STUDIES IN NON-LINEAR OPTICS

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BY

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Abstract

A general method of approach to resonant non-linear optical phenomena involving travelling waves has been developed. Maxwell's equations are solved for the electric field in a spherical Fabry-Perot type optical resonator, which encloses uniaxial anisotropic media. The specific case of propagation perpendicular to the optic axis is considered but the theory can be extended to cover the general case including double refraction. In the presence of more than one optical field of this form, when the medium enclosed in the resonator is non-linear, by expanding the polarization in terms of the electric field in the normal way coupled mode equations are obtained for amplitudes of the eigenmodes concerned.

This general formalism is then used to examine resonant second harmonic generation in the small conversion approximation from a fundamental beam in the lowest order 0-0 mode. Analytical solutions are obtained in three limiting cases, weak focussing, strong focussing and the focus removed infinitely from the non-linear medium. The general case is solved numerically. From the results the values of the variable parameters can be obtained which give the maximum output in any given mode. Graphs are presented giving the output variation in 0-0 and 0-2 modes with focus position, phase matching, focussing and spot size. From there it can be seen that the optimum focussing for the 0-0 mode occurs at \( \frac{1}{z_0} = 5.65 \) (1 crystal length, \( z_0 \) one half the confocal parameter).
Secondly degenerate parametric amplification between two lowest order modes is examined under the approximation that the pump beam is underpleted. An analytic solution is obtained for the single pass amplification in the weak focusing limit and preliminary results of numerical computations for the general case are given. From these results a value for the optimum threshold condition is calculated.
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Chapter 1

1.1 Introduction

The term "non-linear optics" is used in this thesis to describe processes in which energy exchange occurs between a number of optical fields at different frequencies. The discovery and development of lasers made available for the first time extremely high electric fields 100kV/cm at optical frequencies which although still small compared with atomic field strengths 10^5 kV/cm make the classical assumption of a linear response of the material to the field untrue. When the non-linearities are included in the theory they not only modify the linear effects but also give rise to completely new non-linear effects.

The first observation of these effects was made by Franken et al (1) in 1961 using a ruby laser focussed into a quartz crystal. The emergent light was found to contain radiation not only at 0.6943 microns, the ruby laser wavelength, but also at 0.3472 microns the second harmonic wavelength. Since then many other non-linear processes have been observed.2, 3, 4, 5.

If a dielectric medium is subjected to an optical electric field \( E \) which is small it sets up a polarization in the medium that is linear in the field.

\[
P = \beta E \tag{1.1}
\]

(ignoring the vector nature of the electric field)

At higher field strengths we have to consider the right hand side of
The non-linear effects arise from the higher order terms of the non-linear optical effects which are from the general expansion (1.2).

\[ 1.4 \quad \rho = \text{tr} \rho_0 \epsilon_{\text{exc}} \]

where \( \rho \) is the density matrix and \( \epsilon_{\text{exc}} \) is the excitation density of the medium.

The non-linear effects arise from the higher order terms of the non-linear optical effects which are from the general expansion (1.2).

\[ \rho = \rho_0 \epsilon_{\text{exc}} \]

where \( \rho \) is the density matrix and \( \epsilon_{\text{exc}} \) is the excitation density of the medium.
\[ F = -\omega_0^2 r + \lambda r^2 + \ldots \quad \text{(small)} \]

and the displacement of the electron is now governed by the equation.

\[ \frac{d^2 r}{dt^2} + 2\gamma \frac{dr}{dt} + \omega_0^2 r - \lambda r^2 = \frac{-eE}{m} \]

The solution of which for an electric field of the form

\[ E = E_0 e^{-i\omega_1 t} + E_0 e^{i\omega_1 t} \]

is

\[ r = \frac{e}{m} \frac{E(\omega_1) e^{-i\omega_1 t}}{(\omega_0^2 - 2i\gamma\omega_0 - \omega_1^2)} \]

\[ + \frac{e^2}{m^2} \lambda \frac{E^2(\omega_1)}{(\omega_0^2 - 4i\gamma\omega_0 - 4\omega_1^2)(\omega_0^2 - 2i\gamma\omega_0 - \omega_1^2)^2} \]

\[ + \frac{e^2}{m^2} \lambda \frac{E(\omega_1)E^*(\omega_1)}{\omega_0^2 (\omega_0^2 + 2i\gamma\omega_0 - \omega_1^2)(\omega_0^2 + 2i\gamma\omega_0 - \omega_1^2)} \]

\[ + \text{complex conjugate} + \ldots \]

It can be seen that we have new terms over the solution to the linear equation which are quite distinct from the linear terms. The first will give rise to a polarization oscillating at the first harmonic frequency of the applied electric field and the second a constant polarization proportional to the magnitude of the applied electric field.
1.3 The Susceptibility Tensors.

(a) Definition. Since we will be dealing with optical fields which are monochromatic or nearly monochromatic it is more convenient to work with the Fourier transform of the electric field.

\[ E(\mathbf{r}, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, E(\mathbf{r}, t) \exp(i\omega t) \]

than with the electric field itself.

From 1.9 it follows that

\[ E(\mathbf{r}, \omega) = E^*(\mathbf{r}, -\omega) \] since \( E(\mathbf{r}, t) \) is real.

The Fourier transform of the monochromatic field defined in equation 1.7 is given by

\[ E(\omega) = E(\omega_1 - \omega) + E^*(\omega_1 + \omega) \]

The polarization response of the medium in terms of the Fourier transforms of the electric field is given by:

(1) for the linear response

\[ P_{1i}^{(1)}(t) = \int_{-\infty}^{\infty} d\omega \, x_{ij}(\omega) E_j(\omega) \exp(i\omega t) \]

which defines the linear susceptibility tensor.

\( x_{ij}(\omega) \)

for the second order response:

\[ P_{1i}^{(2)}(t) = \int_{-\infty}^{\infty} d\omega_1 d\omega_2 \, x_{ijk}(\omega_1, \omega_2) E_j(\omega_1) E_k(\omega_2) \]

defining the 2nd order susceptibility tensor.

Since \( P_{1i}(t) \) is real we have
1.13 \( x_{ij}(\omega) = x_{ij}^*(-\omega) \)

1.14 \( x_{ijk}(\omega_1, \omega_2) = x_{ijk}^*(-\omega_1, -\omega_2) \)

The order of writing the electric fields in equation 1.12 has no significance, therefore

1.15 \( x_{ijk}(\omega_1, \omega_2) = x_{ijk}(\omega_2, \omega_1) \)

The Fourier transform of the polarization is defined as

1.16 \( P_i(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P_i(t) \exp(i\omega t) \, dt \)

Substituting for \( P_i(t) \) from 1.12 and carrying out two of the integrations we have

1.17 \( P_i^{(1)}(\omega) = x_{ij}(\omega) E_j(\omega) \) and substituting \( \omega^1 \) for \( \omega_1 \)

1.18 \( P_i^{(2)}(\omega) = \int d\omega^1 x_{ijk}(\omega^1, \omega - \omega^1) E_j(\omega^1) E_k(\omega - \omega^1) \)

To show that this corresponds with the normal definition of the non linear susceptibility consider the case of sum mixing of two monochromatic fields.

1.19 \( E_1(\omega_1) = \frac{1}{2} E_1 \omega_1 \delta(\omega - \omega_1) + \frac{1}{2} E_2 \omega_1 \delta(\omega + \omega_1) \)

1.20 \( E_2(\omega_2) = \frac{1}{2} E_1 \omega_2 \delta(\omega - \omega_2) + \frac{1}{2} E_2 \omega_2 \delta(\omega + \omega_2) \)

Substituting into 1.18 and carrying out the integration picking out the coefficients of \( \delta(\omega - \omega_1 + \omega_2) \) and \( \delta(\omega + \omega_1 + \omega_2) \)
\[ P_1(\omega_3) = \left( \frac{i}{2}X_{ijk}(\omega_1, \omega_2) E_j^1 E_k^2 + \frac{i}{2}X_{ijk}(\omega_2, \omega_1) E_j^2 E_k^1 \right) \times \delta(\omega - \omega_1 + \omega_2) + \left( \frac{i}{2}X_{ijk}(\omega_1, \omega_2) E_j^1 E_k^2 + \frac{i}{2}X_{ijk}(\omega_2, \omega_1) E_j^2 E_k^1 \right) \times \delta(\omega + \omega_1 + \omega_2) \]

where \( \omega_3 = \omega_1 + \omega_2 \)

using the intrinsic symmetry of \( X_{ijk} \) and changing the dummy suffices.

\[ P_1(\omega_3) = \frac{1}{2}X_{ijk}(\omega_1, \omega_2) E_j^1 E_k^2 \delta(\omega - \omega_1 + \omega_2) + \frac{1}{2}X_{ijk}(\omega_1, \omega_2) E_j^2 E_k^1 \delta(\omega + \omega_1 + \omega_2) \]

which is just the form for monochromatic polarisation at frequency \( \omega_3 = \omega_1 + \omega_2 \)

and we can write

\[ P_1(\omega_3) = X_{ijk}(\omega_1, \omega_2) E_j^1 E_k^2 \delta(\omega - \omega_3) + \frac{1}{2}X_{ijk}(\omega_1, \omega_2) E_j^2 E_k^1 \delta(\omega + \omega_3) \]

For second harmonic generation

\[ E_j(\omega_1) = \frac{1}{2} X_{ijk}(\omega_1, \omega_2) E_j^1 E_k^2 \delta(\omega - \omega_1 - \omega) + \frac{1}{2} X_{ijk}(\omega_1, \omega_2) E_j^2 E_k^1 \delta(\omega + \omega_1 + \omega) \]

so the polarization at the harmonic frequency is

\[ P_1(2\omega) = \frac{1}{2}X_{ijk}(\omega_1, \omega_2) E_j^1 E_k^2 \delta(\omega - 2\omega_1) + \frac{1}{2}X_{ijk}(\omega_1, \omega_2) E_j^2 E_k^1 \delta(\omega + 2\omega_1) \]

so that \( P_1^2(2\omega) = \frac{1}{2}X_{ijk}(\omega_1, \omega_2) E_j^1 E_k^2 \delta(\omega + 2\omega_1) \)

This definition is the same as Bloembergen, 9, Robinson, 11, but will differ from that of Kleinman, 12, when the \( d_{lm} \) tensor is defined in
if the medium has inversion symmetry the susceptibility tensors must be invariant under this transformation.

\[ A_{ij} = -\delta_{ij} \]

so the second order tensor vanishes identically.

In general the process of finding all the restrictions is rather tedious so we will consider one special case that of Lithium Niobate Li Nb O\(_3\) for reasons which will become apparent later. Lithium Niobate has point group symmetry 3m, \( \text{C}_{3v} \), a threefold axis of symmetry with three reflection planes passing through the rotation axis.

The diagram represents a project of a sphere with points marked on the top surface at the positions marked by the = signs. The axes are taken as shown with the Ox, axis being the axis of threefold symmetry vertical through the centre.

The rotation transformation is given by the matrix

\[ x_{ijk} (\omega_1, \omega_2) = (-1)^3 x_{ijk} (\omega_1, \omega_2) \]
equation 1.36.

b. Spatial Symmetry.

The susceptibility tensors transform between different co-ordinate systems according to the usual transformation laws for polar tensors. If we transform the tensor with a symmetry transformation of the medium the new tensor must be identical with the old one. Hence this restricts the values of the elements of the tensor. A full description of symmetry groups is given in the International Tables for X-ray Crystallography, 13, and Nye, 12.

The transformation laws are easily derived. Consider two co-ordinate systems which are related to each other by a rotation. The co-ordinates of a point with respect to the first set of axes will be related to those of the second set by an orthogonal linear transformation.

\[ X^i' = A_{ij} X^j \]

The electric field and polarization being both polar vectors transform in the same way as the co-ordinates.

\[ P_i'(t) = A_{ij} P_j(t) \]

\[ E_i'(\omega) = A_{ij} E_j(\omega) \]

Thus using 1.11, 1.12, the susceptibility tensors transform as

\[ X^i_{ij}(\omega) = A_{ir} A_{js} X_{rs}(\omega) \]

\[ X^i_{ijk}(\omega_1, \omega_2) = A_{ir} A_{js} A_{kt} X_{rst}(\omega_1, \omega_2) \]

As a general example we consider the inversion transformation
and the reflection transformations

\[ R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

Consider the effect of the reflection on \( X_{ijk} (\omega_1, \omega_2) \)

\[ X_{221} (\omega_1, \omega_2) = R_{2i} R_{2j} R_{1k} X_{ijk} (\omega_1, \omega_2) \]

\[ = R_{2i} R_{2j} \cdot X_{i j 1} (\omega_1, \omega_2) \]

\[ = -X_{221} (\omega_1, \omega_2) \]

1.30 \( \therefore X_{221} (\omega_1, \omega_2) = 0 \)

Similarly any element which has an odd number of subscripts equal to one is identically zero.

This reduces the two tensors to

\[ X^\oplus = \begin{pmatrix} X_{11} & 0 & 0 \\ 0 & X_{22} & X_{23} \\ 0 & X_{32} & X_{33} \end{pmatrix} \]
1.32 \[ X^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & x_{131} & x_{113} & x_{112} & x_{121} \\ x_{211} & x_{222} & x_{223} & x_{233} & x_{232} & 0 & 0 & 0 \\ x_{311} & x_{322} & x_{333} & x_{323} & x_{332} & 0 & 0 & 0 \end{pmatrix} \]

Applying the rotation transformation these reduce further to

1.33 \[ X^{(1)} = \begin{pmatrix} x_{11} & 0 & 0 & 0 \\ 0 & x_{11} & 0 & 0 \\ 0 & 0 & x_{33} & 0 \end{pmatrix} \]

And

1.34 \[ X^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & x_{131} & x_{113} & -x_{222} & -x_{222} \\ -x_{222} & x_{222} & 0 & x_{113} & x_{131} & 0 & 0 & 0 \\ x_{311} & x_{311} & x_{333} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]

For the case of second harmonic generation we are dealing with the tensor \( X_{ijk}(\omega_1, \omega_1) \) which from equation 1.15 is symmetric in \( j \) and \( k \). This symmetry reduces the number of independent elements of \( X^{(2)} \) still further since \( x_{113} = x_{131} \) and we can write the tensor in a reduced form similar to that used for piezoelectric tensors (33).

Define \( d_{im} = x_{ijk}(\omega_1, \omega_1) \) when \( j = k \).

1.35 \[ d_{im} = \frac{1}{2} \left( x_{ijk}(\omega_1, \omega_1) + x_{ikj}(\omega_1, \omega_1) \right) \] when \( j \neq k \)

When \( m \) is given by

\[ m = 1 \ 2 \ 3 \ 4 \ 5 \ 6 \]

When \( jk = 11 \ 22 \ 33 \ 23 \ 31 \ 12 \)
Thus for crystals with symmetry point group 3m as Lithium Niobate

\[
\begin{pmatrix}
0 & 0 & 0 & d_{15} & -d_{22} \\
-d_{22} & d_{22} & 0 & d_{15} & 0 \\
d_{31} & d_{31} & d_{33} & 0 & 0
\end{pmatrix}
\]

The 3 x 6 matrix \( d_{im} \) operates on a column vector \((E E)_m\) where

\[
(E E)_1 = E_1^2, \quad (E E)_2 = E_2^2, \quad (E E)_3 = E_3^2,
\]

\[(E E)_4 = E_1 E_2, \quad (E E)_5 = E_1 E_3 \quad \text{and} \quad (E E)_6 = E_1 E_2.\]

c. **Kleinman's Symmetry Condition**

In addition to these symmetry restrictions there are other restrictions which apply when the non linear polarization is of electronic origin rather than ionic and the crystal is non absorbing for all the optical fields concerned in the interaction.

So the power loss \( P \) of the field \( E(x, t) \) is zero. These restrictions were first put forward by Kleinman, 12, and the argument given here follows that given by him, 15.

The average power loss in time \( 2T \) is given by

\[
P = -\frac{1}{2T} \int_{-T}^{T} dt \int dr \int \frac{d}{dt} E_i^* (r, t) \cdot \frac{d}{dt} E_i (r, t).
\]

and must be zero under our approximations. Replacing \( E \) and \( P \) by their Fourier transforms,

\[
P = \frac{2\pi}{T} \int \int dr \int d\omega \omega^1 E_i^* (r, \omega) P_i (r, \omega^1) \omega^1 \delta (\omega - \omega^1)
\]
where

\[ \text{as } T \to \infty, \frac{\delta_T}{T}(\Omega) \to \delta(\Omega) \]

for \( T \) large enough

\[ P = i \int d^3r \int d\omega \ E_i^*(\xi, \omega) \ p_i(\xi, \omega) \ \omega \]

substituting from 1.18.

\[ 1.38 \ P = i \int d^3r \int d\omega \int d\omega' E_i^*(\xi, \omega) X_{ijk}(\omega^1, \omega - \omega^1) \]

\[ \times E_j(\xi, \omega^1) E_k(\xi, \omega - \omega^1) \]

writing \( \omega = \omega^1 + (\omega - \omega^1) \)

Since \( X \) is symmetric in the last two subscripts, we see that the integral with \( \omega^1 \) is equal to that with \( \omega - \omega^1 \).

so

\[ P = 2i \int d^3r \int d\omega \int d\omega' E_i^*(\xi, \omega) X_{ijk}(\omega^1, \omega - \omega^1) \]

\[ \times E_j(\xi, \omega^1) E_k(\xi, \omega - \omega^1) \]

using \( E(\xi, \omega) = E^*(\xi, -\omega) \) and re-arranging we have

\[ 1.39 \ P = -2i \int d^3r \int d\omega \int d\omega' E_i^*(\xi, \omega) X_{ijk}(-\omega, \omega - \omega^1) \]

\[ \times E_j(\xi, \omega^1) E_k(\xi, \omega - \omega^1) \]

but since \( P = 0 \)

comparing with 1.38, they are both equal to zero, therefore we have

\[ X_{ijk}(-\omega, \omega - \omega^1) = X_{ijk}(\omega^1, \omega - \omega^1) \]

or writing \( X \) with its three associated frequencies.
1.40 \( x_{ijk}(\omega_2, \omega_3, \omega_1) = x_{ijk}(\omega_3, \omega_2, \omega_1) \)

Armstrong et al, 16, and others have carried out Quantum Mechanical calculations for the form of the \( X \)'s under the conditions we have stated and found this symmetry and also the further symmetry.

1.41 \( x_{ijk}(\omega_3, \omega_2, \omega_1) = x_{kji}(\omega_1, \omega_2, \omega_3) \) is predicted.

Ward and Franken, 19, have discussed these symmetries in the presence of dispersion and absorption and concluded that they are relatively insensitive to small amounts of dispersion and absorption. The measurements that have been carried out seem to verify these predictions. Several tables of values appear in the literature, e.g. Robinson, 11, and and Yariv, 20, (Yariv defines his values as Kleinman, 15, et al, Robinson, 11, as Bloembergen et all, 9) 1.4.

**Propagation of E.M. radiation in a Dielectric medium**

The propagation of electromagnetic radiation in a dielectric whose magnetic polarization is negligible is described by solution of Maxwell's equations.

\[
\nabla \times \vec{E} = -\frac{1}{c} \frac{d\vec{H}}{dt}
\]

1.42 \( \nabla \times \vec{H} = \frac{1}{c} \frac{d\vec{D}}{dt} \)

where \( \vec{D} = \vec{E} + 4\pi \vec{P} \)

Eliminating the magnetic field \( \vec{H} \) from these equations in the usual way produces the equation

\[
\nabla \times (\nabla \times \vec{E}) + \frac{1}{c^2} \frac{d^2\vec{E}}{dt^2} = \frac{-4\pi}{c^2} \frac{d^2\vec{P}}{dt^2}
\]
for the electric field.

Where $E$ and $P$ are functions of $\mathbf{r}$ and $t$. Taking the Fourier transform of this equation it produces

$$1.44 \nabla \times (\nabla \times E(\mathbf{r}, \omega)) - \frac{\omega^2}{c^2} E(\mathbf{r}, \omega) = \frac{4\pi \omega^2}{c^2} P(\mathbf{r}, \omega)$$

under the same conditions of convergence as equation 1.9 which are discussed by Butcher, 22, $P(\mathbf{r}, \omega)$ as we have seen can be expanded in terms of the electric field $E$ and the susceptibility tensors.

a. Medium with linear response

The polarization in this case is given by

$$1.45 P'(\mathbf{r}, \omega) = X^\Omega(\omega) E(\mathbf{r}, \omega)$$

substituting into 1.44 and dropping the explicit frequency label, which is an unnecessary complication in the linear regime, we have

$$1.46 \nabla \times (\nabla \times E) - \frac{\omega^2}{c^2} \varepsilon \cdot E = 0$$

where

b. $\varepsilon = 1 + 4\pi X^\Omega$

To investigate the propagation of a plane wave in this medium we write

$$1.47 \mathbf{E} = \mathbf{E}_0 \exp \left( \frac{i}{c} \omega \gamma \mathbf{z} \cdot \mathbf{r} \right)$$

where $\mathbf{r}$ is the position vector, $\mathbf{z}$ the direction of propagation and $\gamma$ the refractive index, $\mathbf{z}$ the direction of the polarization which are to be obtained. Substituting 1.47 into 1.46 we obtain Fresnel's equation

$$1.48 \gamma^2 (\mathbf{z} \times (\mathbf{z} \times \mathbf{E})) + \varepsilon \cdot \mathbf{E} = 0$$
At this point we assume that $\varepsilon$ is real, i.e. the medium is non-absorbing. Equation 1.48 represents a system of simultaneous homogenous linear equations for $a = (a_1, a_2, a_3)$. The equations are simplified by choosing axes along the principal axes of the dielectric tensor, they will be the same as those of $X^\Theta(\omega)$ from 1.46(b). Applying the consistency condition gives a quadratic equation for $\gamma^2$. For an isotropic medium where

$$\varepsilon = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}$$

The equation for $\gamma^2$ reduces to

$$(\gamma^2 - \varepsilon)^2 = 0$$

Substituting this into 1.48 we have $a \cdot \bar{n} = 0$ determining $a$.

In a uniaxial medium, such as the previous example, lithium niobate, the dielectric tensor has the form of equation 1.33

$$\varepsilon = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_2 \end{bmatrix}$$

The orientation of the $x$ and $y$ principal axes is arbitrary and it is convenient to choose them so that $\bar{n}$ lies in the Oxz plane and we can put

$$n_x = \sin \Theta, n_y = 0, n_z = \cos \Theta$$

where $\Theta$ is the angle between the $z$ axis and the direction of propagation. In this case the equation for $\gamma^2$ reduces to
\(1.49 \ (\eta^2 - \varepsilon_1) (\eta^2 (\varepsilon_2 \cos^2 \theta + \varepsilon_1 \sin^2 \theta) - \varepsilon_1 \varepsilon_2) = 0\)

Thus for one solution \(\eta^2 = \varepsilon_1\) which is independent of the direction of propagation. For this solution from 1.48

1.50 \(\vec{a} = (0, 1, 0)\)

i.e. \(\vec{a}\) is orthogonal to both the direction of propagation and the \(z\) axis as in figure 1.1.

The second solution has refractive index given by.

1.51 \(\frac{1}{\eta^2} = \frac{\cos^2 \theta}{\varepsilon_2} + \frac{\sin^2 \theta}{\varepsilon_1}\)

This is the extraordinary wave for which the refractive depends on the direction of propagation. The solution for \(\vec{a}\) has \(a_x = 0\) and

1.52 \(a_x = \frac{\varepsilon_1 - \eta^2 \cos^2 \theta}{\eta^2 \sin \theta \cos \theta}\)

So that \(\vec{a}\) lies in the plane of \(0_x\) and \(\vec{n}\) and is inclined to \(\vec{n}\) at an angle of \(\frac{\pi}{2} - \alpha\) where

1.53 \(\tan \alpha = \frac{1}{2} \eta^2 (\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}) \sin 2\theta\)

The energy flux due to the wave is given by

1.54 \(S = \frac{c}{4\pi} \vec{E} \times \vec{H}\)

from 1.42 \(\vec{H}\) is parallel to \(\vec{n} \times \vec{a}\) so that \(S\) lies in the plane of \(\vec{n}\) and the \(z\) axis but inclined to \(\vec{n}\) at the angle \(\alpha\) given by 1.53. That is the energy of the extraordinary wave propagates at an angle, \(\alpha\), to the direction of phase propagation of the wave (see figure 1.2). Figure 1.3 represents the refractive index as distance
from the origin of both ordinary and extraordinary waves with the direction of propagation for a negative uniaxial crystal.

b. Absorption

The imaginary part of $\varepsilon$ accounts for absorption. Since for any medium of interest absorption will be small we can consider it as a small perturbation on the previous theory. Let the perturbed refractive index be $N$ and polarization vector be $\mathbf{b}$. So Fresnel's equation becomes

$$1.55 \ N^2 \left( n \times (n \times \mathbf{b}) \right) + \varepsilon^1 \mathbf{b} + i \varepsilon^{II} \mathbf{b} = 0$$

where we have written

$$\varepsilon = \varepsilon^1 + i \varepsilon^{II}$$

Taking the scalar product with $\mathbf{a}$, the old polarization vector, and using the symmetry of $\varepsilon^1$ (i.e. $\mathbf{a} \cdot \varepsilon^1 \mathbf{b} = \mathbf{b} \cdot \varepsilon^1 \mathbf{a}$) we can write

$$1.56 \ N^2 \left( \mathbf{a} \cdot n \times (n \times \mathbf{b}) \right) - \gamma^2 \mathbf{b} \cdot (n \times (n \times \mathbf{a})) + i \mathbf{a} \cdot \varepsilon^{II} \mathbf{b} = 0$$

rearranging

$$1.57 \ N^2 = \gamma^2 + \frac{i \mathbf{a} \cdot \varepsilon^{II} \mathbf{b}}{(n \times \mathbf{a})(n \times \mathbf{b})}$$

assuming $\mathbf{a}$ is very nearly equal to $\mathbf{b}$ and $\varepsilon^{II}$ is small we can write using the binomial expansion

$$N = \gamma + iK$$

1.58 where

$$K = \frac{\mathbf{a} \cdot \varepsilon^{II} \mathbf{a}}{2 \gamma (n \times \mathbf{a})^2}$$

Substituting this into the original form for the wave 1.47, we have
1.58 a. \( E = E_0 \exp\left( \frac{i \omega}{c} \mathbf{n} \cdot \mathbf{r} \right) \exp\left( -\frac{\omega a \cdot \mathbf{n} \cdot \mathbf{r}}{2\sigma \gamma (\mathbf{n} \times \mathbf{a})^2} \right) \)

so \( K \) is the extinction coefficient of the wave.

c. **Medium with Quadratic Response**

Including the quadratic response of the polarization to the electric field into Maxwell's equations 1.42 we can write equation 1.46 more generally as

\[ \nabla \times (\nabla \times E(\mathbf{r}, \omega)) - \frac{\omega^2}{c^2} \varepsilon(\omega) E(\mathbf{r}, \omega) = \frac{4\pi \omega^2}{c^2} \mathbf{P}^2(\mathbf{r}, \omega) \]

The solution to this equation for infinite plane waves has been studied by many authors, 16, 17, 18, 21. We will consider here the main points.

We assume that none of the frequencies involved approach the singulatities of the second order susceptibility tensor. This tensor is then real and obeys Kleinman's symmetry conditions and \( \mathbf{P}^2 \) is small and can be regarded as a perturbation on the linear equation which we have seen has solutions of the form

\[ E = a \exp\left( i \frac{\omega}{c} \mathbf{n} \cdot \mathbf{r} \right) \]

We write the perturbed electric field as, 22,

\[ E = (A(\xi)a + \varphi(\xi)) \exp\left( i \frac{\omega}{c} \gamma \xi \cdot \mathbf{r} \right) \]

where \( \xi = \mathbf{n} \cdot \mathbf{r} \) a co-ordinate in the direction of propagation. \( A(\xi) \) is a slowly varying amplitude factor, \( \varphi(\xi) \) is a slowly varying correction term which arises from the components of \( \mathbf{P}^2 \) orthogonal to \( \mathbf{a} \).

Substituting this into equation 1.59 and using Fresnel's equation
and some vector manipulation, 22, we obtain the equation

\[ \frac{d}{ds} A(\mathbf{S}) = \frac{4\pi i \omega}{c} \mathbf{a} \cdot \mathbf{P}^{(2)}(\omega) \frac{\mathbf{a} \cdot \mathbf{P}^{(2)}(\omega)}{(\mathbf{n} \times \mathbf{a})^2} \exp \left( i \frac{\omega}{c} \eta(\omega) \mathbf{S} \right) \]

for the amplitude variation of the field at frequency \( \omega \).

For the simplest case when \( A(\mathbf{S}) \) represents the amplitude of the second harmonic of a field at frequency \( \omega_1 (= \omega/2) \), the polarization takes the form.

\[ \mathbf{a} \cdot \mathbf{P}^{(2)}(\omega) = \frac{E_0^2}{2} a_i x_{ijk} (\omega_1, \omega_1) b_j b_k \]

\[ \times \exp \left( i \frac{2\omega_1}{c} \eta(\omega_1) \mathbf{S} \right) \]

The driving fundamental field being of the form

\[ E_{\omega_1} = E_0 \cos \left( i \frac{\omega_1}{c} \eta(\omega_1) \mathbf{S} \right) \]

Therefore

\[ \frac{d}{ds} A(\mathbf{S}) \propto \frac{E_0^2}{c} \exp \left( i \Delta k \mathbf{S} \right) \quad \text{where} \]

\[ \Delta k = \frac{2\omega_1}{c} \left( \eta(\omega_1) - \eta(2\omega_1) \right) \]

The solution of this equation may be written down immediately if we assume that the second harmonic field is small enough not to affect the amplitude of the fundamental field (i.e. \( E_0 \) is constant), then

\[ A(\mathbf{S}) \propto \frac{\sin \left( \frac{\Delta k \cdot \mathbf{S}}{2} \right)}{\Delta k/2} \]

We see that \( A(\mathbf{S}) \) oscillates sinusoidally when \( \Delta k \neq 0 \) and grows
linearly with \( S \) when \( \Delta k = 0 \). Most transparent optical materials exhibit normal dispersion. Thus the condition of perfect phase matching \( \Delta k = 0 \) cannot be achieved in an isotropic medium or if both waves are the same type in a uniaxial or biaxial medium. However as first pointed out by Giordmaine, 51 and Terhune, 52, the directional dependence of the extraordinary refractive index can be employed to balance out the effects of dispersion in some materials. For potassium dihydrogen phosphate one of the first materials which proved suitable a second harmonic extraordinary wave has the same refractive index as its fundamental ordinary wave when they propagate at an angle of 50° to the optic axis for a fundamental wavelength of 0.6943 microns. Then the polarization vectors can be written \( \mathbf{a} = (0, 1, 0), \)
\( \mathbf{b} = (\cos 50, 0, -\sin 50) \) neglecting the small correction term. We see from equation 1.53 that the energy contained in the second harmonic wave propagates at an angle \( \alpha \) to that of the fundamental wave so the interaction between the two waves will cease after some characteristic length. This is known as the aperture effect. This drawback can be overcome if phase matching could be achieved in a direction perpendicular to the optic axis when \( \alpha = 0 \). This case has also the added advantage that phase matching is not as critical since now the refractive index curves touch rather than intersect. The refractive index of lithium niobate is temperature dependent and obeys the Sellmeier equations, 23,

\[
1.66 \quad \eta_0^2 = 4.19130 + \frac{1.173 \times 10^5 + 1.65 \times 10^{-2} \lambda^2}{\lambda^2 - (2.12 \times 10^2 + 2.7 \times 10^{-5} \lambda^2)^2}
\]
\[ -2.78 \times 10^{-8} \lambda^2 \]

\[ 1.67 \lambda^2 = 4.5567 + 2.605 \times 10^{-7} T^2 \]

\[ + \frac{0.970 \times 10^5 + 2.70 \times 10^{-2} m^2}{\lambda^2 - (2.01 \times 10^2 + 5.4 \times 10^{-5} m^2)^2} - 2.24 \times 10^{-8} \lambda^2 \]

\( \lambda \) wavelength \( 10^{-7} \) cms. (i.e. nms.)

\( T \) Temperature \( \circ K \)

for wavelength between 0.4 and 4.0 microns and phase matching can be achieved in a perpendicular direction to the optic axis for a variety of wavelengths (24, 26).

The polarization vectors for this case are \( \mathbf{a} = (0, 1, 0) \), \( \mathbf{b} = (0, 0, 1) \) and

\[ \frac{1}{2} a_1 X_{ijk}(\omega_1, \omega_1) b_j b_k = \frac{1}{2} d_{31} \text{ from 1.36.} \]

The process involving this tensor element dominates since the others are not phase matched.

For parametric effects when the medium is subjected to two optical fields at frequencies \( \omega_1 \) and \( \omega_2 \) of the form

\[ E^{\omega_1} = E_{01} b_1 \exp \left( \frac{\omega_1}{c} \gamma(\omega_1) \right) \]

\[ E^{\omega_3} = E_{03} b_3 \exp \left( \frac{\omega_3}{c} \gamma(\omega_3) \right) \]

\[ \omega_3 > \omega_1 > 0 \]

the quadratic polarization then contains the difference frequency term
\[ a \cdot p \otimes (\omega_2) = E_{03} E_0^* a_{i} X_{ijk} (\omega_3, -\omega_1) b_{3j} b_{1k} \]
\[ \times \exp \left( i \frac{c}{\lambda} (\omega_3 \gamma(\omega_3) - \omega_1 \gamma(\omega_1)) \right) \]

from equation 1.23

Again the interaction will depend on phase matching term

\[ \Delta k = \frac{1}{c} (\omega_3 \gamma(\omega_3) - \omega_1 \gamma(\omega_1) - \omega_2 \gamma(\omega_2)) \]

and the difference frequency will only be exited to a significant amount if \( \Delta k = 0 \).

Attempts at optical parametric amplification have been made using Lithium Niobate 25, for the degenerate case \( \omega_1 = \omega_2 \). In this case phase matching occurs if the pump wave at \( \omega_3 \) is extraordinary i.e. \( \mathbf{b}_3 = (0, 0, 1) \) and the subharmonic wave is ordinary \( \mathbf{a} = \mathbf{b}_1 = (0, 1, 0) \) then from 1.36

\[ a_{i} X_{ijk} (\omega_3, -\omega_1) b_{3j} b_{1k} = d_{15} \]

as long as \( X_{131} = X_{113} \) which is true under our assumptions that \( \omega_3 \) is much less than the frequencies of the electronic transitions.

Enhancement of Nonlinear Effects

The first and most obvious way to enhance the effect is to use a focussed beam. Several authors, 27-31, 53, 54 have treated second harmonic generation by focussed beams of light; they considered the focussed beam as a sum of plane wave components. Kleinman and Boyd, 15, have treated this in great detail for the case of a gaussian fundamental beam focussed into a nonlinear crystal. They have studied the variation and form of the output with focus position, focussing,
length of crystal, phase matching both theoretically and experimentally and their results are similar to those obtained here.

The second method of enhancing the effect is to place the nonlinear crystal inside a resonator of its own firstly outside the laser cavity ultimately inside. Ashkin, Boyd and Dziedzic, 32, have applied their work on focussed beams to the first case but this method of treating resonant second harmonic generation is rather clumsy and cannot be applied to the general case. We will study this problem here by solving Maxwell's equations for the field in the resonator containing the nonlinear crystal. By basing the study on the modes of the resonator we can avoid the inherent difficulties of the plane wave approach for solving the general case when there is more than one mode present.

This theory can then be applied with minor changes to the theory of parametric amplification and oscillation.
Chapter 2
The Modes of the Optical Resonator

1 Introduction

The open resonator was first proposed as a resonant cavity for lasers by Dicke, 34, and Shawlow and Townes, 35, and Prokhurov, 36, because even at optical frequencies, when the cavity dimensions must be much greater than the wavelength of the contained radiation, they only support relatively few pseudo-eigen modes of oscillation. Only waves which propagate along the axis of the resonator will be supported; any others will have very high losses. Fox and Li, 37, by making self consistent field calculations based on Huygen's principle showed that discrete modes existed both for the plane parallel mirror resonator and the spherical mirror resonator and indicated the field distribution over the mirrors and the losses of the modes. Boyd and Gordon, 38, Boyd and Kogelnik, 39, solved these equations for the spherical mirror resonator and gave an expression for the field in the resonator. I shall give a brief outline of their method to show the way in which the solutions arise and then go on to derive them as solutions to Maxwell's equations for the resonator, firstly for the case when the resonator is filled with an isotropic medium, secondly a uniaxial medium.

* pseudo in that there is a larger possibility that a photon will stay in the mode for a long time but not indefinitely, 40.
Derivation

a. Mirror Field Distribution

We consider the simplest case a confocal resonator (see figure 2.1). Two spherical mirrors, radius of curvature $b$, separated by a distance $b$. Because of the axial symmetry there will be a plane polarized solution and we can use Kirchoff's diffraction theory, 50, to give the field $E(x)$ at the point $P(x)$ due to the distribution $E(x^1)$ over the first mirror surface.

$$E(x) = \frac{1}{4\pi} \iint_S \left( E \frac{\partial \mathcal{U}}{\partial n} - \mathcal{U} \frac{\partial E}{\partial n} \right) \, ds$$

where $\mathcal{U} = \frac{e^{-ikr}}{r}$ and $r = |x - x^1|$, and $\frac{\partial}{\partial n}$ represents differentiation along the outward normal to the mirror surface $S$. Since $\frac{1}{r} << k$ at optical frequencies and

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial r} \cos \phi,$$

we can write

$$2.2 \quad \frac{\partial}{\partial n} \left( \frac{e^{-ikr}}{r} \right) = ik \cos \phi \frac{e^{-ikr}}{r}$$

and to the first order,

$$2.3 \quad \frac{\partial}{\partial n} (E(x^1)) = -ikE(x^1)$$

substituting into 2.1 we have

$$2.4 \quad E(x) = \frac{ik}{4\pi} \iint_S E(x^1) \frac{e^{-ikr}}{r} \left( 1 + \cos \phi \right) \, ds$$

for the resonator $\phi$ never departs very far from zero and $S$ is a square mirror of side $a$. 

By symmetry if \( E(x^1) \) is an eigen mode of the resonator the field distribution will reproduce itself on the second mirror apart from some loss factor. In the usual fashion solving for eigen modes we assume
\[
E(x^1) = E_o \int f_m(x^1) g_n(y^1)
\]
and so the field
\[
E(x) = E_1 \int f_m(x^1) g_n(y^1)
\]
where
\[
E_1 = \sigma_m \sigma_n E_o
\]
\( \sigma_m \sigma_n \) being the eigenvalue of the equation. Therefore
\[
2.7 \quad \sigma_m \sigma_n f_m(x) g_n(y) = \int_{-a}^{a} \int_{-b}^{b} \frac{ik}{2\pi r} e^{-ikr} f_m(x^1) g_n(y^1) dx^1 dy^1
\]
from the diagram 2.1 to the first order; \( \gamma \) is given by
\[
\gamma = b \left( 1 - \frac{xx^1 + yy^1}{b^2} \right) + \ldots
\]
substituting this into 2.7 and separating the integrals
\[
\sigma_m \sigma_n f_m(x) g_n(y) = \frac{ik}{2\pi b} \int_{-a}^{a} \int_{-b}^{b} e^{\frac{ikx^1}{b^2}} f_m(x^1) dx^1
\]
\[
x \int_{-a}^{a} e^{\frac{iky^1}{b^2}} g_n(y^1) dy^1
\]
in terms of dimensionless variables
\[
c = \frac{2k}{b} = \frac{2\pi a}{b\lambda}, \quad X = \frac{x\sqrt{c}}{a}, \quad Y = \frac{y\sqrt{c}}{a}
\]
\( f_m(x) = f_m(x) G_n(Y) = g_n(y) \)
2.9 \[ \sigma_m \sigma_n F_m (X) G_n (Y) = \frac{i e^{-ikb}}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_m (X^1) e^{iXX^1} dX^1 \]
\[ \times \int_{-\infty}^{\infty} G_n (Y^1) e^{iYY^1} dY^1 \]

The solution to this equation will be the product of the solutions to two equations of the form

2.10 \[ F_m (X) = \frac{i e^{ikb}}{\sqrt{2\pi \sigma_m}} \int_{-\infty}^{\infty} F_m (X^1) e^{iXX^1} dX^1 \]

These are given by, 41, the angular and radial wave functions in prolate spheroidal co-ordinates. The field given by these functions falls away quite rapidly from the axis so that we can approximate 2.10 by taking the limits of the integration to infinity and then the solution is given by

2.12 \[ F_m (X) = C H_m (X)e^{-\frac{1}{2}X^2} \]

where \( H_m (X) \) is the Hermite polynomial of degree \( m \), \( C \) a constant. Under this approximation the field distribution over each mirror will be given by

\[ E = E_o H_m (X) H_n (Y) \exp -\frac{1}{2} (X^2 + Y^2) \]

2.13 \[ E = E_o H_m (X \sqrt{\frac{k^2}{b}}) H_n (Y \sqrt{\frac{k^2}{b}}) \exp -\frac{k}{2b} (X^2 + Y^2) \]

(for a two dimensional resonator we can apply the same theory to arrive at

\[ E = E_o H_m (X \sqrt{\frac{k^2}{b}}) \exp (-\frac{k}{2b} X^2) \]
b. Travelling wave in the resonator

Using this solution for the field distribution over the mirror we can now calculate from equation 2.4 the form of the travelling wave which is produced in the cavity. If we assume one of the mirrors is partially transmitting this will also be the form of the wave outside the cavity.

For the simplest case of the lowest order mode, in the approximation of infinite mirrors the defraction integral is

\[ E(x,y,s) = -E_0 \frac{ik}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left( -\frac{k}{2b} (x'^2 + y'^2) \right) \frac{e^{-ikr}}{r} \, dx' \, dy' \]

To the first approximation from diagram 2.2

\[ r = \xi - \frac{W^2}{2b} + \frac{W^2}{2\xi} + \frac{W^1}{2\xi} - \frac{(xx^1 + yy^1)}{\xi} \]

where \( \xi = z + \frac{b}{2} \), \( W^2 = x^2 + y^2 \), \( W^1 = x^{12} + y^{12} \)

substituting into 2.14.

\[ E(x,y) = \frac{E_0 ik}{2\pi \xi} \exp \left\{ -\frac{ikw^2}{2\xi} - ik \xi \right\} \]

\[ x \int_{-\infty}^{\infty} \exp \left\{ -\frac{k}{2b} - \frac{ik}{2b} + \frac{ik}{2\xi} \right\} x^2 + ik xx^1 \, dx \]

\[ x \int_{-\infty}^{\infty} \exp \left\{ -\frac{k}{2b} - \frac{ik}{2b} + \frac{ik}{2\xi} \right\} y^2 + ik yy^1 \, dy \]

taking the first of the integrals, \( I_x \)

\[ I_x = \exp \left\{ -\frac{xx^2}{2\xi^2} \right\} \int_{-\infty}^{\infty} \exp \left\{ x^1 - \left[ \frac{k}{2} \left( \frac{1-i}{b} + \frac{i}{\xi} \right) \right] x^1 \right\} \]

\[ = \exp \left\{ -\frac{xx^2}{2\xi^2} \right\} \int_{-\infty}^{\infty} \exp \left\{ x^1 \right\} \]
changing the variable of integration to

\[ 2.18 \gamma = \frac{k}{2} \left( \frac{1-i}{b} + \frac{i}{\xi} \right)^{\frac{1}{2}} - \frac{ikx}{2 \xi \left( \frac{k}{2} \left( \frac{1-i}{b} + \frac{i}{\xi} \right)^{\frac{1}{2}} \right)} \]

we have

\[ 2.19 I_x = \frac{k}{2} \left( \frac{1-i}{b} + \frac{i}{\xi} \right) \exp \left\{ \frac{-kx^2}{2 \xi^2 \left( \frac{k}{2} \left( \frac{1-i}{b} + \frac{i}{\xi} \right)^{\frac{1}{2}} \right)} \right\} \int_{C} \exp -\gamma^2 d\gamma \]

Where \( C \) is a straight line in the complex \( \gamma \) plane. This integral and also the integral which we obtain for the higher mode case is treated in Appendix I. Using the result

\[ 2.20 I_x = \frac{\sqrt{\pi}}{k} \left( \frac{1-i}{b} + \frac{i}{\xi} \right) \exp \left\{ \frac{-kx^2}{2 \xi^2 \left( \frac{k}{2} \left( \frac{1-i}{b} + \frac{i}{\xi} \right)^{\frac{1}{2}} \right)} \right\} \]

note that the expression for \( E(x, y, z) \) is given as the product of two two-dimensional terms, so the two dimensional result follows immediately. The field \( E(x, y, z) \) is now given by

\[ 2.21 E(x, y, z) = \frac{E_{01}}{\xi} \left( \frac{1-i}{b} + \frac{i}{\xi} \right) \exp \left\{ \frac{-kw^2}{2 \xi^2 \left( \frac{k}{2} \left( \frac{1-i}{b} + \frac{i}{\xi} \right)^{\frac{1}{2}} \right)} \right\} - \frac{ikw^2}{2 \xi} - ik \xi \]

Simplifying and substituting for \( z = \xi - b/2 \)
2.22 \( E(x, y, z) = \frac{E_o (1 + i)}{(1 - \frac{2iz}{b})} \exp \left\{ -\frac{k(x^2 + y^2)}{b(1 + \frac{4z^2}{b^2})} - ik(z + \frac{b}{2}) \right\} \)

Redefining \( E_o \) to absorb the constants

2.23 \( E(x, y, z) = \frac{E_o}{(1 - \frac{2iz}{b})} \exp \left\{ -\frac{k(x^2 + y^2)}{b(1 + \frac{4z^2}{b^2})} - ikz \right\} \)

Separating the real and imaginary parts this can be written alternatively as

2.24 \( E(x, y, z) = \frac{E_o}{(1 + \frac{4z}{b^2})^\frac{1}{2}} \exp \left\{ -\frac{k(x^2 + y^2)}{b(1 + \frac{4z^2}{b^2})} \right\} - \frac{2k(x^2 + y^2)}{b^2(1 + \frac{4z^2}{b^2})} - \frac{b}{2} - \tan^{-1} \left( \frac{2z}{b} \right) \}

We see from this equation for fixed \( z \) the field falls off from the axis of the resonator as

\[ \exp \left\{ -\frac{k(x^2 + y^2)}{2} \right\} \]

\( b(1 + \frac{4z^2}{b^2}) \)

We define the spot size \( W \) of the mode to be the radius where the field has fallen to \( \frac{1}{e} \) of its value on the axis.

2.25 Therefore \( W^2 = \frac{b}{k} \left( 1 + \frac{4z^2}{b^2} \right) \)

which can be written \( W^2 = W_o^2 \left( 1 + \frac{4z^2}{b^2} \right) \)

defining \( W_o = \left( \frac{b}{k} \right)^{\frac{1}{2}} \) the minimum spot size. We can write
\[
\frac{2z}{b} = \frac{2z}{kW_0^2} = \frac{\lambda z}{W_0^2}
\]
where \( \lambda \) is the wavelength and defining \( Z_0 \) and \( \xi \)

\[\frac{b}{2} = Z_0 \quad \text{and} \quad \frac{Z}{Z_0} = \frac{2z}{b} = \xi\]

These definitions will be adhered to throughout this thesis from this point on.

The beam spread is defined by (see Figure 1.3)

\[\Theta = \lim_{Z \to \infty} \left\{ \tan^{-1} \left( \frac{W}{Z} \right) \right\} = \tan^{-1} \left( \frac{W_0}{Z_0} \right)\]

using these definitions we can write the general travelling wave form for the \( n_m \)th mode as

\[E_{n_m}(x, y, z) = E_{n_m}^0 \frac{(1 + i \xi)}{(1 - i \xi)} \frac{n + m}{m + n + 1} \frac{2}{W_0^2} x \left[ \frac{2x^2 + y^2}{W^2} \right] \left( 1 - i \xi \right)\]

or as (39).

\[E_{n_m}(x, y, z) = E_{n_m}^0 \frac{W_0}{W} H_n \left( \frac{2x}{W} \right) H_n \left( \frac{2y}{W} \right) x \exp \left\{ \frac{-k^2z}{W^2} - i \left( \frac{x^2 + y^2}{W^2} \xi - (m + n + 1) \tan^{-1}(\xi) + k\xi \right) \right\}\]

note that these can be split (except for the \( \exp(-ikz) \) term) to give the two dimensional result.
2.4

\[
\begin{align*}
M_1 & \quad \rightarrow \quad z_1 \quad \rightarrow \quad z_2 \quad \rightarrow \quad M_2
\end{align*}
\]

\[
\begin{align*}
d
\end{align*}
\]

2.5

\[
\begin{align*}
\text{negative uniaxial media} \\
\text{positive uniaxial media}
\end{align*}
\]
2.30 \[ E_m(x, z) = E_o \left(1 + i\xi\right)^{m/2} \left(1 - i\xi\right)^{m+1} \frac{H_m}{W} \left(\frac{\sqrt{2x}}{W}\right) \exp \left\{ -\frac{x}{W^2} - i\kappa z \right\} \]

\[ \text{c. The general spherical mirror resonator} \]

Equation 2.28 will also represent the field inside a resonator with arbitrary mirror spacing as long as the mirrors lie on surfaces of constant phase of the electric field. The surfaces of constant phase of the field represented by the equation 2.21 are given by

\[ \frac{x^2 + y^2}{w^2} + \kappa z - (m + n + 1) \tan^{-1} \xi = \text{constant} = \kappa z^1 \]

whereas \( z^1 \) is the point at which the surface intersects the \( z \) axis. Neglecting the term \( (m + n + 1) \tan^{-1} \xi \), since it is small compared with \( \kappa z \), and re-arranging this gives:

\[ x^2 + y^2 = (z^1 - z) 2z (1 + \frac{1}{\xi} 2) \]

which represents a spherical surface, radius of curvature

\[ R = Z (1 + \frac{1}{\xi} 2) = \]

2.31 i.e. \[ R = Z (1 + \frac{Z_o^2}{2z^2}) \]

Thus if we are given two mirrors radius of curvature \( r_1, r_2 \) respectively, placed a distance apart \( d \), distances \( Z_1, Z_2 \) from the beam waist (see figure 2.4) then the beam parameters can be calculated from 2.31.
from this expression we can obtain the values of \( r_1, \ r_2, \ d \) which support resonance. For a more detailed treatment see Bhawalker, 42.

d. Resonant Frequency

The phases of the wave at the two mirrors relative to the origin are (from equation 2.29)

\[
\phi_1 = -kz_1 + (m + n + 1) \tan^{-1} \xi_1 \\
\phi_2 = kz_2 - (m + n + 1) \tan^{-1} \xi_2
\]

Therefore the phase change between the mirrors is

\[ \phi_1 - \phi_2 = kd - \tan^{-1} \xi_2 - \tan^{-1} \xi_1 \]

\[
2.32 \quad \phi_1 - \phi_2 = kd - (m + n + 1) \tan^{-1} \left[ \frac{d (\varphi_1 + \varphi_2 - d)}{(r_1 - d)(r_2 - d)} \right]^{\frac{1}{2}}
\]

using equation 2.31. For resonance \( \phi_1 - \phi_2 \) must be an integer multiple of \( \pi \). Therefore

\[ 2.33 \quad q \pi = kd - (m + n + 1) \tan^{-1} (\psi) \]

\( \psi \) is given by 2.32 and is a function of the cavity parameters only. In general, \( q \) will be large and the resonator will support a number of longitudinal modes the wave number separation of which is given by

\[ \Delta k = k_{q+1} - k_q = \frac{\pi}{d} \]

Equation 2.33 also shows that the transverse modes will have a wave
number separation of

\[ \Delta k = k_m + 1 - k_m = \tan^{-1} \frac{\psi}{d} \]

3. Solution of Maxwell's Equations for the Resonator

(a) Lossless Isotropic Medium

Maxwell's equations for a homogeneous, isotropic, linear and non-magnetic dielectric reduce to

\[ \frac{\nabla^2 \mathbf{E}(r, t)}{c^2} - \frac{\partial^2 \mathbf{E}(r, t)}{\partial t^2} = 0 \]

Taking the Fourier transform we have

\[ \frac{\nabla^2 \mathbf{E}(r, \omega)}{c^2} + \frac{\omega}{c^2} \mathbf{E}(r, \omega) = 0 \]

Considering a plane polarised wave travelling in the \( z \) direction we can write.

\[ \mathbf{E} = \mathcal{E}(x, y, z) \exp(-ikz) \]

where the function \( \mathcal{E} \) will represent the differences between the wave form we are considering and a plane wave. We assume \( \mathcal{E} \) is a slowly varying function and

\[ \left| \frac{d^2 \mathcal{E}}{dz^2} \right| \ll k \left| \frac{d \mathcal{E}}{dz} \right| \]

where \( k = \frac{\omega}{c^2} \)

which is a reasonable assumption since \( k \) is large (\( \sim 10^5 \)) at optical frequencies. Substituting 2.36 into 2.35 and using 2.37 we have

\[ \frac{\partial^2 \mathcal{E}}{\partial x^2} + \frac{\partial^2 \mathcal{E}}{\partial y^2} - 2ik \frac{\partial \mathcal{E}}{\partial z} = 0 \]
This is the basic equation of the subsequent work which was first derived by Kogelnik and Li, 43. The presence of the resonator gives us the boundary conditions for the solution of 2.38. As we have already pointed out the boundary conditions are axially symmetric and so we can assume a plane polarized solution. Secondly they introduce an axis into the problem and so the solution will also have an axis. In the absence of these boundary conditions there will be no axis so that the solutions which we will derive cannot be regarded as solutions of Maxwell's equations in the absence of the resonator.

(i) **Lowest-order mode**

Following the derivation of the lowest order mode given by Kogelnik and Li, 43, we assume \( \mathcal{E} \) can be written in the form

\[
2.39 \quad \mathcal{E} = \mathcal{E}_0 \exp \left( -i \left( P + \frac{k r^2}{2q} \right) \right)
\]

where \( r^2 = x^2 + y^2 \),

\( P \) is a function only of \( z \) and represents a complex phase shift and \( q \) a function only of \( z \) which describes the variation of electric field away from the axis and the curvature of the phase front.

Substituting 2.39 into 2.38 we have

\[
2.40 \quad - \frac{2ik}{q} - \frac{k^2 r^2}{q^2} - 2k \frac{dP}{dz} - \frac{k^2 r^2}{2} \frac{dq}{dz} \left( \frac{1}{q} \right) = 0
\]

Equating coefficients of \( r \)

\[
2.41 \quad (a) \quad \frac{dP}{dz} = -\frac{i}{q} \quad (b) \quad \frac{dq}{dz} = 1
\]

2.41 (b) can be integrated immediately to give

\[
2.42 \quad q = q_0 + z
\]
guided by the previous work (eq. 2.29) we split \( \frac{1}{q} \) into its real and imaginary parts

\[
2.43 \quad \frac{1}{q} = \frac{1}{R} - \frac{i}{2kW^2}
\]

\( R \) will be the radius of curvature of the phase front on the axis and \( w \) a measure of the decrease in field amplitude from the axis. The intensity distribution is gaussian at every cross section and \( W \) can be defined as the spot size or beam radius at a particular \( Z \). We set up our axes at the point where \( R = \infty \) and define the spot size at this point of the \( Z \) axis to be \( W_o \). Therefore from 2.43 and 2.42

\[
q_o = \frac{i k W_o^2}{2}
\]

\[
q = \frac{i k W_o^2}{2} + Z
\]

which gives

\[
2.44 \quad W^2 = W_o^2 \left( 1 + \frac{4 \frac{z^2}{k^2 W_o^4}}{4z^2} \right)
\]

2.45 and

\[
R = Z \left( 1 + \frac{k^2 W_o^4}{4z^2} \right) = \frac{k^2 w^2 W_o^2}{4z}
\]

This result corresponds exactly with our previous definitions.

Substituting for \( q \) into equation 2.41 (a)

\[
\frac{dp}{dz} = -\frac{i}{z + \frac{i k W_o^2}{2}}
\]

\[
iP = \log \left( 1 - \frac{i2z}{kW_o^2} \right)
\]
2.46 \quad = \log \left(1 + \frac{4Z^2}{kW_0^2} \right) - \tan^{-1} \left( \frac{2Z}{kW_0} \right)

substituting back into equation 2.39

\[ \mathcal{E} = \mathcal{E}_0 \exp \left\{ - \log \left(\frac{1 + \frac{1}{4} \frac{Z^2}{W_0^2}}{k^2} \right) + i \tan^{-1} \left( \frac{2Z}{kW_0} \right) - r^2 \left( \frac{1}{W^2} + \frac{ik}{2q} \right) \right\} \]

2.47 \quad \mathcal{E} = \mathcal{E}_0 \frac{W_0}{W} \exp \left\{ - \frac{r^2}{W^2} - \frac{ir^2Z}{kW_0^2} + i \tan^{-1} \left( \frac{2Z}{kW_0^2} \right) \right\}

which corresponds with the lowest order mode of equation 2.29.

\[ r^2 = x^2 + y^2, \quad \frac{2Z}{kW_0^2} = \mathcal{E} \]

(ii) Higher order modes

To derive the higher modes we have to introduce a more general form in the place of equation 2.39. Guided by experience we put

2.48 \quad \mathcal{E} = \mathcal{E}_0 f \left( \frac{\sqrt{x}}{W} \right) g \left( \frac{\sqrt{y}}{W} \right) \exp \left\{ -i \left( p + \frac{kx}{2q} \right) \right\}

substituting this into equation 2.38 and dividing throughout by \( f \ g \)

we have using 2.43

2.49 \quad \frac{1}{f} \left( \frac{2}{W^2} f'' \left( \frac{\sqrt{x}}{W} \right) - 4 \frac{\sqrt{x}}{W^3} f' \left( \frac{\sqrt{x}}{W} \right) \right) + \frac{1}{g} \left( \frac{2}{W^2} g'' \left( \frac{\sqrt{y}}{W} \right) - 4 \frac{\sqrt{y}}{W^3} g' \left( \frac{\sqrt{y}}{W} \right) \right) =
\[- \frac{2i k}{q} - \frac{2k d P}{dz} = 0\]

Separating off the variable \( x \) and then \( y \) and introducing the two constants of separation \(-2m, -2n\) gives the equations

\[2.50 \quad f''(x) - 2X f'(x) + 2mf(x) = 0\]
\[g''(y) - 2Yg'(y) + 2ng(y) = 0\]

for \( f \) and \( g \)

Where \( X = \frac{\sqrt{2x}}{W} \) and \( Y = \frac{\sqrt{2y}}{W} \). These are just the equations for the Hermite polynomials of degree \( m \) and \( n \) respectively so that

\[2.51 \quad f \left( \frac{\sqrt{2x}}{W} \right) = H_m \left( \frac{\sqrt{2x}}{W} \right)\]
\[g \left( \frac{\sqrt{2y}}{W} \right) = H_n \left( \frac{\sqrt{2y}}{W} \right)\]

The equation for \( P \) in this case is

\[\frac{dP}{dz} = -\frac{i}{q} - \frac{2(m + n)}{kW^2}\]

giving

\[2.52 \quad iP = \log \left( 1 + \frac{\frac{J_2}{W}}{k^2 W_o} \right) - i(m + n + 1) \tan^{-1} \left( \frac{2\alpha}{k W_o^2} \right)\]

Thus the general solution for the travelling wave mode in the cavity is given by

\[2.53 \quad E_{mn} = E_o \frac{W}{W} H_m \left( \frac{\sqrt{2x}}{W} \right) H_n \left( \frac{\sqrt{2y}}{W} \right) \exp \left\{ \frac{-x^2}{W^2} - \frac{ir2\alpha}{k W_o^2} \right\}

+ i(m + n + 1) \tan^{-1} \left( \frac{2\alpha}{k W_o^2} \right) \}

changing the notation \( r^2 = x^2 + y^2 \), \( \xi = \frac{2\alpha}{k W_o^2} \)

this corresponds to equation 2.29
(b) **Validity of the approximation**

We have approximated by assuming that

\[
\frac{\frac{d^2 \mathcal{E}}{dz^2}}{< \left( \frac{k \frac{d \mathcal{E}}{dz}}{dz} \right)} \text{ in obtaining equation 2.38. For self consistency our solution 2.53 must obey this condition. For the lowest order mode, calculating the derivatives and substituting into equation 2.54, gives}
\]

\[
2.54 \left| \frac{1}{q^2} \left( 2 - \frac{2 i k r^2}{q} - \frac{k^2 r^4}{4q^2} \right) \right| < \left| \frac{k}{q} (-1 + i \frac{k r^2}{2q}) \right|
\]

\[
(i) \text{ at } r = 0 \text{ we obtain}
\]

\[
\left| \frac{2}{q} \right| < \left| \frac{k}{q} \right|
\]

i.e. \( \frac{\lambda}{\pi} \ll \left| z + i z_0 \right| \)

substituting for q from equation 2.42, \( q_0 = z_0 \). At worst this will give

\[
\lambda \ll \frac{\pi z_0}{\lambda}, \text{ since } z_0 = \frac{\pi \omega_0^2}{\lambda}
\]

we have

\[
2.56 \lambda \ll \frac{\omega_0}{\omega}
\]

(ii) **Finite r** Substituting for \( q = z + i z_0 \) 2.55 becomes

\[
2.57 \left| \frac{2}{k (z^2 + z_0^2)} \right| \left( z^2 - z_0^2 + k r^2 z_0 - \frac{k^2 r^4}{2} \right)
\]
\[-i \left( k r^2 z - 2 r z_0 \right) \ll \left| -z + i \left( k r^2 - \frac{z_0}{2} \right) \right|\]

This breaks down in two regions the first near \( z = 0 \) and

\[kr^2 - \frac{z_0}{2} = 0.\]

The second if \( r \) is large enough. Region one is defined by \((z^2 + (r - W)^2) \ll \frac{1}{k^2}\). This is relatively unimportant since it is so small compared with the beam parameter.

Region two is defined by \( r^2 \gg kW^2 \). The magnitude of the field is proportional to \( \exp(-r^2/w^2) \). So in this region this factor is much less than \( \exp(-k) \) and the field is so small that this inconsistency can be neglected. Some error in the outer regions can be expected since we have approximated the finite mirrors of the resonator by infinite ones. Thus the only significant restriction is given by equation 2.56 the minimum spot size must be much greater than the wavelength of the light of the beam.

(iii) Transverse Nature of the Polarization

Orienting the \( x \) and \( y \) axis so that the field in the resonator is polarized in the \( y \) direction we can write the divergence equation for the field as

\[2.58 \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \]

where \( E_z \) is the component of the field along the direction of propagation. We assume that

\[\frac{\partial E_y}{\partial z} \approx -ikE_z.\]

Taking the lowest order mode (equation 2.39) and calculating the derivative in the \( y \) direction we have
2.59 \( \frac{\partial E_y}{\partial y} = -\frac{\mathbf{k} \cdot \mathbf{y}}{\mathbf{q}} E_y \) so that from equation 2.58 we can write,

\[
\left| \frac{\mathbf{y}}{\mathbf{q}} E_y \right| = \left| E_z \right|
\]

therefore as long as \( |\mathbf{y}| << |\mathbf{q}| \)

\( |E_z| << |E_y| \) and the mode will be traversely polarized.

\[
|\mathbf{q}| = (z^2 + z_0^2)^{1/2} = \frac{z_o}{\mathbf{W}_0} \mathbf{W}_0 \left( 1 + \frac{z^2}{z_0^2} \right) = \frac{\mathbf{k} \mathbf{W}_0 \mathbf{W}}{2}
\]

Therefore the condition reduces to

2.60 \( |\mathbf{y}| << \frac{\mathbf{k} \mathbf{W}_0 \mathbf{W}}{2} \)

and again the ideal condition breaks down far from the axis when the field is small. So that over the region of interest it is a valid approximation to assume there is no component of the field in the direction of propagation.

(iv) Normalization

We assume as before we have a mode propagating along the axis of the resonator polarized in the \( y \) direction.

\( \mathbf{E} = (0, \mathbf{E}, 0) \) to the approximation of the last section. From Maxwell's equations the magnetic field \( \mathbf{H} \) will have the form

\( \mathbf{H} = (\mathbf{H}, 0, 0) \)

The Poynting vector \( \mathbf{S} = \frac{\mathbf{C}}{4\pi} \mathbf{E} \times \mathbf{H} \)

and therefore \( \mathbf{S} = \frac{\mathbf{C}}{4\pi} (0, 0, -\mathbf{E} \mathbf{H}) \)

In the usual way we write

\( \mathbf{E} = \text{Re} \left\{ \mathbf{E} \exp -i (kz - \omega t) \right\} \)
2.61 \[ H = \text{Re} \left\{ X \exp \left[ -i (kz - \omega t) \right] \right\} \]

and using Maxwell's equations

\[ \gamma \mathcal{E} = \mathcal{H} \]

where \( \gamma \) is the refractive index of the medium. Therefore the one non-zero component of the energy flux

\[ 2.62 \quad S_z = \frac{c \gamma}{4\pi} \left\{ \frac{\varepsilon^2}{4} \exp \left[ -2i (kz - \omega t) \right] \right\} + \frac{\varepsilon e^{*}}{2} + \frac{\varepsilon^{*2}}{4} \exp \left[ 2i (kz - \omega t) \right] \}

taking the time average

\[ 2.63 \quad \overline{S}_z = \frac{c \gamma}{4\pi} \cdot \frac{\varepsilon e^{*}}{2} \]

to calculate the total energy flow in the propagating mode we now integrate 2.63 over the cross section.

(i) In two dimensions \( \mathcal{E} \) is given by 2.30 and

\[ 2.64 \quad \overline{S}_{\text{Total}} = \frac{c}{8\pi} \int_{-\infty}^{\infty} \frac{1}{|E_0|^2} \frac{R_n^2 (\sqrt{\frac{2x}{W}})}{1 + \frac{2x^2}{W^2}} dx \]

\[ = \frac{c}{8\pi} \frac{W_0}{\sqrt{2}} \left| E_0 \right|^2 \int_{-\infty}^{\infty} R_n^2 (\xi) \exp (-\xi^2) d\xi \]

where \( \xi = \frac{\sqrt{2x}}{W} \)

Carrying out the integration

\[ 2.65 \quad \overline{S}_{\text{Total}} = \frac{c}{8\pi} \frac{W_0}{\sqrt{2}} \cdot 2^n n! \frac{1}{\sqrt{\pi}} \left| E_0 \right|^2 \]

For unit energy flux
It is convenient at this point to define a new normalised amplitude $\bar{E}_n$ so that

$$2.66 \quad \bar{S}_T = \frac{C}{8\pi} |E_n|^2$$

therefore

$$2.67 \quad |E_o| = |E_n| \left( \frac{1}{W_o} \right)^{\frac{1}{2}} \left( \frac{1}{2\pi} \right)^{\frac{1}{2}} \left( \frac{1 + i\xi}{1 - i\xi} \right)^{n/2}$$

and 2.30 becomes

$$2.68 \quad E_n = \bar{E}_n \left( \frac{1}{W_o} \right)^{\frac{1}{2}} \left( \frac{1}{2\pi} \right)^{\frac{1}{2}} \left( \frac{1 + i\xi}{1 - i\xi} \right)^{n/2} \frac{H_n}{W_o} \left( \sqrt{\frac{2\pi}{W_o}} \right) \exp \left\{ -\frac{x^2}{W_o^2 (1 - i\xi)} \right\}$$

leaving out the exp $(-ikz)$ term

(ii) In three dimensions $E$ is given by 2.28. Using this expression the resulting expression for $\bar{S}_{\text{Total}}$ contains a double integration which is separable into an integration over $y$ and an integration over $x$ each of the form of the integral in equation 2.64. Carrying out these integrations we have

$$2.69 \quad \bar{S}_{\text{Total}} = \frac{C}{16\pi} \left( \frac{w}{w_o} \right)^2 |E_o|^2 |E_o| \left( \frac{1}{2\pi} \right)^2 \left( \frac{1 + m}{2^n m! n!} \right)$$

Again we define a new normalized amplitude $\bar{E}_n$ where

$$|E_o| = |E_n| \left( \frac{1}{W_o} \right)^{\frac{1}{2}} \left( \frac{1}{2^n m! n! \pi^2} \right)^{\frac{1}{2}}$$

and the travelling wave mode is defined as
This normalization is the most convenient and has been used also by Kogelnik, 44.

The energy flux now is equal to

\[
\frac{C}{16 \pi} \frac{|\mathcal{E}|^2}{\varepsilon} \times 10^{-7} \text{ watts}, \text{ and so}
\]

\[
2.17 \quad |\mathcal{E}|^2 = \sqrt{\frac{16 \pi}{C} S_T} x 10^{-7}, \quad S_T \text{ in watts.}
\]

Orthogonality

The normalised modes \( (\mathcal{E}_N = 1) \) are orthogonal since for two dimensions

\[
I = \int_{-\infty}^{\infty} \mathcal{E}_n^* \mathcal{E}_m \, dx
\]

\[
= \frac{1}{W_0^2} \frac{n + m}{2} \left( \frac{n! \cdot m!}{\sqrt{2\pi}} \right) \sqrt{\frac{2}{\tau \pi}} \int_{-\infty}^{\infty} \frac{1}{(1 + \xi^2)^{3/2}} H_n \left( \frac{\sqrt{2\xi}}{W} \right) H_m \left( \frac{\sqrt{2\xi}}{W} \right) \exp \left( -\frac{2\xi^2}{W^2} \right) \, d\xi
\]

\[
= \frac{1}{\sqrt{2^n + m} \cdot n! \cdot m! \cdot \tau} \int_{-\infty}^{\infty} H_n (\xi) H_m (\xi) \exp (-\xi^2) \, d\xi
\]
and for three dimensions

\[ I = \int_{-\infty}^{\infty} E_{n}^* E_{mk} \, dx \, dy \]

\[ = \frac{1}{W_0^2} \frac{n + m + 1 + k}{2^{n/2} m! l! k!} \left(\frac{2}{1 + \xi^2}\right)^{1/2} \]

\[ x \ H_n \left(\sqrt{\frac{2x}{W}}\right) H_m \left(\sqrt{\frac{2x}{W}}\right) \, dx \int_{-\infty}^{\infty} \frac{1}{(1 + \xi^2)^{1/2}} \ H_\ell \left(\sqrt{\frac{2\xi}{W}}\right) \]

\[ x \ H_\ell \left(\sqrt{\frac{2\xi}{W}}\right) \, dy = \delta_{mn} \delta_{\ell \ell} \]

Since the Hermite functions form a complete set over the interval \((-\infty, \infty)\) we can regard the resonator modes as forming a complete set in the cavity and therefore any travelling waveform in the cavity can be expanded as a sum of the modes.

The general solution of the differential equation 2.38 can be written, therefore as

\[ E = \sum_{n} A_n E_n \] for two dimensions and

\[ E = \sum_{n} \sum_{m} A_{nm} E_{nm} \] for three dimensions

where the A's are constants which for a given field can be determined in the normal way using the orthogonality of the \( E_n, (E_{n,m}) \).
4. Propagation in Anisotropic Media

(a) Two dimensions

The propagation of the lowest order two dimensional gaussian mode has been treated independently by Bhawalker, Gambling and Smith, 45, in some detail but here we are only interested in the particular case of propagation perpendicular to the optic axis. Bergstein and Zachos, 48, 49, have also studied propagation in an uniaxial anisotropic medium using a diffraction theory approach.

We consider Maxwell's equations for an infinite uniaxial lossless medium enclosed in a resonator with the optic axis of the medium perpendicular to the axial plane of the resonator. We set up a system of co-ordinates so that the resonator is infinite in the $y$ direction, the optic axis lies in the $x$ direction and the $z$ direction in the direction of propagation. These axes are principal axes of the dielectric tensor which can therefore be written.

$$\varepsilon = \begin{pmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_z & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix}$$

The Fourier transforms of Maxwell's equations (e.g. 1.46) written out in full are

2.73 (a) \[
\frac{\partial^2 E_x}{\partial x^2} - \frac{\partial^2 E_x}{\partial z^2} - \frac{\omega^2}{c^2} \varepsilon_x E_x = 0
\]

(b) \[
\frac{\partial^2 E_y}{\partial z^2} - \frac{\partial^2 E_y}{\partial x^2} - \frac{\omega^2}{c^2} \varepsilon_z E_y = 0
\]
Note that the equations decouple. The \( x \) and \( z \) component describe the propagation of the extraordinary wave whose refraction index varies with direction of propagation and the \( y \) component describes the propagation of the ordinary wave. The \( y \) component has exactly the form of the isotropic equation and hence the foregoing theory applies without change.

Using the divergence equation

\[
2.74 \quad \varepsilon_x \frac{\partial E_x}{\partial x} + \varepsilon_z \frac{\partial E_z}{\partial z} = 0
\]

The \( x \) and \( z \) components can be separated, equation 2.73 (a) becoming

\[
2.75 \quad \frac{\varepsilon_x}{\varepsilon_z} \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial z^2} + \frac{\omega^2}{c^2} \varepsilon_x E_x = 0
\]

We now consider an extraordinary wave propagating in the \( z \) direction. Since we can assume it is transverse it will be governed by this equation.

Let \( E_x = \mathcal{E} \exp(-ikz) \), where \( \mathcal{E} \) is a slowly varying function of \( x \) and \( z \) and \( k = \omega/c \sqrt{\varepsilon_x} \). Substituting into equation 2.75 and using the previous approximation 2.37 we obtain

\[
2.76 \quad \frac{\varepsilon_x}{\varepsilon_z} \frac{\partial^2 \mathcal{E}}{\partial x^2} - 2ik \frac{\partial \mathcal{E}}{\partial z} = 0
\]
Making the scale change $x = (\varepsilon_x/\varepsilon_z)^{1/2} X$ we obtain the two-dimensional analogue of equation 2.38

$$2.77 \quad \frac{\partial^2 \xi}{\partial x^2} - 2i k \frac{\partial \xi}{\partial z} = 0$$

which has solutions

$$\xi_n = \xi_0 \left( \frac{1 + i \xi}{n+1} \frac{2}{\frac{\sqrt{2} X}{W_1}} \right)^{n/2} H_n \left( \frac{\sqrt{2} X}{W_1} \right) \exp \left\{ - \frac{X}{W_o^2 (1 - i \xi)} \right\}$$

Therefore a mode polarized as an extraordinary wave propagates as

$$2.78 \quad \xi_n = \xi_0 \left( \frac{1 + i \xi}{n+1} \frac{2}{\frac{\sqrt{2} X}{W_1}} \right)^{n/2} H_n \left( \frac{\sqrt{2} X}{W_1} \left( \frac{\varepsilon_x}{\varepsilon_z} \right)^{1/2} \right) \exp \left\{ - \frac{\varepsilon_x X^2}{W_o^2 (1 - i \xi)} \right\}$$

$W_o$ no longer represents the spot size of the beam, but we define

$$2.79 \quad W_o = \left( \frac{\varepsilon_x}{\varepsilon_z} \right)^{1/2} W_0$$

and now $W_o$ is the spot size as defined previously (equation 2.25).

also $\xi = \frac{2s}{k W_o^{1/2}} = \frac{2s}{k} \left( \frac{\varepsilon_x}{\varepsilon_z} \right) W_o^2$

$$2.80 \quad \xi = \frac{2s}{k W_o^{1/2}} \left( \frac{\varepsilon_x}{\varepsilon_z} \right)$$

so that the beam spread is altered to

$$2.81 \quad \Theta = \tan^{-1} \left\{ \frac{2}{k W_o} \left( \frac{\varepsilon_x}{\varepsilon_z} \right) \right\}$$
The beam spread therefore is less for an extraordinary beam in a negative uniaxial medium, more for an extraordinary beam in a positive uniaxial medium than the beam spread for the ordinary beam.

The lines of constant phase of equation 2.78 are given by (c.f. section 2 (c)).

\[ x^2 = 2(z^1 - z) z \frac{\varepsilon_x}{\varepsilon_z} (1 - \frac{z_0^2}{z^2}) \]

As before these are circles, now of radius of curvature

\[ R = z \frac{\varepsilon_x}{\varepsilon_z} (1 - \frac{z_0^2}{z^2}) \]

where \[ z_0 = \frac{kW_o^2}{2} \left( \frac{\varepsilon_z}{\varepsilon_x} \right) \]

The spot size, confocal parameters etc. of a set of modes produced by two given mirrors can be calculated from this data as in section 2 (c) (see reference 45 for explicit formulae). Note that the spot sizes of the extraordinarily polarized modes will be different from the ordinary polarized ones for a given resonator.

The normalization of equation 2.78 can be carried out exactly as before and all other conclusions of the earlier work hold for this case.

**Three Dimensions**

Consider a resonator filled with uniaxial lossless dielectric with axes oriented as for the two dimensional case. Then equations 1.46 written out in full are
\[ 2.84 \]

(a) \[
\frac{\partial^2 E_x}{\partial x \partial z} + \frac{\partial^2 E_y}{\partial x \partial y} - \frac{\partial^2 E_x}{\partial z^2} - \frac{\partial^2 E_x}{\partial y^2} - \frac{\omega^2}{c^2} \varepsilon_x E_x = 0
\]

(b) \[
\frac{\partial^2 E_x}{\partial y \partial x} + \frac{\partial^2 E_z}{\partial y \partial y} - \frac{\partial^2 E_y}{\partial x^2} - \frac{\partial^2 E_y}{\partial z^2} - \frac{\omega^2}{c^2} \varepsilon_y E_y = 0
\]

(c) \[
\frac{\partial^2 E_y}{\partial z \partial y} + \frac{\partial^2 E_x}{\partial z \partial z} - \frac{\partial^2 E_z}{\partial x^2} - \frac{\partial^2 E_z}{\partial y^2} - \frac{\omega^2}{c^2} \varepsilon_z E_z = 0
\]

and the divergence equation

\[ 2.85 \]

(a) \[
\frac{\partial E_x}{\partial z} + \varepsilon_x \frac{\partial E_x}{\partial y} + \varepsilon_z \frac{\partial E_z}{\partial z} = 0
\]

Using the divergence equation the others can be reduced to

We assume that the anisotropy is small i.e. \( (1 - \frac{\varepsilon_x}{\varepsilon_z}) \ll 1 \)

and look for a solution representing an almost plane wave travelling
in the $z$ direction. We expect two solutions, extraordinary and ordinary waves, therefore we put

$$2.86 \quad E_x = \varepsilon_x \exp (-i k_e z)$$
$$E_y = \varepsilon_y \exp (-i k_o z)$$
$$E_z = \varepsilon_x^o \exp (-i k_o z) + \varepsilon_z^e \exp (-i k_e z)$$

where $\varepsilon_x^o, \varepsilon_y^o, \varepsilon_z^e$ are slowly varying functions of $x, y, z$.

Substituting these into the divergence equation 2.84 (d) and using the approximations

$$2.87 \left| \frac{\partial \varepsilon_z^o}{\partial z} \right| \ll k_o \left| \varepsilon_z^o \right| \text{ and } \left| \frac{\partial \varepsilon_z^e}{\partial z} \right| \ll k_e \left| \varepsilon_z^e \right|$$

it becomes

$$2.88 \left( \frac{\partial \varepsilon_y^o}{\partial y} - i k_o \varepsilon_z^o \right) \exp (-i k_o z) + \left( \frac{\varepsilon_x}{\varepsilon_z} \frac{\partial \varepsilon_x}{\partial x} - i k_e \varepsilon_z^e \right) \exp (-i k_e z)$$

$$x \exp (-i k_e z) = 0$$

This equation can be split into two equations if the change of the quantities in the brackets is negligible over a period of the exponential term

$$\exp \left\{ -i (k_o - k_e) z \right\} \quad \text{which is } 2\pi / (k_o - k_e)$$

This condition will be true since we have assumed the waves to be nearly plane waves and that $\varepsilon_x^o, \varepsilon_y^o, \varepsilon_z^e$ are slowly varying compared with the $\exp (-i k_o z)$.
2.88 becomes therefore

\[
2.89 \quad (a) \quad \frac{\partial \varepsilon_y}{\partial y} - i k_0 \varepsilon_x^o = 0
\]

\[
(b) \quad \frac{\varepsilon_x}{\varepsilon_z} \frac{\partial \varepsilon_x}{\partial z} - i k_0 \varepsilon_z^o = 0
\]

and these can be used with the substitution of 2.86 to reduce equations 2.85 to

\[
2.90 \quad (a) \quad \frac{\varepsilon_x}{\varepsilon_z} \frac{\partial^2 \varepsilon_x}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial y^2} - 2 i k_0 \frac{\partial \varepsilon_x}{\partial z} = 0
\]

\[
(b) \quad \frac{\partial^2 \varepsilon_y}{\partial x^2} + \frac{\partial^2 \varepsilon_y}{\partial y^2} - 2 i k_0 \frac{\partial \varepsilon_y}{\partial z} = 0
\]

Equation (b) corresponds to that found by, 46, for the more general case, including double refraction.

The third equation is of negligible order. So under these approximations the equations have split as in the two dimensional case, into one governing the ordinary component and the other the extraordinary. The ordinary component 2.90 (b) again propagates as for the isotropic case. Making a scale change in 2.90 (a)

\[
x = \left(\frac{\varepsilon_x}{\varepsilon_z}\right)^{\frac{3}{2}} x
\]

we can reduce it to the isotropic case and thus the solutions of 2.90 (a) will be
2.91 \[ \mathcal{E}_{xmn} = \mathcal{E}_0 \frac{(1 + i \frac{W}{\sqrt{\mathcal{E}_x}})^{\frac{m+n}{2}}}{(1 - i \frac{W}{\sqrt{\mathcal{E}_x}})^{\frac{m+n+2}{2}}} \left( \mathcal{W}_n \left( \frac{\sqrt{2} \mathcal{E}}{\mathcal{W}_x} \frac{\mathcal{E}_z}{\mathcal{E}_x} \right) \right) \]

We see from this that the beam is no longer circular, the curves of constant intensity for a given \( z \) are given by

2.92 \[ \frac{\mathcal{E}_z}{\mathcal{E}_y} x^2 + y^2 = W^2 \] which is an ellipse with semi axes \( W \) and \( W (\frac{\mathcal{E}_x}{\mathcal{E}_z})^{\frac{1}{2}} \).

The surface of constant phase which define the resonator surfaces are given by

\[ \left( \frac{\mathcal{E}_z}{\mathcal{E}_x} x^2 + y^2 \right)^{\frac{1}{2}} + kz = k \frac{Z}{Z} \]

(as for section 2.2 (c))

Rearranging

2.93 \[ \frac{\mathcal{E}_z}{\mathcal{E}_x} x^2 + y^2 = 2 \left( z - z \right) Z \left( 1 + \frac{Z}{Z} \right) \]

This is the surface of an ellipsoid, so that a resonator enclosing a uniaxial medium needs to have ellipsoidal mirrors to achieve resonance for the extraordinary polarized modes. In this situation the ordinary polarized modes will have greater loss due to the wrong curvature of the mirrors and so effectively the resonator will have
only one set of modes. One possible way around the experimental
difficulty of ellipsoidal mirrors in the situation when the non-linear
properties of a particular uniaxial medium are required would be to
include also inside the resonator some medium of the opposite
uniaxial nature to reproduce spherical wave fronts on the mirrors
(see diagram 2.5). This would also enable the ordinary polarized
modes to resonate.

The normalization of the mode will be changed by the factor
\( \frac{\varepsilon_z}{\varepsilon_x} \) which has now appeared. Keeping the definition of
\( \zeta \), the normalized form of \( E_{nm} \) is

\[
2.94 \quad E_{nm} = \frac{\varepsilon_x}{W_0} \left( \frac{\varepsilon_z}{\varepsilon_x} \right)^{\frac{1}{2}} \frac{2z}{kW_0^2} \left( \frac{\omega^2}{c^2} \right) k_{ext} \left( 1 + i \frac{\zeta}{\gamma} \right) \frac{n+m}{n+m+1} (1-i\zeta)^{\frac{n+m+1}{2}}
\]

Effects of Absorption

The derivation of equation 2.38 from equation 2.34 holds without
change when \( \varepsilon \) is complex. The solution to equation 2.38 follows the
same lines except that \( k \) is now complex since

\[
k = \frac{\omega}{c} \varepsilon^{\frac{1}{2}} = k_x - i k_{im}
\]

Instead of equation 2.43 we define

\[
2.95 \quad 1 = \frac{1}{R} - \frac{i}{k_x} \frac{2}{W^2}
\]
and hence

\[ q = \frac{i k r W_o^2}{2} + z \text{ and } z_o = \frac{k r W_o^2}{2} \]

The spot size \( W \) is given by

\[ W^2 = W_o^2 \left( 1 + \frac{k z^2}{k r W_o^4} \right) \]

and the wavefront radius of curvature \( R = z \left( 1 + \frac{k r^2 W_o^4}{4 z^2} \right) \)

\[ = \frac{k r^2 W^2 W_o^2}{4 z} \]

The equation for the factor \( P \) of equation 2.39 for the general case remains unchanged

\[ \frac{dP}{dz} = -i \frac{q}{q} - 2 \frac{(m + n)}{k W^2} \]

but the solution for \( P \), equation 2.52, is altered to

\[ 2.96 \quad iP = \log \left( 1 + \frac{4 z^2}{k r^2 W_o^4} \right) - i \left( 1 + \frac{k r W_o}{k} (m + n) \tan^{-1} \left( \frac{2 z}{k r W_o^2} \right) \right) \]

Thus the general solution for the travelling wave modes after some manipulation to bring out the imaginary part of \( k \) is given by

(c.f. equation 2.53)

\[ 2.97 \quad \mathcal{E}_{nm} = \mathcal{E}_0 \frac{W_o}{W} H_m \left( \frac{\sqrt{2} \chi}{W} \right) H_n \left( \frac{\sqrt{2} \chi}{W} \right) \exp \left\{ -\frac{Z^2}{W^2} \frac{-2 i r^2 z}{k r W_o^2 W^2} \right\} \]

\[ + i (m + n + 1) \tan^{-1} \left( \frac{2 z}{k r W_o^2} \right) \right\} f \left( \frac{k r W_o}{k} \right) \]
where
\[ f(k_{im}) = \exp \left\{ \frac{k_{im}}{k} \left( \frac{ir}{w^2} - \frac{2r^2 z}{kw^2 w_0^2} \right) + \frac{ik_{im}}{k} (m + n)x \right\} \tan^{-1} \left( \frac{2z}{kr w_0^2} \right) \]

The exponential term \( f(k_{im}) \) is due to the presence of absorption. When considering the electric field we will also have the term \( \exp(-k_iz) \) arising from \( \exp(-ikz) \). As we have remarked previously all materials of interest will necessarily have low absorption therefore
\[ k_i << 1 \]

This implies the very much stronger inequality at optical frequencies
\[ k_i << k_r \]

All the terms in the second exponential depend on the factor \( k_{im}/k_r \) and hence can be neglected. This leaves only the term \( \exp(-k_iz) \) which has been introduced by the absorption present.
Chapter 3
Derivation of the Coupled Mode Equation and Calculation of the Coupling Coefficients

1 Two dimensions

We consider now the propagation of the travelling wave modes which were studied in the last chapter in a resonator which contains a uniaxial medium with a non-linear response. From Maxwell's equations for the media coupled equations governing the interchange of energy between the various modes of the resonators are derived. This provides a new direct method for studying the interaction of optical fields due to the non-linear medium when the interaction is inside a resonator. Previously the theory of the non-resonant effect has been adapted to the resonant case.

The optical field at frequency \( \omega \) is subject to equation 1.59.

\[
3.1 \nabla_x \left( \nabla_x \mathbf{E}^\omega \right) - \frac{\omega^2}{c^2} \mathbf{E}^\omega = \frac{4\pi \omega^2}{c^2} P^\omega
\]

and writing \( \mathbf{D} = \mathbf{E} + 4\pi \mathbf{P}^D + 4\pi \mathbf{P}^\omega \)
the divergence equation.

\[
3.2 \nabla \cdot (\mathbf{\varepsilon}(\omega) \cdot \mathbf{E}^\omega) = -4\pi \nabla \cdot \mathbf{P}^\omega
\]

Setting up axes as before, we have the \( x \) axis along the optic axis of the uniaxial medium, the \( z \) axis in the direction of propagation and the \( yz \) plane the axial plane of the resonator. The \( x \) and \( y \) components of equation 3.1 are
The divergence equation

After substituting 3.4 into 3.3 (a) to eliminate \( \frac{\partial E_z}{\partial z} \), 3.3 (a) becomes

\[ 3.5 \quad \epsilon_x \frac{\partial^2 E_x}{\partial x^2} + \epsilon_z \frac{\partial^2 E_z}{\partial z^2} + \frac{\omega^2}{c^2} \epsilon_x E_x = -\frac{4\pi \omega^2}{c^2} p_x \]

The term \( -\frac{4\pi}{\partial x} (\nabla \cdot F) \) has been neglected for the reason that

because \( F \) arises from optical folds propagating in the \( z \) direction which vary slowly in the \( x, y \) plane, the relation

\[ 3.6 \quad \left| \frac{\partial}{\partial x} (\nabla \cdot F) \right| \frac{4\pi \omega^2}{c^2} \left| p_x \right| \]

will hold.

Hence we have the two equations 3.5 and 3.3 (b) governing the propagation of the extraordinary and ordinary waves respectively in the resonator. From here we can proceed as in the linear case putting

\[ E_{x, y} = \mathcal{E}_0, e^{\exp (-ik_{x,y}z)} \]

substituting into 3.5 and 3.3 (b). This gives on approximating as before
We have seen that the general solution to the equation without the non linear polarization can be represented by an infinite sum over the resonator modes.

3.8  \( \mathbf{E} = \sum A_n \mathbf{E}_n \) where the \( A_n \) are constants.

Since the second order polarization is "small" we can regard the solution to the non linear equations as a perturbation of this solution. In general the non linear term couples energy into or out of the wave as it propagates so we introduce 3.8 as a solution of the non linear equation allowing the \( A_n \) to be slowly varying functions of \( z \). This coupled mode approach has been used in the study of systems with small non lineairities.

55, 56,

Let 3.9  \( \mathbf{E}_0^\omega (x,y,z) = \sum_{n=0}^{\infty} A_n^\omega (z) \mathbf{E}_{on}^\omega (x,y,z) \)

\( \mathbf{E}_e^\omega (x,y,z) = \sum_{n=0}^{\infty} B_n^\omega (z) \mathbf{E}_{ne}^\omega (x,y,z) \)

and substituting these into equation 3.7 we have

3.10  (a)  \( \sum_{n=0}^{\infty} -2ik_0 \frac{dA_n}{d\omega} \mathbf{E}_{on}^\omega = -\frac{4\pi \omega^2}{c^2} p_Y \Theta^\omega \exp (ik_0z) \)
Taking the scalar product with \( \mathbf{E}_{\text{om}}^\omega \), \( \mathbf{E}_{\text{em}}^\omega \) respectively we have

\[
\begin{align*}
3.11 & \quad (a) \quad \frac{dA}{dz} = \frac{-2\pi i \omega^2}{k_0 c^2} \int_{-\infty}^{\infty} \mathbf{E}_{\text{om}}^\omega \mathbf{E}_{\text{om}}^* \Omega \mathbf{p}_y \mathbf{dx} \exp (ik_o z) \\
& \quad (b) \quad \frac{dB}{dz} = \frac{-2\pi i \omega^2}{k_0 c^2} \int_{-\infty}^{\infty} \mathbf{E}_{\text{om}}^\omega \mathbf{E}_{\text{em}}^\omega \mathbf{p}_x \mathbf{dx} \exp (ik_o z)
\end{align*}
\]

These equations give us the rate of change of the \( n \)th resonator mode ((a) ordinary (b) extraordinary) for the field at frequency \( \omega \) due to the non linear polarization at that frequency.

(a) Second Harmonic Generation

(1) Coupled Mode Equation:- For second harmonic generation we have to consider two optical fields, frequencies \( \omega \) and \( 2\omega \). Since the ordinary wave at \( \omega \) will be near phase matching to the extraordinary at \( 2\omega \) we need only consider these two components.

From section 1.4 (c) we have

\[
\begin{align*}
\mathbf{p}_y \mathbf{\Omega} \omega &= d_{13} \mathbf{E}_y^* \mathbf{E}_x^2 \\
\mathbf{p}_x \mathbf{\Omega} 2\omega &= 2 d_{31} \mathbf{E}_y^\omega \mathbf{E}_y^\omega
\end{align*}
\]

In terms of the resonator modes.
dropping the explicit $\omega$ subscript and substituting these equations into the appropriate equations from 3.11 we have

\[ 3.13 \quad (a) \quad \frac{dA_m^\omega}{dz} = \frac{-2\pi i \omega^2}{k_1^2} d_{15} A_j^\omega B_k^\omega \int_{-\infty}^{\infty} E_m^\omega E_j^\omega E_k^\omega dx \times \exp \left( i (k - 2k_1) z \right) \]

\[ (b) \quad \frac{d^2 A_m^\omega}{dz^2} = \frac{4\pi i \omega^2}{k_2^2} d_{31} A_j^\omega A_k^\omega \int_{-\infty}^{\infty} E_m^\omega E_j^\omega E_k^\omega dx \times \exp \left( i (k_2 - 2k_1) z \right) \]

which can be written

\[ 3.14 \quad (a) \quad \frac{dA_m^\omega}{dz} = C_{\omega, \omega} A_j^\omega B_k^\omega \]

\[ (d) \quad \frac{d^2 A_m^\omega}{dz^2} = C_{\omega, \omega}^2 A_j^\omega A_k^\omega \]

where the coupling coefficients are defined by 3.13 (a) (b)
(ii) Energy Conservation relation

For a lossless medium there exists a useful relation between the

\[ C_{njk}^{\omega} \text{ and } C_{njk}^{2\omega} \]

The total energy of the system is represented by

\[ \sum_{n} (A_n^{\omega} A_n^{*\omega} + B_n^{2\omega} B_n^{*2\omega}) \] and this must be constant. Thus differentiating this expression for the energy we have

\[ 3.15 \quad \frac{dA_n^{\omega}}{dz} A_n^{\omega} + \frac{dB_n^{2\omega}}{dz} B_n^{2\omega} + \text{ complex conjugate} = 0 \]

substituting from 3.14

\[ C_{njk}^{\omega} A_j^{\omega} A_n^{\omega} A_k^{2\omega} + C_{jkn}^{2\omega} A_j^{*\omega} A_k^{*\omega} B_n^{2\omega} + \text{ complex conjugate} = 0 \]

interchanging dummy suffices we have

\[ (C_{njk}^{\omega} + C_{njk}^{\omega}) A_j^{\omega} A_n^{\omega} B_k^{2\omega} + \text{ complex conjugate} = 0 \]

This relation must be true for all values of the \( A_j \)'s and \( B_j \)'s hence

\[ 3.16 \quad C_{njk}^{\omega} = -C_{njk}^{2\omega} \]

This is an important result which corresponds to Klienman's symmetry condition.
(iii) Calculation of the coefficients $C_{njk}^{2\omega}$

From 3.13 (b) the $C_{njk}^{2\omega}$ depend on the integral

$$I_{jkm} = \int_{-\infty}^{\infty} E_m^{2\omega} E_j^{\omega} E_k^{\omega} \, dx$$

which on substituting for the resonator mode functions introducing the subscript 1, 2 for the fundamental and harmonic quantities respectively is essentially the integral.

$$I_{jkm} = \int_{-\infty}^{\infty} H_m \left( \frac{\sqrt{2x}}{w_2} \right) H_j \left( \frac{\sqrt{2x}}{w_1} \right) H_k \left( \frac{\sqrt{2x}}{w_1} \right)$$

$$\exp \left\{ \frac{-x^2}{w_{02}^2} \left( 1 + i \xi_2 \right) - \frac{2x^2}{w_{01}^2} \left( 1 - i \xi_1 \right) \right\} \, dx$$

where the coefficients will be given by

$$C_{jkm}^{2\omega} = \frac{-4i \omega^2 d_{31} (2\pi)^{\frac{1}{2}} \exp (i\Delta kx)}{k_2 \alpha^2 (2^m + j + k \, m \cdot j \cdot k) \, w_{01}^2 \, w_{02}^{\frac{1}{2}}} \times$$

$$\frac{(1 - i \xi_2)^m}{(1 + i \xi_1)^{\frac{m+1}{2}}} \frac{i + k + 1}{(1 - i \xi_1)^{\frac{m+1}{2}}} \times I_{jkm} \text{ where } \Delta k = k_2 - 2k_1$$

$$\frac{\sqrt{2x}}{w_1} = u \quad \alpha = \frac{w_1}{\sqrt{2w_2}}$$

3.18 becomes
This can be evaluated using the generating function for the Hermite polynomials

$$\exp (2 s x - s^2) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} s^n$$

$I_{jk\alpha}$ will be the coefficient of $t^m s^j p^k$ multiplied by $j! k! m!$ in the expansion of the expression in powers of $t$, $s$, and $p$

$$I = \frac{W_1}{\sqrt{2}} \exp \left(2 \sqrt{2} \alpha t - t^2 + 2us - s^2 + 2pu - p^2 \right)$$

$$- b^2 u^2 \right) \frac{\alpha^2}{u}$$

where $b^2 = \alpha^2 (1 - i \xi_2) + (1 + i \xi_1)$

rearranging

$$I = \frac{W_1}{\sqrt{2}} \int_{-\infty}^{\infty} \exp \left(-b^2 u^2 + 2u \left(\sqrt{2} \alpha t + s + p \right) - b^2 - s^2 - p^2 \right) du$$

To evaluate this integral let $v = ub - (\frac{\sqrt{2} \alpha t + s + p}{b})$
and it reduces to

\[ I = \frac{\sqrt{2}}{\sqrt{\pi}} \exp \left( \frac{(\sqrt{2} \alpha t + s + \nu)^2}{b^2} \right) \int_C \exp \left( -\nu^2 \right) d\nu \]

using the result of appendix 1

\[ \int_C (\exp(-\nu^2)) d\nu = \sqrt{\pi} \]

Expanding the exponential term in 3.24 gives

\[ \exp \left( \frac{(\sqrt{2} \alpha t + s + \nu)^2}{b^2} \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\sqrt{2} \alpha t + s + \nu}{b} \right)^n (t^2 - s^2 - \nu^2) \]

expanding further using the binomial theorem

\[ 3.26 = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(-1)^{n-m}}{n!} \left( \frac{n}{m} \right) \left( \frac{\sqrt{2} \alpha t + s + \nu}{b} \right)^{2m} (t^2 + s^2 + \nu^2)^{n-m} \]

and again using the binomial theorem

\[ 3.27 = \frac{\sqrt{\pi}}{b} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sum_{r=0}^{\infty} \sum_{q=0}^{r} \sum_{l=0}^{q} \sum_{t=0}^{l} \frac{1}{n!} (-1)^{n-m} \frac{(n)}{(m)} \frac{(2m)}{(r)} \frac{(n-m)}{(q)} \frac{(r)}{(l)} \frac{1}{(q-l)} \frac{b^{-2m}}{(\sqrt{2} \alpha)^{2m-r}} \]

\[ x (2m-r-rq) + \frac{1}{2} \]

\[ s (r-1-2q-2l) + (2m-2q) \]

We can rewrite this in a formal way in order to pick out the required coefficient putting \( l + 2l = k \).
\[ 3.28 \quad r + 2q - (1 + 2l) = j \]
\[ 2n - (r + 2q) = m \]

rearranging equation 3.27

\[ 3.29 \quad r = m + j - 2q \]
\[ l = m - 2l \]
\[ n = \frac{m + j + k}{2} \]

\( n \) is an integer, therefore for a non-zero coefficient \( c_{i,j,k}^2 \omega \), \( m + j + k \) must be an even integer. This just expresses the symmetry conditions

odd plus even fundamental mode gives odd S.H. mode
odd " odd " " " even " "
even " even " " " even " "

Picking out the required coefficients we have

\[ \left( -1 \right)^{\frac{m+j+k-m}{2}} x \left( \frac{m+j+k}{2} \right) \left( \frac{2m}{m-j-2q} \right) \left( \frac{m+j-k-m}{2} \right) \left( \frac{m+j-2q}{q} \right) \left( \frac{m-2l}{l} \right) \left( \frac{q}{q} \right) \]

\[ b^{-2\bar{m}} \left( \sqrt{2\lambda} \right)^{2\bar{m}-m-j+2q} \]
Thus the coefficients can be evaluated analytically although the final expression is rather complicated in general. As we would expect the low-order coefficients are much simpler in form eg.

$$I_{000} = \frac{W_1}{b} \sqrt{\frac{n}{2}}$$

$$I_{020} = \frac{W_1}{b} \sqrt{\frac{n}{2}} ( \frac{1}{b} - 1 )$$

$$I_{101} = \frac{W_1}{b} \sqrt{\frac{n}{2}} \frac{1}{2b}$$

The most important set which we will examine more closely are those resulting from the coupling between the lowest order gaussian fundamental mode and the 2nth second harmonic mode.

$$I_{002n} = \frac{W_1}{b} \sqrt{\frac{n}{2}} ( \frac{2\alpha^2}{b^2} - 1 )^n \frac{(2n)!}{n!}$$

From 3.21 and 2.25 we have

$$\alpha = \frac{W_1}{\sqrt{2\nu_2}} = \frac{W_{o1}}{\sqrt{2\nu_{o2}}} \left( \frac{1+\xi_1^2}{\xi_2^2} \right)^\frac{1}{2}$$

and defining

$$\bar{W} = \frac{W_{o1}}{\sqrt{2\nu_{o2}}}$$

$$b^2 = \bar{W}^2 \left( \frac{1+\xi_2^2}{\xi_1^2} \right) \left( 1-i \frac{\xi_1^2}{\xi_2^2} \right) + \left( 1+i \frac{\xi_1^2}{\xi_2^2} \right)$$

$$= \left( \frac{1+i \xi_1^2}{\xi_2^2} \right) \frac{(W^2 + 1 - i (W^2 \xi_1 - \xi_2))}{(1+i \xi_2^2)}$$
and therefore

\[
\frac{2x^2}{b^2} - 1 = w^2 \frac{(1-i \xi_1) (1+i \xi_2) - (1+i \xi_2^2)}{(1-i \xi_2) (w^2+1 - i (w^2 \xi_1 - \xi_2))}
\]

Substituting these expressions into that for \( I_{oo2n} \) and then the resulting expression into equation 3.19 for \( C_{o2n}^\omega \) we have

\[
3.35 \quad C_{oo2n}^\omega = - \frac{4i \omega^2 d_{31}^1}{k_2 c^2 \omega^2 \omega_0^2} \frac{n^3}{2^\frac{1}{2}} \frac{(2n)!}{n!} \frac{1}{(1-i \xi_1)^{\frac{1}{2}}} \exp(i \Delta kz) \frac{1}{(w^2-1-i (w^2 \xi_1 - \xi_2))^{n+\frac{1}{2}}}
\]

The factor \( \frac{(2n)!}{n! 2^n} \) has values

<table>
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<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>6</th>
<th>10 etc</th>
</tr>
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<td>0.6724</td>
<td>0.5590</td>
<td>0.5229</td>
<td>0.4749</td>
<td>0.4197</td>
</tr>
</tbody>
</table>

and so decreases with \( n \) reducing the coupling between the lowest order fundamental mode and 2nth order harmonic mode with increasing \( n \).

Thus far we have only considered the coupling case when the second
harmonic resonator and the fundamental resonator are perfectly aligned and the foci of the beams are at the same position. There will be a reduction of the coupling (i) if the foci are any distance apart (ii) if the axial planes are parallel but a distance apart and (iii) if the axial planes are inclined at a small angle to each other.

(i) The evaluation of the \( C_{oo2n}^2 \) is unaffected by this modification. Considering the fundamental mode to be focussed at the origin and the \( n \)th harmonic mode to be focussed at \( z = f, \xi_j \) remains unchanged but

\[
\xi_2 = 2 \frac{(x - f)}{\kappa_2 W_{o2}}^2
\]

and this introduces the extra term \(-2f/\kappa_2 W_{o2}^2\) into equation 3.35.

(ii) If the axial plane of the fundamental \( j \)th mode is removed from that of the harmonic modes by a distance \( \nu \) of the harmonic modes it will be described by the equation

\[
\xi_j = \left( \frac{1}{W_{o1}^2} \right)^{1/4} \left( \frac{1 + i \xi_j}{1 - i \xi_j} \right)^{1/2} H_j \left( \frac{2}{W_{o1}} (x - \nu) \right)
\]

\[
x \exp \left( -\frac{(x - \nu)^2}{W_{o1}^2} (1 - i \xi_j) \right)
\]

Substituting this form into equation 3.17 equation 3.22 becomes
$$I_{mjk} = \frac{\hat{w}_1}{\sqrt{a^2}} \int_{-\infty}^{\infty} H_m \left( \sqrt{2au} \right) H_j \left( u - S \right) H_k \left( u - S \right) \exp \left\{ -u^2 \alpha^2 (1 - i \xi) \right\} \left\{ -(u - S)^2 (1 + i \xi) \right\} \, du$$

3.38 (b) where $S = \frac{\sqrt{\pi}}{\hat{w}_1}$

using the generating functions as before 3.23 becomes

$$I = \frac{\hat{w}_1}{\sqrt{a^2}} \int_{-\infty}^{\infty} \exp \left\{ -u^2 \alpha^2 + 2u \left( \sqrt{2} \alpha + s + p + S(1 + i \xi) \right) - t^2 - s^2 - p^2 \right\} \, du$$

putting $v = u \alpha - \left( \frac{\sqrt{2} \alpha + s + p + S(1 + i \xi)}{b} \right)$

and carrying out the integration (appendix I) we have

$$I = \frac{\hat{w}_1}{b} \sqrt{\frac{\pi}{2}} \exp \left\{ -S^2 (1 + i \xi) \right\} \left( 1 + \left( \frac{1 + i \xi}{b} \right) \right) +$$

$$\left( \frac{\sqrt{2} \alpha + s + p}{b} \right)^2 - t^2 - s^2 - p^2 + 2 \left( \sqrt{2} \alpha + s + p \right) S$$

Thus the separation of the axial planes has introduced two extra factors into equation

3.24 The first is

$$\exp \left\{ -S^2 (1 + i \xi) \left( 1 - \left( \frac{1 + i \xi}{b} \right) \right) \right\}$$

substituting for $b$ this becomes
The real part of the expression which forms the argument of the exponential function is always less than zero and as this factor will modify each coefficient $C_{jkm}$, they will all decrease with $\xi$ as

$$\exp \left( -\varphi \xi^2 \right),$$

where $$\varphi = \frac{(1 + \xi_1^2)(\xi_1^2 + 1)}{(\xi_1^2 + 1)^2 + (\xi_1 \xi_2 - \xi_2^2)^2} > 0.$$ Thus the coupling between any two modes has an overall decrease with increasing separation but of course there may be local increase.

The second factor

$$\exp \left\{ \frac{2\sqrt{2} \Delta t + s + p_1}{b} \right\} \frac{a}{b}$$

has as the argument of the exponential function an expression which depends on an odd power of $t$, $s$ and $p$ and therefore it will break the symmetry conditions contained in equations 1.29. Consider as a specific example $I_{oo1}$ which is zero under equation 1.29 for the coaxial case. $I_{oo1}$ is the coefficient of $t$ in the expansion of equation 3.40.

$$I_{oo1} = \frac{\bar{w}_1}{b} \sqrt{\frac{\pi}{\alpha}} \exp \left\{ -\frac{\xi^2 \xi_2^2 (1 + \xi_1^2)}{(\xi_1^2 + 1) - i(\xi_1 \xi_2^2 - \xi_2^2)} \right\}$$

so that substituting for $\xi$ from 3.38 (b)
Thus $C_{\omega \omega 1}$ is zero at $v = 0$ as expected and increases with increasing separation $v$, to a maximum when $v = \frac{w_{01}}{2 \sqrt{\epsilon}}$ and thereafter decreases asymptotically approaching zero. Since $\Theta$ depends on $z$ this does not immediately give the separation of the axes which will maximize the coupling in a given crystal except for the case when $z/\lambda_{0} \ll 1$

(iii) For the case when the axial plane of the fundamental resonator is tilted at a small angle $\theta$ to the axial plane of the second harmonic resonator the equation representing the jth fundamental mode is

$$E_{j} = \left( \frac{1}{w_{01}^{2}} \right)^{\frac{1}{2}} \left( \frac{1 + i \xi_{1}}{1 - i \xi_{2}} \right)^{\frac{1}{2}} H_{j} \left( \frac{x}{w_{1}} \right) \exp \left( \frac{-x^{2}}{2 w_{01}^{2}} (1 - i \xi_{1}) - i \theta x \right)$$

if the $z$ axis lies in the axial plane of the second harmonic resonator. The extra term depending on $\theta$ arises from the propagation term $\exp (-ikz)$ of the electric field; all the other correction terms will be negligible in comparison with this one since $k \gg 1$

Using this equation, equation 3.23 becomes

$$I = \frac{W_{1}}{\sqrt{2}} \int_{-\infty}^{\infty} \exp \left( -u^{2} b^{2} + 2u \left( \sqrt{2} \mu t + s + p - i \frac{h \theta \xi_{1}}{\sqrt{2}} \right) \right) du$$
Carrying out the integration as before equation 3.24 becomes

\[ I = \frac{w_1}{b} \sqrt{\frac{\pi}{2}} \exp \left\{ \left( \frac{\sqrt{2}x + s + p}{b^2} \right)^2 \frac{-2ik\theta w_1}{\sqrt{2}} \right\} \left( \sqrt{2}x + s + p \right) \]

\[-\frac{k^2 \theta^2}{2} w_1^2 - \frac{t^2 - s^2 - p^2}{2} \}

As in the previous case the tilt angle has introduced two new factors. The first being \[ \exp\left( \frac{-k^2 \theta^2}{2} \right) \] affecting all the coefficients, the second breaking the symmetry condition of equations 3.29 from 3.47

\[ I_{oo1} = \frac{-w_1}{b} \sqrt{\frac{\pi}{2}} \exp \left( \frac{-k^2 \theta^2}{2} \right) \]

\[ 2ik \theta w_1 \alpha \]

This has exactly the form of the previous coupling when the axes of the resonators were separated, increasing to a maximum and thereafter decreasing and asymptotically approaching zero. The maximum in this case occurs at an angle

\[ \Theta = \frac{1}{kw_1} \]

this again depends on \( z \) and so will not give the maximum coupling condition immediately.

(2) Parametric Amplification

(i) Coupled Mode Equation

We have now three optical fields \( \omega_1, \omega_2, \omega_j \)

where \( \omega_j = \omega_1 + \omega_2 \).
We consider the case of parametric amplification when the pump field, $\omega_j$, propagates as an extraordinary wave and the signal, $\omega_1$, and idler $\omega_2$ fields propagate as ordinary waves. This being the configuration appropriate to a negative uniaxial crystal. The theory can be used for positive uniaxial crystals when the opposite configuration is appropriate by using the appropriate mode function. The relevant second order polarization terms are for this case therefore

$$
\begin{align*}
P_y \otimes \omega_1 &= d_{15} \mathbf{E}_y^* \mathbf{E}_x^2 \mathbf{E}_x \\
P_y \otimes \omega_2 &= d_{15} \mathbf{E}_y^* \mathbf{E}_x^1 \mathbf{E}_x \\
P_x \otimes \omega_3 &= d_{31} \mathbf{E}_y^1 \mathbf{E}_y^2
\end{align*}
$$

Expanding the fields in terms of the resonator modes:

$$
\begin{align*}
P_y \otimes \omega_1 &= d_{15} A_j^* \omega_2 B_k \mathbf{E}_j^* \mathbf{E}_k^3 \exp \left\{ -i (k_3 - k_2) z \right\} \\
P_y \otimes \omega_2 &= d_{15} A_j^* \omega_1 B_k^3 \mathbf{E}_j^* \mathbf{E}_k^2 \exp \left\{ -i (k_3 - k_1) z \right\} \\
P_x \otimes \omega_3 &= d_{31} A_j^1 \omega_2 A_k \mathbf{E}_j^1 \mathbf{E}_k^2 \exp \left\{ i (k_1 + k_2) z \right\}
\end{align*}
$$

Substituting these relations into the appropriate equations from
3.11 (a) and (b) we arrive at the coupled mode equations for the parametric case

\[ \frac{dA_n}{dz} = c_{mjk} A_j \omega_2 A_k \omega_3 \]

\[ \frac{dA_n}{dz} = c_{jlk} A_j \omega_1 A_k \omega_3 \]

\[ \frac{dA_n}{dz} = c_{jkn} A_j A_k \omega_2 \]

where the coupling coefficients are defined in a similar way to the previous case of second harmonic generation.

For example

\[ c_{mjk} \omega_1 = \frac{-2\pi i \omega^2}{k_1 c^2} d_{45} \exp \left\{ -i \Delta k x \right\} \int_{-\infty}^{\infty} E_n^* \omega_1 E_j^* \omega_2 E_k^* \omega_2 dx \]

where \( \Delta k = k_3 - k_1 - k_2 \)

For a lossless medium we can establish a relationship between the coupling coefficients as in the previous case

\[ \sum_n (A_n^1 A_n^1 + A_n^2 A_n^2 + B_n^3 B_n^3) \]

is the total energy present in the system and is constant. By the previous argument used we have
3.53 \[ c_{njk} + c_{njk} = -c_{njk} \]

Note that this result does reduce to 3.16 if \( \omega_1 = \omega_2 \) because of the factor \( \frac{1}{2} \) in front of equation 3.12 (b) produced by the \( \omega_1 = \omega_2 \) symmetry.

(ii) Calculation of the coefficients \( c_{njk} \)

The calculation will follow precisely the lines of the calculation for the second harmonic generation but we have now the added complication of three sets of beam parameters. \( c_{jk\alpha} \) will depend on an integral of the form

\[
  \int_{-\infty}^{\infty} \mathcal{E}_j^* \omega_1 \mathcal{E}_k^* \omega_2 \mathcal{E}_m^* \omega_3 \, dx
\]

substituting for the resonator modes, introducing subscripts 1, 2, 3 to denote the terms arising from the modes at \( \omega_1, \omega_2, \omega_3 \) respectively the integral will reduce to

\[
  I_{jkm} = \int_{-\infty}^{\infty} H_j \left\{ \frac{\sqrt{2\pi}}{\bar{w}_1} \right\} H_k \left\{ \frac{\sqrt{2\pi}}{\bar{w}_2} \right\} H_m \left\{ \frac{\sqrt{2\pi}}{\bar{w}_3} \right\} \exp \left\{ -\frac{x^2}{\bar{w}_0^2(1+i\xi_1)} \right\} \left\{ -\frac{x^2}{\bar{w}_0^2(1+i\xi_2)} \right\} \left\{ \frac{\pi}{\bar{w}_0^2(1-i\xi_3)} \right\}
\]
where \( \omega_{jkm} \) = 

\[
\frac{-2i \omega_1^2 \Delta \omega \exp (i \Delta \omega \chi) (2\pi)^{1/2}}{k_1 c^2 (2^{j} \delta^{j} \omega_1 \omega_2 \omega_3)^{2}} \frac{(1-i \xi_1)^{j}}{(1+i \xi_2)^{k}}
\]

\[\frac{(1+i \xi_2)^{j+1}}{(1-i \xi_2)^{k+1}} \times \Gamma_{jkm}
\]

making the substitutions \( \frac{\sqrt{2x}}{W_3} = u \), \( \frac{\sqrt{2x}}{W_1} = \alpha_1 u \), \( \frac{\sqrt{2x}}{W_2} = \alpha_2 u \)

where \( \alpha_1 = \frac{W_1}{W_3} \), \( \alpha_2 = \frac{W_2}{W_1} \)

the integral becomes

\[3.56(\alpha) \; I_{jkm} = \frac{W_3}{\sqrt{2}} \int_{-\infty}^{\infty} H_j (\alpha_1 u) H_k (\alpha_2 u) H_m (u) \exp \left\{ -u^2 (\alpha_1^2 (1-i \xi_1) + \alpha_2^2 (1-i \xi_2) + (1+i \xi_3)) \right\} du
\]

using the generating function of the Hermite polynomials the integral generating the \( I_{jkm} \) is

\[3.57 \; I = \frac{W_3}{\sqrt{2}} \int_{-\infty}^{\infty} \exp \left\{ -b^2 u^2 + 2u (\alpha_1 t + \alpha_2 s + p) - t^2 - s^2 - p^2 \right\} du
\]

where \( b^2 = \alpha_1^2 (1-i \xi_1) + \alpha_2^2 (1-i \xi_2) + (1+i \xi_3) \)

which becomes, carrying out the integration
If we expand this expression the \( I_{mjk} \) can be picked out as before and the symmetry condition (3.39) \( m + j + k = \text{even integer} \), holds for this case. A few of the low-order coefficients are

\[
I_{000} = \frac{\omega_3}{b} \sqrt{\frac{\pi}{2}}
\]

\[
I_{110} = \frac{\omega_3}{b} \sqrt{\frac{\pi}{2}} \cdot 2 \frac{\alpha_1 \alpha_2}{b^2}
\]

\[
I_{101} = \frac{\omega_3}{b} \sqrt{\frac{\pi}{2}} \cdot 2 \frac{\alpha_1}{b^2}
\]

etc.

We will now evaluate \( c_{000} \) in detail as an example, substituting for \( b \) and \( \alpha_1, \alpha_2 \) we have

\[
I_{000} = \sqrt{\frac{\pi}{2}} \left\{ \frac{\overline{W}_1}{(1 + i \xi_1)} + \frac{\overline{W}_2}{(1 + i \xi_2)} + \frac{1}{(1 - i \xi_3)} \right\}
\]

where we have defined \( \overline{W}_1 = \frac{\omega_3}{w_0^1} \), the ratio of the pump and signal spot sizes,

\( \overline{W}_2 = \frac{\omega_3}{w_0^2} \) the ratio of the pump and idler spot sizes, and substituting this into equation 3.56(a) we arrive at the expression for \( c_{000} \).
This is the result for the perfect case of coincident focus positions and axial planes. As before we can evaluate the result for the various combinations of non-coincident focus positions and axial planes of the three resonators. The method and results will in general be the same as for the case of second harmonic generation which has been considered.

(2) Three Dimensions

Under the approximation considered in chapter II the derivation of the coupled mode equations followed the same lines as for the two dimensional case. The relevant equations derived from Maxwell's equations are (c.f. equation 2.84)

\[
\begin{align*}
\frac{\partial^2 \mathbf{E}}{\partial y \partial z} + \frac{\partial^2 \mathbf{E}}{\partial z \partial y} &= \frac{\partial^2 \mathbf{E}}{\partial x^2} - \frac{\partial^2 \mathbf{E}}{\partial y^2} + \frac{\partial^2 \mathbf{E}}{\partial z^2} - \frac{\varepsilon_0^2}{c^2} \mathbf{E}_{y} \mathbf{E}_{y} = \frac{4\pi \omega^2}{c^2} \mathbf{P}_{y} \\
\frac{\partial^2 \mathbf{E}}{\partial x \partial z} + \frac{\partial^2 \mathbf{E}}{\partial z \partial x} &= \frac{\partial^2 \mathbf{E}}{\partial x^2} - \frac{\partial^2 \mathbf{E}}{\partial y^2} + \frac{\partial^2 \mathbf{E}}{\partial z^2} - \frac{\varepsilon_0^2}{c^2} \mathbf{E}_{x} \mathbf{E}_{x} = \frac{4\pi \omega^2}{c^2} \mathbf{P}_{x}
\end{align*}
\]

and the divergence equation

\[
\varepsilon \frac{\partial^2 \mathbf{E}}{\partial x^2} + \varepsilon \frac{\partial^2 \mathbf{E}}{\partial y^2} + \varepsilon \frac{\partial^2 \mathbf{E}}{\partial z^2} = -4\pi \mathbf{P}_{\mathbf{E}} \mathbf{E}
\]

Using the approximations which were formulated in chapter II for
dealing with the linearized versions of these equations they can be reduced to the equations

\[ 3.63 \ \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} + \frac{\partial^2 \omega}{\partial z^2} + \frac{\omega^2}{c^2} E_y \omega = -\frac{1}{c^2} \frac{\partial \omega}{\partial z} p \omega \]

\[ \epsilon_x \frac{\partial^2 \omega}{\partial x^2} + \epsilon_x \frac{\partial^2 \omega}{\partial y^2} + \frac{\omega^2}{c^2} E_x \omega = -\frac{1}{c^2} \frac{\partial \omega}{\partial x} p_x \omega \]

for the electric fields of the ordinary and extraordinary waves respectively. Making the substitution \( E = \mathcal{E} \exp (-ikz) \) as before reduces these equations to the equations

\[ 3.64 \ \frac{\partial^2 \mathcal{E}_o}{\partial x^2} + \frac{\partial^2 \mathcal{E}_o}{\partial y^2} - 2ik_o \frac{\partial \mathcal{E}_o}{\partial z} = -\frac{1}{c^2} \frac{\partial \omega}{\partial z} p_y \exp (ik_o z) \]

\[ \frac{\epsilon_x}{\epsilon_z} \frac{\partial^2 \mathcal{E}_e}{\partial x^2} + \frac{\partial^2 \mathcal{E}_e}{\partial y^2} - 2ik_e \frac{\partial \mathcal{E}_e}{\partial z} = -\frac{1}{c^2} \frac{\partial \omega}{\partial x} p_x \exp (ik_e z) \]

for the quantities \( \mathcal{E}_{o,e} \). If we now express \( \mathcal{E}_{o,e} \) as a sum over the relevant cavity modes allowing the coefficients to vary with \( z \) as before and substitute the resulting expressions into equations 3.6 we have

\[ 3.65 \ (a) \ \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} -2ik_o \frac{dA_{nm}}{dz} \mathcal{E}_{omn} = -\frac{1}{c^2} \frac{\partial \omega}{\partial z} p_y \exp (ik_o z) \]
If we now apply these results to the theory of second harmonic generation, the expansion of the polarization in terms of the resonator modes can be written

\[ P_y = \frac{d_{15}}{} A_{jk}^{\omega} B_{rs}^{2\omega} E_{j}^{\omega} E_{r}^{\omega} \exp(-i(k_2-k_1)z) \]

\[ P_x = \frac{1}{i} d_{31} A_{jk}^{\omega} A_{rs}^{\omega} E_{j}^{\omega} E_{r}^{\omega} \exp(i(k_2-k_1)z) \]

Substituting these expansions into equations 3.66 produces the simultaneous coupled mode equations

\[ \frac{dA_{nm}^{\omega}}{dz} = c_{nmjkr} A_{jk}^{\omega} B_{rs}^{2\omega} \]
where for the lossless case we have again the connecting relation

\[ 3.69 \quad c_{\alpha \beta \gamma \delta} = -c_{\alpha \beta \gamma \delta}^* \]

between the two coefficients. In three dimension coefficients depend on the double integral

\[ 3.70 \quad I_{\alpha \beta \gamma \delta} = \iint \varepsilon_{\alpha \beta \gamma \delta} \varepsilon_{\alpha \beta \gamma \delta}^* \, dx \, dy \]

which on substituting for the resonator modes from equation 2.91 and 2.28 splits into an integral over \( x \), \( I_x \), and one over \( y \), \( I_y \). \( I_y \) has exactly the form of equation 3.18

\[ 3.71 \quad I_y = \int \frac{H_m \left( \frac{\sqrt{2} x}{w_2} \right) H_k \left( \frac{\sqrt{2} y}{w_1} \right) H_s \left( \frac{\sqrt{2} y}{w_1} \right) \exp \left( \frac{-x^2}{w_{02}^2 (1+i \xi_2)} \right)}{w_{01}^2 (1-i \xi_1)} \, dy \]

whereas \( I_x \) is modified by the \( \varepsilon_{\alpha \beta} / \varepsilon_{\alpha \gamma} \) factor (equation 2.91)

\[ 3.72 \quad I_x = \int \frac{H_m \left( \frac{\sqrt{2} x}{w} \frac{\varepsilon_{\alpha \beta}}{\varepsilon_{\alpha \gamma}} \right) H_k \left( \frac{\sqrt{2} y}{w_1} \right) H_j \left( \frac{\sqrt{2} x}{w_1} \right) \exp \left( \frac{-x^2}{w_{02}^2 (1+i \xi_2)} \frac{\varepsilon_{\alpha \beta}}{\varepsilon_{\alpha \gamma}} \right)}{w_{01}^2 (1-i \xi_1)} \, dy \]
The coupling coefficient is given by the equation

\[ \text{The integrals can be evaluated exactly as before and all the results of the previous theory apply to each integral. It is thus a simple matter to generalize the two dimensional results to three dimensions. The only difficulty being that the expressions are very long and tedious to write out. In the same way as has been done for second harmonic generation the expressions for three dimensional parametric amplification can be evaluated.}

(3) Effect of absorption

It has been shown (section 2.5) that the presence of absorption in the medium only effects the term \( \exp(-ikz) \) of the resonator mode functions. Hence the results of this chapter will be unaffected by the presence of absorption except that \( k \) will now be complex. This introduces a loss term \( \exp(k_{\text{abs}}z) \) into equations 3.35,
Chapter 4

Second Harmonic Generation in the small conversion approximation

4.1 The simplest problem to which the formalism developed in the last chapter can be applied is the problem of resonant second harmonic generation from a travelling wave gaussian beam, 32. In the small conversion approximation there is no reaction back on the fundamental and therefore only the original lowest order (gaussian) mode need be considered of the fundamental beam. We first study the second Harmonic generated in the limiting cases which can be solved analytically and then go on to study the general case numerically. We show that the theory that has been set up gives results consistent with those of Kleinman and Boyd, 15, for the non resonant case.

Consider a suitable resonator containing a uniaxial non linear dielectric orientated with its optic axis perpendicular to the axis of the resonator (fig. 4.1). It is assumed that the mirrors of this cavity are transparent to the fundamental laser frequency and that the alignment of the fundamental beam and the cavity is perfect. The theory can be extended immediately to cover the cases of imperfect alignment as indicated in the last chapter since the fundamental only travels in one direction through the non linear dielectric, second harmonic will only be produced in that direction. In the small conversion approximation the second harmonic produced in a single pass of the non linear dielectric is not dependent on the value of the second harmonic at the starting face of the dielectric and the problem splits into two parts. Firstly the calculation of the various second
harmonic mode amplitudes produced by a single pass of the dielectric, assuming an initial value zero, and secondly using these results to calculate the actual mode amplitudes produced in the resonator and hence the output of the resonator.

Generalizing equation 3.35 to three dimensions the rate of change of the 2n 2m th second harmonic mode amplitude is given by the equation

\[ \frac{dB}{dz} 2n 2m = \omega \frac{\omega}{2n 2m} A_o A_o \]

where

\[ \frac{d^2 \omega}{dz} 2n 2m = 4 i \omega^2 d_{31} \frac{5}{4} \frac{(2n!2m!)^{\frac{3}{2}}}{k_c^2 e^{\frac{1}{2}} \omega_{02} n!m!2^{n+m+\frac{3}{2}}} \exp \left( i \Delta k \right) \]

\[ \left( e^{2 \xi_1 - 1 - i \left( e^{2 \omega^2 \xi_1} - \xi_2 \right)} \right)^n \left( e^{2 \omega^2 + 1 - i \left( e^{2 \omega^2 \xi_1 - \xi_2} \right)} \right)^n \]

\[ \left( e^{2 \omega^2 - 1 - i \left( e^{2 \omega^2 \xi_1 - \xi_2} \right)} \right)^n \left( e^{2 \omega^2 + 1 - i \left( e^{2 \omega^2 \xi_1 - \xi_2} \right)} \right)^n \]

where \( \epsilon_x/ \epsilon_z = e^2 \).

Since

\[ \xi_1 = \frac{2z}{k_1 \omega_{01}^2}, \quad \xi_2 = \frac{2z}{k_2 \omega_{02}^2}, \quad \Delta k = k_2 - 2k_1 \]

we have

\[ \frac{2 \xi_1}{k_1 \omega_{01}^2} - \xi_2 = \frac{2 \xi_1}{k_2 \omega_{02}^2} \]

\[ \Delta k \]

for \( \Delta k < 10 \) the term dependent on \( \Delta k/k_2 \) can be neglected hence

\[ \frac{2 \xi_1}{k_1 \omega_{01}^2} - \xi_2 = 0 \]

4.4

\[ e^{2 \omega^2 \xi_1 - \xi_2} = e^{2 \omega^2 \xi_1} (1 - \frac{1}{e^2}) \]

defining \( c = \frac{1}{e^2} - 1 \) we can write
\[ 4.5 \quad c_{2n 2m}^2 = \frac{\alpha_{nm}}{W_{o2}} \exp(i \Delta k z) \frac{\left( e^{2w^2} - 1 + i e^{2w\xi_1} \right)}{(1 - i \xi_1) \left( e^{2w^2} + 1 + i e^{2w\xi_1} \right)} \left( n + \frac{1}{2} \right) \]

\[ x \frac{(w^2 - 1)^m}{(w^2 + 1)^m + \frac{1}{2}} \]

for \( \Delta k < 10 \) where \( \alpha_{nm} \) is a constant dependent on the mode number.

Since we have assumed \( A_0 \) to be constant the amplitude of the nmth second harmonic mode at the exit face of the dielectric is given by

\[ 4.6 \quad B_{nm}^2 = (A_0)^2 \int_{z_1}^{z_2} c_{2n 2m}^2 dz \]

where the entry face is given by \( z = z_1 \) the exit face \( z = z_2 \)

(figure 4.1).

4.2 New Field approximation

We consider first the case where the dielectric lies in the near fold of both the second harmonic resonator and the fundamental beam i.e. where both

\[ 4.7 \quad \xi_1 \ll 1 \quad \text{and} \quad \xi_2 \ll 1 \]

This is the case treated by Ashkin, Boyd and Dziedzic but for the dielectric potassium Dihydrogen Phosphate, KDP, which does not phase match perpendicular to the optic axis and hence is somewhat less efficient than a material in which this can be done, e.g. Lithium Niobate. This corresponds to the weak focussing limit and can be treated analytically. Two important results are brought out by this case, first the variation of second harmonic output with the relative
spot sizes of the two beams and second, the result that the normal phase matching condition $\Delta k = 0$ no longer holds.

a. **Lowest order mode**

To the first order in $\xi$, we can approximate

$$\left(1 - i \xi \right)^{-1} = \exp \left(i \xi \right)$$

$$\left(1 + i e^{2W^2} + i e^{2W^2} \xi \right)^{-\frac{1}{2}} = \left(e^{2W^2} + 1\right)^{-\frac{1}{2}} \exp \left(-i e^{2W^2} \xi \right)$$

Using these approximations equation 4.5 for the lowest order coupling reduces to

$$4.8 \quad c_{\omega}^{2}\omega = \alpha_{\omega} \exp \left\{ i \left( \frac{\Delta k z}{W_0^2} + \xi \right) \left(1 - \frac{e^{2W^2} \epsilon}{2 \left(e^{2W^2} + 1\right)} \right) \right\}$$

and the second harmonic output in the lowest order mode is given by

$$4.9 \quad b_{\omega}^{2}\omega = \frac{\alpha_{\omega} A_{\omega}^2}{W_0^2} \left(\frac{1}{2 \left(e^{2W^2} + 1\right)} \right)^{\frac{1}{2}} \int_{z_0}^{z_1} \exp \left\{ i \Delta k \right\} dz$$

The variation of this output with the spot sizes of the laser beam and the second harmonic resonator is given to a good approximation by the function

$$4.10 \quad F(\tilde{W}) = \frac{1}{W_0^2 \left(e^{2W^2} + 1\right)^{\frac{1}{2}} \left(W^2 + 1\right)^{\frac{1}{2}}}$$

For constant $W_0^2$ this increases with decreasing $W$ (i.e., decreasing $W_0^2$). As the resonator mode functions are only valid as
long as \( W_{01} \gg \lambda \) this just implies that for maximum output \( W_{01} \), the laser spot size, must be as small as possible as would be expected from physical considerations. For constant \( W_{01} \), \( F(w) \) has a maximum at

\[
4.11 \quad W_{\text{max}} = \frac{1}{\sqrt{e}} = \frac{\sqrt{u_{\text{max}}}}{\sqrt{\gamma_{\text{max}}} (\text{refractive index})}
\]

The behaviour of \( F(w) \) for fixed \( W_{01} \) is shown in graph 1 for \( e^2 = 0.96 \), (an approximate values for Lithium Niobate) when it can be seen that \( W_{\text{max}} = 1.02 \).

The integral in equation 4.7 is of the same form as the integral which arises when considering Second Harmonic Generation from plane waves except that the phase matching condition is now given by

\[
4.12 \quad \Delta k = \frac{-1}{2 k_1 W_{01}} \left( 1 - \frac{e^2 \pi^2 \varepsilon}{2 (e^2 \pi^2 + 1)} \right)
\]

At the maximum value of \( W = 1.02 \)

\[
\Delta k = \frac{-0.99}{k_1 W_{01}} = \frac{-1.98}{k_1 W_{01}}
\]

which is the limit of an infinite laser beam and Second Harmonic resonator tends to the usual \( \Delta k = 0 \) as expected but can be substantially different from this for the usual spot sizes \(~10^{-2} \text{ cms.}\)

b. **Higher Order Modes**

Here we consider first the case when \( eW - 1 \) is not near zero then we can approximate
\[(e^{2W^2} - 1 + i e^{2W^2} \varepsilon_{\frac{1}{2}})^n = (e^{2W^2} - 1)^n \exp \left( \frac{i n e^{2W^2} \varepsilon_{\frac{1}{2}}}{(e^{2W^2} - 1)} \right)\]

Using this and the previous approximations equation 4.5 becomes

4.13 \[\frac{2}{c_2 n 2m} = \frac{\alpha_{nm} (w^2 - 1)^m (e^{2W^2} - 1)^n}{W_{02} (w^2 + 1)^m + \frac{1}{2} (e^{2W^2} + 1)^n + \frac{1}{2}}\]

\[x \exp \left\{ -iz (\Delta k + \frac{1}{\varepsilon_{01}}) (1 + e^{2W^2} \varepsilon_{\frac{1}{2}} (\frac{1}{e^{2W^2} - 1} - \frac{(2n + 1)}{2(e^{2W^2} + 1)}) \right\}\]

The variation with spot size is given again to a good approximation by the function

4.14 \[F_{nm}(W) = \frac{(w^2 + 1)^m (e^{2W^2} - 1)^n}{W_{02} (w^2 + 1)^m + \frac{1}{2} (e^{2W^2} + 1)^n + \frac{1}{2}}\]

again if \(W_{02}\) is held constant this function increases with decreasing \(W_{01}\), the laser spot size, as expected. For \(W_{01}\) held constant the function has two distinct maximum which for \(e = 1\) occur at reciprocal points. The variation of a few of the lower order functions are plotted in graph I for \(e = 0.96\). It can be seen that the value of \(w\) at which the maximum occurs increases with increasing mode numbers. This is brought out by the similar function which occurs in the two dimensional case

4.15 \[F_n(W) = \frac{(w^2 - 1)^m}{W_{02} \frac{1}{2} (w^2 + 1)^n + \frac{1}{2}}\]

which is simpler to handle and has its maxima at

4.16 \[W_{max} = 4_n + 1 \pm \sqrt{(4n + 1)^2 - 1}\]
phase matching for the nth mode occurs when

\[
\Delta k = -\frac{1}{\varepsilon_{01}} \left( 1 + \frac{e^{2w^2}}{\varepsilon_{02}} \left( \frac{n}{(e^{2w^2} - 1)} - \frac{2n + 1}{2(e^{2w^2} + 1)} \right) \right)
\]

Which is dependant on \( n \) the \( x - z \) plane mode number. Hence to some extent modes which are undesired in the output can be discriminated against by varying the relative spot size and the phase matching. For example putting \( W = 1 \) eliminates output in all the higher \( y, z \) plane modes without reducing noticeably the \( 0 - 0 \) output.

If \(|e\, W - 1| \ll 1\) then all the coefficients except \( c_{2, 2m}^{2\omega} \) are of higher order in \( \xi_{1} \), than the first and hence are negligible in the near field approximation. The \( c_{2, 2m}^{2\omega} \) are given by

\[
c_{2, 2m}^{2\omega} = \alpha_{m} \frac{(W^2 - 1)^m i e^{2w^2} \in \xi_{1}}{W_{02} (W^2 + 1)^m + \frac{1}{2} (e^{2w^2} + 1)^n + \frac{1}{2}} \exp \left\{ iz (\Delta k + \frac{1}{\varepsilon_{01}} \left( 1 - \frac{2n + 1}{2(e^{2w^2} + 1)} \right) \right\}
\]

Considering the results so far it can be seen that the region \(|e\, W - 1| \ll 1\) will only be entered when it is the aim to maximise the output of the lowest order modes. From equations 4.16 and 4.6 we have
\[ 4.19 \quad \left| \frac{\alpha_{2\omega}}{\alpha_{00}} \right| = \left| \frac{e^{2\pi^2 \xi_1} \in \alpha_{10}}{(e^{2\pi^2 + 1}) \alpha_{00}} \right| \]

which is approximately equal to \( 3 \times 10^{-2} \left| \xi_1 \right| \)

For \( \left| \xi_1 \right| \ll 1 \) this will be negligible especially since if maximum coupling is required into the lowest order 00 mode than the 0 - 2 mode will neither be perfectly phase matched nor resonate perfectly in the cavity set for the 0 - 0 mode.

### 4.3 Far Field

The second limiting case which can be evaluated analytically when the non linear dielectric has in the far field of both the laser and the S.H. resonator, i.e. both.

\[ \xi_1 \gg 1 \quad \text{and} \quad \xi_2 \gg 1 \quad \text{and} \quad \frac{l}{z_1} \ll 1 \]

Under these approximations the lowest order coefficient \( \alpha_{2\omega} \) can be written

\[ 4.20 \quad \alpha_{2\omega} = \frac{-\alpha_{00}}{\omega_{02} (\omega^2 + 1)^{\frac{1}{2}}} \left( \frac{z_0^3}{-i e^2 \xi_2} \right)^{\frac{1}{2}} \exp \left( i \Delta k \cdot z \right) \]

and hence the S.H. output from a slab of dielectric of length

\[ l = z_2 - z_1 \]

is proportional to

\[ 4.21 \quad I = \int_{z_1}^{z_2} \exp \left( i \Delta k z \right) dz \]

\[ = \left. \frac{e^{i \Delta k z_1}}{z_1} \right\} \int_0^{l} \exp \left( i \Delta k z \right) dz \]

to the first order in \( z/l \). We can see the phase matching condition and the general form of the integral in this limiting case is just that which arises
when considering the interaction of plane waves and hence the output will vary with the usual \( \sin \left( \frac{\Delta k l}{2} \right) / \Delta k \). This is as expected since far from the focus the beam tends to a plane wave.

### 4.4 Infinite Crystal

In the limit of an infinite crystal or equivalently the steep focussing limit \( z_{01} \to 0 \) the expression for the amplitude of the second harmonic output is from equation 4.5.

\[
\epsilon_{2\omega}^{\infty} = \frac{\alpha_{00}}{\epsilon_{02}} \left( \frac{\exp(i\Delta k z)}{(1 - i \xi)(e^{2\xi^2} + 1 + i\xi^2)} \right)
\]

Changing the variable of integration and writing \( \Delta k z = \eta \), \( \xi = \nu \) the integral on which the output in this case depends is

\[
I = \int_{-\infty}^{\infty} \frac{\exp(i\eta \nu) d\nu}{(1 - i \nu)(a + i b \nu)^{3/2}}
\]

where \( a = e^{2\nu^2} + 1 \quad \quad b = e^{2\nu^2} \epsilon \)

This integral can be evaluated under certain conditions by contour integration in the complex \( \nu \) plane. For \( \nu = 0 \) we consider the integral around the contour made up of a semicircle radius \( R \) in the lower half plane and the completing section of the real axis. In the limit \( R \to \infty \) the integral around the semicircle tends to zero and the contour contains the simple pole of the integral at \( \nu = -i \). Hence from Cauchy's theorem...
4.24 \[ I = \frac{-2\pi \exp(y)}{(a + b)^{1/2}} \] for \( y < 0 \)

The integral has a branch point on the upper half plane at \( V = ia \) and so cannot be integrated immediately when \( \Delta k > 0 \). However since \( b \) is small the limit \( b \to 0 \) will give an indication of the behaviour of the integral when \( y > 0 \). In this limit the integral is analytic in the upper half plane and hence integrating around a contour completed in the upper half plane we have by Cauchy's theorem.  

4.25 \[ I = 0 \] for \( y > 0 \) \( b = 0 \)

Thus we expect the integral to exhibit something of a discontinuity at \( \Delta k = 0 \). This general behaviour can be seen in graph 7 which records the variation of the integral \( I \) with \( \Delta k \) for \( \ell/\zeta_0 = 100 \). This shows clearly the discontinuity although it is modified by a periodic fine structure. The integral the limit \( (b = 0) \) has been considered by Kleinman Ashkin and Boyd, 27, in a different context. They obtain a continuous analytic approximation for this fine structure.

4.5 General Formula (a) Lowest Order Mode

The S.H. output in the lowest order mode is given by the equation (from eq. 4.5)

4.26 \[ c_{00} = \frac{\alpha_{00} W z_{01}}{W_{01} (w^2 + 1)^{1/2}} \int_{\xi_1}^{\xi_f} \frac{\exp(iy\xi)}{(1 - i\xi)(a + i b \xi)^{1/2}} d\xi \]

where \( a = e^{2e^2} + 1 \), \( b = e^{2\xi^2} \), \( y = \Delta k \zeta_0 \) and so
contains the mismatch parameter; \( \xi_1 = \frac{z_1}{z_{01}} \), \( \xi_2 = \frac{z_2}{z_{01}} \)

and contains the length 1 of the non-linear medium

\( l = (\xi_2 - \xi_1)z_{01} \) and the position of the foci of the two beams (assumed coincident) \( f = (\xi_2 + \xi_1)z_{01}/2 \). The variation with the focusing parameter \( z_{01} \) of the fundamental beam for constant crystal length 1 can be obtained by rearranging 4.26 to the form

\[
4.17 \quad C_{00} = \frac{\alpha_{00} \frac{W}{(W^2 + 1)^{3/2}}}{\sqrt{k_1} \frac{1}{2}} \sqrt{(\xi_2 - \xi_1)}
\]

and studying the variation of this form with \( \xi_2, \xi_1 \) for a given focus position.

Graphs 2, 3, 4, 6 and 7 show the second harmonic output variation with the phase matching parameter \( \Delta k/2\pi \) for various relative crystal lengths \( \xi_1 = \frac{\ell}{z_{01}} \) with the focus at the centre of the crystal. Graph 2 shows the shortest length has the appearance expected from the new field limit consideration. The curve has the form \( \sin (\pi x)/\pi \) with the absolute maximum slightly off \( \Delta k = 0 \). Away from the absolute maximum all the curves exhibit a periodicity of \( \pi \). (\( \xi \) the plane wave interaction which is proportional to \( \sin (\Delta k 1/2) \) and thus has a period of \( \pi \) on the graph). As the length increases or
equivalently the focussing increases \((z_{01} \text{ decreases})\). The curves become assymetric. The optimum phase matching position moves away from \(\Delta k = 0\) and the central peak moves out on the side \(\Delta k < 0\) engulfing the smaller peaks (graph 4). This marks the increasing effect of the focussing terms in the denominator of the integral \((1 - i \xi), \quad (e^{2\xi^2} + 1 - i e^2 \xi^2)^{1/2}\).

If the gaussian beam is considered as a sum of plane waves, propagating over a range of directions, the reason for the assymetry becomes apparent. When \(\Delta k > 0\) the fundamental and S.H. refraction index surfaces (dgms. 1 - 3) no longer intersect for any direction of propagation hence none of the constituent plane waves is phase matched, but when \(\Delta k < 0\) the index surfaces intersect and so there exist plane waves in the sum which are phase matched. The steep focussing or infinite crystal effect can be seen fully developed in graph 7 as has been previously pointed out. Note that the periodicity of the fine structure for \(\Delta k < 0\) is twice that of that for \(\Delta k > 0\). The width at half length of all the curves in the units of \(\Delta k \ell / 2\) is approximately constant. All these curves are consistent with those of Kleinman and Boyd, 15. Graph 10 shows the variation with focussing at the optimum phase match angle and optimum relative spot size for three positions of the focus. With the focus at the centre the maximum occurs at \(\ell / z_0 = 5.65\) which is consistent with the result given by Kleinman and Boyd, 15, for free S.H.G. \(\ell / z_0 = 5.68\).

If we think of the free second harmonic generated as being
made up of a number of confocal resonator modes, the higher order ones will contribute a small amount to the total and this would be sufficient to explain the difference between the two results. Since the higher modes maximise at a higher degree of focussing the direction of the difference is as expected.

With the forms of the fundamental beam at the entry face, or exit face, of the crystal the maximum occurs at $\frac{1}{z_0} = 3.09$ as shown in the second curve. This figure would of course be the one required in a plano - concave resonator. The third curve represents the case when the focus is one half the crystal length outside the crystal. Here the curve has an early maximum and falls off sharply as the focal region withdraws from the crystal. Also shown on the same graph is the output in the 0.2 mode under the same conditions of phase matching and spot size as the corresponding curve for the 0 - 0 mode.

Graph 8 shown the variation of output with focus position for several lengths at optimum phase match and spot size. Graph 9 shows the variation in shape of the output curve with phase match angle as the focus moves away from the centre of the crystal. As expected from the consideration of the far field limit the optimum phase match position moves towards $\Delta k = 0$ and the shape of the curve approaches $\frac{\sin (\Delta k l/2)}{(\Delta k l/z)}$ as the focus moves away from the centre. A greater range of $\Delta k$ is shown in graph 5 for the case when the focus is at the face of the crystal. It can be seen that the minima have moved up from zero and the curve has taken
on much of the form of the steep focussing limit. These effects can be shown by expanding the integrand in powers of $\varepsilon$ to be due to the increasing effect of the anisotropy as the focus moves from the centre.

**0 - 2 Mode**

The output in the $0 - 2$ mode is given from equation 4.5 by

$$4.28 \ c_{02} = \frac{\kappa_{02}}{w_{02}} \int_{z_0}^{z_1} \frac{\exp(i\Delta k z)(e^{2W^2} - 1 + \frac{1}{2}e^{2W^2}\epsilon_{\gamma})}{(1 - i\epsilon_{\gamma})(e^{2W^2} + 1 + i\frac{1}{2}e^{2W^2}\epsilon_{\gamma})^{3/2}} \ dz$$

This mode of the higher modes will have the greatest output when the various parameters are set for optimisation of the output in the $0 - 0$ mode.

Graphs 11 and 12 show the output variation of this mode with phase matching for $\frac{\lambda}{\lambda_{z0}} = 5.65$ the optimum value for the $0 - 0$ mode.

With the focus at the centre the output exhibits a double peak. This is present at all crystal lengths, the relative heights of the two peaks changing with length. As the focus moves away from the centre (graph 12) the second peak decreases in height and the curve takes on the form of those for the $0 - 0$ mode. The positions of the twin peaks change little with increasing length (focussing) giving rise to the dip in the curve for the $0 - 2$ mode, focus at the centre, on graph 10. This occurs as the position of optimum phase matching for the $0 - 0$ mode moves through the region between the twin peaks of the $0 - z$ mode output.

Although the output variation with relative spot size $W$ for the $0 - 0$ mode has the form of graph 1 for the optimum length
the curve for the output in the 0 - 2 mode has changed and become
very asymmetric about the central position (dotted curve as graph 1).

The relative outputs for the two modes are when the 0 - 0
mode is optimised.

\[
\begin{pmatrix}
C_{00} \\
C_{02}
\end{pmatrix} \approx \begin{pmatrix}
100 \\
50
\end{pmatrix}
\]

for the focus at the centre, and.

Absorption

The presence of absorption can be taken into the foregoing
theory without difficulty. It of course modifies the theory
reducing the output amplitudes, but gives no new effects except
that an optimum crystal length can now be defined which will
depend on the amount of absorption present at each frequency.

4.6 Calculation of the Output from the Resonant Cavity

To calculate the output of the resonant cavity on a given mode
we refer to diagram 4.1 where \( C_0 \) is defined as the amplitude of
the relevant mode just inside the crystal exit face and \( r \exp (i \phi) \)
is the power loss parameter of the cavity including diffraction
losses, reflection losses and the loss due to the output. In the
absence of any second harmonic generation after a single round trip
the amplitude of the mode would be \( C = C_0 r \exp (i \phi) \) since
the second harmonic power is only generated as the wave travels from
left to right the amplitude in the cavity is given by the self
consistency equation.
$4.29 \ C_1 = C_0 r \exp (i \phi) + C_1$

Where $C_1$ is the S.H. generated in a single pass, the quantity we have been considering. Therefore from 4.29,

$4.30 \ C_0 = \frac{C_1}{(1 - r \exp (i \phi))}$

and in terms of the power present

$4.31 \ P = \frac{P_1}{(1 - 2 r \cos \phi + r^2)}$

Therefore the power transmitted from the cavity is given by

$4.32 \ P = \frac{t P_1}{(1 - 2 r \cos \phi + r^2)}$

where $t$ is the transmission coefficient of the exit mirror.

at resonance

$4.33 \ P = \frac{t P_1}{(1 - r)^2}$
As pointed out by Ashkin, Boyd and Dziedzic, 23, this can be much greater than $P_1$, the power obtained in a single pass e.g. the case $r = 0.99$, $t = 0.01$, $P = 10^2 P$

This optimisation process will also discriminate against the higher modes since from equation 2.33 when the $0 - 0$ mode is resonating $\varphi = 0$ the higher modes will not be resonant. For example for a confocal cavity using the data given previously the ratio of the power output from the cavity in the $0 - 2$ mode to that in the $0 - 0$ mode is approximately

$$0.25 \times 10^{-4} \frac{P_{02}}{P_{00}}$$

where $P_{02}$ and $P_{00}$ are the power generated in a single pass of the non linear in the $0 - 2$ and $0 - 0$ mode respectively. Hence the power output in the $0 - 2$ mode is negligible.

In general the fundamental laser beam will consist of a number of longitudinal modes of the laser resonator and hence the fundamental electric field will be of the form

$$E(\omega) = \frac{1}{2} \sum_n (E^{\omega_n} S(\omega - \omega_n) + E^{*\omega_n} S(\omega + \omega_n))$$

where from equation 2.33 $\omega_n$ is given by

$$\omega_n = \frac{c}{d} \left( n\pi + \tan^{-1}(\psi) \right)$$

To a good approximation the spot sizes of each of these modes will be equal and so the general form of each mode will be the same and so in considering the frequency term of the modes we need not consider the mode shape.
Using equation 1.18 the second order polarization will be set up at frequencies.

4.35 \( \omega_q = \omega_m + \omega_n \), where \( \omega_m, \omega_n \) are given by equation 4.34. The resonances of the second harmonic resonator occur at frequencies.

4.36 \( \omega_q = \frac{c}{d_1} \left( q\pi \pm \tan^{-1} (\psi_i) \right) \) where \( d_1 \) is the optical length of the resonator cavity and \( \psi_i \) is calculated from equation 2.32 using the optical length \( d_1 \). Putting together 4.34, 35, 36 resonance occurs when

4.37 \( d_1((m + n) \pi + 2 \tan^{-1} (\psi_i)) = d \left( q\pi \pm \tan^{-1} (\psi_i) \right) \)

From this equation knowing the laser cavity parameters suitable parameters for the second harmonic cavity can be calculated.
Chapter 5

Parametric Amplification

5.1 The second problem to which the formalism developed in the previous chapters has been applied is that of parametric amplification and oscillation from a travelling wave pump optical field. Under the approximation that the pump field is much larger than either the signal or idler fields and therefore may be regarded as constant the problem splits into two parts as with second harmonic generation. First the calculation of the single pass amplification and secondly the resonator calculation which gives the threshold of a given resonator and its output for a given pump field above threshold. Guided by the experience of the last chapter only the interaction between the lowest order, 0-0 modes of each field has been examined, contributions from higher modes have been neglected. Under these approximations an analytical solution to the coupled mode equations is given for the weak focusing - near field limit and preliminary results of numerical computations are given for the general solution.

We consider a non linear dielectric enclosed in a resonator, orientated as before. Impinging on the dielectric will be optical fields at frequencies \( \omega_3, \omega_2, \omega_1 \) which will be referred to as the pump, idler, and signal fields respectively. To achieve phase matching in Lithium Niobate we consider the pump field to be propagating as an extraordinary wave and the signal and idler fields as ordinary waves. The coupled mode equations governing the amplitudes of the signal and idler beams are from equations 3.51.
\[ \frac{dA_1}{ds} = \frac{\omega_1}{A_1 A_2 B} \]
\[ \frac{dA_2}{ds} = \frac{\omega_2}{A_1 B} \]

where \( \omega_1, \omega_2 \) are the coupling coefficients and \( B \) is the pump mode amplitude. The \( \omega \) are given by equation 3.61 or its three dimensional generalization.

5.2 Weak Focusing - Near Field Limit

Under the same approximations as have been used in section 4.2(a) equations 5.1 can be written

\[ \frac{dA_1}{ds} = -i \alpha_1 A_2^* \exp(iyz) \]
\[ \frac{dA_2}{ds} = -i \alpha_2 A_1^* \exp(iyz) \]

where \( \alpha_{1,2} = \frac{\omega_1}{\pi^3} \frac{dA_1}{ds} \frac{dA_2}{ds} \frac{2^2}{B} \frac{w_1}{w_2} \frac{w_2}{w_3} \left( \frac{w_1^2 + w_2^2 + 1}{e^{2} w_1^2 + e^{2} w_2^2 + 1} \right)^{2} \)

and \((d)\) \( y = \Delta k + (e^2 + 1) \left\{ \frac{w_1^2}{w_0^2} \left( \frac{1}{e^2} - \frac{1}{w_0^2} \right) + \frac{w_2^2}{w_0^2} \left( \frac{1}{e^2} - \frac{1}{w_0^2} \right) \right\} \)

and \( w_1 = \frac{w_0^3}{w_0^1}, w_2 = \frac{w_0^3}{w_0^1} \)

All other variables are as defined previously.

From 5.2(c) the variation of the interaction with the various spot
sizes is given by the function

\[ F(w) = \frac{\hat{w}_1 \hat{w}_2}{w_0 (\hat{w}_1^2 + \hat{w}_2^2 + 1)^{\frac{1}{2}}} \]

When \( e = 1 \) the maximum as a function of \( w_0 \) occurs when

\[ \frac{1}{w_0^2} = \frac{1}{w_{02}^2} + \frac{1}{w_{01}^2} \]

and it can be shown by the method of Lagranges undetermined multipliers that the absolute maximum will occur when \( w_{01} = w_{02} \). Since the \( w_1 = w_2 \) symmetry is retained when \( e \neq 1 \) the maximum value would still be expected at \( w_{01} = w_{02} \). Then \( F(w) \) reduces to the function studied in section 4.2(a).

One method of solution of equations 5.2 is to separate the real and imaginary parts and solve the resulting equations 58,16,

Thus putting

\[ A_1 = \hat{A}_1 e^{i\theta_1}, \quad \alpha_1 = \hat{\alpha}_1 e^{i\psi_1} \text{ etc.} \]

and

\[ \phi = \frac{\pi}{2} - (\theta_1 + \theta_2) + \psi + \gamma \]

(Note \( \psi = \psi_1 = \psi_2 \))

from equation 5.2(c)

we have

\[ 5.5 \quad (a) \frac{d\hat{A}_1}{ds} = \hat{\alpha}_1 \hat{A}_2 \cos \phi \]
These equations give immediately information about the maximum possible amplification. From 5.5(c) we see that as long as $y = 0$ all the derivatives of $\Phi$ are identically zero when $\Phi = 0$ and hence $\Phi = 0$ is a solution of 5.5(c). This reduces the problem to solving the two simple coupled equations.

5.6
\[
\begin{align*}
\frac{d\hat{A}_1}{dz} &= \hat{\alpha}_1 \hat{A}_2 \\
\frac{d\hat{A}_2}{dz} &= \hat{\alpha}_2 \hat{A}_1
\end{align*}
\]

which will give the optimum amplification.

The second point which can be deduced from 5.5(c) is that if at the crystal entry face the idler mode amplitude $A_2$ is zero (i.e. non-resonant idler) then the initial value of $\Phi$ is fixed at the optimum value $\Phi = 0$.

Thirdly it can be seen from equation 5.5(a) that the signal is amplified when $-\frac{\pi}{2} < \Phi < \frac{\pi}{2}$ and absorbed otherwise. This has a simple interpretation in terms of the phases of the three interacting waves. From equation 5.4, in the plane wave limit $y = \Delta k$, we have

\[
\Phi = -\frac{\pi}{2} + (\psi + k_3 z) - (\Theta_1 + k_1 z) - (\Theta_2 + k_2 z)
\]

\[
= -\frac{\pi}{2} + \Psi_3 - (\Psi_1 + \Psi_2)
\]
where $\psi_1$, $\psi_2$, $\psi_3$ are the phases of the three interacting waves at any point in the crystal.

Thus if $\psi_1 + \psi_2$ leads $\psi_3$ then both signal and idler are amplified if $\psi_1 + \psi_2$ leads $\psi_3$ then both signal and idler are absorbed. The possibility of signal and idler behaving differently is ruled out since the phase of $\alpha_1$ is equal to that of $\alpha_2$. In the non plane wave case $y$ can still be split into three parts, one associated with each frequency. Then $\psi_1^2$, $\psi_2^2$, $\psi_3^3$ may be interpreted as an average value of the phases of the constituent plane waves of the resonator mode at each frequency.

To obtain the general solution it is simpler to proceed with the complex equations. Multiplying equation 5.2(a) by $\exp(-iyz)$ and differentiating we can then substitute from equation 5.2(b) and obtain the equation

$$5.7 \quad \frac{d^2 A_1}{dz^2} - iy \frac{dA_1}{dz} - \alpha_1^* \alpha_2 A_1 = 0$$

for $A_1$ and by a similar procedure a similar equation for $A_2$. The solution to this equation depends on the familiar (16, 58, 20) gain factor

$$5.8 \quad g^2 = \alpha_1^* \alpha_2 - y^2 / 4$$
The solution for $A_1$ increases exponentially with increasing distance from the entry face when $\beta > 0$ but oscillates sinusoidally when $\beta < 0$.

In terms of the power present in each mode which is directly proportional to $|A_1|^2$, $|A_2|^2$, the solution to equation 5.7 can be written

\[5.9\] (a) for $\beta > 0$.

\[P_1(z) = P_1 + (\alpha_1^2 P_1 + \alpha_1 l^2 P_2 - |\alpha_1| l \sqrt{P_1 P_2} \cos (\tilde{\phi}_0 + \pi) \frac{\sinh^2(2\beta(z-a))}{\beta^2} + \sin (\tilde{\phi}_0 + \pi \frac{l}{2}) |\alpha_1| l \sqrt{P_1 P_2} \frac{\sin^2(2\beta(z-a))}{\beta} \]

(b) for $\beta < 0$.

\[P_1(z) = P_1 + (\alpha_1^2 P_1 + \alpha_1 l^2 P_2 - |\alpha_1| l \sqrt{P_1 P_2} \cos (\tilde{\phi}_0 + \pi) \frac{\sin^2(2\beta(z-a))}{\beta^2} + \sin (\tilde{\phi}_0 + \pi \frac{l}{2}) |\alpha_1| l \sqrt{P_1 P_2} \frac{\sin(2\beta(z-a))}{\beta} \]

where $\alpha_1^2 = \alpha_1 \alpha_2^*$; $P_1, P_2$ are the values of $P_1(z), P_2(z)$ the power present in the signal and idler modes respectively and $\tilde{\phi}_0$ is the value of $\tilde{\phi}$ at the crystal entry face $z = a^*$.

It can be verified that the optimum value of this function with respect to $\gamma$ and $\tilde{\phi}$ occurs at the expected values $\tilde{\phi}_0 = 0$, $\gamma = 0$. It can be seen that for a general value of $\tilde{\phi}_0$, the power output function is not
symmetric with respect to the phase matching parameter $\gamma$. This will only be important in the degenerate case when the signal and idler fields are one, $A_1 = A_2$. Since, in the non-degenerate case, the idler is generated as a side effect and if it is non-resonant, or if it is resonant in an optimised cavity then $\phi_0 = 0$ and the function is symmetric. But in the degenerate case $\phi_0$ is fixed by external parameters and need not be zero. Hence in degenerate parametric amplification it is likely to be difficult to obtain the optimum output experimentally. For the important case $\phi_0 = 0$ we have

$$P_1(z) = P_1 + (\alpha^2 P_1 + |\alpha|^2 P_2) \frac{\sin^2(\beta(z-a))}{\beta^2}$$

\[ + |\alpha|^2 P_1 P_2 \frac{\sinh(\beta(z-a))}{\beta} \]

For the case $P_2 = 0$, non-resonant idler, there is gain for all values of $z$ and $\gamma$. 

(a) $\beta > 0$

$$P_1(z) = P_1 + (\alpha^2 P_1 + |\alpha|^2 P_2) \frac{\sin^2(\beta(z-a))}{\beta^2}$$

\[ + |\alpha|^2 P_1 P_2 \frac{\sin(2\beta(z-a))}{\beta} \]

(b) $\beta < 0$

$$P_1(z) = P_1 + (\alpha^2 P_1 + |\alpha|^2 P_2) \frac{\sin^2(\beta(z-a))}{\beta^2}$$

\[ + |\alpha|^2 P_1 P_2 \frac{\sin(2\beta(z-a))}{\beta} \]
In the other extreme $P_2 = P_1$, degenerate, we have from equation 5.10(b)

$$P_1(z) = P_1(1 + \frac{\beta^2}{2} \sin^2(\beta(z-a)) + \frac{\alpha}{\beta} \sin(2\beta(z-a)))$$

and there is gain as long as

$$\frac{\beta}{\alpha} > -\tan(\beta(z-a))$$

Putting $z-a = 1$, $u = \beta1$ this condition becomes

$$\frac{u}{\alpha1} > -\tan u$$
In the diagram the curves $v = \tan u$ and $v = \frac{-u}{1}$ are sketched and the regions of absorption are shown dotted.

When the initial value of the idler power lies between these extremes the behaviour of the output value of the signal would be expected to lie between these extremes.

5.2 General Case

The single pass amplification is governed by equations 5.1 in the general case where 5.14

$$\omega_1^2 = \frac{2\pi k \omega_1^2 V_1 V_2 \exp(i\Delta k s)}{c^2 k_1 e^{\frac{2}{\tau} \tau_0}}$$

$$x\left(1 + i\xi_3\right)\left(1 - i\xi_2\right) + \frac{\omega_1}{c^2} \left(1 + i\xi_3\right)\left(1 - i\xi_1\right) + \left(1 - i\xi_1\right)\left(1 - i\xi_2\right)$$

$$x(e^2 \omega_1^2(1+i\xi_3^2)(1-i\xi_2^2) + e^2 \omega_2^2(1+i\xi_3^2)(1-i\xi_1^2) + (1-i\xi_1^2)(1-i\xi_2^2))$$

in the three dimensional case. A similar equation defines $C_2$. A computer
program has been developed to solve these equations numerically using
Hilmes predictor-corrector method of integration,\textsuperscript{57}. The program is
described in Appendix III.

Graphs 13 - 17 are preliminary results obtained from this program.
Graphs 13, 14, 15 show the typical variation in amplification of the
signal beam

\[
\text{amplification} = \frac{A_1 - A_{10}}{A_{10}}
\]

with the phase matching parameter \(\Delta k \frac{1}{2}\) for non-degenerate case when the
initial idler amplitude is zero. As has been pointed out this will
correspond to the situation when the idler is non resonant. The pump beam
is focussed at the centre of a 1 cm crystal. Graph 13 shows the behaviour
in a relatively weakly focussed case \(\frac{1}{2}z_0 = 0.5\) and has just the appearance
expected in this limit. As the focussing increases the main peak moves
away from \(\Delta k = 0\) and those peaks on the side \(\Delta k < \Delta k\) (optimum) increase
in height. This behaviour is somewhat different than that which occurs for
second harmonic generation.

Graph 17 shows the variation of the gain with focussing when the
phase matching is optimised both for degenerate parametric amplification
(with optimised initial phase) and for non-resonant idler. Note that the
scales differ for the two curves so that the degenerate amplification is
over ten times greater than the other. The maxima in both cases appears to
be about \( \frac{1}{3.5} \). Although the strongly focussed region is not yet well defined.

Graph 16 represents the variation of the gain in each of these cases, optimised with respect to phase matching; and at the focussing \( \frac{1}{d_{g3}} = 3.33 \) with the effective non-linear coefficient. The effective non-linear coefficient is defined as

\[
5.15 \quad d = 64\pi^2 \frac{d_{15}}{N} \sqrt{\frac{2GP}{c}} \times 10^7
\]

where \( P \) is the power present in the pump beam in watts. This graph can be used to calculate the threshold of a given cavity knowing its losses as is shown in the next section. Note that for this graph also the scales for the gain in the two cases differ by a factor of ten.

### 5.3 Calculation of Threshold and Resonator Output

Consider the schematic diagram of the signal resonator figure 5.2

\( A_1 \) is the signal mode amplitude entering the resonator, \( A \) the amplitude of this mode just inside the crystal entry face. \( C \) is the gain coefficient
which has been examined in the previous section and the transmission loss of the output mirror. In the real case there will be some loss of the input beam as it enters the cavity but this has been neglected. Since there is gain only in one direction we have

\[ A = A_i + A \cos \theta \]

also \[ A_t = \frac{ct \ A_i}{1 - rc} \]

therefore

\[ A_t = \frac{ct \ A_i}{1 - rc} \]

The condition for gain is given by

\[ \left| \frac{A_t}{A_i} \right| \geq 1 \]

Putting \( rc = r \cos^2 \phi \) where \( \phi \) is the change of phase of the amplitude in a round trip and substituting from equation 5.17 the gain condition becomes

\[ Icl^2 (|r|^2 - |t|^2) - 2 |r| \ \text{Id} \ \cos \phi + 1 \leq 0 \]

giving the gain condition

\[ Icl \geq \frac{|r| \ \cos \phi - \sqrt{|r|^2 - |t|^2 \sin^2 \phi}}{|r|^2 - |d|^2} \]
which in the case of resonance, \( \phi = C \), reduces to

\[
|c| > \frac{1}{\sqrt{r_1 + r_2}}
\]

For gain to take place at all, the cavity must be tuned so that the square root in equation 5.19 remains real

\[
\text{i.e. } \sin^2 \phi \leq \frac{\frac{t^2}{r^2}}{2}
\]

For \( r = 0.9 \), i.e. 10\% loss in the resonator and the output mirror having a transmission coefficient of \( t = 0.05 \), i.e. 5\% transmission, we have at resonance from equation 5.20 that \( |c| \geq 1.05 \). Under the conditions of graph 16 this occurs when the effective non-linear coefficient \( d \) is greater than \( 4.6 \times 10^{-7} \) for degenerate parametric amplification and \( 6.6 \times 10^{-3} \) for non-degenerate amplification with a non-resonant idler.

Taking the approximate values for \( d_{15} \), \( 3 \times 10^{-8} \) e.s.u. and for \( c, 3 \times 10^{10} \) cm/sec, and substituting into equation 5.15 it can be seen that threshold occurs for this cavity at \( P = 10 \) milliwatts for the degenerate case and \( P = 1 \) watt in the non-degenerate case. This power of course must be in a single longitudinal pump mode. This method of estimating the threshold of a given cavity should give better results than previous methods 15, 25 since there are fewer doubtful assumptions made.
Appendix I

Evaluation of the Integral \[ I = \int_C \exp(-z^2) \, dz \]

Where the contour \( C \) is a straight line in the complex \( z \) plane given by \( z = bx + c \), where \( b, c \) are complex constants, \( x \) a real variable. As \( x \) runs from \(-\infty\) to \( +\infty \), \( z \) runs on the line \( c \).

Putting \( z = u + iv \)

\[ b = b_1 + ib_2 \]

\[ c = c_1 + ic_2 \]

the equation for the line \( z = bx + c \) is

\[ \frac{v}{b_2} = \left( \frac{u}{b_1} \right) - c_1b_2/b_1 + c_2 \]

Consider \[ I_{c_1} = \int_{c_1} \exp(-z^2) \, dz \]

where \( G \) is the contour given in the diagram \( A1.1 \).

By Cauchy's theorem \( I_{c_1} = 0 \), as the integral has no poles inside the contour.
Therefore we have

\[ A2.2 \quad I_{c_1} = \int_{-\infty}^{\infty} \exp \left( -u^2 \right) \, du - \int_{-\infty}^{\infty} \exp \left( -z^2 \right) \, dz + \int_{-\infty}^{\infty} \exp \left( -z^2 \right) \, dz = 0 \]

Now

\[ |I_{AB}| = \left| \int_{-\infty}^{\infty} \exp \left( -z^2 \right) \, dz \right| \]

\[ = \int_{-\infty}^{\infty} \exp \left( -R^2 \cos^2 \theta - iR^2 \sin^2 \theta \right) \, d\theta \exp \left( -R^2 \cos \theta \right) \, d\theta \]

\[ \leq R \int_{-\infty}^{\infty} \exp \left( -R^2 \cos \theta \right) \, d\theta \]

\[ \leq R \int_{-\infty}^{\infty} \exp \left( -R^2 \cos \theta \right) \, d\theta \quad \text{if} \quad |\alpha| < \frac{\pi}{4} \]

substituting \( \theta = \frac{\pi}{4} - \phi/2 \)

\[ |I_{AB}| \leq \int_{0}^{\pi} \exp \left( -R \sin \phi \right) \, d\phi \]

\[ < \frac{\pi}{4R} \quad \text{by Jordan's Inequality} \]

and as \( R \to \infty \quad |I_{AB}| \to 0 \).

Therefore

\[ A1.3 \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left( -z^2 \right) \, dz = \frac{\sqrt{\pi}}{2} \]

This result can be extended to a line not passing through the origin.
by means of the contour $C_2$

Again by Cauchy's theorem $\int_{C_2} \exp\left(-z^2\right) dz = 0$

$$|I_{BC}| = \left| \int_{\alpha} \exp\left(-x^2 + y^2 - 2ixy\right) idy \right|$$

$$\leq \exp(-x^2) \int_{\alpha} \exp(y^2) dy$$

$$= \exp(-x^2) (1 - \tan^2 x) \int_{0}^{\alpha} \exp(2x\tan u + u^2) du$$

and therefore $|I_{BC}| \to 0$ as $x \to \infty$ if $|x| < \frac{\pi}{4}$

Similarly $|I_{DA}| \to 0$ as $x \to \infty$ and hence

$$\int_{x=\infty}^{\infty} \int_{y=\infty}^{\infty} \exp\left(-z^2\right) dz = \int_{x=\infty}^{\infty} \int_{y=\infty}^{\infty} \exp\left(-z^2\right) dz = \sqrt{\pi}$$

for $|x| < \frac{\pi}{4}$

from A1.1

$$\alpha = \tan^{-1}\left(b_2/b_4\right) = \arg(b)$$

Therefore the condition

$1 \alpha 1 < \frac{\pi}{4}$ implies that
\[
\arg_h(b^2) < \frac{\pi}{2}
\]

ie that the real part of \( b^2 \) is greater than zero which is true for all the integrals which have been considered.
Appendix 2

Numerical Integration of the Coupling Coefficients

The integrals on which the second harmonic output in each mode of the resonator depend are all of the same form, increasing in complexity with increasing mode number. Thus the following method can be used to evaluate any particular one. The integration was carried out on the London University Atlas Computer, the program being written in Fortran V. Use was made of the Fortran statement function procedure in order to facilitate change of the integrand.

The program evaluates the real and imaginary parts of the integral using Simpson's Rule and prints the modulus and argument. It is an extension of one given by Pennington the basic Simpson's Rule integration procedure of which has an error control. The integral is calculated using successively decreasing step lengths until two successive calculations differ by less than a required amount. This procedure only gives good results if the integrand varies consistently over the range of integration because of the approximation introduced by Simpson's Rule. The coupling coefficients satisfy this condition.

The input data for the program was arranged in three nested loops. For each value of the parameters \( W \), spot size, \( a \) and \( b \), crystal and positions, the integral was calculated over a specified range of \( y \) the phase mismatch from \( y_1 \) to \( y_2 \) in steps \( \Delta y \). Having completed this new values of \( a \) and \( b \) were taken and the procedure repeated. This can be seen from the flow diagram for the program figure A2.1. The spot size
parameter \( \xi \) was placed in the outermost loop since this varies slowly near the maximum. The speed of the program varied with the length of the interval \( a, b \) as would be expected but in general it calculated four to eight integrals per second to a specified accuracy of four decimal places. This accuracy was adequate except for determining the maximum values of the focus curve and spot size curve when six places were used.

For the lowest order mode the integral from equation 4.5 of the form

\[
I = \int_a^b \frac{\exp(iyx) \, dx}{(1-ix)(c+idx)^2}
\]

where \( c = a^2w^2 + 1 \quad d = a^2w^2 \)

the real and imaginary parts being

\[
\begin{align*}
I_r &= \int_a^b P_1(x) \, dx \\
I_i &= \int_a^b P_2(x) \, dx
\end{align*}
\]

where

\[
P_1(x) = \frac{\sin(wx)(c+dx^2+(c^2+d^2x^2)^{\frac{3}{2}})+x \cos(wx)(c-d+(c^2+d^2x^2)^{\frac{3}{2}})}{2^\frac{1}{2}(1+x^2)(c^2+d^2x^2)^{\frac{3}{2}}}
\]

The general procedure is unaffected by the inclusion of absorption since Equation A2.1 just becomes

\[
I = \int_a^b \frac{\exp(iyx-kx) \, dx}{(1-ix)(c+idx)^2}
\]
Flow Diagram for S.H. Integration Program

1. Input \( Y < 100 \)
   - yes
   - no
   - stop

2. Input \( Y_1, Y_2, \Delta Y \)
   - \( Y = Y_1 - \Delta Y \)
   - \( Y_1 < 100 \)
   - yes
   - no

3. \( J = 0 \)
   - \( Y = Y_1 + \Delta Y \)
   - \( I_1 = f_1(a) + f_2(b) \)
   - \( I_2 = f_2(a + \Delta x) \)
   - \( I = \Delta x (I_1 + 4I_2) / 3 \)
   - \( I_3 = 0 \)
   - \( J = J + 1 \)
   - yes
   - no
   - \( J = 2 \)
   - \( x = a + \Delta x \)

4. \( J = 1 \)
   - yes
   - no
   - \( x < b \)
   - yes
   - no
   - \( IP = \Delta x (I_1 + 4I_2 + 2I_3) / 3 \)

5. \( f(2) = I \)
   - no
   - yes
   - \( J > 2 \)
   - yes
   - \( I = IP \times 0.0001 \)
   - yes
   - no
   - \( f(t) = I \)
   - yes
   - no
   - \( I = IP \)
   - yes
   - no
   - \( y < y_2 \)
   - yes
   - no

Print \( M, A, y \)
Integration Program

DIMENSION P (2)

11 READ 101, I, B
12 IF (I - 100) 2, 9, 9
8 EP = 1/1/(B*B)
W = W + S

33 READ 103, Y1, Y2, DY
14 IF (Y1 - 100) 29, 11, 11
29 READ 101, A, B
PRRN 101, A, B
IF (A - 100) 31, 33, 33
31 Y = Y1 - DY
32 A0L = SQRT (B - A)
33 Y = Y + DY
1 = 1
25 J = 0
26 J = J + 1
FGR1 (X) = ((W*W + 1 + SQRT ((W*W + 1)*(W*W + 1)))*(COS(X*Y) - X*SIN(X*Y)))/((1 + X*X) + SQRT ((W*W + 1)*(W*W + 1)))*((W*W + 1) + SQRT ((W*W + 1)*(W*W + 1)))*((W*W + 1) + SQRT ((W*W + 1)*(W*W + 1)))*((W*W + 1) + SQRT ((W*W + 1)*(W*W + 1)))*(((W*W + 1) + SQRT ((W*W + 1)*(W*W + 1)))*(W + 1 + SQRT (W*W + 1) + SQRT (W*W + 1)))/(W + 1 + SQRT (W*W + 1) + SQRT (W*W + 1))

1 DX = (B - A)/2
GO TO (21, 22), J
21 FI1 = FGR1 (B) + FGR1 (A)
22 FI2 = FGR1 (A + DX)
GO TO 10
21 FI1 = FGR2 (B) + FGR2 (A)
22 FI2 = FGR2 (A + DX)
GO TO 10
10 FI3 = 0
FI = DX * (FI1 + FI2 + FI3)
2 FI3 = FI2 + FI3
FI2 = 0
TDX = DX
DX = 0.5 * DX
X = A + DX
3 GO TO (12, 13), J
12 FI2 = FI2 + FGR1 (X)
13 FI2 = FI2 + FGR2 (X)
16 X = X + DX
IF (X - B) 3, 3, 4
4 \[ FIP = DX * (PI1 + 4) * (PI2 + 2) / 3 \]
\[ = \text{IF}(|FIP - PI1| < 0.000001) \]
\[ = \text{IF}(DX < 0.0001) \]
5 \[ FIP = FIP \]
6 \[ GO TO 2 \]
7 \[ F(J) = FIP \]
8 \[ \text{IF}(J < 2) \]
9 \[ A10D = \sqrt{F(1)^2 + F(2)^2} \]
10 \[ \text{AN} = \text{ATAN}(F(2)/F(1)) \]
11 \[ BCL = \text{A10D} / \text{AN} \]
12 \[ \text{PRINT 102, A10D, AN, Y, BCL} \]
13 \[ \text{IF}(Y < Y2 < 0.001) \]
14 \[ \text{FORLIT}(2F12, 6, 0.3, F12.6) \]
15 \[ \text{FORLIT}(2F10.3) \]
16 \[ \text{FORLIT}(2F6.3) \]
17 \[ \text{FORLIT}(3F10.3) \]
18 \[ \text{CALL EXIT} \]
19 \[ \text{END} \]
Appendix 3

The parametric coupled differential equations have been integrated using Lilnes predictor-corrector method of numerical integration with a variable step length for speed and accuracy. The program is a generalisation of one given in Pennington 57 written on Fortran V for the London University Atlas Computer, as before. The equations to be solved are of the form

\[ \begin{align*}
\frac{dv_1}{dx} &= \alpha_1 (f_1(x) y_4 - f_2(x) y_3) \\
\frac{dv_2}{dx} &= \alpha_1 (f_1(x) y_3 + f_2(x) y_4) \\
\frac{dv_3}{dx} &= \alpha_2 (f_1(x) y_2 - f_2(x) y_1) \\
\frac{dv_4}{dx} &= \alpha_2 (f_1(x) y_1 + f_2(x) y_2)
\end{align*} \]

where \( f_1(x), f_2(x) \) are defined in the program which follows this section. Since Lilnes method is not self starting the seven starting values required are calculated by the method of Runge-Kutta. This subroutine is also based on the program given in Pennington 57.

The first section of the program calculates all the input data. In order to simplify the input as much as possible the variables chosen as the basic set are those which can be varied directly in an experimental situation. They consist of the wavelength of the three beams, the length
of the crystal, the temperature of the crystal, the confocal parameters of
the three beams, and the effective non-linear coefficients, which includes
the pump power. Section one of the flow diagram shows the way in which this
is accomplished. Equation 1.66, 1.67 are used to define the refractive
index of the crystal with wavelength and temperature. By using the important
secondary parameters the phase matching parameter, the spot sizes, and the
relative length \( \frac{1}{Z_{03}} \) when evaluating the results of the integration, the
results can be regarded as quite general although the equations 1.66 1.67
represent a specific crystal.

The results so far have been restricted to (1) \( \gamma = 0 \) i.e. degenerate
amplification, (2) \( x_1 = -x_2 \) focus at the centre of the crystal and (3)
\( z_1 = z_2 = 1 \) the three confocal parameters equal, although the program
is capable of dealing with these variations.

The first section of the following program also calculates the optimum
phase angle of the initial value of the signal amplitude this being the most
interesting value. By altering this specification any value can be taken
into account.

Flow diagrams for the first two sections of the program are included
but not one for the main part since this is only a minor modification of
the diagram given in Pennington, 57.
Section I

Input and Calculation of Physical Parameters

Input $\lambda_3$, $Y$

$\lambda_1 = \frac{2\lambda_3}{(1 + \alpha_3/\pi)}$

$x = \frac{2\lambda_3}{1 - \alpha_3/\pi}$

Input $x_{03}, z_2$

$x(s) = \frac{x_1}{z_{03}}$

$x_{end} = \frac{x_2}{z_{03}}$

Input $T_1, T_2, \Delta T$

$T_1 < 1000$

no

yes

$T_03 < 100$

no

yes

Calculation of refractive indices; spot sizes; phase match parameter, coefficients $Y_1$, $Y_2$; function parameters; and initial phase angle.
Section II Calculation of the Starting Values of the Milne Program using the Method of Runge-Kutta

1. \( z = f(x, y_1, y_2, y_3, y_4) \) for \( \ell = 1, 4 \)

2. \( x = x_n \)
   \( y = y \)

3. for \( \ell = 1, 4 \)
   \( i = 1 \)
   \( \alpha = 5 \)

4. \( x = x_n + h/2 \)
   \( c_\ell = h \delta \)
   \( y_\ell = y_{n+} + c_\ell \)

5. for \( \ell = 1, 4 \)

6. \( 0^2 = h^2 \)
   \( y_\ell = y_{n+} + c_\ell \)

7. \( x = x_n + h/2 \)
   \( c_\ell = h \delta \)
   \( y_\ell = y_{n+} + c_\ell \)
   for \( \ell = 1, 4 \)
   \( \alpha = 8 \)

8. \( x = x \)
   \( c_\ell = h \delta \)
   \( y_{\ell+} = y_{n+} + (c_1 + 2(c_2 + c_3) + c_4) / 6 \)
   for \( \ell = 1, 4 \)

9. \( y_{\ell+} = y_{n+} \)
   \( y_{\ell+} = z_\ell \)
   \( i \rightarrow i + 1 \)

10. \( i > 7 \)
    yes
    \( y_{\ell+} = y_{n+} \)
    \( y_{\ell+} = z_\ell \)
    \( i \rightarrow i + 1 \)

11. no
Parametric Integration Programme

\( M = 4 \)

Dimension \( Y(7,4), YP(7,4), Q(3,4), QP(3,4), YS(4), C(4), W1(4), WP1(4), W2 \)

Dimension \( Z(4), V(4), YN(4), C1(4), C2(4), C3(4), C4(4), YEND(4) \)

Dimension \( WAV(3), ROF(2) \)

\( \Pi = 3.1415926536 \)

\[
FGR1(X) = \left( (A1+A2*X*X+A3*X**4) + \sqrt{(A1+A2*X*X+A3*X**4)**2+(A4*X+A5*1*X*X*X)**2} \right) \cos(YM*X)+X*(A4+A5*X*X)*\sin(YM*X))/\sqrt{(A1+A2*X*X+A3*X**4)**2+(A4*X+A5*X*X*X)**2}
\]

\[
FGR2(X)=\left( (A1+A2*X*X+A3*X**4)+\sqrt{(A1+A2*X*X+A3*X**4)**2+(A4*X+A5*1*X*X*X)**2} \right) \sin(YM*X)-X*(A4+A5*X*X)*\cos(YM*X))/\sqrt{(A1+A2*X*X+A3*X**4)**2+(A4*X+A5*X*X*X)**2}
\]

\( YT=0.001 \)

\( ERR= 0.00001 \)

\( D = 0.001 \)

READ 110, WAV(3), GA

93 READ 110, XI, X2

IF(X1=100.0)19,44,44

19 READ 112, T1, T2, DT

IF(T1=1000.0)46,93,93

46 READ 112, ZOP, Z1, Z2

47 IF(ZOP=100.0)96,19,19
PRINT 112, ZOP, Z1, Z2
XO=X1/ZOP
XEND=X2/ZOP
PRINT 110, XO, XEND
T=T1-DT
T=T+DT
S=0.001*(T+273.0)
WAV(1)=2*WAV(3)*PI/(PI+GA*WAV(3))
WAV(2)=2*WAV(3)*PI/(PI-GA*WAV(3))
DO 95, I=1, 3
ROF(I)=SQRT(4.9130-0.0278*WAV(I)*WAV(I)+((11.73+1.65*S*S)/(100*WAV 1(I)*WAV(I)-(2.12+0.27*S*S)**2)))
REF=SQRT(4.5567+0.2605*S*S-0.0224*WAV(3)*WAV(3)+((9.7+2.7*S*S)/(10 10*WAV(3)*WAV(3)=(2.01+0.54*S*S)**2)))
WOP=0.01*SQRT(WAV(3)*ZOP/(PI*REF))
W02=WAV(3)*ROF(2)*Z2/(WAV(2)*REF)
W01=WAV(3)*ROF(1)*Z1/(WAV(1)*REF)
YM=2*PI*ZOP*(REF/WAV(3)-ROF(1)/WAV(1)-ROF(2)/WAV(2))*10000.0
AW=W01+W02+1
BW=W01*Z2+W02*Z1-Z1*Z2
CW=Z1+Z2+W01*(Z2-1)+W02*(Z1-1)
E=(REF/ROF(3))**2
EAV=E*W01+E*W02+1
EBW=E*W01*Z2+E*W02*Z1-Z1*Z2
ECW=Z1+Z2+E*W01*(Z2-1)+E*W02*(Z1-1)
\[ A_1 = A_W \cdot E_A W \]
\[ A_2 = E_A W \cdot B_W + A_W \cdot E_B W - C_W \cdot E_C W \]
\[ A_3 = B_W \cdot F_B W \]
\[ A_4 = C_W \cdot E_A W + E_C W \cdot A_W \]
\[ A_5 = E_B W \cdot C_W + E_C W \cdot B_W \]
\[ Y_{TI} = \sqrt{F_{GR1}(X_0)^* + 2 + F_{GR2}(X_0)^*} \]
\[ Y_N(1) = Y_T \cdot \left( \sqrt{1 + F_{GR1}(X)} / Y_{TI} \right) + \sqrt{1 - F_{GR1}(X)} / Y_{TI} \) \cdot 0.5 \]
\[ Y_N(2) = Y_T \cdot \left( \sqrt{1 - F_{GR1}(X)} / Y_{TI} - \sqrt{1 + F_{GR1}(X)} / Y_{TI} \right) \cdot 0.5 \]
\[ Y_N(3) = Y_N(1) \]
\[ Y_N(4) = Y_N(2) \]
\[ A_{P1} = -0.001 \cdot (W_0 \cdot Z_{0P} / (W_0 \cdot \sqrt{S_{QT}(S_{QT}(E)))) \]
\[ A_{P2} = A_{P1} / (W_A(2) \cdot R_0(2)) \]
\[ A_{P1} = A_{P1} / (W_A(1) \cdot R_0(1)) \]
\[ \text{PRINT} \ 110, \ A_{P1}, \ A_{P2} \]

**45**
\[ X_N = X_0 \]
\[ I = 1 \]
\[ D_D = 0.5 \cdot D \]

**22**
\[ X = X_N \]
\[ \text{DO 28 I=1,M} \]

**28**
\[ V(L) = Y_N(L) \]
\[ J = 1 \]

**23**
\[ Z(1) = A_{P1} \cdot (F_{GR1}(X) \cdot V(4) - F_{GR2}(X) \cdot V(3)) \]
\[ Z(2) = A_{P1} \cdot (F_{GR1}(X) \cdot V(3) + F_{GR2}(X) \cdot V(4)) \]
\[ Z(3) = A_{P2} \cdot (F_{GR1}(X) \cdot V(2) - F_{GR2}(X) \cdot V(1)) \]
\[ Z(4) = A_{P2} \cdot (F_{GR1}(X) \cdot V(1) + F_{GR2}(X) \cdot V(1)) \]
\[ \text{IF}(I-1)\ 99, 99, 16 \]
DO 98 L = 1, N
   Y(1, L) = YN(L)
98 XP(1, L) = 7(L)
   I = I + 1
   GO TO 16
16 GO TO (24, 25, 26, 27), J
24 X = XN + DD
   DO 29 L = 1, N
      C1(L) = D * Z(L)
29 V(L) = YN(L) + C1(L) * 0.5
   J = 2
   GO TO 23
25 DO 32 L = 1, N
      C2(L) = D * Z(L)
32 V(L) = YN(L) + C2(L) * 0.5
   J = 3
   GO TO 23
26 X = XN + D
   DO 33 L = 1, N
      C3(L) = D * Z(L)
33 V(L) = YN(L) + C3(L)
   J = 4
   GO TO 23
27 XN = X
   DO 34 L = 1, N
      C4(L) = D * Z(L)
34 \( Y(N(L)) = Y(N(L)) + (C1(L) + 2*(C2(L) + C3(L)) + C4(L))/6 \)

DO 17 \( L=1, M \)

\( Y(I, L) = Y(N(L)) \)

17 \( YP(I, L) = 7(L) \)

\( I = I + 1 \)

IF \( (I - 7) > 22, 91, 91 \)

91 \( X = X0 + 5*D \)

\( ERP = 29*ERR \)

\( ERG = 01*ERP \)

10 \( A = 1.3333333*D \)

\( B = 0.333333*D \)

\( K = 1 \)

1 DO 61 \( L = 1, M \)

61 \( Y(7, L) = Y(3, L) + A*(YP(4, L) + YP(4, L) - YP(5, L) + 2*YP(6, L)) \)

\( X = X + D \)

\( J = 1 \)

2 \( YP(1, 1) = AP1*(FGR1(X)*Y(7, 4) - FGR2(X)*Y(7, 3)) \)

\( YP(7, 2) = AP1*(FGR1(X)*Y(7, 3) + FGR2(X)*Y(7, 4)) \)

\( YP(7, 3) = AP2*(FGR1(X)*Y(7, 2) - FGR2(X)*Y(7, 1)) \)

\( YP(7, 4) = AP2*(FGR1(X)*Y(7, 1) + FGR2(X)*Y(7, 2)) \)

GO TO (3, 4), J

3 DO 51 \( L = 1, M \)

51 \( YS(L) = Y(7, L) \)

51 \( Y(7, L) = Y(5, L) + B*(YP(5, L) + 4*YP(7, L)) \)

\( J = 2 \)

GO TO 2
4 \quad L = 1

52 \quad c(L) = \text{ABS}(Y(7,L) = YS(L))
   \quad \text{IF}(c(L) = \text{ABS}(EKF \times Y(7,L)))53, 53, 21

53 \quad L = L + 1
   \quad \text{IF}(L - M - 1)52, 20, 20

20 \quad \text{IF}(X - XEND + .0001)5, 43, 43

43 \quad \text{IF}(X - XEND - .0001)50, 50, 21

5 \quad L = 1

72 \quad \text{IF}(c(L) = \text{ABS}(EFG \times Y(7,L)))54, 54, 30

54 \quad L = L + 1
   \quad \text{IF}(L - M - 1)72, 7, 7

30 \quad K = 1

6 \quad \text{DO } 55 = 1, K
   \quad \text{DO } 31 I = 1, 6
       \quad Y(I,L) = Y(I + 1,L)
   \quad YP(I,L) = YP(I + 1,L)

31 \quad \text{CONTINUE}
   \quad \text{GO TO } 1

7 \quad \text{GO TO (11, 12, 13, 14, 15), K}

11 \quad \text{DO } 56 L = 1, M
   \quad WP1(L) = Y(1,L)

56 \quad WP1(L) = YP(1,L)
   \quad K = 2
   \quad \text{GO TO } 6

12 \quad K = 3
   \quad \text{GO TO } 6
13 DO 57 L = 1, M
   W2(L) = Y(1, L)
57 WP2(L) = YP(1, L)
   K = 4
   GO TO 6
14 K = 5
   GO TO 6
15 DO 58 L = 1, M
   Y(6, L) = Y(7, L)
   Y(4, L) = Y(3, L)
   Y(3, L) = Y(1, L)
   Y(2, L) = W2(L)
   Y(1, L) = W1(L)
   YP(6, L) = YP(7, L)
   YP(4, L) = YP(3, L)
   YP(3, L) = YP(1, L)
   YP(2, L) = WP2(L)
58 YP(1, L) = WP1(L)
   D = D + D
   GO TO 10
21 IF(D - .0000001) 40, 40, 41
40 PRINT 1001
   IF(T - T2) 94, 46, 46
41 DO 59 L = 1, M
   DO 42 L = 1, 3
Q(I,L) = 0.5 * (Y(5=I,L) + Y(6-I,L)) - 0.0625 * (Y(7-I,L) - Y(6-I,L) - Y(5=I,L) + Y(4-I,L))

QF(I,L) = 0.5 * (YP(5=I,L) + YP(6-I,L)) - 0.0625 * (YP(7-I,L) - YP(6-I,L) - YP(5=I,L) + YP(4-I,L))

Y(6,L) = Y(5,L)
Y(2,L) = Y(3,L)
Y(5,L) = Q(1,L)
Y(3,L) = Q(2,L)
Y(1,L) = Q(3,L)
YP(6,L) = YP(5,L)
YP(2,L) = YP(3,L)
YP(5,L) = QP(1,L)
YP(3,L) = QP(2,L)
YP(1,L) = QP(3,L)

X = X - 2*D
D = 0.5*D
GO TO 10

XEND = X
DO 60 L = 1, M

YEND(L) = Y(7,L)
U1 = (SQRT(YEND(1)**2 + YEND(2)**2) / 0.001) = 1
U2 = SQRT(YEND(3)**2 + YEND(4)**2)
PRINT 101, T, YM, W01, W02, U1, U2
IF(T = T2) 94, 46, 46

CALL EXIT
110 FORMAT(2F10.4)
101 FORMAT(2F12.4,2F10.3,2E15.4)
112 FORMAT(3F10.3)
1001 FORMAT(32HROUND OFF ERROR PREVENTS SOLUTION)
    END
\[
G_{\text{graph}} \frac{1}{2^n m! 2^{nm} m!} F_{nm}
\]
Graph 3

Output relative units

Length \( \frac{\ell}{2} = 2.5 \)

Max = 0.662

Focus at centre
Graph II

Output units

Length, $L_0 = 5.65$
$I_{max} = 0.021$
Focus at centre

$\Delta R_2$

-20
-16
-12
-8
-4
0
4
8
12
16
20
Graph 13

\[ \Delta g/z \]

Gain

Idler = 0 initially
Focus = 0.5
Graph 14

Gain

\( \Delta R / R \)

Idler = 0 Initially

Focus at centre

\( \frac{\Delta R}{R} = 2 \)
Graph 16

Length \( \frac{L}{\alpha} = 3.3 \)

Focus at centre

Degenerate

Idler = 0

Gain

Effective Nonlinear Coefficient \( \times 10^3 \)
Gain \times 10^3(\text{Deg}) \times 10^4(\text{Idler}=0)

Degenerate

Idler

\frac{\theta}{2\pi}
Conclusions

A method has been developed with which non-linear optical problems involving the integration of the nodes of the optical resonator can be studied directly. This enables meaningful approximations to be made about the number of nodes present in any given situation. The method developed here is for travelling waves but the same principles should apply to standing waves.

Two examples of the use of method have been given. In the first the problem of resonant second harmonic generation in the small conversion approximation has been studied: it has been shown that the results are consistent with previous theories and in the general case the results given by Boyd and Kleinman\textsuperscript{15} for non-resonant second harmonic generation.

In applying this method to the second problem, parametric amplification, it has been shown that it gives results which agree with previous theories in the weak focusing limit. The preliminary numerical results indicate the behaviour in the general case. These results extend the previously established theory into the finite focussed region and indicate that further research will prove profitable.
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