure of the crystal and the circumstances in which the Faraday effect may be dependent upon the direction of propagation of the incidental radiation. For crystals with cubic symmetry, such as germanium, the Faraday effect will, like the Hall effect, be isotropic only in relatively weak fields; this situation was first discussed by Stephen and Lidiard (1959), who briefly considered some particular cases arising from a high-frequency extension of the magneto-resistance theory of Abeles and Meiboom (1954). In uniaxial crystals the situation is necessarily more complicated; infra-red measurements have been carried out on bismuth telluride by Austin (1959, 1960) but the experiments and theoretical discussion were restricted to the case of propagation along the optic axis. The anisotropic Faraday effect has been discussed by Gurevich and Ipatova (1959), who derived the infra-red refractive indices

† Now at Department of Physics, Westfield College, London.
for certain specific directions of propagation in hexagonal and in cubic crystals; the behaviour of the Faraday rotation was, however, not considered in detail.

It would thus appear that, as yet, no comprehensive treatment has been published which shows explicitly how the Faraday rotation and ellipticity will depend upon the direction of propagation in a particular crystal. The present paper is concerned with the first stage in the development of such a treatment. The aim of this work is to provide a formal theory, whereby general expressions for the Faraday rotation and ellipticity may be obtained, once the appropriate high-frequency magneto-conductivity tensor is known. The essential feature which emerges is the complexity of the relationship between the rotation and ellipticity on the one hand, and the real and imaginary parts of the propagation constants on the other. This general relationship, which obtains when the incident radiation splits into two elliptically, as opposed to circularly, polarized waves, has not been investigated in any of the papers referred to above.

The fundamental expressions are derived in general form in § 4; in the preceding section an analysis is given of the individual elliptically polarized components into which the initial wave is resolved. Finally, in § 5, two special cases are considered which are of particular significance in cubic crystals. The general expressions are somewhat cumbersome, and not readily amenable to approximation, so that rapid computing facilities are desirable in order to extract quantitative information. A substantial amount of numerical work has, in fact, been carried out on the basis of the present theory and the results will be discussed in subsequent papers.

§ 2. GENERAL FORMULATION

A system of axes is chosen such that the incident radiation is propagated along the positive $z$-direction, and the electric vector is initially parallel to the $x$-direction. The electric field in the medium is then specified by $(E_x, E_y, 0)\exp(i\omega t)$, where $\omega$ is the angular frequency, and the constant magnetic field has components $(0, 0, H)$. These axes will not in general coincide with the principal axes of the dielectric constant and conductivity tensors. Hence in general the electric displacement has components

$$D_x = \epsilon_{xx}E_x + \epsilon_{xy}E_y,$$
$$D_y = \epsilon_{yx}E_x + \epsilon_{yy}E_y,$$  \hspace{1cm} (1)

and for the current density we may write

$$J_x = S_{xx}E_x + S_{xy}E_y,$$
$$J_y = S_{yx}E_x + S_{yy}E_y,$$  \hspace{1cm} (2)

where the $S_{ij}$ will be functions of frequency, field strength $H$, the orientation of $H$ relative to the crystal axes, and the relevant parameters characteristic of the medium. We shall not be concerned here with the calculation of the $S_{ij}$. In principle it is possible to use either a series expansion for $J$ or a closed expression (cf. Abeles and Meiboom, 1954); the conductivity components must be generalized to high frequencies and then transformed to the $(x, y, z)$ axes. Irrespective of the method of calculation it must, however, always be possible to express $J$ in the form (2).

For propagation along the $z$ axis, Maxwell’s equations give (using Gaussian units)

$$\frac{\partial^2 \mathbf{E}}{\partial z^2} = -\frac{\omega^2}{c^2} \mathbf{D} + \frac{4\pi i\omega}{c^2} \mathbf{J},$$
and hence we obtain the two fundamental equations of the present treatment:

\[
\begin{align*}
\frac{\partial^2 E_x}{\partial z^2} &= \mathcal{A} E_x + \mathcal{B} E_y, \\
\frac{\partial^2 E_y}{\partial z^2} &= \mathcal{C} E_x + \mathcal{D} E_y,
\end{align*}
\]

where

\[
\mathcal{A} = -\frac{\omega^2}{c^2} \varepsilon_{xx} + \frac{4\pi i\omega}{c^2} S_{xx},
\]

and \(\mathcal{B}, \mathcal{C}, \mathcal{D}\) are similar expressions corresponding to the suffixes \(xy, yx, \text{and } yy\) respectively. We may note that in the isotropic case the dielectric constant \(\varepsilon\) is a scalar and we have \(\mathcal{A} = \mathcal{D}, \mathcal{B} = -\mathcal{C}\); the argument then proceeds straightforwardly in terms of circularly polarized components (cf. Donovan and Webster 1961).

In the present case the general solution of (3) may be written (omitting the factor \(\exp(i\omega t)\)) as

\[
\begin{align*}
E_x &= a_1 \exp(-\mu_+ z) + a_2 \exp(-\mu_- z), \\
E_y &= b_1 \exp(-\mu_+ z) + b_2 \exp(-\mu_- z),
\end{align*}
\]

where the complex propagation constants are given by

\[
2\mu_\pm^2 = \mathcal{A} + \mathcal{D} \pm [(\mathcal{A} - \mathcal{D})^2 + 4\mathcal{B}\mathcal{C}]^{1/2}.
\]

At this point a complication is encountered which does not arise in the isotropic case, but which occurs at several stages in the present analysis. The difficulty is associated with the appearance of square roots (of quantities which are invariably complex), and the necessity of resolving the consequent ambiguity of sign. Referring to (6), the propagation constants in the isotropic case are given by \(\mu_\pm^2 = \mathcal{A} \pm i\mathcal{B}\), and are therefore dependent upon the sign of \(\mathcal{B}\). If, however, we consider the behaviour of (6) in the limit \(\mathcal{A} - \mathcal{D} \to 0, \mathcal{B} \to -\mathcal{C}\), it is clear that the result will be the same if \(\mathcal{B}\) is replaced by \(-\mathcal{B}\). Some method must therefore be found for choosing the sign of the square root, and here (and elsewhere) the choice is determined by appealing to the limit of isotropic behaviour. Thus in handling (6) it is essential that the real and imaginary parts of the square root shall be given the signs of the real and imaginary parts respectively of \(\mathcal{B}\).

The individual amplitudes may be obtained by applying the boundary condition at the surface \(z = 0\), which necessitates

\[
\begin{align*}
\quad a_1 + a_2 &= E_0, \\
\quad b_1 + b_2 &= 0,
\end{align*}
\]

where \(E_0\) is the initial amplitude of the incident plane-polarized wave. In this way we obtain

\[
\begin{align*}
\quad a_1 &= \frac{E_0}{2} \left[ \frac{\mathcal{A} - \mathcal{D}}{\mu_+^2 - \mu_-^2} + 1 \right], \\
\quad a_2 &= -\frac{E_0}{2} \left[ \frac{\mathcal{A} - \mathcal{D}}{\mu_+^2 - \mu_-^2} - 1 \right], \\
\quad b_1 &= -b_2 = E_0 \frac{\mathcal{C}}{\mu_+^2 - \mu_-^2},
\end{align*}
\]

with \(\mu_+^2 - \mu_-^2 = [(\mathcal{A} - \mathcal{D})^2 + 4\mathcal{B}\mathcal{C}]^{1/2}\).
In dealing with (8) it is convenient to write

$$\frac{\mathcal{A} - \mathcal{D}}{\mathcal{C}} = P + iQ, \quad \frac{\mathcal{B}}{\mathcal{C}} = R + iT,$$

but a further ambiguity of sign is introduced here when division by \(\mathcal{C}^2\) is carried out beneath the square root. The correct sign may be obtained here by ensuring that the amplitudes in (8) tend to their correct values in the isotropic limit. To do this it is necessary to put

$$\frac{[(\mathcal{A} - \mathcal{D})^2 + 4\mathcal{B}^2\mathcal{C}^2]^{1/2}}{\mathcal{C}} = -[(P + iQ)^2 + 4(R + iT)]^{1/2}. \tag{10}$$

Expressing the right-hand side of (10) as \(-(G+iH)\), we have

$$2G^2 = (P^2 - Q^2 + 4R) + V \quad 2H^2 = -(P^2 - Q^2 + 4R) + V \tag{11}$$

where

$$V = [(P^2 - Q^2 + 4R)^2 + (2PQ + 4T)^2]^{1/2},$$

so that

$$GH = PQ + 2T \quad G^2 + H^2 = V. \tag{12}$$

Returning to (8) we can now express the amplitudes in the form

$$a_1 = \frac{E_0}{2} \left[ \frac{(G-P) + i(H-Q)}{G+iH} \right]$$
$$a_2 = \frac{E_0}{2} \left[ \frac{(G+P) + i(H+Q)}{G+iH} \right]$$
$$b_1 = -b_2 = -\frac{E_0}{G+iH}, \tag{13}$$

We now define the four quantities which will enable the final results to be expressed in a convenient and symmetrical form. These are

$$2L_\pm = G \mp P$$
$$2M_\pm = H \mp Q, \tag{14}$$

and with this notation the component wave with propagation constant \(\mu_+\) may be described in terms of \(L_+\) and \(M_+\) and the other wave (with \(\mu_-\) in terms of \(L_-\) and \(M_-\). With these definitions we may obtain from (11) and (12), after a little manipulation, the following auxiliary relations:

$$L_+M_+ + L_-M_- = \frac{1}{2}(GH - PQ)$$
$$L_+M_+ - L_-M_- = \frac{1}{2}(GQ - PH)$$
$$L_+(L_+^2 + M_+^2) = RL_+ + TM_+$$
$$L_+(L_-^2 + M_-^2) = RL_- + TM_-$$
$$M_+(L_+^2 + M_+^2) = TL_+ - RM_+$$
$$M_+(L_-^2 + M_-^2) = TL_- - RM_-$$
$$L_+^2 + M_+^2 = R^2 + T^2, \quad L_-^2 + M_-^2 = R^2 + T^2. \tag{15}$$
These relations are of considerable assistance in simplifying the later work. We are now in a position to analyse the electric field configuration in the component waves and in the resultant wave.

§3. CONSIDERATION OF COMPONENT WAVES

Inside the medium the initially plane polarized wave may be regarded as split into two elliptically polarized component waves, rotating in opposite senses. The component associated with the propagation constant $\mu_+$ rotates in a clockwise sense to an observer looking along the positive $z$ direction; we may refer to this component as the positive ellipse. Using the symbol $\mathcal{E}$ to denote the electric field in the individual waves, the positive ellipse is specified by

$$\mathcal{E}_x = a_1 \exp(i\omega t - \mu_+ z), \quad \mathcal{E}_y = b_1 \exp(i\omega t - \mu_+ z)$$

(19)

where, from (13),

$$a_1 = \frac{E_0}{2} \left[ \frac{G^2 + H^2 - PG - QH + i(PH - GQ)}{G^2 + H^2} \right]$$

and

$$b_1 = -\frac{E_0}{2} \left[ \frac{G - iH}{G^2 + H^2} \right].$$

The negative ellipse is obtained by replacing $a_1, b_1$ and $\mu_+$ by $a_2, b_2$ and $\mu_-$ respectively.

The elimination of $t$ follows standard lines and, writing $\mu_+ = \alpha_+ + i\beta_+$, the equations of the individual ellipses may be derived, using (14), in the form

$$\mathcal{E}_x^2 + (L_2^2 + M_2^2) \mathcal{E}_y^2 \pm 2L_\pm \mathcal{E}_x \mathcal{E}_y = \frac{E_0^2 M_{2\pm}}{G^2 + H^2} \exp(-2\alpha_+ z),$$

(20)

where the upper sign is taken throughout for the positive ellipse and the lower sign for the negative ellipse. It will be seen that the ratio of the respective ‘$x$-axes’ of these two ellipses is $M_2/M_1 \exp[-(\alpha_+ - \alpha_-)z]$.

The combined equations (20) represent ellipses whose principal axes are inclined with respect to the $xy$ axes. We define the individual inclination $\theta$ as the angle between a principal axis and the $x$ axis ($\theta < \pi/2$), measured with a right-hand screw relation with the positive $z$ direction. Whether $\theta$ applies to the major axis or to the minor axis will then be determined by the ellipticity.

From (20) we have

$$\tan 2\theta_\pm = \pm \frac{2L_\pm}{1 - (L_2^2 + M_2^2)}$$

(21)

where the upper and lower signs apply respectively to the positive and negative ellipses.

The individual ellipticities may be dealt with in the following way. We imagine the ellipse to be rotated back through $\theta$ so that its principal axes coincide with the $x$ and $y$ axes; the ellipticity $\delta$ is then defined as the ratio of the principal axis parallel to $y$, to the principal axis parallel to $x$. If $\delta < 1$, the inclination $\theta$ refers to the major axis; if, on the other hand, $\delta > 1$, then $\theta$ refers to the minor axis. The significance of this point may, perhaps, be appreciated more clearly in §5. It should be emphasized that $\delta$ has a significance quite different from the final ellipticity, which is defined to be necessarily less than unity; this difference arises simply because, in dealing with the individual components, it is convenient to restrict $\theta$ to positive values less than $\pi/2$. 
Application of standard methods yields the individual ellipticities in the convenient form

$$\delta_{\pm}^2 = \frac{(1 + L_{\pm}^2 + M_{\pm}^2) + (1 - L_{\pm}^2 - M_{\pm}^2) \cos 2\theta_{\pm} + 2L_{\pm} \sin 2\theta_{\pm}}{(1 + L_{\pm}^2 + M_{\pm}^2) - (1 - L_{\pm}^2 - M_{\pm}^2) \cos 2\theta_{\pm} + 2L_{\pm} \sin 2\theta_{\pm}}, \quad (22)$$

where the upper and lower signs apply respectively to the positive and negative ellipses.

For numerical computation the ellipticities may be expressed in the alternative form

$$\delta_{\pm}^2 = \frac{(1 + L_{\pm}^2 + M_{\pm}^2) \pm N_{\pm}}{(1 + L_{\pm}^2 + M_{\pm}^2) \mp N_{\pm}}, \quad (23)$$

where

$$N_{\pm}^2 = (1 - L_{\pm}^2 - M_{\pm}^2)^2 + 4L_{\pm}^2.$$

All the possibilities may be correctly obtained by adhering to the following rule, which applies to either ellipse. The quantity $N$ is preceded by the positive sign in the numerator of (23) and by the negative sign in the denominator, if the numerator of the expression (21) for $\tan 2\theta$ is positive; if this numerator is negative, the sign of $N$ is reversed in both places.

§ 4. FARADAY ROTATION AND ELLIPTICITY

We now investigate the field configuration in the resultant elliptically polarized wave, and hence derive the two experimentally accessible quantities, namely the Faraday rotation and the ellipticity. We start with the field components (5), including the factor $\exp(i\omega t)$, and to begin with, putting $\mu_{\pm} = \alpha_{\pm} + \beta_{\pm}$ as before, these are written in the general form

$$E_x = \exp(-\alpha_{-} z)[a \cos(\omega t - \beta_{+} z) + b \sin(\omega t - \beta_{-} z)]$$

$$+ \exp(-\alpha_{+} z)[c \cos(\omega t - \beta_{-} z) + d \sin(\omega t - \beta_{+} z)],$$

$$E_y = \exp(-\alpha_{-} z)[e \cos(\omega t - \beta_{+} z) + f \sin(\omega t - \beta_{-} z)]$$

$$- \exp(-\alpha_{+} z)[e \cos(\omega t - \beta_{-} z) + f \sin(\omega t - \beta_{+} z)]. \quad (24)$$

For the elimination of $t$ we introduce the following notation:

$$A^2 = a^2 + b^2, \quad C^2 = c^2 + d^2, \quad E^2 = e^2 + f^2,$$

and

$$\tan \epsilon_1 = \frac{b}{a}, \quad \tan \epsilon_2 = \frac{d}{c}, \quad \tan \epsilon_3 = \frac{f}{e}.$$

Then, by expressing (24) as

$$E_x = J \cos(\omega t - \phi), \quad E_y = K \cos(\omega t - \chi),$$

so that

$$J\left(\frac{\cos}{\sin}\right)\phi = A \exp(-\alpha_{-} z)\left(\frac{\cos}{\sin}\right)(\beta_{+} z + \epsilon_1) + C \exp(-\alpha_{+} z)\left(\frac{\cos}{\sin}\right)(\beta_{-} z + \epsilon_2),$$

$$K\left(\frac{\cos}{\sin}\right)\chi = E \exp(-\alpha_{-} z)\left(\frac{\cos}{\sin}\right)(\beta_{+} z + \epsilon_3) - E \exp(-\alpha_{+} z)\left(\frac{\cos}{\sin}\right)(\beta_{-} z + \epsilon_3), \quad (25)$$

the resultant ellipse may be obtained in the form

$$K^2 E_x^2 + J^2 E_y^2 - 2JKE_x E_y \cos(\chi - \phi) = J^2 E_x^2 \sin^2(\chi - \phi). \quad (26)$$
The Faraday rotation $\Theta$, defined as the angle between the major axis of the ellipse and the initial direction of the electric vector, is then given by

$$\tan 2\Theta = \frac{2JK \cos(\chi - \phi)}{J^2 - K^2}. \quad (27)$$

Using (25), the numerator of (27) may be expressed as

$$2JK \cos(\chi - \phi) = 2E[A \exp(-2\alpha_+x) \cos(\epsilon_1 - \epsilon_3) - C \exp(-2\alpha_-x) \cos(\epsilon_2 - \epsilon_3)
+ \exp[-(\alpha_+ + \alpha_-)x][C \cos(\beta_+z - \beta_-z + \epsilon_3 - \epsilon_3) - A \cos(\beta_+z - \beta_-z + \epsilon_1 - \epsilon_3)]] \quad (28)$$

and the denominator of (27) as

$$J^2 - K^2 = (A^2 - E^2) \exp(-2\alpha_+x) + (C^2 - E^2) \exp(-2\alpha_-x)
+ 2 \exp[-(\alpha_+ + \alpha_-)x][AC \cos(\beta_+z - \beta_-z + \epsilon_1 - \epsilon_2) + E^2 \cos(\beta_+z - \beta_-z)]. \quad (29)$$

It is now necessary to express these results in terms of the notation introduced in § 2. Referring to (13), the coefficients in (24) are given by

$$\begin{align*}
a &= \frac{E_0(G^2 + H^2 - PG - QH)}{2(G^2 + H^2)}, \quad b = \frac{E_0(PH - GQ)}{2(G^2 + H^2)} \\
c &= \frac{E_0(G^2 + H^2 + PG + QH)}{2(G^2 + H^2)}, \quad d = \frac{E_0(GQ - PH)}{2(G^2 + H^2)} \\
e &= -\frac{E_0G}{G^2 + H^2}, \quad f = \frac{E_0H}{G^2 + H^2}. \quad (30)
\end{align*}$$

For the numerator of (27) we require the following expressions, which may be derived from (30) and (14):

$$\begin{align*}
AE \cos(\epsilon_1 - \epsilon_3) &= ae + bf = -\frac{E_0^2L_+}{G^2 + H^2}, \quad (31) \\
CE \cos(\epsilon_2 - \epsilon_3) &= ce + df = -\frac{E_0^2L_-}{G^2 + H^2}, \quad (32) \\
CE \sin(\epsilon_3 - \epsilon_2) &= cf - ed = \frac{E_0^2M_-}{G^2 + H^2}, \quad (33) \\
AE \sin(\epsilon_1 - \epsilon_3) &= be - af = -\frac{E_0^2M_+}{G^2 + H^2}. \quad (34)
\end{align*}$$

Substituting these expressions into (28) we obtain finally

$$\begin{align*}
2JK \cos(\chi - \phi) &= -\frac{2E_0^2}{G^2 + H^2} \left[L_+ \exp(-2\alpha_+x) - L_- \exp(-2\alpha_-x)
+ \exp[-(\alpha_+ + \alpha_-)x][L_+ \cos(\beta_+z + (M_+ + M_+) \sin(\beta)] \right] \quad (35)
\end{align*}$$

where $\beta = \beta_+ - \beta_-.$

The denominator of (27) may be treated in similar fashion, using the following expressions, which may be derived from (30) and (14), with the help of the auxiliary
The Theory of the Faraday Effect in Anisotropic Semiconductors

relations (15)-(18):

\[ A^2 - E^2 = a^2 + b^2 - e^2 - f^2 = \frac{E_0^2}{G^2 + H^2}(L_i^2 + M_i^2 - 1), \]

(36)

\[ C^2 - E^2 = c^2 + d^2 - e^2 - f^2 = \frac{E_0^2}{G^2 + H^2}(L_i^2 + M_i^2 - 1), \]

(37)

\[ AC \cos(\epsilon_1 - \epsilon_2) = ac + bd = \frac{E_0^2}{G^2 + H^2}(L_i L_+ + M_i M_-), \]

(38)

\[ AC \sin(\epsilon_1 - \epsilon_2) = bc - ad = \frac{E_0^2}{G^2 + H^2}(L_i L_- + M_i M_+). \]

(39)

Substituting these expressions, together with \( E^2 = E_0^2/(G^2 + H^2) \), into (29) we obtain finally

\[ J^2 - K^2 = \frac{E_0^2}{G^2 + H^2}[(L_i^2 + M_i^2 - 1) \exp(-2x_z) + (L_i^2 + M_i^2 - 1) \exp(-2x_-z) + 2 \exp[-(x_+ + x_-)z][(L_i L_+ + M_i M_- + 1) \cos \beta z -(L_i M_- - L_+ M_-) \sin \beta z]]. \]

(40)

Combining (35) and (40), the Faraday rotation (27) may thus be expressed as tan \( 2\Theta \)

\[ = \frac{2[L_+ \exp(-\tilde{\alpha}z) - L_- \exp(\tilde{\alpha}z) + [(L_+ L_- + M_+ M_-) \cos \beta z + (M_+ M_-) \sin \beta z]]}{(L_i^2 + M_i^2 - 1) \exp(-\tilde{\alpha}z) + (L_i^2 + M_i^2 - 1) \exp(\tilde{\alpha}z) + 2[(L_i L_+ + M_i M_- + 1) \cos \beta z + (L_i M_- - L_+ M_-) \sin \beta z]], \]

(41)

where \( \tilde{\alpha} = x_+ - x_- \).

The ellipticity \( \Delta \) is defined as the ratio of the minor axis to the major axis of the resultant ellipse; in the notation of (26) it is therefore given by

\[ \Delta^2 = \frac{(K^2 + J^2) - [(K^2 - J^2)^2 + 4J^2K^2 \cos^2(\chi - \phi)]^{1/2}}{(K^2 + J^2) + [(K^2 - J^2)^2 + 4J^2K^2 \cos^2(\chi - \phi)]^{1/2}}. \]

(42)

The terms in the square brackets have already been expressed in final form and may be obtained from (35) and (40). The remaining term may be derived using (36)-(39), with the sign of \( E^2 \) reversed. Hence we have

\[ K^2 + J^2 = \frac{E_0^2}{G^2 + H^2}[(L_i^2 + M_i^2 + 1) \exp(-2x_z) + (L_i^2 + M_i^2 + 1) \exp(-2x_-z) + 2 \exp[-(x_+ + x_-)z][(L_i L_+ + M_i M_- - 1) \cos \beta z + (L_i M_- - L_+ M_-) \sin \beta z]]. \]

(43)

The expressions (41) and (42) are considerably more complicated than their counterparts in the isotropic theory and it does not seem possible to arrive at any useful approximations which are generally applicable. The essential feature which emerges is that the real and imaginary parts of the propagation constants both enter into the expressions for \( \Theta \) and \( \Delta \), in contrast to the simple separation which occurs in the isotropic theory. This is a direct consequence of the splitting of the original plane-polarized wave into two ellipses; if two circles are obtained (which is the case for certain directions of propagation), then \( \Theta \) and \( \Delta \) are given by the simple 'isotropic' formulae and this is, in fact, the only situation which has previously been discussed in the literature.
§ 5. SPECIAL CASES

In dealing with crystals of practical interest, considerations of symmetry may lead to a simplification of the general expressions (41) and (42) for certain orientations of the magnetic field. We consider now two special cases which are of particular importance in crystals with cubic symmetry when the magnetic field is restricted to certain crystal planes. The detailed implications will be discussed in a later paper, but it may be mentioned here that the application of these special cases is determined not by the details of the energy surfaces used in calculating the magneto-conductivity tensors, but only by their overall symmetry properties. It turns out, in fact, that with cubic symmetry special case (i) is obtained when the magnetic field is restricted to the (110) plane, and special case (ii) is obtained when the field is restricted to the (100) plane. This applies, for instance, to the multi-ellipsoid models associated with the conduction bands in both germanium and silicon.

Special case (i) \( \mathcal{A} = \mathcal{D} \); \( \mathcal{B} \) and \( \mathcal{C} \) unrestricted.

Here we have \( P = Q = 0 \) and hence, from (14)

\[
\begin{align*}
L_+ &= L_- = \frac{1}{2} G, \\
M_+ &= M_- = \frac{1}{2} H.
\end{align*}
\]

Reference to the combined equations (20) shows that the left-hand sides are the same apart from the sign of the \( \mathfrak{E}_x \mathfrak{E}_y \) term. The incident wave thus splits into two ellipses, having the same eccentricity, the scale factor being \( \exp(-\alpha z) \).

The inclinations of these component ellipses are given by (21), whence we obtain

\[
\tan 2\theta_+ = -\tan 2\theta_-,
\]

so that

\[
\theta_+ + \theta_- = \frac{\pi}{2}.
\]

Considering the individual ellipticities, we have, from (22)

\[
\begin{align*}
\delta_+^2 &= \frac{(1 + L^2 + M^2) - (1 - L^2 - M^2) \cos 2\theta_+ - 2L \sin 2\theta_+}{(1 + L^2 + M^2) + (1 - L^2 - M^2) \cos 2\theta_+ + 2L \sin 2\theta_+} \\
&= \frac{1}{\delta_+^2}.
\end{align*}
\]

The ellipticities in this case are reciprocally related and the component ellipses are symmetrically disposed with respect to the \( x \)-axis. A schematic diagram of the pattern of the field configuration in the \( xy \) plane is shown in the figure (a).

The expression for the Faraday rotation may be appreciably simplified, and, substituting (44) into (41) we obtain

\[
\tan 2\Theta = \frac{2[L \sinh \alpha x - 2M \sin \beta x]}{(L^2 + M^2 - 1) \cosh \alpha x + (L^2 + M^2 + 1) \cos \beta x}.
\]

Special case (ii) \( \mathcal{B} = -\mathcal{C} \); \( \mathcal{A} \) and \( \mathcal{D} \) unrestricted.

Here we have \( R = -1 \), \( T = 0 \), so that (16) and (18) reduce to

\[
\begin{align*}
L_-(L_+^2 + M_+^2) &= -L_+, \\
L_+(L_-^2 + M_-^2) &= -L_-,
\end{align*}
\]

and

\[
(L_+^2 + M_+^2)(L_-^2 + M_-^2) = 1.
\]
Schematic diagram showing electric field configuration in component waves: (a) for special case (i), (b) for special case (ii). The ellipses are shown for simplicity as identical, whereas their sizes will, of course, be different for \( z \neq 0 \) owing to unequal attenuations.

When we apply these results to the component waves, if the positive ellipse is given by (20) the negative ellipse may be written as

\[
(L_+^2 + M_+^2) \varepsilon^2_x + \varepsilon^2_y + 2L_+ \varepsilon_x \varepsilon_y = \frac{E_0^2 M_+ M_-}{G^2 + H^2} \exp(-2\alpha z),
\]

which shows that, in this case also, two similarly proportioned ellipses are obtained. From (21) we have, again,

\[
\tan 2\theta_- = -\tan 2\theta_+,
\]

so that

\[
\theta_+ + \theta_- = \frac{\pi}{2}.
\]

Substituting (46) in (22) we obtain, for the ellipticities,

\[
\delta_-^2 = \frac{(L_+^2 + M_+^2 + 1) - (L_+^2 + M_+^2 - 1) \cos 2\theta_+ + 2L_+ \sin 2\theta_+}{(L_+^2 + M_+^2 + 1) + (L_+^2 + M_+^2 - 1) \cos 2\theta_+ - 2L_+ \sin 2\theta_+},
\]

\[
= \delta_+^2.
\]

In this case the ellipticities are thus equal, and the two component ellipses are symmetrically disposed with respect to the line \( y = x \), as illustrated in the figure (b).

Although expression (41) does not assume a markedly simpler form here, we may reduce the denominator somewhat by making use of the relations

\[
L_+ M_- = -L_- M_+,
\]

and

\[
L_+ L_- + 1 = M_+ M_-,
\]
which follow for this case from (11) and (15). In this way we obtain:
\[ \tan 2\theta = \frac{2[L_+ \exp(-\tilde{a}z) - L_- \exp(\tilde{a}z) + [(L_- - L_+) \cos \beta z + (M_- + M_+) \sin \beta z)]}{(L_+^2 + M_+^2 - 1) \exp(-\tilde{a}z) + (L_-^2 + M_-^2 - 1) \exp(\tilde{a}z) + 4M_+M_- \cos \beta z - L_- \sin \beta z}. \] (47)

**Isotropic case.**

In conclusion we show briefly how the results of the present treatment go over to the isotropic case, which is in effect a combination of special cases (i) and (ii). Here we have \( P = Q = T = 0, R = -1 \), so that

\[ L_+ = L_- = 0, \quad M_+ = M_- = 1. \]

The combined equations (20) for the component waves reduce to two circles, the ratio of the radii being \( \exp(-\tilde{a}z) \).

Considering the coefficients in equation (26), we have, from (35), (40) and (43):

\[ J^2 + K^2 = \frac{E_0^2}{2} \left[ \exp(-2x_+z) + \exp(-2x_-z) \right], \]

\[ J^2 - K^2 = E_0^2 \exp[-(x_+ + x_-)z] \cos \beta z, \]

\[ 2JK \cos(\chi - \phi) = -E_0^2 \exp[-(x_+ + x_-)z] \sin \beta z. \]

The Faraday rotation as given by (27) thus becomes

\[ \tan 2\theta = -\tan \beta z. \]

The square root term in expression (42) for the ellipticity reduces to

\[ E_0^2 \exp[-(x_+ + x_-)z] \]

and hence we obtain

\[ \Delta = \frac{\exp(-x_+z) - \exp(-x_-z)}{\exp(-x_+z) + \exp(-x_-z)}, \]

where the sign is entirely arbitrary and has no physical significance.

§ 6. **Conclusion**

In this paper the formal treatment of the Faraday effect has been extended, with the object of providing an interpretation for experimental results on anisotropic semiconductors. General expressions have been derived for the rotation and ellipticity, which are applicable to any crystal symmetry and any orientation of magnetic field. The essential physical content lies in the tensor components \( \epsilon_{ij} \) and \( S_{ij} \) defined in §2; if these are known the quantitative implications of the present treatment may be worked out for a given specimen. In particular, given the form of the energy surfaces, it is possible to investigate the dependence of the Faraday rotation upon the orientation of the magnetic field. Measurements on suitable single crystals could thus yield information relevant to band structure in a manner comparable with the results of d.c. magneto-resistance and cyclotron-resonance experiments.
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Note on the Faraday Effect in Anisotropic Semiconductors

In a recent paper (Donovan and Webster 1962, to be referred to as I) a formal theory was developed for the Faraday effect due to free charge carriers in anisotropic semiconductors. This paper unfortunately contains certain minor algebraic errors, which have led to incorrect signs for various terms in the essential expressions. The purpose of the present letter is to rectify these errors and to show that, with small alterations to the notation, the equations in I may be restated correctly with relatively little modification. Apart from the redefinitions the notation of I will be used, without further explanation, and the amended equations will be numbered as in I, but distinguished by primes.

The first error occurs in the paragraph following equation (6) in I: in fact, the propagation constants in the isotropic case are given by Thus the real and imaginary parts of the square root in (6) must be given the signs of the real and imaginary parts respectively of $-i\beta$. On dividing by beneath the square root, we find the amplitudes in (8) remain unchanged in the isotropic limit; hence the negative sign on the right-hand side of equation (10) is incorrect. Writing

$$[(P+iQ)^2 + 4(R+iT)]^{1/2} = G+iH$$

as before, (11) and (12) remain unchanged, but the amplitudes are now expressed as

$$a_1 = E_0 \frac{G+P}{2G+iH}$$
$$a_2 = E_0 \frac{G-P}{2G+iH}$$
$$b_1 = -b_2 = \frac{E_0}{G+iH}$$

At this stage we now redefine $L_\pm$ and $M_\pm$ so that the positive suffix is still associated with the component wave with propagation constant $\mu_+$. Hence we write

$$2L_\pm = G \pm P$$
$$2M_\pm = H \pm Q$$

and all subsequent equations must be understood in terms of these new definitions. The auxiliary relations (15) to (18) are, however, unaffected, with one exception, namely

$$L_+M_- - L_-M_+ = \frac{1}{2}(PH-GQ).$$

The second error occurs in the derivation of (20) from the field components (19) and is also implicit in (24). In combining the real parts of $\varepsilon_x$ and $\varepsilon_y$ the terms containing $\sin(\omega t - \beta_\pm z)$ were given the wrong sign. When this is corrected, the equations of the individual ellipses become

$$\varepsilon_x^2 + (L_\pm^2 + M_\pm^2)\varepsilon_z^2 \mp 2L_\pm \varepsilon_x \varepsilon_y = \frac{E_0^2 M_\pm^2}{G^2 + H^2} \exp(-2\alpha z)$$

and the individual inclinations are given by

$$\tan 2\theta_\pm = \frac{\mp 2L_\pm}{1 - (L_\pm^2 + M_\pm^2)}.$$
The individual ellipticities may still be obtained from (23), provided that the sign convention stated in I is referred to (21'). It is, perhaps, worth mentioning here that a simpler form is available:

$$\delta_\pm = \frac{2M_\pm}{(1 + L^2_\pm + M^2_\pm) \mp N_\pm}$$

(23')

where the same sign convention is required for the denominator only.

The correct form of (24) is obtained when the coefficients $b$, $d$ and $f$ are preceded by negative signs. Here we have written $a_1 = a + ib$, $a_2 = c + id$, $b_1 = e + if$, and the six expressions given in (30) must be replaced by the correct versions obtained from (13').

The analysis then proceeds exactly as in I and the Faraday rotation $\Theta$ is given by

$$\tan 2\Theta = \frac{2[L_+ \exp(-\bar{\alpha}z) - L_- \exp(\bar{\alpha}z) + (L_- - L_+) \cos \beta z - (M_- + M_+) \sin \beta z]}{[(L_+^2 + M_+^2 - 1) \exp(-\bar{\alpha}z) + (L_-^2 + M_-^2 - 1) \exp(\bar{\alpha}z) + 2((L_+L_- + M_+M_- + 1) \cos \beta z + (L_-M_+ - L_+M_-) \sin \beta z)].}$$

(41')

The ellipticity $\Delta$ may be obtained from (42), using the corrected forms for the three terms required; two of these follow immediately from the numerator and denominator of (41') and the remaining term is

$$K^2 + J^2 = \frac{E_0^2}{G^2 + H^2} \exp(- (\alpha_+ + \alpha_-)z)[(L_+^2 + M_+^2 + 1) \exp(-\bar{\alpha}z) + (L_-^2 + M_-^2 + 1) \exp(\bar{\alpha}z) + 2((L_+L_- + M_+M_- - 1) \cos \beta z + (L_-M_+ - L_+M_-) \sin \beta z)].$$

(43')

The factor outside the square bracket will, of course, cancel in the final expression for $\Delta$.

Regarding the discussion of the special cases in I, § 5, we may note that the relations between the individual ellipses remain unchanged, and simplifications of the expressions for the various parameters may be made as before, with obvious modifications. In special case (i), for example, the Faraday rotation is given by

$$\tan 2\Theta = - \frac{2[L \sinh \bar{\alpha}z + M \sin \beta z]}{(L^2 + M^2 - 1) \cosh \bar{\alpha}z + (L^2 + M^2 + 1) \cos \beta z}. \quad (45')$$

Department of Physics, Westfield College, London, N.W.3.

8th March 1962.

B. Donovan.

Janet Webster.


† The factor 2 appearing before $M$ in the numerator of (45) is a mistake.
The Theory of the Faraday Effect in Anisotropic Semiconductors
II: Application to n-type Germanium

By B. DONOVAN AND JANET WEBSTER

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The Theory of the Faraday Effect in Anisotropic Semiconductors

II: Application to n-type Germanium

BY B. DONOVAN AND JANET WEBSTER

Department of Physics, Westfield College, London

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Abstract. The theory of the anisotropic free-carrier Faraday effect in semiconductors, previously developed by the authors, is applied to n-type germanium, taking into account the detailed structure of the conduction band. The high-frequency magnetoconductivity tensors are derived from a closed solution of the Boltzmann equation and are evaluated for a non-degenerate system of electrons, considering lattice scattering only.

The Faraday rotation $\Theta$ and ellipticity $\Delta$ are evaluated numerically, as functions of the direction of propagation in the crystal, for two cases, in which the magnetic field is confined to the (100) and (110) planes respectively. Over the range of field strengths considered (up to 30 kilo-oersteds), anisotropic changes in $\Theta$ and $\Delta$ of the order of 30% may be obtained in the microwave region, but decrease with increasing frequency, becoming negligible in the infra-red.

For all directions of propagation, except [100] and [111], it is shown that $\Theta$ and $\Delta$ depend also upon the orientation of the initial plane of polarization with respect to the crystal axes. This effect is similar to the directional effect in magnitude, frequency dependence and field dependence, and is most pronounced for propagation along [110]-type directions.

§ 1. INTRODUCTION

CONSIDERABLE attention has, in recent years, been directed towards the Faraday effect in semiconductors, and the particular manifestations due to free charge carriers have been investigated experimentally in several cases. So far, however, attention has been restricted to those situations which can be discussed in terms of theoretical expressions derived for isotropic media. The generalization of the theory of the free-carrier effect, appropriate to anisotropic semiconductors, was first given by Donovan and Webster (1962; the second of these papers corrects minor algebraic errors in the first and contains the definitive formulae; the two will be referred to jointly as I). This formal treatment leads to general expressions for the rotation of the plane of polarization and the ellipticity of the radiation for an arbitrary direction of propagation in the crystal.

The object of the present paper is to apply the results of I to the case of n-type germanium, using the multi-ellipsoid model for the surfaces of constant energy in momentum space for the conduction electrons. The method is based essentially on a high-frequency generalization of magnetoresistance theory; for ellipsoidal energy surfaces a solution of the Boltzmann equation exists which enables the current density to be expressed in closed form (see, for example, Blatt 1957, p. 273). It may be mentioned here that an alternative approach was originally examined, in which the current density was obtained as an expansion in ascending powers of the magnetic field strength $H$. To the first order in $H$ the Faraday effect is, like the Hall effect, isotropic in crystals.
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with cubic symmetry (cf. Stephen and Lidiard 1959). The inclusion of an additional term in $H^2$ proved totally inadequate for a detailed description of anisotropic behaviour; the results obtained were markedly different from those of the strong-field theory, from which it is clear that a small number of terms gives a very poor approximation to the closed expression for the current density.

In the present investigation two special cases are considered, in which the magnetic field (along which the radiation is propagated) is restricted to the (100) and (110) planes respectively. These two situations are shown to correspond to the two special cases discussed in I, in which a simplification of the general expressions is possible. The basic formulation is given in §2 and the essential quantitative information, relevant to a typical specimen of n-type germanium, is presented in §3. An additional aspect of anisotropic behaviour is implicit in the present treatment, but has not, so far as the authors are aware, been discussed previously. The Faraday rotation and ellipticity depend not only on the direction of propagation, but also on the orientation of the initial plane of polarization with respect to the crystal axes. This effect, which offers interesting experimental possibilities, is dealt with in §4.

The magnitude of the anisotropic variation in both the Faraday rotation and the ellipticity depends, in general, upon the frequency of the radiation, the magnetic field strength and the specimen thickness. Extensive numerical calculations have been carried out in order to determine the effects of varying these parameters and the main conclusions are briefly summarized, to enable the experimental significance of the anisotropic behaviour to be assessed.

It may be pointed out that, in a rigorous treatment of the Faraday effect, account should be taken of the wave reflected from the surface of the specimen. This would, of course, apply equally to isotropic and anisotropic cases and would, if pursued in the present treatment, require modification of the boundary conditions in I. Preliminary investigation of this correction shows that its effect is negligible, so far as the results of the present paper are concerned, but may well be significant when considering, for instance, the frequency dependence of the rotation and ellipticity for semiconductors in which the refractive index varies rapidly with frequency.

§2. CALCULATION OF CONDUCTIVITY TENSORS

In order to apply the theory given in I to a particular case, an explicit expression for the current density must first be obtained. This will now be calculated for the electrons in the conduction band in germanium, for which the surfaces of constant energy in momentum space consist of four ellipsoids of revolution, oriented along the [111] directions. Thus, choosing the principal axes of an ellipsoid as the coordinate system, the energy $E$ may be expressed in terms of the momentum $p$ as

$$ E = \frac{1}{2} \left[ \frac{p_1^2}{m_1} + \frac{p_2^2}{m_2} + \frac{p_3^2}{m_3} \right] $$

where the principal effective masses are $m_1 = m_2$ (transverse) and $m_3$ (longitudinal).

The calculation follows closely the strong-field formulation of Abeles and Meiboom 1954, referred to as AM) and is, therefore, given only in outline. With respect to the coordinate system of (1), we specify the time-dependent electric field in the medium having angular frequency $\omega$ by components $\mathbf{E}_i$ and the static magnetic field by components $\mathbf{H}_i$ ($i = 1, 2, 3$). The contribution to the current density from one ellipsoid
is given by

\[ j = \frac{e}{4\pi^3\hbar^3} \int \text{grad} E \frac{\partial f_0}{\partial E} \phi \, dp \quad (2) \]

where \( f_0 \) is the Fermi distribution function and \( \phi \) is the high-frequency generalization of expression (4.1) of AM. Thus, replacing the relaxation time \( \tau \) by \( \tau/(1+i\omega \tau) \), we have (using Gaussian units)

\[ \phi = -\frac{e\tau}{m_1D(1+i\omega \tau)} \left( \left. \frac{p_1 + e\tau}{c(1+i\omega \tau)} \right| \frac{p_2H_3}{m_2} - \frac{p_3H_2}{m_3} \right) 
  + \frac{H_1}{m_2m_3} \left( \frac{e\tau}{c(1+i\omega \tau)} \right)^2 \left[ p_1H_1 + p_2H_2 + p_3H_3 \right] \delta_1 \quad (3) \]

+ two further terms obtained by cyclic permutation of the suffixes, where

\[ D = 1 + \left( \frac{e\tau}{c(1+i\omega \tau)} \right)^2 \left[ \frac{H_1^2}{m_2m_3} + \frac{H_2^2}{m_3m_1} + \frac{H_3^2}{m_1m_2} \right]. \]

The current density (2) may be expressed in tensor notation as

\[ j_i = s_{ik}(\omega, \mathbf{H}) \delta^i_k \quad (4) \]

and the \( s_{ik} \) are given formally by the equations (4.9) of AM if the functions \( a(\omega), \beta(\omega), \gamma(\omega) \) contained therein are appropriately modified. As a typical illustration we consider:

\[ s_{11} = -\frac{e}{4\pi^3\hbar^3} \int \frac{\partial f_0}{\partial E} \frac{p_i^2}{m_1} \frac{e\tau}{D(1+i\omega \tau)} \left[ 1 + \frac{H_1^2}{m_1m_3} \left( \frac{e\tau}{c(1+i\omega \tau)} \right)^2 \right] \, dp. \quad (5) \]

In the present instance, we consider only lattice scattering and we write the relaxation time as

\[ \tau = a(T/E)^{1/2} \quad (6) \]

where \( a \) is a function of temperature. Intervaly scattering is neglected. We regard the electrons as constituting a non-degenerate system, described by the distribution function

\[ f_0(E) = \frac{nh^3}{2(2\pi kT)^{3/2}m_1m_3^{1/2}} \exp \left( -\frac{E}{kT} \right) \quad (7) \]

where \( n \) is the carrier density for a single ellipsoid.

The general procedure follows that of AM, but the integral over the energy is now complex. Using the notation

\[ x = \frac{E}{kT}, \quad u = \frac{4ne^2a}{3\pi^{1/2}m_3}, \quad K = \frac{m_3}{m_1} \]

the first integral in (5) leads to the expression

\[ uK \int_0^\infty \frac{[x^2(x+a^2\omega^2+a^2Q^2)-ia\omega x^{3/2}(x+a^2\omega^2-a^2\Omega^2)]}{(x-a^2\omega^2+a^2\Omega^2)^2+4a^2\omega^2x} e^{-x} dx \quad (8) \]

where

\[ \Omega = \frac{eK^{1/2}}{m_3c}(H_1^2 + H_2^2 + KH_3^2)^{1/2}. \quad (9) \]

The parameter \( \Omega \) is the cyclotron resonance frequency associated with the ellipsoid for a general orientation of the magnetic field, and has the value \( eH/m_3c \) for \( \mathbf{H} = (0, 0, H_3) \).
We now define the complex quantity \( \alpha(\omega, \mathbf{H}) \) by writing (8) as

\[
u K \alpha(\omega, \mathbf{H})
\]

and for \( \omega = 0 \) this is easily seen to reduce to the term \( uK \alpha(\omega) \) which occurs in the expression for \( s_{11} \) in AM. Similar considerations show that all the components \( s_{ij}(\omega, \mathbf{H}) \) in (4) are given by (4.9) in AM, provided that \( \alpha(\omega) \), \( \beta(\omega) \) and \( \gamma(\omega) \) are replaced by their respective high frequency generalizations \( \alpha(\omega, \mathbf{H}), \beta(\omega, \mathbf{H}) \) and \( \gamma(\omega, \mathbf{H}) \). For the two latter quantities we find, after some manipulation

\[
\begin{align*}
\beta(\omega, \mathbf{H}) &= \int_0^\infty \frac{x^3 e^{-x}}{x-a_2^2 + a^2 \Omega^2} - 2 \omega x^2 \int_0^\infty \frac{e^{-x}}{x-a_2^2 + a^2 \Omega^2} \\
\gamma(\omega, \mathbf{H}) &= \int_0^\infty \frac{\omega x^3}{x-a_2^2 + a^2 \Omega^2} + \frac{\omega x}{a^2 \Omega^2} (3x-a_2^2 + a^2 \Omega^2)^2 e^{-x} dx
\end{align*}
\]

These integrals can be expressed, by the method of partial fractions, as suitable combinations of two basic integrals \( I_1 \) and \( I_2 \), which appear in the theory of the isotropic Faraday effect (Donovan and Webster 1961). If we write

\[
\begin{align*}
I_1(\xi) &= \int_0^\infty \frac{x^2 e^{-x}}{x+a_2^2} dx, \\
I_2(\xi) &= \int_0^\infty \frac{x^2 e^{-x}}{x+a_2^2} dx
\end{align*}
\]

the expressions in (8) and (10) may be put in the form

\[
\begin{align*}
\alpha(\omega, \mathbf{H}) &= \frac{1}{2}[I_1(\omega + \Omega) + I_1(\omega - \Omega)] - \frac{i \omega}{2} [(\omega + \Omega)I_2(\omega + \Omega) + (\omega - \Omega)I_2(\omega - \Omega)] \\
\beta(\omega, \mathbf{H}) &= \frac{1}{2\Omega} [(\omega + \Omega)I_2(\omega + \Omega) - (\omega - \Omega)I_2(\omega - \Omega)] + \frac{i}{2a_2^2} [I_1(\omega + \Omega) - I_1(\omega - \Omega)] \\
\gamma(\omega, \mathbf{H}) &= \frac{1}{2a_2^2 \Omega^2} [2I_1(\omega) - I_1(\omega + \Omega) - I_1(\omega - \Omega)] \\
&\quad - \frac{i}{2a_2^2 \Omega^2} [2\omega I_2(\omega) - (\omega + \Omega)I_2(\omega + \Omega) - (\omega - \Omega)I_2(\omega - \Omega)]
\end{align*}
\]

The transformation to the cubic axes of the crystal and the summation over the four ellipsoids can now be carried out, following the procedure of AM. We use the suffixes 1, 2, 3 henceforth to signify the cubic axes. For each ellipsoid an appropriate 'back transformation' must be performed on \( \mathbf{H} \) to compensate for the effect of rotating the field along with the axes when transforming to the (1, 2, 3) system. This leads, in general, to a different contribution from each ellipsoid to any given tensor component, which is thus obtained as a sum over the four possible values of \( \alpha_j(\omega, \mathbf{H}), \beta_j(\omega, \mathbf{H}) \) and \( \gamma_j(\omega, \mathbf{H}) \), the \( q_j \) being coefficients which are dependent upon \( \mathbf{H} \). The different values of \( \alpha_j(\omega, \mathbf{H}) \) etc. occur as a consequence of the four cyclotron frequencies \( \Omega_j \) into which expression (9) transforms.

We thus arrive ultimately at the total current density in the form

\[
J_i = S_{ik}(\omega, \mathbf{H}) e_k
\]

where

\[
\begin{align*}
S_{11} &= \sum_{j=1}^{4} \left( \left( \frac{2K+1}{3} \right) \alpha_j(\omega, \mathbf{H}) + \left( \frac{e a H_1}{m_1 c} \right)^2 \gamma_j(\omega, \mathbf{H}) \right) \\
S_{12} &= \sum_{j=1}^{4} \left( \left( \frac{K-1}{3} \right) (-1)^j \alpha_j(\omega, \mathbf{H}) - \left( \frac{e a}{3m_1 c} \right) q_j \beta_j(\omega, \mathbf{H}) + \left( \frac{e a}{m_1 c} \right)^2 H_1 H_2 \gamma_j(\omega, \mathbf{H}) \right)
\end{align*}
\]

(14)
the four cyclotron frequencies are

$$\begin{align*}
\Omega_1 &= \lambda[(K+2)|H|^2 + 2(K-1)(H_1 H_2 + H_2 H_3 + H_3 H_1)]^{1/2} \\
\Omega_2 &= \lambda[(K+2)|H|^2 + 2(K-1)(-H_1 H_2 - H_2 H_3 - H_3 H_1)]^{1/2} \\
\Omega_3 &= \lambda[(K+2)|H|^2 + 2(K-1)(H_1 H_2 - H_2 H_3 - H_3 H_1)]^{1/2} \\
\Omega_4 &= \lambda[(K+2)|H|^2 + 2(K-1)(-H_1 H_2 + H_2 H_3 - H_3 H_1)]^{1/2}
\end{align*}$$

(15)

with

$$\lambda = \frac{e}{m_0 c} \left( \frac{K}{3} \right)^{1/2}$$

and the coefficients $q_j$ are

$$\begin{align*}
q_1 &= (K-1)(H_1 + H_2) + (K+2)H_3 \\
q_2 &= (K-1)(H_1 - H_2) + (K+2)H_3 \\
q_3 &= (K-1)(-H_1 + H_2) + (K+2)H_3 \\
q_4 &= (K-1)(-H_1 - H_2) + (K+2)H_3
\end{align*}$$

(16)

The other components of $S_{ik}$ are obtained by cyclic permutation of the suffixes on $H$ in (14), (15) and (16).

We now introduce a system of axes $(x, y, z)$ such that the incident radiation is propagated along the positive $z$ direction, and the electric vector is initially parallel to the $x$ axis.† Relative to these axes, the magnetic field, therefore, has the form $(0, 0, H)$ and the required current density components may be written as

$$\begin{align*}
J_x &= S_{xx} \mathcal{E}_x + S_{xy} \mathcal{E}_y \\
J_y &= S_{yx} \mathcal{E}_x + S_{yy} \mathcal{E}_y
\end{align*}$$

(17)

It therefore remains to transform the conductivity tensor components in (14) from the $(1, 2, 3)$ axes to the $(x, y, z)$ axes. Then, following the treatment in I, we can investigate the Faraday rotation and ellipticity as functions of the orientation of $H$ with respect to the crystal axes. Referring to I, equation (4), the four fundamental complex quantities are (since the dielectric constant has the form $\varepsilon_{xx} = \varepsilon$, $\varepsilon_{xy} = 0$, etc.)

$$\mathcal{A} = -\frac{\omega^2}{c^2} + \frac{4\pi i \omega}{c^2} S_{xx}$$

(18)

with similar expressions for $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$.

The complex propagation constants $(\mu_{\pm} = \alpha_{\pm} + i\beta_{\pm})$ for the two elliptically polarized component waves are given by

$$2\mu_{\pm}^2 = \mathcal{A} + \mathcal{D} \pm [(\mathcal{A} - \mathcal{D})^2 + 4\mathcal{B}\mathcal{C}]^{1/2}.$$  

(19)

As shown in I, it is convenient to work with the four real quantities $P, Q, R, T$, defined by

$$\frac{\mathcal{A} - \mathcal{D}}{\mathcal{C}} = P + iQ, \quad \frac{\mathcal{B}}{\mathcal{C}} = R + iT.$$  

(20)

Finally, the Faraday rotation $\Theta$ and ellipticity $\Delta$ may be obtained from general expressions of the form

$$\tan 2\Theta = \frac{2JK \cos \Phi}{J^2 - K^2}$$

(21)

† It should be pointed out here that in the present section, and in § 3, the initial direction of polarization is fixed for a given direction of propagation. The general case of an arbitrary direction of polarization is dealt with in § 4.
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\[ \Delta = \frac{2JK \sin \Phi}{J^2 + K^2 + \left[(K^2-J^2)^2+4J^2K^2 \cos^2 \Phi\right]^{1/2}} \]  

(22)

where \( J, K \) and \( \Phi \) are complicated functions of \( \alpha_x, \beta_x, P, Q, R, T \) and the specimen thickness \( z \).

We may show briefly how the above formulation goes over to the isotropic theory when the energy surfaces are given spherical symmetry. Setting \( K = 1 \), we have \( m_1 = m_3 = m^* \) and \( \Omega = eH/m^*c \). The conductivity tensors are invariant under any rotation of axes and have the form

\[
S_{11} = S_{22} = S_{xx} = S_{yy} = 4ux(\omega, \mathbf{H}) \\
S_{21} = -S_{12} = S_{yx} = -S_{xy} = 4u\Omega a\beta(\omega, \mathbf{H}).
\]

Using (12), the right-hand sides are easily seen to be equal respectively to the quantities \( M \) and \( \Omega \) occurring in a previous treatment of the isotropic effect (Donovan and Webster 1961, § 2; in this paper the cyclotron resonance frequency is defined as \( \Omega = -eH/m^*c \)).

The expressions (21) and (22), in their general form, present a formidable problem of numerical evaluation; we turn, therefore, to a discussion of two special cases, in which symmetry considerations enable various simplifications to be achieved.

2.1. \( H \) Confined to (110) Plane

We first consider the situation in which the magnetic field is confined to the (110) plane and is inclined at an angle \( \phi \) to the [001] direction (i.e. the 3-axis). The \((x, y, z)\) and \((1, 2, 3)\) systems are coincident at \( \Theta = 0 \); for any given orientation the transformation matrix is:

\[
\begin{pmatrix}
\frac{1}{2} & 1 + \cos \phi & -1 + \cos \phi & \sqrt{2} \sin \phi \\
-1 + \cos \phi & 1 + \cos \phi & \sqrt{2} \sin \phi \\
-\sqrt{2} \sin \phi & -\sqrt{2} \sin \phi & 2 \cos \phi
\end{pmatrix}
\]

It should be pointed out that for any \( \phi \neq 0 \) the field components \( S'_x \) and \( S'_y \) are displaced from the cubic axes, but always remain equally inclined to the 1 and 2 axes respectively.

The tensor components required for (17) are found to be (writing \( \alpha_j(\omega, \mathbf{H}), etc.)

\[ S_{xx} = S_{yy} = \frac{u}{3}(2K+1) \sum_j \alpha_j \]

\[
-\frac{u}{3}(K-1) \left[ \sin^2 \beta_j \sum_j (-1)^j \alpha_j + \sqrt{2} \sin \phi \cos \phi (\alpha_3 - \alpha_1) \right]
\]

\[ S_{xy} = \frac{u}{3}(K-1) \left[ \frac{(1 + \cos^2 \beta_j)}{2} \sum_j (-1)^j \alpha_j - \sqrt{2} \sin \phi \cos \phi (\alpha_3 - \alpha_1) \right]
\]

\[
-\frac{u}{3} \left( \frac{e\alpha}{m_1c} \right) H_0(K+2) \sum_j \beta_j - (K-1) \sin^2 \beta_j \sum_j (-1)^j \beta_j + 2 \sqrt{2} \sin \phi \cos \phi (\beta_1 - \beta_3)
\]

and \( S_{yx} \) is identical with \( S_{xy} \) except that the second term is preceded by a positive sign.

The number of different cyclotron frequencies is three, in general, since the second and fourth ellipsoids are always symmetrically disposed with respect to the magnetic
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field. From (15) we have

\[
\begin{align*}
\Omega_1 &= \lambda H_0 [(K + 2) + (K - 1) (\sin^2 \phi + 2 \sqrt{2} \sin \phi \cos \phi)]^{1/2} \\
\Omega_2 &= \Omega_4 = \lambda H_0 [(K + 2) - (K - 1) \sin^2 \phi]^{1/2} \\
\Omega_3 &= \lambda H_0 [(K + 2) + (K - 1) (\sin^2 \phi - 2 \sqrt{2} \sin \phi \cos \phi)]^{1/2}
\end{align*}
\]

Since \( S_{xx} = S_{yy} \), irrespective of the value of \( \phi \), we have always \( \omega = \Omega \), from (18), which demonstrates that this situation is identical with the special case (i) discussed in I.

Further simplifications are apparent, and are exactly comparable with those encountered in cyclotron resonance experiments (cf. Dresselhaus, Kip and Kittel 1955). Thus, when \( \cos \phi = 1/\sqrt{3} \), i.e. when \( H \) lies along the [111] direction, we have

\[
\Omega_2 = \Omega_3 = \Omega_4
\]

and the first term in \( S_{xy} \) and \( S_{yx} \) vanishes. For this particular orientation, therefore, we have \( \mathcal{A} = \mathcal{B} \), referring to (18), and \( \Theta \) and \( \Delta \) are obtainable from the simple 'isotropic' formulae (cf. Donovan and Webster 1961). The same applies also, of course, to \( \phi = 0 \) (\( H \) parallel to cubic axis), when all the \( \Omega \)'s are equal.

2.2. \( H \) Confined to (100) Plane

We next consider the situation in which the magnetic field is confined to the (100) plane and we again use \( \phi \) to denote the angle between \( H \) and the 3-axis. The transformation of axes in this case is simply a rotation about the 1-axis (to which \( S_x \) remains parallel for all \( \phi \)); the transformation matrix is

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \phi & \sin \phi \\
0 & -\sin \phi & \cos \phi
\end{pmatrix}
\]

and hence we find

\[
\begin{align*}
S_{xx} &= \frac{2u}{3} (2K + 1)(x_1 + x_2) \\
S_{yy} &= S_{xx} + \frac{4u}{3} (K - 1) \sin \phi \cos \phi (x_1 - x_2) \\
S_{xy} &= -S_{yx} = -\frac{2u}{3} \left( \frac{ea}{m_\text{t}c} \right) H_0 [(K + 2)(\beta_1 + \beta_2) + 2(K - 1) \sin \phi \cos \phi (\beta_1 - \beta_2)]
\end{align*}
\]

The cyclotron frequencies are equal in pairs and from (15) we have

\[
\begin{align*}
\Omega_1 &= \Omega_4 = \lambda H_0 [(K + 2) + 2(K - 1) \sin \phi \cos \phi]^{1/2} \\
\Omega_2 &= \Omega_3 = \lambda H_0 [(K + 2) - 2(K - 1) \sin \phi \cos \phi]^{1/2}
\end{align*}
\]

Since \( S_{xy} = -S_{yx} \) we have \( \mathcal{A} = \mathcal{B} \) for all \( \phi \), and the situation, therefore, corresponds to the special case (ii) discussed in I. For \( \phi = 0 \) or \( \pi/2 \) we have also \( S_{xx} = S_{yy} \) (all the \( \Omega \)'s are equal), so that \( \mathcal{A} = \mathcal{B} \) and the 'isotropic' expressions for \( \Theta \) and \( \Delta \) may be used.

§ 3. GENERAL RESULTS

We now apply the results of the previous section to a quantitative investigation of the anisotropic effects in a typical non-degenerate specimen of n-type germanium.
Theory of the Faraday Effect in Anisotropic Semiconductors

The following representative data (applying at $T = 300 \degree K$) have been used:

- d.c. conductivity $\sigma_0 = 9 \times 10^{10}$ c.s.u.,
- carrier density ($4n$ in present notation) $= 1.3 \times 10^{14}$ cm$^{-3}$, (cf. Debye and Conwell 1954, sample 51), together with the dielectric constant $\epsilon = 16$.

The effective mass parameters have been well established by cyclotron resonance experiments (Dresselhaus, Kip and Kittel 1955, Levinger and Frankl 1961) and we take

$$m_1 = 0.08m, \quad m_2 = 1.6m, \quad K = 20.$$  

Substituting these values in the appropriate expression for $\sigma_0$, we find $a = 4.3 \times 10^{-13}$ sec.

It should be noted that the relaxation time in the Drude–Zener treatment corresponds to $4a/3\pi^{1/2}$.

The magnitude of the anisotropy considered here is determined by the two quantities $\Omega a$ and $\Omega/\omega$. It is well known that the Faraday effect may be exploited experimentally under considerably less stringent conditions than those required for cyclotron resonance (viz. $\Omega a > 1, \Omega/\omega = 1$). While it is clearly not possible to state precise conditions under which the anisotropy attains experimentally significant proportions, we may mention as a rough guide for n-type germanium that, to obtain anisotropic changes of at least several per cent, we require

$$\Omega a \gtrsim \frac{1}{4} \quad \text{and} \quad \frac{\Omega}{\omega} \gtrsim \frac{1}{4}.$$  

Referring to the above data, $\Omega$ for the [100] direction is $1.33 \times 10^8$ Hz, and therefore both conditions may be amply satisfied in the microwave region by using fields of the order of $2 \times 10^4$ Oe. At infra-red frequencies, however, the second condition requires very strong fields ($H \sim 2 \times 10^6$ Oe), in which case the first condition is automatically satisfied.

In the present discussion we focus attention principally on the microwave region, and we consider magnetic fields up to $3 \times 10^4$ Oe. Further calculations based on higher field strengths, referring in particular to the situation in the infra-red, will be presented in a later paper.

3.1. Basic Anisotropy Patterns

Figure 1 shows the Faraday rotation plotted as a function of the angle $\phi$ in the two crystal planes considered in §2, with $\omega = 6.28 \times 10^{10}$ sec$^{-1}$ (3 cm radiation), $H_0 = 2 \times 10^4$ Oe and $\alpha = 5$ mm. As $\phi$ increases from zero, the magnitude of $\Theta$ initially decreases and the complete patterns display the correct symmetry for a cubic lattice, namely a periodicity of $\pi$ in the (110) plane and $\pi/2$ in the (100) plane. The general patterns illustrated by curves $a$ and $b$ are obtained over a wide range of $\omega$ and $H_0$ and are characteristic of their respective planes. Orientations of $H_0$ along [110]-type directions appear on both curves, but give different values for $\Theta$. This is due to the difference in the orientations of the initial plane of polarization in the two cases, and is considered in detail in §4. The ellipticity has also been evaluated, but, since the anisotropic behaviour closely resembles that of the rotation, it will not be discussed separately.

It should be borne in mind that these anisotropic phenomena arise (i) because the propagation constants are dependent upon the direction of propagation, and (ii) as a consequence of the elliptical polarization of the component waves, which is responsible for the complexity of formulae (21) and (22). In order to illustrate the effect of these two contributions, the broken lines in figure 1 (curves $c$ and $d$) show the corresponding variation of $-\frac{1}{2}\beta z(\beta = \beta_+ - \beta_- \text{as in I})$. In the (100) plane the effect of applying formula (21)
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is to reduce the anisotropy, and this holds generally for both \( \Theta \) and \( \Delta \) for all values of \( \omega \) and \( H_0 \). In the (110) plane the effect of formula (21) is to make the initial variation of \( \Theta \) more rapid and a sharp peak is obtained at \( \phi \sim 38^\circ \). It will be seen that curves \( a \) and \( c \) intersect at the [111] direction, which, as pointed out in § 2, is a special case where (21) reduces to the 'isotropic' formula. For \( \phi > \cos^{-1}(1/\sqrt{3}) \), \( \Theta \) falls below \(-\frac{1}{2}\beta z\); in this region the difference between the values of \( \Theta \) at \( \phi = \pi/2 \) and \( \phi = 0 \) is a decreasing function of \( H_0 \) and vanishes for \( H_0 \sim 5000 \text{ Oe} \).

![Figure 1](image1.png)

Figure 1. Variation of Faraday rotation with orientation of magnetic field: \( a \), in (110) plane; \( b \), in (100) plane. The broken lines represent \(-\frac{1}{2}\beta z\); \( c \), in (110) plane; \( d \), in (100) plane. \( \lambda = 3 \text{ cm}, H = 2 \times 10^4 \text{ Oe}, z = 5 \text{ mm} \).

The basic patterns illustrated in figure 1 are obtained at all frequencies, with the exception of a comparatively narrow band in the region where \( \beta \) changes sign. This occurs at \( \omega \sim 3 \times 10^{12} \) for \( H_0 \sim 2 \times 10^4 \text{ Oe} \); since \( \Theta \) is very small, this region is not experimentally significant in the present context. It should, however, be pointed out that at optical frequencies, where \( \Theta \) is positive, the patterns of figure 1 are obtained,

![Figure 2](image2.png)

Figure 2. Variation of Faraday rotation with frequency for various orientations of magnetic field: \( a \), along [100]; \( b \), along [111]; \( c \), along [011], with \( \psi = 0 \); \( d \), along [011], with \( \psi = 45^\circ \). \( H = 2 \times 10^4 \text{ Oe}, z = 5 \text{ mm} \).
but without inversion, i.e. $\Theta$ initially increases with increasing $\phi$. Similar considerations apply to the ellipticity, bearing in mind that $\varepsilon$ passes through zero at $\omega \sim 2 \times 10^{11}$.

The frequency dependence of the Faraday rotation is shown in figure 2, for $H_0 = 2 \times 10^4$ Oe and $z = 5$ mm. Curves $a$ and $b$ apply respectively to propagation along the [001] and [111] axes, and emphasize that anisotropic behaviour is most prominent in the microwave region. Curve $c$ applies to $\phi = 45^\circ$ in the (100) plane and the maximum anisotropy in this plane is thus conveyed by the separation of curves $a$ and $c$; this is always less than the maximum anisotropy in the (110) plane. In the frequency range between the crossing of the various curves and the change in sign of $\Theta$ the patterns of figure 1 are obtained in inverted form.

![Figure 3. Variation of ellipticity with frequency for various orientations of magnetic field: $a$ and $b$, along [100]; $c$, along [011], with $\psi = 0$; $d$, along [011], with $\psi = 45^\circ$. $H = 3 \times 10^4$ Oe for curve $b$, and $2 \times 10^4$ Oe for curves $a$, $c$ and $d$; $z = 5$ mm throughout.]

Figure 3 shows the frequency dependence of the ellipticity, with $z = 5$ mm. Curves $a$ and $c$ (for $H_0 = 2 \times 10^4$ Oe) apply respectively to $\phi = 0$ and $45^\circ$ in the (100) plane. The effect of increasing the field strength is illustrated by curve $b$, which applies to the [001] direction with $H_0 = 3 \times 10^4$ Oe. In comparison with curve $a$, the peak is sharpened and the slope is diminished in the region where $\Delta \simeq 0$. The curves in figures 2 and 3 are similar to those obtained in the isotropic case (Donovan and Webster 1961, figures 1 and 4); thus the main characteristics of the frequency dependence of $\Theta$ and $\Delta$ are not materially affected by an analysis based on the detailed band structure.

### 3.2. Relative Anisotropy

Some quantitative aspects of the anisotropy may conveniently be discussed by expressing the maximum variation of $\Theta$ or $\Delta$, in either of the crystal planes considered, as a percentage of the value for the [001] direction. The relative anisotropy $\omega$ so defined will be denoted by $\rho_R$ for the rotation and by $\rho_E$ for the ellipticity. For both $\Theta$ and $\Delta$ the limiting values in the (100) plane occur at $\phi = 0$ and $45^\circ$, virtually without exception, and those in the (110) plane at $\phi = 0$ and $\sim 35-40^\circ$.

Considering the behaviour of $\Theta$, the calculations show that $\rho_R$ is, for a given field, almost independent of frequency for $\omega < 10^{13}$, i.e. up to the region of the intersection of the curves in figure 2. In the vicinity of this cross-over $\rho_R$ is very small, but increases rapidly as $|\Theta|$ tends to zero. Similar behaviour is exhibited by the ellipticity (but the
cross-over occurs at a lower frequency, as shown in figure 3); in the range \( \omega < 10^{13} \), 
\( \rho_{E} \) is virtually identical with \( \rho_{R} \). Both \( \rho_{R} \) and \( \rho_{E} \) tend to zero rapidly as the frequency
increases above \( \omega \sim 6 \times 10^{12} \) (with the fields considered in figures 2 and 3, \( \Omega / \omega \) is too
small to satisfy the condition previously referred to).

Representative values of \( \rho_{R} \) (for a path length of 1 mm) are given in the accompanying
table, and illustrate the consistency at low frequencies.

<table>
<thead>
<tr>
<th>Crystal plane ( \omega ) (sec(^{-1}))</th>
<th>( H_{0} ) ( 10^{4} ) Oe</th>
<th>( 2 \times 10^{4} ) Oe</th>
<th>( 3 \times 10^{4} ) Oe</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^{9} )</td>
<td>15.7</td>
<td>12.1</td>
<td>25.8</td>
</tr>
<tr>
<td>( 10^{10} )</td>
<td>15.7</td>
<td>12.0</td>
<td>25.9</td>
</tr>
<tr>
<td>( 10^{11} )</td>
<td>16.3</td>
<td>11.5</td>
<td>25.7</td>
</tr>
<tr>
<td>( 6.3 \times 10^{11} )</td>
<td>14.3</td>
<td>10.0</td>
<td>24.9</td>
</tr>
<tr>
<td>( 10^{12} )</td>
<td>10.8</td>
<td>6.7</td>
<td>24.4</td>
</tr>
</tbody>
</table>

3.3. Influence of \( H_{0} \) and \( z \)

In the microwave region \( \Theta \) and \( \Delta \) are linear functions of field strength only in weak
fields and, for all orientations, pass through a broad maximum between 20 and 30 kOe. Figure 4 shows the field dependence of the Faraday rotation, with \( \omega = 10^{11} \) sec\(^{-1}\) and

\( z = 5 \) mm, for various orientations of the magnetic field. The maximum, which occurs
in the region \( \Omega a \sim 1 \), is highest and most pronounced for the [100] direction. At
infra-red frequencies \( |\Theta| \propto H_{0} \) over the range of figure 4.

![Figure 4. Variation of Faraday rotation with field strength for various orientations of magnetic field: a, along [100]; b, along [111]; c, along [011], with \( \psi = 0 \); d, along [011], with \( \psi = 60^\circ \). \( \omega = 10^{11} \) sec\(^{-1}\), \( z = 5 \) mm.](image)
On the basis of calculations carried out with \( z = 1 \) and 5 mm, it is concluded that, in this range, the anisotropy does not depend sensitively upon path length. It is found that the \( \rho \) values decrease slightly with increasing \( z \) when \( \bar{\alpha} z \) and \( \bar{\beta} z \) approach unity. If \( \bar{\alpha} z \) and \( \bar{\beta} z \) are both small compared with unity, the \( \rho \) values are practically independent of \( z \), except in the neighbourhood of a sign change in either \( \bar{\alpha} \) or \( \bar{\beta} \).

§4. ANISOTROPY DUE TO ORIENTATION OF PLANE OF POLARIZATION

We have already drawn attention to the fact that, in general, the Faraday rotation and ellipticity depend not only on the direction of propagation, but also on the orientation of the initial plane of polarization with respect to the crystal axes. In the two transformations previously considered, the initial electric field \( E_x \) is always parallel to the [100] axis when \( H \) is confined to the (100) plane, but is inclined at an angle

\[
\cos^{-1}\left[\frac{1}{2}(1 + \cos \phi)\right]
\]

to the [100] axis when \( H \) lies in the (110) plane. The directional dependence of \( \Theta \) and \( \Delta \) shown in figure 1 relates to these specific dispositions of field components; while some arbitrariness of presentation is unavoidable, this particular choice preserves a conveniently symmetrical arrangement of field components in the lattice. For a complete understanding of anisotropic behaviour we need to know how \( \Theta \) and \( \Delta \) vary when the propagation direction is fixed and the electric vector is rotated in the \( xy \) plane.

To investigate this effect we introduce the azimuthal angle \( \phi \), measured from the initial direction of \( E_x \), and carry out a further transformation to a system of axes \((x', y', z)\) specified by a clockwise rotation through \( \phi \), looking along the positive \( z \) direction. It can easily be shown that the propagation constants \( \mu_{\pm} \) are invariant under this transformation, and this result is independent of any consideration due to crystal symmetry. The dependence of \( \Theta \) and \( \Delta \) upon \( \phi \) arises since the quantities \( P, Q, R, T \) in equations (20) are not invariant under the transformation. The only exceptions occur for propagation along the [100] or [111] directions, when \( \Theta \) and \( \Delta \) are given by the ‘isotropic’ formulae and are independent of \( \phi \). This particular anisotropic manifestation is, in fact, a direct consequence of elliptically, as opposed to circularly, polarized component waves.

The conductivity tensors may be transformed to the \((x', y', z)\) axes and the required components may be expressed in the form

\[
\begin{align*}
S_{x'x'} &= S_{xx} \cos^2 \phi + S_{yy} \sin^2 \phi + (S_{xy} + S_{yx}) \sin \phi \cos \phi \\
S_{y'y'} &= S_{xx} \sin^2 \phi + S_{yy} \cos^2 \phi - (S_{xy} + S_{yx}) \sin \phi \cos \phi \\
S_{x'y'} &= S_{xy} \cos^2 \phi - S_{yx} \sin^2 \phi + (S_{yy} - S_{xx}) \sin \phi \cos \phi \\
S_{y'x'} &= S_{yx} \cos^2 \phi - S_{xy} \sin^2 \phi + (S_{yy} - S_{xx}) \sin \phi \cos \phi
\end{align*}
\]

(27)

For the particular applications considered here, these expressions may be simplified in accordance with the properties of the \( S_{ik} \) derived in §§ 2.1 and 2.2.

By means of (27) \( \Theta \) and \( \Delta \) may be evaluated for any required combination of propagation and polarization directions; a significant variation with \( \phi \) is obtained, provided \( \omega \) and \( \Omega \) satisfy the conditions mentioned in § 3. Typical results for the microwave region are illustrated in figure 5, in which \( \Theta \) is plotted as a function of \( \phi \) for various directions of propagation, with the same values of \( \omega, H_0 \) and \( z \) as in figure 1. The curves are seen to be roughly sinusoidal in character, with a periodicity of \( \pi \) and an approximate separation of \( \pi/2 \) between maxima and minima. In order to clarify the diagram
somewhat, the broken lines \((H \text{ in } (1\overline{1}0) \text{ plane})\) have been displaced through 45°, which corresponds to the \(y'\) axis parallel to the \([\overline{1}10]\) direction for all values of \(\phi\).

This azimuthal variation is most pronounced in the case of propagation along \([110]\)-type directions and here the experimentally significant quantity, namely the maximum change in a cycle divided by the mean value, is of the order of 30% with a field of 20 kOe. An effect of this magnitude clearly suggests that useful information may be obtained by merely rotating the plane of polarization. Confirmatory evidence on the band structure, for example, could be obtained from a comparison of experimental and theoretical curves; any marked lack of agreement might throw light on anisotropy originating elsewhere, e.g. in the relaxation time or due to intervalley scattering. Numerical calculations have also been carried out for the ellipticity, which exhibits similar behaviour to that shown in figure 5.

![Figure 5. Variation of Faraday rotation with orientation of plane of polarization for various orientations of magnetic field in the planes indicated: \(a\), \(\phi = 15°\); \(b\), \(\phi = 30°\); \(c\), \(\phi = 45°\); \(d\), \(\phi = 70°\). \(\lambda = 3 \text{ cm}, H = 2 \times 10^4 \text{ Oe}, z = 5 \text{ mm.}\)](image)

Referring back to figure 2, curves \(c\) and \(d\) illustrate the frequency dependence of \(\Theta\) for the orientations \(\psi = 0\) and 45° respectively on curve \(c\) of figure 5; the corresponding effect for the ellipticity is shown by curves \(c\) and \(d\) of figure 3. The variation of \(\Theta\) with field strength is shown in figure 4, in which curves \(c\) and \(d\) refer to \(\psi = 0\) and 60° respectively on curve \(c\) of figure 5. These examples indicate that, with suitable values of \(\omega\) and \(H_0\), a substantial azimuthal effect may be obtained. This effect does not depend significantly upon path length, but decreases slightly as \(\alpha z\) and \(\beta z\) approach unity, when the maximum and minimum values of \(\Theta\) (or \(\Delta\)) move to \(\psi \approx 60°\) and 150° (as in figure 5). For smaller thicknesses, such that \(\alpha z\) and \(\beta z\) are both small compared with unity, the extreme values occur almost exactly at \(\psi = \pi/4\) and 3\(\pi/4\).

**§ 5. CONCLUSION**

In this paper the theory of the anisotropic Faraday effect has been applied to the case of n-type germanium, using the multi-ellipsoid model for the conduction band. The object has been to obtain quantitative information on the anisotropic variation of
the rotation and ellipticity, and its dependence upon frequency and field strength. The
numerical calculations were performed on the University of London Mercury computer
and the programmes may readily be adapted, if necessary, to deal with different values
of the intrinsic parameters of the material.

With field strengths up to about 30 kOe, the chief interest attaches to the microwave
region, and the detection of anisotropic behaviour would appear to be eminently feasible
with the techniques currently available (cf. Furdyna and Broersma 1960). For infra-red
studies, with the same field strengths, anisotropic complications are negligible (in cubic
crystals), but with fields of order 200 kOe the anisotropic theory would be essential.
It may be mentioned that an effect strikingly similar to figure 5, curve c, has been
observed in the course of experiments on thin nickel films, with the light travelling
along the [110] direction (Dr. R. F. Miller, private communication). The rotation of
the plane of polarization provides an experimental method of relative simplicity and its
exploitation should have interesting consequences in magneto-optical studies of aniso-
tropic semiconductors.

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