PROBLEMS IN RELATIVITY THEORY AND RELATIVISTIC COSMOLOGY

Thesis submitted to the University of London

Dedicated to my Mother by

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ABSTRACT

The thesis consists of the following three parts:

PART I

Chapter I: Møller's Theory on Energy and Its Localization and Its Application to Static Fields

The difficulties of the Einstein canonical momentum-energy pseudo-tensor are discussed. Møller's new theory on the concept of energy and its localization in general relativity is summarized (and its application by Møller criticized) and applied to find that the energy of the Schwarzschild fields is equal to the gravitational mass of, and resides inside, the material system associated with the fields.

Chapter II: The Electromagnetic Energy and the Gravitational Mass of a Charged Particle in General Relativity

The electromagnetic energy $\mathcal{E}$ of the field of a charged particle is calculated using Møller's new theory. The contribution of $\mathcal{E}$ to the gravitational mass of the particle is investigated. Contrary to currently accepted ideas it is shown that $\mathcal{E}$ increases the (newtonian) gravitational mass of the particle by an amount which is precisely the mass-equivalence of $\mathcal{E}$.

PART II

Energy in Plane Gravitational Waves of Finite Duration

The result that the passage of plane gravitational waves impart a relative velocity to test particles
originally at relative rest, first obtained by Bondi, Pirani and Robinson using groups of motions, is obtained here by more direct and mathematically easier methods using only the geodesic equations. This effect shows that these waves must carry energy. Møller's result that these waves carry no energy is discussed.

PART III

OBSERVABLE RELATIONS IN RELATIVISTIC COSMOLOGY

A new observational criterion likely to solve the "cosmological problem" is formulated. It incorporates the fundamental property that an evolving expanding universe must be more congested at great distances than it is in the cosmic neighbourhood of the observer, while a steady-state universe must exhibit the same congestion at all distances. It is shown that this congestion, measured in suitable statistical terms by the ratio of the angular separation of galaxies from their neighbour galaxies to the angular diameter of the galaxies themselves, is proportional to \((1+z)^{-1}\) in an evolving universe; it is independent of \(z\), the red-shift, in a steady-state universe.

The applicability of the criterion and the angular diameter of a galaxy in special relativity are also discussed.
Acknowledgements

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\[ v = \frac{c}{\sqrt{1 - \frac{v^2}{c^2}}} \]

The cosmological metric tensor is used throughout.

\[ \Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\rho\sigma} \left( \partial_{\nu} g_{\sigma\lambda} + \partial_{\lambda} g_{\nu\sigma} - \partial_{\sigma} g_{\nu\lambda} \right) \]  

(Christoffel symbol of the second kind)

The Newtonian gravitational constant \( (G) \) is taken to be unity throughout.
The following notation has consistently been used throughout this thesis:

Greek indices $\alpha, \beta, \ldots \mu, \nu, \ldots$ assume the values 1, 2, 3, 4.

Latin indices $a, b, \ldots m, n, \ldots$ assume the values 1, 2, 3, ONLY.

Einstein's summation convention is used throughout. $\eta_{\mu\nu}$, the metric (of signature \(-2\)) of a general Riemannian 4-space.

$\mathcal{G}_{\mu\nu}$, the fundamental metric tensor

$g$, Determinant $|G_{\mu\nu}|$

$x^\mu (t)$, Temporal coordinate

$x^a$, Spatial coordinates

Comma (,) denotes partial differentiation (e.g. $g_{\mu\nu,\rho} = \frac{\partial g_{\mu\nu}}{\partial x^\rho}$)

$c$, Fundamental velocity (sometimes taken to be unity).

$\{\xi_{\mu\nu}\} = \frac{1}{2} g^{\gamma\lambda} (g_{\mu\lambda,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda})$ (Christoffel symbol of the second kind)

The newtonian gravitational constant ($G$) is taken to be unity throughout.
CHAPTER I

Möller's Theory on Energy and its Localization in General Relativity and its Application to Static Fields

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Chapter II

The Electromagnetic Energy and the Gravitational Mass of a Charged Particle in General Relativity

We first give a brief summary of Möller's theory in §3. Some of the advantages and limitations of the theory are also pointed out. The theory is applied in §4.
CHAPTER I

Möller's Theory on Energy and its Localization in General Relativity and its application to Static Fields

I. Introduction

The difficulties of the Einstein canonical momentum-energy pseudo-tensor, \( \Theta^\nu_\mu = \sqrt{-g} \left( T^\nu_\mu + \frac{1}{2} g^\nu_\mu \mathcal{T} \right) \), have often been pointed out in various books on relativity theory (1), (2) and explicitly been discussed in recent papers by Bergman (3), Möller (4) and Komar (5) and are briefly summarized in §2. An attempt to overcome many of these difficulties was made by Möller (4) in 1958. Möller applied his theory (4) to find the energy of the exterior Schwarzschild field. But, although he obtains the expected results, his application is somewhat incomplete and unsatisfactory; it also gives no results at all as to the localization of this energy which is one of the merits of this theory. It is the object of this chapter to obtain the same results in a more explicit way and, at the same time, to find where the energy is localized.

We first give a brief summary of Möller's theory in §3. Some of the advantages and limitations of the theory are also pointed out. The theory is applied in §4.
and §5 to find the energy of the exterior and interior Schwarzschild fields respectively. It is found that the energy of the exterior field is zero while the energy of the interior field is equal to the gravitational mass of the material system associated with the fields. The conclusion is that all the gravitational energy resides wholly inside the material system associated with the fields.

§2. Difficulties of the Einstein momentum-energy pseudo-tensor, $\Theta^\gamma_\beta$

In this theory the total momentum-energy, $(-P_\alpha, E)$, inside a 3-space $\Omega$, is given by (6)

$$(-P_\alpha, E) = \int_\Omega \Theta^\gamma_\beta dx' dx'^2 dx'^3$$

(2.1)

where $\Theta^\gamma_\beta$ is the momentum-energy pseudo-tensor density defined by

$$\Theta^\gamma_\beta = \sqrt{-g} \left( T^\gamma_\beta + t^\gamma_\beta \right)$$

(2.2)

$T^\gamma_\beta$ is the momentum-energy tensor of the material system and $t^\gamma_\beta$ is referred to as the "potential gravitational momentum-energy" of the field. $t^\gamma_\beta$ is defined in terms of the Lagrangian

$$\mathcal{L} = \sqrt{-g} g^{\mu\nu} \left[ \{ \beta_\mu \} \{ \alpha_\nu \} - \{ \alpha_\nu \} \{ \beta_\mu \} \right]$$

(2.3)
The quantities $\Theta^\alpha_\beta$ can be expressed in the form of "plain divergence" given by (7) or in the more useful form

$$\Theta^\alpha_\beta = \h^\alpha_\beta, \gamma$$

(2.6)

where

$$\h^\alpha_\beta = - \h^\beta_\alpha = \frac{g_{\beta \gamma}}{2k\sqrt{-g}} \left[ (-g) \left( g^{\alpha \nu} g_{\mu \gamma} - g^{\nu \gamma} g_{\alpha \mu} \right) \right]_{\mu \nu}$$

(2.7)

Expressions (2.6), (2.7) were first discovered by Freud (8) and, apparently, re-discovered by Möller (4).

Using (2.6) in (2.1) we get

$$(-P, \Sigma) = \int_2 \h^\alpha_\beta, \gamma d\chi' d\chi^2 d\chi^3$$

$$= \int_2 \h^\mu_\gamma, \gamma d\chi' d\chi^2 d\chi^3$$

$$= \int_5 \h^\mu_\gamma, \gamma d\Sigma$$

(2.8)
by Gauss's theorem. \( \eta_{\nu} \) is a unit vector normal to the surface element \( ds \), and \( S \) is a closed surface enclosing \( S \). Equation (2.8) states that to find \(-P_{\nu},E\) we only need the values of \( h^{\mu\nu}_{\beta} \) (i.e. the values of \( \eta_{\nu} \) and its derivatives) on the surface \( S \). \( \Theta_{\nu}^{\mu} \) satisfies the conservation law
\[
\Theta_{\nu}^{\mu} = 0
\] (2.9)
as is also apparent from (2.6) and the antisymmetry of \( h^{\mu\nu}_{\beta} \) in \( \alpha, \nu \). The identification expressed by equation (1.1) follows from the conservation law (2.9).

A closer examination of the above formulation, however, reveals a number of difficulties which make the use of the theory very limited.

It follows directly from equation (2.9) that \( \Theta_{\nu}^{\mu} \) is not a true tensor density; i.e. it does not behave like a tensor density under general space-time coordinate transformations, not even under purely spatial transformations (i.e. transformations not involving the time coordinate). The non-tensorial character of \( \Theta_{\nu}^{\mu} \) also follows from equations (2.3), (2.4) defining the pseudo-tensor \( t_{\nu}^{\mu} \) since \( \Theta \) is not a true scalar density. This is the greatest disadvantage of the theory and, in the last analysis, the origin of the difficulties of this theory. In particular \( \Theta_{\nu}^{\mu} \) does not behave like
a scalar density under general space-time or even under purely spatial coordinate transformations. It also follows from equations (2.3), (2.4) that $t^\alpha_\beta$ is a function of the derivatives of $g_{\mu\nu}$ of order not greater than the first. This implies the existence of a coordinate system (normal coordinates) in which $t^\alpha_\beta$ is zero at any arbitrary world point. Schrodinger (9) has proved that when the exterior Schwarzschild field is written in the form

$$ds^2 = -(dx^1)^2 -(dx^2)^2 -(dx^3)^2$$

$$- \frac{2m}{r^2(r-2m)} \left( x'dx' + x^2 dx^2 + x^3 dx^3 \right) + \left(1 - \frac{2m}{r} \right) dt^2$$

all the components of $t^\alpha_\beta$ vanish identically.

A number of examples were given to demonstrate the defects of the theory. Bauer (10), for example, has shown that by expressing the Minkowski space-time in spherical polar coordinates we get an infinite value for the total energy. The same absurd result is obtained when we calculate the energy of the exterior Schwarzschild field when the latter is expressed in the form

$$ds^2 = -\frac{dr^2}{(1 - \frac{2m}{r})} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + \left(1 - \frac{2m}{r} \right) dt^2$$

The situation is confused even more by the
different pseudo-tensors introduced by Landau and Lifschitz ([11]) and Goldberg ([12]). Bergmann's work has shown ([2]) that the components of these canonical momentum-energy pseudo-tensors may be thought of as the generators of infinitesimal coordinate transformations. Corresponding to the rigid parallel displacement of the coordinate origin.

As a result of the above mentioned difficulties it became doubtful whether the expression (2.1) or (2.8) has any physical meaning at all.

Einstein, however, has shown ([13]) that by restricting the coordinate system to be quasi-galilean equation (2.8) gives physically well defined results for the whole of the 3-space \( \mathbb{R}^3 \), i.e. when \( S \) is the surface at spatial infinity. By a quasi-galilean system of coordinates we mean a coordinate system in which \( g_{\mu\nu} \) approach \( \eta_{\mu\nu} \), the Minkowski space-time metric tensor, at spatial infinity. \( \eta_{\mu\nu} \) is defined by

\[
\eta_{\mu\nu} = \text{diag}(-1, -1, -1, +1)
\]

More precisely when \( g_{\mu\nu} \) satisfy the so-called Einstein-Klein conditions

\[
g_{\mu\nu} = \eta_{\mu\nu} + O\left(\frac{1}{r}\right)
\]

\[
g_{\mu\nu,\rho} = O\left(\frac{1}{r^2}\right)
\]

\[\text{(2.10)}\]
then \((I4)\)
\[
\int_S \Theta^I_\beta \eta^I_\epsilon dS = 0
\] (2.11)

From equation (2.9) we get
\[
\frac{d}{dx^u} \int_\Omega \Theta^x_\beta dx'dx'^2dx'^3 + \int_\Omega \Theta^x_\beta, \epsilon dx'dx'^2dx'^3 = 0
\]

Transforming the second integral into a surface integral by Gauss's theorem and using (2.11), we get (with \(x'^u = t\))
\[
\frac{d}{dt} \int_\Omega \Theta^x_\beta dx'dx'^2dx'^3 = 0
\]
Therefore
\[
\int_\Omega \Theta^x_\beta dx'dx'^2dx'^3 = (-P_b, \xi) = \text{Constant} \quad (2.12)
\]

Therefore under the conditions (2.10) \((-P_b, \xi)\) has the following desirable properties

1. \((-P_b, \xi)\) is a constant of "time". Also,
2. \((-P_b, \xi)\) is a free vector, i.e. it transforms as a vector under the linear coordinate transformations
\[
\chi^\alpha = A^\alpha_\beta \chi^\beta
\]
where \(A^\alpha_\beta\) are constants.
3. \((-P_b, \xi)\) is invariant under all "spatially localized transformations", i.e. transformations under which conditions (2.10) remain unchanged.

(For a proof of these properties see, for example, Einstein (13), Möller (6), Frautman (14).)
It is appropriate to quote here the following results obtained by the present author.

The most general spherically symmetric metric, in isotropic coordinates, is of the form

$$ds^2 = -\alpha(r) \left( dx^2 + dy^2 + dz^2 \right) + \beta(r) dt^2 \quad (2.13)$$

where

$$r^2 = x^2 + y^2 + z^2 \quad (2.14)$$

and $\alpha(r)$, $\beta(r)$ are functions of $r$ only. We assume, further, that $\alpha(r)$, $\beta(r)$ satisfy the Einstein-Klein conditions (2.10) as $r \to \infty$. We shall find the total energy of the field described by (2.13).

Using equations (2.5) or (2.6) and (2.7) we find that, for the metric (2.13), is given by

$$\Theta_\mu^\nu = \frac{1}{k} \left\{ \frac{\partial}{\partial x} (h r_x) + \frac{\partial}{\partial y} (h r_y) + \frac{\partial}{\partial z} (h r_z) \right\} \quad (2.15)$$

where

$$h = -\frac{c \alpha}{dr} \left( \frac{\beta}{\alpha} \right)^{1/2} \quad (2.16)$$

and

$$r_x = \frac{\partial r}{\partial x} = \frac{x}{r}, \quad r_y = \frac{y}{r}, \quad r_z = \frac{z}{r} \quad (2.17)$$
No approximation is involved in these results.

Substituting $\Theta^{\mu}_{\nu}$ in (2.12) we get for the total energy $E$,

$$E = \frac{1}{k} \int \left[ \frac{\partial}{\partial x} (hr_x) + \frac{\partial}{\partial y} (hr_y) + \frac{\partial}{\partial z} (hr_z) \right] dx dy dz \quad (2.18)$$

where $\mathcal{O}$ is the whole of 3-space. Changing (2.18) into a surface integral by Gauss's theorem, we get

$$E = \frac{1}{k} \int_{\mathcal{S}} h \, ds$$

In "spherical polar" coordinates this reads

$$E = \frac{1}{k} \int_{\mathcal{S}} h r^2 \sin \theta d\theta d\phi \quad (2.19)$$

Returning to the Einstein formulation of momentum-energy we see that although the properties of $(-p, E)$ listed above are very satisfactory they are valid only in a quasi-galilean coordinate system. Also, the theory gives an answer only for the total momentum-energy of the whole of the 3-space $\tau = \text{constant}$.

Further, according to modern radiation theory conditions (2.10) are too stringent to allow radiation. This limitation, however, can be avoided by imposing different conditions on the metric tensor, so that radiation is allowed, while the properties (1), (2), (3) are still
valid. This was done by Trautman (14) but, again, the above properties hold for a specific coordinate system. In the following section we summarize the theory proposed by Møller which overcomes many of the above difficulties of the Einstein canonical momentum-energy pseudo-tensor, is applicable to any coordinate system and can be applied to find the energy of any part of the 3-space \( t = \text{constant} \), i.e. it deals with the localization of energy.
§3. Summary of Möller's theory on energy and its localization in general relativity

The theory proposed by Möller in 1958 to overcome many of the difficulties of the Einstein momentum-energy pseudo-tensor, \( \Theta^{\alpha}_{\beta} \), pointed out in §2, as presented in his first paper (4), is the following:

He observed that the condition (2.9) i.e. \( \Theta^{\alpha}_{\beta,\alpha} = 0 \)
does not uniquely determine \( \Theta^{\alpha}_{\beta} \). \( \Theta^{\alpha}_{\beta} \) is determined within a quantity \( S_{\beta}^{\alpha} \) with identically vanishing divergence (i.e. \( S_{\beta,\alpha}^{\alpha} = 0 \)). He uses this freedom to define a new momentum-energy pseudo-tensor, \( J_{\beta}^{\alpha} \), by

\[
J_{\beta}^{\alpha} = \Theta_{\beta}^{\alpha} + S_{\beta}^{\alpha}
\]

where \( S_{\beta}^{\alpha} \) is chosen so that

(i) \( S_{\beta,\alpha}^{\alpha} = 0 \)

(ii) \( \int_{\Omega} S_{\alpha}^{\alpha} dx' \delta x^0 dx^1 dx^2 dx^3 = 0 \) when the coordinates used are quasi-galilean and \( \Omega \) is the whole of 3-space \( t = \) constant.

(iii) \( J_{\alpha}^{\alpha} \) behaves like a 4-vector density under all spatial coordinate transformations

\[
\tilde{x}^{\alpha} = x^{\alpha}(x', x^1, x^2, x^3)
\]

\[
\tilde{x}^{4} = x^{4}
\]

\( (\alpha = 1, 2, 3) \)

(3.1)

Condition (ii) ensures that this theory agrees
with Einstein's theory when the latter is applicable at all (i.e. when the coordinates used are quasi-galilean).

As the quantity $S_\beta^\alpha$ satisfying conditions (i), (ii), (iii) Möller took

$$S_\beta^\alpha = h_\beta^\alpha,\gamma - \delta_\beta^\gamma h_\mu^\mu,\nu + \delta_\beta^\nu h_\mu^\mu,\gamma,$$

where $h_\beta^\alpha$ is defined by (2.7) i.e.

$$h_\beta^\alpha = -h_\beta^\alpha = \frac{g_{\beta\gamma}}{2\sqrt{-g}} \left[ (g_{\alpha\nu}g_{\nu\mu} - g_{\nu\nu}g_{\nu\mu}) \right]_{\gamma,\mu}.$$

$S_\beta^\alpha$ added to $\Theta_\beta^\alpha$ the new momentum-energy pseudo-tensor $\mathcal{T}_\beta^\alpha = \Theta_\beta^\alpha + S_\beta^\alpha$ can finally be reduced to

$$\mathcal{T}_\beta^\alpha = \chi_\beta^\alpha,\nu,$$

where

$$\chi_\beta^\alpha = \chi_\beta^{\alpha,\nu} = \frac{\sqrt{-g}}{k} \left( g_{\beta\mu}\alpha - g_{\beta\mu}\nu \right) g_{\alpha,\gamma} g_{\nu,\mu}$$

(3.3)

From (3.2) and the antisymmetry of $\chi_\beta^\alpha$ in $\alpha, \nu$ we have the conservation law $\mathcal{T}_\beta^\alpha,\alpha = 0$, from which follows the usual interpretation that the momentum-energy $(-p_\beta, E)$ is given by

$$(-p_\beta, E) = \int_{\Sigma} \mathcal{T}_\beta^\alpha dx^1 dx^2 dx^3$$

(3.4)

where $\Sigma$ is now any part of the system or any part of
the 3-space surrounding the system. This in essence is Møller's theory.

**Discussion and Criticisms**

Møller has studied the transformation properties of $\mathcal{J}_\mu^\alpha$ in great detail (15). He found that, for the spatial transformations (3.1), $\mathcal{J}_\mu^\alpha$ transforms according to the law

$$\mathcal{J}_\mu^\alpha = \left| \frac{\partial x^\mu}{\partial x^\nu} \right| \frac{\partial x^\alpha}{\partial x^\beta} \mathcal{J}_\nu^\beta$$

(3.5)

or

$$\mathcal{J}_\mu^\alpha = \left| \frac{\partial x^m}{\partial x^n} \right| \frac{\partial x^\alpha}{\partial x^b} \mathcal{J}_b^b$$

(3.6a)

and

$$\mathcal{J}_\mu^\mu = \left| \frac{\partial x^\mu}{\partial x^\nu} \right| \mathcal{J}_\nu^\nu$$

(3.6b)

as is easily seen from the equations (3.1), (3.5). It follows that the main advantages of the new theory over Einstein's theory are the following:

$\mathcal{J}_\mu^\alpha$ has the great merit in behaving like a 4-vector density (see (3.5)) under all spatial coordinate transformations (3.1). In particular, as
seen from (3.6) $\mathbf{J}_\mu$ behaves like a scalar density and the momentum density $\mathbf{J}_\mu$ behaves like a vector density under the transformations (3.1). This means that in changing the coordinate system, for example, from "cartesian" coordinates to "spherical polar" coordinates, we no longer get the absurd results that we would otherwise get by using Einstein's theory. Also, the volume throughout which the energy is calculated, need no longer be the whole of the 3-space $t =$ constant. It is, therefore, meaningful in this theory to talk about "localization of energy".

On the other hand, from the point of view of general relativity theory, Møller's theory has the great disadvantage that $\mathbf{J}_\mu$ behaves like a 4-vector density only under spatial coordinate transformations of the type (3.1). As a result it seems that the theory can be used with certainty only when the fields are static. The theory becomes less certain (if applicable at all) in the case of non-static fields (see part II). In this part of the thesis we shall only be concerned with static fields.

Although the theory as presented in (4) seems rather arbitrary the same results were obtained later by Møller (15) in a most elegant way from a variational principle using the method of infinitesimal transforma-
tions. He proved that the $\pi^\alpha_\beta$ in (2.7) follow from a variational principle in which the integrand is the Lagrangian $\mathcal{L}$ defined by (2.3) while $\chi^\alpha_\beta$ defined by (3.3) follow from a variational principle in which the integrand is the scalar curvature density $\mathcal{R}$ defined by

$$\mathcal{R} = \sqrt{-g} R = \sqrt{-g} g^{\mu \nu} R_{\mu \nu}$$

where $R_{\mu \nu}$ is the Ricci tensor.

The vector character of $\mathcal{J}^\alpha_a$ under the spatial coordinate transformations (3.1) is also proved very clearly in (15). The theory was applied by Møller (16) in investigating the energy in gravitational waves. These fields being non-static, however, and in view of the effect gravitational waves have on test particles, discussed by Bondi, Pirani and Robinson and obtained in a different way in part II of this thesis, the results obtained by Møller do not seem altogether satisfactory.
§4. The energy of the exterior Schwarzschild field

The exterior Schwarzschild field is described by the spherically-symmetric static metric

$$ds^2 = -\frac{dr^2}{(1 - \frac{2m}{r})} - r^2(\text{d}\theta^2 + \text{sin}^2\theta\text{d}\phi^2) + (1 - \frac{2m}{r})dt^2 \quad (4.1)$$

This is the field associated with a "spherical" material system situated at the origin of the spatial coordinates, and it holds only throughout the space exterior to the material system. The solution (4.1) was first obtained by Schwarzschild (17) (see also (1)).

To avoid discussing the singularity at $r = 2m$ and its possible physical meaning we shall assume that the material system has a "radius" $\alpha$, say, greater than $2m$. This is in agreement with the restrictions which the interior solution (see §5) imposes on the size of a material system. In view of the discussion in chapter II (part I) we would like to emphasize that the $m$ appearing in (4.1) is the (Newtonian) gravitational mass of the material system. This is proved by considering the behaviour of a neutral test particle moving in the field (4.1) at large distances from the system, where the $3$-space $t = \text{constant}$ is approximately Euclidean, and then comparing it with the corresponding classical case (18).
With these preliminary remarks we proceed to calculate the energy of the field described by (4.1) using the theory summarized in §3.

From (3.2), (3.4) we get for the energy $E$

$$E = \int_0^{\infty} \int_0^\pi \int_0^{2\pi} \chi_{\mu\nu}^{\alpha\gamma} r^4 d\rho d\omega d\phi$$

(4.2)

where we have taken $(x', x', n, x') = (r, \theta, \phi, t)$.

Using (3.3) to evaluate $\chi_{\mu\nu}^{\alpha\gamma}$ with $g_{\mu\nu}$ given by (4.1) we find that

$$\chi_{\mu\nu}^{\alpha\gamma} = \frac{2m \sin \theta}{r}$$

(4.3)

Therefore,

$$\chi_{\mu\nu}^{\alpha\gamma} = 0$$

(4.4)

It follows from (4.2), (4.4) that $E = 0$ i.e. the energy of the field outside the material system is zero.

Criticisms of Møller's application

Møller uses his theory to find the energy of the field (4.1) in the following way (4):

He uses the expression (3.3), for the momentum-
energy pseudo-tensor $\mathcal{T}_\beta^\alpha$, in (3.4) and transforms the integral for the momentum-energy (3.4) into a surface integral, by Gauss's theorem, in the form

$$(-p, e) = \int_S \left( \chi^{\mu \nu} k_{-\frac{1}{2}} \right) dS_e$$

(4.5)

where he takes $dS_e$ as

$$dS_e = \sqrt{\mathcal{J}} (d\theta d\phi, 0, 0)$$

(4.6)

$\mathcal{J}$ is the determinant of the spacial metric tensor $\mathcal{g}_{ab}$ (or $\mathcal{g}_{ab}$ in the case of (4.1)).

Then Møller calculates $(\chi^{\mu \nu} k_{-\frac{1}{2}})$, which is equal to

$$\frac{2m \sin \theta}{\chi \sqrt{\mathcal{J}}} \mathcal{J}^{\frac{1}{2}}$$

By expanding $\mathcal{J}^{\frac{1}{2}}$ and neglecting terms in $r^{-i}$ for $i>2$ he obtains

$$\chi^{\mu \nu} k_{-\frac{1}{2}} = \left( \frac{2m}{kr^2}, 0, 0 \right)$$

(4.7)

Substituting this in (4.5) he finds for the energy $E$,

$$E = \int_S \left( \chi^{\mu \nu} k_{-\frac{1}{2}} \right) dS_e = \lim_{r \to \infty} \int \frac{2m}{kr^2} \sqrt{\mathcal{J}} \ d\theta d\phi$$

By expanding $\sqrt{\mathcal{J}}$ he obtains, for large $r$,

$$E = \lim_{r \to \infty} \int_S \frac{2m}{kr^2} r^{-2} \sin \theta \ d\theta d\phi$$

$$= m$$

using $k = 8\pi$. 

We have proved earlier in this section that the
(This is worked out on page 362 of his paper (4).)

It is not difficult to see, however, that this calculation is erroneous and misleading. If we leave all the approximations to be carried out at the end of our calculations we find that

\[ E = \int_S \left( \chi_{\lambda}^\nu y^{-\frac{3}{2}} \right) \frac{1}{\sqrt{y}} \sin \theta \cos \phi \, d\theta \, d\phi \]

using (4.5), (4.6)

\[ = \int_0^{2\pi} \int_0^\pi \chi_{\lambda}^\nu \sin \theta \, d\theta \, d\phi \]

i.e. does not enter into this integral at all. Using (4.3) for \( \chi_{\lambda}^\nu \), the energy is now given by

\[ E = \int_S \frac{2m}{\sqrt{y}} \sin \theta \cos \phi \, d\theta \, d\phi \]

\[ = m \]

, the same as before. In this case, however, no approximations in \( \frac{1}{\sqrt{y}} \) were carried out.

In other words the energy \( E \) is independent of the particular "spherical" surface over which the integral is carried out. This absurd result was masked in Möller's calculations by taking two successive approximations, one in \( y^{-\frac{3}{2}} \) and the other in \( y^{-\frac{1}{2}} \).

It will be remembered that Möller's theory can be applied to find the energy inside any "spherical" surface, not necessarily inside the surface at spatial infinity.

We have proved earlier in this section that the
energy outside the material system is zero. This was due to the fact that \( \mathcal{J}^\mu_\nu = \chi_{\mu, \nu}^\nu = 0 \). This important result that \( \mathcal{J}^\mu_\nu = 0 \) everywhere outside the material system was also masked in Møller's work by changing the volume integral (3.4) into a surface integral.

To find the energy of the Schwarzschild field using Einstein's theory we must first express the metric (4.1) into isotropic coordinates in the form

\[
d s^2 = -\left(1 + \frac{m}{2\rho}\right)^4 \left[ d\rho^2 + \rho^2 (d\theta^2 + \sin^2\theta d\phi^2) \right] + \frac{(1 - \frac{m}{2\rho})^2}{(1 + \frac{m}{2\rho})^2} dt^2 \quad (4.9)
\]

or

\[
d s^2 = -\alpha(\rho) \left( dx^2 + dy^2 + dz^2 \right) + \beta(\rho) dt^2 \quad (4.10)
\]

where

\[
\alpha(\rho) = \left(1 + \frac{m}{2\rho}\right)^4
\]

\[
\beta(\rho) = \left(1 - \frac{m}{2\rho}\right)^2 \left(1 + \frac{m}{2\rho}\right)^{-2}
\]

(4.9) is obtained from (4.1) by the transformation

\[
\rho = \rho \left(1 + \frac{m}{2\rho}\right)^2 \quad (4.11)
\]

Conditions (2.10) are easily seen to be satisfied and
therefore Einstein's theory is applicable. It follows from (2.16) that

\[ h = \frac{2m}{F^2} \left( 1 - \frac{m}{2F} \right) \]

Hence equation (2.19) gives, as \( r \to \infty \)

\[ E = m \]

(4.12)

which is the well-known result obtained by Einstein.

It is the total energy of the whole of the 3-space

\( t \) = constant. At first sight this result seems to contradict our result (4.4), based on Møller's theory, that the energy of the field is zero. There is no contradiction, however, because we shall prove, in the following section, that the energy of the interior Schwarzschild field is, indeed, equal to \( m \) and it resides wholly inside the material system.
§5. The energy of the interior Schwarzschild field

The interior Schwarzschild field is described by the spherically-symmetric static metric

$$ds^2 = -(1 - \frac{r^2}{R^2})^{-1} dt^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

$$+ \left[ A - B \sqrt{1 - \frac{r^2}{R^2}} \right]^2 dt^2$$ (5.1)

where

$$A = \frac{3}{2} \sqrt{1 - \frac{a^2}{R^2}}$$

$$B = \frac{4}{3}$$

$$R^2 = \frac{3}{8 \pi \rho} = \frac{a^3}{2m}, \quad m = \frac{4}{3} \tau \rho a^3$$ (5.2)

This is the field associated with a "spherical" material system, the material being an incompressible perfect fluid with constant proper density $\rho$. The constant $a$ is the "radius" of the spherical system and the origin of spatial coordinates is at the "centre", $r=0$, of the system. The field as described by (5.1) holds only inside the material system; i.e. for $r<\alpha$. At $r=\alpha$ (5.1) agrees with (4.1) which describes the field outside the material system. In fact, the various constants occurring in (5.1) are chosen to satisfy this condition. From this "boundary condition" at $r=\alpha$ it also follows that the $\omega (= \frac{4}{3} \tau \rho a^3)$ in (5.1), (5.2)
is the same as the \( m \) occurring in (4.1); in other words \( m \) is the gravitational mass of the incompressible perfect fluid. The solution (5.1) imposes an upper and a lower limit on the possible size of the "spherical" fluid, the lower limit being \( a > 2m \). This is in agreement with the assumption made in §4.

The solution (5.1) was first obtained by Schwarzschild (19), (see also Tolman (20)).

As before (§4) the expressions (3.2), (3.4) give, for the energy \( E \),

\[
E \equiv \int \int \int \int \chi^\nu_\mu \, dr \, d\theta \, d\phi
\]

(5.3)

where, as in (84), \( (x^1, x^2, x^3, x^4) = (r, \theta, \phi, t) \).

Using (3.3) with \( g_{\mu \nu} \) given by (5.1) we find that \( \chi^\nu_\mu \) are given by

\[
\chi^1_4 = \frac{2 B r^3 \sin \theta}{K R^2} \\
\chi^2_4 = \chi^3_4 = 0
\]

(5.4)

Substituting (5.4) in (5.3) we get, for the energy \( E \),

\[
E = \int \int \int \left[ \frac{2 B r^3 \sin \theta}{K R^2} \right] dr \, d\theta \, d\phi
\]

\[
= \frac{8 \pi B a^3}{K R^2}
\]

\[
= \frac{a^3}{2 R^2}
\]

, using \( K = 8 \pi, \beta = \frac{1}{2} \)

\[
= m
\]

, from (5.2)
We have, therefore, that the total energy of the material system throughout the volume occupied by the system is equal to the gravitational mass of the system.

Further it was proved in §4 that the energy of the exterior field is zero. We conclude, therefore, that all the energy of the system associated with the Schwarzschild fields is equal to the gravitational mass of the system and resides inside the material system.

Even if we accept Møller's result that the energy of the system is equal to its gravitational mass, a result which he obtained by considering only the exterior Schwarzschild field in an unsatisfactory way, his work makes no reference as to the localization of this energy. It will be remembered that one of the merits of Møller's theory is that it is meaningful to talk about the "localization of energy". Although at the very end of his paper (4) Møller explicitly remarks that "in any static or only stationary system the energy density is zero in the empty space surrounding the matter" he makes no reference whatever where the energy is localized when he worked out the energy of the exterior Schwarzschild field. Our results conform to the above remark of Møller but the way they were obtained seems far more satisfactory and gives, at the same time, the localization of this energy (according to the new theory).
CHAPTER II

The electromagnetic energy and the gravitational mass of a charged particle in general relativity

1. Introduction

The question whether the electromagnetic energy of a charged system contributes to the gravitational mass of the system has often been discussed (see for example Bonnor (21)) but a satisfactory answer is yet to be found. In the case of a charged particle (which for convenience we shall refer to as an "electron"), with which we shall be concerned in this chapter, it is commonly accepted that the electromagnetic energy contributes nothing to the gravitational mass of the electron. The main argument in favour of this point of view is the following: In the solution given in §2 describing the field of an electron, \( \mu' \) is identified with the "gravitational mass" of the electron and, as such, it is assumed to be independent of \( e \), the charge of the electron. It is argued, then, that since \( e \) does not appear in any term in \( \frac{1}{r} \) (where \( r \) is the "distance" from the electron) in which \( \mu' \), the "mass" manifests itself, \( e \) cannot contribute anything to the gravitational mass of the electron. We show in this chapter that this argument, and, especially the
interpolation of \( \mu \) as being the "gravitational mass" of the electron, is erroneous and that, in fact, when \( \mu \) is properly interpreted, the electromagnetic energy, \( \mathcal{E} \), increases the (Newtonian) gravitational mass (i.e. the mass of the central particle when \( e=0 \)) of the electron by an amount which is precisely the mass-equivalence of \( \mathcal{E} \).

We obtain first the solution of the Maxwell-Einstein field equations associated with the electro-gravitational field of a charged particle in §2. Möller's theory summarized in chapter I (§3) is applied in §3 to calculate the electromagnetic energy of the field of a charged particle. A brief discussion on the finiteness of the structure of an (actual) electron is also given. §4 deals with the behaviour of a neutral test particle moving in the electro-gravitational field of the electron. By comparing this behaviour with the corresponding behaviour in classical theory at large distances from the electron a new interpretation is given to \( \mu \), and the contribution of the electromagnetic energy to the Newtonian gravitational mass is determined. It turns out that this contribution is precisely the mass-equivalence of the electromagnetic energy. This result, however, contrary to the currently accepted ideas, is not altogether unexpected from the point of view of relativity theory.
In §5 we give a brief discussion on Jeffery's work on the motion of a charged particle in an electro-gravitational field. It is shown that our results can also be obtained from Jeffery's work if we work to a higher approximation than Jeffery did. The chapter ends with the application of Einstein's theory on momentum-energy to find the energy of the field of an electron. We find, in §6, that this energy is equal to $\mu$, and deduce, again, that the old interpretation of $\mu$ is meaningless.
§2. The gravitational field of a charged particle (electron)

The exact solution of this problem was first given by Nordstrom (22), using the calculus of variations, and by Jeffery (23), using the Maxwell-Einstein field equations. It can be found in any book on relativity theory (see, for example, Tolman (20)). The solution contains two arbitrary constants of integration \( \mu, \epsilon \).

\( \mu \) is identified with the (Newtonian) gravitational mass (i.e. with the constant \( m \) occurring in equation (4.1) of chapter I) of the electron and \( \epsilon \) with the electric charge of the electron. We consider the argument leading to the identification of \( \mu \) as inaccurate (see page 44). In §4 we put forward a different argument which enables us to give a new interpretation to \( \mu \). Our interpretation differs from the usual one that \( \mu \) is the gravitational mass of the electron. To show that the argument leading to the old interpretation of \( \mu \) is inaccurate and that the new interpretation is possible it is imperative to examine critically how \( \mu \) and \( \epsilon \) arise in the solution of the problem. We therefore give a brief derivation of the solution.

Taking the "spherical" charged particle at rest
at the spatial origin of our coordinate system we can assume a static spherically-symmetric solution described by the metric

\[ ds^2 = -e^{\lambda} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + e^{\nu} dt^2 \]  

(2.1)

where

\[ \lambda = \lambda(r) , \quad \nu = \nu(r) \]

and

\[ \lambda \rightarrow 0 , \quad \nu \rightarrow 0 , \quad \text{as} \quad r \rightarrow \infty \]

To find \( \lambda(r) , \nu(r) \) we have to use Maxwell's equations in empty space (with no currents)

\[ \mathcal{F}_\alpha^\beta = 0 \]  

(2.2)

together with Einstein's field equations for matter-free space

\[ R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -8\pi\sigma E_{\alpha\beta} \]  

(2.3a)

where

\[ \mathcal{F}_\alpha^\beta = \sqrt{-g} F_\alpha^\beta \]

\[ F_{\alpha\beta} = -F_{\beta\alpha} = (\Phi_{\alpha,\beta} - \Phi_{\beta,\alpha}) \]  

(2.4)

is the electromagnetic tensor, \( \Phi^\alpha \) is the generalized
potential, \( E^\alpha_\beta \) is the electromagnetic momentum-energy stress tensor given by
\[
E^\alpha_\beta = \frac{1}{4\pi} \left(- F^\alpha_\gamma F^\gamma_\beta + \frac{1}{2} \delta^\alpha_\beta F^{\mu\nu} F^\mu_\nu \right) \quad (2.5)
\]
and \( R^a_b \) is the Ricci tensor.

Because of (2.5) \( E^\alpha_\beta = 0 \) and, therefore, (2.3a) becomes
\[
R^a_b = -\delta^a_b E^\alpha_\beta \quad (2.3)
\]

Assuming \( \phi^\alpha \) to be a function of \( r \) only it follows from (2.4) that the only possible non-zero components of \( F^a_b \) would at most be \( F^2_2, F^3_3, F^4_4 \).

Using (2.1), (2.2) we get for \( F^2_2 \),
\[
\frac{\partial}{\partial r} \left( F^2_2 e^{-\frac{i}{2}(\lambda-\nu) \sin \theta} \right) = 0 \quad (2.6)
\]

This is integrated to
\[
F^2_2 = \text{constant} \cdot e^{\frac{i}{2}(\lambda-\nu)} \quad (2.6)
\]

with a similar expression for \( F^3_3 \).

It will be remembered that when the space-time is flat
\[ F^2_2 = H z , \quad F^3_3 = 4 y \]  

i.e. \( F^2_2, F^3_3 \) are components of the magnetic field which are zero in the case of a charged particle at rest in an inertial frame of reference. Therefore, since \( \lambda(r) \rightarrow 0, \quad \nu(r) \rightarrow 0 \) as \( r \rightarrow \infty \) i.e. the space-time (2.1) is approximately
flat for large \( r \) the constants occurring in (2.6) and in the similar expression for \( F_{31} \) must be zero. Therefore \( F_{12} = F_{13} = 0 \).

To obtain \( F_{\mu 1} \) we again use (2.1), (2.2). We get,

\[
\frac{\partial}{\partial x^\alpha} (\sqrt{-g} F^\mu{}_{\alpha}) = 0
\]

i.e.

\[
\frac{\partial}{\partial x'} (\sqrt{-g} g^{\mu'\nu'} F_{\nu 1}) = 0
\]

since all the other \( F_{\mu \nu} = 0 \).

This integrates to

\[
-e^{-\frac{i}{2} (\lambda + \nu)} F_{\mu 1} = \text{constant} \frac{1}{r^2}
\]

or

\[
F_{\mu 4} = -F_{41} = \text{Constant} \frac{1}{r^2} e^{\frac{i}{2} (\lambda + \nu)} \quad (2.7)
\]

At large distances \( r \)

\[
F_{\mu 4} = \text{Constant} \frac{1}{r^2}
\]

since \( \lambda \to 0 \), \( \nu \to 0 \),

and at these distances \( F_{\mu 4} \) must be the same as the \( F_{\mu 4} \) of special relativity, i.e. \( \varepsilon / r^2 \).

Therefore the constant occurring in (2.7a) must be equal to \( \varepsilon \), the electric charge of the electron, measured in ordinary e.s.u.

(2.7a) reduces, then, to

\[
F_{\mu 4} = -F_{41} = \frac{\varepsilon}{r^2} e^{\frac{i}{2} (\lambda + \nu)} \quad (2.7)
\]
With these values of $F_{\alpha \beta}$ (2.5) gives, for $E^\alpha_{\beta}$,

$$E_1 = -E_2 = -E_3 = E_4 = \frac{e^2}{\pi r^4}$$

(2.8)

Using (2.8) and (2.1) in (2.3) (for details see (19)) the only surviving equations are

$$\frac{e^2}{r^2} = -e^{-\lambda} \left( \frac{\nu' + \frac{1}{r^2}}{r^2} + \frac{1}{r^2} \right)$$

(2.9)

$$\frac{e^2}{r^2} = e^{-\lambda} \left( \frac{\nu'' - \frac{\lambda'}{r^2}}{r^2} + \frac{\nu' + \nu'' - \lambda'}{r^2} \right)$$

(2.10)

$$\frac{e^2}{r^2} = e^{-\lambda} \left( \frac{\lambda'}{r^2} - \frac{1}{r^2} \right) + \frac{1}{r^2}$$

(2.11)

where $\lambda' = \frac{d\lambda}{dr}$, $\nu' = \frac{d\nu}{dr}$ etc.

Subtracting (2.9) from (2.11) we get

$$\lambda' + \nu' = 0$$

and since we want $\lambda$, $\nu$ to tend to zero as $r \to \infty$ we must have

$$\lambda + \nu = 0$$

(2.12)

From (2.12) and (2.9) we get

$$\frac{e^2}{r^2} = -e^{-\lambda} \left( \frac{\nu' + \frac{1}{r^2}}{r^2} \right) + \frac{1}{r^2}$$
where
\[ y = e^\mu, \quad y' = \frac{ds}{dr} \]
The last differential equation can be written in the form
\[ \frac{d}{dr} (y r) = 1 - \frac{e^2}{r^2} \]
This can be integrated at once to give
\[ y = e^\mu = e^{-\lambda} = 1 - \frac{2\mu}{r} + \frac{e^2}{r^2} \quad (2.13) \]
where \( \mu \) is a constant of integration. This is the precise way in which \( \mu, e \) arise in the solution. It is easy to see that \( y \) satisfies all the three equations (2.9), (2.10), (2.11). Therefore the required solution is
\[ ds^2 = -y'^2 dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + y dt^2 \quad (2.14) \]
with
\[ y = 1 - \frac{2\mu}{r} + \frac{e^2}{r^2} \quad (2.13) \]

Discussion
We have said earlier that \( \mu \) is commonly accepted as the gravitational mass of the electron.
But as we have seen $\mu$ arises as a constant of integration and there is nothing in the derivation of (2.14) to suggest that $\mu$ is the mass of the electron. Without further investigation it is impossible to identify $\mu$. Because (2.14) reduces to the Schwarzschild field (chapter I, equation (4.1)) when we put $e=0$ it was assumed that $\mu$ must be the gravitational mass. But we are not at liberty to use this kind of argument, simply because $\mu$ can equally well be a constant of the form $\mu, \mu + ce^2$, where $\mu_1, \mu_2$ are constants not containing $e$ (such a relationship might be possible since $\mu$ and $e$ are constants of integration of a system of simultaneous differential equations). When we put $e=0$, again (2.14) reduces to the Schwarzschild field but now it is $\mu_2$ that has to be identified with the gravitational mass. We conclude that without further investigation $\mu$ remains completely unidentified, except for the almost intuitive notion that $\mu$ must be related with the (Newtonian) gravitational mass of the electron in some way.

Perhaps it will be useful to say, here, what we mean by the term "(Newtonian) gravitational mass" of a particle. In classical theory this term is usually taken to be a measure of the power of the particle in
causing a field of acceleration around it. We shall not attempt to give a precise definition of the above term in general relativity. It will be sufficient for our purpose to take \( m \), the constant appearing in the Schwarzschild exterior solution, to represent the "(Newtonian) gravitational mass" of a neutral particle.

Looking at the metric (2.13), (2.14) we see that, for large \( r \) (neglecting terms in \( \frac{1}{r^2} \)) the metric reduces to the Schwarzschild metric

\[
d s^2 = -\frac{c^2 r^2}{\sqrt{1 - \frac{2m}{r}}} - r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) + \left(1 - \frac{2m}{r}\right) dt^2
\]

In this sense, and only in this sense, we could call \( \mu \) the "(Newtonian) gravitational mass" of the electron. But as we used this term in the case of a neutral particle, we shall call \( \mu \) the "effective gravitational mass" of the electron, in the sense that at large distances has the same significance as \( m \).

On the other hand, as we have seen above, \( \epsilon \) can unquestionably be identified with the electric charge of the electron. In what follows we shall treat \( \epsilon \) as known while \( \mu \) as an (as yet) undetermined constant till we come to §4 where a new interpretation is given to \( \mu \).
§3. The electromagnetic energy of the field of a charged particle ("electron")

We have seen in §2 that the field associated with a charged particle, of charge $\xi$, is described by the spherically-symmetric static metric

$$ds^2 = -(1 - \frac{2\mu}{r} + \frac{\xi^2}{r^2})^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

$$+ (1 - \frac{2\mu}{r} + \frac{\xi^2}{r^2}) dt^2 \quad (3.1)$$

This holds for the matter-free space outside the electron.

We use Møller's theory summarized in chapter I (§3) to find the energy, $E_{AB}$, of the field (3.1) inside a 3-space $\mathcal{Q}$ given by $r = r_A, r = r_B, (r_A < r_B)$

From (I, (3.2)), (I, (3.4)) we get for the energy $E_{AB}$ inside $\mathcal{Q}$,

$$E_{AB} = \int_{\mathcal{Q}} \chi_{\mu\nu} \, dr \, d\theta \, d\phi \quad (3.2)$$

where we denote $(x^i, x^j, x^k, x^\ell) = (r, \theta, \phi, t)$

With $g_{\mu\nu}$ given by (3.1) the values of $\chi_{\mu\nu}$ given by (I, (3.3)) are:

$$\chi_{\ell r} = \frac{2\xi \mu}{\lambda} \left( \mu - \frac{\xi^2}{r^2} \right)$$

$$\chi_{\ell \ell} = \chi_{r r} = 0 \quad , \ (\chi_{\ell \ell} = 0) \quad (3.3)$$
Substituting in (3.2) we get,

\[ \mathcal{E}_{A B} = \frac{2}{K} \int \frac{\partial}{\partial r} \left( \mu - \frac{e^2}{r} \right) \sin \theta \, d\theta \, d\phi \]

\[ = -\frac{2}{K} \int \frac{\partial}{\partial r} \left( \frac{e^2}{r} \right) \sin \theta \, d\theta \, d\phi \quad (3.4) \]

\[ = -\frac{2e^2}{K} \int_{r_A}^{r_B} \int_0^{\pi} \int_0^{2\pi} \frac{\partial}{\partial r} \left( \frac{1}{r} \right) \sin \theta \, d\theta \, d\phi \]

\[ = e^2 \left( \frac{1}{r_A} - \frac{1}{r_B} \right) \quad (3.5) \]

where we have put \( K = 8\pi \).

It follows from (3.5) that the energy of the field is purely electromagnetic and that \( \mu \) (usually called the "gravitational mass" of the electron) contributes nothing to \( \mathcal{E}_{A B} \). Now when \( e = 0 \) (3.1) reduces to the Schwarzschild field (\( \mu \) not necessarily remaining unchanged); therefore when \( e = 0 \) (3.5) should give the energy of the Schwarzschild field. But we see that when \( e = 0 \), \( \mathcal{E}_{A B} = 0 \) for all \( r_A \), \( r_B \). This is in complete agreement with the results obtained in chapter I that the energy of the exterior Schwarzschild field is zero.
It is interesting to compare (3.5) with the corresponding classical electromagnetic energy $\mathcal{E}_c$. This is given by

$$\mathcal{E}_c = \frac{1}{8\pi} \int_{\Omega} (E^2 + H^2) \, d\Omega,$$

in the usual notation,

$$= \frac{1}{8\pi} \int_{\Omega} E^2 \, d\Omega,$$

since $H = 0$ in this case,

$$= \frac{1}{8\pi} \int_{r_A}^{r_B} \int_{\theta_A}^{\theta_B} \int_{\phi_0}^{2\pi} \frac{e^2}{r^4} \, r^2 \sin \theta \, dr \, d\theta \, d\phi,$$

since $|E| = \frac{e}{r^2}$

$$= \frac{1}{2} e^2 \left( \frac{1}{r_A} - \frac{1}{r_B} \right) \quad (3.6)$$

We see, therefore, that the energy in the two cases (3.5), (3.6) is the same apart from the factor 2. This is not surprising because the factor 2 turns up quite often in problems of this nature; as Papapetrou pointed out (24) "we must accept that the factor 2 expresses a fundamental property of the static electro-gravitational field". It may well be due to non-Maxwellian stresses of the field, (see, also, Bonnor (21), Callaway (25)).

If we let $r_B \to \infty$ we get from (3.5) that the energy $\mathcal{E}_{AB} \to \mathcal{E}_A$ where

$$\mathcal{E}_A = \frac{e^2}{r_A} \quad (3.7)$$
This is the total energy outside $r=r_A$.

We divert, now, from the main work of this chapter to discuss, on the basis of (3.7), the finiteness of the structure of the electron.

**On the finiteness of the structure of an electron**

The solution (3.1) has often been taken to represent a model of an (actual) electron by various authors (1), (2), (3) and on the basis of (3.1) the finiteness of the structure of the electron has been discussed. The arguments of these authors are either obscure or incomplete. Our formulation, above, when (3.1) is taken to represent an (actual) electron, offers a satisfactory theory on the finiteness of the structure of the electron.

It follows from the metric (3.1) that, for sufficiently large values of $E$ (satisfying the inequality $\frac{E^2}{\mu^2} > 1$), the field described by (3.1) has only one singularity at $r=0$. This seems to imply that an electron can exist with no finite structure (i.e. a mathematical point) and yet having a definite mass and charge. But, as Eddington remarked in his book (1), page 186) it is more likely that an electron is of finite structure in which case the solution (3.1) holds only outside this structure of "radius" $\alpha$, say. Then (1) "the total energy of the electromagnetic
field beyond this radius would be equal to the mass of the electron determined by observation. For this reason \( a \) is usually taken as the radius of the electron". Plausible as the above remarks by Eddington may be, it is not at all clear to the present author why "the total energy of the electromagnetic field beyond this radius would be equal to the mass of the electron .....". In any case Eddington proceeded in a rather obscure way to find that the radius \( a \) of the electron is related to the (Newtonian) gravitational mass, \( m \), of the electron by

\[
\alpha = \frac{e^2}{m} \quad (3.8)
\]

Eddington, of course, took it for granted that \( m \) appearing in (3.8) was identical to \( \mu \) appearing in (3.1). Yet his arguments leading to (3.8) demanded no relation at all between \( m \) and \( \mu \).

If we accept Eddington's remark that "the total energy of the electromagnetic field ..... would be equal to the mass of the electron ..... ", it follows directly from equation (3.7) that the energy \( E_A \) is equal to the gravitational mass \( m \) when \( r = \alpha \), where

\[
\alpha = \frac{e^2}{m}
\]

This is exactly the equation (3.8) given by Eddington. In our work it is an immediate consequence of equation (3.7).
Jeffery (22) in discussing the concept of "a point electron" takes as the radius, \( \alpha \), of the electron the value of \( r \) for which

\[
\gamma = 1 - \frac{2\mu}{r} + \frac{e^2}{r^2}
\]

takes the same value as its value at \( r = \infty \), i.e. the value \( \gamma = 1 \). It is easily seen that the value of \( r \) for which \( \gamma = 1 \) is given by

\[
r = \alpha = \frac{e^2}{2\mu}
\]

(3.9)

Jeffery interprets \( \mu \) as the (Newtonian) gravitational mass of the electron and gets the right order of magnitude for the radius \( \alpha \). We fail, however, to see any connection at all between the radius of the electron and the value of \( r \) for which \( \gamma \) takes the value 1, (the value of \( \gamma \) at \( r = \infty \)). As we said in the introduction we are going to give a different interpretation to \( \mu \) in §4. According to this interpretation equation (3.9) gives a negative value for the radius \( \alpha \) (assuming that the (Newtonian) gravitational mass \( m \) is positive). Accordingly, equation (3.9) is to be disregarded. The right order of magnitude which Jeffery's work gives for the value of the radius seems to be accidental.

Another reason for believing that the electron is of finite structure i.e. that the radius \( \alpha \) is finite (not necessarily equal to \( \frac{e^2}{\mu m} \)) is provided by
equation (3.7). If we assume that the radius of the electron is not finite (i.e. that the electron is a mathematical point) then the field (3.1) holds down to \( r = 0 \). It follows from (3.7), however, that the energy \( E_A \) tends to infinity as \( r_A \to 0 \). Since this is physically impossible we must postulate an infinite negative energy at the singularity \( r = 0 \), itself, to cancel the infinite positive energy \( E_A \) (as \( r_A \to 0 \)). We consider this unsatisfactory and we take the view that the electron is of finite structure, i.e. \( \alpha \) is non-zero. The field (3.1), then holds for \( r > \alpha \) only.

Our discussion does not give any explicit value for the radius \( \alpha \). It shows, however, in a very clear way that \( \alpha \) must be finite. The most our discussion can give is the exact relation that exists between the radius, \( \alpha \), of the electron and the total electromagnetic energy \( E_A \) of the field of the electron and is given by

\[
E_A = \frac{\varepsilon^2}{\alpha} \tag{3.10}
\]

Coming back to the main discussion of this chapter it follows from (3.7) that the total electromagnetic energy of a charged particle of radius \( \alpha \) is given by

\[
E_A \to E = \frac{\varepsilon^2}{\alpha} \tag{3.11}
\]
§4. Behaviour of a neutral test particle in the field of an electron

We have now come to an important point of our work; the interpretation of \( \mu \) occurring in the metric describing the field of an electron. This we shall do by comparing the behaviour of a neutral test particle moving in the field of an electron and at large distances from it with the corresponding behaviour in classical theory.

The world line of a neutral test particle moving in a gravitational field in empty space is a geodesic. This is no longer an additional postulate of the theory of relativity. It is one of the greatest achievements of relativity theory that the geodesic motion of a neutral test particle, and the equations of motion of any system of massive particles in general, follow from the field equations of the theory. Several proofs of the geodesic motion of a neutral test particle in empty space were given in many papers (1), (26), (27), (28), (29). In this section we are only interested in the motion of a neutral test particle moving in an electro-gravitational field. It is by no means obvious that the world-line of such a particle is a geodesic, because from the point of view of relativity theory
the space is not empty. Energy (electromagnetic), of density \( \frac{e^2}{4\pi r^4} \), as seen from (3.4), is spread throughout the space surrounding the electron. The work of Robertson (26), however, can be extended to give the motion of a test particle in an electro-gravitational field. In fact it is almost a corollary of Robertson's work (see also section 5) that the world-line of a neutral test particle moving in an electro-gravitational field in matter-free regions is a geodesic.

The motion of a charged particle (which is not a geodesic) has been examined in great detail by Jeffery (22). Although his work has nothing to do with the present discussion we shall show in §5 that his results, taken to a higher approximation than the one Jeffery took, supports our work in this chapter.

We shall now find the geodesics of the space-time described by the metric

\[
 ds^2 = - \frac{dr^2}{\gamma} - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) + \gamma dt^2
\]  

(4.1)

\[
 \gamma = 1 - \frac{2\mu}{r} + \frac{e^2}{r^2}
\]

(4.2)

As we remarked above these geodesics are the world-lines of neutral test particles in the field of an electron described by (4.1), (4.2).
Write
\[ 2T = -\frac{\dot{r}^2}{y} - r^2(\dot{\phi}^2 + \sin^2 \theta \dot{\phi}^2) + \dot{r} \dot{t}^2 \]  \tag{4.3}

where
\[ \dot{x}^\alpha = \frac{dx^\alpha}{ds}, \quad (\alpha \neq 0) \]
\[ \alpha = 1, 2, 3, 4 \quad \text{and} \quad x^\alpha = (r, \theta, \phi, t) \]

Then the geodesics are given by the ordinary (non-null) Lagrange-Euler equations
\[ \frac{d}{ds} \left( \frac{\partial T}{\partial \dot{x}^\alpha} \right) - \frac{\partial T}{\partial x^\alpha} = 0 \]  \tag{4.4}

These equations possess the first integral given by
\[ -\frac{\dot{r}^2}{y} - r^2(\dot{\phi}^2 + \sin^2 \theta \dot{\phi}^2) + \dot{r} \dot{t}^2 = 1 \]  \tag{4.5}

For \( \alpha = 2 \), (4.4), (4.3) give
\[ \frac{d}{ds} (r^2 \dot{\theta}) - r^2 \sin \theta \cos \theta \dot{\phi}^2 = 0 \]

Therefore, without loss of generality, we can consider the motion in the hyper-plane \( \theta = \frac{\pi}{2}, \dot{\theta} = 0 \).

Then the other equations (4.4) corresponding to \( \alpha = 1, 3, 4 \) respectively, can be reduced to
\[ \frac{d}{ds} \left( \frac{\dot{r}}{y} \right) + \frac{\dot{r}^2}{y^2 r} (\mu - \frac{\dot{\phi}^2}{r}) - r \ddot{\phi}^2 + \frac{\dot{\phi}^2}{r^2} (\mu - \frac{\dot{r}^2}{r}) = 0 \]  \tag{4.6}
\frac{d}{ds}(r^2 \dot{\phi}) = 0 \quad (4.7a)

\frac{d}{ds}(\nu \dot{\epsilon}) = 0 \quad (4.8a)

together with the first integral
\[ -\frac{r^2}{r} - r^2 \dot{\phi}^2 + \nu \dot{\epsilon}^2 = 1 \]

Equations (4.7a), (4.8a) give, on integration,
\[ r^2 \dot{\phi} = \kappa \]
\[ \nu \dot{\epsilon} = \kappa \]

where \( \kappa \), \( \kappa \) are constants.

From (4.7), (4.8), (4.9) and writing \( \nu = \frac{1}{r} \) we get the equation
\[ \left( \frac{du}{d\phi} \right)^2 + u^2 = \frac{k^2 - 1}{h^2} + \frac{2\mu}{h^2} u - \frac{e^2}{h^2} u^2 + 2\mu u^3 - e^2 u^4 \]

which after differentiation w.r.t. \( u \) reduces to
\[ \frac{d^2 u}{d\phi^2} + u = \frac{1}{h^2} \left( \mu - e^2 u \right) + 3\mu u^2 - 2e^2 u^3 \] \( (4.11) \)

This is the general differential equation of the worldline of the neutral test particle. As a check, (4.11) reduces exactly to the corresponding equation in a Schwarzschild field when we put \( e = 0 \) (see, for example (1)). (The advance of the perihelion implied by (4.11)
has also been worked out but it is irrelevant in the present discussion.

To be able to compare the relativistic behaviour of the neutral test particles with the behaviour in the corresponding classical case we consider a particle at rest in the 3-space $t = \text{constant}$, in the coordinate system $(r, \theta, \phi, t)$, and at large distances from the origin.

With \( \frac{dr}{ds} = \frac{d\phi}{ds} = 0 \) equations (4.6), (4.9) give

\[
\frac{\dot{r}^2}{r^2} + \frac{\dot{t}^2}{r^2} = 0
\]

(4.12)

Substituting \( \dot{t}^2 = 1 \)

(4.13)

Combining (4.12), (4.13) we get

\[
\frac{d^2r}{dt^2} = -\frac{\dot{r}}{r^2} \left( \mu - e^2 r^{-1} \right)
\]

(4.14)

Substituting \( \dot{r} = 1 - \frac{2}{r} + \frac{e^2}{r^2} \) equation (4.14) reduces, for large \( r \), to

\[
\frac{d^2r}{dt^2} = -\frac{1}{r^2} \left( \mu - e^2 r^{-1} \right)
\]

(4.15)

where terms of the order \( \frac{\mu}{r^3} \), \( \frac{\mu \varepsilon^2}{r^4} \) and higher have been neglected.

For these values of \( r \) the 3-space
constant is very nearly flat and the parameter appearing in equation (4.15) becomes the same as the Newtonian universal time. For these values of \( r \) (4.15) must, therefore, be the same as the corresponding classical formula.

Before discussing the classical analogue of the above problem it will be useful to consider briefly the "circular" geodesics of the metric (4.1). We shall compare these with the corresponding circular paths of a neutral test particle in the corresponding classical problem. Equation (4.6), with \( r = 0, r = \rho \), say, where \( \rho \) is a constant, gives

\[ \rho^2 \dot{\phi}^2 = \frac{1}{\rho} \left( \mu - \frac{e^2}{\rho} \right) \dot{t}^2 \]  

\( \dot{\phi} \) as given by (4.16) is easily seen to be consistent with the remaining equations (4.7), (4.8); therefore the solution (4.16) describes a possible "circular" motion. Dividing both sides of (4.16) by \( \dot{t}(\neq 0) \) we get

\[ \rho \left( \frac{d\phi}{dt} \right)^2 = \frac{1}{\rho^2} \left( \mu - \frac{e^2}{\rho} \right) \]  

(4.17)

Again, for large values of \( \rho \) this must be identical with the corresponding classical equation.
The classical analogue

We see at once that there is no strictly classical problem corresponding to the above relativistic problem. In the relativistic case the neutral test particle moves under the influence of a gravitational mass at the spatial origin of the coordinates and under the influence of the electromagnetic energy, of density $\frac{e^2}{\gamma^2}$, of the field. This is clearly seen from the equations (4.3) to (4.17) describing the motion of the test particle. The result that the electromagnetic energy does influence the behaviour of the neutral test particle is only to be expected in relativity theory because of the equivalence of energy and mass. No such equivalence exists between energy and mass in classical theory, and, therefore, no strictly classical problem corresponds to the relativistic problem. To construct a classical problem corresponding to the relativistic one we shall assume the equivalence of energy and mass to be valid in classical theory as well. This is not an unreasonable assumption in view of the wide acceptance of the equivalence of energy and mass. A similar assumption was made by McCrea in generalizing the Newtonian models of the universe in cosmology (30). The classical problem
then, would be: Determine the motion of a test particle moving under the influence of a spherical central body of (Newtonian) gravitational mass \( m \) and radius \( \alpha \), say, together with a mass distribution of density \( e^{\nu} \) extending throughout space from \( r=\alpha \) to \( r=\infty \).

We shall first find the gravitational intensity of the field at a point \( P \) distance \( r \) from the origin. Considering the mass distribution only we shall find the intensity at \( P(r) \) due to the mass distribution inside the sphere of radius \( b_1 \), say, where \( b_1<r \) and the intensity at \( P(r) \) due to the mass distribution outside the sphere of radius \( b_2 \), say, where \( b_2>r \) (at the end \( b_1, b_2 \) will both be put equal to \( r \)). To do this we first find the potential at \( P(r) \).

**Case I \( r>b_1 \).**

We choose the spherical polar coordinates \((r, \theta, \phi)\) (with pole at the origin) so that \( P \) lies on \( \theta=0 \). Then the potential at \( P \) due to a volume element \( dV \) at \((r, \theta, \phi)\) is

\[
\frac{e^2 dV}{4 \pi \rho^4 R} = \frac{e^2 \sin \theta d\rho d\theta d\phi}{4 \pi \rho^2 R}
\]

where \( R^2 = r^2 - 2rp \cos \theta + \rho^2 \) (see figure 1)

Therefore the total potential, \( V_i(r) \), at \( P(r) \) is given by

\[
-V_i(r) = \frac{e^2}{4 \pi} \int_{\alpha}^{b_1} \int_{0}^{\pi} \int_{0}^{2\pi} \frac{\sin \theta d\rho d\theta d\phi}{\rho^2 [r^2 - 2rp \cos \theta + \rho^2]^{\frac{3}{2}}}
\]
Classical Analogue.

Case I \((r > b_1)\)

Case II \((b_2 > r)\)

Figure 1.
Integrating w.r.t. $\phi$ and $\theta$ first we get

$$-V_i(r) = \frac{e^2}{2r} \int_{\alpha}^{b_f} \left( \frac{r^2 - 2r^2 \cos \theta + r^2 \theta}{\rho^2} \right) \frac{d\theta}{\rho^2} \varphi$$

Therefore we get for the potential $V_i(r)$

$$-V_i(r) = \frac{e^2}{r} \left( \frac{1}{a} - \frac{1}{b_i} \right)$$

(4.A)

By a similar integration we find that the mass, $M$, between $r = a$, $r = b_i$ is given by

$$M = e^2 \left( \frac{1}{a} - \frac{1}{b_i} \right)$$

(4.B)

From (4.A), (4.B) we get the usual result that

$$-V_i(r) = \frac{M}{r}$$

From (4.A) we see that the intensity, $I$, of the field at $P(r)$ is given by

$$I = -\frac{\partial V_i}{\partial r} = -\frac{e^2}{r^2} \left( \frac{1}{a} - \frac{1}{b_i} \right)$$

(4.C)

Case II $b_2 > r$

Choosing the coordinates as before we get for the total potential, $V_2(r)$, at $P(r)$ due to the mass distribution outside $b_2 (> r)$.
Therefore

\[ V_2(r) = \frac{e^2}{2b_2^2} \]  

(4.18)

It follows from (4.18) that \( V_2 \) is independent of \( r \) and, therefore, the intensity due to \( V_2 \) at \( P \) is zero. This, again, is the result we should expect.

Taking \( b_v = r \) we conclude that the intensity at \( P \) due to the mass distribution only is

\[ \frac{e^2}{r^2} \left( \frac{1}{a} - \frac{1}{r} \right) \]

Therefore the total intensity of the field at \( P \) due to the mass \( m \) and the mass distribution of density \( \frac{e^2}{4\pi r^2} \) is given by

\[ \frac{m}{r^2} + \frac{e^2}{r^2} \left( \frac{1}{a} - \frac{1}{r} \right) \]  

(4.19)

and acts towards the origin.

Therefore its acceleration at distance \( r \) is given by

\[ \frac{d^2r}{dt^2} = \left\{ \frac{m}{r^2} + \frac{e^2}{r^2} \left( \frac{1}{a} - \frac{1}{r} \right) \right\} \]

(4.19)

In the case of circular orbits of radius \( r = \rho \), say,
described under the influence of the force given by

\[ p \left( \frac{d\phi}{dt} \right)^2 = \frac{m}{\rho^2} + \frac{e^2}{\rho^2} \left( \frac{1}{\alpha} - \frac{1}{\rho} \right) \]  

\[ (4.20) \]

where \( \frac{d\phi}{dt} \) is the angular velocity of the particle.

We are now in a position to compare the relativistic results with the classical ones and thus identify \( \mu \). We remarked above that at large distances from the electron the two results must be identical.

A comparison of \((4.15)\) with \((4.19)\) gives

\[ \frac{\mu}{r^2} - \frac{e^2}{r^3} = \frac{m}{r^2} + \frac{e^2}{r^2} \left( \frac{1}{\alpha} - \frac{1}{r} \right) \]

It follows, therefore, that

\[ \mu = m + \frac{e^2}{\alpha} \]  

\[ (4.21) \]

It will be remembered that equation \((4.15)\) was obtained \((4.14)\) by neglecting terms of the order \( \mu^2/r^3 \), \( \mu e^2/r^4 \). Another, exact, relation between, \( \mu \), \( e \) and \( m \) is obtained by comparing the circular paths of the particle in the relativistic and classical case.

A comparison of \((4.17)\) with \((4.20)\) gives the exact relationship

\[ \frac{\mu}{\rho^2} = \frac{e^2}{\rho^3} = \frac{m}{\rho^2} + \frac{e^2}{\rho^2} \left( \frac{1}{\alpha} - \frac{1}{\rho} \right) \]
which gives

\[ \mu = m + \frac{e^2}{\alpha} \]

, as in (4.21).

Although the relation obtained in this way is exact (i.e. it holds for all \( r \) and \( t \)) the comparison has meaning only when we take \( r \) to be large enough. For these values of \( r \) the space-time is flat and \( t \) becomes the same as the Newtonian time. Yet another way of obtaining a relation between \( \mu \), \( e \) and \( m \) is to compare the general differential equation (4.11), the world-line of a neutral test particle, with the corresponding equation in our classical example. We have shown that, in our classical example, the test particle, at distance \( r \), moves under a central force given by (4.18), i.e.

\[ \frac{m}{r^2} + \frac{e^2}{r^2} \left( \frac{1}{\alpha} - \frac{1}{r} \right) \]

Hence the classical orbit of the test particle is given by the differential equation

\[ \frac{d^2 \mu}{d\phi^2} + \mu = \frac{1}{\hbar^2} \left[ m + e^2 \left( \frac{1}{\alpha} - \mu \right) \right] \]

At large distances from the electron, where the 3-space is approximately flat, equation (4.11) must be identical to the above equation. Neglecting terms in \( \mu^2 \) and of higher order in (4.11) and comparing it with the
above equation we get
\[ \mu - e^2 u = m + \frac{e^2}{\alpha} - e^2 \]
which leads to the relation \( \mu = m + e^2 \alpha \) which is precisely equation (4.21).

We have proved in §3 that the total electromagnetic energy of the field of a charged particle, charge \( e \), radius \( \alpha \) is \( e^2 \alpha \) (equation (3.11)). We come, therefore, to the important conclusion that
\[ \mu = (\text{Newtonian}) \text{ gravitational mass} + \text{mass-equivalence of the electromagnetic energy}. \]

Contrary to the currently accepted ideas that the electric charge \( e \) contributes nothing to the Newtonian gravitational mass of the particle, we find that this is no longer true. What was previously called the Newtonian gravitational mass we now call the effective gravitational mass of the electron and it is the sum of the Newtonian gravitational mass (i.e. when \( e = 0 \)) of the central particle and the mass-equivalence of the electromagnetic energy due to the charge \( e \).

Substituting for \( \mu = m + \frac{e^2}{\alpha} \) in (4.1) we get
\[
\begin{align*}
\text{d}s^2 &= - \left[ 1 - \frac{2}{r} \left( m + \frac{e^2}{\alpha} \right) + \frac{e^2}{r^2} \right] \text{d}r^2 - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \\
&\quad + \left[ 1 - \frac{2}{r} \left( m + \frac{e^2}{\alpha} \right) + \frac{e^2}{r^2} \right] \text{d}t^2 \tag{4.22}
\end{align*}
\]
When \( e = 0 \) (4.22) reduces to the Schwarzschild field

\[
ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2) + \left(1 - \frac{2m}{r}\right)dr^2
\]

(4.23)

where we know, now, that \( m \) is, indeed, the (Newtonian) gravitational mass of the central (neutral) particle.

Results following from the new interpretation of \( \mu \)

The new interpretation of \( \mu \) enables us to answer some puzzling questions which could not satisfactorily be answered on the basis of the old interpretation that \( \mu \) is the "gravitational mass" of the electron. I. "Since \( \mu \) and \( \varepsilon \) are arbitrary independent constants of integration why can we not put \( \mu \) equal to zero?" Indeed, in the usual interpretation of \( \mu \) there is no reason at all why \( \mu \) cannot be put equal to zero. Our new interpretation shows that \( \mu \) cannot be zero. For, it follows from (4.21) that \( \mu \) is zero only when

\[m = -\frac{\varepsilon^2}{\alpha}\]

Since \( \varepsilon \) is real and \( \alpha \) is always positive, \( m \), the gravitational mass, must be negative. According to the usual outlook this is impossible; the existence of negative mass is far from
being demonstrated. It follows, therefore, that $\varphi$ cannot be put equal to zero.

II. In discussing the finiteness of the structure of an electron we mentioned (on page 51) that Jeffery took as the radius of the electron the value of $r$ for which $\nu = 1 - \frac{2e^2}{r^2}$ takes the same value as its value at $r = \infty$. i.e. Jeffery took for the radius $\alpha$ the value (see (3.9))

$$\alpha = \frac{e^2}{2\mu}$$

(3.9)

According to the new interpretation of $\mu$ (3.9) gives

$$\alpha = -\frac{e^2}{2m} = \frac{e^2}{2(m + e^2a^2)}$$

Therefore $\alpha = -\frac{e^2}{2m}$. Again, $e$ is real and $\alpha$ is always positive. This relation demands that $m$ be negative. This we reject as before and we consider the relation (3.9) as meaningless, even if it gives, accidentally, a value for which $\alpha$ is of the right order of magnitude.

III. Another interesting observation that can be made is the following: In the new interpretation of $\mu$ if we put $m$, the (Newtonian) gravitational mass of the
electron, equal to zero all the equations describing the
behaviour of a neutral test particle (equations (4.3)
to (4.20)) give perfectly logical and physical results.
Also the electromagnetic energy remains the same as
before (as we have seen the energy is independent of \(m\)).
This seems to imply the possibility of having a charged
particle with zero (Newtonian) gravitational mass.
However, this is all our present theory can say about
this possibility.

\[\text{The equation (5.1) is given by (see}
\text{eq. (28), page 260)}\]

\[
\frac{\gamma}{\lambda^2} + \left( \frac{\bar{v}}{\lambda} \right) \frac{d\gamma}{ds} \frac{dx^\alpha}{ds} + \frac{\gamma}{\lambda^2} \gamma^\alpha \frac{dx^\alpha}{ds} = 0 \quad (5.1)
\]

\(\gamma^\alpha \gamma_\alpha\) is the electromagnetic tensor.

(5.1) is a tensor equation and it reduces to the Lorentz
equation of motion (11).

\[
\frac{d\gamma}{ds} = \left[ \hat{E} + (\gamma \times \hat{H}) \right]
\]

In space-time is flat. \(\hat{E}\) is the electric vector,
\(\hat{H}\) the magnetic vector, and \(\gamma\) the
relativistic momentum of the particle. The equation (5.1) was, at the early
stage of relativity theory, an additional postulate of
the theory. It was employed because of its tensorial
character and because it reduces, in a flat space-time,
to the Lorentz force. It must be emphasised, however,
§5. Discussion of Jeffery's work

We give in this section a brief discussion of Jeffery's work (23) on the motion of a charged particle (mass $M$, charge $e$) moving in the field of a charged particle described by $(4.1)$, $(4.2)$.

It is well known that the motion of such a particle (when its influence on the background field in which it is moving is neglected) is given by (see e.g. (20), page 260)

$$\frac{d^2 x^\nu}{ds^2} + \left\{ \frac{\nu}{\alpha \beta} \right\} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} - \frac{e}{M} F^\nu_{\alpha \beta} \frac{dx^\alpha}{ds} = 0 \quad (5.1)$$

where $F_{\alpha \beta}$ is the electromagnetic tensor.

$(5.1)$ is a tensor equation and it reduces to the Lorentz equation of motion $(11)$,

$$\frac{dp}{dt} = e \left[ E + (v \times H) \right]$$

when the space-time is flat. $E$ is the electric vector, $v$ the velocity of the test particle, $H$ the magnetic vector and $p$ the relativistic momentum of the particle. The equation $(5.1)$ was, at the early stages of relativity theory, an additional postulate of the theory. It was employed because of its tensorial character and because it reduces, in a flat space-time, to the Lorentz force. It must be emphasized, however,
that equation (5.1) is no longer an additional postulate of the theory. Robertson (26) has proved, as early as 1933, that equation (5.1) is a consequence of the field equations of the theory.

Incidentally it follows from (5.1) that when the test particle is neutral i.e. $\xi = 0$ (5.1) reduces to the geodesic equation of the metric (4.1). This is in accordance with what we remarked in $\xi$ 4.

Jeffery found in great detail the general equations of motion as given by (5.1), which, for a charged particle at rest in the coordinate system used,

$$\frac{dr}{ds} = \frac{d\theta}{ds} = \frac{d\phi}{ds} = 0,$$

reduce to the simple equations

$$\frac{d^2r}{ds^2} + \frac{1}{r^2} \gamma' \left(\frac{dt}{ds}\right)^2 = \frac{e\xi}{Mr^2} \gamma \frac{dt}{ds}$$

(5.2)

$$\gamma' \left(\frac{dt}{ds}\right)^2 = 1$$

(5.3)

where

$$\gamma' \equiv \frac{dy}{dr} = \frac{2}{r^2} \left(\mu - \frac{e^2}{r}\right)$$

(5.4)

Eliminating $\frac{dt}{ds}$ from (5.2), (5.3) we get

$$\frac{d^2r}{dt^2} = -\frac{1}{2} \gamma' \gamma' + \frac{e\xi}{Mr^2} \gamma'^{3/2}$$

(5.5)

At large distances from the central electron (5.4), (5.5) give

$$M \frac{d^2r}{dt^2} = -\frac{\mu M}{r^2} + \frac{e\xi}{r^2}$$

(5.6)
where terms of order higher than \( r^{-2} \) are neglected. From (5.6) Jeffery deduced that \( \mu, e \) are the "gravitational mass" and the charge of the central particle respectively.

By expanding (5.5) to the same order of magnitude in \( \frac{1}{r} \) as we have expanded equation (4.14) however, we obtain

\[
\mathcal{M} \frac{d^2 r}{dt^2} = - \frac{\mathcal{M}}{r^2} (\mu - \frac{e^2}{r}) + \frac{e e}{r^2}
\]

(5.7)

The term \( \frac{e e}{r^2} \) denotes the electromagnetic interaction while the term \( -\frac{1}{r^2}(\mu - \frac{e^2}{r}) \) denotes the gravitational interaction per unit mass (of the test particle).

If we compare this gravitational interaction with the corresponding gravitational interaction of the classical problem discussed in §4, i.e. with equation (4.18) we get

\[
\mu = m + e^2 a^{-1}
\]

This is precisely the expression we obtained in §4 (4.21).

It is to be emphasized, however, that Jeffery's paper was concerned only with the motion of a charged particle in the electro-gravitational field of an electron and had nothing at all to do with our present work.
The object of this section is to show that Einstein's theory on momentum-energy gives the right result for the energy of the electrogravitational field of a charged particle if, and only if, \( \mu \), the constant appearing in the metric (3.1), is given the interpretation we proposed in §4 i.e.,

\[
\mu = m + e^2a^{-1} \tag{6.1}
\]

We start with the metric (3.1) describing the field of a charged particle,

\[
ds^2 = -\left(1 - \frac{2\mu}{r} + \frac{e^2}{r^2}\right)dt^2 - \frac{r^2}{\left(1 - \frac{2\mu}{r} + \frac{e^2}{r^2}\right)}(d\phi^2 + e^2 \sin^2 \theta d\phi^2) + \left(1 - \frac{2\mu}{r} + \frac{e^2}{r^2}\right)dr^2 + r^2(d\phi^2 + \sin^2 \theta d\phi^2) \tag{6.2}
\]

It is easily seen from (6.2) that Einstein-Klein conditions (2.10) are not satisfied. Hence Einstein's theory is not applicable in this coordinate system.

If we apply the transformation (see Møller (6))

\[
\frac{dF}{r} = \frac{1}{(1 - \frac{2\mu}{r} + \frac{e^2}{r^2})^{\frac{1}{2}}} \frac{dr}{r} \tag{6.3}
\]

then the metric (6.2) transforms to the isotropic form

\[
ds^2 = -\frac{r^2}{r'^2} \left(\frac{dF'^2}{F'^2} + F'^2d\phi^2 + r'^2 \sin^2 \theta d\phi^2\right) + \left(1 - \frac{2\mu}{r} + \frac{e^2}{r^2}\right)dt^2
\]
Changing to "cartesian coordinates" \((x,y,z)\) this reduces to

\[
\begin{align*}
    ds^2 &= -\frac{r^2}{\nu^2} \left( dx^2 + dy^2 + dz^2 \right) + \left( 1 - \frac{2m}{r} + \frac{\nu^2}{r^2} \right) dt^2 \tag{6.4}
\end{align*}
\]

Integrating (6.3) we get, after some rearrangement, the explicit transformation

\[
    r = \frac{c^2 - \frac{C}{r}}{\nu^2} \left\{ \left( 1 + \frac{\mu}{c^2} \right)^2 - \frac{\epsilon^2}{\nu^2 c^2} \right\} \tag{6.5}
\]

where \(C\) is a constant. We shall choose \(C\) so that when \(\epsilon = 0\) (6.4) reduces to the Schwarzschild field and (6.5) to the corresponding equation

\[
    r = \left( 1 + \frac{\mu}{\nu^2} \right)^2 \frac{c^2}{\nu^2} \tag{6.6}
\]

Therefore \(\mu\) must be equal to \(\frac{\epsilon^2}{\nu^2}\). (6.5), then, reduces to

\[
    r = \frac{c^2}{\nu^2} \left\{ \left( 1 + \frac{\mu}{\nu^2} \right)^2 - \frac{\epsilon^2}{\nu^2 c^2} \right\} \tag{6.6}
\]

Equations (6.6), (6.4) give, in the \(F\)-system,

\[
    ds^2 = -\alpha(F) \left( dx^2 + dy^2 + dz^2 \right) + \beta(F) \, dt^2 \tag{6.7}
\]

where

\[
    \alpha(F) = \left[ \left( 1 + \frac{\mu}{\nu^2} \right)^2 - \frac{\epsilon^2}{\nu^2 c^2} \right]^2 \tag{6.8}
\]

\[
    \beta(F) = \frac{\left[ 1 - \frac{\mu^2}{\nu^2 c^2} + \frac{\epsilon^2}{\nu^2} \right]^2}{\left[ \left( 1 + \frac{\mu}{\nu^2} \right)^2 - \frac{\epsilon^2}{\nu^2 c^2} \right]^2} \tag{6.9}
\]
As a check, when $\varepsilon = 0$ (6.7) reduces to the Schwarzschild metric (4.9) of chapter I.

Differentiating (6.8) w.r.t. $\bar{r}$ we get

$$\frac{d\alpha}{d\bar{r}} = -\frac{2}{\bar{r}^2} \left[ \left( 1 + \frac{\mu}{2\bar{r}} \right)^2 - \frac{\varepsilon^2}{4\bar{r}^2} \right] \left[ \mu \left( 1 + \frac{\mu}{2\bar{r}} \right) - \frac{\varepsilon^2}{2\bar{r}} \right]$$ (6.10)

It is easily seen from (6.8), (6.9) that

$$\alpha(\bar{r}) = 1 + O\left( \frac{1}{\bar{r}^2} \right)$$

$$\beta(\bar{r}) = 1 + O\left( \frac{1}{\bar{r}^2} \right)$$

and

$$\frac{d\alpha}{d\bar{r}} = O\left( \frac{1}{\bar{r}^2} \right), \quad \frac{d\beta}{d\bar{r}} = O\left( \frac{1}{\bar{r}^2} \right)$$

Hence Einstein-Klein conditions (2.10) of chapter I are satisfied. Therefore we can apply Einstein's theory to find the total energy of (6.7). From (2.16), (2.19) of chapter I we get, for the energy $E$,

$$E = \lim_{\bar{r} \to \infty} - \frac{1}{k} \int_{S} \frac{d\alpha}{d\bar{r}} \left( \frac{\beta}{\alpha} \right)^{\prime 2} \bar{r}^{-2} \sin \theta \, d\theta \, d\phi$$

Using equations (6.8), (6.9), (6.10) this reduces to

$$E = \lim_{\bar{r} \to \infty} \frac{8\pi}{k} \left[ \frac{\mu (1 + \frac{\mu}{2\bar{r}}) - \frac{\varepsilon^2}{2\bar{r}^2}}{1 - \frac{\mu^2}{4\bar{r}^2} + \frac{\varepsilon^2}{4\bar{r}^2}} \right]$$

$$= \frac{8\pi}{k} \mu$$

since $k = 8\pi$. 

(6.11)
That is, the total electro-gravitational energy of the field is equal to $\mu$. According to the old interpretation that $\mu = m$, the (newtonian) gravitational mass, (6.11) states that the electric charge $\varepsilon$ contributes nothing to the total energy of the field. This is clearly absurd. Equation (6.11) is sufficient to show most definitely that the interpretation that $\mu$ is equal to the (newtonian) gravitational mass is wrong. It makes nonsense of Einstein's theory and our physical conception about energy.

According to our new interpretation of $\mu$ (6.11) would read

$$E = m + \frac{\varepsilon^2}{\alpha}, \left[ \frac{\varepsilon^2}{\alpha} = \text{Electromagnetic energy} \right]$$  

(6.12)

for the total energy of the field. When $\varepsilon = 0$ (6.12) reduces to $E = m$ in agreement with the results of chapter I.

As far as the present writer knows the result expressed by equation (6.11) has not been noticed before. It gives the final blow to the old interpretation of $\mu$ and gives the best possible support of our theory. Besides the physically well-defined results that we have obtained the whole logical structure of our work must be a definite indication of the correctness of our new interpretation of $\mu$. 
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PART II

Energy in plane gravitational waves

of finite duration.
PART II

Energy in plane gravitational waves of finite duration

1. Gravitational waves is a purely relativistic concept. There is no classical analogue. The study of gravitational waves originated by Einstein (1) as early as 1918 in the approximate solution of the Einstein vacuum field equations for weak fields. Exact cylindrical gravitational waves were investigated by Einstein and Rosen (2), Rosen (3) and finally solved by Marder (4), (5). Plane gravitational waves, with which we shall be concerned in this part of the thesis, were also investigated by Einstein and Rosen (2), Rosen (6) but it was not till 1957, 1959 when Bondi (7), Bondi, Pirani and Robinson (8) gave the final solution to this problem. The final solution for plane gravitational waves was made possible only through the work of Lichnerowicz (9) on the regularity conditions which the coordinate system and the metric tensor must satisfy if the solution is to have any physical meaning.

In the recent paper by Bondi, Pirani and Robinson (8) where plane gravitational waves are investigated in great detail, plane gravitational waves are defined as follows:
A plane wave metric

(A) is a non-flat solution of the empty space-time field equations \( R_{\mu\nu} = 0 \).

(B) admits a 5-parameter group of motions of the space-time into itself.

Condition (B) is demanded by analogy from the invariant properties of plane electromagnetic waves under certain groups of transformations. (8).

The important question which immediately arises is whether plane gravitational waves carry energy. For a quantitative answer use must be made of the momentum-energy tensor and in particular the formulae (1.12) in Einstein's theory, or (3.4) in Möller's theory (see chapter I) for the total momentum-energy. But for reasons stated in chapter I (part I) and the end of §3 of this chapter the formulation of momentum-energy as given in chapter I seems inapplicable in the case of plane gravitational waves.

It seems that no quantitative answer can be given at present. Bondi, Pirani and Robinson (B.P.R. for short) have shown (8) that a qualitative answer can be given to this problem by considering plane gravitational waves of finite duration (Sandwich waves). They have shown that the passage of such a wave produces a relative acceleration in freely moving test particles.
This effect can, in principle, be used to extract energy from the wave. Although this shows that the waves must carry energy it gives no answer as to the amount of energy they carry. This relative acceleration of the test particles set up by the passage of the wave was obtained by the above authors by using the theory of continuous groups of motions. In this chapter we shall obtain the same results as the above authors from a more elementary standpoint using only the geodesic equations. We do not claim that our method is logically simpler than the one given by B.P.R. but we do claim that it is mathematically easier and physically clearer.

In §2 we describe the metric representing plane gravitational waves of finite duration. In §3 we obtain the relative velocity of two freely moving particles produced by the passage of the wave. The chapter ends with a discussion of these results and the results obtained by Möller (10).
2. Metric of the plane gravitational wave of finite duration

For the simplest solution satisfying conditions (A) and (B) of §1 B.P.R. found the metric

\[ ds^2 = -e^{2\phi} d\xi^2 - u^2 \left( e^{2\beta} d\eta^2 + e^{-2\beta} d\xi^2 \right) + e^{2\phi} du^2 \]  \hspace{1cm} (2.1)

where

\[ u = \tau - \xi \]
\[ \phi = \phi(u) \]
\[ \beta = \beta(u) \]  \hspace{1cm} (2.2)

where \( \phi(u), \beta(u) \) are arbitrary functions of \( u \).

If we calculate the Ricci tensor \( R_{\mu\nu} \) for the metric (2.1) we find that the only non-identically vanishing components are given by

\[ R_{\mu\mu} = 2 \left( \beta' - \frac{2\phi'}{u} \right) \]
\[ = R_{\mu\mu} = -R_{14} \]  \hspace{1cm} (2.3)

where \( \phi' = \frac{d\phi}{du} \), \( \beta' = \frac{d\beta}{du} \).

It follows from (2.3) that the empty space-time field equations \( R_{\mu\nu} = 0 \) is satisfied if \( \phi, \beta \) satisfy the equation

\[ 2\phi' = u\beta'^2 \]  \hspace{1cm} (2.4)
But if the solution (2.1) is to represent a non-flat solution of the equations $R_{\mu \nu} = 0$, $\phi$, $\beta$ must satisfy a further condition which is obtained by considering the components of the Riemann tensor $R_{\mu \nu \rho \sigma}$. Calculating for the metric (2.1) we find that the only non-vanishing components are given by

$$R_{1212} = e^{2\beta} \left\{ u^2 \beta'^2 + 2u \beta' + u^2 \beta'' - 2u^2 \beta' \phi' - 2u \phi' \right\}$$

$$= R_{2424} = R_{1224}$$

and

$$R_{1313} = -e^{-2\beta} \left\{ u^2 \beta'' + 2u \beta' - u^2 \beta'^2 + 2u \phi' - 2u^2 \phi' \right\}$$

$$= R_{1334} = R_{3434}$$

Using equation (2.4) these reduce to

$$R_{1212} = R_{2424} = R_{1224}$$

$$= u e^{2\beta} \left\{ 2\beta' + \beta'' - u^2 \beta'^3 \right\}$$

$$R_{1313} = R_{1334} = R_{3434}$$

$$= -u e^{-2\beta} \left\{ u \beta'' + 2\beta' - u^2 \beta'^3 \right\}$$

It follows from (2.5a), (2.5b) that the solution
(2.1) is flat only if
\[ \beta'' + 2u'\beta' - u\beta^3 = 0 \] (2.5)
(If we exclude the singularity at \( u = 0 \).)

The simplest function \( \beta(u) \) satisfying equation (2.5) is \( \beta(u) = \beta_0 \) where \( \beta_0 \) is a constant. From the relation (2.4) it follows that \( \phi(u) = \phi_o \) where \( \phi_o \) is another constant. When \( \beta = \beta_0 \), \( \phi = \phi_o \) the metric (2.1) reduces to
\[ ds^2 = -e^{2\phi_0}d\xi^2 - u^2(e^{2\phi_0}d\eta^2 + e^{-2\phi_0}d\xi^2) + e^{2\phi_0}d\tau^2 \] (2.6)
which, as we said above, is flat (\( R_{\mu\nu\rho\sigma} = 0 \)).

It is easily seen that the equations
\[ \tau - \xi = \xi = t - x \]
\[ \tau + \xi = e^{-2\phi_0}\left[ t + x - u^{-1}(y^2 + z^2) \right] \] (2.7)
transform the metric (2.6) into the Minkowski space-time
\[ ds^2 = -dt^2 - dy^2 - dz^2 + dt^2 \] (2.8)

Returning to the metric (2.1) it can be shown that if we perform the coordinate transformations
\[ \tau - \xi = t - x \]
\[ \tau + \xi = e^{-2\phi_0}\left[ t + x - u^{-1}(y^2 + z^2) \right] \]
\[ \eta = e^{-\beta_0}y u^{-1} \]
\[ \xi = e^{\beta_0}z u^{-1} \] (2.9)
then the metric (2.1) reduces to

\[ ds^2 = c dt^2 - dx^2 - dy^2 - dz^2 + 2\beta'(y dy - z dz)(dt - dx) \]

\[ -\left\{ (t^2 - x^2)\beta'^2 + 2\mu^{-1}(y^2 - z^2)\beta' \right\}(dt - dx)^2 \] (2.10)

In this form the deviation of the wave space-time from the Minkowski space-time is clearly shown.

We shall denote the coordinate system \((x, y, z, t)\) by \(S_1(x, y, z, t)\) and the coordinate system \((\xi, \eta, \xi, \tau)\) by \(S_2(\xi, \eta, \xi, \tau)\).

Bondi, Pirani and Robinson were interested in finding the effect of the passage of the plane-waves on freely moving particles. To do this they considered plane gravitational waves of finite duration, what they have called sandwich waves, bounded by the planes

\[ t - x = t - \xi = u, \]

and \[ t - x = t - \xi = u_2, \] say, where \(u, < u_2\). The consideration of such waves is possible because of Lichnerowicz's regularity conditions on the metric tensor. To avoid the singularity of the field at \(u=0\) we take \(0 < u, < u_2\).

The sandwich wave separates space-time into three regions \(R_1, R_2, R_3\), as follows (see figure 2).

\(R_1\) : \(0 < u < u_1\), \(R_{\text{uro}}=0\), and the constants
\( \phi_0, \beta_0 \) are chosen, for convenience, as \( \phi_0 = \beta_0 = 0 \).

\( R_2 : u_1 < u < u_2, R_{uv} \neq 0 \); \( \phi(u), \beta(u) \)
are arbitrary functions related only by the equation
\[ 2\phi' = u\beta'^2. \]
\( \beta(u) \) does not satisfy equation (2.5).

\( R_3 : u_2 < u < \), \( R_{uv} = 0 \), and \( \phi(u), \beta(u) \)
are given by \( \phi = \frac{\alpha}{2}, \beta = 0 \) where \( \alpha \) is a
constant. \( R_{uv} \) is, of course, zero throughout the
space-time.

In the \((t, x)\) plane the three regions are
shown as in the figure (2).
Plane gravitational waves of finite duration.

figure 2.
3. Relative velocity imparted to freely moving test particles by the passage of the wave

We shall consider, with B.P.R., two test particles at relative rest in an inertial frame in the $S_i(x, y, z, t)$ coordinate system in region $R$, i.e. before the arrival of the wave. We shall show that after the passage of the wave the two particles are in relative motion in an inertial frame of reference in the system $S_i(x, y, z, t)$. This effect of the plane waves on test particles can, in principle, be used to extract energy from the wave.

We shall first consider the geodesic equations (i.e. paths of neutral test particles) in the general space-time

$$ds^2 = -e^{2\phi} dt^2 - u^2 (e^{2\beta} d\eta^2 + e^{-2\beta} d\xi^2) + e^{2\phi} dr^2$$

where $\phi, \beta$ are arbitrary functions of related by the equation $2\Phi' = u^2$. We write

$$2T = -e^{2\Phi} - u^2 (e^{2\beta} \dot{\eta}^2 + e^{-2\beta} \dot{\xi}^2) + e^{2\phi} \dot{r}^2$$

where, writing $\chi^\alpha = (\xi, \eta, \eta, \tau), \alpha = 1, 2, 3, 4$,

$$\dot{\chi}^\alpha = \frac{d\chi^\alpha}{ds} \quad (ds \neq 0)$$
Then the non-null geodesics of the metric (3.1) are given by the Euler-Lagrange equations

\[ \frac{d}{ds} \left( \frac{\partial T}{\partial \dot{\chi}} \right) - \frac{\partial T}{\partial \chi} = 0 \]  \hfill (3.3)

These equations possess the first integral

\[ -e^{2\Phi} \dot{\xi}^2 - u^2 (e^{2\beta} \dot{\eta}^2 + e^{-2\beta} \dot{\xi}^2) + e^{2\Phi} \dot{\xi}^2 = 1 \]  \hfill (3.4)

For \( \alpha = 1, 2, 3 \), respectively, equations (3.3), (3.2) give

\[ \frac{d}{ds} \left( -e^{2\Phi} \dot{\xi} \right) = \left\{ \Phi' e^{2\Phi} \dot{\xi}^2 + u (1 + u\beta') e^{2\beta} \dot{\eta}^2 \right\} + u (1 - u\beta') e^{-2\beta} \dot{\xi}^2 - \Phi' e^{2\Phi} \dot{\xi}^2 \]  \hfill (3.5)

Using equations (3.7), (3.8), (3.9) for

\[ \frac{d}{ds} \left( e^{2\Phi} \dot{\xi} \right) + \left\{ \Phi' e^{2\Phi} \dot{\xi}^2 + u (1 + u\beta') e^{2\beta} \dot{\eta}^2 \right\} + u (1 - u\beta') e^{-2\beta} \dot{\xi}^2 - \Phi' e^{2\Phi} \dot{\xi}^2 \]  \hfill (3.6)

\[ \frac{d}{ds} \left( u^2 e^{2\beta} \dot{\eta} \right) = 0 \]  \hfill (3.7a)

\[ \frac{d}{ds} \left( u^2 e^{-2\beta} \dot{\xi} \right) = 0 \]  \hfill (3.8a)

where \( \Phi' = \frac{d\Phi}{du} \), \( \beta' = \frac{d\beta}{du} \).

(3.7a), (3.8a) give, on integration,

\[ u^2 e^{2\beta} \dot{\eta} = k \]  \hfill (3.7)

\[ u^2 e^{-2\beta} \dot{\xi} = \lambda \]  \hfill (3.8)
where $k$, $l$ are constants.

Adding (3.5), (3.6) we get

$$\frac{d}{ds} \left[ e^{2\phi} (\dot{\xi} - \dot{\eta}) \right] = 0$$

which gives, on integration,

$$e^{2\phi} (\dot{\xi} - \dot{\eta}) = h,$$

where $h$ is a constant.

Therefore

$$\dot{\xi} - \dot{\eta} = \dot{u} = h e^{-2\phi} \quad (3.9)$$

The first integral (3.4) can be written in the form

$$e^{2\phi} (\dot{\xi} - \dot{\eta}) (\dot{\xi} + \dot{\eta}) - u^2 (e^{2\phi} \ddot{\eta}^2 + e^{-2\phi} \ddot{\xi}^2) = 1$$

Using equations (3.7), (3.8), (3.9) for $\dot{\eta}$, $\ddot{\xi}$, $\dot{\xi} - \dot{\eta}$ we get

$$\dot{\xi} + \dot{\eta} = \frac{1}{h} \left\{ 1 + \frac{k^2}{u^2} e^{-2\beta} + \frac{\lambda^2}{u^2} e^{2\beta} \right\} \quad (3.10)$$

Solving (3.9) and (3.10) we get $\dot{\eta}$, and $\dot{\xi}$.

Therefore the 4-velocity vector $(\dot{\xi}, \dot{\eta}, \dot{\xi}, \dot{\eta})$ is given by

$$\frac{\dot{\xi}}{\dot{\eta}} = \frac{1}{2h} \left\{ -h e^{-2\phi} + (1 + \frac{k^2}{u^2} e^{-2\beta} + \frac{\lambda^2}{u^2} e^{2\beta}) \right\} \quad (3.11a)$$

$$\dot{\xi} = \text{constant}$$
To get the only use we made of the equations (3.5), (3.6) was their sum and we have, therefore, to verify that the above solution for \((\xi, \eta, \xi, \tau)\) satisfies (3.5), (3.6) separately. It is not difficult to see that the solution (3.11) does, indeed, satisfy (3.5) and (3.6).

We do not need to integrate the last four equations to find the complete solution \((\xi, \eta, \xi, \tau)\) of the geodesics. For our purpose all we need is the 4-vector \((\xi, \eta, \xi, \tau)\).

In a Minkowski space-time the world line of a free particle will be of the form

\[
\begin{align*}
x &= \text{Constant} \\
y &= \text{Constant} \\
z &= \text{Constant} \\
t &= s
\end{align*}
\]
where $ds$ is given by

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 \quad (3.13)$$

We start, with B.P.R., with two test particles at relative rest in region $\mathcal{R}$, i.e. before the arrival of the wave. The question of what we mean by saying that two particles are at relative rest in a general space-time (even if it is flat but expressed in general curvilinear coordinates) is by no means easy. The only case in which we can say with certainty that two particles are at relative rest is when it is possible to find an inertial frame of reference in which if one of the particles is at rest then so is the other.

Accordingly we start with two particles, $A, B$, in the Minkowski space-time (3.13) with spacial coordinates given by $A(0, 0, 0), B(0, -\Lambda, 0)$, in the coordinate system $S, (x, y, z, t)$; $\Lambda$ is a constant.

It follows from (3.13) that the world-line of the particle $B$ in the region $\mathcal{R}$, in the coordinate system $S$, is

$$x = 0, \quad y = -\Lambda, \quad z = 0, \quad t = s \quad (3.14)$$

We recall that the metric (3.1) in region $\mathcal{R}$,
in the $S_2(\xi, \eta, \zeta, \tau)$ system is given by
\[ ds^2 = -a^2 - a^2 \left( d\eta^2 + d\zeta^2 \right) + d\tau^2 \] (3.15)
\[ (\phi = \beta = 0) \], and the transformations taking
(3.15) into (3.13) now reduce to (see (2.7))
\[ \tau - \xi = u = t - x \]
\[ \tau + \xi = t + x - u^{-1} (y^2 + z^2) \]
\[ \eta = yu^{-1}, \zeta = zu^{-1} \] (3.16)
Therefore the world-line of B in region R in the
system $S_2(\xi, \eta, \zeta, \tau)$ is, by (3.14), (3.16)
\[ \tau = s - \frac{\Lambda^2}{2u} \]
\[ \xi = -\frac{\Lambda^2}{2u} \]
\[ \eta = -\Lambda u^{-1}, \zeta = 0 \] (3.17)
This shows that the particle B is no longer at rest w.r.t. the coordinate system $S_2(\xi, \eta, \zeta, \tau)$.
From the first two equations of (3.17) we get
\[ u = \tau - \xi = s \]
giving
\[ \dot{u} = \dot{\tau} - \dot{\xi} = 1 \] (3.18)
Using (3.18), (3.17) the 4-velocity vector 
\((\dot{\xi}, \dot{\eta}, \dot{\zeta}, \dot{\nu})\) of the particle B in \(R\) is

\[
\begin{align*}
\dot{\xi} &= \frac{\Lambda^2}{2\mu^2} \mu = \frac{\Lambda^2}{2\mu^2} \\
\dot{\eta} &= \Lambda \mu^{-2} \\
\dot{\zeta} &= 0 \\
\dot{\nu} &= 1 + \frac{\Lambda^2}{2\mu^2}
\end{align*}
\]  

(3.19)

With this 4-velocity B moves till it meets the wave at \(\mu = \mu_2\). Then for \(\mu > \mu_2\), B will move with a 4-velocity given by equations (3.11).

We shall assume that the 4-velocity vector is continuous across the boundary \(\mu = \mu_1\) (and also at \(\mu = \mu_2\)).

By comparing (3.9) and (3.18) at \(\mu = \mu\), where \(\phi = \beta = 0\) and equations (3.7), (3.8) with equations (3.19) for \(\dot{\eta}, \dot{\zeta}\) respectively we get

\[
\begin{align*}
\hat{\eta} &= 1 \\
\hat{\zeta} &= 0 \\
\hat{\lambda} &= \Lambda
\end{align*}
\]

With these values of \(\hat{\eta}, \hat{\lambda}, \hat{\zeta}\) it is easily seen that \((\dot{\xi}, \dot{\eta}, \dot{\zeta}, \dot{\nu})\) is continuous across \(\mu = \mu_1\).

Hence the 4-velocity of the particle B inside the wave, i.e. \((\dot{\xi}, \dot{\eta}, \dot{\zeta}, \dot{\nu})\) can be found from

\[
\begin{align*}
\dot{\nu} + \dot{\xi} &= 1 + \frac{\Lambda^2}{u^2} e^{-2\beta} \\
\dot{\eta} - \dot{\xi} &= e^{-2\phi} \\
\hat{\eta} &= \frac{\Lambda}{u^2} e^{-2\beta} \\
\hat{\zeta} &= 0
\end{align*}
\]

(3.20)
It will move with this 4-velocity till it reaches the boundary at \( u = u_2 \) where \( \phi = \frac{\alpha}{2} \), \( \beta = 0 \).

The 4-velocity at \( u = u_2 \) is, from (3.20),

\[
\begin{align*}
\dot{t} - \dot{x} &= e^{-\alpha} \\
\dot{t} + \dot{x} &= 1 + \frac{\Lambda^2}{u_2^2} \\
\dot{\eta} &= \frac{\Lambda}{u_2^2} \\
\dot{\zeta} &= 0
\end{align*}
\]

(3.21)

(No need to find \( \dot{\xi}, \dot{\eta}, \dot{\zeta}, \dot{\tau} \) separately yet.)

(3.21) gives the 4-velocity with which B starts moving in the region \( \mathcal{R}_3 \) which, in the \( S_\alpha(\xi, \eta, \zeta, \tau) \) system, is described by the metric

\[
d\xi^2 = -e^\alpha d\xi^2 - u^2 (d\eta^2 + d\zeta^2) + e^\alpha d\tau^2 \quad (3.22)
\]

This, it will be recalled, is a flat space-time and it transforms to the Minkowski space-time (3.13) by the equations (see (2.7))

\[
\begin{align*}
\tau - \xi &= \omega = t - x \\
e^\alpha (\tau + \xi) &= t + x - u^{-1}(y^2 + z^2) \\
\eta &= y u^{-1}, \quad \zeta = z u^{-1}
\end{align*}
\]

(3.23)

(3.26a)

The 4-velocity vector \( (\dot{\xi}, \dot{\eta}, \dot{\zeta}, \dot{\tau}) \) in \( \mathcal{R}_3 \) is obtained from the equation (3.11) if we put \( \phi = \frac{\alpha}{2} \),
Therefore we have, in the coordinate system \( S_2 \),
\[
\begin{align*}
\dot{t} - \dot{\xi} &= \dot{u} = k_1 e^{-\alpha} \\
\dot{t} + \dot{\xi} &= \frac{1}{h_1} \left( 1 + \frac{k_1^2}{u^2} + \frac{\lambda^2}{u^2} \right) \\
\dot{\eta} &= k_2 u^{-2} \\
\dot{\xi} &= \lambda_3 u^{-2}
\end{align*}
\] (3.24)
where \( k_1, k_2, \lambda_3, h_1 \) are constants.

To find \( k_1, k_2, \lambda_3, h_1 \), we compare (3.24) with (3.21) at \( u = u_2 \) where the \( (\dot{t}, \dot{\xi}, \dot{\eta}, \dot{\xi}) \) in \( R_3 \) must be the same as the \( (\dot{t}, \dot{\xi}, \dot{\eta}, \dot{\xi}) \) in \( R_2 \).

We find that
\[
\begin{align*}
h_1 &= 1, \\
k_1 &= 1 \\
\lambda_3 &= 0
\end{align*}
\]

It follows that the 4-velocity of B in region \( R_3 \) is obtained from
\[
\begin{align*}
\dot{t} - \dot{\xi} &= \dot{u} = e^{-\alpha} \\
\dot{t} + \dot{\xi} &= 1 + \lambda^2 u^{-2} \\
\dot{\eta} &= \lambda_3 u^{-2} \\
\dot{\xi} &= 0
\end{align*}
\] (3.25)

Equation (3.25a) can be integrated immediately to give
\[
\eta = t - \xi = e^{-\alpha} \xi + C
\] (3.26a)

where \( C \) is a constant. In the final analysis \( C \) depends on the behaviour of the particle inside the
wave. From (3.25b) we have
\[
\frac{d}{ds}(\tau + \tau) = 1 + \Lambda^2 u^{-2} = 1 - \Lambda^2 \frac{d}{du} (u^{-1})
\]
\[
= 1 - \Lambda^2 \frac{ds}{du} \frac{d}{ds} (u^{-1})
\]
Using (3.25a) for \( \frac{ds}{du} \) we get
\[
\frac{d}{ds}(\tau + \tau) = 1 - \Lambda^2 e^{-\alpha} \frac{d}{ds} (u^{-1})
\]
which gives, on integration,
\[
\tau + \tau = s - \Lambda^2 e^{-\alpha} \frac{u}{u} + D
\]
where \( D \) is another constant.

In the same way we get for \( \eta, \xi \),
\[
\eta = -\frac{\Lambda}{u} e^{\alpha} + E
\]
\[
\xi = 0
\]
where \( E \) is a constant.

Hence the world-line of the particle B after the passage of the wave is given by
\[
\tau - \tau = u = e^{-\alpha} + C
\]
\[
\tau + \tau = \xi - \frac{\Lambda}{u} e^{\alpha} + D
\]
\[
\eta = -\frac{\Lambda}{u} e^{\alpha} + E
\]
\[
\xi = 0
\]
We can now find the world-line of B in \( R^3 \), in the coordinate system \( S_j(x,y,z,t) \) by making use of
the transformation equations (3.23). The final result is

\[ x = s \sinh \alpha - \frac{N}{2} \]

\[ y = 0 \]

\[ z = 0 \]

\[ t = s \cosh \alpha + \frac{N}{2} \] (3.28)

Looking at the equations (3.27) we see that they are straight lines, as we should expect in a Minkowski space-time. It also seems as though this world-line is independent of \( \mu_1, \mu_2 \); i.e. it is independent of the duration of the wave; this is not so, however, because the constants \( C, D, E \) depend, in the final analysis, on \( \phi(u), \beta(u) \) in \( \mathbb{R}^2 \) as well as on \( \mu_1, \mu_2 \).

We now consider the particle at rest at the spacial origin \((0, 0, 0)\) in the system \( S_1(x, y, z, t) \) in region \( \mathbb{R}_1 \); i.e. before the arrival of the wave. Proceeding in exactly the same way as before (for the particle B) it can be shown that the world-line of A in \( \mathbb{R}_3 \), after the passage of the wave, in the coordinate system \( S_1 \), is

\[ x = s \sinh \alpha - \frac{N}{2} \]

\[ y = 0 \]

\[ z = 0 \]

\[ t = s \cosh \alpha + \frac{N}{2} \] (3.28)
where $N$ is a constant of integration.

It follows from (3.28) that the effect of the wave on A is to give A a 4-velocity $(\dot{x}, \dot{y}, \dot{z}, \dot{t})$ given by

$$(\dot{x}, \dot{y}, \dot{z}, \dot{t}) = (\sinh \alpha, 0, 0, \cosh \alpha)$$

(3.29)

But this is not an invariant property because by a Lorentz transformation of the form

$$\begin{align*}
\tilde{x} &= -(t - N/2) \sinh \alpha + (x + N/2) \cosh \alpha \\
\tilde{y} &= y \\
\tilde{z} &= z \\
\tilde{t} &= (t - N/2) \cosh \alpha - (x + N/2) \sinh \alpha
\end{align*}$$

(3.30)

the particle A can be reduced at rest at the spatial origin $(0, 0, 0)$ of an inertial frame of reference with coordinates $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$.

But what is important is the relative velocity imparted to the two particles A, B by the passage of the wave. To find this relative velocity (if any) we must refer the world-line of B in $\mathbb{R}^3$ in the coordinate system $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$ in which A is at rest at the spatial origin.

Equations (3.30), (3.27) give for the world-line of B in $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$
where the constants $K_1$, $K_2$ are given by

$$K_1 = \frac{N}{2} - \frac{1}{2} \left( D e^a - 2\Lambda e^a E + C e^{a^2} + C \right)$$

$$K_2 = \frac{N}{2} + \frac{1}{2} \left( D e^a - 2\Lambda e^a E + C e^{a^2} - C \right)$$

It follows that the particle B has a 4-velocity

$$\frac{\dot{x}}{c} = e^{-2a}$$

$$\frac{\dot{y}}{c} = e^{-a}$$

$$\frac{\dot{z}}{c} = 0$$

$$\frac{\dot{t}}{c} = 1 + \frac{E^2}{2} e^{-2a}$$

The 4-velocity as given by (3.33) is equivalent to a Newtonian velocity $(v_x, v_y, v_z)$ which is given by
Also since the particle A is at rest at the origin of the spatial coordinates \((\vec{x}, \vec{y}, \vec{z})\) the Newtonian velocity of B given by (3.34) is the relative velocity of the two particles A, B. We conclude, therefore, that the effect of the passage of the wave on the two particles A, B which initially are at relative rest is to set them in relative motion, the Newtonian relative velocity being given by (3.34).

The result obtained by B.P.R. is that the relative Newtonian velocity of A, B is

\[
\nu_x = \frac{e^{-2a}}{2} \nu^2 \rho^2 \left(1 + \frac{\nu^2 \rho^2}{2} e^{-2a}\right)^{-1}
\]

\[
\nu_y = e^{-a} \nu \rho \left(1 + \frac{\nu^2 \rho^2}{2} e^{-2a}\right)^{-1}
\]

\[
\nu_z = 0
\]

where \(\nu(z, \lambda), \rho\) are constants. Our result agrees with this if we take \(\nu = \nu \rho\).
Discussion

We have mentioned in chapter I (part I) that Möller (10) applied his theory on "momentum-energy and its localization" to find the energy in the plane gravitational waves. This is the only theory at present that may be used to calculate the energy of a field when the field is described by a metric in any coordinate system we like. Einstein's theory is not applicable at all because the coordinates used in (2.10) or (3.1) are not quasi-galilean (see chapter I). It will be remembered that doubts were expressed in chapter I as to the applicability of Möller's theory to non-static fields. Möller in his work used the metric (2.10) and showed that the wave travels with the fundamental velocity in the x-direction. But when he calculated the total energy of the waves he found that $\mathcal{E}$ is zero.

We have shown, on the other hand, that when the wave passes the freely moving particles A, B which are initially at rest (relative), it sets A, B in relative motion according to the equations (3.34). Bondi (7) suggested a simple device which, making use of this relative motion, can be used to extract energy
from the wave. The device consists of a "rigid" rod and a bead which can slide on the rod with some friction. If the rod is placed in a suitable direction transverse to the direction of the wave propagation and the bead is at rest relative to the rod and away from the rod's centre of gravity, then the passing of the wave will result in some relative motion of the rod and the bead (to a first approximation both bead and rod's mass centre will each move on geodesics). This relative motion will generate heat and hence energy. Since there are no sources of energy we must conclude that this energy is extracted from the wave itself. This shows in a very simple way that the wave must carry energy.

In view of this result Møller's conclusion that the wave carries no energy cannot be very convincing. We see, therefore, that Møller's theory which was successfully applied in part I to find the energy of static fields may be inapplicable in the case of non-static fields. It was because of the above result that doubts were expressed in chapter I that Møller's theory may be inapplicable to non-static fields. Although we feel that this limitation of Møller's theory may be due to the fact that $J^\mu$ behaves like a 4-vector only under spatial coordinate transformations we have been unable to prove it.
References

(9) Lichnerowicz A., Theories relativistes de la gravitation et de l'electromagnetisme, Paris: Mason et Cie, Chapter I.
(10) Møller C., Max-Planck Festschrift, Berlin (1958), 139.
Observable Relations in Relativistic Cosmology

1. Introduction

Current cosmological theories fall into two distinct classes:

(A) Evolutionary theories
(B) Steady-state theories

Class (A) includes the relativistic (1), (2), and the Newtonian (2), (3), (4), (6) cosmologies and the relativistic (equivalent to those based on kinematic relativity), (5), (7), (8), (9). Class (B) includes the steady-state theories of Bondi and Gold (10), and Hoyle (11). The hypotheses on which these theories are based and their general features are discussed in §2.

Much work has been done to find out the observable differences of the two theories (A), (B) and decide by actual observations which of these theories best represents the actual universe. It turns out that the cases in which the two theories agree are more than the cases in which they give different results. Even in the case in which the two theories differ from each other the observable difference is so minute and the actual observations so uncertain that no final
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Class (A) includes the relativistic (1), (2), and the Newtonian (3), (4), (5), (6) cosmologies and the cosmologies based on the theory of special relativity (equivalent to those based on Kinematic relativity), (6), (7), (8), (9). Class (B) includes the steady-state theories of Bondi and Gold (10), and Hoyle (11). The hypotheses on which these theories are based and their general features are discussed in §2.

Much work has been done to find out the observable differences of the two theories (A), (B) and decide by actual observations which of these theories best represents the actual universe. It turns out that the cases in which the two theories agree are more than the cases in which they give different results. Even in the case in which the two theories differ from each other the observable difference is so minute and the actual observations so uncertain that no final
solution has been given to this so-called "cosmological problem". Most of the observable relations formulated concern the relation between the "spectral red-shift and the apparent magnitude of galaxies", the relation between "number of galaxies and apparent magnitude of galaxies" etc. (see (1), (2), (12), (13)).

In this section we seek a well-formulated observational criterion for the fundamental property that an evolving expanding universe must appear to be more congested at great distances than it is in the cosmical neighbourhood of the observer, while a steady state universe ought to exhibit the same congestion at all distances. As the measure of congestion we take the ratio of a statistical measure of the angular separation between galaxies (or cluster of galaxies) and their suitably specified neighbours in the sky, to a statistical measure of the angular diameter of the galaxies themselves. This is discussed in §3. In the remaining sections this ratio is worked out for the Newtonian, special relativity, general relativity and steady-state cosmologies. It is shown with an unexpected degree of accuracy that if \( z \) is the spectral red-shift of these galaxies the above ratio would be proportional to \( (z+1)^{-\gamma} \) in an evolving universe; it
would be independent of $z$ in a steady-state universe. This seems the simplest available criterion to see whether the universe is evolving or not. The feasibility of applying this criterion to the actual universe is discussed in §9. Unfortunately, in spite of its simplicity in principle, rather obvious reasons render its useful application somewhat difficult (if not improbable). However, since it concerns a first-order effect of the expansion, and there is some choice in the objects to which it might be applied, and since the results hold for $\theta$ (the angular separation of galaxies) not necessarily being a small angle, some feasible statistical adaptation of the method may be devised in the future.

In the appendix I we work out certain integrals arising from sections 5, 6, 7, 8 and in appendix II the problem of (special) relativistic apparent size is further discussed for the sake of its intrinsic interest.

A brief discussion of this work was published jointly with Professor W.H. McCrea in 1959, a copy of which is supplied at the end of this thesis.
2. Main features of the cosmological models

All the cosmological models, in practice, ignore local irregularities of the actual universe. They treat it as being a smoothed-out universe, its contents behaving like a continuous fluid. The object of all cosmological theories is to discover a theoretical model representing the smoothed-out universe and such that their predictions and their theoretical observable relations should be the same as the ones actually observed in the actual universe. There are many competing cosmological models each claiming superiority over the others in certain respects. The relativistic models claiming superiority on its logical structure having as its basis the edifice of the theory of general relativity; the steady-state theory on its mathematical simplicity and on certain other philosophical aspects. It seems that only by actual observations can the present controversy be settled. To find the simplest observational criterion likely to differentiate between the various theories we briefly mention the main features of the two classes of theories (A) and (B) of §1:

In the evolutionary theories the universe is assumed to be homogeneous and isotropic about any spatial
point at a given epoch $\tau$, although the general picture of the universe changes with $\tau$. The contents of the universe (galaxies, cluster of galaxies) are moving radially away from each other (their magnitudes remaining approximately constant), so that the universe is expanding. The contents of the universe are supposed to have expanded away from a single point or from an initial highly congested, unstable state (e.g. from the Einstein state in relativistic cosmology (14)), at some distant time in the past. According to the conservation laws implied by the physical theories applicable in the appropriate evolutionary models the total matter of the universe must remain constant. Accordingly no new matter is being created. Since the universe is expanding it follows that the universe must get less and less congested with the passage of time. The point of interest in our discussion is that an evolutionary expanding universe must have been more congested in the past than it is now.

In the steady-state theories homogeneity and isotropy of the universe is also assumed but the theories also demand that the general picture of the universe remains unchanged with the passage of time. This is the fundamental difference between the two classes of theories. In the steady-state theories
galaxies are also moving away from each other and, therefore, we have, again, an expanding universe. But if the steady-state character of the universe over large regions is to be preserved we cannot but postulate the continual creation of matter. Out of this newly created matter new galaxies are formed thus keeping the steady-state character of the universe unchanged. Accordingly, in this theory, the congestion of the universe must be the same everywhere and at all times.

We are not going to decide here which of the theories best represents the actual universe on philosophical, logical or mathematical grounds. We shall leave actual observations to differentiate between the theories. Accordingly we proceed to propose our new observational criterion.
3. The Observational Criterion

It follows directly from the brief summary of the cosmological models given in §2, that if the universe as a whole is evolving and a region of it $\tau$ light-years distant is observed (now), this region as seen (now) must be $\tau$ years younger than our own neighbourhood. In particular, if the universe is expanding and no new matter is assumed to be created, as in models (A), this region must be seen to be more congested than our own cosmic neighbourhood; the degree of congestion must be increasing with increasing $\tau$. On the other hand, if the universe is in a steady-state, a region of it at any distance must be seen to be not different from our neighbourhood (always provided that we are speaking of sufficiently large regions). This gives, in principle, the simplest observational means of discovering whether the actual universe is evolving or not; it is literally a question of looking to see if it is. Further this criterion makes use of the most fundamental features of the cosmological theories as summarized in §2.

The essence of the required procedure may be indicated as follows. Let $\Delta$ be some observable
measure of the angular size, as seen by an observer \( O \), of an object \( Q \) belonging to some standard category (e.g. some type of cluster of galaxies). Let \( P \) be \( Q \)'s nearest neighbour in space and receding from \( Q \) as a manifestation of the general cosmical expansion (that is, \( P \) is not gravitationally bound to \( Q \)). Let \( \Theta \) be the observable angular separation of \( P \), \( Q \) in the sky as seen by \( O \). Taking a number of objects like \( Q \) all at the same "distance" (as judged by some convenient criterion) within specified limits, we find the average values of \( \Delta \), \( \Theta \) for these objects. Let \( \Delta^* \), \( \Theta^* \) denote the average values of \( \Delta \), \( \Theta \). \((\Delta^*)^-\) itself may be taken as a measure of the distance of the objects concerned. The important point we note is that the ratio \( \Theta^*/\Delta^* \) is a measure of the congestion of the universe at that particular distance \(((\Delta^*)^-)\). If the ratio \( \Theta^*/\Delta^* \) is found to decrease with increasing distance (e.g. with decreasing \( \Delta^* \)) we should infer that the universe is expanding and evolving. More generally, any dependence of \( \Theta^*/\Delta^* \) on the "distance" would contradict the steady-state hypothesis.

Since \( \Theta^* \), \( \Delta^* \) are related in the same way to the average actual distance \( PQ \) and the average actual diameter of the objects concerned,
respectively, (apart from a purely numerical factor allowing for the fact that distances are projected on the sky), it follows that the ratio of these two actual distances is given by \( \frac{\theta^*}{\Delta^*} \) independently of the particular geometry of space-time. Therefore the suggested procedure demands no knowledge of this geometry. This, also, means that although our criterion will differentiate between cosmological models of class (A) and (B) it cannot distinguish between the various evolutionary models of the class (A). We do not consider this as a limitation of our procedure. Our object is to find whether the universe is evolving or not. To try to distinguish between the various evolutionary models at the same time it means giving up the above criterion and with it the great advantage that the above procedure demands no knowledge of the geometry of space-time.

The procedure proposed above is only the gist of the method. A proper mathematical formulation is needed with an expression of the results in suitable statistical terms. In particular, since the spectral red-shift, \( \Delta \), is the most convenient criterion of distance, we shall discuss the possible dependence of \( \frac{\theta^*}{\Delta^*} \) on \( \Delta \), rather than on \( \Delta^* \) itself.

In what follows we shall first give the
mathematical formulation of the above criterion. It appears that no such formulations has been given before the publication of paper (16), (to be called paper I henceforth), which was published in collaboration with Professor McCrea. The formulation possesses some interest of its own, in particular the unexpected property that the results obtained are exact; they do not depend on the angle $\Theta$ being a small angle.

The above criterion was first proposed by W.H. McCrea after commenting on J. Neyman and E. Scott's lecture on "Statistical approach to problems of cosmology" (17). In his own words: "Define a parameter $S$ that is to be some statistical measure of the apparent (angular) size of a cluster, and another parameter $\ell$ that is to be some statistical measure of the apparent (angular) separation between "neighbouring" clusters. "Neighbouring" is to be judged with the aid of red-shifts as a criterion of distance from us. Then any dependence of $S/\ell$ upon distance would contradict the steady-state hypothesis. The advantage of this method is that it is independent of a knowledge of the geometry of space-time".
4. Cosmological Models and Assumptions

As is usual in the solution of the "cosmological problem" the theory can be given only for theoretical models of the universe. Their main features were discussed in §2. We formulate our criterion, below, for the models in Newtonian, special relativity (here equivalent to kinematic relativity), general relativity, and steady-state cosmologies. There is some gain in insight, as well as some mathematical interest in treating these cases separately. It is also shown that for all the evolutionary models the results are of exactly the same form in an appropriate coordinate system.

As we pointed out in §2 all the cosmological models are "smoothed-out" models. But for the purpose of the theoretical treatment we shall suppose that any model under consideration contains "galaxies" which are the only objects to be considered.

We assume that
(a) each galaxy has no motion except that of the cosmical expansion; that is, galaxies are not gravitationally bound.
(b) each galaxy has a (proper) size which
does not vary with time.  

(c) each galaxy has a nearest-neighbour galaxy (to be called simply its "neighbour") and that the directions joining the galaxies to their neighbours are randomly distributed in space.  

(d) at a particular cosmic epoch, the distances between the galaxies and their neighbours are distributed according to some statistical law, but that there is no statistical correlation between these distances and the sizes of the galaxies.  

(e) in the evolutionary models galaxies are neither created nor destroyed; it follows that a galaxy that is the neighbour to a given galaxy at any epoch is its neighbour at any other epoch.  

In the steady-state models we retain assumptions (a) to (d) but not (e); instead we admit the appearance of new galaxies so that, statistically, a galaxy has a nearest neighbour at a fixed distance not depending on the epoch.  

It must be emphasized that these assumptions serve mainly to provide a terminology for the mathematical formulation. Any attempt to endow the models with some sort of physical reality would automatically ensure that they must contain objects of some kind that behave in the manner described above
without further special assumptions. When we discuss the possible correspondence between the model and the actual universe the objects mentioned above will not be literally individual galaxies.

After our paper (16) was published, Davidson published a paper (18) dealing with more or less the same topics as we have done in (16). His work differs widely from our own in detail. Davidson covers a wider programme in the sense that he dispenses with assumptions (a), (b) and that his work may differentiate between the various evolutionary models too. We do not consider assumption (a) as being stringent in any way. What we are interested in is the general cosmical expansion of the universe. Assumption (b) on the constancy of the size of the galaxies might be considered as a limitation of our theory but it is unlikely that any variation of the size of the galaxies w.r.t. time would mask the overall behaviour of the universe as considered here. On the other hand, Davidson's formulation is only approximate; in particular, the angle $\theta$ must be assumed to be a small angle.

We proceed to give the mathematical formulation of our criterion.
5. **Newtonian Cosmology**

The relevant features of Newtonian cosmology have been summarized by Bondi (6) and McCrea (15) and no attempt to recover their results is made here. The space used is euclidean and the time is the newtonian universal time \( t \). The galaxy \( O \) will mean the galaxy (from where observations are made), in the sense of \( \mathbb{R}^3 \), whose centre is \( O \), and an observer \( O \) will mean an observer at this centre. Similarly for other galaxies. Let \( O \) be the observer whose observations we wish to discuss, and let \( Q \) be another galaxy. In this cosmology all its features are deduced from the hypothesis that the space is homogeneous and isotropic about any observer. In particular, it follows that the position vector \( \overrightarrow{OA} \) of \( Q \) referred to \( O \) is of the form

\[
\overrightarrow{OA} = a \cdot R(t) \tag{5.1}
\]

where \( a \) is a fixed vector permanently characterizing the particular galaxy \( Q \), and \( R(t) \) is a function of \( t \) only and the same for all \( Q \). Let \( P \) be \( Q \)'s nearest neighbour in the sense of section 3. The position vector \( \overrightarrow{OP} \) of \( P \) relative to \( Q \) is

\[
\overrightarrow{OP} = b \cdot R(t) \tag{5.2}
\]
where \( \mathbf{b} \) is another fixed vector characterizing \( P \).

It follows from (5.1), (5.2) that the position vector \( \mathbf{q}_{P} \) of \( P \) relative to \( O \) is given by

\[
\mathbf{q}_{P} = (\mathbf{a} + \mathbf{b}) R(t) \tag{5.3}
\]

Let \( \theta \) be the angle \( QO\mathbf{P} \), and \( \bar{\theta} \) be the angle between the vectors \( \mathbf{a}, \mathbf{b} \), as shown in the figure. Then, as is easily seen from figure 3

\[
\tan \theta = \frac{b \sin \bar{\theta}}{a + b \cos \bar{\theta}} \tag{5.4}
\]

where \( a = |\mathbf{a}|, b = |\mathbf{b}| \).

It follows from (5.4) that for given galaxies \( O, Q, P \) the angles \( \theta, \bar{\theta} \) are independent of \( t \) as it should be expected from the cosmological model and the assumption that \( O, Q, P \) are not gravitationally bound. Since light is supposed to be propagated along euclidean straight lines in this model, the angle \( \theta \) is the required angular separation of \( Q, P \) as seen from \( O \). We now suppose, for the moment, that \( \mathbf{b} \) is given, with \( a > b \), but that all directions of \( \mathbf{b} \) in space are equally probable in accordance with assumption (c) in §4. Let \( \bar{\theta} \) be the mean value of \( \theta \) corresponding to all possible positions of \( P \).
Then
\[ \Theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{0}^{\pi} \theta \sin \theta \, d\theta \, d\phi. \]

Therefore
\[ 2\Theta = \int_{0}^{\pi} \theta \sin \theta \, d\theta. \quad (5.5a) \]

Integrating (5.5a) partially we get
\[ 2\Theta = \left[ -\theta \cos \theta \right]_{0}^{\pi} + \int_{0}^{\pi} \frac{d\theta}{d\theta} \cos \theta \, d\theta. \]

But in view of (5.4) for the value of \( \Theta \) in terms of \( \theta \) and the assumption that \( a > b \), the expression in the bracket vanishes. \( \Theta \) is, therefore, given by
\[ 2\Theta = \int_{0}^{\pi} \frac{d\theta}{d\theta} \cos \theta \, d\theta. \quad (5.5) \]

To evaluate this integral we use (5.4) to find \( d\theta/d\theta \) in terms of \( \theta \). Substituting this value of \( d\theta/d\theta \) in (5.5) the integral can be evaluated exactly (appendix I) to give
\[ \Theta = \frac{\pi}{2} \frac{b}{a}. \quad (5.6) \]

This very simple dependence of \( \Theta \) on \( b/a \) would hardly have been foreseen as an exact result. In particular it follows that we can now let \( b \) have...
whatever its appropriate distribution-law as mentioned in section A(d). Substituting the mean value \( b^* \) of \( b \) in (5.6) we get the mean value \( \Theta^* \) of \( \Theta \). Thus the formula (5.6) reduces to

\[
\Theta^* = \frac{\pi}{2} \frac{b^*}{a}
\]  

(5.7)

and gives the mean angular separation, as seen from \( O \), between any galaxy with a given \( \alpha \)-value and its neighbours. This simple formula involves no approximations.

Newtonian cosmology in its simplest presentation requires no assumption about the speed of light. But, if it is to yield a self-consistent model involving light propagation, we must postulate that the speed \( c(t) \) of light measured locally by an observer attached to a galaxy depends at most upon the time \( t \). It follows that, in accordance with the assumptions (homogeneity and isotropy) on which Newtonian cosmology is based, that the speed of light measured locally and at the same time is the same for all observers. It is necessary to assume that the Newtonian law of addition of velocities holds even if one of the velocities is the velocity of light \( c(t) \).

It follows from (5.1) that the velocity \( \nu \) of
a galaxy at distance \( \mathcal{g} \) (\( = \epsilon R(t) \), say) from the origin is given by

\[
\mathbf{v} = \frac{d \mathcal{g}}{dt} = \epsilon \frac{R'(t)}{R(t)} \mathcal{g} , \quad (R'(t) = \frac{d R}{dt})
\]

It follows that the local radial velocity \( \frac{d \mathcal{g}}{dt} \) of light is given by

\[
\frac{d \mathcal{g}}{dt} = \frac{R'(t)}{R(t)} \mathcal{g} + c(t) \]

or

\[
R(t) \frac{d}{dt} \left[ \frac{\mathcal{g}}{R(t)} \right] = t \cdot c(t) \quad (5.8)
\]

for outward or inward propagation, respectively. We shall adapt the expression (5.8) for the velocity of light in the simple case in which \( c(t) = c \), a constant (Bondi (6)). The consistency of the results we shall obtain below serves as a justification of the above assumptions.

Let \( t_0 \) be the epoch of observation by \( O \).

If \( t_Q \) is the epoch of the emission by \( Q \) of light reaching \( O \) at \( t_0 \), then, by (5.8)

\[
\frac{\mathcal{g}_Q}{R(t_0)} = c \int_{t_Q}^{t_0} \frac{dt}{R(t)}
\]

Using equation (5.1) we get

\[
\int_{t_Q}^{t_0} \frac{dt}{R(t)} = \frac{\alpha}{c} \quad (5.9)
\]
Since the R.H.S. is constant it follows by differentiation that, if $dt_o$ is the time-interval during which radiation is emitted by $Q$ at time $t_o$ and $dt_o$ is the time-interval during which this radiation is received by $O$ at time $t_o$:

$$\frac{dt_o}{dt_o} = \frac{R(t_o)}{R(t_o)} = \frac{R_o}{R_o}$$

Therefore, if $\lambda_o$, $\lambda_q$ are the wavelengths of radiation as measured by $O$, $Q$, respectively, then

$$\lambda_o = \frac{\lambda_o}{\lambda_q} = \frac{R_o}{R_q} \quad (5.10)$$

This is the well-known formula for the red-shift $z$.

Therefore, as Bondi has remarked, the above assumption on the propagation of light in Newtonian cosmology lead, through equation (5.8), to exactly the same form (5.10) as the corresponding result in relativistic cosmology.

We come now to the evaluation of the angular diameter of $Q$. In section §4 we have made the assumption (b) that the proper size of a galaxy is independent of $t$. Let $\Delta$ be the angular diameter of $Q$ as seen by $O$ at the epoch $t_o$. We assume that $\Delta$ is a small angle and that results are required only to the first order in $\Delta$ (see appendix II).
Treating $Q$ as a sphere of diameter $l$, this angle is given by
\[ \Delta = \frac{l}{\alpha R_0} \] (5.11)
since $\alpha R_0$ is the distance of $Q$ from $O$ at time $t_Q$ when light leaves $Q$. (Were the galaxy $Q$ an elongated body of length $l$, we should have $\frac{\pi l}{\alpha}$ in place of $l$ in (5.11)). If $\ell^*$ is the mean value of $l$ for all galaxies, then the mean value $\Delta^*$ of $\Delta$ for a given $\alpha$-value is
\[ \Delta^* = \frac{\ell^*}{\alpha R_0} \] (5.12)
$R_0$ remains the same because by (5.9), if $\alpha, t_0$ are given, then $t_Q$ is fixed and so $R_0$ is also fixed.

From (5.7), (5.12) we get for the ratio $\Theta^*/\Delta^*$ (our measure of congestion of the universe at a given distance),
\[ \frac{\Theta^*}{\Delta^*} = \frac{\pi b^*}{\ell^*} R_0 \]
Using (5.10) to find $R_0$ we finally get
\[ \frac{\Theta^*}{\Delta^*} = \left( \frac{\pi}{\alpha} \frac{b^*}{\ell^*} R_0 \right) \frac{1}{1+z} \] (5.13)
According to the postulated properties of the models the red-shift $z$ is the only quantity in the right-hand member of (5.13) that depends upon the
distance of the galaxies under observation at epoch to .. Naturally, we do not need to employ any measure of distance other than \( z \). Thus if a number of galaxies all exhibiting the same red-shift \( z \) are observed to have mean angular diameter \( \Delta^* \) and to have mean angular separation \( \Theta^* \) from their neighbours in space, then for the present model (5.13) asserts that \( \Theta^*/\Delta^* \) decreases with increasing \( z \) in proportion to \( (1+z)^{-\gamma} \).

This is the precise expression for the model of the increasing congestion of the universe to be seen at increasing distance from the observer which was described qualitatively in section 2.
Fig. 3. Newtonian Cosmology.

Fig. 4. Special Relativity.
6. Cosmology of Special (Kinematical) Relativity

A model of the expanding universe that can be treated by special relativity was first described by Kermack and McCrea (7) and has recently been treated by Synge ((8), p.150). For the present purposes it is the same as that given by kinematical relativity (9). As we shall show below the cosmological model based on special relativity is a particular model of those of general relativistic cosmology. It is interesting, however, to treat this model by itself using only the basic results of special relativity.

The special relativity model corresponds to the Newtonian model described in §5 in the particular case in which the galaxies move radially away from each other with uniform relative velocities, i.e. as seen from (5.1), in the particular case in which \( \mathcal{R}(t) = t \). Newtonian theory is abandoned, of course, and use is made of the kinematics and optics of special relativity instead. We postulate that a frame moving with some galaxy is inertial. It then follows from above (i.e. from the uniform relative motion of the galaxies) that there is an inertial frame moving with every galaxy. All the quantities we shall be considering are referred
to one or another of such frames.

We start with the galaxies $O$, $Q$, $P$ as in section 5. Replacing $\mathcal{R}(t)$ by $t$ the expressions (5.1), (5.2) now become

$$\frac{\mathcal{Q}}{\mathcal{P}} = a \frac{t}{t}$$  \hspace{1cm} (6.1)

$$\frac{\mathcal{P}}{\mathcal{P}} = b \frac{t}{t}$$  \hspace{1cm} (6.2)

where $t$ is time measured in the inertial frame $S$ moving with $O$ and $\mathcal{T}$ is time in the inertial frame $\mathcal{S}$ moving with $Q$. It follows that $a$ is the velocity of $Q$ relative to $O$ in $S$, and $b$ is the velocity of $P$ relative to $Q$ in $\mathcal{S}$.

Let $Ox$, $Oy$ be rectangular axes in $S$, and $Qx$, $Qy$ be rectangular axes in $\mathcal{S}$, chosen so that $Ox$, $Qx$ lie along $OQ$, and $Oy$, $Qy$ lie in the plane $OQP$. We measure time in $S$, $\mathcal{S}$ at the instant at which $O$, $Q$ coincide.

Let $\varphi QP = \mathcal{P}$, as measured in $\mathcal{S}$; this is a fixed angle determined by the fixed vector $b$. Since $P$ moves with constant velocity $\mathcal{P}$ relative to $Q$ in $\mathcal{S}$, its world-line, in $\mathcal{S}$, is given by

$$\frac{\mathcal{X}}{\mathcal{T}} = b \cos \varphi \mathcal{P}$$, $$\frac{\mathcal{Y}}{\mathcal{T}} = b \sin \varphi \mathcal{P}$$  \hspace{1cm} (6.3)
To find the world-line of $P$ in the inertial frame $S$ we apply the Lorentz transformations in their simple form

$$
\begin{align*}
\bar{x} &= \beta (x + at) \\
\bar{y} &= \bar{y}, \quad \bar{z} = \bar{z} \\
\bar{t} &= \beta (x + \frac{at}{c^2})
\end{align*}
$$

(6.4)

where

$$\beta = \left(1 - \frac{a^2}{c^2}\right)^{-\frac{1}{2}}$$

since $\bar{S}$ moves relative to $S$ with velocity $a$ along $0x$. It is easily seen from (6.3), (6.4) that the world-line of $P$ in $S$ is given by

$$
\begin{align*}
\frac{x}{t} &= \frac{\alpha + b\cos \theta}{\left[1 + (ab/c^2)\cos \theta\right]} \\
\frac{y}{t} &= \frac{b\sin \theta}{\beta\left[1 + (ab/c^2)\cos \theta\right]}
\end{align*}
$$

(6.5)

It follows from (6.5) that the galaxy $P$ moves, in $S$, with uniform radial velocity in accordance with the general feature of the special (kinematical) relativity cosmology. The fixed line through $0$ in which $P$ moves makes an angle $\theta$, say, with the $0x$ axis

where, from (6.5),

$$\tan \theta = \frac{b\sin \theta}{\beta\left[\alpha + b\cos \theta\right]}$$

(6.6)
Equation (6.5) provides the special relativity analogue of (5.3), and (6.6) is the analogue of (5.4) in Newtonian cosmology. It is seen that, in the limit when \( c \) tends to infinite, equations (6.5), (6.6) reduce to the Newtonian equations (5.3), (5.4) respectively. From the optics of special relativity it follows that the angle \( \theta \) is again the fixed angular separation of \( Q, P \) as seen from \( O \).

Again we temporarily assume that \( b \) is given, with \( a > b \), but that all directions of \( b \) in space in \( \bar{S} \) are equally probable, i.e. that \( Q \) is equally likely to see his neighbour \( P \) in any direction in space (assumption (c) in §24). Then \( \Theta \), the mean value of \( \theta \) as defined in §5, is again formally given by (5.5) but with \( \theta \) now given in terms of \( \bar{\theta} \) by (6.6). The integral (5.5) is now somewhat different and more complicated owing to the factor \( \beta \) in (6.6), but it can still be evaluated exactly (appendix I). We find that \( \Theta \) is now given by

$$
\Theta = \frac{1}{2} \tau \frac{c^2}{\beta ab} \left[ 1 - \left( 1 - \frac{b^2}{c^2} \right)^{1/2} \right]
$$

(6.7)

As a check, we notice that this agrees with the Newtonian result (5.6) in the limit when \( c \) tends to
infinity, as we should expect. But we shall consider the exact correspondence between (6.7) and (5.6) in a different way at the end of this section.

Light propagation in special relativity presents no difficulties. It is one of the principles of the theory that the speed of light in an inertial frame is \( c \). Since time is measured (in \( S, \tilde{S} \)) at the event at which \( O, Q \) coincide and \( a \) is the velocity of \( Q \) relative to \( O \) in \( S \), the distance of \( Q \) from \( O \) at time \( t_Q \), as measured in \( S \), is \( at_Q \). Therefore, light emitted by \( Q \) at time \( t_Q \) reaches \( O \) at time \( t_0 \) if

\[
c(t_0 - t_Q) = at_Q
\]

giving

\[
c t_0 = (c + a) t_Q \quad (6.8)
\]

Also, when the standard formula for the doppler effect in special relativity is applied to \( Q \) receding from \( O \) with speed \( a \), we get for the red-shift \( z_{(8)} \),

\[
z = \left( \frac{\lambda_0}{\lambda_Q} \right) = \beta \left( 1 + \frac{a}{c} \right) \quad (6.9)
\]

where, as before, \( \lambda_0 \) is the wave-length of the radiation received at \( O \), as measured in \( S \),
and $\lambda_a$ is the wave-length of radiation emitted by $Q$ as measured in $\overline{S}$.

Since, at time $t_0$, observer $O$ sees $Q$ at distance $a t a$, he sees it as having angular diameter $\Delta$, where

$$\Delta = \frac{\ell}{a t a} = \frac{\ell}{a t_0} \left(1 + \frac{a}{c}\right)$$  \hspace{1cm} (6.10)

by using (6.8). Here we have again treated $Q$ as a sphere of proper-diameter $\ell$, noting that the diameter perpendicular to the sight-line is not affected by the recession of $Q$ from $O$, i.e. by the transformation (6.4) from $\overline{S}$ to $\overline{S}$. Also we have again assumed $\Delta$ to be a small angle. Although an exact expression for $\Delta$ is given in appendix II it is unlikely that, in practice, $\Delta$ will be required to a higher approximation than the one in (6.10). As before, we may now insert mean values for $\ell$ and by the same argument as in §5 we have for the mean value $\Delta^*$ of $\Delta$,

$$\Delta^* = \frac{\ell^*}{a t_0} \left(1 + \frac{a}{c}\right)$$ \hspace{1cm} (6.11)

From (6.7), taking mean values, and (6.11) we get for our measure of congestion $\Theta^* \Delta^*$,
Using (6.9) for the expression $\beta(1 + \frac{a}{c})$ we finally get

$$\frac{\Theta^*}{\Delta^*} = \left\{ \frac{\pi c^2}{2} \left[ \frac{1 - (1 - b^2/c^2)^{1/2}}{b} \right]^* \frac{a a_0}{\ell^* \beta(1 + \frac{a}{c})} \right\} \frac{1}{1 + z} \quad (6.12)$$

where the asterisk on the right denotes the mean value of the quantity in square-brackets. The fact that equation (6.12) is a more complicated function of than in the Newtonian cosmology case is of no immediate significance. All that matters here is that the quantity in curly brackets in (6.12) is certainly independent of $z$. It follows that the dependence of $\Theta^*/\Delta^*$ upon $z$ is precisely the same as before and so the conclusion stated at the end of section 5 applies also to the special relativity model.

**Transformation of Parameters.** The metric of the space-time of special relativity used here with $\gamma$, $\Theta$, $\phi$ as polar coordinates referred to $O$ is

$$ds^2 = -d\gamma^2 - \gamma^2(d\Theta^2 + \sin^2 \Theta d\phi^2) + c^2 dt^2 \quad (6.13)$$
Under the allowable space-time coordinate transformation
\[ t = \frac{1 + \frac{c^2}{r^2}}{1 - \frac{c^2}{r^2}} \tau \] (6.14)
\[ \varrho = \frac{r}{1 - \frac{c^2}{r^2}} c \tau \]
it is easily seen that the metric (6.13) transforms into
\[ ds^2 = c^2 d\tau^2 \left( 1 - \frac{c^2}{r^2} \right)^2 \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \] (6.15)

Anticipating the results of the following section (6.15) describes a particular general relativistic model of (7.1) with \( R(\tau) \equiv \tau, k = -1 \).
This was first proved by Kermack and McCrea (7) and shows that the special relativity model can be deduced from the general relativistic models as we have remarked earlier on in this section.

We saw above that the world-line of a typical galaxy is of the form (6.1), that is, say, \( \varrho = \hat{\omega} t \), where \( \hat{\omega} \) is a constant vector. We may write this in the form
\[ \varrho = \hat{\omega} t, \theta = \text{Constant}, \phi = \text{Constant} \] (6.16)
Using (6.14) equations (6.16) become
\[ t = \text{Constant}, \theta = \text{Constant}, \phi = \text{Constant} \] (6.18)
where the constant value of \( r \) is given by
\[
\frac{cr}{1 + \frac{1}{2} r^2} = h_\Phi \tag{6.17}
\]

It follows that all the galaxies have simply fixed spatial coordinates in the \((r, \theta, \phi, \tau)\) system. Moreover, the \( r, \theta, \phi \)-space (i.e. the 3-space \( \tau = \text{constant} \)) is seen to be a space of constant curvature at every instant \( \tau \) and so the form of (6.15) is invariant when the origin of these coordinates is transferred from \( O \) to any other galaxy (see following section).

In particular, if \( \bar{b} \) is the \( r \)-coordinate of \( P \) referred to \( Q \) then, since from (6.2) the \( k \)-parameter is \( b \), corresponding to (6.17) we have
\[
\frac{b}{c} = \frac{\bar{b}}{1 + \frac{1}{2} \bar{b}^2}
\]
whence
\[
\left(1 - \frac{b^2}{c^2}\right)^{\frac{\tau}{2}} = \frac{1 - \frac{1}{2} \bar{b}^2}{1 + \frac{1}{2} \bar{b}^2}
\]

It follows that the function \( \frac{c}{b} \left[1 - \left(1 - \frac{b^2}{c^2}\right)^{\frac{\tau}{2}}\right] \) occurring in (6.7), (6.12) is equal to \( \bar{b}^{\frac{\tau}{2}} \), i.e.
\[
\frac{c}{b} \left[1 - \left(1 - \frac{b^2}{c^2}\right)^{\frac{\tau}{2}}\right] = \frac{1}{2} \bar{b} \tag{6.18}
\]
Referring to the variables in (6.14), we see that along the world-line of $0$, since $r = 0$, we have $t = \tau$. Hence in place of $t_0$ in (6.10), (6.11), (6.12) we may write $\tau_0$, which is the epoch of observation in the new variables. Using this and (6.18), the formula (6.12) becomes

$$\frac{\Theta^*}{\Delta^*} = \left( \frac{1}{\mu} \right) \, \frac{b_{\xi}}{\ell_{\xi}^*} \, R_0 \frac{1}{1 + z}$$

(6.19)

where $R_0 = c \, \tau_0$.

(6.19) is now of exactly the same form as (5.13). We shall return to this feature in section 7.
7. General Relativistic Cosmology

The metrics describing the expanding general relativistic models are all included in

\[ ds^2 = c^2 d\tau^2 \frac{R^2(\tau)}{(1 - \frac{\tau^2}{2})^2} \left[ d\tau^2 + \tau^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \] (7.1)

(We write \( \tau \) in place of the more usual symbol \( \tau \) simply because \( \tau \) has been used in another sense in sections 5 and 6.) The metric (7.1) can be, and has been, obtained from some general kinematic and geometric considerations independently of the theory of relativity. The first derivation of (7.1) was given by Robertson (19), (20) and by Walker (21) and is usually called the Robertson-Walker metric. Their derivation is based on the assumption that the universe is homogeneous and isotropic about any (fundamental) observer who is at rest with respect to the mean motion of the matter in its own neighbourhood and on the assumption that the world-lines of the fundamental observers (particles) do not intersect except possibly at one singular point in the past (Weyl's postulate). The coordinates used in (7.1) are so-called co-moving coordinates and fundamental observers (substratum) are characterized as having fixed spatial coordinates \( (r, \theta, \phi) \).
Tolman ((1), page 364) gave another derivation of the metric \((7.1)\) based on the single assumption that the universe at large is spherically symmetric about any fundamental observer. Perhaps the best derivation of \((7.1)\) from the point of view of general relativity is due to McVittie ((2), page 138). His derivation is based on the assumption that the universe is smoothed-out and has spherical symmetry about any fundamental observer. With these assumptions only and by employing co-moving coordinates and making use of the field equations of general relativity he was able to obtain the metric \((7.1)\). Yet another derivation of \((7.1)\) was given very recently by Eisenhart (22) using the theory of continuous groups of motions.

The variable \(\tau\) occurring in \((7.1)\) is referred to as the cosmic or world-time and it is the time registered by a clock carried by a fundamental observer. The function \(R(\tau)\) is a disposable function of \(\tau\) only whose geometrical interpretation is that \(\kappa/R^2(\tau)\) is the constant curvature of the 3-space \(\tau = \text{constant}\). The curvature is, of course, constant for any given value of \(\tau\) but it generally varies as \(\tau\) varies. The symbol \(\kappa\) is a constant and we can, without loss of generality, take it to be equal to \(1, 0, -1\) according as to whether the
curvature is positive, zero, or negative respectively. Sometimes it will be convenient to write \( k = \frac{1}{k^2} \)
where \( k \) is a constant.

We shall assume that the galaxies we are dealing with are members of the "substratum" and therefore they have fixed spatial coordinates \( (r, \theta, \phi) \), i.e.
\[
    r = \text{Constant}, \quad \theta = \text{Constant}, \quad \phi = \text{Constant} \tag{7.2}
\]

We shall find it more useful to employ the metric (7.1) in the form
\[
    ds^2 = c^2 dt^2 - \left[ \frac{d\rho^2}{1 - k\rho^2} + \rho^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \tag{7.3}
\]
where we have taken the fundamental velocity \( c \) to be unity. (7.2) is obtained from (7.1) by the purely spatial coordinate transformation
\[
    \rho = \frac{r}{1 + \frac{k}{c^2} r^2} \tag{7.4}
\]

It follows from (7.2), (7.4) that the galaxies have fixed spatial coordinates in the \( (\rho, \theta, \phi) \) system too.

Once again we consider the galaxies 0, P, Q as in the previous sections. We take 0 to be the
point $\rho = 0$ and $Q$ to be the point $\rho = a$, $\theta = 0$ (remembering that, as in spherical polar coordinates, such particular values do define single points in the 3-space $\tau = \text{constant}$ without mentioning the other coordinates; in fact, $\phi$ has no meaning when $\theta = 0$, and $\theta$, $\phi$ have no meaning when $\rho = 0$).

We shall show that it is possible to transfer the spatial origin of these coordinates, i.e. the point $\rho = 0$ or $r = 0$ in the 3-space, from one galaxy (e.g. 0) to any other galaxy (e.g. Q) leaving the form (7.1) or (7.3) invariant. This is a fundamental property of (7.1) or (7.3) and is due to the fact that the 3-spaces $\tau = \text{constant}$ have constant curvature for any given value of $\tau$. The required transformation affecting this change of origin is discussed in Tolman's book ((1), page 370) but because of its importance and the wider use we are going to make of it we shall discuss it in detail. To avoid introducing square roots we shall write $k = \frac{1}{\kappa^2}$.

The metric (7.3), then, reduces to

$$ds^2 = d\tau^2 - R^2(\tau)\left[\frac{d\rho^2}{1 - \rho^2/k^2} + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)\right](7.5)$$
The values of \( k(1, 0, -1) \) now correspond to \( \omega_0^2 = 1, \infty, -1 \) respectively.

In the \((\rho, \theta, \phi, \tau)\) system with \(0\) at the spatial origin we have, then, the following table of coordinates for \(0, Q\) (dots denoting indeterminate values).

<table>
<thead>
<tr>
<th>((\rho, \theta, \phi))</th>
<th>(\rho)</th>
<th>(\theta)</th>
<th>(\phi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Galaxy (0)</td>
<td>(\alpha)</td>
<td>(\alpha)</td>
<td>(\alpha)</td>
</tr>
<tr>
<td>Galaxy (Q)</td>
<td>(\alpha)</td>
<td>(\alpha)</td>
<td>(\alpha)</td>
</tr>
</tbody>
</table>

*Table 1*

We change from the \((\rho, \theta, \phi)\) to the \((z_1, z_2, z_3, z_4)\)-system by the purely spatial transformations

\[
\begin{align*}
    z_1 &= R_0 \left( 1 - \rho^2 / R_0^2 \right)^{1/2} \\
    z_2 &= \rho \sin \theta \cos \phi \\
    z_3 &= \rho \sin \theta \sin \phi \\
    z_4 &= \rho \cos \theta
\end{align*}
\]

(7.6) changes the metric (7.5) to

\[
    ds^2 = d\tau^2 - R^2(\tau) \left[ dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2 \right]
\]

(7.7)
which shows that the 3-spaces \( \tau = \text{constant} \) described by (7.5) can be regarded as embedded in a euclidean space of dimensionality 4.

From table 1 and equations (7.6) it is easily seen that the new z-coordinates of O, Q are given as in the following table

<table>
<thead>
<tr>
<th>Galaxy 0</th>
<th>( Z_1 )</th>
<th>( Z_2 )</th>
<th>( Z_3 )</th>
<th>( Z_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Galaxy Q</td>
<td>( R_w )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Galaxy Q</td>
<td>( R_w (1 - \frac{a^2}{k^2}) )</td>
<td>0</td>
<td>0</td>
<td>( \alpha )</td>
</tr>
</tbody>
</table>

Table 2

We apply now the transformations, corresponding to a rotation in the \( z,z' \)-plane,

\[
\begin{align*}
Z_1 &= z_1 \cos \alpha + z_2 \sin \alpha \\
Z_2 &= z_2 \\
Z_3 &= z_3 \\
Z_4 &= -z_1 \sin \alpha + z_2 \cos \alpha
\end{align*}
\]

where

\[
\sin \alpha = \frac{\alpha}{k}, \quad \cos \alpha = \left(1 - \frac{a^2}{k^2}\right)^{\frac{1}{2}}
\]

It can easily be verified that under these transformations
the metric (7.7) remains invariant, i.e.

\[ ds^2 = d\tau^2 - R^2(\tau) \left[ d\bar{Z}_1^2 + d\bar{Z}_2^2 + d\bar{Z}_3^2 + d\bar{Z}_4^2 \right] \]  

(7.9)

From the table 2 and the equations (7.8) it follows that the coordinates of 0, Q in the \( \bar{Z} \)-system are given by

\[
\begin{array}{c|cccc}
(\bar{Z}_1, \bar{Z}_2, \bar{Z}_3, \bar{Z}_4) & \bar{Z}_1 & \bar{Z}_2 & \bar{Z}_3 & \bar{Z}_4 \\
Galaxy 0 & R_\omega \left( 1 - \chi^2 / k_\omega^2 \right)^{\frac{1}{2}} & 0 & 0 & -a \\
Galaxy Q & R_\omega & 0 & 0 & 0 \\
\end{array}
\]

Table 3

Now, the metric (7.7) was obtained from (7.5) by the transformation equations (7.6), and since the metric (7.9) is of exactly the same form as the metric (7.7), we can use transformations of the form (7.6) again to obtain (7.5) in a coordinate system \((\bar{\rho}, \bar{\theta}, \bar{\phi})\), say, in which its form remains invariant. Applying, therefore, the transformations

\[
\begin{align*}
\bar{Z}_1 &= R_\omega \left( 1 - \bar{\rho}^2 / k_\omega^2 \right)^{\frac{1}{2}} \\
\bar{Z}_2 &= \bar{\rho} \sin \bar{\theta} \cos \bar{\phi} \\
\bar{Z}_3 &= \bar{\rho} \sin \bar{\theta} \sin \bar{\phi} \\
\bar{Z}_4 &= \bar{\rho} \cos \bar{\theta}
\end{align*}
\]  

(7.10)
the metric (7.9), and through it the metric (7.5), transforms to
\[ ds^2 = d\tau^2 - R^2(\tau) \left[ \frac{d\bar{\rho}^2}{1 - \bar{\rho}^2 R^2} + \bar{\rho}^2 (d\bar{\sigma}^2 + \sin^2 \bar{\sigma} d\bar{\phi})^2 \right] \] which is of exactly the same form as (7.5). Further, the \((\bar{\rho}, \bar{\sigma}, \bar{\phi})\) coordinates of 0, Q, as obtained from table 3 and equations (7.10), are given by

<table>
<thead>
<tr>
<th>( (\bar{\rho}, \bar{\sigma}, \bar{\phi}) )</th>
<th>( \bar{\rho} )</th>
<th>( \bar{\sigma} )</th>
<th>( \bar{\phi} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Galaxy 0</td>
<td>( \alpha )</td>
<td>( \pi )</td>
<td>.</td>
</tr>
<tr>
<td>Galaxy Q</td>
<td>0</td>
<td>.</td>
<td>.</td>
</tr>
</tbody>
</table>

**Table 4**

It follows from table 4 that 0 has, in the new coordinate system \((\bar{\rho}, \bar{\sigma}, \bar{\phi})\), \( \bar{\rho} = \alpha, \bar{\sigma} = \pi \) as its coordinates, while Q is now at the spatial origin of \((\bar{\rho}, \bar{\sigma}, \bar{\phi})\). Hence by the above sequence of transformations the spatial origin at 0 is changed to another galaxy Q without changing the form of the metric (7.5).

Combining the transformation equations (7.6), (7.8), (7.10) we find, therefore, that the
transformation equations which changes the origin of spatial coordinates from 0 to Q without changing the form of the metric are given by

\[ R_\omega \left( 1 - \rho^2 \right)^{1/2} = R_\omega \left( 1 - \rho^2 \left( 1 - a^2 / R^2 \right) \right)^{1/2} + \rho a \cos \theta \] (a)

\[ \rho \sin \theta \cos \phi = \rho \sin \theta \cos \phi \] (b)

\[ \rho \sin \theta \sin \phi = \rho \sin \theta \sin \phi \] (c)

\[ \rho \cos \theta = \alpha \left( 1 - \rho^2 \right)^{1/2} + \rho \cos \theta \left( 1 - a^2 / R^2 \right)^{1/2} \] (d)

These can be solved in the form

\[ R_\omega \left( 1 - \rho^2 \right)^{1/2} = R_\omega \left( 1 - \rho^2 \left( 1 - a^2 / R^2 \right) \right)^{1/2} - \rho a \cos \theta \] (a)

\[ \rho \sin \theta \cos \phi = \rho \sin \theta \cos \phi \] (b)

\[ \rho \sin \theta \sin \phi = \rho \sin \theta \sin \phi \] (c)

\[ \rho \cos \theta = \alpha \left( 1 - \rho^2 \right)^{1/2} + \rho \cos \theta \left( 1 - a^2 / R^2 \right)^{1/2} \] (d)

It is seen from (7.12), (7.13) that we can, without loss of generality, take \( \phi = \phi \). Then the equations (7.13) reduce to

\[ R_\omega \left( 1 - \rho^2 \right)^{1/2} = R_\omega \left( 1 - \rho^2 \left( 1 - a^2 / R^2 \right) \right)^{1/2} - \rho a \cos \phi \] (a)

\[ \rho \sin \theta = \rho \sin \theta \] (b)

\[ \rho \cos \theta = \alpha \left( 1 - \rho^2 \right)^{1/2} + \rho \cos \theta \left( 1 - a^2 / R^2 \right)^{1/2} \] (c)

\( \phi = \phi \) (d)
These four equations are not all independent; in fact (7.14a) can be obtained from (7.14b), (7.14c). As we said before equations (7.14) leave the metric invariant while they transfer the spatial origin of coordinates from O to Q.

With Q as the spatial origin of coordinates O has the spatial coordinates \((\alpha, \pi, \cdot)\). Let the galaxy P (Q's neighbour) have coordinates \((\vec{\rho}, \vec{\sigma}, \vec{\tau})\), in this coordinate system, with \(\vec{\rho} = b\). It follows from equations (7.14b), (7.14c) that

\[
\tan \theta = \frac{b \sin \vec{\sigma}}{\alpha (1 - \frac{b^2}{R^2})^{1/2} + b (1 - \frac{\alpha^2}{R^2})^{1/2} \cos \vec{\sigma}} 
\]  

(7.15)

The angle \(\theta\) is, of course, the \(\theta\) -coordinate of P in the coordinate system in which O is at the spatial origin of coordinates.

It is a simple property of light-tracks in the space-time (7.1), or (7.5) that radiation received at the spatial origin must travel along radial null-geodesics (see (2), page 145). It follows that the angle \(\vec{\sigma}\) defines the direction in which Q sees P, and the angle \(\theta\) defined by (7.15) is the angular separation of Q, P as seen from O.

Equation (7.15) is the most general formula
defining $\theta$ in the problem we have been concerned with. All the formulae corresponding to (7.15) in the various cosmological models we have considered (or can be obtained from (7.1)) are particular cases of (7.15) in the appropriate coordinate system (see end of this section).

Equation (7.15) shows that the angle $\theta$ is a constant of time in accordance with our assumptions in §4 and the general features of the cosmological model under discussion.

Again we keep $b$ fixed for the moment, with $a > b$, but assume as before that all directions, in space, of $P$ as seen by $Q$ are equally probable. Then $\bar{\theta}$, the mean value of $\theta$ as defined in §5, is again given formally by (5.5) but with $\theta$ now given in terms of $\bar{\theta}$ by the much more complicated formula (7.15). The resulting integral can still be evaluated exactly to give, for the mean value $\bar{\theta}$ of $\theta$,

$$\bar{\theta} = \frac{1}{2} \frac{\pi}{ab} R^2 \left[ \frac{1}{1 - \left(1 - \frac{b^2}{R^2}\right)^{1/2}} \right]$$

(7.16)

This is the mean angular separation, as seen from $Q$, between any galaxy with a given $\alpha$-value and its neighbour. It is the most general formula for all cosmological models represented by, or derived from, the space-time...
(7.1). It involves no approximation and includes all space-times whose \( \tau \) = constant are of positive, zero or negative curvature. The results published in (16) and the results of Newtonian and special relativity cosmology can be derived from the above formula (see end of this section).

Spectral Red-shift. Generalizing results of the special theory of relativity concerning light-tracks, we shall assume that light, in matter-free regions, propagates on null-geodesics. Our cosmological model (7.1) or (7.5) is not, of course, free of matter but we shall assume, with all the other writers, that the light-tracks in the space-time (7.1), or (7.5) are the null-geodesics \( ds = 0 \) of these metrics. For radial null-geodesics \( d\theta = d\phi = 0 \) (7.5) gives, with \( ds = 0 \),

\[
d\tau^2 = \frac{R(\tau) d\rho}{(1 - \rho^2/\rho_0^2)^{1/2}}
\]

for inward light-propagation. It follows that, if radiation leaves \( Q \) at \( \tau_0 \) and reaches \( 0 \) at \( \tau_0 \), then \( \tau_0, \tau_0 \) must satisfy the equation

\[
\int_0^a \frac{d\rho}{(1 - \rho^2/\rho_0^2)^{1/2}} = \int_{\tau_0}^{\tau_0} \frac{d\tau}{R(\tau)} = F(\tau_0) - F(\tau_0), \quad \text{say,} \quad (7.17)
\]
where \( \frac{dF}{d\tau} = \frac{1}{R(\tau)} \).

Since \( a, R_w \) are constants it follows that the integral on the left-hand side of (7.17) must be a constant while the integral on the right-hand side is a function of \( \tau_0 \) and \( \tau_\alpha \) only, \( I(\tau_0, \tau_\alpha) \), say, where \( I \) is, of course, equal to the constant on the L.H.S. of (7.17). It follows that if radiation of wave-length \( \lambda_\alpha \), as measured at Q, is emitted during the time-interval \( \delta\tau_\alpha \) at \( \tau_\alpha \) and all this radiation is received at O during the time-interval \( \delta\tau_0 \) at time \( \tau_0 \) and at wave-length \( \lambda_0 \), as measured at O, \( \delta\tau_0 \) and \( \delta\tau_\alpha \) must be related by

\[
\delta I = \frac{\partial I}{\partial \tau_0} \delta\tau_0 + \frac{\partial I}{\partial \tau_\alpha} \delta\tau_\alpha = 0
\]

or, since, from (7.17), \( \frac{\partial I}{\partial \tau_0} = \frac{1}{R(\tau_0)} \) and \( \frac{\partial I}{\partial \tau_\alpha} = -\frac{1}{R(\tau_\alpha)} \)

\[
\frac{\delta\tau_0}{\delta\tau_\alpha} = \frac{R_0}{R_\alpha}
\]

where \( R_0 = R(\tau_0), R_\alpha = R(\tau_\alpha) \). Using formula (7.18) for the red-shift we can write

\[
{1 + z} = \frac{\lambda_0}{\lambda_\alpha} = \frac{R_0}{R_\alpha}
\]

(7.18)

Since the number of wave-crest emitted at Q (during \( \delta\tau_\alpha \)) are all received at O (during \( \delta\tau_0 \)) the last relation gives the well-known red-shift \( z \),
Let the galaxy $Q$ subtend a small angle $\Delta$ at $O$ at the epoch of observation $\tau_0$, so that $\tau_Q$ is again the epoch at which the radiation leaves $Q$. Treating $Q$ as a "sphere" of diameter $\ell$ and expressing from (7.5) that $\ell$ is in particular the proper-diameter perpendicular to the sight-line, we get (cf. (23) equation 5)

$$\ell = R(\tau_0) \alpha \Delta$$  \hspace{1cm} (7.19)

where $\alpha$ is the radial coordinate of $Q$. Taking mean values we have for the mean value $\Delta^*$ of $\Delta$,

$$\Delta^* = \frac{\ell^*}{R(\tau_0)\alpha}$$  \hspace{1cm} (7.20)

Inserting mean values in (7.16) and using (7.20) we have for the measure of congestion of the universe $\Theta^*/\Delta^*$,

$$\frac{\Theta^*}{\Delta^*} = \frac{\pi}{2} R\omega^2 \left[ \frac{1 - (1 - b^2 R\omega^2)^{1/2}}{b} \right] \frac{R(\tau_0)}{\ell^*}$$

Using formula (7.18) for the red-shift we can write $R_Q = \frac{R_0}{1+z}$; hence $\Theta^*/\Delta^*$ reduces to

$$\frac{\Theta^*}{\Delta^*} = \left\{ \frac{\pi}{2} R\omega^2 \left[ \frac{1 - (1 - b^2 R\omega^2)^{1/2}}{b} \right] \right\} \frac{R_0}{\ell^*} \frac{1}{1+z}$$  \hspace{1cm} (7.21)

where the asterisk on the block-brackets has the same
formal meaning as in §6. Although (7.21) is more complicated than in the two previous cases we notice that the quantity in curly-brackets is certainly independent of $z$. It follows that, even in this most general evolutionary model of the expanding universe, the dependence of $\Theta^*/\Delta*$ upon $z$ is precisely the same as before, namely $(1+z)^{-1}$. All the remarks at the end of §5 apply to this general case too.

When the formulation of $\Theta^*/\Delta*$ was given in paper I, in the case of general relativistic cosmology, the ratio $\Theta^*/\Delta*$ was worked out separately for the three cases $k=1$, 0, $-1$ i.e. when the 3-space $\tau = \text{constant}$ is of positive, zero and negative curvature (of course we gave the working only in the case of $k=1$; the case of $k=0,-1$ is obtained in a similar way). In this thesis we made a definite generalization in the sense that $\Theta^*/\Delta*$ has been worked out for a 3-space with an arbitrary curvature $(k=1,0,-1)$. In fact the expression (7.21) shows explicitly how $\Theta^*/\Delta*$ depends on the curvature of the 3-space. The particular case in which $k=1,0,-1$ can readily be obtained from the general formula (7.21). We shall obtain the results given in paper I (section 5) and we shall also show that the Newtonian and special
relativity results of sections 5 and 6 can all be obtained from the general relativistic results obtained in this section.

(a) Results of paper I (section 5).

With \( \kappa^2 = \frac{1}{\kappa} \) equation (7.21) reduces to

\[
\frac{\Theta^*}{\Delta^*} = \left\{ \frac{1}{\kappa^2} \left[ \frac{1 - (1 - b^2)^{1/2}}{b} \right] \right\} \frac{2}{1 + \frac{1}{4} b^2}
\]

Applying the transformation (7.4) for \( b \), i.e.

\[
b = \frac{b}{1 + \frac{1}{4} b^2}
\]

we get

\[
(1 - b^2)^{1/2} = \frac{1 - \frac{1}{4} b^2}{1 + \frac{1}{4} b^2}
\]

Therefore

\[
\frac{1 - (1 - b^2)^{1/2}}{b} = \frac{b}{\kappa^2}
\]

Therefore, we finally get for \( \frac{\Theta^*}{\Delta^*} \),

\[
\frac{\Theta^*}{\Delta^*} = \left( \frac{\kappa}{\kappa^2} \frac{b^*}{\ell^*} \frac{2}{1 + \frac{1}{4} b^2} \right) \frac{1}{1 + \frac{1}{4} b^2} \]

(7.22)

This is precisely equation (5.15) given in I.

Not only this final result, but also the transformation equations (5.4), (5.5), (5.7), (5.8) given in I transforming the spatial origin of the
coordinates from galaxy 0 to galaxy Q, the expressions (5.9) for \( \tan \theta \), and (5.10) for the mean value \( \bar{\theta} \) can be obtained from the corresponding formulae of this section.

With \( k = 1 \), write

\[
\rho = \sin \chi
\]  \hspace{1cm} (7.23)

Then the metric (7.3) reduces to

\[
ds^2 = d\tau^2 - R^2(\tau) \left\{ dx^2 + \sin^2 x (d\theta^2 + \sin^2 \theta d\phi^2) \right\}
\]  \hspace{1cm} (7.24)

which is the metric (5.3) of I.

Writing \( \alpha = \sin \chi \), in accordance with (7.23) equations (7.12a), (7.14b), (7.12d), (7.14c) reduce, respectively, to

\[
\begin{align*}
\cos \bar{x} &= \cos \chi \cos \alpha + \sin \chi \sin \alpha \cos \bar{\phi} \\
\sin \bar{x} \sin \bar{\phi} &= \sin \chi \sin \theta \\
\sin \bar{x} \cos \bar{\phi} &= -\sin \chi \cos \alpha + \sin \chi \cos \alpha \cos \theta \\
\sin \chi \cos \bar{\phi} &= -\sin \chi \cos \alpha + \sin \chi \cos \alpha \cos \bar{\phi}
\end{align*}
\]

These are precisely the equations (5.4), (5.5), (5.7), (5.8), respectively, given in I.

Applying equation (7.23) to (7.15), with \( R^2 = 1 \) and \( b = \sin \gamma \), we get

\[
\tan \theta = \frac{\sin \gamma \sin \bar{\phi}}{\sin \alpha \cos \gamma + \sin \gamma \cos \alpha \cos \bar{\phi}}
\]
which is equation (5.9) of I.

With the above values of $b$, $R_w$ equation (7.16) gives

$$\Theta = \frac{\pi}{2} \cos \alpha \tan \left( \frac{\alpha}{2} \right)$$

This is equation (5.10) of I.

(b) Derivation of Newtonian cosmology

Since the space of Newtonian cosmology is euclidean $k = 0$, i.e. $R_w^2 = \infty$. Putting $R_w^2 = \infty$ equation (7.15) reduces to

$$\tan \Theta = \frac{b \sin \Theta}{a + b \cos \Theta}$$

This is equation (5.4) of the Newtonian cosmology.

From (7.16) with $R_w^2 \rightarrow \infty$ we get

$$\Theta = \lim_{R_w^2 \rightarrow \infty} \frac{\pi}{2ab} R_w^2 \left[ 1 - \left( 1 - \frac{b^2}{2R_w^2} + O \left( \frac{1}{R_w^2} \right) \right) \right]$$

$$= \frac{\pi b}{2a}$$

This is equation (5.6).

Again, by taking the limit as $R_w^2 \rightarrow \infty$, equation (7.21) gives

$$\frac{\Theta^*}{\Delta^*} = \left( \frac{\pi}{4} - \frac{b^*}{\ell^*} R_0 \right) \frac{1}{1 + z}$$

which is equation (5.13).
We note from (7.4) that when \( k = \frac{1}{R_0^2} \rightarrow 0 \) the coordinates \( \rho, r \) are equal. Therefore it makes no difference if we find the \textit{various} limits above as \( R_0^2 \rightarrow 0 \) in the \( \rho \) or \( r \)-system of coordinates.

The fact that Newtonian results, in this case, can be derived from the general relativistic results is an additional indication that Newtonian cosmology gives results similar to those of general relativistic cosmology.

\( \textbf{c) Derivation of the special relativity cosmology} \)

Let us apply the space-time coordinate transformations (see (7))

\[
\begin{align*}
\tau &= \tau \cosh x \\
\varphi &= \tau \sinh x
\end{align*}
\]

\[\text{(7.25)}\]

to the special relativity metric

\[ ds^2 = -dt^2 - g^2(d\theta^2 + \sin^2 \theta d\phi^2) + d\tau^2 \]

\[\text{(7.26)}\]

of section 6. It is easily seen that (7.26) transforms to

\[ ds^2 = d\tau^2 - \tau^2 \left\{ d\chi^2 + \sinh^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2) \right\} \]

\[\text{(7.27)}\]
This shows that the 3-space $\tau = \text{constant}$ is of constant negative curvature at any given value of $\tau$.

It follows from (6.16), i.e. $\varphi = \lambda t$, the equation representing the world-line of the galaxies, and equations (7.25) that

$$h = \frac{g}{t} = \tanh \chi$$

(7.28)

In the metric (7.3) we put $k = -1$ and we perform the transformation

$$\rho = \sinh \chi$$

(7.29)

It is easily seen that (7.3) reduces to

$$ds^2 = d\tau^2 - R^2(\tau) \left\{ d\chi^2 + \sinh^2 \chi \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right\}$$

(7.30)

It is seen that (7.27) is a special case of (7.30) with $R(\tau) = \tau$. This we have shown in section 6 too.

In the $(\chi, \theta, \phi, \tau)$ system, equation (7.15) for $\tan \theta$ becomes, with $k = -1$,

$$\tan \theta = \frac{\sinh \chi \sin \theta}{\sinh \chi \cosh \gamma + \sinh \gamma \cosh \chi \cos \theta}$$

(7.31)

where, according to (7.29), we have put

$$a = \sinh \alpha$$

$$b = \sinh \beta$$
Equation (7.31) can be written in the form
\[
\tan \theta = \frac{\tanh \gamma \sin \varphi \, \sech \alpha}{\tanh \gamma + \tanh \alpha \cos \varphi}
\]

Hence, using equation (7.28) and the identity
\[
\sech \alpha = \left(1 - \tanh^2 \alpha\right)^{1/2}
\]
this reduces to
\[
\tan \theta = \frac{b \sin \varphi}{\beta \left[b + a \cos \varphi\right]}
\]
where \(\beta = \left(1 - \alpha^2\right)^{-1/2}\)

This is precisely equation (6.6) of special relativity.

Using equation (7.29) and equation (7.16) for the mean angular separation \(\Theta\), with \(R_w = -1\), we get
\[
\Theta = \frac{\pi}{\tanh \alpha} \left[1 - \cosh \gamma \right]^{1/2}
\]
\[
\Theta = \frac{\pi}{2 \beta} \left[1 - \tanh \gamma \right]^{1/2}
\]

Using equations (7.32), (7.28) we finally get
\[
\Theta = \frac{\pi}{2 \beta b} \left[1 - (1 - b^2)^{1/2}\right]
\]
which is equation (6.7). In a similar way all results of the special relativity cosmology obtained in section 6 can be deduced from the general relativistic results.
8. Steady-state Cosmology

The metric describing the steady-state cosmological model is given by

$$ds^2 = d\tau^2 e^{2\tau/T} \left\{ dr^2 + r^2 (d\phi^2 + \sin^2 \phi d\phi^2) \right\}$$  \hspace{1cm} (8.1)

where $T$ is fixed, being the "Hubble time" for the model. (8.1) is a particular case of the Robertson-Walker metric which, as was remarked in section 7, can be derived from purely kinematical and geometrical considerations. In (8.1) the function $R(\tau)$ is equal to $e^{\tau/T}$ and the curvature, $k$, of the 3-space $\tau = \text{constant}$ is zero. These values of $R(\tau)$ and $k$ are obtained by demanding, further, that the universe must be in a steady-state (see Bondi (6), page 145, McCrea (24)). Any particular "galaxy" is again fixed in the $(r, \phi)$ coordinates, which expresses the assumption (see Section 8.4 (a)) that it shares in the cosmical expansion only. But despite this expansion new galaxies are continually coming into existence from newly created matter so as, in fact, to maintain the postulated steady-state character of the universe. A galaxy once formed is assumed to remain in existence and, of course, to remain fixed in the $(r, \phi)$ coordinates.
Let a galaxy Q play the same part as in the previous sections. Transforming to Q as origin of the spatial coordinates \((\bar{r}, \bar{\theta}, \bar{\phi})\) (the possibility of such transformation was proved in §7) the metric becomes

\[
dS^2 = d\bar{r}^2 - e^{2\bar{\tau}/T} \left\{ d\bar{r}^2 + r^2 (d\bar{\theta}^2 + \sin^2\bar{\theta} d\bar{\phi}^2) \right\} \tag{8.2}
\]

To find the ratio \(\Theta^*/\Delta^*\) we proceed in three distinct ways.

(a) In this case we can idealize the steady-state concept as follows. Q is the origin of \((\bar{r}, \bar{\theta}, \bar{\phi})\) in the space-time \((\bar{r}, \bar{\theta}, \bar{\phi})\). At epoch \(\tau = 0\) let a galaxy \(P_0\) come into existence with \(\bar{r} = \bar{b}\) and suppose \(P_0\) is then the nearest neighbour of Q. At epoch \(\tau = \tau_1\), say, let another galaxy \(P_2\) come into existence with \(\bar{r} = \bar{b} e^{-\tau_1/T}\); at \(\tau = 2\tau_1\) let \(P_2\) come into existence with \(\bar{r} = \bar{b} e^{-2\tau_1/T}\), and so on.

Consider the transformations

\[
\begin{align*}
\tau &= \tau' + n \tau_1 \\
\bar{r} &= \bar{r}' e^{-n \tau_1/T}
\end{align*} \tag{8.3}
\]

where \(\tau_1\) is fixed. It is easily seen that (8.3) transforms the metric (8.2) into

\[
dS^2 = d\tau'^2 - e^{2\tau'/T} \left\{ d\tau'^2 + \tau'^2 (d\theta^2 + \sin^2\theta d\phi^2) \right\} \tag{8.4}
\]
which shows that (8.3) leaves the metric (8.2) invariant. It follows, therefore, that during the interval \( n \tau_1 < \tau < (n+1) \tau_1 \), the galaxy \( P_1 \) bears to \( Q \) the same relationship as that borne by \( P_0 \) during the interval \( 0 < \tau < \tau_1 \). Thus in successive intervals \( \tau_i \), the part of \( Q \)'s nearest neighbour is taken successively by \( \ldots, P_0, P_1, P_2, \ldots, P_n \ldots \) and each of these in turn behaves in exactly the same way, during the corresponding interval of time, as seen from \( Q \).

This is the simplest way in which \( Q \)'s possession of a nearest neighbour can be rendered "steady" (in the mean).

Finally, suppose \( \tau_i \) is small compared with \( \tau \).

Then, from what we have proved above, we can say that the galaxy which is nearest neighbour to \( Q \) at \( \tau_q \) has

\[
F = b e^{-\frac{\tau_q}{\tau}}
\]

Also we suppose, as we have done before, that \( P \) is equally likely to be in any direction in space as seen from \( Q \). Then it is easily seen, without going into details, that, corresponding to equation (7.22) we should now have

\[
\frac{\Theta^*}{\Delta^*} = \left[ \frac{1}{4} \tau \frac{b}{c^*} R_0 e^{-\frac{\tau_q}{\tau}} \right] \frac{1}{1+z}
\]  

(8.5)

The red-shift \( z \) can be calculated in the same way as in §7. It follows from (7.18), with

\[
R(\tau) = e^{\frac{\tau}{\tau}} \quad \text{that}
\]

\[
1+z = R_0 e^{-\frac{\tau_q}{\tau}}
\]  

(8.6)
Thus (8.5), (8.6) give for the measure of congestion

\[ \frac{\Theta^*}{\Delta^*} \]

\[ \frac{\Theta^*}{\Delta^*} = \frac{1}{\pi} \frac{b^*}{c^*} \]  \hspace{1cm} (8.7)

As we expect in the case of the steady-state universe, this ratio is a \textbf{constant} which is independent both of the distance of the system observed and also of the epoch of observation.

(b) In the above method observations are concentrated on a particular galaxy, partaking in the general cosmical expansion, and the steady-state. Concept was merely interpreted as maintaining a neighbour P at a fixed proper distance from Q. Here observation is concentrated on a Q-object at a particular fixed proper-distance from O. Then we shall interpret the steady-state as maintaining a Q-object at this proper distance from O and also maintaining a P neighbour of Q at a fixed proper distance from Q. Then by the same argument as in (a) we can say that the distances \( \tilde{r}_p \), \( r_Q \) of P, Q from Q, O, respectively, at time \( \tau_Q \) are given by

\[ \tilde{r}_p = b e^{-\tau_Q/\tau} \]

\[ r_Q = a e^{-\tau_Q/\tau} \]  \hspace{1cm} (8.8)
Then, with these coordinates, \( \Theta \) the mean value of \( \theta \) is given by (see equation (7.16))

\[
\Theta = \lim_{R^2 \to \infty} \frac{\pi R^2}{2ab} e^{-2\tau_a/T} \left[ 1 - \left( 1 - \frac{\tau^2}{R^2} \right)^{1/2} \right]
\]

\[
= \frac{1}{\pi} \frac{b}{a}
\]

(8.9)

If the angular diameter \( \Delta \) of \( Q \) is small and \( \ell \) is the proper diameter of \( Q \) then we get from (8.8), (8.1)

\[
\ell = e^{\tau_a/T} a e^{-\tau_a/T} \Delta
\]

\[
= a \Delta
\]

(8.10)

If \( \Delta \) is used as a criterion of distance from \( O \) then (8.10) confirms our initial assumption that we are making observations at a fixed proper distance from \( O \).

Taking means for \( \bar{b} \), \( \ell \) and dividing (8.9) by (8.10) we get

\[
\frac{\Theta^*}{\Delta^*} = \frac{1}{\ell} \frac{\tau^*}{b^*}
\]

which is precisely equation (8.7).
(c) In this method we shall assume that \( \theta \), the angular separation, is a small angle, and regard the steady-state concept as merely maintaining a neighbour \( P \) at a fixed proper-distance from \( Q \) (as in (a)). Then analogously to (7.19) we have

\[
\vec{b} \sin \vec{\theta} = R(\tau_0) \alpha \theta \tag{8.11}
\]

as seen from the metric (8.1). Here, in the coordinate system \((r, \theta, \phi)\), \(0\) is at the spatial origin \( r = 0 \), and \( Q \) has \( r = \alpha \); \( \vec{\theta} \) is the inclination of \( QP \) to the sight-line \( OQ \). If \( \Theta \) is the average value of \( \theta \) for all directions of \( QP \) in the 3-space of \( Q \), then (8.11) gives

\[
\bar{\alpha} = \frac{1}{\pi} \bar{b} \tag{8.12}
\]

For the angular magnitude, \( \Delta \), of \( Q \) we have, corresponding to (7.19)

\[
R(\tau_0) \alpha \Delta = \ell \tag{8.13}
\]

Averaging over \( \bar{b} \), \( \ell \) and dividing (8.12) by (8.13) we obtain

\[
\frac{\Theta^*}{\Delta^*} = \frac{1}{\pi} \bar{b} \frac{\ell^*}{\ell^*}
\]

which is the expression (8.7). But this derivation holds only when \( \Theta^* \) as well as \( \Delta^* \) is a small angle.

These simple derivations of (8.7) suffice to
confirm, by comparison with the evolutionary formula (7.21) or (7.22), that there is the expected difference between the steady-state and the evolutionary models in regard to the dependence of $\Theta^*_{/\Delta^*}$ upon distance (as denoted by the red-shift $\mathcal{z}$).

The first derivations (a), (b) of (8.7) are designed to indicate the lines of a more general treatment. Instead of the fixed interval $\mathcal{T}$ we ought to consider a random interval and the mean value of this interval ought not necessarily to be small compared with $\mathcal{T}$. We have not obtained a satisfactory general treatment; we are not certain that such a treatment would yield precisely (8.7). It would, however, be mainly of mathematical interest only since the present simplified treatment suffices to show the essential difference between the steady-state and the evolutionary models. The result that for an evolutionary expanding universe $\Theta^*_{/\Delta^*}$ depends on the epoch of observation as well as on the distance (as denoted by $\mathcal{z}$) of the observed system while $\Theta^*_{/\Delta^*}$ is the same at all times and at any distance from the observer was what we set out to prove. We now discuss the possibility of applying our results to the actual universe.
9. Possibility of application

General considerations
(a) The sole purpose of any application would be to find whether, in the actual universe, the ratio $\Theta^*/\Delta^*$ depends upon $Z$ (or any other criterion of distance). An observational investigation need not, in practice, deal with many $Z$-values. In the first instance, it would be natural to try to determine a value of the ratio for objects at the greatest possible distance at which they can be studied, and then to compare the results for similar objects at the smallest convenient distance. Then the main consideration would be for there to be, at both distances, observable objects of the sort required in sufficient numbers for the determination of significant average values of $\Theta^*, \Delta^*$. The accuracy demanded is very considerable unless a very large value of $Z$ can be used. But one interest of the foregoing results is to show in the simplest possible way how small is the observable difference between a steady-state and an evolving expanding universe (being measured simply by the factor $/\pm Z$) except at large red-shifts.

The advantage of our criterion is that the effect we are discussing is a first order effect in $Z$. 
Basically, therefore, no other observable means of discriminating between the steady-state and other theories can demand less accuracy. As regards any proposed application it would therefore obviously be for consideration as to whether it could be achieved with less labour than the methods of discrimination so far proposed, most of which demand observations out to some value of $z$ instead of, as here, observations at selected values of $z$.

(b) The objects typified by "galaxy Q" in the theory must belong to some recognizable single category, but they need not be actual individual galaxies. They must have a measurable size and, in order to apply the theory as formulated here, they must have spectral feature from which the red-shift can be determined.

(c) The objects typified by "galaxy P" must also belong to a single category, but this need not be the same as for Q. In order to apply the theory in its simple form, it is not explicitly required that either the size or the red-shift of a P-object be measurable. It is, of course, necessary to be able to infer that anything used as a P-object is not gravitationally bound to the corresponding Q-object.
(d) There would be advantages in selecting categories of objects that are not very numerous. This should make it easier to know that pairs of the objects are not gravitationally bound to each other, and would probably make it easier to recognize "neighbours". It may be recalled here that $\Theta^*$ in (7.22) is not necessarily a small angle, and so we may deal with neighbours that are well separated in the sky.

(e) Clearly, there are possible adaptations and variants of the simple theory that might be more feasible for use in practice. For example, it would be natural to measure the separation of each Q-object from several neighbours rather than from a single one. Again, for application to P-objects for which red-shifts can be obtained, the theory could be formulated so as to apply to P-objects whose red-shifts differ by less than a certain amount from the red-shift of the corresponding Q-object. This might help in giving a more more usable definition of "neighbours".

(f) In all observational problems of the present kind there is the ever-present difficulty of selection effects. As regards the present problem these have been
cogently stated by Neyman and Scott (17). But we shall see below how some of these difficulties might be overcome.

(g) There is a complication in that, if the universe is evolving, objects otherwise in the same category may exhibit evolutionary effects correlated with distance. In particular, the intrinsic size may depend upon age and so upon distance. This has also been stressed recently by Davidson (18) whose solution of the "angular size," "angular separation" problem takes account of any possible evolutionary effects. Our formula (7.22) cannot be strictly applied in this case. However, in the first place, there is no reason why any such dependence of the intrinsic size upon distance should exactly mask the calculated dependence of \( \Theta^* / \Delta^* \) upon \( z \). Consequently, were \( \Theta^* / \Delta^* \) found observationally to be independent of \( z \), this would still be strong support for the steady-state model; on the other hand, any dependence upon \( z \), even if not precisely that predicted by the simple theory of evolutionary models, would still count against the steady-state theory. In the second place, even in evolutionary models, certain "size"-parameters would be expected to depend very little upon age, for example,
the distance between the components of a binary galaxy.

**Possible applications**

(a) The now obvious type of object to consider is clusters of galaxies. In principle, a feasible procedure might be the following. Consider first clusters at the greatest possible distance. As the size of a cluster, i.e. the quantity $\Delta$, we may take the mean angular separation of, say, the five brightest galaxies taken in pairs. As a measure of $\Theta$, we may take the mean angular separation of these galaxies from the first brightest galaxies, taken in pairs, in the nearest cluster. The averages $\Theta^*$, $\Delta^*$ would then be got from as many pairs of clusters as possible at about the same distance. Attention would then be transferred to clusters at the least possible distance. From the empirical relation between red-shift and apparent magnitude ($25$), it ought to be possible to determine which of the nearby clusters contain galaxies as bright as those observed in the remote clusters. Only such nearby clusters should be considered. In this way, the selection effects foreseen by Neyman and Scott ($17$) should be avoided.

(b) Probably the only possibility, even in
principle, of using an individual galaxy as a Q-object would be to take it to be, say, the brightest galaxy in the nearest cluster. But it is unlikely that the angular sizes of individual galaxies could be measured for sufficiently remote systems. A slightly better hope would be to use, instead of the size of individual galaxies, the angular separation between the components of binary galaxies, provided these could be recognized in sufficiently distant clusters.

(c) It is conceivable that future development in radio-astronomy may provide the sort of observations required, either alone or in combination with official results. As a case of such a combination, the Q-objects might be large clusters of galaxies observed optically while the P-objects might be radio-sources. But for this radio-astronomy would have to find some means of measuring red-shifts (or some feature that can be correlated with the red-shift) sufficiently accurately to aid a decision as to which radio sources are neighbours of any one of the optical sources employed.
10. The smoothed-out universe

All the models that we have been considering are models of the smoothed-out universe. That is to say, in comparing theory and observation, we make the hypothesis that the behaviour of the actual universe on a sufficiently large scale is the same as it would be were its contents in the form of a uniform ideal fluid. It is necessary briefly to consider the implications of this procedure.

Consider in this context a cluster of galaxies. If, as usual, we regard this as a system held together by its own gravitation, then its member galaxies do not recede from each other in consequence of the cosmical expansion. Except as a result of relaxation (in the sense of the dynamical theory of gases) or of loss of mass (by radiation etc.), the clusters will preserve a constant size. It follows that "a sufficiently large scale" in the above sense must mean at any rate a scale greater than the dimensions of a single cluster. The discussion in section \( \phi \) is based upon the implicit assumption that this is all that is required. More precisely, it was assumed that the clusters are randomly distributed in space and that their mutual recession is the same as if the universe were filled with a
uniform fluid.

Recently, however, it has been suggested that the actual universe may exhibit a phenomenon of second-order clustering, or of clusters of clusters (17). The existence of the phenomenon is not established; such evidence as there is may be merely a consequence of comparing the actual universe with a statistical model based upon unacceptable physical assumptions (see McCrea (17)). Nevertheless, it is worth asking how such a phenomenon, if real, would affect our present discussion.

The phenomenon might mean that the actual universe possesses an hierarchical structure. If so, the smoothed-out universe would be meaningless. However, at present there is no indication that we need pursue this possibility.

Supposing then the smoothed-out universe still to have significance on a sufficiently large scale, it is clear in the first place that a clustering of clusters might require some elaboration of the meaning of "neighbouring clusters" for the purposes contemplated in section 9. Further, with any particular definition of neighbouring clusters, it must be asked whether a pair of neighbours would recede from each other with the rate of cosmical expansion calculated for the
smoothed-out universe. (For various reasons, it appears unnecessary to consider the possibility of the clusters being gravitationally bound to each other; that being so, their mutual recession is the cosmical expansion.) The rate might depend, say, upon whether the pair belongs to the same cluster of clusters or to two different ones.

If the rate of expansion does vary from place to place in the universe, depending upon the distribution of clusters, then we should expect that the empirical relation between the red-shift and the apparent magnitude of galaxies would show differences from one direction in space to another. For the sight-line would encounter different groupings of the clusters in different directions. Actually, no differences have been found to within the accuracy of the observations. There may, moreover, be a physical reason for this, since the universe may contain much more diffuse matter distributed through space between galaxies and clusters than there is matter in the galaxies. So the contents of the universe may actually be much more nearly uniform than appears from the distribution of visible matter alone. For this reason, even a clustering of clusters, if real, need not denote a serious departure of the universe from uniformity.
To sum up, some inadequacy of the smoothed-out representation of the universe has to be kept in mind as a possibility in any discussion of observable relations, but as yet there is nothing to indicate that it is not adequate for such relations as we have considered in this thesis.
Appendix I. Evaluation of Integrals

We consider first the integral leading to the general equation (7.16) for $\theta$.

is given by the integral (5.5), i.e.

$$2\theta = \int_0^\pi \frac{d\theta}{d\bar{\theta}} \cos \bar{\theta} \, d\bar{\theta}$$  \hspace{1cm} (1)

where

$$\tan \theta = \frac{b \sin \bar{\theta}}{a(1 - \frac{b^2}{\alpha^2})^{1/2} + b(1 - \frac{a^2}{\alpha^2})^{1/2} \cos \bar{\theta}}$$  \hspace{1cm} (2)

(For convenience we write $\Omega^2 = R^2$.)

Differentiating (2) to obtain $a\xi^2 \frac{d\xi}{d\bar{\theta}}$ (which is identically equal to $(1 + \tan^2 \theta) \frac{d\theta}{d\bar{\theta}}$) and again using (2) to evaluate $(1 + \tan^2 \theta)$ we find

$$\frac{d\theta}{d\bar{\theta}} = \frac{\Omega^{-2} \left[ ab(1 - \frac{b^2}{\alpha^2})^{1/2} \cos \bar{\theta} + b^2(1 - \frac{a^2}{\alpha^2})^{1/2} \right]}{1 - \left[(1 - \frac{a^2}{\alpha^2})^{1/2}(1 - \frac{b^2}{\alpha^2})^{1/2} - \frac{ab \cos \bar{\theta}}{\alpha^2} \right]^2}$$  \hspace{1cm} (3)

Write

$$A = (1 - \frac{a^2}{\alpha^2})^{1/2}$$

$$B = (1 - \frac{b^2}{\alpha^2})^{1/2}$$  \hspace{1cm} (4)
Substituting (3) in (1) we get for the integrand \(\frac{d\theta}{d\bar{\theta}} \cos \bar{\theta}\)

\[
\frac{d\theta}{d\bar{\theta}} \cos \bar{\theta} = \frac{\Omega^{-2} b \cos \bar{\theta} \left[ \alpha B \cos \bar{\theta} + b A \right]}{1 - \left[ AB - \frac{ab \cos \bar{\theta}}{\Omega^2} \right]^2}
\]

This can be written in the form

\[
\frac{d\theta}{d\bar{\theta}} \cos \bar{\theta} = \frac{\Omega^2}{2ab} \left[ \frac{(1-AB)(B-A)}{(1-AB + \frac{ab \cos \bar{\theta}}{\Omega^2})} + \frac{(1+AB)(B+A)}{(1+AB - \frac{ab \cos \bar{\theta}}{\Omega^2})} \right] - 2B
\]

Remembering that

\[
\int_0^\pi \frac{du}{N + NC\cos u} = \frac{\pi}{(M^2 - N^2)^{1/2}} \quad (M > N)
\]

we find that

\[
\frac{\Omega^2}{2ab} \left( 1-AB \right) \left( B-A \right) \int_0^\pi \frac{d\bar{\theta}}{\left[ (1-AB) + \frac{ab \cos \bar{\theta}}{\Omega^2} \right]} = \frac{\Omega^2}{2ab} \left( 1+AB \right) \left( B-A \right) \left[ \frac{\pi}{\left( 1-2AB + A^2B^2 - \frac{a^2b^2}{\Omega^2} \right)^{1/2}} \right]
\]

But from (4) we get

\[
A^2 = \left( 1 - \frac{a^2}{\Omega^2} \right)
\]

\[
\alpha^2 = \Omega^2 \left( 1 - A^2 \right)
\]

which gives
Similarly \( b^2 = \Omega^2(1 - B^2) \) \hspace{1cm} (6)

Substituting (6) in (5) we finally get, for (5), the value

\[
\frac{\Omega^2}{2ab} \left(1 - AB\right) \left(B - A\right) \frac{\pi}{(B - A)} = \frac{\Omega^2}{2ab} \left(1 - AB\right) \pi \hspace{1cm} (7)
\]

Similarly

\[
\frac{\Omega^2}{2ab} \left(1 + AB\right) \left(B + A\right) \int_{0}^{\pi} \frac{d\bar{\theta}}{\left(1 + AB\right) - \frac{ab}{\Omega^2} \cos \bar{\theta}}
\]

\[
= \frac{\Omega^2}{2ab} \left(1 + AB\right) \pi \hspace{1cm} (8)
\]

Therefore \( \Theta \) as given by (1) is equal to

\[
\Theta = \frac{1}{2} \frac{\pi}{ab} \Omega^2 (1 - B)
\]

\[
= \frac{1}{2} \frac{\pi}{ab} R_\omega^2 \left[1 - \left(1 - \frac{b^2}{R_\omega^2}\right)^{1/2}\right] \hspace{1cm} (9)
\]

by using (4).

(9) is precisely equation (7.16).

Putting

\[
R_\omega^2 = 1, \quad a = \sin \alpha, \quad b = \sin \beta \quad \text{in (9)}
\]
\[
\theta = \frac{\pi}{2} \frac{1 - \cos \phi}{\sin \alpha \sin \phi} = \frac{\pi}{2} \sec \alpha \tan \left( \frac{\phi}{2} \right)
\]

This is equation (5.10) of paper I.

Similarly putting

\[ R^2 = -1, \quad a = \sinh \alpha, \quad b = \sinh \beta \quad \text{in (9)} \]

we get

\[
\theta = \frac{\pi \text{sech} \alpha}{2 \tanh \alpha \tanh \beta} \left[ 1 - \text{sech} \beta \right]
\]

Putting \( a = \tanh \alpha, \quad b = \tanh \beta \) (see 7.28)
and using the identity

\[
\text{sech} \alpha = (1 - \tanh^2 \alpha)^{1/2} = (1 - a^2)^{1/2}
\]

we get

\[
\theta = \frac{\pi \beta}{2 a b} \left[ 1 - (1 - b^2)^{1/2} \right], \quad \beta = (1 - a^2)^{-1/2}
\]

This is equation (6.7) of Special Relativity Cosmology.

Finally if we let \( R^2 \to \infty \) equation (9) reduces to

\[
\theta = \frac{\pi b}{4a}
\]

which is equation (5.6)

of Newtonian Cosmology. But the integral arising from (5.4) is also easy to evaluate directly.
Appendix II. Relativistic Apparent Size

Special relativity. As we remarked earlier, we treat this problem not because of its possible practical use but mainly because of its intrinsic interest. We consider first the case of special relativity and we use the notation of section 6. Let "galaxy Q" be a sphere with centre Q and proper diameter \( \delta \); as before, it is fixed in the inertial frame \( \overline{S} \). We wish to determine the angular radius \( \delta \) as seen by observer O at time \( t_0 \). Once again, as in figure 4, it is sufficient to consider events in the spatial plane \( Oxy \) or \( Q\overline{xy} \) and, in particular, to consider the circular section \( \Gamma \) of the "galaxy Q" in the plane \( Q\overline{xy} \).

Suppose the observer O sees galaxy Q by means of some illumination beyond Q. Then the angle \( \delta \) is determined by a photon that O would describe as grazing the galaxy Q. Therefore observer O must describe the path of this photon as being a tangent to \( \Gamma \) at some point \( L \) (figure 5). (This statement can also be proved analytically.)

Let \( (\overline{x}_L, \overline{y}_L, t_L) \) be the event of the photon leaving \( L \), where

\[
\overline{x}_L = -\varrho \sin \delta, \quad \overline{y}_L = \varrho \cos \delta
\] (1)
Apparent size in Special Relativity.

Figure 5.
and \( \delta \) is the angle between the tangent at \( L \) and the \( \bar{x} \)-axis. Let \((\bar{x}_0, \bar{y}_0, \bar{t}_0)\) be the event of the photon reaching the \( \bar{x} \)-axis at \( \bar{O} \), say. Then, from the geometry of the figure in the \( \bar{x}\bar{y} \)-plane

\[
\begin{align*}
\bar{x}_0 &= -q \cosec \delta, \quad \bar{y}_0 = 0 \\
\bar{c} \bar{t}_0 &= \bar{c} \bar{t}_L + q \cot \delta
\end{align*}
\]  

Let the same events referred to be \((x_L, y_L, t_L), (x_0, y_0, t_0)\) respectively.

The condition for the photon to reach \( O \) at \( t = t_0 \) is \( O \equiv \bar{O} \), or \( x_0 = 0 \). The Lorentz transformation gives

\[
\begin{align*}
\bar{x}_0 &= \beta (x_0 - at_0), \quad \bar{y}_0 = y_0, \quad \bar{t}_0 = \beta (t_0 - \frac{a x_0}{c^2})
\end{align*}
\]

with \( \beta = (1 - \frac{a^2}{c^2})^{-\frac{1}{2}} \). Using (2) and putting \( x_0 = 0 \) these yield

\[
\beta at_0 \sin \delta = q \tag{3}
\]

and hence

\[
a \bar{t}_L = q \cosec \delta \left[ \frac{1}{\beta} - \frac{a}{c} \cos \delta \right] \tag{4}
\]

Then the Lorentz transformation gives also

\[
\begin{align*}
x_L &= \beta (x_L + a \bar{t}_L), \quad y_L = \bar{y}_L
\end{align*}
\]

Using (1), (3), (4) these become
\[ x_L = \gamma \beta \csc \delta \cos \delta \left( \cos \delta - \frac{a}{c} \right) \] \tag{5}

\[ y_L = g \cos \delta \]

From the geometry in we have

\[ \tan \delta = \frac{y_L}{x_L} \]

and from (5) this becomes

\[ \tan \delta = \frac{\sin \delta}{\beta \left[ \cos \delta - \frac{a}{c} \right]} \] \tag{6}

After some reduction using (3) this may be written

\[ \tan \delta = \frac{g \left( 1 - \frac{a^2}{c^2} \right)}{\left[ c^2 t_0^2 - g^2 \left( 1 - \frac{a^2}{c^2} \right) \right]^{1/2}} - \frac{a^2 t_0}{c} \] \tag{7}

On putting \( \alpha = \tan \alpha \) as in the appendix I, this is more compactly expressed by

\[ \tan \delta = \left( \frac{g}{c t_0} \right) \left\{ \sinh \alpha \cos \alpha \left[ \left( 1 - \frac{g^2}{c^2 t_0^2} \cosec^2 \alpha \right)^{1/2} - \tanh \alpha \right] \right\}^{-1} \] \tag{8}

To the first order in \( \left( \frac{g}{c t_0} \right) \), if this is small, (7) becomes

\[ \delta = \frac{g}{a t_0} \left( 1 + \frac{a}{c} \right) \] \tag{9}

Putting \( \Delta = 2 \delta, \ l = 2g \) in accordance with the
definitions of all these quantities, we see that
this is in agreement with (6.10). Equation (7) provides
the exact formula for $\Delta$ if this is required.

It is unlikely that $\delta$ would not be small in
any application, but it is satisfactory to note from
the form of (7) that (6.10) is correct to within an
error of the order of $(9/\alpha_0)^3$.

Discussion. The calculation has been given in the
present form because of several corrolaries that at
least possess academic interest.

(i) It follows from (1), (3) that

$$\left| \vec{x}_L \right| = \frac{q^2}{\beta \alpha_0^3}$$

(10)

The elementary classical result omits the factor $\beta$.
Thus, since $\beta > 1$, the relativistic value of $\left| \vec{x}_L \right|$ is less than the classical value. For example, if an
observer travels vertically upwards from the surface of
the Earth, then at any instant he can see some of the
Earth's surface beyond the classical horizon.

(ii) It may be noted that (6) is the relativistic
aberration-formula.

(iii) From (7) we see that

$$\tan \delta \rightarrow \frac{2q}{c \alpha_0} \quad \alpha \rightarrow \alpha \rightarrow c$$

On the other hand, we see from (6.6) that
Thus according to the mathematics, the "galaxy P" would be seen inside the "galaxy Q" if $\alpha$ is sufficiently large. This is entirely correct according to the assumptions. For the assumption (6.2) implies that P is inside galaxy Q for sufficiently small $\epsilon$ and, by selecting a sufficiently large value of $\alpha$, we should see Q at arbitrarily early stage in its history. However, the assumptions are not meant to be applied to such extreme cases. Besides, the values of $\alpha$ for which observations of the actual universe are possible come nowhere near the values for which these peculiarities arise.

**General relativity.** The reasons we were able to obtain an exact value of $\tan \delta$ for all values of $\theta$ in special relativity are that there proper-length has an unambiguous meaning. This is not the case in general relativity. Therefore, in order to extend formula (7.22) to the case where $\Delta$ is not small, we should have to consider various possible interpretations of the statement that galaxy Q is a sphere of fixed radius. The resulting investigation would have little interest in the present context. In any case, the physical
features that can result from the more general treatment are sufficiently well-illustrated by the use of special relativity.
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Observable relations in relativistic cosmology III

By

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With 3 figures in the text

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A well-formulated observational criterion is sought for the property that an evolving expanding universe must appear to be more congested at great distances than it is in the cosmical neighbourhood of the observer, while a steady-state universe ought to exhibit the same congestion at all distances. The ratio of a statistical measure of the angular separation between objects (such as clusters of galaxies) and suitably specified neighbours in the sky to a statistical measure of the angular diameter of the objects themselves is considered. If $z$ is the spectral red-shift of these objects, it is shown that, with an unexpected degree of generality, the ratio would be proportional to $(1 + z)^{-1}$ in an evolving universe; it would be independent of $z$ in a steady-state universe. The feasibility of applying this criterion to the actual universe is discussed. Some related questions concerning the smoothed-out representation of the universe are briefly considered. In an appendix, the problem of relativistic apparent size is further discussed for the sake of its intrinsic interest.

1. Introduction

If the universe as a whole is evolving and a region of it $T$ light-years distant is observed, this region as seen is $T$ years younger than our own neighbourhood. In particular, if the universe is expanding and matter is assumed to be conserved, the region must be seen to be more congested than our neighbourhood, the degree of congestion increasing with increasing $T$. On the other hand, if the universe is in a steady-state, a region at any distance must be seen to be not different from our neighbourhood (always provided that we are speaking of sufficiently large regions). This gives in principle the simplest observational means of discovering whether the universe is evolving or not; it is literally a question of looking to see if it is.

The essence of the required observational procedure may be indicated as follows. Let $\Delta$ be some observable measure of the angular size of an object $Q$ belonging to some standard category (for example, some type of cluster of galaxies). Let $P$ be $Q$'s nearest neighbour in space that recedes from $Q$ as a manifestation of the general cosmical expansion (that is, $P$ is not gravitationally bound to $Q$). Let $\Theta$ be the observable angular separation of $P, Q$ in the sky. Taking a number of objects like $Q$ all at the same distance (as judged by some convenient criterion) to within specified limits, let $\Delta^*, \Theta^*$ be the average values of $\Delta, \Theta$ for these
objects. Then \((A^*)^{-1}\) is itself a measure of the distance concerned, while \((\Theta^*/A^*)^{-1}\) is a measure of the congestion of the universe at that distance. Then, if the ratio \(\Theta^*/A^*\) is found to decrease with decreasing \(A^*\), we should infer that the universe is expanding and evolving. More generally, any dependence of \(\Theta^*/A^*\) on \(A^*\) would contradict the steady-state hypothesis.

Since \(\Theta^*, A^*\) are related in the same way to the average actual distance \(PQ\) and the average actual diameter of the objects concerned (apart from a purely numerical factor allowing for the fact that distances are seen projected on the sky), the ratio of these two actual distances is given by \(\Theta^*/A^*\) independently of the particular geometry of spacetime. Therefore the suggested procedure demands no knowledge of this geometry.

All this is only the gist of the method. A proper mathematical formulation is needed, with an expression of the results in suitable statistical terms. In particular, since the spectral red-shift \(z\) is the most convenient criterion of distance, it is natural to discuss the possible dependence of \(\Theta^*/A^*\) on \(z\), rather than on \(A^*\) itself.

We shall first give the mathematical treatment which appears not to have been given before. It possesses some interest of its own, in particular the unexpected property that the results do not depend upon the angle \(\Theta\) being a small angle. We shall then briefly discuss the applicability of the method. Unfortunately, in spite of its simplicity in principle, rather obvious reasons render its useful application somewhat improbable. However, since it concerns a first-order effect of the expansion and since there is some choice in the objects to which it might be applied, some feasible statistical adaptation of the method may be devised in the future. This discussion then gives rise to some consideration of the interpretation of smoothed-out models of the universe.

The previous two papers of this series appeared a long time ago (McCrea, 1935, 1939); the first will be referred to as paper I. A brief account of the present work was first given in a discussion at the Royal Statistical Society in 1957 (McCrea, 1958).

2. Cosmological models

The theory can, of course, be given only for theoretical models of the universe. We give it below for the models in newtonian, special relativity (here equivalent to kinematic relativity) general relativity and steady-state cosmologies. There is some gain in insight, as well as some mathematical interest, in treating these cases separately.

These are all "smoothed-out" models. But for purposes of the theoretical treatment we shall suppose that any model under consideration contains "galaxies" which are the only objects to be considered, that
each galaxy has no motion except that of the cosmical expansion, and
that each has a size which does not vary with time. We assume further
that each galaxy has a nearest-neighbour galaxy (to be called simply
its “neighbour”), the directions joining the galaxies to their neighbours
being randomly distributed in space. At a particular cosmic epoch, the
distances between the galaxies and their neighbours are assumed to be
distributed according to some statistical law, but with no statistical
correlation between these distances and the sizes of the galaxies. In
agreement with our general assumption, the distance between any
particular galaxy and its neighbour varies with epoch strictly in accord­
ance with the cosmical expansion. In all but the steady-state model, we
assume that galaxies are neither created nor destroyed; it follows that
the galaxy that is neighbour to a given galaxy at any epoch is its
neighbour at any other epoch. In the steady-state model, we retain all
the other assumptions, but we admit the appearance of new galaxies;
in this case, statistically a galaxy has a neighbour at a fixed distance not
depending on the epoch.

It must be emphasized that these assumptions serve mainly to provide
a terminology for the mathematics. Any attempt to endow the models
with some sort of physical reality would almost automatically ensure
that they must contain objects of some kind that behave in the manner
described without further special assumptions. When we discuss the
possible correspondence between the models and the actual universe, the
objects will not be literally individual galaxies.

3. Newtonian cosmology

The relevant features of newtonian cosmology have been summarized
by McCreA (1953) and by Bondi (1952). The space used is euclidean
and the time is universal newtonian time \( t \). The galaxy \( O \) will mean the
galaxy, in the sense of section 2, whose centre is \( O \), and the observer \( O \)
will mean an observer at this centre, and similarly for other galaxies.
Let \( O \) be the observer whose observations we wish to discuss, and let \( Q \)
be another galaxy. In this cosmology the position vector of \( Q \) referred
to \( O \) is of the form

\[
q_Q = a R(t) ,
\]

where \( a \) is a fixed vector characterizing the particular galaxy \( Q \), and
\( R(t) \) is a function of \( t \) only, the same for all \( Q \). Let galaxy \( P \) be the
neighbour of \( Q \) in the sense of section 2. The position vector of \( P \) relative
to \( Q \) is

\[
\tilde{q}_P = b R(t) ,
\]

where \( b \) is another fixed vector. Then the position of \( P \) relative to \( O \) is

\[
q_P = (a + b) R(t) .
\]
Let $\theta$ be the angle $QOP$, and let $\tilde{\theta}$ be the angle between the vectors $a$, $b$. Then it is easily seen from figure 1 that

$$\tan \theta = \frac{b \sin \tilde{\theta}}{a + b \cos \tilde{\theta}}, \quad (3.4)$$

where $a = |a|$, $b = |b|$. For given galaxies $O$, $Q$, $P$ the angles $\theta$, $\tilde{\theta}$ are independent of $t$, as they must be in accordance with the cosmological model. Since light is supposed to be propagated along euclidean straight lines in this model, the angle $\tilde{\theta}$ is the angular separation of $Q$, $P$ as seen from $O$.

We now suppose, for the moment, that $b$ is given, with $b < a$, but that all directions of $b$ in space are equally probable. Let $\tilde{\Theta}$ be the mean value of $\tilde{\theta}$ corresponding to all possible positions of $P$. Then

$$2\tilde{\Theta} = \int_0^\pi \tilde{\theta} \sin \tilde{\theta} \, d\tilde{\theta} = \int_0^\pi \frac{d\tilde{\theta}}{d\tilde{\Theta}} \cos \tilde{\theta} \, d\tilde{\Theta} \quad (3.5)$$

on integrating by parts and noting that the integrated part vanishes at the limits of integration by virtue of (3.4) and the assumption $b < a$. Using (3.4) to obtain $d\tilde{\theta}/d\tilde{\Theta}$ in terms of $\tilde{\Theta}$ we find that the integral in (3.5) can be evaluated exactly (appendix I), giving

$$\tilde{\Theta} = \frac{1}{4} \pi b/a. \quad (3.6)$$

This very simple dependence on $b/a$ would hardly have been foreseen as an exact result. In particular, it follows that we can now let $b$ have whatever is its appropriate distribution-law as mentioned in section 2, and if we take its mean value $b^*$, say, in (3.6) we shall have the mean value $\tilde{\Theta}^*$ of $\tilde{\Theta}$. Thus the formula

$$\tilde{\Theta}^* = \frac{1}{4} \pi b^*/a \quad (3.7)$$

gives the mean angular separation, as seen from $O$, between any galaxy with a given $a$-value and its neighbour. This simple formula involves no approximation.

Newtonian cosmology in its simplest presentation requires no assumption about the speed of light. But, if it is to yield a self-consistent model involving light-propagation, we must postulate that the speed $c(t)$ as measured locally by an observer attached to a galaxy depends at most upon the time $t$. Bondi (1952) has pointed out that this postulate, coupled with the use of the appropriate newtonian kinematics, requires
the radial velocity of light $dq/dt$ at distance $q$ from the observer to be given by

$$R(t) \frac{dq}{dt} = \pm c(t)$$ (3.8)

for outward or inward propagation, respectively. We adopt this in the simplest case of $c(t) = c$, a constant.

Let $t_o$ be the epoch of observation by $O$. If $t_Q$ is the epoch of emission by $Q$ of light reaching $O$ at $t_o$, then by (3.8)

$$\frac{q_o}{R(t_o)} = c \int_{t_Q}^{t_o} \frac{dt}{R(t)}.$$  

Using (3.1), this is

$$\int_{t_Q}^{t_o} \frac{dt}{R(t)} = \frac{a}{c}.$$  

(3.9)

If $\lambda_o, \lambda_Q$ are the wavelengths of the radiation as measured by $Q, O$, it follows from (3.9) by a familiar argument (e.g. Bondi (1952) p. 87) that

$$1 + z = \frac{\lambda_o}{\lambda_Q} = \frac{R_0}{R_Q},$$  

where $R_0 = R(t_o), R_Q = R(t_Q)$. This is the well-known formula for the red-shift $z$; as Bondi has remarked, the postulate (3.8) leads to exactly the same form of (3.10) as the corresponding result in relativistic cosmology.

In section 2 we have made the hypothesis that the proper-size of a galaxy is independent of $t$. Let $\Delta$ be the angular diameter of $Q$ as seen by $O$ at epoch $t_Q$. We assume that $\Delta$ is a small angle and that results are required only to the first order in $\Delta$ (see appendix II). Then, if $Q$ be treated as a sphere of diameter $l$, this angle is

$$\Delta = l/aR_Q$$  

(3.11)

since $aR_Q$ is the distance of $Q$ from $O$ when the light leaves $Q$. [Were galaxy $Q$ an elongated body of length $l$, then we should have $\frac{1}{4} \pi l$ in place of $l$ in (3.11).] If $l^*$ is the mean value of $l$ for all galaxies, then the mean value $\Delta^* of \Delta$ for a given $a$-value is

$$\Delta^* = l^*/aR_Q$$  

(3.12)

since (3.9) shows that, if $a, t_o$ are given, then $t_Q$ is fixed and so $R_Q$ is fixed.

Combining (3.7), (3.10), (3.12) we have finally

$$\frac{\Theta^*}{\Delta^*} = \left(\frac{1}{4} \pi \frac{l^*}{l^*} R_Q\right) \frac{1}{1+z}.$$  

(3.13)
According to the postulated properties of the model, the red-shift \( z \) is the only quantity in the right-hand member of (3.13) that depends upon the distance of the galaxies under observation at epoch \( t_0 \). Naturally, we do not need to employ any measure of distance other than \( z \). Thus, if a number of galaxies all exhibiting the same red-shift \( z \) are observed to have mean angular diameter \( A^* \) and to have mean angular separation \( \Theta^* \) from their neighbours in space, then for the present model (3.13) asserts that \( \Theta^*/A^* \) decreases with increasing \( z \) in proportion to \( (1 + z)^{-1} \).

This is the precise expression for the model of the increasing congestion of the universe to be seen at increasing distance from the observer which was described qualitatively in section 1.

4. Cosmology of special relativity or kinematical relativity

A model of the expanding universe that can be treated by special relativity was first described by Kermack and McCrea (1933) and has recently been treated by Synge (1956, p. 156). For present purposes it is the same as that given by kinematical relativity (Milne, 1948). It corresponds to the newtonian model of section 3 in the particular case in which the galaxies move with uniform relative velocities, i.e. the case \( R(t) = t \), but with the use of the kinematics and optics of special relativity in place of those of newtonian theory. We postulate that a frame moving with some galaxy is inertial and then it follows that there is an inertial frame moving with every galaxy. All the quantities to be considered are referred to one or another of such frames.

We start with the galaxies \( O, Q, P \) as in section 3 and (3.1), (3.2) have now to be replaced by

\[
\begin{align*}
q_Q &= at \quad (4.1) \\
\bar{q}_P &= b\tilde{t}, \quad (4.2)
\end{align*}
\]

where \( t \) is time in the inertial frame \( S \) moving with \( O \) and \( \tilde{t} \) is time in the inertial frame \( \bar{S} \) moving with \( Q \). Then \( a \) is the velocity of \( Q \) relative to \( O \) in \( S \), and \( b \) is the velocity of \( P \) relative to \( Q \) in \( \bar{S} \).

Let \( Ox, Oy \) be the rectangular axes in \( S \), and \( Q\bar{x}, Q\bar{y} \) those in \( \bar{S} \), such that \( Ox, Q\bar{x} \) lie along \( OQ \), and \( Oy, Q\bar{y} \) lie in the plane \( OQP \).

Let \( \bar{x}\bar{Q}\bar{P} = \bar{\theta} \), measured in \( \bar{S} \); this is a fixed angle determined by the fixed vector \( b \). Then \( P \)'s world-line referred to \( \bar{S} \) is

\[
\bar{x}/\bar{t} = b \cos \bar{\theta}, \quad \bar{y}/\bar{t} = b \sin \bar{\theta}. \quad (4.3)
\]
A simple application of the appropriate Lorentz transformation then shows that \( P \)'s world-line in \( S \) is

\[
\frac{x}{t} = \frac{a + b \cos \tilde{\theta}}{1 + (ab/c^2) \cos \tilde{\theta}}, \quad \frac{y}{t} = \frac{b \sin \tilde{\theta}}{\beta [1 + (ab/c^2) \cos \tilde{\theta}]} \tag{4.4}
\]

where

\[
\beta = (1 - a^2/c^2)^{-^{1/2}}.
\]

Thus in \( S \) the galaxy \( P \) moves with uniform velocity in a fixed line through \( O \) that makes angle \( \theta \) with \( Ox \) where, from (4.4)

\[
\tan \theta = \frac{b \sin \tilde{\theta}}{\beta (a + b \cos \tilde{\theta})}. \tag{4.5}
\]

Equations (4.4) provide the special relativity analogue of (3.3), and (4.5) is the analogue of (3.4). The angle \( \theta \) is again the fixed angular separation of \( Q, P \) as seen from \( O \).

Again we temporarily assume that \( b \) is given, with \( b < a \), but that all directions of \( b \) in space in \( \bar{S} \) are equally probable, i.e. that \( Q \) is equally likely to see his neighbour in any direction in space. Then \( \Theta \), defined as before, is again given formally by (3.5) but with \( \theta \) now given in terms of \( \tilde{\theta} \) by (4.5). The integral in (3.5) is now somewhat different owing to the factor \( \beta \) in (4.5), but it can still be evaluated exactly (appendix I). We obtain

\[
\Theta = \frac{1}{2} \pi \frac{c^2}{\beta ab} \left[ 1 - (1 - b^2/c^2)^{1/2} \right]. \tag{4.6}
\]

As a check, we notice that this agrees with the newtonian result (3.6) in the limit when \( c \) tends to infinity, as we should expect. But we shall consider the correspondence with (3.6) in a different way at the end of this section.

In special relativity the light-speed is \( c \) in every inertial frame. In \( S \) the distance of \( Q \) from \( O \) at time \( t_Q \) is \( at_Q \) and, therefore, light emitted by \( Q \) at \( t_Q \) reaches \( O \) at \( t_Q \) if

\[
c(t_Q - t_O) = at_Q, \quad \text{giving} \quad ct_Q = (c + a)t_Q. \tag{4.7}
\]

Also, the standard formula for the doppler effect in special relativity, applied to \( Q \) receding from \( O \) with speed \( a \), gives

\[
1 + z = \frac{\lambda_o}{\lambda_Q} = \beta \left( 1 + \frac{a}{c} \right). \tag{4.8}
\]

Since, at \( t_Q \), observer \( O \) sees galaxy \( Q \) at distance \( at_Q \) he sees it as having angular diameter \( A \), where

\[
A = \frac{\lambda}{at_Q} = \frac{\lambda}{at_Q} \left( 1 + \frac{a}{c} \right), \tag{4.9}
\]

using (4.7). Here we have again treated \( Q \) as a sphere of proper-diameter \( \ell \), noting that the diameter perpendicular to the sight-line is not affected by the recession, i.e. by the transformation from \( S \) to \( \bar{S} \). Also we have again
assumed $\Delta$ to be a small angle (see appendix II). As before, we may now
insert mean values and write

$$\Delta^* = \frac{l^*}{a_0} \left( 1 + \frac{a}{c} \right).$$  \hspace{1cm} (4.10)

Combining (4.6), (4.8), (4.10), we obtain

$$\frac{\Theta^*}{\Delta^*} = \left( \frac{1}{2} \pi c^2 \left[ \frac{1}{b} - \left( 1 - \frac{b^2}{c^2} \right)^{1/2} \right] \right)^* t_0 \left( \frac{1}{1 + z} \right).$$  \hspace{1cm} (4.11)

where the asterisk on the right denotes the mean value of the quantity
in square brackets. The fact that this is a more complicated function
of $b$ than in the previous case is of no immediate significance. All that
matters here is that the quantity in \{ \} in (4.11) is certainly independent
of $z$. Thus the dependence of $\Theta^*/\Delta^*$ upon $z$ is precisely the same as before
and so the conclusion stated at the end of section 3 applies also to the special
relativity model.

Transformation of parameters. The metric of the space-time of special
relativity used here with $q, \theta, \varphi$ as polar coordinates referred to $O$ is

$$ds^2 = c^2 dt^2 - dq^2 - q^2 d\theta^2 - q^2 \sin^2 \theta d\varphi^2.$$  \hspace{1cm} (4.12)

Under the transformation

$$t = \frac{1 + \frac{1}{2} r^2}{1 - \frac{1}{2} r^2} \tau, \quad q = \frac{r}{1 - \frac{1}{2} r^2} c \tau, \quad \theta = \theta, \quad \varphi = \varphi$$  \hspace{1cm} (4.13)

this becomes

$$ds^2 = c^2 d\tau^2 - \frac{c^2 \tau^2}{(1 - \frac{1}{2} r^2)^2} (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2).$$  \hspace{1cm} (4.14)

The world-line of a typical galaxy is of the form (4.1), that is, say,
$q = h t$, where $h$ is a constant vector, which may be written

$q = h t, \quad \theta = \text{constant}, \quad \varphi = \text{constant}.$  \hspace{1cm} (4.15)

Using (4.13), this becomes

$r = \text{constant}, \quad \theta = \text{constant}, \quad \varphi = \text{constant},$

where the value of $r$ is given by

$$\frac{c r}{1 + \frac{1}{2} r^2} = h.$$  \hspace{1cm} (4.16)

Thus all the galaxies are simply fixed in the $r, \theta, \varphi$ system. Moreover,
the $r, \theta, \varphi$-space is seen to be a space of constant curvature and so the
form of (4.14) is invariant when the origin of these coordinates is trans­
ferred from $O$ to any other galaxy.

In particular, if $\tilde{b}$ is the $r$-coordinate of $P$ referred to $Q$ then, since
from (4.2) the $h$-parameter is $b$, corresponding to (4.16) we have

$$\frac{b}{c} = \frac{\tilde{b}}{1 + \frac{1}{2} \tilde{b}^2}, \quad \text{whence} \quad \left( 1 - \frac{b^2}{c^2} \right)^{1/2} = \frac{1 - \frac{1}{2} \tilde{b}^2}{1 + \frac{1}{2} \tilde{b}^2}.$$
Thus, referring to the function of \( b \) occurring in (4.6)

\[
\frac{c}{b} \left[ 1 - \left(1 - \frac{b^2}{c^2} \right)^{1/4} \right] = \frac{1}{2} b.
\]

(4.17)

Returning to the variables in (4.13), we see that along the world-line of \( O \), since \( r = 0 \), we have \( t = \tau \). Hence in place of \( t_0 \) in (4.9), etc., we may write \( \tau_0 \), which is the epoch of observation in the new variable. Using this and (4.17), the formula (4.11) becomes

\[
\frac{\theta^*}{\tau} = \left( \frac{1}{4} \pi \frac{\tilde{b}^*}{\tau} \right) \frac{1}{1+z} \quad (R_0 = c \tau_0).
\]

This is now of exactly the same form as (3.13). We shall return to this feature in section 5.

5. General relativistic cosmology

The metrics of relativistic cosmology are all included in

\[
ds^2 = c^2 d\tau^2 - \frac{R^2(\tau)}{1 + \frac{1}{k} \tau^2} \left( d\tau^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\varphi^2 \right),
\]

(5.1)

where \( k = 1, 0, \) or \(-1 \). (We write \( \tau \) in place of the more usual symbol \( t \) simply because \( t \) has been used in another sense in sections 3, 4.)

We consider first the case \( k = 1 \), and we make the transformation from \( r \) to \( \chi \) defined by

\[
r = 2 \tan \frac{1}{2} \chi.
\]

(5.2)

This puts (5.1) into the form

\[
ds^2 = c^2 d\tau^2 - R^2(\tau) \left( d\chi^2 + \sin^2 \chi \, d\theta^2 + \sin^2 \theta \, d\varphi^2 \right).
\]

(5.3)

The "galaxies" are still those postulated in section 2. Then the centre of any galaxy has fixed coordinates \( r, \theta, \varphi \) or \( \chi, \theta, \varphi \). A fundamental property of (5.1), (5.3) is that \( ds^2 \) is invariant in form when we transfer the origin of these coordinates, i.e. the point \( r = 0 \) or \( \chi = 0 \) in the 3-space, from any one galaxy to any other. This is because the 3-space has constant curvature for any given value of \( \tau \).

Once again we consider the galaxies \( O, Q, P \) as in the previous sections. Let \( O \) be the point \( \chi = 0 \) and let \( Q \) be the point \( \chi = \alpha, \theta = 0 \) (remembering that, as in spherical polar coordinates, such particular values do define single points in the 3-space without mentioning the other coordinates; in fact, \( \varphi \) has no meaning when \( \theta = 0 \), and \( \theta, \varphi \) have no meaning when \( \chi = 0 \)). We suppose \( 0 < \alpha < \frac{1}{2} \pi \).

Consider then the transformation from \( \chi, \theta, \varphi \) to \( \tilde{\chi}, \tilde{\theta}, \tilde{\varphi} \) defined by

\[
cos \tilde{\chi} = \cos \alpha \cos \chi + \sin \alpha \sin \chi \cos \theta
\]

(5.4)

\[
sin \tilde{\chi} \sin \tilde{\theta} = \sin \chi \sin \theta,
\]

(5.5)

\[
\tilde{\varphi} = \varphi.
\]

(5.6)
These equations possess a unique real solution that reduces to the identity transformation if \( a = 0 \). It is useful to note that (5.4), (5.5) yield

\[
\sin \chi \cos \theta = \sin \alpha \cos \chi + \cos \alpha \sin \chi \cos \theta
\]  
(5.7)

and

\[
\sin \chi \cos \theta = \sin \alpha \cos \chi + \cos \alpha \sin \chi \cos \theta.
\]  
(5.8)

We see that in the new coordinates \( O \) is the point \( \chi = \alpha, \theta = \pi \) and \( Q \) is the point \( \bar{\chi} = 0 \). Also it can be verified that

\[
d\bar{\chi}^2 + \sin^2 \bar{\chi} d\bar{\theta}^2 + \sin^2 \bar{\chi} \sin^2 \bar{\theta} d\varphi^2 = d\chi^2 + \sin^2 \chi d\theta^2 + \sin^2 \chi \sin^2 \theta d\varphi^2.
\]

The transformation is therefore that from coordinates with \( O \) as origin to corresponding coordinates with \( Q \) as origin.

Let \( P \) be the point \((\bar{\chi}, \bar{\theta}, \varphi)\) with \( \bar{\chi} = \gamma \). Then from (5.5), (5.8)

\[
\tan \theta = \frac{\sin \gamma \sin \bar{\theta}}{\sin \alpha \cos \gamma + \cos \alpha \sin \gamma \cos \bar{\theta}}.
\]  
(5.9)

Also, from simple properties of light-tracks in the space-time (5.3), \( \bar{\theta} \) defines the direction in which \( Q \) sees \( P \), and \( \theta \) is the angular separation of \( Q, P \) as seen from \( O \). Therefore (5.9) plays the same part as (3.4), (4.5) in the previous cases.

Again we keep \( \gamma \) fixed for the moment, with \( \gamma < \alpha \), but assume that all directions in space of \( P \) as seen from \( Q \) are equally probable. Then, by a calculation (appendix I) like those in the preceding cases, we find for the mean value of \( \theta \) in (5.9)

\[
\Theta = \frac{1}{2} \pi \cosec \alpha \tan \frac{1}{2} \gamma.
\]  
(5.10)

If \( \bar{b} \) is the \( \tau \)-coordinate of \( P \) referred to \( Q \), we have analogously to (5.2)

\[
\bar{b} = 2 \tan \frac{1}{2} \gamma.
\]  
(5.11)

Substituting in (5.10) and taking mean values when \( \gamma \) is allowed to range over whatever be its supposed distribution, we obtain

\[
\Theta^* = \frac{1}{4} \pi \bar{b}^* \cosec \alpha.
\]  
(5.12)

Along an inward radial light-track (null-geodesic) in (5.3)

\[
c d\tau = - R(\tau) d\chi.
\]

Hence, if radiation leaves \( Q \) at \( \tau_Q \) and reaches \( O \) at \( \tau_0 \),

\[
\chi = c \int_{\tau_0}^{\tau_Q} \frac{d\tau}{R(\tau)}
\]

giving the usual formula for the red-shift

\[
1 + z = \frac{\lambda_0}{\lambda_Q} = \frac{R_0}{R_Q},
\]  
(5.13)

where \( R_0 = R(\tau_Q), R_Q = R(\tau_0) \).
Let the galaxy $Q$ subtend a small angle $\Delta$ at $O$ at epoch of observation $\tau_0$, so that $\tau_Q$ is again the epoch at which the radiation leaves $Q$. Treating $Q$ as a sphere of diameter $l$ and expressing from (5.3) that $l$ is in particular the proper-diameter perpendicular to the sight-line, we get (cf. Paper I, equation (5))

$$l = R(\tau_Q) \sin \Delta .$$

(5.14)

Combining (5.12), (5.13), and (5.14) as applied to mean values, we get finally

$$\Theta^w = \left( \frac{1}{4} \pi \frac{b^*}{l^*} R(0) \right) \frac{1}{1 + z} .$$

(5.15)

The calculation in the case $k = -1$ proceeds similarly using hyperbolic functions of the variable corresponding to $\chi$. That in the case $k = 0$ is formally the same as in section 3. Thus the result (5.15) applies to all cases.

The result is also exactly as in (3.13), and so in particular we verify that the postulate (3.8) with $c(t) = c$ yields agreement with the relativistic result. It is also the same as (4.18); in fact (4.14) is the particular case of (5.1) with $R(\tau) = ct$, $k = -1$, but the derivation of (4.18) from (4.11) is instructive. With regard to the latter, it must be pointed out however, that it is not the dependence upon the parameter $b$ or $\bar{b}$ that is of immediate physical interest. The main interest is in the fact that the parameter $a$ or $z$ does not appear explicitly in the final result.

The observational interpretation of (5.15) given at the end of section 3 now applies to all the models.

6. Steady-state cosmology

The metric of the steady-state model may be written

$$ds^2 = c^2 d\tau^2 - \exp(2\tau/T) \left( d\tau^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right) ,$$

(6.1)

where $T$ is fixed, being the “Hubble time” for the model. Any particular “galaxy” is again fixed in the $r, \theta, \varphi$ coordinates, which expresses the assumption that it shares in the cosmical expansion. But in this case new galaxies are continually coming into existence so as, in fact, to maintain the steady state despite this expansion. A galaxy once formed is assumed to remain in existence and, of course, to remain fixed in the $r, \theta, \varphi$ coordinates.

Let a galaxy $Q$ play the same part as in the previous sections. Transferring to $Q$ as origin of coordinates $\tilde{r}, \tilde{\theta}, \tilde{\varphi}$ the metric becomes

$$ds^2 = c^2 d\tau^2 - \exp(2\tau/T) \left( d\tilde{r}^2 + \tilde{r}^2 d\tilde{\theta}^2 + \tilde{r}^2 \sin^2 \tilde{\theta} d\tilde{\varphi}^2 \right) .$$

(6.2)

Then we can idealize the steady-state concept as follows. At epoch $\tau = 0$ let a galaxy $P_0$ come into existence with $\tilde{r} = \tilde{b}$, and suppose $P_0$ is then the nearest neighbour of $Q$. At epoch $\tau = \tau_0$, let another galaxy $P_1$
come into existence with $\bar{r} = \bar{b} \exp(-\tau_1/T)$; at $\tau = 2\tau_1$ let $P_2$ come into existence with $\bar{r} = \bar{b} \exp(-2\tau_1/T)$, and so on. Then, during the interval $n\tau_1 < \tau < (n + 1)\tau_1$, the galaxy $P_n$ bears to $Q$ the same relationship as that borne by $P_o$ during $O < \tau < \tau_1$. For the transformation $\tau = \tau' + n\tau_1$, $\bar{r} = r' \exp(-n\tau_1/T)$, where $\tau_1$ is fixed, transforms (6.2) into the same form with $\tau'$, $r'$ in place of $\bar{r}$, $\bar{b}$. Thus in successive intervals $\tau_1$, the part of $Q$'s nearest neighbour is taken successively by $P_1$, $P_2$, $\ldots$, $P_n$, $\ldots$, and each of these in turn behaves in exactly the same way as seen from $Q$. This is the simplest way in which $Q$'s possession of a nearest neighbour can be rendered "steady" (in the mean). Finally, suppose $\tau_1$ is small compared with $T$. Then we can say that the galaxy which is nearest neighbour to $Q$ at $\tau_Q$ has $\bar{r} = \bar{b} \exp(-\tau_Q/T)$. Also we suppose it to be equally likely to be in any direction in space as seen from $Q$. Then, corresponding to (5.15) we should now have

$$\frac{\Theta^*}{A^*} = \left[ \frac{1}{4} \pi \frac{\bar{b}^*}{l^*} R_0 \exp(-\tau_Q/T) \right] \frac{1}{1+z} .$$

(6.3)

In the present case we may again use (5.13) with now $R(\tau) = \exp(\tau/T)$, giving

$$1 + z = R_o \exp(-\tau_Q/T) .$$

(6.4)

Thus (6.3), (6.4) give

$$\frac{\Theta^*}{A^*} = \frac{1}{4} \pi \frac{\bar{b}^*}{l^*} .$$

(6.5)

As we expect in the case of the steady-state universe, this ratio is a constant which is independent both of the distance of the systems observed and also of the epoch of observation.

Alternatively, suppose we merely regard the steady-state as maintaining a neighbour $P$ at a fixed proper-distance from $Q$, and suppose $QP$ subtends a small angle $\theta$ at $O$. Then analogously to (5.14) we have

$$\bar{b} \sin \bar{\theta} = R(\tau_0) a \theta$$

(6.6)

where, in the coordinate system of (6.1), $O$ is at $r = 0$, and $Q$ has $r = a$, and where $\bar{\theta}$ is the inclination of $QP$ to the sight-line $OQ$. If $\Theta$ is the average value of $\theta$ for all directions of $QP$ in the 3-space of $Q$, then (6.6) gives

$$R(\tau_0) a \Theta = \frac{1}{4} \pi \bar{b} .$$

The relation corresponding to (5.14) is

$$R(\tau) a A = l .$$

Averaging over $\bar{b}$, $l$ and dividing, we obtain

$$\frac{\Theta^*}{A^*} = \frac{1}{4} \pi \frac{\bar{b}^*}{l^*}$$

as in (6.5). But this derivation holds only when $\Theta^*$ as well as $A^*$ is a small angle.
These simple derivations of (6.5) suffice to confirm, by comparison with (5.15), that there is the expected difference between the steady-state and the other models in regard to the dependence of $\theta^*/\Delta^*$ upon distance (as denoted by the redshift $z$). The first derivation of (6.5) above is designed to indicate the lines of a more general treatment. Instead of the fixed interval $\tau_1$ we ought to consider a random interval and the mean value of this interval ought not necessarily to be small compared with $T$. We have not obtained a satisfactory general treatment; it is not certain that such a treatment would yield precisely (6.5). It would, however, be mainly of mathematical interest only since the present simplified treatment suffices to show the essential difference between the steady-state and the other cases.

7. Possibility of application

General considerations. a) The sole purpose of any application would be to find whether, in the actual universe, the ratio $\theta^*/\Delta^*$ depends upon $z$ (or upon any other criterion of distance). An observational investigation need not deal with many $z$-values. In the first instance, it would be natural to try to determine a value of the ratio for objects at the greatest distance at which they can be studied, and then to compare the result for similar objects at the smallest convenient distance. Then the main consideration would be for there to be, at both distances, observable objects of the sort required in sufficient numbers for the determination of significant average values $\theta^*$, $\Delta^*$. The accuracy demanded is very considerable unless a very large value of $z$ can be used. But one interest of the foregoing results is to show in the simplest possible way how small is the observable difference between a steady-state and an evolving universe (being measured simply by the factor $1 + z$) except at large red-shifts.

The effect we are discussing is a first order effect in $z$. Basically, therefore, no other observational means of discriminating between the steady-state and other theories can demand less accuracy. As regards any proposed application it would therefore obviously be for consideration as to whether it could be achieved with less labour than the methods of discrimination so far proposed, most of which demand observations out to some value of $z$ instead of, as here, observations at selected values of $z$.

b) The objects typified by “galaxy $Q$” in the theory must belong to some recognizable single category, but they need not be actual individual galaxies. They must have a measurable size and, in order to apply the theory as formulated here, they must have some spectral feature from which the red-shift can be determined.

c) The objects typified by “galaxy $P$” must also belong to a single category, but this need not be the same as for $Q$. In order to apply the
theory in its simple form, it is not explicitly required that either the size or the red-shift of a \( P \)-object be measurable. It is, of course, necessary to be able to infer that anything used as a \( P \)-object is not gravitationally bound to the corresponding \( Q \)-object.

d) There would be advantages in selecting categories of objects that are not very numerous. This should make it easier to know that pairs of the objects are not gravitationally bound to each other, and would probably make it easier to recognize "neighbours". It may be recalled that \( \Theta^* \) in (5.15) is not necessarily a small angle, and so we may deal with neighbours that are well separated in the sky.

e) Clearly, there are possible adaptations and variants of the simple theory that might be more feasible for use in practice. For example, it would be natural to measure the separation of each \( Q \)-object from several neighbours rather than from a single one. Again, for application to \( P \)-objects for which red-shifts can be obtained, the theory could be formulated so as to apply to \( P \)-objects whose red-shifts differ by less than a certain amount from the red-shift of the corresponding \( Q \)-object. This might help in giving a more usable definition of "neighbours".

f) In all observational problems of the present kind there is the ever-present difficulty of selection effects. As regards the present problem, these have been cogently stated by Neyman and Scott (1958). But we shall see below how some of these difficulties might be overcome.

g) There is a complication in that, if the universe is evolving, objects otherwise in the same category may exhibit evolutionary effects correlated with distance. In particular, the intrinsic size may depend upon age and so upon distance. In that case the formula (5.15) could not be strictly applied. However, in the first place, there is no reason why any such dependence should exactly mask the calculated dependence of \( \Theta^*/\Delta^* \) upon \( z \). Consequently, were \( \Theta^*/\Delta^* \) found observationally to be independent of \( z \), this would still be strong support for the steady-state model; on the other hand, any dependence upon \( z \), even if not precisely that predicted by the simple theory of evolutionary models, would still count against the steady-state theory. In the second place, even in evolutionary models, certain "size"-parameters would be expected to depend very little upon age, for example, the distance between the components of a binary galaxy.

Possible applications. a) The most obvious type of object to consider is clusters of galaxies. In principle, a feasible procedure might be the following. Consider first clusters at the greatest possible distance. As the size of a cluster, i.e. the quantity \( \Delta \), take the mean angular separation of, say, the five brightest galaxies taken in pairs. As a measure of \( \theta \), take the mean angular separation of these galaxies from the five brightest in the nearest cluster. The averages \( \Delta^*; \Theta^* \) would then be got from as many

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pairs of clusters as possible at about the same distance. Attention would then be transferred to clusters at the least possible distance. From the empirical relation between red-shift and apparent magnitude (Humason, Mayall and Sandage, 1956), it ought to be possible to determine which of the nearby clusters contain galaxies as bright as those observed in the remote clusters. Only such nearby clusters should be considered. In this way, the selection effects foreseen by Neyman and Scott (1958) should be avoided.

b) Probably the only possibility, even in principle, of using an individual galaxy as a Q-object would be to take it to be, say, the brightest galaxy in a cluster and to take as the P-object, the brightest galaxy in the nearest cluster. But it is unlikely that the angular sizes of individual galaxies could be measured for sufficiently remote systems. A slightly better hope would be to use, instead of the size of individual galaxies, the angular separation between the components of binary galaxies, provided these could be recognized in sufficiently distant clusters.

c) It is conceivable that future developments in radioastronomy may provide the sort of observations required, either alone or in combination with optical results. As a case of such a combination, the Q-objects might be large clusters of galaxies observed optically while the P-objects were radio-sources. But for this radio-astronomy would have to find some means of measuring red-shifts (or some feature that can be correlated with the red-shift) sufficiently accurately to aid a decision as to which radio sources are neighbours of any one of the optical sources employed.

8. The smoothed-out universe

All the models that we have been considering are models of the smoothed-out universe. That is to say, in comparing theory and observation, we make the hypothesis that the behaviour of the actual universe on a sufficiently large scale is the same as it would be were its contents in the form of a uniform ideal fluid. It is necessary briefly to consider the implications of this procedure.

Consider in this context a cluster of galaxies. If, as usual, we regard this as a system held together by its own gravitation, then its member-galaxies do not recede from each other in consequence of cosmical expansion. Except as a result of relaxation (in the sense of the dynamical theory of gases) or of loss of mass (by radiation, etc.), the cluster will preserve a constant size. It follows that “a sufficiently large scale” in the above sense must mean at any rate a scale greater than the dimensions of a single cluster. The discussion in section 7 is based upon the implicit assumption that this is all that is required. More precisely, it was assumed that the clusters are randomly distributed in space and that their
mutual recession is the same as if the universe were filled with a uniform fluid.

Recently, however, it has been suggested that the actual universe may exhibit a phenomenon of second-order clustering, or of clusters of clusters (see Neyman and Scott, 1958, and references there given). The existence of the phenomenon is not established; such evidence as there is may be merely a consequence of comparing the actual universe with a statistical model based upon unacceptable physical assumptions (McCrea, 1958). Nevertheless, it is worth asking how such a phenomenon, if real, would affect our present discussion.

The phenomenon might mean that the actual universe possesses an hierarchical structure. If so, the smoothed-out universe would be meaningless. However, at present there is no indication that we need to pursue this possibility.

Supposing then the smoothed-out universe still to have significance on a sufficiently large scale, it is clear in the first place that a clustering of clusters might require some elaboration of the meaning of "neighbouring clusters" for the purposes contemplated in section 7. Further, with any particular definition of neighbouring clusters, it must be asked whether a pair of neighbours would recede from each other with the rate of cosmical expansion calculated for the smoothed out universe. (For various reasons, it appears unnecessary to consider the possibility of the clusters being gravitationally bound to each other; that being so, their mutual recession is the cosmical expansion.) The rate might depend, say, upon whether the pair belongs to the same cluster of clusters or to two different ones.

If the rate of expansion does vary from place to place in the universe, depending upon the distribution of clusters, then we should expect that the empirical relation between the red-shift and the apparent magnitude of galaxies would show differences from one direction in space to another. For the sight-line would encounter different groupings of the clusters in different directions. Actually, no differences have been found to within the accuracy of the observations. There may, moreover, be a simple physical reason for this, since the universe may contain much more diffuse matter distributed through space between galaxies and clusters than there is matter in the galaxies. So the contents of the universe may actually be much more nearly uniform than appears from the distribution of visible matter alone. For this reason, even a clustering of clusters, if real, need not denote a serious departure of the universe from uniformity.

To sum up, some inadequacy of the smoothed-out representation of the universe has to be kept in mind as a possibility in any discussion of observable relations, but as yet there is nothing to indicate that it is not adequate for such relations as we have considered in this paper.
Appendix I. Evaluation of integrals

We consider first the integral leading to (5.10). From (3.5), (5.9) it is

\[ 2\Theta = \int_0^\pi \frac{d\theta}{\cos \theta} \, d\bar{\theta} \]

where

\[ \tan \theta = \frac{\sin \gamma \sin \bar{\theta}}{\sin \alpha \cos \gamma + \cos \alpha \sin \gamma \cos \bar{\theta}}. \]  

(2)

Differentiating (2) to obtain \( \sec^2 \theta \, d\theta/d\bar{\theta} \) and again using (2) to evaluate \( \sec^2 \theta \), the integrand in (1) is found to be

\[ \sin \gamma \cos \bar{\theta} \left( \cos \alpha \sin \gamma + \sin \alpha \cos \gamma \cos \bar{\theta} \right) \]

\[ 1 - (\cos \alpha \cos \gamma - \sin \alpha \sin \gamma \cos \bar{\theta})^2. \]

This may be written in the form

\[ \frac{1}{2 \sin \gamma \sin \gamma} \left[ \frac{(\cos \gamma - \cos \alpha)(1 - \cos \alpha \cos \gamma)}{1 - \cos \alpha \cos \gamma + \sin \alpha \sin \gamma \cos \bar{\theta}} + \right. \]

\[ \left. + \frac{(\cos \gamma + \cos \alpha)(1 + \cos \alpha \cos \gamma)}{1 + \cos \alpha \cos \gamma - \sin \alpha \sin \gamma \cos \bar{\theta}} - 2 \cos \gamma \right]. \]  

(3)

Remembering that

\[ \int_0^\pi \frac{d\theta}{M + N \cos \theta} = \frac{\pi}{(M^2 - N^2)^{1/2}} \quad (M > N) \]

we find

\[ \int_0^\pi \frac{d\bar{\theta}}{1 + \cos \alpha \cos \gamma \pm \sin \alpha \sin \gamma \cos \bar{\theta}} = \frac{\pi}{\cos \gamma \pm \cos \alpha}. \]  

(4)

Using (3), (4) in (1) we obtain

\[ \Theta = \frac{1}{2} \pi \frac{1 - \cos \gamma}{\sin \alpha \sin \gamma} = \frac{1}{2} \pi \csc \alpha \tan \frac{1}{2} \gamma. \]  

(5)

This is (5.10).

Returning to (4.5) and writing

\[ a = c \tanh A, \quad b = c \tanh B \]

so that \( A > B \) and \( \beta = \cosh A \), we obtain

\[ \tan \theta = \frac{\sinh B \sin \bar{\theta}}{\sinh A \cosh B + \cosh A \sinh B \cos \bar{\theta}}. \]  

(7)

This is of the form (2) but with the constants expressed as hyperbolic functions instead of circular functions. Corresponding to (5) we have therefore

\[ \Theta = \frac{1}{2} \pi \frac{\cosh B - 1}{\sinh A \sinh B} = \frac{1}{2} \pi \frac{1 - \sech B}{\cosh A \tanh \alpha \tanh B} \]

\[ = \frac{1}{2} \pi \frac{c^2}{\beta ab} \left[ 1 - (1 - b^2/c^2)^{1/2} \right]. \]  

(8)

using (6). This is (4.6).
Finally, if we let \( c \to \infty \) in (4.5), so that \( \beta \to 1 \), we obtain (3.4). Also in this limiting case (8) becomes

\[
\Theta = \frac{1}{4} \pi \frac{b}{a}.
\]

This is (3.6). But the integral arising from (3.4) is also easy to evaluate directly.

**Appendix II. Relativistic apparent size**

**Special relativity.** We consider first the case of special relativity and we use the notation of section 4. Let "galaxy \( Q' \)" be a sphere with centre \( Q \) and proper-radius \( q \); as before, it is fixed in the frame \( S \). We wish to determine the angular radius \( \delta \) as seen by observer \( O \) at time \( t_0 \). Once again, as in figure 2, it is sufficient to consider events in the spatial plane \( Oxy \) or \( Q\bar{xy} \) and, in particular, to consider the circular section \( \Gamma \) of the "galaxy \( Q' \)" in the plane \( q\bar{xy} \).

Suppose that observer \( O \) sees galaxy \( Q \) by means of some illumination beyond \( Q \). Then the angle \( \delta \) is determined by a photon that \( O \) would describe as grazing the galaxy. Therefore observer \( Q \) must describe the path of this photon as being a tangent to \( \Gamma \) at some point \( L \) (figure 3).

Let \((\bar{x}_L, \bar{y}_L, \bar{t}_L)\) be the event of the photon leaving \( L \), where

\[
\bar{x}_L = -q \sin \delta, \quad \bar{y}_L = q \cos \delta
\]

and \( \delta \) is the angle between the tangent at \( L \) and the \( \bar{z} \)-axis. Let \((\bar{x}_0, \bar{y}_0, \bar{t}_0)\) be the event of the photon reaching the \( \bar{x} \)-axis at \( \bar{O} \), say. Then, from the geometry of the figure in the \( \bar{x}, \bar{y} \)-plane

\[
\bar{x}_0 = -q \cosec \delta, \quad \bar{y}_0 = 0, \quad c\bar{t}_0 = c\bar{t}_L + q \cot \delta.
\]

Let the same events referred to \( S \) be \((x_L, y_L, t_L), (x_0, y_0, t_0)\), respectively.

The condition for the photon to reach \( O \) at \( t = t_0 \) is \( \bar{O} = O \), or \( x_0 = 0 \). The Lorentz transformation gives \( \bar{x}_0 = \beta(x_0 - at_0), \quad \bar{y}_0 = y_0, \quad \bar{t}_0 = \beta(t_0 - a x_0/c^2) \) with \( \beta = (1 - a^2/c^2)^{-1/2} \). Using (2) and putting \( x_0 = 0 \), these yield

\[
\beta at_0 \sin \delta = q
\]

and hence

\[
a\bar{t}_L = q \cosec \delta [1 - (a/c) \cos \delta]. \tag{4}
\]

Then the Lorentz transformation gives also \( x_L = \beta(\bar{x}_L + a\bar{t}_L), \quad y_L = \bar{y}_L \).

Using (1), (3), (4), these become

\[
x_L = q \beta \cosec \delta \cos \delta (\cos \delta - a/c), \quad y_L = q \cos \delta. \tag{5}
\]
From the geometry in $S$ we have
\[ \tan \delta = \frac{y_L}{x_L} \]
and from (5) this becomes
\[ \tan \delta = \frac{\sin \delta}{\beta (\cos \delta - a/c)} . \]  
(6)

After some reduction using (3) this may be written
\[ \tan \delta = \frac{q (1 - a^2/c^2)}{[a^2 t_0^2 - q^2 (1 - a^2/c^2)]^{1/2} - a^2 t_0/c} . \]  
(7)

On putting $a = c \tanh A$ as in appendix I, this is more compactly expressed by
\[ \tan \delta = \frac{q}{ct_0} \{ \sinh A \cosh A [(1 - q^2 \cosech^2 A/c^2 t_0^2)^{1/2} - \tanh A] \}^{-1} . \]  
(8)

To the first order in $(q/ct_0)$, if this is small, (7) becomes
\[ \delta = \frac{q}{ct_0} \left( 1 + \frac{a}{c} \right) . \]  
(9)

Putting $A = 2 \delta$, $l = 2q$ in accordance with the definitions of all these quantities, we see that this is in agreement with (4.9). Equation (7) provides the exact formula for $\Delta$ if this is required.

It is unlikely that $\delta$ would not be small in any application, but it is satisfactory to note from the form of (7) that (4.9) is correct to within an error of the order of $(q/ct_0)^2$.

Discussion. The calculation has been given in the present form because of several corollaries that at least possess academic interest.

(i) It follows from (1), (3) that
\[ |\vec{x}_L| = q^2/\beta ct_0 . \]  
(10)

The elementary purely classical result omits the factor $\beta$. Thus, since $\beta > 1$, the relativistic value of $|\vec{x}_L|$ is less than the classical value. For example, if an observer travels vertically upwards from the surface of the Earth, then at any instant he can see some of the Earth’s surface beyond the classical horizon.

(ii) It may be noted that (6) is the relativistic aberration-formula.

(iii) From (7) we see that
\[ \tan \delta \to \frac{2q}{ct_0} \quad \text{as} \quad a \to c . \]

On the other hand, we see from (4.5) that
\[ \theta \to 0 \quad \text{as} \quad a \to c \quad (\text{all } \tilde{\theta}) . \]

Thus, according to the mathematics, the “galaxy $P$” would be seen inside the “galaxy $Q$”, if $a$ is sufficiently large! This is entirely correct according to the assumptions. For the assumption (4.2) implies that $P$ is inside galaxy $Q$ for sufficiently small $\tilde{t}$ and, by selecting a sufficiently large value of $a$, we should see $Q$ at an arbitrarily early stage in its
history. However, the assumptions are not meant to be applied to such extreme cases. Besides, the values of $a$ for which observations of the actual universe are possible come nowhere near the values for which these peculiarities would arise.

**General relativity.** The reason we were able to obtain an exact value of $\tan \delta$ for all values of $q$ in special relativity is that there proper-length has an unambiguous meaning. This is not the case in general relativity. Therefore, in order to extend formula (5.14) to the case where $\Delta$ is not small, we should have to consider various possible interpretations of the statement that galaxy $Q$ is a sphere of fixed radius $q$. The resulting investigation would have little interest in the present context. In any case, the physical features that can result from the more general treatment are sufficiently well-illustrated by the use of special relativity.

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**References**


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