HAUSDORFF MEASURE FUNCTIONS

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ABSTRACT

In many works on Hausdorff Measure Theory it has been the practice to place certain restrictions on the measure functions used. These restrictions usually ensure both the monotonicity and the continuity of the functions. The aim of the first four chapters of this thesis is to find conditions under which the restrictions of continuity and monotonicity may be relaxed.

In the first chapter we deal with the monotonicity condition with respect to both measures and pre-measures. The second and third chapters are concerned with an investigation of the continuity condition with regard to measures and pre-measures, respectively. Then, having found conditions under which these restrictions may or may not be relaxed, we are able, in the fourth chapter, to generalise some known results to the case of discontinuous and non-monotone functions.

Some of the results of the first four chapters prompted an investigation of the properties of measures corresponding to sequences of measure functions, and this is incorporated in the fifth chapter.

The main purpose of the final chapter is to determine whether or not some of the results of the earlier chapters may be extended to Hilbert space.
PREFACE

I should like to thank my supervisor, Professor H. G. Eggleston, for the advice he has given me in our many discussions on this work. The results contained in this thesis are, to the best of my knowledge, new, although many of them have been proved in less general cases; in these situations the known results are acknowledged in the introductions to the relevant chapters. Finally, I should like to express my gratitude to the Science Research Council for my studentship.
DEFINITIONS AND NOTATION

For any function \( h(x) \) we define \( h(x^-) \) and \( h(x^+) \) as follows,

\[
\begin{align*}
  h(x^-) &= \lim_{y \to x^-} h(y) \\
  h(x^+) &= \lim_{y \to x^+} h(y).
\end{align*}
\]

We note that if \( h(x) \) is a monotonic increasing function we have,

\[
\begin{align*}
  h(x^-) &= h(x-0) \\
  h(x^+) &= h(x+0)
\end{align*}
\]

If \( x \) is a point of discontinuity of \( h(x) \) we shall define the size of the discontinuity at \( x \) to be \((h(x^+)-h(x^-))\).

We say that \( h(x) \) is a Hausdorff measure function if it satisfies the following conditions,

i). \( h(x^-) > 0 \quad \text{for} \quad x > 0 \)

ii). \( h(x^+) \to 0 \quad \text{as} \quad x \to 0 \)

If \( h(x) \) satisfies i) and ii) above as well as,

iii). \( \lim_{x \to 0} \frac{h(x)}{x^q} > 0 \)

for some positive integer \( q \), we say that \( h(x) \) is a \( q \)-dimensional Hausdorff measure function.

If \( S \) is any set in a metric space \((X, d)\) say, we shall denote by \( d(S) \) the diameter of \( S \) that is,
Also we denote the closure of $S$ by $\overline{S}$.

By $\mathcal{F}$ we shall mean the space of all real number sequences $\{x_n\}$ such that $\sum x_n^2$ is convergent. We shall denote the points of $\mathcal{F}$ by $x_n$ where $x_n = \{x_n\}$. Wherever indices occur, we shall write, for example, $x^{(n)} = \{x^{(n)}_n\}$. If $S$ is a set in $\mathcal{F}$ then we write,

$\mathcal{F} + S = \{x + y : y \in S\}$

where the addition is performed component-wise. We make $\mathcal{F}$ into a metric space by introducing the metric, $\rho$ such that for $x_n, y_n \in \mathcal{F}$,

$\rho(x_n, y_n) = \left[ \sum_{n=1}^{K} \left| x_n - y_n \right| ^2 \right]^{1/2}$

If $X, Y$ are sets in $\mathcal{F}$ then,

$\rho(x, X) = \inf \{ \rho(x, x_n) : x_n \in X \}$

and,

$\rho(X, Y) = \inf \{ \rho(x_n, y_n) : x_n \in X, y_n \in Y \}$

A set of points $S$ in Euclidean space or $\mathcal{F}$ is said to be convex, if whenever two points $x, y$ belong to $S$ all the points of the form,

$\lambda x + (1-\lambda)y$

where $0 < \lambda < 1$ also belong to $S$. If $A$ is any set, then by $\text{conv} A$ we mean the smallest convex set which contains $A$. The following results
will be assumed wherever necessary,

a). \( (\mathbb{E}^n, \rho) \) is a complete metric space (see, for example, Sierpinski (10)),

b). \( d(A) = d(\text{conv } A) \) (see, for example, Eggleston (4)),

and, c). \( d(\text{conv } A) = d(\overline{\text{conv } A}) \).

For any point \( \alpha \) and any positive real number \( r \) we shall write \( S(\alpha, r) \) for the sphere centre \( \alpha \), radius \( r \). Suppose \( P \) is a set in a metric space \( K \) and \( r, d, \alpha \) are positive real numbers. Then for every point \( \alpha \in P \) we define \( N(\alpha, n, d, \rho) \) to be the largest number of disjoint spheres of the form \( S(\rho, d, \alpha) \) with \( \rho \in \alpha \) which can meet \( S(\alpha, d) \). We write,

\[
N(n, d, \rho) = \sup_{\alpha \in P} N(\alpha, n, d, \rho).
\]

We say that a function \( h(\alpha) \) is blanketed if for all \( \alpha > 0 \) there exist positive real numbers \( k(\alpha, h) \) and \( K(\alpha, h) \) which satisfy,

\[
k(\alpha, h) h(\beta) \leq h(\alpha \beta) \leq K(\alpha, h) h(\beta)
\]

for all \( \beta > 0 \). Then, if \( h(\alpha) \) is a monotonic, increasing, blanketed, Hausdorff measure function we write,

\[
h(\rho, \alpha) = \lim_{\epsilon \to 0} \lim_{d \to 0} \lim_{\mu \to \alpha} \frac{h(d, \epsilon)}{N(\mu, d, \rho \cap S(\alpha, \epsilon))}
\]

for each point \( \alpha \) of \( K \). Finally, we write,

\[
h(\rho) = \sup_{\alpha \in P} h(\rho, \alpha).
\]

A set \( A \) is said to have finite dimension if there exists a monotonic
increasing, blankets, Hausdorff measure function \( \lambda(x) \) such that, \( \lambda(A) = 0 \).

A metric space \( \mathcal{M} \) is said to be a \( \beta \)-space if there exist positive real numbers \( \delta, \kappa (\geq \kappa_0) \) and \( N = N(\kappa) \) such that, for all \( r < \delta \), at most \( N(\kappa) \) disjoint open spheres of radius \( \kappa r \) can meet any given open sphere of radius \( \kappa \). Lerman (6) has shown that a compact set \( A \) in a metric space has finite dimension if and only if \( A \) is a \( \beta \)-space.

If \( \lambda(x) \) is a Hausdorff measure function and \( S \) is a set in Euclidean space or \( \mathbb{I}^d \), then, following Hausdorff (5), we define the corresponding Hausdorff pre-measure of \( S \) denoted by \( \Lambda^h_j(S) \) as follows,

\[
\Lambda^h_j(S) = \inf \sum_{U_i \supseteq S} \lambda(d(U_i)),
\]

where the lower bound is taken over all coverings of \( S \) by open convex sets each of diameter strictly less than \( \delta \). We then define the Hausdorff measure \( \Lambda^h(S) \) of \( S \) as follows,

\[
\Lambda^h(S) = \lim_{\delta \to 0} \Lambda^h_j(S).
\]

This will be referred to as the \( h \)-measure of \( S \). We will write \( \Lambda^h_j(S) \) and \( \Lambda^h(S) \) when \( \lambda(x) = \lambda_0 \), and \( \Lambda^h_j(S) \) and \( \Lambda^h(S) \) when \( \lambda(x) = \lambda_\infty \). The measure \( \Lambda^h_{R\delta}(S) \) is defined in a similar manner to \( \Lambda^h(S) \) but here we restrict the coverings to be open \( q \)-dimensional rectangles.

Similarly \( \Lambda^h_{C\delta}(S) \) refers to coverings by open \( q \)-dimensional cubes. \( \Lambda^h_j(S) \) and \( \Lambda^h(S) \) will refer to coverings by closed convex sets \( U_i \) where, for \( \Lambda^h_j(S) \) we insist that \( d(U_i) \leq \delta \) for all \( i \).

Two Hausdorff measure functions \( \lambda(x) \) and \( \phi(x) \) will be said to be measure equivalent whenever, for all sets \( S \), \( \Lambda^h(S) \) is positive and finite if and only if \( \Lambda^\phi(S) \) is positive and finite. Also, if \( \lambda(x) \) and
functions we write $h < q$ if,

$$\lim_{x \to 0} \frac{q(x)}{h(x)} = 0.$$  

For any set $S$ we denote the complement of $S$ by $\mathcal{C}S$ and $X \setminus Y$ by $X \setminus Y$. If $x$ is any real number we denote the greatest integer less than or equal to $x$ by $[x]$. Finally, we shall call $\{x_n\}$ a null sequence if $x_n \to 0$ as $n \to \infty$.  


Introduction

The first theorem of this chapter shows that in all subsequent investigations of the Hausdorff measures of sets in metric as well as in Euclidean spaces it is sufficient to prove theorems only for the case of monotonic Hausdorff measure functions. Theorems 2 and 3 show that this result cannot always be extended to the case of Hausdorff pre-measures. Thus throughout this work the theorems concerning Hausdorff measures will be seen to be true for both monotonic and non-monotonic measure functions. Whereas those concerning pre-measures will only be proved for the case of monotonic measure functions.

Theorem 1

Given any Hausdorff measure function \( h(x) \) there exists a monotonic increasing Hausdorff measure function \( H(x) \) say, such that for any set \( S \), we have,

\[
\bigvee^H(S) = \bigvee^h(S).
\]

Proof

We know that \( h(x^+) > 0 \) for all positive values of \( x \), let \( X \) be a fixed positive real number. Define \( H(x) \) as follows,

\[
H(x) = \inf_{y \in (x,X]} h(y) \quad \text{and} \quad H(x) = h(x) \quad \text{for} \quad x > X \quad -(1)
\]

\( i \). \( H(x) \) is monotonic increasing. For, if \( x \) and \( y \) are such that \( x > y \), then,

\[
H(y) = \inf_{z \in (y,X]} h(z) \leq \inf_{z \in (x,X]} h(z) = H(x),
\]

\( z \in (y,X) \quad z \in (x,X] \)
that is, \( H(y) \leq H(x) \).

ii). Clearly \( H(x') < H(x) \) for all \( x' \), and so, from the definition of a Hausdorff measure function, we have,

\[
H(x') \to 0 \quad \text{as } x' \to x
\]  

Also from the statement made at the beginning of the proof, we know that,

\[
H(x) > 0 \quad \text{whenever } x > 0
\]  

Now let \( S \) be any set (in any metric space), we certainly have, by ii).

\[
H^H(S) \leq H^A(S)
\]  

Clearly if \( H^H(S) = \infty \), then we have \( H^H(S) = H^A(S) \). So we may assume that \( H^H(S) < \infty \) for the remainder of the proof.

Given any \( x > 0 \) and any \( \varepsilon > 0 \) choose \( \varepsilon' (\leq \varepsilon) \) such that,

\[
H(x) < \varepsilon H(\varepsilon) \quad \text{whenever } x < \varepsilon'
\]  

this is possible by (2) and (3).

Further, choose a covering of \( S \) by open convex sets \( \{ U_{i}^\varepsilon \} \) such that,

\[
S \subset \bigcup_{i} U_{i}^\varepsilon
\]  

\[
\mu(U_{i}^\varepsilon) < \varepsilon' \quad \text{for all } i
\]  

and

\[
\sum_{i} H(d(U_{i}^\varepsilon)) < H^H(S) + \varepsilon'/2.
\]  

We now define a new open convex covering of \( S \) by sets \( \{ V_{i} \} \) as follows;

For each \( i \),

\[
\text{if } H(d(U_{i}^\varepsilon)) = H(d(U_{i}^\varepsilon')) \quad \text{we put } V_{i} = U_{i}^\varepsilon'
\]  

\[
\text{otherwise, put } V_{i} = U_{i}^\varepsilon' \quad \text{for all } i
\]
if \( h(\alpha(v_{c}')) = H(\alpha(u_{c}')) \) then we have, from (5),

\[
H(\alpha(v_{c}')) = \inf_{u_{c} \in [\alpha(u_{c}'), \alpha(v_{c})]} \sup_{u \in [\alpha(v_{c}'), \alpha(u_{c})]} h(u) = \inf_{u_{c} \in [\alpha(u_{c}'), \alpha(v_{c})]} \sup_{u \in [\alpha(v_{c}'), \alpha(u_{c})]} h(u)
\]

since \( \alpha(u_{c}') < \varepsilon' \).

Thus we can choose \( V_{c} \) to satisfy the following conditions,

\[
\alpha(V_{c}) > \alpha(u_{c}')
\]

and \( V_{c} \) open, convex

\[
\alpha(V_{c}) < \varepsilon \quad \text{for all } \xi
\]

and

\[
\alpha(V_{c}) < \varepsilon \quad \text{for all } \xi
\]

and \( V_{c} \) is an open convex covering of \( S \) with

\[
\alpha(V_{c}) < \varepsilon \quad \text{for all } \xi
\]

and from (8) and (12) we have,

\[
\sum_{c} h(\alpha(V_{c})) < \sum_{c} H(\alpha(u_{c}')) + \frac{\varepsilon}{2}
\]

\[
< \sum_{c} H(\alpha(V_{c})) + \frac{\varepsilon}{2}
\]

that is,

\[
\sum_{c} H(\alpha(V_{c})) < \sum_{c} H(\alpha(V_{c})) + \varepsilon
\]

Now since \( \varepsilon' < \varepsilon \) we know that,

\[
\varepsilon' \to 0 \quad \text{as } \varepsilon \to 0
\]

Thus from (14) we have,
\[ \lambda^h(S) \leq \lambda^H(S) + \varepsilon, \]

but the \( \varepsilon \) was an arbitrary positive real number, and so,

\[ \lambda^h(S) \leq \lambda^H(S) \]

which, combined with (4) gives us the required result,

\[ \lambda^h(S) = \lambda^H(S). \]

Thus we have shown that as far as Hausdorff measures are concerned we can always replace a non-monotonic function by a monotonic one which is below the original function. Theorem 2 shows us that this replacement cannot be effected by using a function which is above the original one. The result is actually a little stronger in that it is proved using pre-measures.

**Theorem 2**

There exists a Hausdorff measure function \( \lambda(x) \), such that if \( H(x) \) is any monotonic Hausdorff measure function with \( H(x) \geq \lambda(x) \) for all \( x \), then,

\[ \lambda^H(S) \geq \lambda^h(S) \]

for arbitrarily small positive numbers \( \varepsilon \) and for all sets \( S \) on the real line.

**Proof**

Define \( \lambda_\alpha = \lambda_{\alpha} \) for \( \alpha = 0, 1, 2, \ldots \).
Put, $h(x) = x_a$ for $x \in (x_{a+1}, x_a]$ with $\n$ even or zero,
and $h(x) = 4x_a$ for $x \in (x_{a+1}, x_a]$ with $\n$ odd.

Now let $H(x)$ be a monotonic function with $H(x) \geq h(x)$ for all $x$. Then clearly we must have,

$$H(x) \geq 2x \quad \text{for all } x.$$

Thus for all $S$ and for all sets $S$ we must have,

$$\bigwedge_j^H (S) \geq 2 \bigwedge_j (S) \quad \text{-(16)}$$

But it is easy to see that for arbitrarily small positive values of $\n$ and for all linear sets $S$,

$$\bigwedge_j^H (S) = \bigwedge_j (S) \quad \text{-(17)}$$

Thus combining (16) and (17) we have,

$$\bigwedge_j^H (S) \geq 2 \bigwedge_j^H (S)$$

for arbitrarily small values of $\n$ and for all linear sets $S$. Hence the theorem is proved.
Theorem 3 now shows us that as far as Hausdorff pre-measures are concerned, there exist functions which cannot be replaced by monotonic ones.

**Theorem 3**

There exists a Hausdorff measure function \( H(x) \) and a set \( S \) such that if \( H(x) \) is any monotonic Hausdorff measure function then,

\[
\mathcal{H}_\delta (S) \neq \mathcal{H}_\delta (S),
\]

for arbitrarily small positive values of \( \delta \).

**Proof**

We define the sequence \( \{\alpha_n\} \) of positive real numbers as follows:

Put \( \alpha_1 = 1 \) and assume that \( \alpha_1, \ldots, \alpha_{n-1} \) have been defined, then put,

\[
\alpha_{2n} = \frac{1}{2} \alpha_{2n-1}
\]

and,

\[
\alpha_{2n+1} = \left( \frac{25}{128} \right) \alpha_{2n-1}
\]

We now define the function \( h(x) \) as follows,

\[
h(x) = \begin{cases} 
\alpha_{2n} & \text{for } x \in \left( \frac{\alpha_{2n}}{25}, \frac{\alpha_{2n}}{25} \right] \\
\alpha_{2n-1} & \text{for } x \in \left( \frac{\alpha_{2n-1}}{25}, \frac{\alpha_{2n-1}}{25} \right] 
\end{cases}
\]
Figure 1 is a rough sketch of the curve \( h(x) \), and we can see that \( h(x) \) is a Hausdorff measure function.

Let \( S_n \) be the unit interval \([0, 1]\).

Now, for any integer \( n \),

\[
\frac{h(x_{2n})}{x_{2n}} = 2^{\frac{1}{2}n_2} x_{2n-1} = 4 \, x_{2n-1}^{\frac{1}{2}n_2}.
\]

And,

\[
\frac{h(x_{2n+1})}{x_{2n+1}} = x_{2n+1}^{\frac{1}{2}n_2} = \left( \frac{128}{64} \right)^{\frac{1}{2}n_2} x_{2n-1}^{\frac{1}{2}n_2}.
\]
also,
\[ \frac{h(x_{2n})}{x_{2n}} < \frac{h(x_{2n+1})}{x_{2n+1}} \]  \hspace{1cm} (23)

and,
\[ \frac{h(x_{2n})}{x_{2n}} \leq \frac{h(x)}{x} \quad \text{whenever} \quad 0 < x < x_{2n} \]  \hspace{1cm} (24)

Let \( \{U_i\} \) be any sequence of open intervals such that,
\[ d(U_i) < x_{2n} \quad \text{for all} \quad i \]

and,
\[ S_1 \subseteq \bigcup_{i=1}^{\infty} U_i. \]

Then we have, by (24),
\[ \sum_{i=1}^{\infty} h(d(U_i)) > \frac{h(x_{2n})}{x_{2n}} \sum_{i=1}^{\infty} d(U_i). \]

But, since \( S_1 \subseteq \bigcup U_i \), we must have,
\[ \sum_{i=1}^{\infty} d(U_i) > 1, \]

thus, combining these results,
\[ \sum_{i=1}^{\infty} h(S_i) > \frac{x_{2n}^{-1}}{} h(x_{2n}) \]  \hspace{1cm} (25)
Now, let $H(x)$ be any monotonic increasing Hausdorff measure function.

We assume that, for all large values of $\alpha$,

$$\bigwedge_{x_{z_{in}}}^{H}(s_{i}) = \bigwedge_{x_{z_{in}}}^{h}(s_{i}),$$

and show that this assumption leads to a contradiction.

Now,

$$\bigwedge_{x_{z_{in}}}^{H}(s_{i}) \leq (1 + x_{z_{in}}^{-1}) H(x_{z_{in}}^{-1})$$

and so, by (26) and (25),

$$x_{z_{in}}^{-1} H(x_{z_{in}}) \leq (1 + x_{z_{in}}^{-1}) H(x_{z_{in}})$$

that is,

$$H(x_{z_{in}}) \geq \frac{1}{1 + x_{z_{in}}} H(x_{z_{in}})$$

for all large values of $\alpha$.

Thus, by the monotonicity of $H(x)$ we must have,

$$H(x) \geq \frac{2}{1 + x_{z_{in}}^{-1}} \text{ for } x \in \left[\frac{x_{z_{in}}}{2}, \frac{x_{z_{in}}^{2}}{2} \right],$$

Now consider any value, $x$, in the open interval $\left(\frac{2}{2+b} x_{z_{in-1}}, x_{z_{in}}\right)$ we have,

from (23),

$$\frac{\lambda(x)}{\lambda^{'}} = 4 x_{z_{in-1}}^{-1} < \frac{\lambda(x_{z_{in}})}{\lambda(x_{z_{in+1}})}.$$
Thus, for any such \( \nu \),

\[
\Lambda^h_{\nu} (S_1) \geq 4 \nu^{-\frac{1}{4}}.
\]

Again, we may assume that for all such small values of \( \nu \), we have,

\[
\Lambda^H_{\nu} (S_1) = \Lambda^h_{\nu} (S_1),
\]

therefore,

\[
4 \nu^{-\frac{1}{4}} \leq (1 + \nu^{\frac{1}{4}}) H(\nu),
\]

that is,

\[
H(\nu) \geq \frac{1}{1 + \nu^{\frac{1}{4}}} \nu H(\nu).
\]

Now, define the function \( H'(\nu) \) as follows,

\[
H'(\nu) = \frac{2}{1 + \nu^{\frac{1}{4}}} \nu^{\frac{1}{6}} \text{ for } \nu \in \left[ \nu, \nu^{\frac{1}{4}} \right]
\]

\[
= \frac{4 \nu^{\frac{1}{4}} \nu}{1 + \nu} \text{ for } \nu \in \left( \frac{2}{5} \nu^{\frac{1}{4}}, \nu \right]
\]

\[
= \frac{2}{1 + \nu^{\frac{1}{4}}} \nu^{\frac{1}{6}} \text{ for } \nu \in \left( \nu^{\frac{1}{4}}, \frac{2}{5} \nu^{\frac{1}{4}} \right]
\]

We have shown that if \( H(\nu) \) is a monotonic function such that for any set \( S_1 \),

\[
\Lambda^H_j (S_1) = \Lambda^h_j (S_1),
\]
for all small values of $\delta$, then,
\[ H(x) \geq H'(x) \quad \text{for all } x \quad -(29) \]

Also, we can see from the above definition that,
\[ \lim_{n \to 0} \frac{H'(x)}{x^n} = \frac{5}{4}. \]

Thus, we have, for all sets $S$,
\[ \bigwedge_{x \in S} \frac{H'(x)}{x^n} \geq \frac{5}{4} \bigwedge_{x \in S} \frac{\mu}{x^n}. \quad -(30) \]

But we know that,
\[ \lim_{n \to 0} \frac{\mu}{x^n} = 1. \]

Now let $\{A_n\}$ be a sequence of positive integers such that $\sum A_n^{-1}$ is convergent and,
\[ \prod_{n=1}^{\infty} (1 - A_n^{-1}) > \frac{q}{10} \quad -(31) \]

Let $\{B_m\}$ be an increasing sequence of positive real numbers such that,
\[ B_m \to \infty \quad \text{as } m \to \infty \quad -(32) \]
and
\[ B_m > q \quad \text{for all } m. \quad -(33) \]

Further, we can choose a null-sequence $\{\epsilon_n\}$ say, such that each $x_n$ is a point of continuity of $\mu(x)$ and,
\[ \frac{\mu(x_n)}{x_n} \to 1 \quad \text{as } n \to \infty \quad -(34) \]

We inductively define a sequence $\{t_n\}$ of real numbers as follows,
choose \( t_0 \) arbitrarily such that \( t_0 \in \{ x_n \} \) now assume that \( t_0, \ldots, t_m, \) have been defined.

Choose \( t_m \) such that,

1. \( 0 < t_m < t_{m-1} \) and \( t_m \in \{ x_n \} \)

2. \( C_m \cdot t_m = t_{m-1} \), \( C_m > A_m \)

3. \( B_m \cdot C_m \cdot t_m < t_{m-1} \).

Denote by \( K_m \) the integral part of \( C_m \). Let \( S_0 \) be the set of points of a closed interval of length \( t_0 \) on the real line. In \( S_0 \), construct \( K_1 \) closed intervals of length \( t_1 \), equally spaced distance \( \eta_1 \) apart and such that there is an interval of length \( t_1 \) at each end of \( S_0 \). Denote by \( S_1 \) the \( K_1 \) closed intervals so formed. In each interval of \( S_1 \), construct \( K_2 \) closed intervals of length \( t_2 \), equally spaced distance \( \eta_2 \) apart with an interval of length \( t_2 \) at each end of the interval of \( S_1 \). Denote by \( S_2 \) the \( K_1 \cdot K_2 \) closed intervals so formed. In general \( S_\infty \) is a set of \( K_1 \cdot \ldots \cdot K_m \) closed intervals of length \( t_m \) such that in any interval of \( S_\infty \), there are \( K_m \) intervals of \( S_\infty \) equally spaced distance \( \eta_m \) apart with an interval of length \( t_m \) at each end of the intervals of \( S_\infty \). We write,

\[
S = \bigcap_{n=0}^{\infty} S_n .
\]

Then clearly,

\[
\bigcap_{n=1}^{\infty} (S) \leq K_1 \cdot \ldots \cdot K_m \cdot t_m \quad \text{for all } n
\]

\[
\leq t_0 .
\]

Since \( S \) is compact we need only consider finite open coverings. Also, if
\{V_i\} is a finite covering of \(S\) by means of intervals of \(S_n\) for different values of \(n\), then we have,

\[
\sum_i (d(V_i))^{1/n} \leq \frac{c_1 \cdot \ldots \cdot c_{n-1}}{t_n}^{1/n}
\]

for all large integers \(N\).

This is because we may replace an interval of \(S_n\) by the \(k_{n+1}\) intervals of \(S_{n+1}\) which it contains. So we have, by (31),

\[
\sum_i (d(V_i))^{1/n} \geq \prod_{i=1}^{N} (1 - c_{i-1})^{1/n} \geq \frac{q_i}{t_i^{1/n}}.
\]

Now let \(\{U_i\}\) be any sequence of open intervals which form a covering of \(S\). Consider a particular interval \(U_i\) of this covering. There is a least integer \(n_i\), say, such that \(S_n \cap U_i\) is contained in one interval of \(S_{n-1}\) but has points in common with at least two different intervals of \(S_{n-1}\). Let \(l_i\) be the length of the interval \(U_i\) and \(r\) the number of intervals of \(S_{n-1}\) which intersect \(S_n \cap U_i\). Then we have,

\[
l_i \leq t_{n-1} \quad \text{-(37)}
\]

and,

\[
l_i \geq (r-2)t_{n-1} + (r-1)q_{n-1} \quad \text{-(38)}
\]

From (37) we have,

\[
t_{n-1}^{1/n} \leq \frac{l_i^{1/n}}{t_{n-1}}. \quad \text{-(39)}
\]

Also we know that,

\[
t_{n-1} = k_{n-1} t_n + (k_{n-1}-1) q_n \quad \text{-(40)}
\]

therefore, by iii), we have,
Thus, combining (33) and (40),

$$\frac{l_i}{t^m_{m-1}} \geq \frac{(r-2) t_m + (r-1) y_m}{K_m t_m + (K_m - 1) y_m}$$

so that,

$$r t^m_m \leq t^m_{m-1} \leq r \left[ \frac{K_m t_m + (K_m - 1) y_m}{(r-2) t_m + (r-1) y_m} \right]^{\frac{1}{r}} l_i$$

$$\leq \left[ \frac{1}{1 - \frac{2}{r + (r-1) y_m / t_m}} \right]^{\frac{1}{r}} l_i$$

$$\leq \left[ \frac{1}{1 - \frac{2}{2 + y_m / t_m}} \right]^{\frac{1}{r}} l_i$$

$$\leq \left( \frac{\beta_m + 1}{\beta_m - 1} \right)^{\frac{1}{r}} l_i$$

by (41)

where $M$ is the greatest integer $m$ such that,

$$d(U_i) < y^m_m$$

for all $i$. 
Therefore we have, by (36),

\[
\sum_{i} (c(u_i))^{n_k} \geq \left( \frac{b_m - 1}{b_m + 1} \right)^n \eta_0 \leq \varepsilon_0^{\frac{1}{10}}
\]

that is,

\[
\lambda^{n_k}(S) \geq \eta_0 \leq \varepsilon_0^{\frac{1}{10}}
\]

since \( b_n \to \infty \) as \( m \to \infty \).

Also we have,

\[
\lambda^{n_k}(S) = (k_1 \cdots k_n \eta_0^n) \leq (k_1 \cdots k_n \eta_0^n) / \eta_0^{n_k}
\]

Therefore, by (30) and (42),

\[
\lambda^{n_k}(S) \leq \varepsilon_0^{\frac{1}{10}}
\]

Thus, combining (30), (42) and (43), we have,

\[
\lambda^{n_k}(S) \geq \eta_0 \geq \eta_0^{n_k} \geq \eta_0^{n_k} \lambda^k(S) > \lambda^k(S)
\]

Hence, for all small values of \( \delta \) we have,

\[
\lambda^H_{\delta}(S) > \lambda^k_{\delta}(S)
\]

as required. Thus, if we have \( \lambda^H_{\delta}(S) = \lambda^k_{\delta}(S) \) for all small values of \( \delta \), then we must also have statement (44) and hence the theorem is proved.
CHAPTER 2

INTRODUCTION

The ideas discussed in this chapter came initially, from a study of Dvoretzky's paper (2) in which he proves that, given any continuous, monotonic Hausdorff measure function, a necessary and sufficient condition for there to be a set in q-dimensional Euclidean space with the property that $\mathcal{H}(S)$ is positive and finite is that $\mathcal{H}(x)$ should be a q-dimensional Hausdorff measure function. In his remarks at the end of the paper, Dvoretzky explains how, in the one-dimensional case, this result can be extended to discontinuous functions. Following these results it seemed interesting to investigate whether discontinuous functions need any special consideration with regard to Hausdorff measures, or whether, in fact, it is sufficient to consider only continuous functions. If we alter the definition of Hausdorff measure so that we consider either, just closed convex coverings, or coverings consisting of any convex sets, then Dvoretzky's result generalises directly to discontinuous functions for sets in q-dimensional Euclidean space. In fact we see, from Theorem 4, that Dvoretzky's result does apply to discontinuous functions in q-dimensional Euclidean space, when only convex open coverings are permitted in the definition of Hausdorff measure. Theorems 5 and 6 show that when considering a particular set of finite, positive measure it is necessary only to consider continuous functions. Theorems 7 and 8 show that for any discontinuous one-dimensional Hausdorff measure function $\mathcal{H}(x)$ there is a continuous one-dimensional Hausdorff measure function $\mathcal{H}(x)$ such that, for all linear sets $S$, $\mathcal{H}(S)$ is equal to $\mathcal{H}(S)$. But, under certain conditions in q-dimensional space $(q > 1)$ the discontinuous functions
require special consideration. This latter result suggests that some of the theorems which have been established for continuous functions may not generalise to the discontinuous case.

Before proving these theorems we need to prove a lemma.

Lemma 1

For any set $S$ in $q$-dimensional Euclidean space and any Hausdorff measure function $\mu(x)$, we have,

\[
\mu_R(S) \geq \mu^b(S) \geq \left(\sqrt[q]{\frac{q}{2}} + 1\right)^{-q} \mu_R(S)
\]

Proof.

Given any $\varepsilon > 0$ and $\delta > 0$ let $\{U_i^\delta\}$ be an open covering of $S$ such that,

\[
d(U_i^\delta) < \varepsilon \quad \text{for all } i \quad -(1)
\]

and,

\[
\sum_i \mu(U_i^\delta) < \mu^b(S) + \left(\sqrt[q]{\frac{q}{2}} + 1\right)^{-q} \varepsilon \quad -(2)
\]

Now replace each set $U_i^\delta$ by $\left(\sqrt[q]{\frac{q}{2}} + 1\right)^{-q}$ open $q$-dimensional rectangles $R_i, \ldots, R_i(\sqrt[q]{\frac{q}{2}} + 1)^q$, with sides parallel to the coordinate axes, and such that,

\[
d(R_i) = d(U_i^\delta) \quad \text{for } s = 1, \ldots, (\sqrt[q]{\frac{q}{2}} + 1)^q \quad \text{and for each } i
\]

\[
U_i \subseteq \bigcup_{s=1}^{(\sqrt[q]{\frac{q}{2}} + 1)^q} R_i^s \quad \text{for each } i
\]

In this manner we get a covering of $S$ by open rectangles $R_i$ such that,
Thus, by (2), and the definition of Hausdorff measure,
\[ \sum_{i,j} h(d(R_{i,j})) = \left(\left\lceil \frac{d}{\sqrt{a}} \right\rceil + 1 \right)^q \sum_i h(d(u_i^j)). \]

But, this result is true for arbitrary \( d \) and \( \varepsilon \), so we have,
\[ \sum_{i,j} h(R_{i,j}) (S) = \left(\left\lceil \frac{d}{\sqrt{a}} \right\rceil + 1 \right)^q \sum_i h(S) + \varepsilon. \]

The inequality \( \sum_{i,j} h(R_{i,j}) (S) \geq \sum_i h(S) \) is trivial and thus we have the required result.

**Theorem 4**

Let \( \mu(x) \) be any Hausdorff measure function. Then a necessary and sufficient condition for there to exist a set \( S \) in q-dimensional Euclidean space with \( \mu(S) \) positive and finite is,
\[ \lim_{x \to 0} \frac{\mu(x)}{x^q} > 0. \]

that is, \( \mu(x) \) is a q-dimensional Hausdorff measure function.

**Proof**

Assume,
\[ \lim_{x \to 0} \frac{\mu(x)}{x^q} = 0. \]

Let \( R \) be any q-dimensional rectangle with longest side \( d \), say. Choose a
sequence \( \{x_n\} \) of real numbers such that,
\[
x_n \to 0 \quad \text{as} \quad n \to \infty
\]
and,
\[
\frac{h(x_n)}{x_n^q} \to 0 \quad \text{as} \quad n \to \infty
\]
For any integer \( n \), we may cover \( R \) with \( \left( \left\lceil \frac{d(x_n^q)}{x_n} \right\rceil + 1 \right)^q \) squares of diameter \( x_n \). Thus, for any positive number \( \delta \), we have,
\[
\wedge_{\delta}^h(R) \leq \left( \left\lceil \frac{d(x_n^q)}{x_n} \right\rceil + 1 \right)^q \frac{h(x_n)}{x_n^q}
\]
for all large integers \( n \). That is,
\[
\wedge_{\delta}^h(R) \leq \left( 2d(x_n^q) \right)^q \frac{h(x_n)}{x_n^q}
\]
But,
\[
\frac{h(x_n)}{x_n^q} \to 0 \quad \text{as} \quad n \to \infty,
\]
so \( \wedge_{\delta}^h(R) \) is zero for every rectangle \( R \) and for every \( \delta > 0 \), thus \( \wedge_{\delta}^h(R) = 0 \) for every set \( S \) in \( R \) and so by Lemma 1, \( \wedge^h(S) = 0 \) for all sets \( S \). Hence the necessity of the condition is proved.

We now prove the sufficiency of the condition.

**Case 1:**
\[
\lim_{x \to 0} \frac{h(x)}{x^q} = \kappa,
\]
where \( \kappa \) is a finite, positive constant.

Now, given any \( \varepsilon > 0 \), there exists an \( x_0 \) such that,
\[
\frac{h(x)}{x^q} > \kappa - \varepsilon \quad \text{for all} \quad x \leq x_0,
\]
also we know that,
\[
\frac{h(x)}{x^q} < \kappa + \varepsilon
\]
for infinitely many small values of $p$.

Now let $R$ denote not only a rectangle, but also its $q$-dimensional volume (that is, the product of the lengths of its sides). Let $\{V_i\}$ be any open covering of $R$ with $d(V_i) < \delta < \infty$ for all $i$, then,

$$
\bigwedge_{\delta}^h(R) = \bigwedge_{\delta} \sum_{i} h(d(V_i)) \geq \bigwedge_{\delta} \sum_{i} (p-1) d(V_i)^q.
$$

Each set $\{V_i\}$ can be enclosed in a $q$-dimensional cube of side $d(V_i)$ with sides parallel to the coordinate axes. These cubes then cover $R$ and so we have,

$$
\sum_{i} (d(V_i))^q \geq R
$$

thus we have,

$$
\bigwedge_{\delta}^h(R) \geq (p-1) R.
$$

Now let $\{a_1, \ldots, a_q\}$ denote the side-lengths of the rectangle $R$, then,

$$
\prod_{i=1}^q a_i = R.
$$

Also, for any $x < \delta$,

$$
\bigwedge_{\delta}^h(R) \leq \prod_{i=1}^q \left( \left\lfloor \frac{a_i}{x} \right\rfloor + 1 \right) h(x).
$$

Now choose $x$ so small that,

$$
x < \delta
$$

$$
\min_{1 \leq i \leq q} \left\lfloor \frac{a_i}{x} \right\rfloor > \frac{1}{2},
$$

and,

$$
\frac{h(x)}{x^q} < x + \varepsilon.
$$
Then we have,

$$\mathcal{L}_\delta^h(R) \leq (1+\varepsilon)^a \left( \sum_{i=1}^a \frac{\alpha_i}{\alpha} \right) R \cdot \frac{h(x)}{\infty^a}$$

$$< (1+\varepsilon)^a \left( \sum_{i=1}^a \frac{\alpha_i}{\alpha} \right) R$$

The inequalities (3) and (4) hold for all positive values $\delta$ and $\varepsilon$. Thus we have,

$$\alpha R \leq \mathcal{L}_\delta^h(R) \leq \alpha \left( \sum_{i=1}^a \frac{\alpha_i}{\alpha} \right) R.$$ 

Hence the sufficiency is proved for $\lim_{\alpha \to 0} \frac{h(x)}{\infty^a} = \alpha$ where we have, $0 < \alpha < \infty$.

**Case 2:**

$$\lim_{\alpha \to 0} \frac{h(x)}{\infty^a} = \alpha$$ and hence $\lim_{\alpha \to 0} \frac{h(x)}{\infty^a} = \alpha$.

We shall now construct a set $S$ in a similar manner to Dvoretzky's construction and prove that this has the property that $\mathcal{L}^h(S)$ is positive and finite.

Since $h(x)$ may be assumed to be monotonic increasing, it has only a countable number of points of discontinuity. Let these points be denoted by $d_1, d_2, \ldots$.

Thus from (5) we have,

$$\lim_{\alpha \to 0} \frac{h(x)}{\infty^a} = \alpha$$

$$\lim_{\alpha \to 0} \frac{h(x)}{\infty^a} = \alpha$$ for any $x \neq d_i$. for any $i$. 

27.
From (6) it follows that there exist arbitrarily small positive numbers \( \varepsilon \), satisfying,

\[
x + \varepsilon \quad \text{for any} \quad i, \quad \frac{h(x)}{x^a} < \frac{2h(x)}{x^b} \quad \text{(7)}
\]

for all \( t \) satisfying \( 0 < t < x \) and \( t \neq a \) for any \( i \).

Now let \( \{A_n\} \) be a sequence of positive numbers such that all the terms of the sequence are greater than two and such that the series \( \sum \frac{1}{A_n} \) converges. Also, let \( B \) be a positive number greater than or equal to two.

We now proceed to construct the sequence \( \{y_n\} \) of positive numbers, as follows:

choose \( y_0 \) to be any positive number such that \( y=y_0 \) satisfies (7).

Having chosen \( y_n \) for \( 0 < n < n-1 \), we choose \( y_{n+1} (0 < y_{n+1} < y_{n-1}) \) such that,

\[\begin{align*}
\text{a).} & \quad (7) \text{ holds for } y=y_{n+1} \\
\text{b).} & \quad h(y_{n+1})=y_{n+1} h(y_n) \text{ with } C_n > A_n \\
\text{c).} & \quad b_c y_n < y_{n+1}.
\end{align*}\]

It is clear that these choices are permissible from the restrictions placed on \( h(x) \).

We assume that \( x \to 0 \) as \( n \to \infty \).

Denote by \( y \) the integral part of \( y_n \). Let \( S_o \) be the set of points of a closed cube \( J_o \) with sides parallel to the coordinate axes and each of length \( y \). Each side of this \( q \)-dimensional cube is a closed interval of length \( y \). From each side we remove \( y_{n-1} \) open intervals each of length \( y \). So that there remain \( y_n \) closed intervals each of length \( y \). This is possible since,
We do this in such a way that when considering any one face of the cube, the opposite face is dissected in an exactly symmetrical fashion. We now form \( K^a \) closed cubes of side \( \frac{x_0}{\sqrt{a}} \), inside \( T_0 \), by taking cartesian products of the intervals of length \( \frac{x_0}{\sqrt{a}} \). Each one of these closed cubes is denoted by \( I_j \) and \( S_j \) denotes the set of points contained in the \( K^a \) cubes \( I_j \). In each \( I_j \), we construct \( K^a \) cubes \( I_k \) each of side \( \frac{x_0}{\sqrt{a}} \) in a similar manner, and denote by \( S_k \) the set of points contained in the \( K^a \) closed cubes \( I_k \). Continuing in this way we define \( S \) as the set of points contained in all \( S_n \) \((n=0,1, \ldots)\). Then, \( S \) is a perfect nowhere dense set.

We have now to show that \( \lambda^*(S) \) is positive and finite.

Given any \( \delta > 0 \), there is a sufficiently large \( n \) for which \( x_n < \delta \). The set \( S \) being included in \( S_n \) can be covered by the \( K^a \) closed cubes \( I_n \).

\[
\sum_{U_i \in S_n} \lambda(U_i) = K^a \cdots K^a \lambda(x_n)
\]

where, the \( U_i \) are the cubes \( I_n \) which form the set \( S_n \).

Using b) and the fact that \( K_n \leq C_n \) in (8), we get,

\[
K^a \cdots K^a \lambda(x_n) \leq K^a \cdots K^a \lambda(x_{n-1}) \leq \cdots \leq \lambda(x_0)
\]

Since each of the points \( x_n \) is a point of continuity of \( \lambda(x) \) we can replace each of the closed sets \( U_i \) by open sets \( V_i \) containing \( U_i \) and
such that \( \sum_i \mu_i \delta(V_i) \) differs from \( \sum_i \mu_i \delta(U_i) \) by an arbitrarily small amount. Combining this fact with (8) and (9) and the definition of Hausdorff measure, we have,

\[
\Lambda^h_\delta(S) \leq \sum_i \mu_i \delta(V_i) < 2 \mu(x_0)
\]

and therefore, since (10) holds for all \( J > 0 \),

\[
\Lambda^h_\delta(S) < 2 \mu(x_0) < \infty.
\]

Thus we have proved that \( \Lambda^h(S) \) is finite.

Before proving \( \Lambda^h(S) > 0 \) we note the following. If we cover \( S \) with the \( \kappa_1^a, \ldots, \kappa_n^a \) cubes \( I_i \), then \( \sum_i^\nu \mu(x) \) will be,

\[
\kappa_1^a \ldots \kappa_n^a \mu(x_0) > (c_i - 1)^a \ldots (c_n - 1)^a \mu(x_0)
\]

\[
= \frac{\mu(x_0)}{c_1^a} \ldots \frac{\mu(x_0)}{c_n^a}
\]

\[
= \prod_{i=1}^\nu \left( \frac{c_i - 1}{c_i^a} \right) \mu(x_0).
\]

Now we know that \( \sum \sqrt[\nu]{A_\nu} \) converges and that \( c_\nu > A_\nu \) so \( \sum \sqrt[\nu]{c_\nu} \) converges and hence so does \( \prod (1 - c_\nu)^a/ \). Denoting this product by \( \rho \), we have, for every \( \nu \),

\[
\sum_{i=1}^\nu \mu(x_0) \geq \rho \mu(x_0)
\]

This inequality still holds if we enclose \( S \) in a finite number of cubes \( I_m, I_n, \ldots \) not necessarily bearing the same index. This follows from (9)
which shows that if any \( \overline{J}_{a_{-1}} \) is replaced by the \( \mathcal{K}_{a} \) cubes \( \overline{J}_{a} \) included in it the contribution to \( \sum \ell \mathcal{U}(\mathcal{C}) \) cannot increase.

We now prove that \( \mathcal{A}^\mathcal{K}_a(S) > 0 \).

We show that if we enclose \( S \) in any finite number of open rectangles \( \mathcal{R}_1, \mathcal{R}_2, \ldots \) and if \( \sum \ell \mathcal{U}(\mathcal{C}) \) for these rectangles is \( \mathcal{H} \), then there is a covering of \( S \) by cubes \( \overline{J}_{a} \) for which \( \sum \ell \mathcal{U}(\mathcal{C}) < \mathcal{H} \), where \( \mathcal{C} \) is a constant. Hence by virtue of our previous remarks, \( \mathcal{H} > \mathcal{C} \mathcal{U}(\mathcal{C}) \) and consequently \( \mathcal{A}^\mathcal{K}_a(S) > 0 \).

If each one of the rectangles \( \mathcal{R} \) which does not contain any point of \( S \) is deleted, and every other rectangle \( \mathcal{R} \) is replaced by the largest closed rectangle \( \mathcal{R} \) the interior of which is contained in \( \mathcal{R} \) and such that each one of its edges contains at least one point of \( S \) then the rectangles \( \mathcal{R} \) still cover \( S \) and \( \sum \ell \mathcal{U}(\mathcal{C}) \) is only diminished by this replacement. It is sufficient to show that if \( \mathcal{R} \) is any one of the rectangles thus obtained, then it can be replaced by \( r^q \) cubes \( \overline{J}_{a} \) with \( r^q \ell \mathcal{U}(\mathcal{C}) \) where \( l \) is the diameter of \( \mathcal{R} \).

Consider \( \mathcal{R} \) and the sets \( S_1, S_2, \ldots \); there must be a first set \( S_\infty \) such that \( \mathcal{R} \) contains points not belonging to \( S_\infty \). Then \( \mathcal{R} \) is contained in one cube \( \overline{J}_{a_{-1}} \) but has points in common with at most \( r^q \) cubes \( \overline{J}_{a} \) contained in \( \overline{J}_{a_{-1}} \) where \( r \) is the greatest number of cubes \( \overline{J}_{a} \) met by any one edge of \( \mathcal{R} \) \((r \geq 2) \). This is because every side of \( \mathcal{R} \) contains at least one point of \( S \).

Since \( l > a_{-1} \), we have that,

\[
\frac{\ell \mathcal{U}(\mathcal{C})}{a_{-1}^q} < \frac{2 \mathcal{U}(\mathcal{C})}{l^q}, \quad \text{if } l > a_{-1} \text{ for any } i.
\]

If \( l = a_{-1} \) for some \( i \), then there is an \( \mathcal{R}' \) such that \( \mathcal{R} \prec \mathcal{R}' \prec \mathcal{R} \) and \( \mathcal{R}' \) is a point of continuity of \( \mathcal{U}(\mathcal{C}) \). Then \( \mathcal{R}' \prec a_{-1} \), and hence,

\[
\frac{\ell \mathcal{U}(\mathcal{C})}{a_{-1}^q} < \frac{2 \mathcal{U}(\mathcal{C})}{l'^q}.
\]
Also we have,

\[ l > \frac{1}{2} \left( (r-2) \frac{x_n}{\sqrt{q}} + (r-1) \frac{y_n}{\sqrt{q}} \right) \]

where \( \frac{y_n}{\sqrt{q}} \) is the distance between two adjacent cubes of \( S_n \),
and therefore,

\[ l' > \frac{1}{2\sqrt{q}} \left\{ (r-2) x_n + (r-1) y_n \right\} \]

Now,

\[ x_{n-1} = K_n x_n + (K_n - 1) y_n \]

and therefore,

\[ y_n = \frac{x_{n-1} - K_n x_n}{K_n - 1} > \frac{B K_n x_n - K_n x_n}{K_n - 1} \] by \( c \).

\[ > x_c. \]

Thus,

\[ l' > \frac{1}{2\sqrt{q}} \left[ (r-2) x_n + (r-1) y_n \right] \]

\[ > \frac{1}{2\sqrt{q}} \frac{(r-1)}{2K_n}. \]

Therefore,

\[ r^4 h(x_m) < \frac{r^4}{K_n} h(x_{n-1}) < \frac{2r^4 x_{n-1}}{K_n} h(l') \]

\[ < \frac{r^4}{K_n} \left[ \frac{2K_n - 1}{2\sqrt{q} (r-1)} \right]^q h(l') \]
So we have,
\[ \sum h(l') > \lambda c \rho h(x_0). \]

But, in every case \( \lambda' \leq \lambda \) and hence,
\[ \sum h(l) \geq \sum h(l'). \]

Therefore,
\[ \lambda, \rho(x) \geq \lambda c \rho h(x_0). \]

Thus, using Lemma 1,
\[ \lambda h(x) \geq \frac{1}{c(L^k+1)^2} \rho h(x_0) > 0. \]

Hence the theorem is proved.

We now proceed to investigate whether it is always possible to replace discontinuous functions by continuous ones.

**Theorem 5**

If \( \lambda(x) \) is any one-dimensional Hausdorff measure function and \( S \) is a set on the real line, such that \( \lambda^h(S) \) is positive and finite, then there exists a continuous one-dimensional Hausdorff measure function \( \lambda^H(S) \)

say, such that,
\[ \lambda^h(S) = \lambda^H(S). \]

**Proof**

Let \( \bigcup X \) be any covering of \( S \) by open intervals \( X_i \). Take any
one of these intervals \( X_i \); if it overlaps another interval of the cover \( X_i \), say, then let,

\[
X_i = (a_i, \eta_i), \quad X_j = (a_j, b_j) \quad \text{and} \quad a_j < b_i \quad \text{say.}
\]

Then there exist points \( \eta_j \) and \( \eta_i \) such that,

\[
a_j < \eta_j < \eta_i < b_i.
\]

and both \((\eta_i - a_i)\) and \((b_i - \eta_j)\) are points of continuity of \( \mathcal{W}(x) \).

Replace \( X_i \), \( X_j \) by the open intervals \( X_i', X_j' \) such that,

\[
X_i' = (a_i, \eta_i) \quad \text{and} \quad X_j' = (\eta_j, b_j).
\]

Then, these two intervals cover as much of \( S \) as the original two did, their diameters are less than the original and are both at points of continuity of \( \mathcal{W}(x) \).

If, however, \( X_i \) does not overlap any other \( X_j \) for \( j \neq i \) then, we proceed as follows; given any \( \varepsilon > 0 \) there exists a \( \delta_i \) such that, for integral values of \( i \),

\[
\mathcal{W}(\delta_i) < \varepsilon/2;
\]

and,

\[
0 < \delta_i < (b_i - a_i).
\]

The intervals \( X_i = (a_i, b_i) \) and \((b_i - \delta_i/\varepsilon, b_i + \delta_i/\varepsilon)\) overlap and cover as much of \( S \) as \( X_i \) did. We now replace these intervals, as before, by \((a_i, \eta_i)\) and \((\eta_i, b_i + \delta_i/\varepsilon)\) with,

\[
-\delta_i/\varepsilon = \eta_i - b_i < \eta_i < \eta_i + \delta_i/\varepsilon < b_i.
\]

such that \((\eta_i - a_i)\) and \((b_i + \delta_i/\varepsilon - \eta_i)\) are points of continuity of \( \mathcal{W}(x) \).
In this way, we get a covering \( \bigcup_{i=1}^{k} X_i' \) of \( S \) such that,

\[
\sum_{i} h(d(X_i')) < \sum_{i} h(d(X_i)) + \varepsilon
\]

for any given \( \varepsilon > 0 \) and with the property that \( d(X_i') \) is a point of continuity of \( h(x) \).

Let \( x_1, x_2, \ldots \) be an enumeration of all the points of discontinuity of \( h(x) \). Given any \( \varepsilon > 0 \) and any \( \delta > 0 \), there exists a covering \( \bigcup X_i \) of \( S \) such that,

\[
\sum_{i} h(d(X_i)) < \sum_{i} h(d(X_i')) < \sum_{i} h(X_i) + \varepsilon
\]

and \( d(X_i') < \delta \) for all \( i \).

Replace this covering by the corresponding covering \( \bigcup X_i' \) of \( S \).

Then, we have, by (11) and (12),

\[
\sum_{i} h(X_i) < \sum_{i} h(d(X_i')) < \sum_{i} h(X_i) + 2\varepsilon
\]

with \( d(X_i') < \delta \) for all \( i \), and \( d(X_i') \) is a point of continuity of \( h(x) \) for each \( i \).

We now choose \( \delta \) such that,

\[
\sum_{i} h(X_i) - \varepsilon < \sum_{i} h(d(X_i')) < \sum_{i} h(X_i) + 2\varepsilon
\]

Thus, combining (13) and (14) we see that, given any \( \varepsilon > 0 \) there exists \( \delta' > 0 \) such that for all \( \delta < \delta' \) there is a covering \( \bigcup X_i' \) of \( S \) such that \( d(X_i') < \delta \) for all \( i \), \( d(X_i') \) is a point of continuity of \( h(x) \) for each \( i \), and,

\[
\sum_{i} h(X_i) - \varepsilon < \sum_{i} h(d(X_i')) < \sum_{i} h(X_i) + 2\varepsilon
\]
Now choose a sequence \( \{d_n\} \) of positive numbers satisfying the following conditions,

\[
\begin{align*}
&d_n > d_{n+1} \quad \text{for all } n \quad - (16) \\
&d_n \to 0 \quad \text{as } n \to \infty \quad - (17) \\
&d_n < \delta' \quad \text{for all } n \quad - (18)
\end{align*}
\]

and such that each \( d_n \) is a point of continuity of \( h(x) \). Then, as shown above, for any \( n \), there exists a covering \( \bigcup \chi'_n \) of \( S \) such that \( d(\chi'_n) < d_n \) for all \( n \), \( d(\chi'_n) \) is a point of continuity of \( h(x) \) for each \( n \), and,

\[
\lambda^h(S) - \varepsilon < \sum \lambda^h(d(\chi'_n)) < \lambda^h(S) + 2\varepsilon \quad - (19)
\]

Since, for any \( n \), \( \sum \lambda^h(d(\chi'_n)) \) is convergent we must have,

\[
h(d(\chi'_n)) \to 0 \quad \text{as } n \to \infty
\]

and therefore, since \( h(x) \to 0 \) for \( x \to 0 \),

\[
d(\chi'_n) \to 0 \quad \text{as } n \to \infty.
\]

Thus zero is the only possible limit point of the sequence of diameters \( \{d(\chi'_n)\}_{n=1}^{\infty} \). Because of this, we can now enclose each \( \chi'_n \) in an interval \( \lambda_{ij} = (\alpha_{ij}, \beta_{ij}) \) such that no \( \lambda_{ij} \) contains a \( d(\chi'_n) \). If at any stage we find that a point of discontinuity \( x \) is already included in an \( \lambda_{ij} \) for some \( j \neq i \) we leave it alone. We, further, insist that \( \alpha_{ij} \) and \( \beta_{ij} \) should be points of continuity of \( h(x) \) for each \( i \) and that no \( \lambda_{ij} \) should contain any \( d_n \). This last restriction is permissible as zero
is the only limit point of the sequence \( \{L_n\} \).

We now define the continuous function \( H^{(n)}(x) \) as follows,

\[
H^{(n)}(x) = \begin{cases} 
    \psi(x) & \text{if } x \notin \Lambda_i \text{ for each } i, \\
    \psi(x_i) & \text{if } x \in [x_i, P_i], \\
    H^{(n)}(\phi_i^{(m)}) = \psi(\phi_i^{(m)} + \alpha) & \text{for } m = 1, 2, \ldots 
\end{cases}
\]

where \( \phi_i^{(m)} = \alpha_i + \frac{1}{2^m}(\alpha_i - \alpha_i) \) for \( m = 0, 1, 2, \ldots \).

Finally define \( H^{(n)}(x) \) to be linear and continuous in \([\phi_i^{(m)}, \phi_i^{(m+1)}]\) for \( m = 2, 3, \ldots \) and in \([\phi_i^{(n)}, x_i]\).

Thus, we have defined a continuous, increasing function \( H^{(n)}(x) \) such that

\[
H^{(n)}(x_i) = \psi(x_i) \text{ and }, \quad H^{(n)}(x') = \psi(x') \text{ for all } x \text{ and,}
\]

\[
H^{(n)}(d(X_{1i})) = \psi(d(X_{1i})) \text{ for all } i.
\]

So we have,

\[
\Lambda \left( S - \tau \right) < \sum_{i} H^{(n)}(d(X_{1i})) < \Lambda(S) + 2\tau \quad - (10)
\]

and,

\[
d(X_{1i}) < \delta_i \text{ for all } i.
\]

We now consider the covering \( \cup \alpha_{1i} \) of \( S \). If necessary, we shrink any of

the intervals \( \lambda_{1i} \) to intervals \( \lambda'_{1i} \) contained in \( \lambda_{1i} \) with \( \lambda_{1i} \equiv (\alpha_{1i}, P_{1i}) \)

and such that no \( \lambda_{1i} \) contains a \( d(X'_{1i}) \). For those values of \( i \) where

\( \lambda_{1i} \not\equiv \alpha_{1i} \), we insist that \( \alpha_{1i} > \phi_i^{(n)} \). Also, if any of the

discontinuities of \( \psi(x) \) becomes 'exposed' by this shrinking we similarly

enclose them in a suitable \( \lambda_{1i} \) which does not contain any \( \phi_i^{(m)} \) unless
it is the discontinuity itself. Again we define the continuous, increasing function $H^{(\nu)}(x)$ as follows,

$$H^{(\nu)}(x) = \begin{cases} H(x) & \text{if } x \neq \frac{1}{2^n}, \text{ for each } n \\ H\left(\frac{1}{2^n}\right) & \text{if } x \in \left[\frac{1}{2^n}, \frac{1}{2^{n+1}}\right) \end{cases}$$

where

$$H^{(m)}\left(x_{n+1}\right) = H\left(x_{n+1} + \phi_{n+1}^{(m)}\right) \text{ for } m = 0, 1, 2, \ldots$$

Finally define $H^{(\nu)}(x)$ to be linear and continuous in $[\phi_{n+1}^{(m)}, x_{n+1}]$ for $m = 0, 1, 2, \ldots$ and in $[\phi_{n+1}^{(m)}, x_{n+1}]$.

Thus, we see that,

$$H^{(\nu)}(x) \geq H^{(\nu)}(x) \geq H(x) \text{ for all } x,$$

and

$$H^{(\nu)}(\delta(X_{n+1})) = H(\delta(X_{n+1})) \text{ for all } i.$$

Again we have,

$$\Lambda^k(S) - \varepsilon < \sum_{i} H^{(\nu)}(\delta(X_{n+1})) < \Lambda^k(S) + 2\varepsilon \quad -(21)$$

and

$$\delta(X_{n+1}) < \delta_{n} \text{ for all } i.$$

Continuing in this manner we get continuous, increasing functions $H^{(m)}(x)$ such that,

$$H^{(m)}(x) \geq H^{(m)}(x) \geq H(x) \text{ for all } x,$$

$$H^{(m)}(\delta(X_{n+1})) = H(\delta(X_{n+1})) \text{ for all } i.$$

$$\Lambda^k(S) - \varepsilon < \sum_{i} H^{(m)}(\delta(X_{n+1})) < \Lambda^k(S) + 2\varepsilon \quad -(22)$$

and

$$\delta(X_{n+1}) < \delta_n \text{ for all } i.$$
We now define the function $H(\alpha)$ as follows,

$$H(\alpha) = H^{(\alpha)}(\alpha')$$

for $x \in \mathbb{C}_{\alpha'}^\alpha$.

We see that $H(\alpha)$ is continuous, increasing and $H(\alpha) \to 0$ as $\alpha \to 0$. Also we have,

$$H(\alpha) \geq H(\alpha')$$

for all $\alpha'$ - (23)

and

$$H(\alpha) \leq H^{(\alpha)}(\alpha')$$

for $x \in (0, \delta_{\alpha})$ - (24)

From (23), we deduce that $H(\alpha)$ is also a one-dimensional Hausdorff measure function. For any integer $n$, we have, using (24) and (22),

$$\mathcal{L}^H(\alpha) \leq \mathcal{L}^{H^{(\alpha)}}(\alpha) \leq \sum_{i=0}^{\infty} H^{(\alpha)}(x_{\alpha}';)$$

$$\leq \mathcal{L}^H(\alpha) + 2\varepsilon.$$

But,

$$\lim_{n \to \infty} \mathcal{L}^H(\alpha) = \mathcal{L}^H(\alpha),$$

thus,

$$\mathcal{L}^H(\alpha) \leq \mathcal{L}^H(\alpha) + 2\varepsilon.$$

From (23) we see that,

$$\mathcal{L}^H(\alpha) \geq \mathcal{L}^H(\alpha).$$

Hence we have shown that, given any $T > 0$ there is a continuous one-dimensional Hausdorff measure function $H(\alpha)$ such that,

$$\mathcal{L}^H(\alpha) \leq \mathcal{L}^H(\alpha) \leq \mathcal{L}^H(\alpha) + 2\varepsilon.$$

Thus we get the required continuous function merely by multiplying $H(\alpha)$ by the appropriate constant, that is,
We see that the proof of this theorem relied very heavily on the fact that we were working with sets on the real line. The next theorem shows that the result is also true in q-dimensional Euclidean space ($q > 1$).

**Theorem 6**

If $\mathcal{H}(\alpha)$ is any q-dimensional Hausdorff measure function and $S$ is a set in q-dimensional Euclidean space such that $\mathcal{H}(S)$ is positive and finite, then there exists a continuous q-dimensional Hausdorff measure function $\mathcal{H}(\alpha)$ such that,

$$\mathcal{H}(S) = \mathcal{H}(S).$$

**Proof**

Let $\{d_n\}$ be a sequence such that $d_n \downarrow 0$ as $n \to \infty$ and such that each $d_n$ is a point of continuity of $\mathcal{H}(\alpha)$. Given any $\varepsilon > 0$, then corresponding to each $d_n$ there exists a covering $\bigcup_{i=1}^{\infty} X_{n_i}$ of $S$ such that,

$$d(X_{n_i}) < d_n$$

for all $i$,

and,

$$\mathcal{H}_{d_n}(S) < \sum_i \mathcal{H}(d(X_{n_i})) < \mathcal{H}_{d_n}(S) + \varepsilon \quad -(25)$$

We may assume that the $X_{n_i}$ are ordered such that,

$$d(X_{n_{i+1}}) < d(X_{n_i}).$$
Define the function \( F^{(0)}(x) \) as follows,
\[
F^{(0)}(x) = \min(d(X_{i;i}), d(X_{i;i})), \quad \text{for } x \in \left( d(X_{i;i}), d(X_{i;i}) \right]
\]
for \( i = 1, 2, \ldots \).

Then \( F^{(0)}(x) \geq h(x) \) for \( x \in (0, d(X_{i;i})) \) and,
\[
\bigwedge_{d_{X_i}} F^{(0)}(S) \leq \sum_{i} h(d(X_{i;i})) = \sum_{i} h(d(X_{i;i})).
\]

Thus, by (25), we have,
\[
\bigwedge_{d_{X_i}} F^{(0)}(S) < \bigwedge_{d_{X_i}} h(S) + \varepsilon \tag{26}
\]
Define \( F^{(1)}(x) \) as follows,
\[
F^{(1)}(x) = F^{(0)}(x) \quad \text{for } x \in \left( d(X_{i;i}), d(X_{i;i}) \right]
\]
\[
F^{(1)}(x) = h(d(X_{i;i})) \quad \text{if } x \in \left( d(X_{i;i}), d(X_{i;i}) \right]
\]
and \( F^{(1)}(x) \geq h(d(X_{i;i})) \)
and,\[
F^{(1)}(x) = F^{(0)}(x) \quad \text{if } x \in \left( d(X_{i;i}), d(X_{i;i}) \right]
\]
and \( F^{(1)}(x) < h(d(X_{i;i})) \).

Then,
\[
\min(d(X_{i;i}), d(X_{i;i})) \leq F^{(1)}(x) \leq F^{(0)}(x) \quad \text{in } (0, d(X_{i;i}))]
\]
and,
\[
\bigwedge_{d_{X_i}} F^{(1)}(S) < \bigwedge_{d_{X_i}} h(S) + \varepsilon. \tag{27}
\]
Continuing in this manner we get \( F^{(n)}(x) \) satisfying,
\[
h(x) \leq F^{(n)}(x) \leq F^{(n-1)}(x) \tag{28}
\]
and,
\[ \mathcal{L}^F(S) < \mathcal{L}^h(S) + \varepsilon. \]  \hfill (29)

Now define \( F(x) \) as follows,
\[ F(x) = F^{(n)}(x) \quad \text{for} \quad x \in (d(x_{m+1}), d(x_m)) \]

Then, by (28),
\[ F(x) \geq \mathcal{H}(x) \quad \text{for all} \ x \]
and thus \( F(x) \) is a q-dimensional Hausdorff measure function. Also, for each \( n \),
\[ F(x) \leq F^{(n)}(x) \quad \text{for} \quad x \in (0, d(x_m)) \]
So for each \( n \), we have,
\[ \mathcal{L}^F(S) < \mathcal{L}^{F^{(n)}}(S) < \mathcal{L}^h(S) + \varepsilon, \]
and therefore,
\[ \mathcal{L}^h(S) \leq \mathcal{L}^{F(S)} < \mathcal{L}^h(S) + \varepsilon. \]

Thus, for the proof of the theorem, we need only consider those discontinuous functions which are step functions with zero as the only possible limit point of their points of discontinuity, and at any one of the discontinuities \( x \), say, \( \mathcal{H}(x) = \mathcal{H}(x - a) \).

Now let \( \mathcal{H}(x) \) be such a function and let \( S \) be a set such that \( \mathcal{L}^h(S) \) is positive and finite. We show that this can be replaced by a continuous function \( \mathcal{H}(x) \) such that \( \mathcal{L}^h(S) = \mathcal{L}^h(S) \).

Let \( \bigcup \mathcal{X}_i \) be a covering of \( S \) by open q-dimensional rectangles with sides parallel to the coordinate axes. Then if, for some \( i \), \( d(x_i) \) is a
point of discontinuity of \( h(x) \) we can replace this rectangle \( X_i \) by two
other rectangles \( Y_j, Y_k \) such that,

\[
d(Y_j) < d(X_i) \quad \text{and} \quad d(Y_k) < d(X_i)
\]

\[
h(d(Y_j)) = h(d(Y_k)) = h(d(X_i))
\]

and, \( Y_j \) and \( Y_k \) together cover at least as much of \( S \) as \( X_i \) did. Finally,
we can also guarantee that \( d(Y_j) \) and \( d(Y_k) \) will be points of continuity
of \( h(x) \).

Thus, if \( \bigcup_{i \in I} X_i \) is any covering of \( S \) by open rectangles we can replace
it by another such covering \( \bigcup_{i \in I} Y_i \) of \( S \), such that \( d(Y_j) \) is a point of
continuity of \( h(x) \) for each \( j \) and,

\[
\sum_j h(d(Y_j)) < 2 \sum_i h(d(X_i)).
\]

Further, if \( d(X_i) < \delta \) for all \( i \), then \( d(Y_j) < \delta \) for all \( j \). We now
proceed to the construction of the continuous function \( H(x) \) by a
diagonal argument similar to the one used in the proof of Theorem 5.

Let \( \{ \delta_n \} \) be a sequence of positive numbers such that \( \delta_n \to 0 \) as \( n \to \infty \)
and each \( \delta_n \) is a point of continuity of \( h(x) \). Corresponding to each \( \delta_n \) and
to any given \( \varepsilon > 0 \), there is a covering \( \bigcup X_{n,i} \) of \( S \), such that,

\[
\sum_{n,i} h(d(X_{n,i})) < \sum_i h(d(X_i)) < \sum_{n,i} h(d(X_{n,i})) + \varepsilon
\]

with \( d(X_{n,i}) < \delta_n \) for all \( i \).

Replace this covering by the corresponding covering \( \bigcup Y_{n,j} \) so that,
given any \( \varepsilon > 0 \) we have,

\[
\sum_{n,j} h(d(Y_{n,j})) < \sum_{i} h(d(Y_i)) < 2 \sum_{n,j} h(d(Y_{n,j})) + 2\varepsilon \quad (*)
\]
with \( d(Y_{j,i}) < \delta_j \) for all \( j \), and \( d(Y_{j,i}) \) is a point of continuity of \( \lambda(x) \) for each \( i \). Let \( x_1, x_2, \ldots \) be an enumeration of all the points of discontinuity of \( \lambda(x) \).

Enclose each \( x_i \) in an open interval \( l_{i,j} = (\alpha_{i,j}, \beta_{i,j}) \) such that, both \( \alpha_{i,j} \) and \( \beta_{i,j} \) are points of continuity of \( \lambda(x) \), there are no \( x_j \) in \( l_{i,j} \) with \( j < i \) and there are no \( d(Y_{j,i}) \) nor \( \delta_j \) in \( l_{i,j} \). Define the continuous function \( H^{(i)}(x) \) as follows,

\[
H^{(i)}(x) = \lambda(x) \quad \text{if } x \not\in l_{i,j} \quad \text{for each } i.
\]

\[
H^{(i)}(x) = \lambda(x_j + \delta_j) \quad \text{for } x \in (x_{i,j}, \beta_{i,j}].
\]

and define \( H^{(i)}(x) \) to be continuous, increasing and greater than or equal to \( \lambda(x) \) in the interval \([\alpha_{i,j}, x_{i,j}]\).

So we have,

\[
H^{(i)}(x) \geq \lambda(x) \quad \text{for all } x,
\]

and,

\[
\sum_{j=1}^{\infty} H^{(i)}(d(Y_{j,i})) = \sum_{j=1}^{\infty} \lambda(d(Y_{j,i}))
\]

therefore, by (30), we have,

\[
\sum_{j=1}^{\infty} H^{(i)}(d(Y_{j,i})) < 2 \sum_{j=1}^{\infty} \lambda(d(Y_{j,i})) + 2 \delta.
\]

Now consider the covering \( \bigcup Y_{i,j} \) of \( S \). Enclose each \( x_{i,j} \) in an open interval \( l_{i,j} = (\alpha_{i,j}, \beta_{i,j}) \) contained in \( l_{i,j} \) such that, both \( \alpha_{i,j} \) and \( \beta_{i,j} \) are points of continuity of \( \lambda(x) \), there are no \( x_j \) in \( l_{i,j} \) with \( j < i \) and there are no \( d(Y_{j,i}) \) in \( l_{i,j} \). Define the continuous function \( H^{(i)}(x) \) as follows,
and define $H^y(x)$ to be continuous, increasing, greater than or equal to $\omega(x)$ and less than or equal to $H^\alpha(x)$ in the interval $[\alpha_1, \alpha]$. 

So we have,

$$H^y(x) \geq H^\alpha(x) \geq \omega(x) \quad \text{for all } x,$$

and,

$$\mathcal{M} H^y, R(q_y) (S) < 2 \mathcal{M} H, R(q_y) (S) + 2 \varepsilon.$$

Continuing in this manner we define the continuous function $H^\omega(x)$ such that,

$$\omega(x) \leq H^\omega(x) \leq H^{\omega_x}(x) \quad \text{for all } x,$$

and,

$$\mathcal{M} H^\omega, R(q_\omega) (S) < 2 \mathcal{M} H, R(q_\omega) (S) + 2 \varepsilon.$$

Finally, we define the continuous function $H(x)$ as follows,

$$H(x) = H^{\omega_x}(x) \quad \text{for } x \in [\delta_{x+1}, \delta_{x})$$

Now we see that $H(x)$ is a continuous q-dimensional Hausdorff measure function with

$$\omega(x) \leq H(x) \leq H^{\omega_x}(x) \quad \text{for all } x \quad \text{and for all } x$$

Thus,

$$\mathcal{M}^\omega (S) \leq \mathcal{M}^H (S).$$
Also, for each value of \( n \),
\[
\mathcal{H}^{R_{(\alpha)}}(S) \leq \mathcal{H}^{R_{(\alpha)}}(S) \leq 2 \mathcal{H}^{R_{(\alpha)}}(S) + 2 \varepsilon
\]
\[
\leq 2 \mathcal{H}^{R_{(\alpha)}}(S) + 2 \varepsilon.
\]

So we have,
\[
\mathcal{H}(S) \leq \mathcal{H}^{R_{(\alpha)}}(S) \leq 2 \mathcal{H}^{R_{(\alpha)}}(S) + 2 \varepsilon.
\]

Therefore, using Lemma 1,
\[
\mathcal{H}(S) \leq \mathcal{H}(S) \leq \mathcal{H}(S) + 2 \varepsilon.
\]

The remainder of the proof is trivial, since we have only to multiply \( \mathcal{H}(x) \) by a constant to get the required continuous function.

From the proofs of Theorems 5 and 6 we see that the continuous functions, \( \mathcal{H}(x) \), are dependent on the set \( S \) under consideration. We shall now see that these results can, under certain restrictions, be extended to give a continuous function which is independent of the set \( S \).

**Theorem 7**

If \( \mathcal{H}(x) \) is any \( q \)-dimensional Hausdorff measure function with the property that,
\[
\frac{\mathcal{H}(x)}{x^q} \rightarrow 0 \quad \text{as} \quad x \rightarrow 0
\]

then there exists a continuous \( q \)-dimensional Hausdorff measure function \( \mathcal{H}(x) \) such that,
\[
\mathcal{H}(S) = \mathcal{H}(S)
\]
for all sets \( S \) in \( q \)-dimensional Euclidean space.
Proof.

Let $C$ be a convex set of diameter $d$ in $q$-dimensional Euclidean space. Denote by $C'$ the set $\{x: s(x, C) > \delta\}$ for some $\delta > 0$, where $s$ is the metric in the space. Then we can cover the set $C \setminus C'$ with $[K \cdot d^{q-1}/3^{q-1}]$ $q$-dimensional cubes of diameter $\delta$, where $K$ is a constant dependent on $q$. (See Appendix 1, page 143.)

Let $\{\rho_n\}$ be a decreasing sequence of positive real numbers such that $\lim_{n \to \infty} \rho_n = 0$. Let $\{\rho_n\}$ be a strictly decreasing sequence of positive numbers such that $\rho_n \to 0$ as $n \to \infty$ and each point $\rho_n$ is a point of continuity of $h(x)$. Let $x^{(i)}_1, \ldots, x^{(i)}_n$ be those points of discontinuity of size greater than $\rho_n^{q-1}/2^{q-1}$ with $x^{(i)}_1 > \rho_n$ and $x^{(i)}_i > x^{(i)}_{i-1}$ for $i = 1, \ldots, n$.

In general let $x^{(i)}_{n_{i+1}}, \ldots, x^{(i)}_{n_i}$ be those discontinuities of $h(x)$ of size greater than $\rho_{n_i}^{q-1}/2^{q-1}$ with $\rho_{n_{i+1}} > x^{(i)}_{n_i} > \rho_{n_{i}}$ and $x^{(i)}_{n_i} > x^{(i)}_{n_{i+1}}$ for $i = n_{i+1}, \ldots, n_i$.

Corresponding to each $x^{(i)}_i$ define $\eta^{(i)}_i$ such that $(\eta^{(i)}_i + 1, \leq i \leq n_i)$,

$$\eta^{(i)}_k \subset \{x^{(i)}_k\}_{k=\eta^{(i)}_i, \ldots, n_i}$$ and $x^{(i)}_i - \eta^{(i)}_i > \max(\eta^{(i)}_i, x^{(i)}_i, \rho_n)$ \hspace{1cm} (31)

$x^{(i)}_i - \eta^{(i)}_i$ is a point of continuity of $h(x)$ \hspace{1cm} (32)

$$h(\eta^{(i)}_i)/(\eta^{(i)}_i) < \frac{1}{2^{q-1} K^{q-1}}$$ \hspace{1cm} (33)

$$\eta^{(i)}_k \subset (x^{(i)}_i - \eta^{(i)}_i, x^{(i)}_i)$$ \hspace{1cm} for all $s \leq j$ \hspace{1cm} (34)

and $k = n_{s+1}, \ldots, n_i$ for $s < j$$

and $k = n_{s+1}, \ldots, n_{s-1}$ for $s = j$.

Define the continuous $q$-dimensional Hausdorff measure function $H(x)$ as follows,

1. for $x \in (\rho_{n_{i+1}}, \rho_{n_{i}})$ and $x \notin [x^{(i)}_i - \eta^{(i)}_i, x^{(i)}_i]$ for all $i = n_{i+1}, \ldots, n_i$
define \( H(x) \) to be continuous, increasing, greater than or equal to \( \lambda(x) \) and such that,

\[
H(x) \leq \lambda(x) + \epsilon_i \rho_i \]

this is possible because of the definition of \( \lambda^{(i)} \).

ii). \( H(x^{(i)} - \eta^{(i)}_c) = \lambda(x^{(i)} - \eta^{(i)}_c) \) for all \( j \) and for all \( \epsilon = \eta^{(i)}_{j+1}, \ldots, \eta^{(i)}_j \).

iii). \( H(\rho_j) = \lambda(\rho_j) \) for all \( j \).

iv). for \( x \in (x^{(i)}_c - \eta^{(i)}_c, x^{(i)}_c) \) define \( H(x) \) to be increasing, greater than or equal to \( \lambda(x) \), continuous in \( (x^{(i)}_c - \eta^{(i)}_c, x^{(i)}_c) \) and continuous on the left at \( x^{(i)}_c \) with \( H(x^{(i)}_c) = \lambda(x^{(i)}_c + 0) \).

Then \( H(x) \) is continuous and \( H(x) \geq \lambda(x) \) for all \( x \), thus,

\[
\Lambda^H(S) \geq \Lambda^\lambda(S)
\]

for all sets \( S \). - (35)

Now let \( S \) be a set in \( q \)-dimensional Euclidean space, contained in a \( q \)-dimensional cube of diameter \( \lambda \).

Let \( \delta \) be a positive number such that,

\[
x^{(i)}_{c+1} < \delta < x^{(i)}_c - \eta^{(i)}_c\]

for some \( j \) and some \( i \),

\[
\eta^{(i)}_{j+1} \leq i \leq \eta^{(i)}_j.
\]

Let \( \{U_i^\delta\} \) be an open covering of \( S \) with,

\[
\rho(U_i^\delta) < \delta
\]

for all \( i \).
and, \[ h^b(S) + \varepsilon_{I_0} > \sum_{i} h(d(U_i^5)) \] - (34)

where \( I_0 \) is the least integer \( j \) such that,
\[ x_{k_j}^i < j \]
for all \( k = \eta_{i-1}, \ldots, \eta_j \),

(clearly \( I_0 \to \infty \) as \( J \to \infty \)).

Now, for each \( i \),
\[ d(U_i^5) \in [\rho_j, \rho_{j+1}) \]
for some \( j \).

If \( d(U_i^5) \ll \langle x_{k_i}^i, x_{k_i}^{i+1} \rangle \) for all \( k = \eta_{i-1}, \ldots, \eta_j \), then by i), ii), and iii),
\[ h(d(U_i^5)) \leq h(d(U_i^5)) + \frac{\varepsilon_{I_0} \rho_j^2}{2^{i+1}} \]

If \( d(U_i^5) \ll \langle x_{k_i}^i, x_{k_i}^{i+1} \rangle \) for some \( k_i \), then we have, by (33),
\[ h(d(U_i^5)) > h(x_{k_i}^i, x_{k_i}^{i+1}) + h(\eta_i^{i+1} - \eta_{k_i}^i) + h(\eta_i^{i+1} - \eta_{k_i}^i) - \frac{\varepsilon_{I_0} (\eta_i^{i+1} - \eta_{k_i}^i) q_{i+1}}{2^{i+1} K} \] - (37)

Now, \( \eta_{k_i}^i \in [\rho_j, \rho_{j+1}) \) for some \( j \geq s \), and,
\[ \eta_{k_i}^i \ll \langle x_{k_i}^i - \eta_{i}^i, x_{k_i}^i \rangle \] for all \( i = \eta_{s+1}, \ldots, \eta_j \).

Therefore,
\[ h(\eta_{k_i}^i) \leq h(\eta_{k_i}^i) + \frac{\varepsilon_{I_0} \rho_j^2}{2^{i+1}} \] - (38)

Also,
\[ x_{k_i}^i - \eta_{k_i}^i \in [\rho_j, \rho_{j+1}) \]

and,
\[ h(x_{k_i}^i - \eta_{k_i}^i) = h(x_{k_i}^i - \eta_{k_i}^i) \] - (39)
Replace each \( U_i \) which satisfies \( d(U_i) \in (x_k^{(i)}, y_k^{(i)}, x_k^{(i)}) \) for some \( k_j \), by a set of diameter \( x_k^{(i)} - y_k^{(i)} \) together with \( \left[ K \left( \frac{x_k^{(i)}}{y_k^{(i)}} \right)^{q - 1} \right] \) q-dimensional open cubes of diameter \( y_k^{(i)} \), denote these replacement sets by \( V_i \). If \( d(U_i) \in (x_k^{(i)} - y_k^{(i)}, x_k^{(i)}) \) for all \( k_j, \ldots, k_1 \), we put \( V_i = U_i \). Thus, we get another open covering \( \bigcup V_i \) of \( S \) with \( d(V_i) < \delta \) for all \( i \). Now, since \( S \) is contained in a cube of diameter \( \delta \), we may suppose that there are less than \( \left( \frac{\delta}{2} \right)^{-q} \) values \( d(U_i) \) in \( (x_k^{(i)} - y_k^{(i)}, x_k^{(i)}) \) and less than \( \rho_j^{-q} \) values \( d(U_i) \) in \( [\rho_j, \rho_j^{-1}] \). This is because \( (\rho_j^{(i)})^{-q} \) such values of \( d(U_i) \) or \( \rho_j^{-q} \) such values of \( d(U_i) \) could arise from a collection of sets \( \{U_i\} \) which would be sufficient to cover the whole cube.

Thus, for those sets \( V_i \) for which \( V_i \neq U_i \) we have, from (37),

\[
\sum_i h(d(U_i)) > \sum_i h(d(V_i)) - \sum_i (x_k^{(i)})^{-q} \cdot K \left( \frac{x_k^{(i)}}{y_k^{(i)}} \right)^{q - 1} \cdot \frac{\xi_j}{\rho_j} \cdot \frac{\xi_j}{\rho_j^{-1}}
\]

So we must have the following inequality, involving all the sets \( U_i \) and \( V_i \),

\[
\sum_i h(d(U_i)) > \sum_i h(d(V_i)) - \frac{\xi_j \xi_j}{2^{-q} + 1}
\]

Now, by the definitions of the sets \( V_i \) and (38) and (39), we have,

\[
\sum_i h(d(V_i)) > \sum_i h(d(V_i)) - \sum_j \rho_j^{-q} \left( \frac{\xi_j \rho_j}{2^{j+1}} \right) - \sum_i (x_k^{(i)})^{-q} \cdot K \left( \frac{x_k^{(i)}}{y_k^{(i)}} \right)^{q - 1} \cdot \frac{\xi_j}{\rho_j} \cdot \frac{\xi_j}{\rho_j^{-1}}
\]

\[
\geq \sum_i h(d(V_i)) - \frac{\xi_j \xi_j}{2^{j+1} + 1} - \frac{\xi_j \xi_j}{2^{j+1} + 1}
\]

Therefore,

\[
\sum_i h(d(U_i)) > \sum_i h(d(V_i)) - \frac{\xi_j \xi_j}{2^{j+1} + 1} - \frac{\xi_j \xi_j}{2^{j+1} + 1}
\]
Thus we have shown that for any \( \delta \) of the prescribed form,

\[
\Lambda_k(S) \geq \Lambda_d^H(S) - \frac{\delta_{\frac{1}{2^d}}}{2^{T_4}} - \frac{K\delta_{\frac{1}{2^{T_4+1}}}}{2^{T_4}}.
\]

But there are arbitrarily small \( \delta \) of this form, and so,

\[
\Lambda_k(S) \geq \Lambda^H(S)
\]

since \( \xi_i \to 0 \) as \( i \to \infty \). Hence, combining this with (35) we have the required result for sets \( S \) lying in cubes of diameter \( \frac{1}{\alpha} \).

Now let \( S \) be any set in \( q \)-dimensional Euclidean space. We may divide the space up into a countable set of closed cubes \( \{c_i\} \), each of diameter \( \frac{1}{\alpha} \). Since we have,

\[
\frac{h(x)}{x^{q-1}} \to 0 \quad \text{as} \quad x \to 0,
\]

those points of \( S \) which lie on the intersection of two of these cubes form a set \( S_0 \) such that \( \Lambda^H(S_0) = 0 \). Hence we may write,

\[
S = \bigcup_{i=0}^{\infty} S_i,
\]

where \( \Lambda^H(S_i) = \Lambda^H(S) \) for \( i = 1, 2, \ldots \) and \( \Lambda^H(S_0) = \Lambda^H(S_0) = 0 \).

Also we have,

\[
\Lambda^H(S) = \sum_{i=1}^{\infty} \Lambda^H(S_i) = \sum_{i=1}^{\infty} \Lambda^H(S_i) = \Lambda^H(S),
\]

and hence the result extends to all sets \( S \) in \( q \)-dimensional Euclidean
If $\mathcal{H}(x)$ is any one-dimensional Hausdorff measure function, then there is a continuous one-dimensional Hausdorff measure function $\mathcal{H}(x)$ such that,

$$\mathcal{H}^p(S) = \mathcal{H}^1(S)$$

for all linear sets $S$.

The next theorem shows that it is not always possible to replace discontinuous functions by continuous ones. That is to say that the restrictions imposed in Theorem 7 cannot be relaxed.

**Theorem 8**

There exists a two-dimensional Hausdorff measure function $\mathcal{H}(x)$, say, with,

$$\frac{\mathcal{H}(x)}{x} \to \infty \quad \text{as} \quad x \to 0,$$

such that, for any continuous two-dimensional Hausdorff measure function $\mathcal{H}(x)$ say, there is a set $S$ with the property that,

$$\mathcal{H}^p(S) \neq \mathcal{H}^1(S).$$

**Proof**

Let $x_n = \frac{1}{n}$ for $n = 0, 1, 2, \ldots$

Define $\mathcal{H}(x)$ as follows,

$$\mathcal{H}(x) = 1 \quad \text{for} \quad x \geq x_0.$$
\[ h(x_n) = \alpha^{-1} h(x_{n-1}) \quad \text{for } n = 1, 2, \ldots \]
where \( 1 < \alpha < 2 \)

and
\[ h(x) = h(x_n) \quad \text{for } x \in (x_{n+1}, x_n]. \]

Then, clearly,
\[ \frac{h(x)}{x} \to \alpha \quad \text{as } x \to 0. \]

Also,
\[ \frac{h(x_n)}{x_n} \leq \frac{h(t)}{t} \quad \text{for all } t \in (0, x_n] \]
and for \( n = 1, 2, \ldots \)

and
\[ \frac{h(x)}{x} \leq \alpha \frac{h(t)}{t} \quad \text{for all } t \in (0, x] \]
and for all \( x \).

Now let \( H(x) \) be any continuous Hausdorff measure function. Then for each \( n \) we have,

either \[ H(x_n) \geq \alpha^{\frac{1}{n}} h(x_n) \]
or \[ H(x_n) < \alpha^{\frac{1}{n}} h(x_n) \]

We, firstly assume that,
\[ H(x_n) < \alpha^{\frac{1}{n}} h(x_n) \quad \text{for infinitely many } n. \]

So, there exists a subsequence \( \{x_{n_i}\} \) of \( \{x_n\} \) such that,
\[ H(x_{n_i}) < \alpha^{\frac{1}{n_i}} h(x_{n_i}) \quad \text{for all } i. \]
Because of the continuity of \( H(x) \), there exist \( \varepsilon > 0 \) such that, for all \( i \),

\[
H(x) < x^{-1/4} \quad \text{for} \quad x \in (x_{n_i}, x_{n_i} + \varepsilon_i),
\]

we may, further, insist that,

\[
x_{n_i} + \varepsilon_i < x_{n_{i-1}}.
\]

Now let \( \{a_n\} \) be a sequence of positive numbers such that, \( \sum A_n^{-1} \) is convergent and,

\[
\prod_{n=1}^{\infty} (1 - 2/A_n) > x^{-1/4}.
\]

Also let \( \{b_n\} \) be a sequence of positive numbers such that,

\[
b_n \to 0 \quad \text{as} \quad n \to \infty \quad \text{and} \quad b_n > 2 \quad \text{for all} \quad n.
\]

We now construct a sequence \( \{u_n\} \) as follows:

choose \( u_0 \) arbitrarily from the open interval \( (x_{n_1}, x_{n_1} + \varepsilon_1) \).

Having chosen \( u_n \) for \( n = 1, \ldots, n-1 \), choose \( u_n \) such that,

1. \( 0 < u_n < \frac{1}{2} u_{n-1} \).
2. if \( u_{n-i} \in (x_{n_j}, x_{n_j} + \varepsilon_j) \) then \( u_n \in (x_{n_j}, x_{n_j} + \varepsilon_j) \) with \( j > i \).
3. \( u_{n-i} - 2 u_n > x_{n_i} \) where \( u_{n-i} \in (x_{n_j}, x_{n_j} + \varepsilon_j) \).
4. \( C_n \L(x, u_n ) = \L(x, u_{n-1}) \) with \( C_n > A_n \).
5. \( b_n C_n u_n < u_{n-1} \).
Write \( K_\alpha = 2^\lceil \alpha / 2 \rceil \).

We now proceed to construct the set \( S \) in two-dimensional Euclidean space. Denote by \( S_\circ \) the closed circle with centre at the origin and of diameter \( u_s \). Draw the diameters of \( S_\circ \) at angles \( \Theta_1, 2\Theta_1, \ldots, (2\alpha - 1)/\Theta_1 \) to the positive \( x \)-axis, where \( \Theta_1 = \frac{2\pi}{K_1} \). At each end of these diameters and at the intersections of the \( x \)-axis with the perimeter of the circle we draw a closed circle of diameter \( u_y \), inside \( S_\circ \), having one point of contact with the circumference of \( S_\circ \) and having centre on the diameter of \( S_\circ \). Denote by \( S \) the union of these \( K_1 \) closed circles. Inside each circle of \( S \), we draw \( K_1/2 \) diameters at an angle \( \Theta_2 \) apart, where \( \Theta_2 = \frac{2\pi}{K_1} \). At the ends of these diameters we draw closed circles of diameter \( u_y \) in a similar manner to that described above. Thus, we have \( K_1 \) closed circles of diameter \( u_y \) and these we denote by \( S_y \). Continuing in this manner we get sets \( S_1, S_2, \ldots \) with \( S_n \) consisting of \( K_1 \) closed circles of diameter \( u_y \). Also we have \( S_n \supset S_{n+1} \) for all \( n \).

We define the set \( S = \bigcap_{n=0}^{\infty} S_n \).

Now since each \( u_n \) is a point of continuity of \( h(x) \), and since \( u_n \to 0 \) as \( n \to \infty \) we have,

\[
m^h(S) \leq K_1 \ldots K_n h(u_n) \quad \text{for all } n
\]

\[
\leq h(u_0).
\]

We, also, note that,

\[
K_1 \ldots K_n h(u_n) > (C_1 - \frac{1}{2}) \ldots (C_n - \frac{1}{2}) h(u_n)
\]

\[
= \prod_{i=1}^{n} \left( 1 - \frac{1}{2C_\alpha} \right) h(u_0).
\]
Now, given any $\varepsilon > 0$, choose $\delta > 0$ such that,

$$u^\varepsilon > \frac{1}{2}$$

whenever $u_m < \delta$.  \((43)\)

Let $\{U_i\}$ be an arbitrary open covering of $S$ such that,

$$d(U_i) < \delta$$

for all $i$.  \((44)\)

Since $S$ is compact, we may assume that $\{U_i\}$ consists of a finite number of open sets.

Let $n$ be such that,

$$u_m < d(U_i)$$

for all $i$.  \((45)\)

If $U_i$ is such that,

$$u_m < d(U_i) \leq u_{m-1}$$

then $U_i$ intersects at most one circle of $S_{m-1}$ (by v).  \((46)\)

If $d(U_i) > u_{m-1} - 2u_m$, then,

$$\omega(d(U_i)) = \omega(u_{m-1})$$

by iii).  \((47)\)

and so we can replace $U_i$ by the circle of $S_{m-1}$ which it intersects, without increasing $\Sigma_i \omega(d(U_i))$ and the circle covers at least as much of $S$ as $U_i$ did.

Now assume $d(U_i) < u_{m-1} - 2u_m$ and that $U_i$ intersects more than one circle of $S_m$. Then $U_i$ intersects at most $\omega$ circles of $S_m$, where,
Since \( U_i \) intersects more than one of the circles of \( S_n \),

\[
d(U_i) \geq (y_{n-1} - y_n) \sin \Theta_n / y_n - y_n
\]

\[
\geq (y_{n-1} - y_n) / k_n - y_n, \quad \text{since } \sin \theta \geq \frac{2\pi}{\pi} \quad -(45)
\]

for \( 0 \leq x \leq \pi/\theta_n \).

So we have,

\[
m(h(y_n)) \leq \left( \frac{k_n}{\pi} \left( \frac{d(U_i) + y_n}{y_{n-1} - y_n} \right) + 1 \right) h(y_n) \quad \text{by (44)}
\]

\[
\leq \left( \frac{k_n}{2} \left( \frac{d(U_i) + y_n}{y_{n-1} - y_n} \right) + 1 \right) h(y_n) \quad \text{by (45)}
\]

\[
\leq \left( \frac{d(U_i) + y_n}{y_{n-1} - y_n} \right) k_n h(y_n) \quad \text{by iv), and v) ...}
\]

Now, if \( d(U_i) \leq \frac{x_n}{\theta_n} \), where \( y_n \in (x_{n-1}, x_n + \varepsilon) \) then,

\[
h(y_{n-1}) / y_{n-1} \leq h(d(U_i)) / d(U_i)
\]

thus,

\[
m(h(y_n)) \leq \left( \frac{R_n C_n (d(U_i) + y_n)}{R_n C_n - 1 d(U_i)} \right) h(d(U_i))
\]
\[
< \left\{ 1 + \frac{1}{r_{\kappa-1}} \left( 1 + \frac{y_{\kappa-1}}{d(U_i)} \right) \right\} h(d(U_i)) \leq \left\{ 1 + \frac{1}{r_{\kappa-1}} \left( 1 + \frac{y_{\kappa-1}}{d(U_i)/r_{\kappa-1} - y_{\kappa-1}} \right) \right\} h(d(U_i)) \quad \text{by (45)}
\]
\[
< \left\{ 1 + \frac{1}{r_{\kappa-1}} + \frac{2}{r_{\kappa-1} - y_{\kappa-1}} \right\} h(d(U_i)) \quad \text{by (42) and since } r_{\kappa-1} \geq 1.
\]

So we have shown that in this case we can replace \( U_i \) by \( \kappa \) circles of \( S_{\kappa} \) causing \( h(d(U_i)) \) to increase by a factor of at most \((1 + \varepsilon)\) and the circles cover at least as much of \( S \) as \( U_i \) did. If \( U_i \) meets only one circle of \( S_{\kappa} \) then since \( d(U_i) > y_{\kappa} \) we may replace it by this circle of \( S_{\kappa} \).

Now, if,
\[
\frac{d(U_i)}{\varepsilon} > y_{\kappa}/\varepsilon,
\]
then,
\[
h(d(U_i)) = \frac{\varepsilon}{y_{\kappa}} h(y_{\kappa}) > \frac{\varepsilon}{y_{\kappa}} h(y_{\kappa-1}).
\]

Also, since we have assumed that \( d(U_i) < y_{\kappa-1} - 2y_{\kappa} \), we know that \( U_i \)
intersects less than \( r_{\kappa}/\varepsilon \) circles of \( S_{\kappa} \).

Now, if there is a \( U_j \) with \( j \neq i \) and,
\[
\frac{\kappa}{\varepsilon} < d(U_j) < y_{\kappa-1} - 2y_{\kappa},
\]
such that \( U_j \) intersects the same circle of \( S_{\kappa-1} \) as \( U_i \) does, then we may replace \( U_i \) and \( U_j \) by the circle of \( S_{\kappa-1} \), since,
\[ \lambda(d(U_i)) + \lambda(d(U_j)) > \lambda(S_n), \]

and \( U_i, U_j \) together intersect less than \( \kappa_n \) circles of \( S_n \). Also if we have such a \( U_i, U_j \) say, and if the remainder of the circle of \( S_n \) which \( U_i \) intersects is covered by members of \( \{U_n\} \) all with,

\[ \eta_n < d(U_i) < \eta_{n-1}, \]

then we may replace all these \( U_i \)'s together with \( U_j \) by the circle of \( S_{n-1} \). Thus, so far, we have replaced each \( U_i \) satisfying,

\[ \eta_n < d(U_i) < \eta_{n-1}, \]

except those such that,

\[ x_n / \lambda < d(U_i) < \eta_{n-1} - 2\eta_n \]  \hspace{1cm} (48)

and such that there is no other \( U_j \) meeting the same circle of \( S_{n-1} \) as \( U_i \) does and satisfies (48) and (43).

We now make similar replacements with respect to those \( U_i \) for which,

\[ \eta_{n-1} < d(U_i) < \eta_{n-2} \]

and repeat the procedure up to and including the case where,

\[ \eta_1 < d(U_i) < \eta_n. \]

Clearly we may assume that,

\[ d(U_i) < \eta_n \]  \hspace{1cm} for all \( i \).
Now assume that there is a $U_i$ which has not been replaced, let $\omega$ be such that,
\[ \omega_{\alpha m} < \alpha(U_i) \leq \omega_{\alpha m-1}, \]
then if the circle of $S_{\alpha m}$ which it intersects has been used to replace a different member of $\{U_i\}$ then we may ignore the set $U_i$. We now assume that there are still some $U_i$ remaining which cannot be ignored, as explained above and which have not been replaced. Let there be such a $U_i$ with,
\[ \omega_{\alpha m} < \alpha(U_i) \leq \omega_{\alpha m-1}. \]

Then if $S_{\alpha m}^i$ is the circle of $S_{\alpha m}$ which $U_i$ intersects, we know that $S_{\alpha m}^i$ is partially covered by a $U_j$ with $\alpha(U_j) > \omega_{\alpha m-1}$ and $U_j$ has not been replaced. Therefore, if $\omega_{\alpha m} < \alpha(U_j) \leq \omega_{\alpha m-1}$ then $S_{\alpha m}^i$ is partially covered by a $U_k$ with $\alpha(U_k) > \omega_{\alpha m-1}$ and $U_k$ has not been replaced. Thus we see that if there were to be such a $U_i$ then given any integer $t$, there is a $U_j$ belonging to $\{U_i\}$ such that $\alpha(U_j) > \omega_{t-1}$ and $U_j$ has not been replaced. If we take $t=1$ we see that this is a contradiction since we can assume that,
\[ \alpha(U_i) \leq \omega_0, \]
for all $i$.

Thus we conclude that all $U_i$ with
\[ \omega_{\alpha m} < \alpha(U_i) \leq \omega_{\alpha m-1} \]
have been replaced.

We now consider those remaining $U_i$ for which,
Then \( S_{n-1} \) is partially covered either,
a) by \( U_j \) with \( u_{n-1} < d(U_j) \leq u_{n-1} \) and which have not been replaced, or,
b) by \( U_k \) with \( d(U_k) > u_{n-1} \) and such that \( U_k \) has not been replaced.

It has just been shown that a) is impossible and we can get a similar contradiction from b).

Continuing in this manner we see that all the \( U_i \) have been replaced by a collection \( \{c_i\} \) of circles of \( S_0 \cup \ldots \cup S_n \) such that,

\[
\bigcup c_i \supseteq S \quad \text{and} \quad (1+\varepsilon) \sum_i h(d(c_i)) \geq \sum_i h(d(c_i)).
\]

We further note that the inequality (41) holds for any finite collection of circles of \( \bigcup S_n \) which covers \( S \) - this is because we may replace any circle of \( S_{n-1} \) by the \( \kappa_n \) circles of \( S_n \) which it contains.

Thus,

\[
(1+\varepsilon) \sum_i h(d(c_i)) > \sum_i h(d(c_i)) \geq \rho h(u_0),
\]

where \( \rho = \prod_{n=1}^{\infty} (1-\frac{1}{A_n}). \)

Therefore, since \( \{U_i\} \) was an arbitrary covering of \( S \), we have,

\[
\Lambda^h(S) \geq (1+\varepsilon)^{-1} \rho h(u_0),
\]

thus,

\[
\Lambda^h(S) \geq (1+\varepsilon)^{-1} \rho h(u_0).
\]

but the \( \varepsilon \) was an arbitrary positive number and so,

\[
\Lambda^h(S) \geq \rho h(u_0) \quad \text{-(49)}
\]
Also, we have,

$$\mathcal{N}^H(S) \leq K_1 \cdots K_n H(y_n)$$

for all $n$

$$< \alpha^{-H_0} K_1 \cdots K_n h(y_n)$$

$$\leq \alpha^{-H_0} h(y_n)$$

$$< \beta h(y_n).$$

Therefore,

$$\mathcal{N}^H(S) = \mathcal{N}^h(S).$$

Thus if $H(x_n) < \alpha^{-H_0} h(x_n)$ for infinitely many small $x_n$, then we have constructed a set $S$ such that,

$$\mathcal{N}^H(S) = \mathcal{N}^h(S).$$

Now assume that,

$$H(x_n) \geq \alpha^{-H_0} h(x_n)$$

for all small $x_n$.

So we can assume that,

$$H(x_n) \geq \alpha^{-H_0} h(x_n)$$

for all $n$.

Then there exists $\varepsilon_n > 0$ with,

$$x_n - \varepsilon_n > x_{n+1} \quad \frac{h(x_n)}{x_n - x_n} < \frac{h(x_{n+1})}{x_{n+1} - x_{n}}$$

such that,

$$H(x) > \alpha^{-H_0} h(x)$$

for $x \in [x_n - \varepsilon_n, \varepsilon_n]$. 
Now define $H'(x)$ such that,

$$H'(x) = \alpha^{\frac{3}{4}} h(x_{n-1})$$

for $x \in (x_{n-1}, x_n)$. Then $H'(x) < H(x)$ and,

$$\frac{H'(x_{n+1} - x_n)}{x_{n+1} - x_n} = \frac{x_{n+1} - x_n}{x_{n+1} - x_n} = \frac{\alpha^{\frac{3}{4}} h(x_{n+1})}{x_{n+1} - x_n} < \frac{\alpha^{\frac{3}{4}} h(x_n)}{x_n - x_n} = \frac{H'(x_n - x_n)}{x_n - x_n}$$

Therefore,

$$\frac{H'(x_{n+1} - x_n)}{x_{n+1} - x_n} = \frac{H'(t)}{t}$$

for all $t \in (0, x_n - x_n)$. Also,

$$\frac{H'(x_{n+1} - x_n)}{x_{n+1} - x_n} = \frac{x_{n+1} - x_n}{x_{n+1} - x_n} = \frac{\alpha^{\frac{3}{4}} h(x_{n+1})}{x_{n+1} - x_n} > \frac{\alpha^{\frac{3}{4}} h(x_n)}{x_n - x_n}$$

therefore,

$$\frac{H'(x_{n+1} - x_n)}{x_{n+1} - x_n} \to \infty$$

as $x_n - x_n \to \infty$.

that is,

$$H'(x) \to \infty$$

as $x \to \infty$.

Further,

$$\alpha H'(x_{n+1} - x_n) = H'(x_{n+1} - x_n).$$

Thus, as in the previous part of the proof we can construct a sequence $\{u_n\}$ and a set $S$ such that,

$$H'(u_n) \geq \Lambda h(S) \geq \rho H'(u_n).$$

and,

$$\Lambda h(S) < \rho H'(u_n).$$
But $H'(x) \leq H(x)$ and so we have,

$$\lambda^H(S) \geq P H'(y_0),$$

therefore,

$$\lambda^L(S) \neq \lambda^H(S).$$

Hence the theorem is proved.
CHAPTER 3

INTRODUCTION

We saw in the last chapter, how, under certain conditions we could replace discontinuous functions by continuous ones without altering the corresponding Hausdorff measures. In this chapter we investigate the possibility of extending these results to the case of Hausdorff pre-measures. Theorems 9, 10, 11, 12, and 13 are concerned with the extension of some results of Sion and Sjerve (11) to the case of discontinuous functions. Theorem 14 shows us some conditions under which discontinuous functions can be replaced by continuous ones. Finally, Theorem 15 shows that the replacement used in Theorem 7 of Chapter 2 cannot be used in the case of Hausdorff pre-measures.

Theorem 9

If \( I(x) \) is any monotonic increasing \( q \)-dimensional Hausdorff measure function with the property that,

\[
\frac{I(x)}{x^{q-1}} \to 0 \quad \text{as} \quad x \to 0
\]

then for any \( d > 0 \) and any increasing sequence \( \{S_n\} \) of sets in \( q \)-dimensional Euclidean space we have,

\[
\overline{\lambda}_d \left( \bigcup_{n=1}^{\infty} S_n \right) = \lim_{n \to \infty} \overline{\lambda}_d (S_n).
\]

Proof

Let \( C \) be any convex set, in \( q \)-dimensional Euclidean space, with diameter \( d \). Write,

\[
C' = \{ x : I(x, E_C) > d \}
\]
where \( \varrho \) is the metric in the space. Then, as in Theorem 7 of Chapter 2, we can cover the set \( C \setminus C' \) with \( \left[ K \left( d/\varrho \right)^{q-1} \right] \) sets of diameter \( \delta \), where \( K \) is a constant.

Consider any set \( S \) and any positive number \( \varepsilon \). Given any \( \varepsilon > 0 \) let \( \{ U_i^j \} \) be a sequence of open sets such that,

\[
S \subset \bigcup_i U_i^j \quad \text{and} \quad d(U_i^j) < \delta \quad \text{for all } i
\]

and,

\[
\sum_i h(d(U_i^j)) < \Lambda_j^k(S) + \varepsilon.
\]

Clearly we have,

\[
S \subset \bigcup_i \overline{U_i^j} \quad \text{and} \quad d(\overline{U_i^j}) < \delta \quad \text{for all } i
\]

and,

\[
\sum_i h(d(\overline{U_i^j})) < \Lambda_j^k(S) + \varepsilon.
\]

Thus,

\[
L_j^k(S) < \Lambda_j^k(S) + \varepsilon.
\]

Hence, since \( \varepsilon \) was arbitrary and positive, we have shown that,

\[
\Lambda_j^k(S) > L_j^k(S) \quad - (1)
\]

Now, given any \( \varepsilon > 0 \), we can choose a closed covering \( \{ U_i^j \} \) of \( S \) with \( d(U_i^j) < \delta \) for all \( i \), and,

\[
\sum_i h(d(U_i^j)) < L_j^k(S) + \varepsilon \quad - (2)
\]
For each $i$, choose $\eta > 0$ such that,
\[
\frac{\kappa \{ d(U_i^j) \}^{q-1}}{\eta^{q-1}} \kappa(\eta) < \frac{\eta}{2^{i+1}}
\]
and,
\[\eta < \varepsilon.\]

Then we can replace $U_i^j$ by an open set of diameter $(d(U_i^j) - 2\eta)$ together with $\left[ \kappa \{ d(U_i^j) \}^{q-1} \right] \text{open sets of diameter } \eta$. Hence we get a new open covering $\{ V_i^j \}$ of $S$ with,
\[
d(V_i^j) < \varepsilon
\]
for all $i$.

And,
\[
\sum_i \kappa(d(V_i^j)) < \sum_i \kappa(d(U_i^j)) + \frac{\varepsilon}{2} - (3)
\]

Thus, combining (2) and (3), we have,
\[
\bigwedge_{\delta}^h(S) < \sum_i \kappa(d(V_i^j)) < \bigwedge_{\delta}^h(S) + \varepsilon.
\]

Combining this result with (1) we see that,
\[
\bigwedge_{\delta}^h(S) = \bigwedge_{\delta}^h(S),
\]

and so it is sufficient to prove that,
\[
\bigwedge_{\delta}^h(\bigcup_{n=1}^{\infty} S_n) = \lim_{n \to \infty} \bigwedge_{\delta}^h(S_n).
\]

We now define a pseudo-metric on the space of subsets of $q$-dimensional Euclidean space. Denote by $S(\varepsilon)$ the set,
\[
\{ x : \varepsilon(x, s) < \varepsilon \}
\]
for any set $S$. Define the distance between two subsets $S, T$ by,
\[
\]
\[ s(S, T) = \min \{ t : S \subseteq T(t) \text{ and } T \subseteq S(t) \} \]

We then write \( S_n \to S \) when,

\[ s(S_n, S) \to 0 \quad \text{as } n \to \infty. \]

Firstly, we suppose that \( \bigcup_{n=1}^{\infty} S_n \) is bounded in \( q \)-dimensional Euclidean space.

Now, for each \( n \), we consider a sequence \( \{ U_i^n \} \) of closed sets with the following properties,

1. \( S_n \subseteq \bigcup_{i=1}^{\infty} U_i^n \)

2. \( 0 < \alpha_n \leq \alpha(U_i^n) \leq \beta \) for all \( i \)

3. \( \sum_{i=1}^{\infty} \kappa(d(U_i^n)) \leq \lambda(S_n) + \frac{\mu_n}{\beta} \)

4. \( U_i^n \to V_i \) as \( n \to \infty \) (\( V_i \) compact).

We can satisfy condition 4), because of Blaschke's Selection Theorem and from the fact that we may assume that the sets \( U_i^n \) are uniformly bounded (e.g. see Eggleston (4)).

Now, since \( \lambda(x) > 0 \) for all \( x > 0 \), we see from 3), that,

\[ d(U_i^n) \to 0 \quad \text{as } i \to \infty. \]

Now let,

\[ a = \lim_{n \to \infty} \sum_{i=1}^{\infty} \kappa(d(U_i^n)) \]

then, given any \( \varepsilon > 0 \) we can find a strictly increasing sequence \( \{ n_k \} \) of
Integers such that,

\[ \sum_i h(d(U_i^K)) < a + \varepsilon \quad \text{for all } k. \]  
\[-(4)\]

By iv), we have,

\[ \sum_i h(d(V_i) - o) < a \]  
\[-(5)\]

and so,

\[ d(V_i) \to 0 \quad \text{as } i \to \infty. \]

Now define,

\[ b = a - \sum_i h(d(V_i) - o). \]  
\[-(6)\]

By the argument given at the beginning of the proof, using the fact that,

\[ \frac{h(x)}{x^{a-1}} \to 0 \quad \text{as } x \to 0, \]

we can find open sets \( W_i^j, j = 1, \ldots, m_i \) such that for each \( i, \)

\[ d(W_i^j) < d(V_i) \quad \text{for } j = 1, \ldots, m_i; \]

\[ V_i = \bigcup_{j=1}^{m_i} W_i^j. \]

\[ \sum_{j=1}^{m_i} h(d(W_i^j)) < h(d(V_i) - o) + \frac{\varepsilon}{m_i}. \]  
\[-(7)\]

and

\[ d\left( \bigcup_{j=1}^{m_i} W_i^j \right) \to 0 \quad \text{as } i \to \infty. \]

Given any \( \varepsilon > 0 \), choose an integer \( I \) such that for all \( i \geq I, \)

\[ d\left( \bigcup_{j=1}^{m_i} W_i^j \right) < \varepsilon. \]
and, 
\[ \sum_{i \geq 1} h(d(V_i) - 0) < \varepsilon \]  

Further, given any integer \( n \), choose \( N = n_k (\geq n) \) for some \( k \), such that for \( i = 1, \ldots, I \),
\[ U_i^n \subseteq W_i^I \cup \cdots \cup W_i^n \]

and,
\[ \sum_{i = 1}^I \{ h(d(V_i) - 0) - h(d(U_i^n)) \} < \varepsilon. \]  

Then,
\[ \bigcup_{i = 1}^I \bigcup_{j = 1}^m W_i^j \subseteq \bigcup_{i = 1}^I U_i^n, \]

but, for \( i > I \),
\[ d(U_i^n) \leq d(U_i^I) \leq d \left( \bigcup_{j = 1}^m W_i^j \right) < \varepsilon. \]

So we have,
\[ \sum_{i \geq 1} h(d(U_i^n)) \leq \sum_{i \geq 1} h(d(U_i^I)) \leq \sum_{i \geq I} h(d(U_i^I)) \]
\[ = \sum_{i \geq 1} h(d(U_i^I)) - \sum_{i \geq I} h(d(V_i) - 0) + \sum_{i \geq I} h(d(V_i) - 0) \]
\[ - \sum_{i \geq I} h(d(U_i^I)) + \sum_{i \geq I} h(d(V_i) - 0) \]
\[ < b + 3 \varepsilon, \quad \text{using (4), (6), (9) and (8)}. \]

Thus, letting \( p \to 0 \), we have,
\[ \sum_{i \geq 1} h \{ \bigcup_{j = 1}^m W_i^j \} \leq b + 3 \varepsilon. \]
Therefore, we have,

$$L^k \{ \bigcup_{i=1}^I \bigcup_{j=1}^M \bigcup_{c=1}^M \bigcup_{j=1}^M \bigcup_{i=1}^I V_i \} \leq b + 3 \varepsilon,$$

because we could choose for $U \bigcup_{j=1}^M W_j$ a descending sequence of open sets whose intersection is $V_i$, for the cases $i = 1, \ldots, I$. (It is well known that for any ascending sequence of sets $\{E_n\}$ we have,

$$L^k (\bigcup_{n} E_n) = \lim_{n \to \infty} L^k (E_n).$$

So we have,

$$L^k \{ \bigcup_{i=1}^I \bigcup_{j=1}^M \bigcup_{c=1}^M \bigcup_{j=1}^M \bigcup_{i=1}^I V_i \} \leq b + 3 \varepsilon.$$

Thus,

$$L^k \{ \bigcup_{i=1}^I \bigcup_{j=1}^M V_i \} \leq b + 3 \varepsilon + \sum_{i=1}^I \sum_{j=1}^M h(d(W_j^c)) \leq b + 5 \varepsilon \quad \text{using (7) and (8).}$$

So, letting $\varepsilon \to 0$ and $\varepsilon \to 0$, we have,

$$L^k \{ \bigcup_{i=1}^I U V_i \} \leq b.$$

That is, by iii), and the fact that $L^k \{ U \cap V_i \} = \lim_{n \to \infty} L^k \{ U \cup V_i \}$,

$$\sum_{i=1}^I h(d(V_i)) + L^k \{ U \cap V_i \} \leq \lim_{n \to \infty} L^k \{ S_n \}.$$ 

Now, we know that $d(V_i) \leq \varepsilon$ for all $i$. Thus, given any $\varepsilon > 0$, we can cover $U \cup V_i$ by closed sets $\{W_i\}$ such that,

$$U \cup V_i \subset \bigcup_{i=1}^I W_i.$$
\[ \ell(\lambda, \omega) \leq \delta \quad \text{for all } \lambda \]

and,

\[ \sum_{i} k(\lambda, \omega^{i}) < \sum_{i} k(\lambda, \omega^{i}) - 0 + \varepsilon. \]

Thus,

\[ L_{\delta}(U \cup S_{n}) \leq L_{\delta}(U \cup \omega^{i}) + \varepsilon \]

\[ < L_{\delta}(U \cup \omega^{i}) + \varepsilon \]

\[ \leq \lim_{n \to \infty} L_{\delta}(S_{n}) + \varepsilon. \]

This is true for arbitrary positive \( \varepsilon \), and so we have,

\[ L_{\delta}(U \cup S_{n}) \leq \lim_{n \to \infty} L_{\delta}(S_{n}). \]

Clearly,

\[ L_{\delta}(U \cup S_{n}) \geq \lim_{n \to \infty} L_{\delta}(S_{n}), \]

thus we have shown that,

\[ L_{\delta}(U \cup S_{n}) = \lim_{n \to \infty} L_{\delta}(S_{n}). \]

Hence we have proved the theorem when \( U \cup S_{n} \) is bounded. We can, now, extend this result to the case of unbounded sets by a method of Davies (1).

Suppose that \( U \cup S_{n} \) is unbounded. The result is obvious if \( \lim_{n \to \infty} \Lambda_{\delta}(S_{n}) \) is infinite, so we must now assume that the limit is finite.

Let \( C \) be a q-dimensional cube, sides length \( \delta \), parallel to the coordinate axes. Let \( \{ c_{i} \} \) denote \( \Lambda^{q} \) cubes of side \( \delta \) into which \( C \) may be divided. Let \( \{ c_{i} \} \) be an enumeration of all the distinct cubes which may
be obtained from $C$ by translations whose components are integral multiples of $2^j$.

For each $i$, the cubes $C^i_r \ (r=1, 2, \ldots)$ are a distance not less than $\delta$ from one another. Thus we have,

$$\Lambda^h_\delta \left( \bigcup_{n} s_n \cap \bigcup_{r=1}^{R} C^i_r \right) = \sum_{r=1}^{R} \Lambda^h_\delta \left( s_n \cap C^i_r \right) \quad \text{(11)}$$

Suppose that the series in (11) were divergent for at least one value of $i$. In that case we could choose $R$ so large that,

$$\Lambda^h_\delta \left( \bigcup_{n} s_n \cap \bigcup_{r=1}^{R} C^i_r \right) > \lim_{n \to \infty} \Lambda^h_\delta \left( s_n \cap \bigcup_{r=1}^{R} C^i_r \right). \quad \text{(12)}$$

Then clearly,

$$\lim_{n \to \infty} \Lambda^h_\delta \left( s_n \cap \bigcup_{r=1}^{R} C^i_r \right) > \lim_{n \to \infty} \Lambda^h_\delta \left( s_n \cap \bigcup_{r=1}^{R} C^i_r \right),$$

and so, from (12),

$$\Lambda^h_\delta \left( \bigcup_{n} s_n \cap \bigcup_{r=1}^{R} C^i_r \right) > \lim_{n \to \infty} \Lambda^h_\delta \left( s_n \cap \bigcup_{r=1}^{R} C^i_r \right),$$

contradicting the theorem for bounded sets. Thus, for each $i$, the series in (11) is convergent. Given any $\varepsilon > 0$ choose a value of $R$ such that,

$$\sum_{r=1}^{R} \sum_{i=1}^{n} \Lambda^h_\delta (\bigcup_{n} s_n \cap C^i_r) < \varepsilon.$$

Then we have,

$$\Lambda^h_\delta \left( \bigcup_{n} s_n \right) \leq \Lambda^h_\delta \left( \bigcup_{n} s_n \cap \bigcup_{r=1}^{R} C^i_r \right) + \Lambda^h_\delta \left( \bigcup_{n} \bigcup_{r=1}^{R} C^i_r \right)$$

$$< \Lambda^h_\delta \left( \bigcup_{n} s_n \cap \bigcup_{r=1}^{R} C^i_r \right) + \varepsilon$$
but the bounded case of the theorem gives us,
\[ \mathcal{L}^h_d\left( \bigcup_{n=1}^{\infty} U_n \cup U \cup C \right) = \lim_{n \to \infty} \mathcal{L}^h_d\left( S_n \cup U \cup C \right) \]
\[ \leq \lim_{n \to \infty} \mathcal{L}^h_d\left( S_n \right). \]

Hence,
\[ \mathcal{L}^h_d\left( \bigcup_{n=1}^{\infty} U_n \right) < \lim_{n \to \infty} \mathcal{L}^h_d\left( S_n \right) + \varepsilon \quad \text{for every } \varepsilon > 0, \]
and therefore,
\[ \mathcal{L}^h_d\left( \bigcup_{n=1}^{\infty} U_n \right) \leq \lim_{n \to \infty} \mathcal{L}^h_d\left( S_n \right). \]

The reverse inequality is trivial and hence the theorem is proved.

**Corollary**

If \( \lambda(x) \) is any monotonic increasing one-dimensional Hausdorff measure function, then for any \( \delta > 0 \) and any increasing sequence of sets \( \{S_n\} \) on the real line, we have,
\[ \mathcal{L}^h_\delta\left( \bigcup_{n=1}^{\infty} S_n \right) = \lim_{n \to \infty} \mathcal{L}^h_\delta\left( S_n \right). \]

Next, instead of considering sequences of sets we look at convergent sequences of values of \( \delta \).

**Theorem 10**

If \( \lambda(x) \) is any monotonic increasing \( q \)-dimensional Hausdorff
measure function with the property that,

\[ \frac{\lambda(\mathcal{C})}{\mathcal{C}^{\alpha-1}} \to 0 \quad \text{as } \mathcal{C} \to 0 \]

then for any \( \delta > 0 \) any set \( S \) in \( q \)-dimensional Euclidean space and any sequence \( \{\mathcal{C}_n\} \) with \( \mathcal{C}_n \to 0 \) as \( n \to \infty \) we have,

\[ \lambda^h(S) = \lim_{n \to \infty} \lambda_{\delta + \mathcal{C}_n}^h(S). \]

Proof

Clearly we have,

\[ \lambda_{\delta}^h(S) \geq \lambda_{\delta + \mathcal{C}_n}^h(S) \quad \text{for all } n. \quad \text{(13)} \]

We now assume that the set \( S \) is bounded. Thus we can assume that \( S \) is contained in a \( q \)-dimensional cube of side length \( \mathcal{C}_0 \), say. Given any \( \varepsilon > 0 \), for each integer \( n \), let \( \{U^n_\varepsilon\} \) be a sequence of open sets such that,

\[ S \subset \bigcup_{k=1}^{\infty} U^n_\varepsilon \]

and,

\[ \mathcal{A}(U^n_\varepsilon) < \delta + \mathcal{C}_n \quad \text{for all } \varepsilon. \]

Choose \( n \) so large that,

\[ \mathcal{C}_n < \delta \quad \text{(15)} \]

and,

\[ \left( \left[ \frac{\mathcal{C}_n}{\delta} \right] + 1 \right)^\alpha \kappa \left( \frac{2 \delta}{\mathcal{C}_n} \right)^{q-1} \lambda(\mathcal{C}_n) < \varepsilon/2 \quad \text{(16)} \]

where \( \kappa \) is the constant introduced in Theorem 9. Now we can replace each \( U^n_\varepsilon \) with the property \( \mathcal{A}(U^n_\varepsilon) \geq \delta \) by a set \( V^n_\varepsilon \) with \( \mathcal{A}(V^n_\varepsilon) < \delta \).
together with \( k \left( \frac{2^{2^{d-1}}}{\varepsilon_n} \right)^{a-1} \) sets of diameter \( \varepsilon_n \). Thus, we get a new covering of \( S \), \( \{W_i\} \) say, with,

\[
\delta(W_i) < \delta \quad \text{for all } i.
\]

There are at most \( \left( \left[ \frac{c^{2d}}{\delta} \right] + 1 \right)^a \) sets \( U_i \) with \( \delta(U_i) \geq \delta \), since, this number of such sets would be sufficient to cover \( S \).

So we have,

\[
\sum_i k(\delta(W_i)) < \sum_i k(\delta(U_i)) + \left( \left[ \frac{c^{2d}}{\delta} \right] + 1 \right)^a k \left( \frac{2^{d-1}}{\varepsilon_n} \right)^{a-1} k(\varepsilon_n)
\]

\[
\leq \sum_i k(\delta(W_i)) + \frac{c}{\varepsilon_n} \quad \text{by (16)}
\]

\[
< \sum_{j=1}^n k(S_j) + \tau. \quad \text{by (14).}
\]

Therefore, for all large \( n \),

\[
\sum_{j=1}^n k(S_j) < \sum_{j=1}^n k(S_j) + \tau
\]

Hence we have the required result from (18), (13) and the fact that \( \tau \) was an arbitrary positive number.

Thus we have proved that, for bounded sets,

\[
\sum_{j=1}^n k(S_j) \rightarrow \sum_{j=1}^n k(S_j) \quad \text{as } n \rightarrow \infty.
\]

Now let \( S \) be an arbitrary set in \( q \)-dimensional Euclidean space. Then we write,

\[
S = \bigcup_{i=1}^\infty S_i
\]

with each \( S_i \) bounded and \( S_i \subset S_{i+1} \), for all \( i \).
Then by Theorem 9,

\[ \mathcal{L}^h_j(S) = \lim_{\varepsilon \to 0} \mathcal{L}^h_j(S; \varepsilon) \]

\[ = \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \mathcal{L}^h_j(S; \delta \varepsilon) \]

\[ = \lim_{\delta \to 0} \mathcal{L}^h_j(S; \delta) \]  

(19)

Thus by (13) and (19), we have,

\[ \lim_{\varepsilon \to 0} \mathcal{L}^h_j(S) = \mathcal{L}^h_j(S) \] as required.

Corollary

If \( \mathcal{L}(r) \) is any monotonic increasing one-dimensional Hausdorff measure function then for any \( \delta > 0 \) any linear set \( S \) and any sequence \( \{r_n\} \) with \( r_n \to 0 \) as \( n \to \infty \) we have,

\[ \mathcal{L}^h_j(S) = \lim_{n \to \infty} \mathcal{L}^h_j(S; r_n) \]

Davies (1) shows that the result of Theorem 9 sometimes breaks down even in the case of continuous functions when we don't insist on the property,

\[ \frac{\mathcal{L}(r)}{a^{r-1}} \to 0 \] as \( r \to 0 \).
But Sion and Sjerve (11) have shown that the result is true for the pre-measure \( L^h_f \) in the continuous case even without the above property. Theorem 11 now shows that the latter result does not extend to the discontinuous case.

**Theorem 11**

There exists a discontinuous two-dimensional Hausdorff measure function \( h(x) \) say, with,

\[
\frac{h(x)}{x} \not\to 0 \quad \text{as } x \to 0
\]

and a positive number \( \delta \) and an increasing sequence of sets \( \{S_i\} \) in two-dimensional Euclidean space such that,

\[
\lim_{i \to \infty} L^h_f (S_i) \neq L^h_f (\bigcup S_i).
\]

**Proof**

Let

\[
x_n = \frac{1}{2^n}
\]

for \( n \geq 1, 2, \ldots \).

Define \( h(x) \) as follows,

\[
h(x) = x_n \quad \text{for } x \in (x_{n+1}, x_{n+2})
\]

and,

\[
h(x_n) = \frac{3}{2} x_n.
\]

Then, clearly, \( h(x) \) is a two-dimensional Hausdorff measure function. Take \( \delta = x_n \) for some positive integer \( N \).

Denote by \( S_n \) the common part of the closed discs.
\[ x^2 + y^2 \leq \frac{1}{6} \, d^2 \quad \text{and} \quad (x - \frac{1}{6})^2 + y^2 \leq \frac{1}{6} \, d^2. \]

Then we have,

\[ S_n \subset S_{n+1} \quad \text{for all } n. \]

Also we see that, \( \bigcup_{n=1}^{\infty} S_n \) is the open disc \( x^2 + y^2 < \frac{1}{6} \, d^2 \) together with that part of the circumference which lies to the right of the \( y \)-axis.

So that we have,

\[ L^d(S_n) < 2n \quad \text{for all } n. \]

It is clear that,

\[ L^d(\bigcup_{n=1}^{\infty} S_n) \leq \frac{3}{2} \, k_n, \]

since we can cover \( \bigcup_{n=1}^{\infty} S_n \) by its own closure. We now assume that all the sets of the covering have diameters strictly less than \( \delta \). Thus, let \( \{ \mathcal{U}_i \} \)
be any closed covering of \( \bigcup_{n=1}^{\infty} S_n \) such that,

\[ d(\mathcal{U}_i) < \delta \quad \text{for all } i. \]

Clearly, we see that no set \( \mathcal{U}_i \) can contain points on the boundary of \( \bigcup_{n=1}^{\infty} S_n \) which are diametrically opposite. Let \( \{ \mathcal{U}_n \} \) be a subsequence of \( \{ \mathcal{U}_i \} \) such that each \( \mathcal{U}_n \) has at least one point in common with the boundary of \( \bigcup_{n=1}^{\infty} S_n \).

Let the intersection of each \( \mathcal{U}_n \) with the boundary of \( \bigcup_{n=1}^{\infty} S_n \) subtend an angle \( \phi_i \) at the centre of \( \bigcup_{n=1}^{\infty} S_n \).

Then,

\[ \sin \phi_i \leq \frac{d(\mathcal{U}_n)}{2n}. \]
and \( 0 \leq \phi_i \leq \frac{n}{2} \) for each \( i \).

Clearly we must have,

\[
\sum_i \phi_i > 2 \pi
\]

in order that the sets \( U_i \) form a covering of \( \cup S_n \).

Also, we know that,

\[
\sum_i \omega_i \phi_i > \frac{2 \phi_i}{n}
\]

Thus,

\[
\sum_i d(U_i) > \sum_i d(U_{i*}) > \frac{\chi_n}{n} \sum_i 2 \phi_i > 2 \chi_n.
\]

Hence, since \( h(\omega) \geq \chi \) for all \( \omega \), we must have,

\[
L^h_0 (\cup S_n) = \frac{3}{4} \chi_n.
\]

Thus the theorem is proved.

We now show that we cannot always relax the conditions imposed in Theorem 10.

**Theorem 12**

There exists a discontinuous, two-dimensional Hausdorff measure function \( h(\omega) \), say, with,

\[
\frac{h(\omega)}{\omega} \to 0 \quad \text{as} \quad \omega \to \infty,
\]

a \( \delta > 0 \) and a set \( S \) in two-dimensional Euclidean space, such that,
Proof

Let,

\[ x_n = \left(\frac{2}{3}\right)^n \quad \text{for } n = 1, 2, \ldots \]

Define,

\[ h(x) = x_n \quad \text{for } x \in (x_{n-1}, x_n]. \]

Choose \( d = x_N \) for some positive integer \( N \), and denote by \( S \) the open disc \( x^2 + y^2 < \frac{1}{4}d^2 \) together with that part of the circumference which lies to the right of the \( y \)-axis.

Then for all integers \( n \),

\[ \int_{J_{x_n}}^h (S) \leq x_{n-1} = \frac{3}{2} x_n, \]

and, by a similar argument to that given in the previous theorem we see that,

\[ \int_{h}^h (S) = 2x_n. \]

Hence the theorem is proved.

Next, we extend the result of Theorem 10.

Theorem 13

If \( h(x) \) is any monotonic increasing \( q \)-dimensional Hausdorff
measure function with the property that,

\[
\lim_{\lambda \to \infty} \frac{h(\lambda)}{\lambda^{d-1}} = 0
\]

then for any \( d > 0 \), any set \( S \) in \( q \)-dimensional Euclidean space and any sequence \( \{T_n\} \) with \( T_n \downarrow 0 \) as \( n \to \infty \) we have,

\[
\bigwedge_{j}^{d}(S) = \lim_{n \to \infty} \bigwedge_{j}^{d-T_n}(S).
\]

**Proof.**

Clearly we have,

\[
\bigwedge_{j}^{d}(S) \leq \bigwedge_{d-T_n}^{d}(S)
\]

for all \( n \). \hspace{1cm} (2a)

So that, if \( S \) is such that \( \bigwedge_{d-T_n}^{d}(S) = \infty \), then we have,

\[
\lim_{n \to \infty} \bigwedge_{d-T_n}^{d}(S) = \bigwedge_{d}^{d}(S).
\]

So it is sufficient to prove the theorem for sets \( S \) such that \( \bigwedge_{d}^{d}(S) \) is finite.

Given any \( \varepsilon > 0 \), let \( \{U_i\} \) be a sequence of open sets with the following properties,

\[
S \subseteq \bigcup_{i=1}^{\infty} U_i , \quad d(U_i) < d
\]

and,

\[
\sum_{i=1}^{\infty} h(d(U_i)) < \bigwedge_{d}^{d}(S) + \varepsilon. \hspace{1cm} (21)
\]

From (21) and the fact that \( h(\lambda) > 0 \) for all \( \lambda > 0 \) we have,

\[
d(U_i) \to 0 \hspace{1cm} \text{as} \hspace{0.5cm} i \to \infty
\]

Thus, for some constant \( C \), there are at most \( C \) sets \( U_i \) with \( d(U_i) > d - \varepsilon \).
Now choose $N$ such that for all $n > N$,

$$\tau_n < \frac{d}{2}$$  \hfill (21)

and,

$$CK\left(\frac{d}{\tau_n}\right)^{q-1}h(\tau_n) < \frac{\varepsilon}{2}$$  \hfill (23)

where $K$ is the constant introduced in Theorem 9. For each $n > N$, replace each $U_i$ with the property $d(U_i) \geq d - \tau_n$ by an open set $V_i^\circ$ with $d(V_i^\circ) < d - \tau_n$ together with $[K\left(\frac{d}{\tau_n}\right)^{q-1}]$ open sets of diameter $\tau_n$. Thus we get another open covering of $S$ by sets $W_i$, say, such that,

$$d(W_i^\circ) < d - \tau_n$$  \hfill (24)

for all $i$.

Also, since there are at most $C$ sets $U_i$ with $d(U_i) \geq d - \tau_n$ we must have,

$$\sum_i h(d(W_i^\circ)) \leq \sum_i h(d(V_i^\circ)) + CK\left(\frac{d}{\tau_n}\right)^{q-1}h(\tau_n)$$

$$\leq \sum_i h(d(V_i^\circ)) + CK\left(\frac{d}{\tau_n}\right)^{q-1}h(\tau_n)$$

$$< \int_S^h + \varepsilon,$$

by (21) and (23).

Thus, for all $n > N$,

$$\int_{d - \tau_n}^h(S) < \int_{d}^h(S) + \varepsilon$$  \hfill (24)

Therefore, using (20) and (24) we have,
which completes the proof of the theorem.

Corollary of Theorems 10 and 13

If \( h(x) \) is any monotonic increasing q-dimensional Hausdorff measure function with the property that,\
\[
\frac{h(x)}{x^{q-1}} \to 0 \quad \text{as} \quad x \to 0
\]
then for any set \( S \) in q-dimensional Euclidean space and for any sequence \( \{d_n\} \) with \( d_n \to d \) as \( n \to \infty \) for some positive real number \( d \), we have,
\[
\lim_{n \to \infty} \chi_{d_n}(S) = \chi_d(S).
\]

Proof

Given any \( \varepsilon > 0 \), we know from Theorems 10 and 13 that there exists a positive integer \( N' \) such that for all \( n \geq N' \),
\[
\chi_{d_n}(S) > \chi_d(S) - \varepsilon,
\]
and,
\[
\chi_{d_n}(S) < \chi_d(S) + \varepsilon.
\]
Now, since \( d_n \to d \) as \( n \to \infty \) there exists a positive integer \( N \) such that, for all \( n \geq N \),
\[
d_n - \frac{1}{N'} < d_n < d_n + \frac{1}{N'}.
\]
So we have, for all \( n \geq N \),
Thus we have proved that,
\[ \lim_{n \to \infty} \mathcal{H}_n^h(S) = \mathcal{H}^h(S). \]

We now give conditions under which it is possible to replace
discontinuous functions by continuous ones without altering the
corresponding Hausdorff pre-measures.

**Theorem 14**

Let \( h(x) \) be any monotonic increasing q-dimensional Hausdorff
measure function with the property that,
\[ \frac{h(x)}{x^{q-1}} \to 0 \quad \text{as} \quad x \to 0 \]
and such that its points of discontinuity have zero as their only limit
point. Then there is a continuous Hausdorff measure function \( H(x) \), say,
such that for any \( \delta > 0 \), and any set \( S \) in q-dimensional Euclidean space,
\[ \mathcal{H}_\delta^h(S) = \mathcal{H}_\delta^H(S). \]

**Proof**

Let \( \{v_i\} \) be an enumeration of all the points of discontinuity of \( h(x) \)
We may assume that $x_i > x_{i+1}$ for all $i$, and that,

$$x_i \to 0 \quad \text{as} \quad i \to \infty.$$

Choose $\varepsilon_i$ arbitrary positive such that,

$$\varepsilon_i < \frac{1}{2} x_i \quad \text{and} \quad \varepsilon_i \leq \frac{1}{2} x_i.$$

Assume that we have chosen $\varepsilon_1, \ldots, \varepsilon_i$, we then choose $\varepsilon_i$ to be positive and such that,

$$x_i + \varepsilon_i < x_{i-1} \quad \text{for} \quad j = 1, \ldots, i-1 \quad (25)$$

$$\varepsilon_i < x_i \quad (26)$$

$$\varepsilon_i < x_i - (x_{i+1} + \varepsilon_{i-1}) \quad \text{for} \quad j = 1, \ldots, i-1 \quad (27)$$

$$h(x_i + \varepsilon_i) \leq h(2x_i) \quad (28)$$

$$\varepsilon_i \leq \frac{1}{3} x_i \quad \text{and} \quad \varepsilon_i \leq \frac{1}{3} x_i \quad (29)$$

and,

$$h(2\eta) < \left( \frac{\eta}{2x_i} \right)^{q-1} \left[ h(x_i + \varepsilon_i) - h(x_i - \varepsilon_i) \right] h_i \quad (30)$$

for all $\eta \in (0, \varepsilon_i)$, where $h_i$ is the constant introduced in Theorem 9.

Define $H(x)$ as follows,

$$H(x) = h(x) \quad \text{for} \quad x \in [x_i + \varepsilon_i, x_{i-1}) \quad \text{for some} \quad i$$

$$H(x_i) = h(x_i - \varepsilon_i)$$

in the intervals $(x_i, x_i + \varepsilon_i)$ define $H(x)$ to be continuous and monotonic increasing so that,
\[ H(x_i + \eta) = H(x_i) + \kappa \left( \frac{2x_i}{\eta} \right)^{\alpha-1} H^{*}(\eta) \quad \text{for } 0 < \eta < x_i, \]

where \( H^{*}(x) \) is a continuous increasing function with the following properties,

\[ H^{*}(x) \geq \kappa (2x)^{\alpha-1} \quad \text{for all } x \]
\[ H^{*}(x) \rightarrow 0 \quad \text{as } x \rightarrow 0 \]

and

\[ H^{*}(x_i) = \frac{1}{\kappa} \left( \frac{x_i}{2x_i} \right)^{\alpha-1} \left\{ \kappa (x_i + \eta) - \kappa (x_i - \eta) \right\} \quad (33) \]

This definition makes \( H(x) \) continuous at \( x_i \) because of (32); and continuity at \( x_i + \xi \) follows from (33).

Also we have,

\[ H^{*}(x_i) \geq \frac{1}{\kappa} \left( \frac{x_i}{2x_i} \right)^{\alpha-1} \left\{ \kappa (x_i + \eta) - \kappa (x_i - \eta) \right\} \]

\[ > \kappa (2x_i) \quad \text{by } (30). \]

Hence the equations (31) and (33) are consistent. It is clear that (31) and (32) are consistent since,

\[ \frac{\kappa (2x)}{x^{\alpha-1}} \rightarrow 0 \quad \text{as } x \rightarrow 0 \]

Finally, we need to show that we can choose such an \( H^{*}(x) \) and ensure that,

\[ H(x_i + \eta) \leq \kappa (x_i + \eta) \quad \text{for } 0 < \eta < x_i. \]

Now, using (30),
\[ h(x_\alpha - \alpha) + h\left(\frac{2x_\alpha}{\eta}\right) h(2\eta) < h(x_\alpha + \alpha) \leq h(x_\alpha + \eta). \]

Hence we can choose such a function \( h_\alpha(\tau) \). Thus, we see that

\[ h(\alpha \tau) \leq h(\alpha \tau) \quad \text{for all } \alpha, \]

and so we have,

\[ \bigwedge_{\mathcal{C}} h_\alpha(S) \leq \bigwedge_{\mathcal{C}} h_\alpha(S), \]

for all sets \( S \) and positive numbers \( \delta \). Let \( S \) be a set in \( q \)-dimensional Euclidean space and \( \delta \) a positive number, then given any \( \varepsilon > 0 \) we can choose a sequence of open sets such that,

\[ S \subset \bigcup_{\mathcal{C}} U_{\varepsilon}^{\delta} \quad \text{and} \quad d(\bigcup_{\mathcal{C}} U_{\varepsilon}^{\delta}) < \delta \quad \text{for all } \mathcal{C}, \]

and,

\[ \varepsilon \cdot h(\bigcup_{\mathcal{C}} U_{\varepsilon}^{\delta})) < \bigwedge_{\mathcal{D}} h_\alpha(S) + \varepsilon. \]

Now we assume that for some \( \mathcal{C} \),

\[ h(\bigcup_{\mathcal{C}} U_{\varepsilon}^{\delta})) \neq h(\bigcup_{\mathcal{C}} U_{\varepsilon}^{\delta})), \]

then we must have,

\[ d(\bigcup_{\mathcal{C}} U_{\varepsilon}^{\delta}) \subset \{x_j, x_j + \varepsilon\} \quad \text{for some } j. \]

Let \( d(\bigcup_{\mathcal{C}} U_{\varepsilon}^{\delta}) \subset \{x_j, x_j + \varepsilon\} \) where \( 0 < \eta < \varepsilon \). Then if \( \eta \in \bigcup_{\mathcal{K}} (x_k, x_{k-1}) \) for some \( k \) ( \( k > j \) ) then there is a \( \lambda \) such that,

\[ \eta + \lambda \in \{x_k + \varepsilon, x_{k-1}\}. \]
with \( 0 < \lambda \leq \tau_K \).

Choose open sets \( \{ V_{i,s}^{\delta} \} \) with \( s = 1, 2, \ldots, [K \left( \frac{x_i + \eta}{\eta + \lambda} \right)^{q^{-1}}] + 1 \), such that,

\[
\gamma(V_{i,1}^{\delta}) = x_i - \lambda
\]

\[
\gamma(V_{i,s}^{\delta}) = \eta + \lambda \quad \text{for} \quad s = 2, \ldots, [K \left( \frac{x_i + \eta}{\eta + \lambda} \right)^{q^{-1}}] + 1
\]

and,

\[
\bigcup_{i}^{\delta} \gamma \left( \frac{x_i + \eta}{\eta + \lambda} \right)^{q^{-1}} + 1 \subseteq \bigcup_{s=1}^{\delta} V_{i,s}^{\delta}.
\]

Then,

\[
H(\gamma(V_{i,s}^{\delta})) = H(x_i + \eta) = H(x_i) + K \left( \frac{x_i + \eta}{\eta + \lambda} \right)^{q^{-1}} H(x)(\eta)
\]

\[
\geq H(x_i + \lambda) + K \left( \frac{x_i + \eta}{\eta + \lambda} \right)^{q^{-1}} h(\eta + \lambda)
\]

\[
\geq H(x_i - \lambda) + K \left( \frac{x_i + \eta}{\eta + \lambda} \right)^{q^{-1}} h(\eta + \lambda)
\]

\[
\geq H(x_i - \lambda) + K \left( \frac{x_i + \eta}{\eta + \lambda} \right)^{q^{-1}} H(\eta + \lambda)
\]

\[
\geq \sum_{s=1}^{\delta} H(\gamma(V_{i,s}^{\delta})).
\]

Also we see that,
\[
x_{j} - \lambda = x_{j} - \tau_{k} > x_{j+1} + \tau_{j+1}
\]
by (27)

therefore,

\[
x_{j} - \lambda \in (x_{j+1} + \tau_{j+1}, x_{j})
\]

and so we have,

\[
H(\mathcal{A}(V_{i,s}^{d})) = h(\mathcal{A}(V_{i,s}^{5}))
\]
for all \( i \) and \( s \).

Further, we note that,

\[
\mathcal{A}(V_{i,s}^{d}) < \delta
\]
for all \( i \) and \( s \).

Thus we have,

\[
\mathcal{J}_{d}^{h}(S) = \sum_{i} \sum_{s} h(\mathcal{A}(V_{i,s}^{d})) \leq \sum_{i} H(\mathcal{A}(U_{i}^{\delta})) < \mathcal{J}_{d}^{H}(S) + \varepsilon.
\]

So, since the \( \varepsilon \) was arbitrarily small we have,

\[
\mathcal{J}_{d}^{h}(S) = \mathcal{J}_{d}^{H}(S).
\]

Hence the theorem is proved.

In Theorem 7 of Chapter 2 we showed that as far as Hausdorff measures are concerned, any discontinuous one-dimensional Hausdorff measure function \( \mathcal{H}(x) \) can be replaced by a continuous function \( H(x) \) with \( H(x) \geq \mathcal{H}(x) \). We now show that this is not possible for the pre-measures.
Theorem 15

There is a discontinuous one-dimensional Hausdorff measure function \( \mathcal{H}(x) \), say, such that if \( \mathcal{H}(x) \) is a continuous function with \( \mathcal{H}(x) \geq \mathcal{H}(x) \) for all \( x \), then there exists a positive number \( \delta \) and a set \( S \) on the real line such that,

\[
\mathcal{H}_{\delta}(S) \neq \mathcal{H}_{\delta}(S).
\]

Proof

Let

\[ x_n = \frac{1}{6^n} \quad \text{for} \quad n = 1, 2, \ldots \]

Define \( \mathcal{H}(x) \) as follows,

\[ \mathcal{H}(x) = x_n^{\frac{1}{2^n}} \quad \text{for} \quad x \in (x_{n+1}, x_n] \]

Denote by \( S_n \) the closed interval \([0, x_n]\). Then, clearly,

\[
\mathcal{H}_{\delta}(S_n) = \mathcal{H}(x_n) = x_n^{\frac{1}{2^n}} = 2^{-n}.
\]

Now let \( \mathcal{H}(x) \) be any continuous function such that,

\[ \mathcal{H}(x) \geq \mathcal{H}(x) \quad \text{for all} \quad x. \]

Then we must have,

\[ \mathcal{H}(x_n) \geq \frac{1}{x_n^{n-1}}. \]

So, there is a positive real number \( \delta \) such that,

\[ \mathcal{H}(x) \geq \frac{3}{2} x_n \quad \text{for} \quad x \in [x_n - \delta, x_n]. \]

\[ \delta < \frac{1}{2} x_n \]
Now let \( \{ U_i^{\infty} \} \) be any open covering of \( S_n \) such that,

\[
d(U_i^{\infty}) < x_n \quad \text{for all } i.
\]

Then if,

\[
d(U_i^{\infty}) \in [x_n - \delta, x_n) \quad \text{for some } i,
\]

we have,

\[
H(d(U_i^{\infty})) \geq \frac{3}{2}^{2/n} = 3 \bigwedge_{x_n}^H(S_n).
\]

Finally, we assume that, for all \( i \),

\[
d(U_i^{\infty}) < x_n - \delta.
\]

Then,

\[
\sum_i H(d(U_i^{\infty})) \geq \bigwedge_{x_n - \delta}^H(S_n)
\]

\[
\geq (x_n - \delta)^{1/n} + \delta^{1/n}
\]

\[
\geq \frac{x_n}{(x_n - \delta)^{1/n}} = \left( \frac{x_n}{x_n - \delta} \right)^{1/n} \bigwedge_{x_n}^H(S_n).
\]

Thus, in either case,

\[
\sum_i H(d(U_i^{\infty})) \geq c \bigwedge_{x_n}^H(S_n),
\]

where \( c > 1 \).

But the covering \( \{ U_i^{\infty} \} \) was arbitrary and so we have,
\[ \bigwedge_{x_n}^H (S_n) \geq c \bigwedge_{x_n}^h (S_n), \]

that is,
\[ \bigwedge_{x_n}^H (S_n) \neq \bigwedge_{x_n}^h (S_n). \]

Hence the theorem is proved.
CHAPTER 4

INTRODUCTION

In this chapter, rather than considering the exact values of the Hausdorff measure of certain sets, we will only be interested in whether or not the measure is positive and finite. The first theorem gives us sufficient conditions to ensure the measure equivalence of two Hausdorff measure functions. The following four theorems are concerned with an investigation into the necessity of these conditions. In the last five theorems we use the results of the first half of the chapter to extend some work of Rogers (9) and Laman (6,7,8) to the case of discontinuous functions, and to show that a result of Eggleston (3) does not remain true for discontinuous functions.

Theorem 16

Let \( \mathcal{H}(\infty) \) and \( \mathcal{H}(\infty) \) be two q-dimensional Hausdorff measure functions. If there exists a decreasing sequence \( \{x_n\} \) of positive real numbers such that,

1. \( x_n \to 0 \) as \( n \to \infty \)
2. \( \lim_{n \to \infty} \frac{x_{n+1}}{x_n} > 0 \)

and

3. \( \frac{H(x_n)}{\mathcal{H}(x_n)} \to l \) as \( n \to \infty \) where \( 0 < l < \infty \).

Then the functions \( \mathcal{H}(\infty) \) and \( \mathcal{H}(\infty) \) are measure equivalent, for sets in q-dimensional Euclidean space.

Proof

We know that for any set \( S \) in q-dimensional Euclidean space, \( \mathcal{H}(S) \)
is positive and finite if and only if \( \bigwedge_{i=1}^{h,c(q)}(S) \) is positive and finite.

Now let,

\[
\lim_{\alpha \to 0} \frac{x_{n+1}}{x_n} = 2^\alpha \quad (\alpha > 0).
\]

Let \( S \) be a set in \( q \)-dimensional Euclidean space such that \( \bigwedge_{j=1}^{h,c(q)}(S) \) is positive and finite. Let \( \varepsilon \) and \( \delta \) be two given positive numbers, then there exists an open covering \( \{ U_i^\delta \} \) of \( S \) by cubes such that,

\[
\bigwedge_{j=1}^{h,c(q)}(S) - \varepsilon < \sum_i h(d(U_i^\delta)) < \bigwedge_{j=1}^{h,c(q)}(S) + \varepsilon - \varepsilon
\]

and

\[
d(U_i^\delta) < \delta \quad \text{for all } i.
\]

Now assume that \( \delta \) is so small that,

\[
\frac{x_{n+1}}{x_n} \geq \delta \quad \text{for all } \varepsilon \text{ such that } \varepsilon < \delta
\]

Then, for each \( i \), using (2) and (3), we have,

\[
x_{n+1} < d(U_i^\delta) < x_n < 1/\delta \quad x_{n+1}
\]

for some integer \( n \).

So we can replace each cube \( U_i^\delta \) by \( \left( \left[ \frac{1}{\delta} \right] + 1 \right)^q \) cubes \( V_i \) of diameter \( x_{n+1} \). Thus there exists another open cover \( \{ V_i \} \) of \( S \) by cubes such that,

\[
d(V_i) \in \{ x_n \} \quad \text{for all } i
\]

\[
d(V_i) < \delta \quad \text{for all } i
\]

and,

\[
\sum_i h(d(V_i)) \leq \left( \left[ \frac{1}{\delta} \right] + 1 \right)^q \sum_i h(d(U_i^\delta))
\]
Now we know that,
\[ \frac{H(x_n)}{h(x_n)} \rightarrow \lambda \quad \text{as} \quad n \rightarrow \infty, \]
therefore there exists an integer \( N \) such that,
\[ l - \varepsilon < \frac{H(x_n)}{h(x_n)} < l + \varepsilon \quad \text{for all} \quad n \geq N. \]

Also, there exists a real number \( \delta' > 0 \) such that for all \( \delta < \delta' \), we have,
\[ \frac{H(x_n)}{h(x_n)} < l + \varepsilon \quad \text{whenever} \quad x_n < \delta. \]

Thus, since (4) holds for arbitrarily small values of \( \delta \), we have,
\[ \sum_i H(d(v_i)) < (l + \varepsilon) \sum_i h(d(v_i)) < \left( \left\lceil \frac{1}{\alpha} \right\rceil + 1 \right)^\alpha (l + \varepsilon) \left( \sum_i h^c(x)/x^\alpha \right) \]
therefore,
\[ \sum_i h^c(x)/x^\alpha \leq \left( \left\lceil \frac{1}{\alpha} \right\rceil + 1 \right)^\alpha (l + \varepsilon) \sum_i h^c(x)/x^\alpha \]

Hence the theorem is proved because of the symmetry of condition iii).

**Corollary**

For any discontinuous Hausdorff measure function \( H(x) \), there
exists a continuous Hausdorff measure function \( H(x) \) such that, for sets
in Euclidean space, \( H(x) \) and \( H(x) \) are measure equivalent.
We note that the above results can easily be extended to compact finite dimensional metric spaces. We see this from the following;

If we have,

$$
\kappa \kappa_{n+1} < o(U^S) \leq \kappa \kappa_n < \sqrt[n]{\kappa \kappa_{n+1}}
$$

for some set \(U^S\) of an open covering of \(S\), then,

$$
U^S \subseteq S(\kappa \kappa_n, \kappa \kappa_{n+1}) \quad \text{for some } \kappa \in U^S.
$$

Now there exist at most \(N(\kappa \kappa_n)\) disjoint spheres of radius \(\kappa \kappa_{n+1}\) meeting \(S(\kappa \kappa_n, \kappa \kappa_{n+1})\). Thus, \(U^S\) is contained in \(N(\kappa \kappa_n)\) spheres of radius \(\kappa \kappa_{n+1}\), which in turn are contained in \(N(\kappa \kappa_n)\) sets of diameter \(\kappa \kappa_{n+1}\). The remainder of the proof is analogous to that given in Theorem 16.

We now show that Theorem 16 would not hold true if we dropped the condition ii).

**Theorem 17**

For every decreasing sequence of positive numbers \(\{\kappa_n\}\) with \(\kappa_n \to 0\) as \(n \to \infty\) and,

$$
\lim_{n \to \infty} \frac{\kappa \kappa_{n+1}}{\kappa \kappa_n} = 0,
$$

there exist two one-dimensional Hausdorff measure functions \(\lambda(\kappa)\) and \(H(\kappa)\) with,

$$
\frac{H(\kappa \kappa_n)}{\lambda(\kappa \kappa_n)} \to 1 \quad \text{as } n \to \infty
$$

and a set \(S\) such that \(\lambda^H(S)\) is positive and finite whilst \(\lambda^H(S)\) is zero.
Proof

Since,

\[ \lim_{n \to \infty} \frac{x_{n+1}}{x_n} = 0, \]

we may write,

\[ x_{n+1} = \phi(n) x_n. \]

where,

\[ \phi(i) \to 0 \quad \text{as} \quad i \to \infty. \]

Define,

\[ h(x) = (\phi(1))^{1/e} \quad \text{for} \quad x > x_3, \]

\[ = (\phi(1) \phi(2))^{1/e} \quad \text{for} \quad x_3 > x > x_4, \]

\[ = (\phi(1) \ldots \phi(m-1))^{1/e} \quad \text{for} \quad x_m > x > x_{m+1}, \]

and,

\[ H(x) = (\phi(1))^{1/e} \quad \text{for} \quad x > x_4, \]

\[ = (\phi(1) \phi(2))^{1/e} \quad \text{for} \quad x_4 > x \leq x_5, \]

\[ = (\phi(1) \ldots \phi(m-1))^{1/e} \quad \text{for} \quad x_{m+1} > x \leq x_{m+2}. \]

Clearly these functions satisfy the postulates of the theorem.

For all \( n \), there exists \( u_n \in (x_{n+1}, x_n) \) such that,

\[ \frac{H(u_n)}{u_n} < \frac{H(t)}{t} \quad \text{and} \quad \frac{h(u_n)}{u_n} < \frac{h(t)}{t} \quad \text{for all} \quad t \in (0, u_n]. \]

We now construct the set \( S \) as in Theorem 4 of Chapter 2, by means of
a sequence \( \{x_n\} \subset \{y_n\} \) with respect to the function \( H(x) \). We see that 
\( \lambda^*(S) \) is positive and finite, but \( \lambda^s(S) \) is zero.
Hence the theorem is proved.

Clearly if \( w(x) \) and \( H(x) \) are continuous functions then either,
1. \( \frac{H(x)}{w(x)} \to 0 \) as \( x \to 0 \)
2. \( \frac{H(x)}{w(x)} \to \infty \) as \( x \to 0 \)

or
3. there exists a sequence \( \{x_n\} \) such that \( x_n \to 0 \) as \( n \to \infty \)
and,
\[
\frac{H(x_n)}{w(x_n)} \to 0 \quad \text{as} \quad n \to \infty \quad \text{where} \quad 0 < \lambda < \infty.
\]

We now see that this is not true if the functions are discontinuous.

**Theorem 18**

There exist two q-dimensional Hausdorff measure functions \( H(x) \) and \( w(x) \) such that,
\[
\lim_{x \to 0} \frac{H(x)}{w(x)} = \kappa, \quad \text{and} \quad \lim_{x \to 0} \frac{H(x)}{w(x)} = 0,
\]
and with the property that there are no convergent sequences \( \{H(x_n)/w(x_n)\} \)
with non-zero limit, where \( \{x_n\} \) is a null sequence. For these two functions there are sets \( S_1, S_2 \) such that,
\[
0 < \lambda^*(S_1) \leq \kappa \quad \lambda^s(S_1) = 0.
\]
end

Proof

To prove this theorem it suffices to define two appropriate functions, the constructions of the sets $S_1, S_2$ can then be carried out using the methods of Theorem 4 of Chapter 2.

Define,

$$\begin{align*}
H(x) &= \frac{1}{(2^m-1)!} \quad \text{for} \ x \in \left[\frac{(2m+1)!}{(2m)!}, \frac{(2m)!}{(2m-1)!}\right] \\
\lambda(x) &= \frac{1}{(2^n)!} \quad \text{for} \ x \in \left[\frac{(2n)!}{(2n-1)!}, \frac{(2n-1)!}{(2n-2)!}\right].
\end{align*}$$

It is easy to see that these functions have the required properties.

Next, we prove that it is possible to have measure equivalence even when the conditions in Theorem 16 are contradicted.

Theorem 19

There exist two measure equivalent $\varphi$-dimensional Hausdorff measure functions $H(x), \lambda(x)$ such that if

$$\left\{\frac{H(x_n)}{\lambda(x_n)}\right\}$$

is convergent for some null sequence $\{x_n\}$ then,

$$\lim_{n \to \infty} \frac{x_{n+1}}{x_n} = 0.$$

Proof

Define,
\[ h(x) = \frac{1}{n!} \quad \text{for } x \in \left[\left(\frac{1}{n+1}\right)^{\frac{1}{n}}, \left(\frac{1}{n}\right)^{\frac{1}{n}}\right] \]

\[ H(x) = 2h(2n)! \quad \text{for } x \in \left[\left(\frac{1}{2n+1}\right)^{\frac{1}{2n}}, \left(\frac{1}{2n}\right)^{\frac{1}{2n}}\right] \]

\[ = \frac{1}{2. (2n+1)!} \quad \text{for } x \in \left[\left(\frac{1}{2n+1}\right)^{\frac{1}{2n}}, \left(\frac{1}{2n}\right)^{\frac{1}{2n}}\right]. \]

We see from the definition that,

\[ \frac{1}{2} h(x) \leq H(x) \leq 2h(x) \quad \text{for all } x, \]

hence we clearly have measure equivalence and the theorem is proved.

Clearly we can see that if,

\[ \lim_{x \to 0} \frac{H(x)}{h(x)} < \infty \quad \text{and} \quad \lim_{x \to 0} \frac{H(x)}{h(x)} > 0, \]

then \( h(x) \) and \( H(x) \) are measure equivalent. But, by considering the following example we see that there exist two \( q \)-dimensional Hausdorff measure functions \( h(x) \) and \( H(x) \) with,

\[ \lim_{x \to 0} \frac{H(x)}{h(x)} = \infty \]

and for any set \( S \), \( \lambda^h(S) \) is positive and finite if and only if \( \lambda^H(S) \) is positive and finite.

Define,

\[ h(x) = \frac{1}{n!} \quad \text{for } x \in \left[\left(\frac{1}{n+1}\right)^{\frac{1}{n}}, \left(\frac{1}{n}\right)^{\frac{1}{n}}\right] \]

\[ H(x) = 1 \quad \text{for } x \in \left[\left(\frac{1}{2n+1}\right)^{\frac{1}{2n}}, \left(\frac{1}{2n}\right)^{\frac{1}{2n}}\right]. \]

\( \frac{\lambda^h(S)}{\lambda^H(S)} \) is always positive and finite, since \( H(x) \) is less than any
continuous function which is greater than \( h(x) \) and because for any set \( S \) we can always find a continuous function \( g(x) \) greater than \( h(x) \) for which,

\[
\mathcal{L}^1(S) < 2 \left( \left[ \frac{h(x)}{g(x)} \right] + 1 \right)^2 \mathcal{C}^1(S)
\]

(this fact was proved in Theorems 5 and 6 of Chapter 2).

It is interesting to investigate whether or not measure equivalence implies the existence of a null sequence on which the ratio of the functions is convergent to a non-zero limit. It is easy to see that this is the case if we are only considering continuous functions. For, if there is no null sequence with the required property we are left with only two possibilities,

1. \( \lim_{x \to a} \frac{h(x)}{g(x)} = 0 \)

or, 2. \( \lim_{x \to a} \frac{h(x)}{g(x)} = \infty \).

Clearly both these possibilities are inconsistent with measure equivalence.

The next theorem shows that the opposite result is true for discontinuous functions.

Theorem 20

There exist two measure equivalent one-dimensional Hausdorff measure functions \( h(x) \) and \( g(x) \) such that there is no convergent sequence \( \{ \frac{h(x_n)}{g(x_n)} \} \) (where \( \{ x_n \} \) is a null sequence), with non-zero limit.
Proof

Define

\[ x_n = \frac{1}{2^n} \quad \text{for } n = 0, 1, 2, \ldots \]

\[ h(x) = \frac{1}{1 + k_n(x+1)} \quad \text{for } x \in \left[ x_{k_n(x+1)}, x_{k_n(x)} \right] \]

\[ H(x) = \frac{1}{2 + k_n(x+1)(1+n)} \quad \text{for } x \in \left[ x_{k_n(x+1)}, x_{k_n(x)} \right] \]

Then,

\[ \frac{h(x)}{x_{k_n(x+1)}} = \frac{1}{2} \quad \text{for all } n = 3, 5, 7, \ldots \]

and,

\[ \frac{H(x)}{x_{k_n(x+1)}} = \frac{1}{2} \quad \text{for all } n = 2, 4, 6, \ldots \]

Thus,

\[ \lim_{x \to 0} \frac{h(x)}{x} = \lim_{x \to 0} \frac{H(x)}{x} = \frac{1}{2} \]

So, we must have,

\[ m^*(S) \geq \frac{1}{2} \Lambda(S) \quad \text{for all sets } S \]

Now let \( S \) be any set on the real line and, given any \( \varepsilon > 0 \), let \( \{ U_i^j \} \)
be a sequence of open intervals such that,

\[ S \subseteq \bigcup_{i=1}^{\infty} U_i^j \]

\[ \Lambda(U_i^j) < \varepsilon \quad \text{for all } i \]
and, \[ \Lambda_j(S) \leq \sum_i d(U^5_i) < \Lambda_j(S) + \varepsilon. \]

Let \( \{x_n\} \) be a sequence of positive real numbers such that,
\[ \frac{h(x_n)}{x_n} \rightarrow V_\varepsilon \]
as \( n \rightarrow \infty \)
and let \( \{x_{n_i}\} \) be a subsequence of \( \{x_n\} \) such that, for each \( i \),
\[ d(U^5_i) \in (x_{n_{i+1}}, x_{n_i}]. \]

We can replace each \( U^5_i \) by \( \left( \left( \frac{d(U^5_i)}{x_{n_{i+1}}} \right) + 1 \right) \) open intervals of length \( x_{n_{i+1}} \). Thus, we get another open covering \( \{V^5_i\} \) of \( S \) such that,
\[ d(V^5_i) < \varepsilon \]
for all \( i \),
and,
\[ \sum_i (d(V^5_i)) = \sum_i \left( \left( \frac{d(U^5_i)}{x_{n_{i+1}}} \right) + 1 \right) x_{n_{i+1}} \leq \sum_i (d(U^5_i) + x_{n_{i+1}}) \leq 2 \sum_i d(U^5_i) < 2 \Lambda_j(S) + 2 \varepsilon. \]

Thus, for all \( \varepsilon > 0 \), there is an open covering \( \{V^5_i\} \) of \( S \) such that,
\[ d(V^5_i) < \varepsilon \]
for all \( i \),
and,
\[ d(V^5_i) \in \{x_{n_i}\} \]
and \( \Lambda_\delta(S) \leq \sum_i d(V_i^\delta) < 2 \Lambda_\delta(S) + 2 \pi \).

We can choose a positive real number \( \delta' \), say, such that,
\[
\frac{h(d(V_i^\delta))}{d(V_i^\delta)} < 1
\]
for all \( i \) whenever \( 0 < \delta < \delta' \).

Thus, for \( 0 < \delta < \delta' \), we have,
\[
\Lambda_\delta(S) \leq \sum_i h(d(V_i^\delta)) \leq \sum_i d(V_i^\delta)
< 2 \Lambda_\delta(S) + 2 \pi.
\]

Hence we have,
\[
\frac{1}{2} \Lambda(S) \leq \Lambda_\delta(S) \leq 2 \Lambda(S).
\]

Clearly, we can get the same result with regard to \( \Lambda^H(S) \) and so we have proved that the functions \( k(x) \) and \( H(x) \) are measure equivalent for sets on the real line. Also it is clear that there are no convergent sequences \( \left\{ \frac{h(x_n)}{H(x_n)} \right\} \) with non-zero limit. Hence the theorem is proved.

Eggleston (3) has shown that given any positive number \( \delta \) and any function \( k(x) \) satisfying,

i). \( k(x) \) continuous and strictly increasing

ii). \( \frac{x^n}{k(x)} \) is an increasing function of \( x \).
and, iii) \( h(0) = 0 \), \( \lim_{x \to 0^+} \frac{x^n}{h(x)} = 0 \),

we can construct a set \( A \) in \( n \)-dimensional Euclidean space so that \( \mathcal{H}^n(A) = \infty \). It is now easily possible to extend this result to functions satisfying,

a) \( h(x) > 0 \) for \( x > 0 \)

b) \( \lim_{x \to 0} h(x) = 0 \)

and, c) \( \lim_{x \to 0} \frac{h(x)}{x^n} = \infty \).

In the same paper Eggleston defines two functions (satisfying i), ii), and iii),) to be incomparable when,

\[
\lim_{x \to 0^+} \frac{h(x)}{H(x)} = \lim_{x \to 0^+} \frac{H(x)}{h(x)} = 0.
\]

He shows that for two incomparable functions we can construct a set \( A \) such that \( \mathcal{H}^n(A) \) is positive and finite whilst \( \mathcal{H}^n(A) = \infty \). Our next theorem shows that this result does not extend to the case of discontinuous functions.

**Theorem 21**

There are two incomparable \( q \)-dimensional Hausdorff measure functions \( h(x) \) and \( H(x) \), say, such that if \( S \) is a set in \( q \)-dimensional Euclidean space, then if \( \mathcal{H}^q(S) \) is finite we must also have \( \mathcal{H}^q(S) \) finite.

**Proof**

Define the decreasing sequence \( \{\mu_x\} \) as follows,
Define the function \( q(x) \) as follows,

\[
q(x) = \begin{cases} 
1 & \text{for } x > x_n \\
\frac{q(x_{n-1})}{n} & \text{for } x \in (x_{n-1}, x_n] 
\end{cases}
\]

Then \( q(x) \) is a \( q \)-dimensional Hausdorff measure function.

Define \( h(x) \) such that,

\[
h(x) = q(x) \quad \text{for } x \neq \{x_i\}
\]

\[
h(x_n) = nq(x_n) \quad \text{for all } n.
\]

Define \( H(x) \) such that,

\[
H(x) = q(x) \quad \text{for } x \neq \{x_i\}
\]

\[
H(x_{n+1}) = q(x_{n+1}) \quad \text{for all } n
\]

Then both \( h(x) \) and \( H(x) \) are \( q \)-dimensional Hausdorff measure functions.

Also,

\[
\frac{h(x_{n+1})}{H(x_{n+1})} = \frac{1}{2n+1}
\]
and,
\[ \frac{H(x_{2n})}{h(x_{2n})} = \frac{g(x_{2n})}{2n} = \frac{1}{2n} \]

thus,
\[ \lim_{x \to 0^+} \frac{h(x)}{H(x)} = \lim_{x \to 0^+} \frac{H(x)}{h(x)} = 0, \]

that is, the functions \( h(x) \) and \( H(x) \) are incomparable.

Now let \( S \) be any set such that \( \mathcal{H}^1(S) \) is finite, then since \( h(x) > g(x) \)
we have \( \mathcal{H}^2(S) \) is finite. From Theorems 5 and 6 of Chapter 2 we see
that, given any \( r > 0 \), there is a continuous Hausdorff measure function
\( G(x) \) say, such that \( G(x) > g(x) \) for all \( x > 0 \) and,
\[ \mathcal{H}^2(S) \leq \mathcal{G}(S) < 2 (\left\lceil \frac{1}{a} \right\rceil + 1) \mathcal{H}^2(S) + \varepsilon. \]

Now we have \( h(x) > g(x) \) and \( H(x) > g(x) \) and both \( h(x) \) and \( H(x) \)
are less than any continuous function which is greater than \( g(x) \), that
is,
\[ h(x) \leq G(x) \quad \text{and} \quad H(x) \leq G(x). \]

Thus,
\[ \mathcal{H}^2(S) < \mathcal{H}(S) \leq \mathcal{G}(S) < 2 (\left\lceil \frac{1}{a} \right\rceil + 1) \mathcal{H}^2(S) + \varepsilon, \]

and,
\[ \mathcal{H}^2(S) < \mathcal{H}(S) \leq \mathcal{G}(S) < 2 (\left\lceil \frac{1}{a} \right\rceil + 1) \mathcal{H}^2(S) + \varepsilon. \]

Hence the theorem is proved.
Combining this result with the corollary to Theorem 16 of this chapter, we see that Eggleston's theorem does not hold true for functions \( h(x) \), satisfying:

1. \( h(x) \) is continuous and strictly increasing
2. \( h(0) = 0 \), \( \lim_{x \to 0^+} \frac{x^\alpha}{h(x)} = 0. \)

Since, if it held true in the continuous case it would also be true for discontinuous functions, and we have just seen that this is not so. Thus we have a negative answer to the problem of whether we can always replace any Hausdorff measure function by another one \( h(x) \), say, with the property that \( \frac{h(x)}{x^\alpha} \) is a decreasing function of \( x \).

We now generalize a result of Rogers (9) to the case of discontinuous functions.

**Theorem 22**

Let \( h(x) \) be a \( q \)-dimensional Hausdorff measure function and \( E \) a compact set of non-\( \sigma \)-finite \( g \)-measure in a Euclidean space (or in a compact, finite-dimensional metric space). Then there is a continuous Hausdorff measure function \( q(x) \) with \( h < q \) and such that \( E \) is of non-\( \sigma \)-finite \( g \)-measure.

**Proof**

Let \( \{x_n\} \) be an enumeration of all the discontinuities of \( h(x) \). Define the decreasing sequence \( \{u_n\} \) as follows. Choose \( u_1 \) arbitrarily such that \( u_1 \in \{x_n\} \), having chosen \( u_1, \ldots, u_{n-1} \), choose \( u_n \) such that,

\[
\frac{1}{2} u_{n-1} < u_n < \frac{1}{2} u_{n-1}.
\]
Define \( H(\mathbf{x}) \) to be continuous increasing and,

\[
H(u_n) = h(u_n) \quad \text{for all } n,
\]

with,

\[
H(u) \leq h(u) \quad \text{for all } u.
\]

Then we have,

1). \( u_n \to 0 \) as \( n \to \infty \)

2). \( u_{n+1} > u_n \) for all \( n \)

3). \( \frac{H(u_n)}{h(u_n)} \to 1 \) as \( n \to \infty \).

Thus all the conditions of Theorem 16 are satisfied. So, for any set \( S \), \( \mathcal{M}^H(S) \) is positive and finite if and only if \( \mathcal{M}^h(S) \) is positive and finite. Now, since \( E \) is a compact set of non-\( \sigma \)-finite \( h \)-measure it must be of non-\( \sigma \)-finite \( H \)-measure. Thus, from Rogers' result (9), there exists a continuous Hausdorff measure function \( g(x) \) with \( H < g \) such that \( E \) is of non-\( \sigma \)-finite \( g \)-measure. But \( H(x) \leq h(x) \) for all \( x \) and thus \( H < q \). Hence the theorem is proved.

We, next, generalise some results of Laman (6, 7).

**Theorem 23**

Let \( E \) be a finite dimensional compact metric space, and suppose
that $\mathcal{H}(\varepsilon)$ is a Hausdorff measure function such that $\mathcal{H}(\varepsilon)$ is infinite. Then it is possible to select from $E$ a closed subset of any given $h$-measure.

Proof

If $\alpha$ is any given positive number, it is sufficient to find a closed subset $\rho$ such that $\mathcal{H}(\rho) \geq \alpha$. Laman (7) proves this result for the case when $\mathcal{H}(\varepsilon)$ is a continuous (on the right) function. From Theorem 16 we can find a continuous function $\mathcal{H}(\varepsilon)$, with $\mathcal{H}(\varepsilon) \leq \mathcal{H}(\varepsilon)$ and such that $\mathcal{H}(\varepsilon)$ and $\mathcal{H}(\varepsilon)$ are measure equivalent.

Thus $\mathcal{H}(\varepsilon) = \alpha$ and therefore from Laman's result we can find a closed subset $\rho$ such that,

$$\mathcal{H}(\rho) \geq \alpha.$$

But $\mathcal{H}(\varepsilon) \geq \mathcal{H}(\varepsilon)$ for all $\varepsilon$, and therefore,

$$\mathcal{H}(\rho) \geq \alpha.$$

This completes the proof of the theorem.

Theorem 24

Let $\mathcal{H}(\varepsilon)$ be a q-dimensional Hausdorff measure function. Then it is possible to construct in $\mathbb{R}$ a closed set $A$ such that,

$$0 < \mathcal{H}(A) < \infty.$$

Proof

Let $\mathcal{H}(\varepsilon)$ be a continuous function with $\mathcal{H}(\varepsilon) \geq \mathcal{H}(\varepsilon)$ for all $\varepsilon$ and such that $\mathcal{H}(\varepsilon)$ and $\mathcal{H}(\varepsilon)$ are measure equivalent for sets in
compact finite dimensional metric spaces. Then since $H(x) > h(x)$ we
know that $H$ is a q-dimensional measure function. In particular, we know
that,

$$
\frac{H(x)}{x^q} \rightarrow k \quad \text{as } x \rightarrow 0.
$$

Thus, there exists a decreasing sequence \(\{x_n\}\) such that,

$$
x_n \downarrow 0 \quad \text{as } n \rightarrow \infty,
$$

and for each $n$,

$$
\frac{H(x_n)}{x_n^q} \leq \frac{H(x)}{x^q} \quad \text{for all } x \in (0, x_n],
$$

since $H(x)$ is continuous.

In the closed interval $[x_{n+1}, x_n]$ define the continuous function $H'(x)$
as follows,

$$
H'(x) = \inf_{x_{n+1} \leq y \leq x_n} \left\{ \frac{H(y)}{y^{q+1}} \right\}.
$$

Then $H'(x) < H(x)$ for all $x$ and $H'(x)/x^{q+1}$ increases from $H(x_n)/x_n^{q+1}$
to $H(x_{n+1})/x_{n+1}^{q+1}$ as $x$ decreases from $x_n$ to $x_{n+1}$.

Now consider two real numbers $x, z$ such that $x > z$ then because of the
continuity of $H(x)$ we have,

$$
\frac{H(x)}{x^q} = \frac{H(y_n)}{y_n^{q+1}},
$$

where $H'(y_n) = H(y_n)$ and $y_n \in [x_{n+1}, x_n]$ for some integer $n$, and,

$$
\frac{H(z)}{z^q} = \frac{H(y_z)}{y_z^{q+1}},
$$
where \( H'(u_2) = H(u_2) \) and \( u_2 \in [x_{n+1}, x] \) for the same integer \( n \).

Now, if

\[
\frac{H'(x)}{x^{q+1}} = \frac{H'(z)}{z^{q+1}},
\]

we must have \( H'(x) \geq H'(z) \).

If,

\[
\frac{H'(x)}{x^{q+1}} < \frac{H'(z)}{z^{q+1}},
\]

then we must have, \( u_{x_2} \in [z, x] \).

In the interval \([z, u_{x_2}]\), the function \( \frac{H(x)}{x^{q+1}} \) takes all values from \( \frac{H(z)}{z^{q+1}} \) to \( \frac{H(u_2)}{u_2^{q+1}} \) which is greater than or equal to \( \frac{H(u_2)}{u_2^{q+1}} \).

Therefore, there exists \( t \in [z, u_{x_2}] \) such that,

\[
\frac{H(t)}{t^{q+1}} = \frac{H(u_2)}{u_2^{q+1}}.
\]

Thus, we have, because of the monotonicity of \( H(n) \),

\[
H'(x) \leq H(t) \leq H(u_{x_2}) \leq H'(x)
\]

and so we have proved that the function \( H'(x) \) is monotonic increasing.

Also, \( H'(x) \) is continuous and \( \frac{H'(x)}{x^{q+1}} \) increases to infinity as \( x \) decreases to zero. Laman (6) shows that for functions of this type it is possible to construct, in \( \lambda^r \), a compact, perfect set \( A \) such that,

\[
0 < H'(A) < \infty,
\]

(for definition, see p.4)

and,

\[
0 < \int H'(A) < \infty.
\]
Now since \( H(x) \geq H'(x) \) for all \( x \), we have,

\[
\mathcal{M}^H(A) \geq \mathcal{M}^H(A) > 0.
\]

Define the function \( q(x) \) as follows,

\[
q(x) = x^{H'(x)}
\]

then we have,

\[
q(A) = 0,
\]

and so \( A \) is a compact finite dimensional metric space.

So, if \( \mathcal{M}^H(A) \) is finite, then we must have,

\[
0 < \mathcal{M}^H(A) < \infty
\]

because of the measure equivalence of \( H(x) \) and \( H'(x) \).

Now, if \( \mathcal{M}^H(A) = \infty \) we can use the result of Theorem 23 to select a closed subset \( \beta \) of \( A \) such that,

\[
0 < \mathcal{M}^H(\beta) < \infty,
\]

and again we have,

\[
0 < \mathcal{M}^H(\beta) < \infty.
\]

Hence the theorem is proved.

Finally, we state a theorem of Larson (8) which can easily be generalized to the discontinuous case using the corollary to Theorem 16.
Theorem 25

Let $\omega(x)$ be a Hausdorff measure function and $A$ an analytic set of non-$\sigma$-finite $h$-measure in a compact finite dimensional metric space, then we can construct $2^{\aleph_0}$ disjoint closed subsets of $A$ which have non-$\sigma$-finite $h$-measure.
CHAPTER 5

INTRODUCTION

In Chapter 3 we obtained some results relating Hausdorff pre-measures and convergent sequences \{\varepsilon_n\} of positive real numbers. Following this, it seemed interesting to investigate the properties of the Hausdorff measures of a set with regard to functions \( h_n(x) \) where \( \{h_n(x)\} \) is a convergent sequence of functions.

Theorem 26

There exists a sequence of Hausdorff measure functions \( \{h_n(x)\} \) such that,

\[
h_n(x) \rightarrow h(x) \quad \text{uniformly as } n \rightarrow \infty
\]

where \( h(x) \) is a Hausdorff measure function, and a set \( S \) with the property,

\[
\lim_{n \rightarrow \infty} h_n(S) = h(S).
\]

Proof

We shall in fact show that there exists such a sequence of functions \( \{h_n(x)\} \) with limit \( h(x) \) such that,

\[
\lim_{n \rightarrow \infty} h_n(S) = \infty \quad \text{whenever } h(S) > 0.
\]

Let

\[ x_n = \frac{1}{n} \quad \text{for } n = 1, 2, \ldots \]

and,

\[ \alpha_n = \frac{1}{\alpha} \quad \text{for } \omega = 1, 2, \ldots \]

Define
\[ \mathcal{H}(x) = \alpha_n \quad \text{for} \quad x \in (x_{n-1}, x_n]. \]

We note that \( \alpha_n \) is such that for all positive integers \( q \),

\[ \frac{\alpha_n}{x_n^q} \to \infty \quad \text{as} \quad n \to \infty. \]

This ensures that \( \mathcal{H}(x) \) is a q-dimensional Hausdorff measure function.

For each integer \( n \), define \( \mathcal{H}_n(x) \) as follows,

\[ \mathcal{H}_n(x) = \left(1 + \frac{n}{x_n^q}\right) \alpha_n \quad \text{for} \quad x \in (x_{n-1}, x_n]. \]

Then each \( \mathcal{H}_n(x) \) is a q-dimensional Hausdorff measure function.

Clearly,

\[ \mathcal{H}_n(x) \to \mathcal{H}(x) \quad \text{uniformly as} \quad n \to \infty. \]

Choose any set \( S \) such that,

\[ 0 < \mathcal{H}(S) < \infty. \]

Consider the function \( \mathcal{H}_n(x) \) for some fixed positive integer \( n \). Then, given any real number \( A \), there exists an integer \( M = M(A) \) such that,

\[ \frac{\mathcal{H}_n(x_n)}{\mathcal{H}(x_n)} > A \quad \text{for all} \quad n \geq M. \]

Now, given any \( x > 0 \), choose a sequence \( \{ U_{n,i}^x \} \) of open sets such that,

\[ S \subset \bigcup_{i=1}^{k} U_{n,i}^x \]

\[ d(U_{n,i}^x) < x_n \quad \text{for all} \quad i. \]
and,

$$\mathcal{H}_n(S) = \sum_i h_n(d(U_{n,i}^n)) < \mathcal{H}_n(S) + \varepsilon,$$

Then, for all $n \geq N$, we have,

$$\mathcal{H}_n(S) < \sum_i h(d(U_{n,i}^n)) < \frac{1}{A} \left( \sum_i h_n(d(U_{n,i}^n)) \right)$$

$$< \frac{1}{A} (\mathcal{H}_n(S) + \varepsilon),$$

that is,

$$\mathcal{H}_n(S) > A \mathcal{H}_n(S) - \varepsilon.$$

Thus, since $n, A$ and $\varepsilon$ were arbitrary and because,

$$0 < \mathcal{H}_n(S) < \infty,$$

we have,

$$\lim_{n \to \infty} \mathcal{H}_n(S) = \infty.$$ 

Hence the theorem is proved.

**Corollary**

There exists a sequence of Hausdorff measure functions $\{h_n(x)\}$ and a function $h(x)$ such that,

$$\frac{h_n(x)}{h(x)} \to 1 \quad \text{as} \quad n \to \infty.$$
Thus, we see that uniform convergence of the functions is not sufficient to ensure that the limit operation commutes with the Hausdorff measure. We, now, establish sufficient conditions for this property to hold true.

**Theorem 27**

For any Hausdorff measure function \( h(x) \) and any sequence \( \{h_n(x)\} \) of Hausdorff measure functions such that,

\[
\frac{h_n(x)}{h(x)} \rightarrow 1 \quad \text{uniformly as } \ n \rightarrow \infty
\]

we have,

\[
\lim_{n \rightarrow \infty} \lambda^h(S) = \lambda^h(S),
\]

for all sets \( S \) and for any positive real number \( \delta \).

**Proof**

Given any \( \varepsilon > 0 \) we can choose a positive integer \( N \) such that,

\[
(1 - \varepsilon) < \frac{h_n(x)}{h(x)} < (1 + \varepsilon),
\]

for all \( n \geq N \) and for all \( x \).

Thus, for all \( n \geq N \), any set \( S \) and any positive real number \( \delta \), we have,

\[
(1 - \varepsilon) \lambda^h(S) \leq \lambda^h(S) \leq (1 + \varepsilon) \lambda^h(S),
\]
that is,

\[(1-\varepsilon)\Lambda_{\delta}^{h}(S) \leq \lim_{n \to \infty} \Lambda_{\delta}^{h}(S) \leq (1+\varepsilon)\Lambda_{\delta}^{h}(S)\].

Hence, since the \(\varepsilon\) was arbitrary, we have,

\[\lim_{n \to \infty} \Lambda_{\delta}^{h}(S) = \Lambda_{\delta}^{h}(S)\],

as required.
CHAPTER 6

INTRODUCTION

In this chapter we work in the space $\mathbb{Y}$ and investigate whether or not some of the theorems of previous chapters can be extended to this non-Euclidean space. In Theorems 4 and 24 of Chapters 2 and 4, respectively, we showed that, corresponding to any $q$-dimensional Hausdorff measure function, firstly there is a set in $q$-dimensional Euclidean space with positive, finite $h$-measure, and secondly there is a set in $\mathbb{Y}$ with positive, finite $h$-measure. The first theorem of this chapter shows that there are Hausdorff measure functions such that,

$$\lim_{x \to 0} \frac{h(x)}{x^q} = 0,$$

for all positive integers $q$, and sets $S$ in $\mathbb{Y}$ such that $\lambda^h(S)$ is positive and finite. Clearly, the sets $S$ could not be embedded in any Euclidean space. The second theorem shows that there are discontinuous functions such that for sets in $\mathbb{Y}$ there are no measure equivalent continuous functions. Finally, we show that Theorem 10 of Chapter 3 does not extend to the space $\mathbb{Y}$.

Theorem 28

There exists a compact set $S$ in $\mathbb{Y}$ and a Hausdorff measure function $h(x)$ such that, for any positive integer $q$,

$$\lim_{x \to 0} \frac{h(x)}{x^q} = 0,$$

for which,

$$0 < \lambda^h(S) < \infty.$$
Proof

Define the sequence \( \{x_n\} \) of positive real numbers so that,

\[ x_n = \frac{1}{2^n} \quad \text{for} \quad n = 0, 1, 2, \ldots \]

Define the function \( h(x) \) as follows,

\[ h(x) = x^n \quad \text{for} \quad x \in (x_{n+1}, x_n] \]

then, clearly, \( h(x) \) satisfies the conditions of the theorem.

Let \( \{A_n\} \) be a sequence of integers such that \( \sum A_n^{-1} \) is convergent and \( A_n \geq 2 \) for all \( n \). We now inductively define a sequence \( \{t_n\} \) of real numbers; choose \( t_0 \) to be an arbitrary positive number such that,

\[ t_0 < x_1 \]

and, \( t_0 \not\in \{x_n\} \)

we assume that \( t_0 \in (x_{n+1}, x_n) \)

for some positive integer \( n_0 \).

We now suppose that \( t_{n_0}, \ldots, t_{n_m} \) have been defined and that,

\[ t_{n_m} \in (x_{n_m+1}, x_{n_m}) \]

for some positive integer \( n_{m+1} \).

Choose \( t_m \) as follows,

a). \quad 0 < t_m < \frac{1}{3} t_{m-1}

b). \quad c_m h(t_m) = h(t_{m-1}) \quad \text{with} \quad c_m > A_{m-1}. \]
\( t_{m-1} - 2t_m > x_{m-1} \),

and

\( t_m \notin \{x_k\} \).

Now put,

\[ \kappa_m = [e_m] \quad \text{for } m = 1, 2, \ldots \]

Let \( S(0) \) be the collection of all points of the form,

\[ (0, \ldots, 0, t_i / \sqrt{n}, 0, \ldots) \]

Put,

\[ S(0) = \text{conv} [\alpha^{(0)} + S'(0)] \]

where \( \alpha^{(0)} = 0 \) for all \( j \).

Let \( S'(1) \) be the collection of all points of the form,

\[ (0, \ldots, 0, t_i / \sqrt{n}, 0, \ldots) \]

Put,

\[ S(1, i_1) = \text{conv} [\alpha^{(i_1)} + S'(1)] \]

where \( \alpha^{(i_1)} = 0 \) for \( j \neq i_1 \)

and

\[ \alpha^{(i_1)} = t_i - t_{i_1} / \sqrt{n} \]

Then define,

\[ S(1) = \bigcup_{i_1=1}^{k_i} S(1, i_1) \]
In general, let \( S'(\omega) \) denote the set of all points of the form,

\[
(0, \ldots, 0, t_n, 0, \ldots).
\]

Put,

\[
S(n, i_1, \ldots, i_n) = \cup_{\alpha} \left[ \alpha^{(i_{n-1}, \ldots, i_1)} \right] + S'(\omega),
\]

where,

\[
\alpha^{(i_{n-1}, \ldots, i_1)} = \frac{t_0 - t_i}{\sqrt{n}},
\]

\[
\alpha^{(i_{n-1}, \ldots, i_1)} = \frac{t_i - t_{i+1}}{\sqrt{n}},
\]

\[
\alpha^{(i_{n-1}, \ldots, i_1)} = \frac{t_{n-1} - t_n}{\sqrt{n}},
\]

and, \( \alpha^{(i_{n-1}, \ldots, i_1)} = 0 \) for all other values of \( j \).

Then define,

\[
S(\omega) = \bigcup_{i=1}^{K_n} S(n, i_1, \ldots, i_n).
\]

Having defined \( S(\omega) \) for \( \omega = 0, 1, \ldots \) we need, firstly, to show that

\[
S(\omega+1) \subset S(\omega)
\]

for all integers \( \omega \). To this end, it is sufficient to prove that, for each \( \omega \),

\[
S(n, i_1, \ldots, i_{n-1}, i_n) \subset S(n-1, i_1, \ldots, i_{n-1}),
\]

for \( i_n = 1, \ldots, K_n \).

Consider a point of \( S(n, i_1, \ldots, i_n) \) of the form,
\[ \alpha (i_1, \ldots, i_n) + (0, \ldots, t_{i_n}^\alpha / \lambda, 0, \ldots) \]

where the value \( t_{i_n}^\alpha \) appears in the position \( \rho \). We can write this in the form,

\[ \lambda [\alpha (i_1, \ldots, i_{n-1}, 0, \ldots, t_{i_n}^\alpha / \lambda, 0, \ldots)] + (1-\lambda) [\alpha (i_1, \ldots, i_{n-1}, 0, \ldots, t_{i_n}^\alpha / \lambda, 0, \ldots)] \]

where the term \( t_{i_n}^\alpha \) in the first square bracket is in position \( \rho \) and that in the second square bracket in position \( k_1 + \cdots + k_{n-1} + i_n \). If we put \( \lambda = t_{i_n}^\alpha \). Thus by the convexity of \( S(n-1, i_1, \ldots, i_{n-1}) \) we have,

\[ S(n, i_1, \ldots, i_{n-1}, i_n) \subset S(n-1, i_1, \ldots, i_{n-1}) \]

for \( i_n = 1, \ldots, k_n \) as required.

Also we can see that,

\[ d[S(n, i_1, \ldots, i_n)] = t_{i_n}^\alpha \quad \text{for all } n. \]

Thus, since \( t_{i_n}^\alpha \to 0 \) as \( n \to \infty \) and because \( \ell^1 \) is complete, we can define the non-empty set,

\[ S = \bigcap_{n=0}^\infty S(n). \]

We next prove that \( S \) is compact. Let \( \{x_n\} \) be an infinite sequence of points of \( S \). We now construct a convergent subsequence, using the fact that for any integer \( n \) there are only finitely many sets \( S(n, i_1, \ldots, i_n) \).

Choose \( x_n \in \{x_n\} \) such that,

\[ x_n \in S(1, i_1) \quad \text{for some } i_1 \text{ with } 1 \leq i_1 \leq k_1. \]
and such that $S(1, i_1) \subseteq \{x_n\}$ contains infinitely many points of the sequence.

Now, assume that $x_{m-1}$ has been chosen so that,

$$x_{m-1} \subseteq S(m-1, i_1, \ldots, i_{m-1}),$$

where $1 \leq i_j \leq n_j$ for $j = 1, \ldots, m-1$.

Choose $x_m$ so that,

$$x_m \subseteq \frac{1}{2}x_n \quad x_m \neq x_{m-1},$$

and,

$$x_m \subseteq S(m, i_1, \ldots, i_{m-1}, i_m) \text{ for some } i_m \text{ with } 1 \leq i_m \leq n_m$$

and such that $S(m, i_1, \ldots, i_m)$ contains infinitely many points of the sequence $\{x_n\}$. Thus we have defined a subsequence $\{x_{n_m}\}$ of $\{x_n\}$.

Now, given any $\varepsilon > 0$, there exists a positive integer $M$ such that,

$$t_m < \varepsilon \quad \text{for all } m \geq M.$$

But, from the construction of the subsequence, there exists, for every $m \geq M$ a set $S(m, i_1, \ldots, i_m)$ such that,

$$x_{n_m} \subseteq S(m, i_1, \ldots, i_m) \quad \text{for all } j \geq m.$$

Thus,

$$\varepsilon(x_{n_m}, 2x_{n_{m+1}}) \leq t_m < \varepsilon,$$

for all $m \geq M$ and for any positive integer $i$. Thus $\{x_{n_m}\}$ is a Cauchy sequence, and, by the completeness of $\mathbb{R}$ has a limit point. Also, we
know that this limit point must be in $S$ since $S$ is, clearly, closed.

Hence $S$ is compact.

Now define,

$$T(m, i_1, \ldots, i_m) = \{ x : \delta(x, S(m, i_1, \ldots, i_m)) < \frac{a_m}{2} (x_n - t_n) \}.$$ 

Then we have the following,

- $T(m, i_1, \ldots, i_m)$ is open and convex
- $S(m, i_1, \ldots, i_m) \subset T(m, i_1, \ldots, i_m)$
- $d[T(m, i_1, \ldots, i_m)] = \frac{1}{2} a_m + \frac{1}{2} (x_n - t_n) < a_m$,

and therefore,

$$h[d[T(m, i_1, \ldots, i_m)]] = h[d[S(m, i_1, \ldots, i_m)]] = h(t_n).$$

Also we have,

$$d[T(m, i_1, \ldots, i_m)] \to 0 \quad as \quad m \to \infty,$$

and thus,

$$\mathcal{L}^k(\{S\}) \leq k_1 \cdots k_m h[d(T(m, i_1, \ldots, i_m))] \quad for \ all \ m$$

$$\leq c_1 \cdots c_m h(t_m) = h(t_0), \quad by \ (b).$$

Thus, we have shown that,

$$\mathcal{L}^k(\{S\}) < \infty.$$

Now, let $\{U_i\}_{i=1}^N$ be any covering of $S$ by a sequence of open
convex sets, we need only consider finite coverings because of the compactness of $S$.

Consider a particular set $U_i$, there exists a positive integer $m = m(i)$ such that all the points of $S \cap U_i$ belong to the same $S_{l^{-1}, i, \ldots, i_{m-1}}$ but to at least two different sets $S_{l^{-1}, i_{m}, \ldots, i_m}$.

Thus, we may assume that,

$$d(U_i) \leq t_{m-1}.$$

Now, for $i_m \neq i_0$,

$$\varepsilon \left[ S(m, i, \ldots, i_{m-1}, i_m), S(m, i, \ldots, i_{m-1}, i_m) \right] = t_{m-1} - t_m,$$

thus, we must have,

$$d(U_i) \geq t_{m-1} - t_m,$$

and so,

$$\lambda(d(U_i)) = \eta^{m-1} = \lambda(t_{m-1}), \quad \text{by } \varepsilon.$$

Hence, we may replace the set $U_i$ by the corresponding set $T(m^{-1}, i, \ldots, i_{m-1})$.

Thus, since the $U_i$ was arbitrary we may assume that any covering of $S$ consists of sets of the form $T(m^{-1}, i, \ldots, i_m)$ for finitely many values of $n$. Let $n^*$ be the largest such value of $n$, then from b), we may assume that the covering consists of the $K_1, \ldots, K_{n^*}$ sets $T(n^*, i, \ldots, i_{n^*})$.

Now,

$$K_1, \ldots, K_{n^*}, \lambda \{ \delta(T(n^*, i, \ldots, i_{n^*})) \} = K_1, \ldots, K_{n^*}, \lambda(t_{n^*})$$

$$\leq (c_1 \cdots c_{n^* - 1}) \lambda(t_{n^*})$$
But $\sum A_i^{-1}$ is convergent, thus $\prod (1 - \frac{1}{A_i})$ is convergent with product $\beta$, say, and so we have,

$$\lambda^h(S) > \beta k(t_o) > 0.$$ 

Thus, we have shown that,

$$0 < \lambda^h(S) < \infty.$$ 

Also, we note that since $\beta$ can be made arbitrarily close to one by appropriate choice of $\{A_i\}$ we can construct sets $S$ in $\mathcal{L}$ of $h$-measure arbitrarily close to any given value.

It can be seen from the proof of Theorem 28 that the only property of $\lambda(x)$ used in the construction of the set $S$ is that $\lambda(x)$ is a monotonic increasing step function. Now, if $\lambda(x)$ is any monotonic increasing continuous function we can always find a step function $H(x)$, say, such that $\lambda(x) = H(x) < 3\lambda(x)$. Hence for all sets $S$ we will have,

$$\lambda^h(S) \leq H^h(S) \leq 3\lambda^h(S).$$ 

Thus, we see that for any continuous Hausdorff measure function $\lambda(x)$, there exists a set $S$ in $\mathcal{L}$ such that,

$$0 < \lambda^h(S) < \kappa.$$
We have shown that, in Euclidean space and, in fact, in compact, finite
dimensional metric spaces, given any discontinuous Hausdorff measure
function there is a continuous measure equivalent function. The next
theorem shows us that this is not the case in the space $\mathbb{L}^n$.

Theorem 29

There exists a discontinuous Hausdorff measure function $\nu(x)$ such
that for sets in $\mathbb{L}^n$ there is no continuous measure equivalent function.

Proof

Let,

$$\gamma_n = \frac{1}{n^n}$$

for $n = 0, 1, 2, \ldots$.

Define $\nu(x)$ as follows,

$$\nu(x) = \frac{1}{(n+1)!}$$

for $x \in (x_{n+1}, x_n]$. Now, let $H'(x)$ be any continuous monotonic increasing function. Then, either,

1. $H'(\gamma_n) > \frac{1}{(n-1)!}x_n$ for infinitely many values of $n$,

or, 2. $H'(\gamma_n) \leq \frac{1}{(n-1)!}x_n$ for all large values of $n$.

Consider case 1), since $H'(x)$ is continuous, there exists a positive
real number $\tilde{r}_n$, and a subsequence $\{\gamma_{n_k}\}$ of $\{x_n\}$ such that,

$$\gamma_{n_{k+1}} < \gamma_{n_k} - \tilde{r}_n.$$
and, \[ H'(x) > \frac{1}{(n-1)! \cdot n!} \quad \text{for} \quad x \in (y_n - x_n, y_n). \]

Define the function \( H(x) \) as follows,
\[ H(x) = \frac{1}{(n-1)! \cdot n!} \quad \text{for} \quad x \in (y_n - x_n, y_n - x_{n-1}). \]

Then we have,
\[ H'(x) > H(x) \quad \text{for all} \quad x. \]

Let \( \{A_n\} \) be a given sequence of integers such that \( \sum A_n \) is convergent.

We now define a decreasing sequence \( \{t_n\} \) of positive real numbers; choose \( t_0 \) arbitrarily in the open interval \( (u_n - x_n, u_n) \) for some positive integer \( n \); assume that \( t_0, \ldots, t_{m-1} \) have been chosen and that
\[ t_{m-1} \in (u_n - x_{m-1}, u_n). \]

Choose \( t_m \) such that,
\[
\begin{align*}
& a). \quad t_m < \frac{1}{2} \cdot t_{m-1}, \\
& b). \quad c_m \cdot H(t_m) = H(t_{m-1}) \quad \text{with} \quad c_m \geq A_m, \\
& c). \quad t_{m-1} - 2 \cdot t_m > u_n - x_{m-1}, \\
& d). \quad t_m \in (u_n - x_m, u_n). 
\end{align*}
\]

Construct the set \( S \) with respect to the sequence \( \{t_n\} \) just as in the proof of Theorem 28. Again, we have,
\[ 0 < \mathcal{M}(S) < \kappa. \]

Now we also know that,
\[ \mathcal{H}_\varepsilon^h(S) \leq K_1 \ldots K_m h \left[ d \{ T(m, i_1, \ldots, i_m) \} \right] \]

for arbitrary positive real numbers \( \varepsilon \) and for all large integral values of \( m \).

Thus we have,

\[ \mathcal{H}_\varepsilon^h(S) \leq c_1 \ldots c_m h(t_m) \]

\[ = \frac{1}{N_m} \sum_{m} c_1 \ldots c_m H(t_m) = \frac{1}{N_m} H(t_0), \]

where \( c_m = \frac{1}{N_m} \).

But,

\[ N_m \to \infty \quad \text{as} \quad m \to \infty, \]

and so we have,

\[ \mathcal{H}_\varepsilon^h(S) = 0. \]

Thus, the theorem is proved for the case i); it is easy to see that an analogous proof will deal with case ii).

We have shown that if \( h(x) \) is any monotonic \( q \)-dimensional Hausdorff measure function with the property that,

\[ \frac{h(x)}{x^{q-1}} \to 0 \quad \text{as} \quad x \to 0, \]

then for any set \( S \) in \( q \)-dimensional Euclidean space and for any sequence \( \{ d_n \} \) with \( d_n \to \sigma \) as \( n \to \infty \) for some positive real number \( \sigma \), we have,

\[ \lim_{n \to \infty} \mathcal{H}_{d_n}^h(S) = \mathcal{H}_\sigma^h(S). \]
The next theorem shows that this result does not extend to the space $I^1$.

**Theorem 30**

There exists a Hausdorff measure function $\lambda(x)$, a compact set $S$ in $I^1$ and for arbitrarily small positive values of $\alpha$, a sequence $\{d_n\}$ such that $d_n \to 0$ as $n \to \infty$ with the following properties,

1. $\frac{\lambda(x)}{x^q} \to 0$ as $n \to \infty$ for all positive integers $q$,

2. $0 < \lambda^h(S) < \infty$,

and, $\lim_{n \to \infty} \lambda_{d_n}^h(S) \neq \lambda^h(S)$.

**Proof**

Let,

$$x_n = \frac{1}{d_n}$$

for all positive integers $n = 0, 1, 2, \ldots$.

Define $\lambda(x)$ as follows,

$$\lambda(x) = (x!)^{-2}$$

for $x \in [x_n, x_{n-1})$.

Thus we see that,

$$\lambda(x) > 0$$

for $x > 0$,

$$\lambda(x) \downarrow 0$$

as $x \to 0$,

and,

$$\frac{\lambda(x)}{x^q} \to 0$$

as $x \to 0$ for all positive integers $q$.

Choose $t_o$ to be an arbitrary positive number such that $t_o \in \{x_n\}$. Now
assume that \( t_0, \ldots, t_{m-1} \) have been chosen and that \( t_{m+1} = x_{n_{m+1}} \in \mathbb{R} \).

We choose \( t_m \) as follows,

i). \( t_m = x_{n_m} \in \mathbb{R} \) with \( n_m > n_{m-1} + m \),

ii). \( K_m \cdot h(t_m) = (1 + \frac{1}{n_m}) \cdot h(t_{m-1}) \) with \( K_m \) a positive integer,

iii). \( t_{m-1} - 2 < t_m < x_{n_m + 1} \).

Condition ii) can be satisfied since,

\[
\frac{(1 + \frac{1}{n_m}) \cdot h(x_{n_{m-1}})}{h(x_{n_m})} = \frac{\frac{m^2 + 1}{m^2} \cdot (\frac{n_m}{n_{m-1}})^2}{\frac{m^2}{m^2} \cdot (\frac{n_m}{n_{m-1}})^2} = \frac{m^2 + 1}{m^2} \left( \frac{n_m}{n_{m-1}} \right)^2,
\]

and this must be integral because of condition i).

We now proceed to the construction of the set \( S \).

Choose \( K_1 \) points of the form,

\[
(0, \ldots, 0, \frac{t_0}{\sqrt{t}}, 0, \ldots)
\]

where the entry \( \frac{t_0}{\sqrt{t}} \) is in position \( i \), for \( i = 1, \ldots, K_1 \). Denote these points by \( \xi(\alpha_i) \) with \( \alpha_i = 1, \ldots, K_1 \).

Now, choose \( K_2 \) points of the form,

\[
(0, \ldots, 0, \frac{t_0 - t_{i-1}}{\sqrt{t}}, 0, \ldots, 0, \left[\frac{t_0 - t_{i-1}}{\sqrt{t}}\right]^{\alpha_i}, 0, \ldots, 0, \frac{t_0}{\sqrt{t}}, 0, \ldots)
\]

where \( \frac{t_0 - t_{i-1}}{\sqrt{t}} \) is in position \( i \), \( \left[\frac{t_0 - t_{i-1}}{\sqrt{t}}\right]^{\alpha_i} \) is in position \( K_1 + i \).
and $t_i/j_i$ is in position $2K_i + i, K_i + i_2$, for $i, i_2 = 1, \ldots, K_i$.
Denote these points by $\xi(\alpha_1, \alpha_2)$ with $\alpha_1 = 1, \ldots, K_i$ and $\alpha_2 = 1, \ldots, K_i$.

We note the following facts,

$$\varphi[\xi(\alpha_1), \xi(\beta_1)] = t_o$$
for $\alpha_1 \neq \beta_1$,;

$$\varphi[\xi(\alpha_1, \alpha_2), \xi(\alpha_1, \beta_2)] = t_i$$
for $\alpha_1 = \beta_2$,

$$\varphi[\xi(\alpha_1, \alpha_2), \xi(\beta_1, \beta_2)] = t_o$$
for $\alpha_1 \neq \beta_1$,

$$\varphi[\xi(\alpha_1, \alpha_2), \xi(\alpha_1, \alpha_2)] = t_i$$
for $\alpha_1 \neq \alpha_2$,

and,

$$\varphi[\xi(\alpha_1, \alpha_2), \xi(\beta_1)] = t_o$$
for $\alpha_1 \neq \beta_1$.

Now choose $K_1, K_2, K_3$ points of the form,

$$\left(0, \ldots, 0, \frac{t^i_0 - t^i_1}{t_0}, \ldots, 0, \frac{t^i_0 - t^i_1}{t_0}, 0, \ldots, 0, \frac{t^i_0 - t^i_1}{t_0}, \ldots, 0, \frac{t^i_0 - t^i_1}{t_0}, 0, \ldots, 0, \frac{t^i_0 - t^i_1}{t_0}, \ldots\right)$$

where, $\frac{t^i_0 - t^i_1}{t_0}$ is in position $i_1$; $\left[\frac{t^i_0 - t^i_1}{t_0}\right]^{K_i}$ is in position $K_i + i_1$;

$\frac{t^i_0 - t^i_1}{t_0}$ is in position $2K_i + i, K_i + i_2$; $\left[\frac{t^i_0 - t^i_1}{t_0}\right]^{K_i}$ is in position $2K_i + i, K_i + i_2 + i_3$, and $t^i_0$ is in position

$$2(K_i + K_1, K_i + K_2 + i, K_3(K_1 + 1) + i_3),$$
for $i_1 = 1, \ldots, K_i$, $i_2 = 1, \ldots, K_i$, $i_3 = 1, 2, 3$.

Denote these points by $\xi(\alpha_1, \alpha_2, \alpha_3)$ where $\alpha_j = 1, \ldots, K_i$, $j = 1, 2, 3$.

We note the following facts,

$$\varphi[\xi(\alpha_1, \alpha_2, \alpha_3), \xi(\beta_1)] = t_o$$
for $\alpha_1 \neq \beta_1$,

$$\varphi[\xi(\alpha_1, \alpha_2, \alpha_3), \xi(\alpha_1, \beta_2)] = t_i$$
for $\alpha_1 = \beta_2$,

$$\varphi[\xi(\alpha_1, \alpha_2, \alpha_3), \xi(\alpha_1, \alpha_2, \beta_3)] = t_o$$
for $\alpha_3 = \beta_3$.
Now assume that $\mathcal{V}_1, \ldots, \mathcal{V}_{n-1}$ points $\xi(\alpha_1, \ldots, \alpha_n)$ with $\alpha_i = 1, \ldots, \mathcal{V}_i$ for $i = 1, \ldots, n-1$ have been defined. We define the $\mathcal{V}_n$ points $\xi(\alpha_1, \ldots, \alpha_n)$ with $\alpha_i = 1, \ldots, \mathcal{V}_i$ for $i = 1, \ldots, n$ in a similar manner to that described above, so that we have,

$$\phi\left[\xi(\alpha_1, \ldots, \alpha_n), \xi(\alpha_1, \ldots, \alpha_{n-1}, \beta_{n-1})\right] = t_{n-1},$$

where $\alpha_{n-1} \neq \beta_{n-1}$ for $i = 0, \ldots, n-1$,

and,

$$\phi\left[\xi(\alpha_1, \ldots, \alpha_n), \xi(\alpha_1, \ldots, \alpha_{n-1}, \beta_{n-1}, \beta_{n-2}, \ldots, \beta_n)\right] = t_{n-2},$$

where $\alpha_{n-1} \neq \beta_{n-1}$ for $i = 0, \ldots, n-1$.

Now, suppose that this selection of points has been carried out for every positive integer $n$. Then, if $\{\alpha_i\}$ is any sequence of integers with

$$1 \leq \alpha_i \leq \mathcal{V}_i$$

for $i = 1, 2, \ldots$ the corresponding sequence of points

$$\{\xi(\alpha_1, \ldots, \alpha_n)\}_{n=1,2,\ldots}$$

has the following property,

$$\phi\left[\xi(\alpha_1, \ldots, \alpha_n), \xi(\alpha_1, \ldots, \alpha_{n+r})\right] = t_n,$$

for any positive integers $n$ and $r$. Thus, since we know that $t_n \to 0$ as $n \to \infty$ we see that $\{\xi(\alpha_1, \ldots, \alpha_n)\}$ is a Cauchy sequence, and by the completeness of $\mathbb{Q}$ must converge to a point, which we denote by $\xi(\alpha_1, \alpha_2, \ldots)$

We note that,
\[ \varepsilon \left[ \varepsilon (x_1, \ldots, x_n, x_{n+1}, \ldots), \varepsilon (x_1, \ldots, x_n, y_{n+1}, \ldots) \right] = t_n, \]

if \( k_{n+1} \neq b_{n+1} \).

Now let,

\[ S = \left\{ x \in \mathbb{R}^n : x = \varepsilon (x_1, \ldots, x_n) \right\}. \]

Define,

\[ S(x_1, \ldots, x_n) = \text{conv} \{ x : x = \varepsilon (x_1, \ldots, x_n, y_{n+1}, \ldots) \}, \]

where the \( y_i \) are integers such that \( 1 \leq y_i \leq r_i \) for \( i = n+1, n+2, \ldots \).

Then,

\[ d [S(x_1, \ldots, x_n)] = t_n. \]

Define,

\[ T(x_1, \ldots, x_n) = \{ x : x \in \mathbb{R}^n, S(x_1, \ldots, x_n) < \gamma t_n \}, \]

then we have,

\[ \mathcal{L} \left[ d (T(x_1, \ldots, x_n)) \right] = \mathcal{L} \left[ d (S(x_1, \ldots, x_n)) \right]. \]

We see that \( S \) can be covered by the \( U_1, \ldots, U_n \) open convex sets \( T(x_1, \ldots, x_n) \) for any positive integer \( n \).

Take an infinite sequence \( \{ y_n \} \) of points \( y \) in \( S \) and write,

\[ y_n = \varepsilon (x^1, x^2, \ldots) \quad \text{for} \quad n = 1, 2, \ldots \]

then \( x^1 = x^2 \), say, for infinitely many values of \( n \).
Choose the subsequence \( \{x^n_i\} \) of \( \{x^n\} \) such that,

\[
x^n_i = \varepsilon (v_1, x^n_1, x^n_2, \ldots)
\]

for \( i = 1, 2, \ldots \),

then \( x^n_i = \alpha^n_i \) say, for infinitely many values of \( i \). Choose the subsequence \( \{x'^n_i\} \) of \( \{x^n_i\} \) such that,

\[
x'^n_i = \varepsilon (x'^n_1, x'^n_2, \ldots)
\]

for \( j = 1, 2, \ldots \).

In this manner we generate a convergent subsequence of \( \{x^n\} \) and hence we prove the compactness of \( S \).

Now, let \( \delta \) be an arbitrary positive number such that \( \delta \neq \{x^n_i\} \). Let \( \alpha \) be a positive number such that,

\[
t_m < \delta < t_{m-1}.
\]

Let \( \{U_i\} \) be a sequence of open convex sets, such that,

\[
d(U_i) < \delta
\]

for all \( i \),

and,

\[
S \subseteq \bigcup_{i} U_i.
\]

We want to find the lower bound of the sum \( \sum_i \lambda(d(U_i)) \) over all such sequences of sets. Firstly, because of the compactness of \( S \) we need only consider finite coverings. Now, consider a set \( U_i \in \{U_i\} \) such that,

\[
d(U_i) \in (t_m, t_{m-1})
\]

for some integer \( m \) with \( n > m \).

Now \( U_i \) can only contain points of \( S \) which lie in the same set \( T(v_1, \ldots, v_n) \).

But, we know that \( \lambda(d(U_i)) \) is greater than or equal to \( \lambda(d(T(v_1, \ldots, v_n))) \),

and so we may replace the \( U_i \) by the corresponding set \( T(v_1, \ldots, v_n) \).
Thus we need only consider coverings consisting of sets of the form
\( \mathcal{T}(\alpha_1, \ldots, \alpha_n) \) for finitely many values of \( n \). Let \( N \) be the greatest value of these integers \( n \). Then there must be \( \rho K_N \) sets \( \mathcal{T}(\alpha_1, \ldots, \alpha_N) \) where \( \rho \) is an integer, assuming none of the covering sets is redundant. Now,

\[
K_N \left[ d \left\{ \mathcal{T}(\alpha_1, \ldots, \alpha_N) \right\} \right] = K_N h \left[ d \left\{ SM(\alpha_1, \ldots, \alpha_N) \right\} \right]
\]

\[
= K_N h(t_N)
\]

\[
= (1 + \frac{1}{N}) h(t_{N-1}) \quad \text{by ii).}
\]

\[
= (1 + \frac{1}{N}) h \left[ d \left\{ \mathcal{T}(\alpha_1, \ldots, \alpha_{N-1}) \right\} \right].
\]

Thus, we should replace each block of \( K_N \) sets \( \mathcal{T}(\alpha_1, \ldots, \alpha_N) \) by the single set \( \mathcal{T}(\alpha_1, \ldots, \alpha_{N-1}) \) which they all intersect. Continuing in this manner we eventually get,

\[
\mathcal{N}_d^h (S) = K_1 \ldots K_m h \left[ d \left\{ \mathcal{T}(\alpha_1, \ldots, \alpha_N) \right\} \right]
\]

\[
= K_1 \ldots K_m h(t_m)
\]

\[
= \prod_{i=1}^{m} \left(1 + \frac{1}{i \nu} \right) h(t_0).
\]

Hence,

\[
\mathcal{N}_d^h (S) = \lim_{n \to \infty} \prod_{i=1}^{m} \left(1 + \frac{1}{i \nu} \right) h(t_0),
\]

which is positive and finite.

Now, consider \( d = t_N \) for some integer \( N \), since we are only interested in sets of diameter less than \( t_N \) we get,
Thus we have,

$$\sum_{j}^{k}(S) = \prod_{i=1}^{N+1} \ln(t_{\theta(i)})$$

Now, let \( \{d_{n}\} \) be any strictly decreasing sequence such that,

$$d_{n} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$ 

Then for all large values of \( n \) we have,

$$\sum_{d_{n}}^{k}(S) = \prod_{i=1}^{N} (1 + \frac{1}{c_{i}}) \ln(t_{\theta(i)})$$

That is,

$$\lim_{n \rightarrow \infty} \sum_{d_{n}}^{k}(S) = \prod_{i=1}^{N} (1 + \frac{1}{c_{i}}) \ln(t_{\theta(i)}) = \frac{1}{1 + (NH)^{2}} \sum_{d}^{k}(S) \neq \sum_{d}^{k}(S).$$

Hence the theorem is proved.
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APPENDIX 1

Let $C$ be a convex set of diameter $d$ in $q$-dimensional Euclidean space. Denote by $C'$ the set $\{x: \varrho(x, C') > \frac{d}{q}\}$ for some small $\delta > 0$, where $\varrho$ is the metric in the space.

Assertion

We can cover $C \setminus C'$ with $K \left(\frac{d}{\delta}\right)^2$ cubes of diameter $\delta$, where $K$ is a constant dependent only on $q$.

Proof

Let $P, Q$ be polytopes such that,

$P \subset C$ and \( \{ x: \varrho(x, C) < \frac{d}{q}\} \subset P \),

$C' \subset Q$ and \( \{ x: \varrho(x, Q) < \frac{d}{q}\} \subset C' \).

Then,

\[ P \cap Q \supset \{ x: \varrho(x, Q) > 2\delta \} \]

Since, if $p \in \{ x: \varrho(x, Q) > 2\delta \}$ and $x \in C$, then there exists $q \in \partial P$ such that,

\[ \varrho(x, q) < \frac{d}{q} \]

So we have,

\[ \varrho(p, x) + \varrho(x, q) \geq \varrho(p, q) > 2\delta \]

that is,

\[ \varrho(p, x) > 2\delta - \frac{d}{q} = 7\delta \frac{1}{4}. \]
Hence $p \in C'$.

Now, if $p \in C' \setminus Q$ then there exists $s \in C \setminus C'$ such that,

$$\varrho(p, s) < \frac{\delta}{4}.$$  

Also, there exists $x \in C \setminus C$ such that $\varrho(s, x) \leq \delta$ and hence,

$$\varrho(p, x) < \frac{\delta}{4} + \delta = \frac{5\delta}{4},$$

that is, $p \not\in C' \setminus Q$ and therefore $p \in Q$.

Hence $\{x : \varrho(x, P) > 2\delta \} \subseteq Q$ and clearly $Q < \mathcal{P}$.

Now let $x \in C \setminus C'$ then there exists at least one point $q$ on the frontier of $\mathcal{P}$ such that,

$$\varrho(p, x) = \min \{\ell(q, x) : q \in \text{frontier of } \mathcal{P}\}.$$  

Let $s = \varrho(p, x)$ then $s > 0$, since $x$ is an interior point of $\mathcal{P}$. Then from the definition of $p$, we have $S(x, s) \subseteq \mathcal{P}$. Also, since $p$ lies on the frontier of $\mathcal{P}$ there is a support hyperplane $H$ of $\mathcal{P}$ through $p$. Clearly $H$ must also support $S(x, s)$ and hence $H$ is the unique support hyperplane through $p$. Further, we see from this argument that if we erect a right-cylinder of height $2\delta$ on each facet of $\mathcal{P}$ we will have a covering of $C \setminus C'$.

Now let $\{V_i\}_{i=1}^N$ be a finite covering of $C \setminus C'$ by disjoint cubes $V_i$ each of diameter $\delta$. Then we must have,

$$\bigcup_{i=1}^N V_i < \{x : \varrho(x, C) \leq 3 \delta \setminus \{x : \varrho(x, C') > 3 \delta \}.$$  

So, by a similar argument to the one above we have,

$$N \left(\frac{\delta}{\delta^2}\right)^q \leq a_{\alpha} (\alpha d)^{q-\alpha-6} \delta$$
where, \( a_q \) is the surface area of the unit \( q \)-dimensional sphere.

Hence we may choose \( K = 6a_q2^{q-1}\sqrt{a_q} \) and the assertion is proved.
Correction to Pages 79 and 80.

To prove that \( \int_{\Delta}^{\infty} L_n \left( \bigcup_{n=1}^{\infty} S_n \right) = \frac{3}{2} x_N \).

It is clear that we can cover \( \bigcup_{n=1}^{\infty} S_n \) by its own closure. We now assume that all the sets of the covering have diameter strictly less than \( \delta \). Thus, let \( \{ U_i \} \) be a closed covering of \( \bigcup_{n=1}^{\infty} S_n \) such that,

\[
d(U_i^\delta) < \delta \quad \text{for all } i.
\]

If there is more than one \( U_i \) such that \( d(U_i) > x_{N+1} \) then clearly,

\[
\sum_i d(U_i) \geq 2x_N.
\]

Now assume that there is at most one such \( U_i \) then,

\[
d(U_i) < \delta - \epsilon \quad \text{for all } i \quad \text{and for all small } \epsilon > 0.
\]

Choose one such \( \epsilon \), let \( \{ U_i \} \) be a subsequence of \( \{ U_i \} \) such that each \( U_i \) has at least one point in common with the circle \( x^2 + y^2 = \frac{1}{4}(\delta - \epsilon)^2 \).

Clearly, no \( U_i \) can contain diametrically opposite points of this circle. Let the intersection of \( U_i \) with the circle subtend an angle \( 2\phi \) at the origin then,

\[
\sin \phi_i = \frac{d(U_i)}{x_N - \epsilon} \quad \text{and} \quad 0 \leq \phi_i < \frac{\pi}{2}.
\]
Since the circle must be covered we have,

\[ \sum 2\phi_i \geq 2\pi \]

and using the fact that \( \sin \phi_i \geq \frac{2\phi_i}{\pi} \), we get,

\[ \sum d(U_i) \geq \sum d(U_{n_i}) > \frac{(x_N - \epsilon)}{\pi} \sum 2\phi_i \geq 2(x_N - \epsilon) \]

But this is true for all small values of \( \epsilon \), so that,

\[ \sum d(U_i) \geq 2x_N \cdot \]

Hence, since \( h(x) \geq x \) for all \( x \), we must have,

\[ L^h_0 \left( \bigcup S_n \right) = \frac{3}{2} x_N \]