PRESENTATIONS OF GENERAL LINEAR GROUPS

John Richard Silvester

ABSTRACT

Let R be an associative ring with a 1. Denote by $GL_n(R)$ the group of invertible $n \times n$ matrices over R, and by $GE_n(R)$ the subgroup of $GL_n(R)$ generated by the elementary and invertible diagonal matrices. Certain specified relations between these generators hold universally, that is, for any ring R. We call a ring R universal for GE_n if $GE_n(R)$ has these relations as defining relations, and we shew that if R is a local ring (i.e. a ring in which the set of all non-units is an ideal) or the ring of rational integers, then R is universal for GE_n, for all n. This both generalizes known results for n=2, and includes the classical case where R is a field, possibly skew.

By adding further relations to those already considered we obtain in a similar way the concept 'quasi-universal for GE_n', giving a class of rings which strictly includes the class of all rings universal for GE_n, but which is better behaved than the latter under certain ring constructions. We shew that every semi-local ring (i.e. every ring R such that R modulo its Jacobson radical has the minimum condition on right ideals) is quasi-universal for GE_n, for all n.

Finally we shew how to obtain a presentation of $GE_n(R)$ for any R. This is unwieldy, but simplifies greatly for a certain class of rings called GE_2-reducible rings, which includes all Euclidean rings. We shew that for such rings R a set of defining relations for $GE_n(R)$, for $n \geq 3$, is obtained by taking the universal relations together with certain relations in $GE_3(R)$.

CONTENTS

ACKNOWLEDGEMENTS

I should like to thank Professor P.M.Cohn for much wise counsel and constant encouragement. I should also like to thank the following, who provided grants: The Science Research Council, The United States-United Kingdom Educational Commission, and Rutgers, The State University.

John R. Silvester

Bedford College, London

Queen Mary College, London

Rutgers, The State University

19th May 1969

1. Introduction

The general linear groups $GL_n(K)$ (where K is a field) and their subgroups and automorphisms have received much attention, and the theory is well-developed, even when K is skew. Comparatively little is known about the groups $GL_n(R)$ of invertible $n \times n$ matrices over an arbitrary ring R. In [5] the question of finite generation of $GL_n(R)$ is investigated for certain types of Dedekind domain R. In [6], all the automorphisms of $GL_n(R)$, $n \geq 3$, are found, where R is any integral domain; this is all the more remarkable in view of the fact that not every integral domain is a GE-ring (see below). In [7], certain characteristic subgroups and isomorphisms of subgroups of $GL_n(R)$, $n \geq 3$, are studied; in spite of the very general title, the rings are all integral domains, or even principal ideal domains, and with characteristic $\neq 2$.

In [1], the structure of $GL_2(R)$ for quite wide classes of rings R is examined, and here we attempt to follow the same line of investigation for $GL_n(R)$, for general n. The main tool in [1] was a presentation of $GL_2(R)$ for certain rings called universal GE_2-rings. Finding a presentation of $GL_n(R)$, for $n \geq 3$, is so much more difficult that the present work is confined almost exclusively to that task.

Over a field, every invertible matrix is a product of elementary and diagonal matrices. Over a ring, this need not be true; indeed, we define $GE_n(R)$ to be the subgroup of $GL_n(R)$ generated by the elementary and invertible diagonal $n \times n$ matrices, and we say R is a GE_n-ring if $GE_n(R) = GL_n(R)$. A ring R is universal for GE_2 if $GE_2(R)$ has a certain presentation (see chapter 2); if R is also a GE_2-ring, it is a universal GE_2-ring.

In [1], it was shewn that local rings (in particular, fields) are universal for GE_2; our main result in chapter 3 is a presentation of $GE_n(R)$ for any local ring R and any n, and it seems natural to take this as the basis for the definition of 'universal for GE_n'. With the help of results in [3] and [4] it is then shewn that the ring Z of rational integers is universal for GE_n.

If R is universal for GE_{nm}, this tells us something about the structure of $GE_n(R_m)$, where R_m is the ring of $m \times m$ matrices over R. In particular, we can ask: if R is universal for GE_{nm}, is R_m universal for GE_n? With certain restrictions, the answer is yes. The restrictions can be removed by considering instead a wider class of rings called quasi-universal for GE_n. We shew (chapter 4) that every semi-simple Artinian ring is quasi-universal for GE_n, and we give a simple sufficient condition for such a ring to be universal for GE_n.

Let J(R) be the Jacobson radical of R. The structure of $GE_n(R/J(R))$ is closely related to that of $GE_n(R)$; indeed we prove (chapter 4) that if R/J(R) is quasi-universal for GE_n, so is R. Thus every semi-local ring is quasi-universal for GE_n, and as before there is a simple sufficient condition for such a ring to be universal for GE_n.

We conclude (chapter 6) by giving a presentation of $GE_n(R)$ for any R whatsoever; it is however a rather clumsy presentation. Nonetheless it simplifies greatly for certain rings, in particular for Euclidean rings.

Notation. The following notation is used throughout.
Let R be a ring, associative and with a 1, and denote by U(R) the multiplicative group of units of R. Elements of U(R) are denoted by Greek letters.
Let R_n be the ring of $n \times n$ matrices over R. R_n has identity I_n, and its group of units is the general linear group $GL_n(R)$.
Let e_{ij} be the usual 'matrix units' (1 in the i,j position, 0 elsewhere). For $i \neq j$ and $x \in R$, put $B_{ij}(x) = I_n + xe_{ij}$. Clearly $B_{ij}(x) \in GL_n(R)$. Put $[\alpha]_i = I_n + (\alpha-1)e_{ii}$ = the diagonal matrix with α in the ith diagonal place and 1 elsewhere.
Put $[\alpha,\beta]_{ij} = [\alpha]_i[\beta]_j$, $D_{ij}(\alpha) = [\alpha,\alpha^{-1}]_{ij}$ and $[\alpha_1,\alpha_2,\ldots \alpha_n] = \Pi_i [\alpha_i]_i$.
Define $GE_n(R)$ as the subgroup of $GL_n(R)$ generated by all $[\alpha]_i$ and all $B_{jk}(x)$ ($\alpha \in U(R)$, $x \in R$, $1 \leq i,j,k \leq n$, $j \neq k$).
If $GE_n(R) = GL_n(R)$ we say R is a GE_n-ring. R is a GE-ring if it is a GE_n-ring for all n. GE stands for 'Generalized Euclidean': note that every Euclidean ring is a GE-ring.

2. Universal GE_2-rings

In $GE_2(R)$, put $E(x) = B_{12}(1-x)B_{21}(-1)B_{12}(1)$

$$= \begin{pmatrix} x & 1 \\ -1 & 0 \end{pmatrix}$$

$GE_2(R)$ is generated by all the $E(x)$ and $[\alpha,\beta]$ ($x \in R$, $\alpha,\beta \in U(R)$). Then the following 'universal' relations always hold:

$$\text{(A)} \quad \left| \begin{array}{ll} E(x)E(0)E(y) = -E(x+y) & x,y \in R \\ E(\alpha)E(\alpha^{-1})E(\alpha) = -D_{12}(\alpha) & \alpha \in U(R) \\ E(x)[\alpha,\beta] = [\beta,\alpha]E(\beta^{-1}x\alpha) & \alpha,\beta \in U(R),\ x \in R \\ [\alpha_1,\alpha_2][\beta_1,\beta_2] = [\alpha_1\beta_1,\alpha_2\beta_2] & \alpha_i,\beta_i \in U(R) \end{array} \right.$$

Following [1], we say that R is <u>universal for GE_2</u> if (A) is a complete set of defining relations for $GE_2(R)$. If in addition R is a GE_2-ring, we say that R is a <u>universal GE_2-ring</u>. In this case, (A) is a complete set of defining relations for $GL_2(R)$.

In [1], the following rings are shewn to be universal for GE_2:

a. Local rings (in particular, fields).
b. Discretely normed rings (in particular, k-rings with a degree-function).
c. Discretely ordered rings (in particular, the ring Z of rational integers).

Since a local ring (i.e. a ring in which the non-units form an ideal) is a GE-ring, it is a universal GE_2-ring.

Our first question is: do any rings fail to be universal for GE_2? Corollary (2.3) (below) answers this in the affirmative.

<u>Lemma (2.1)</u>. In any ring R, $E(a)E(b)E(-a)E(-b) = I$

$$\iff ab = 0 = ba \quad (a,b \in R).$$

<u>Proof</u>. For any $a,b \in R$, $E(a)E(b)E(-a)E(-b)$

$$= \begin{bmatrix} abab-ab+1 & -aba \\ -bab & ba+1 \end{bmatrix}$$

The result is now clear. □

(Note: the symbol □ will be used to indicate the conclusion of a proof.)

Definition: Let R,S be rings. A <u>U-homomorphism</u> $f:R \to S$ is a homomorphism of the additive group of R into the additive group of S such that

$$f(1) = 1$$

$$\text{and} \quad f(\alpha x \beta) = f(\alpha)f(x)f(\beta) \qquad x \in R,\ \alpha,\beta \in U(R).$$

The following theorem is proved in [1:(11.2)]: Given R,S,f as above, with R universal for GE_2, then f induces a homomorphism $f^*:GE_2(R) \to GE_2(S)$ by the rules:

$$f^*(E(x)) = E(f(x)) \qquad x\in R$$

$$f^*([\alpha,\beta]) = [f(\alpha),f(\beta)] \qquad \alpha,\beta\in U(R)$$

Proposition (2.2). If R,S are rings, and R is universal for GE_2, and if $f:R \to S$ is a U-homomorphism, then $xy = 0 = yx$ $(x,y\in R) \Rightarrow f(x)f(y) = 0$.

Proof. Construct the homomorphism f^*, as above. By (2.1), if $xy = 0 = yx$, then

$$E(x)E(y)E(-x)E(-y) = I$$

Apply f^* to each side.

$$E(f(x))E(f(y))E(-f(x))E(-f(y)) = I$$

By (2.1), $f(x)f(y) = 0$. □

Corollary (2.3). Let k be a field, and let R be the ring formed by adjoining to k two commuting indeterminates x,y with the added relation $xy = 0$. Then R is not universal for GE_2.

Proof. A normal form for a general element t of R is

$$t = xf(x)+yg(y)+a \quad (f(x)\in k[x],\ g(y)\in k[y],\ a\in k).$$

Then
$$\begin{aligned} tt_1 &= (xf(x)+yg(y)+a)(xf_1(x)+yg_1(y)+a_1) \\ &= x(xf(x)f_1(x)+f(x)a_1+af_1(x)) \\ &\quad + y(yg(y)g_1(y)+g(y)a_1+ag_1(y)) + aa_1 \end{aligned}$$

If $tt_1 = 1$ we must have

(i) $aa_1 = 1$
(ii) $xf(x)f_1(x)+f(x)a_1+af_1(x) = 0$
(iii) $yg(y)g_1(y)+g(y)a_1+ag_1(y) = 0$

(ii) is an equation in $k[x]$, and by examining the degrees of the three terms,we see that $xf(x)f_1(x) = 0$, and so one of $f(x)$, $f_1(x)$ must be zero; hence both are zero. Similarly $g(y)$, $g_1(y)$ are both zero.

So $U(R) = k^*$, the non-zero elements of k. Now put $S = k[x]$. R,S are both free k-modules of countably infinite rank. Define $f:R \to S$ by

$$f(x^n) = x^{2n} \quad (n \geqslant 0) \quad (\text{and so } f(1) = 1)$$

$$f(y^m) = x^{2m-1} \quad (m > 0)$$

and extend by linearity. f is an isomorphism of k-modules, and since $f(1) = 1$ and $U(R) = k^*$, f is a U-homomorphism. But $xy = 0 = yx$, and $f(x)f(y) = x^2.x = x^3 \neq 0$. By (2.2), R is not universal for GE_2. □

Proposition (2.4). R as in (2.3). Then R is a GE-ring.

Proof. Let $A\in GL_n(R)$. Then $A = A_0+xA_1+yA_2$, where $A_0\in k_n$, $A_1\in k[x]_n$, and $A_2\in k[y]_n$.

Now $\det(A)\in U(R)=k^*$. Therefore $\det(A)=\det(A_0+yA_2)$ (i.e. replacing x by 0 does not affect the value of $\det(A)$). So $A_0+yA_2 \in GL_n(R)$.

Put $B = A(A_0+yA_2)^{-1}$. Then $B = B_0+xB_1+yB_2$, where $B_0\in k_n$, $B_1\in k[x]_n$, $B_2\in k[y]_n$.

$$B_0+yB_2 = A(A_0+yA_2)^{-1}\Big|_{x=0} = I.$$

Therefore $B_2=0$ and $B=B_0+xB_1$.
So $A = (B_0+xB_1)(A_0+yA_2)$

$$\begin{aligned} &\in GL_n(k[x])GL_n(k[y]) \\ &\subseteq GE_n(k[x])GE_n(k[y]) \\ &\subseteq GE_n(R). \quad \square \end{aligned}$$

We note in passing the rather special structure of the group $GE_n(R)$:

$$GE_n(R) = GE_n(k[x])GE_n(k[y])$$

$$GE_n(k[x])\cap GE_n(k[y]) = GE_n(k)$$

R itself is a sort of direct product of $k[x]$ and $k[y]$, amalgamating k.

Thus we have found a fairly easy example of a GE-ring R which is not universal for GE_2. R contains zero-divisors; we proceed to find a ring which is a GE-ring and an integral domain (it is even a principal ideal domain) but which is not universal for GE_2. The method is a generalization of the above.

<u>Lemma (2.5)</u>. Let R be any ring; a_i $(i=1..4)\in R$ and $\alpha,\beta\in U(R)$.
Then (*) $E(a_1)E(a_2)E(a_3)E(a_4) = [\alpha,\beta^{-1}]$

$$\Longleftrightarrow \left|\begin{array}{lll} \text{(i)} & a_1a_2 &= 1-\alpha \\ \text{(ii)} & a_2a_1 &= 1-\beta \\ \text{(iii)} & a_3 &= -a_1\beta^{-1} \\ \text{(iv)} & a_4 &= -\beta a_2 \end{array}\right.$$

<u>Proof</u>. (*) is true $\Longleftrightarrow$
$$\left|\begin{array}{ll} a_1a_2a_3a_4-a_1a_2-a_3a_4-a_1a_4+1 = \alpha & \text{(a)} \\ a_1a_2a_3-a_1-a_3 = 0 & \text{(b)} \\ a_2a_3a_4-a_2-a_4 = 0 & \text{(c)} \\ a_2a_3-1 = -\beta^{-1} & \text{(d)} \end{array}\right.$$

Suppose (a)-(d) hold. From (d) $a_2a_3 = 1-\beta^{-1}$.
In (c) $(1-\beta^{-1})a_4-a_2-a_4 = 0$, i.e. $a_2+\beta^{-1}a_4 = 0$, whence (iv).
In (b) $a_1(1-\beta^{-1})-a_1-a_3 = 0$, i.e. $a_1\beta^{-1}+a_3 = 0$, whence (iii).
From (iii), $a_2a_1 = -a_2a_3\beta = (\beta^{-1}-1)\beta = 1-\beta$, i.e. (ii).
From (iii) and (iv), $a_3a_4 = a_1a_2$,
and from (iv), $a_1a_4 = -a_1\beta a_2$.

In (a), $a_1a_2a_1a_2-2a_1a_2+a_1\beta a_2 = \alpha-1$
i.e. $a_1(a_2a_1-2+\beta)a_2 = \alpha-1$.

From (ii), $a_1(1-\beta-2+\beta)a_2 = \alpha-1$
i.e. $a_1a_2 = 1-\alpha$, so (i) holds.

Conversely, if (i)-(iv) hold,

then $a_2a_3 = -a_2a_1\beta^{-1}$ by (iii)

$= -(1-\beta)\beta^{-1}$ by (ii)

$= 1-\beta^{-1}$, so (d) holds.

So $a_1a_2a_3 = a_1-a_1\beta^{-1} = a_1+a_3$, whence (b).

Also $a_2a_3a_4 = a_4-\beta^{-1}a_4 = a_4+a_2$, whence (c).

Then $a_1a_2a_3a_4-a_1a_2-a_3a_4-a_1a_4 = a_1(1-\beta^{-1})a_4+a_1\beta^{-1}a_4+a_1\beta^{-1}a_4-a_1a_4$

$= a_1\beta^{-1}a_4$

$= -a_1a_2$

$= \alpha-1$, whence (a). □

<u>Proposition (2.6)</u>. Let R,S be rings, where R is universal for GE_2, and let $f:R \to S$ be a U-homomorphism. Then if $\exists\ a_1,a_2\in R$ and $\alpha\in U(R)$ such that $a_1a_2 = 1-\alpha$, we deduce $f(a_1a_2) = f(a_1)f(a_2)$.

<u>Proof</u>. Define β by $a_2a_1 = 1-\beta$.

β is a unit, since $(1-a_2a_1)(1+a_2\alpha^{-1}a_1)$

$= 1-a_2a_1+a_2(1-a_1a_2)\alpha^{-1}a_1 = 1$

and $(1+a_2\alpha^{-1}a_1)(1-a_2a_1)$

$= 1-a_2a_1+a_2\alpha^{-1}(1-a_1a_2)a_1 = 1$

Construct the homomorphism $f^*:GE_2(R) \to GE_2(S)$ as before.

By (2.5), $E(a_1)E(a_2)E(-a_1\beta^{-1})E(-\beta a_2) = [\alpha,\beta^{-1}]$.

Apply f* to each side:

$E(f(a_1))E(f(a_2))E(-f(a_1)f(\beta)^{-1})E(-f(\beta)f(a_2))$

$= [f(\alpha),f(\beta)^{-1}]$.

From (2.5), $f(a_1)f(a_2) = 1-f(\alpha) = f(1-\alpha) = f(a_1a_2)$. □

<u>Corollary (2.7)</u>. Let k be a field not containing a square root of −1. Let x be an indeterminate, and $X = \{(1+x^2)^n,\ n=0,1,2,\ldots\}$. Let R be the localization $k[x]_X$: then R is a commutative integral domain, and is not universal for GE_2.

<u>Proof</u>. $U(R) = \{a(1+x^2)^n;\ a\in k^*,\ n\in Z\}$.

For if $(1+x^2)^n p(x).(1+x^2)^m q(x) = 1$, where $p(x),q(x)\in k[x]$ and are not divisible by $1+x^2$, then $n+m \leq 0$, since $U(k[x]) = k^*$. If $n+m < 0$ then $1+x^2 \mid p(x)q(x)$ in $k[x]$: but $1+x^2$ is a prime of $k[x]$. Therefore $n+m = 0$, and so $p(x)q(x) = 1$ and $p(x),q(x)\in k^*$.

Let y,z be commuting indeterminates, and put $Y = \{(1+y)^n,\ n=0,1,2,\ldots\}$. Let S be the localization $k[y,z]_Y$.

Define $f:R \to S$ by $\left| \begin{array}{l} f(x^{2n}) = y^n \\ f(x^{2n+1}) = y^n z \end{array} \right.$

and extend in the obvious way: thus if $r \in R$ we can write

$$r = (1+x^2)^n (p(x^2)+q(x^2).x)$$

and then $f(r) = (1+y)^n (p(y)+q(y).z)$.

Now the restriction of f to $k[x^2]_X$ is clearly an additive homomorphism, and we then have

$$f(ax+b) = f(a)z+f(b) \quad (a,b \in k[x^2]_X)$$

whence f itself is an additive homomorphism. Further, $f(1) = 1$ and $f(\alpha r) = f(\alpha)f(r)$ $(\alpha \in U(R),\ r \in R)$, whence, since R and S are commutative, f is a U-homomorphism.

Now put $a_1=x$, $a_2=-x$, $\alpha=1+x^2 \in U(R)$. Then $a_1 a_2 = 1-\alpha$.

By (2.6), if R is universal for GE_2, $f(a_1 a_2) = f(a_1)f(a_2)$.

But $f(a_1 a_2) = f(-x^2) = -y$, and $f(a_1)f(a_2) = -z^2$.

Therefore R is not universal for GE_2. □

Note that since R is a localization of k[x], which is both a principal ideal domain and a GE-ring, R itself is a principal ideal domain and a GE-ring. Other examples of such rings have been found by P.M.Cohn in [2]:

<u>Corollary (2.8)</u>. (P.M.Cohn) The ring R of integers in $Q(\sqrt{-2})$ (Q=rationals) is not universal for GE_2.

N.B. A similar result holds for the rings of integers in $Q(\sqrt{-7})$ and $Q(\sqrt{-11})$.

<u>Proof</u>. $U(R) = \{\pm 1\}$, so since the map $f: a+b\sqrt{-2} \mapsto a$ is additive, it is a U-homomorphism.

But if $a_1 = \sqrt{-2} = -a_2$, we have $a_1 a_2 = 1-\alpha$, where $\alpha = -1$.

Then $f(a_1) = 0 = f(a_2)$, but $f(a_1 a_2) = 2$ since $a_1 a_2 = 2$, and so by (2.6), R is not universal for GE_2. □

Note that this R is a Euclidean ring.

Thus we see that a ring need not be pathological in order to fail to be universal for GE_2. It is natural to ask: what other relations can be added to the relations (A) so as to give a complete set of defining relations for a wider class of rings? An answer sufficient to cover (2.8) is given in [2]: namely that the extra 'universal' relation

$$\{E(a)E(b)\}^m_r = -I \quad (\text{all } a,b \in R \mid ab=m=ba,\ m=2 \text{ or } 3)$$

gives, with (A), a set of defining relations for $GE_2(R)$, where R is the ring of integers in $Q(\sqrt{-d})$, d = 2, 7 or 11. (The same paper shews that when d = 1 or 3 the corresponding R is universal for GE_2. The other values of d are covered in [1].) A second way of widening the class of rings is the subject of chapter 4, where, however, we are dealing with $GE_n(R)$. Chapter 3 is concerned with the formulation of the definition of 'universal for GE_n'; in preparation for this we prove the following:

Proposition (2.9). A ring R is universal for GE_2 iff $GE_2(R)$ has the following presentation:

Generators: $B_{ij}(x)$, $[\alpha,\beta]$ $(x\in R,\ \alpha,\beta\in U(R),\ 1\leq i,j\leq 2,\ i\neq j)$

Relations:

1. $B_{ij}(x)B_{ij}(y) = B_{ij}(x+y)$ $(x,y\in R)$
2. $B_{ij}(\alpha-1)B_{ji}(1) = D_{ij}(\alpha)B_{ji}(\alpha)B_{ij}(1-\alpha^{-1})$ $(\alpha\in U(R))$
3. $B_{ij}(x) = B_{ji}(-1)B_{ij}(1)B_{ji}(-x)B_{ij}(-1)B_{ji}(1)$ $(x\in R)$
4. $B_{ij}(x)[\alpha_1,\alpha_2] = [\alpha_1,\alpha_2]B_{ij}(\alpha_i^{-1}x\alpha_j)$ $(x\in R,\ \alpha_k\in U(R))$
5. $[\alpha_1,\alpha_2][\beta_1,\beta_2] = [\alpha_1\beta_1,\alpha_2\beta_2]$ $(\alpha_i,\beta_i\in U(R))$

Proof. We have

$$E(x) = B_{12}(1-x)B_{21}(-1)B_{12}(1) \qquad (a)$$

and then $$B_{12}(x) = E(-x)E(0)^{-1} \qquad (b)$$

and $$B_{21}(x) = E(0)^{-1}E(x) \qquad (c)$$

Now suppose R is universal for GE_2. We shew first that the above relations 1.-5. (which are true in any ring) imply the universal relations (A) (page 6), using the definitions (a), (b) and (c).

$$\begin{aligned} E(0)^2 &= B_{12}(1)B_{21}(-1)\{B_{12}(2)B_{21}(-1)\}B_{12}(1) && \text{by 1.} \\ &= B_{12}(1)B_{21}(-1)\{-B_{21}(1)B_{12}(-2)\}B_{12}(1) && \text{by 2. and 1.} \\ &= -B_{12}(1)B_{12}(-2)B_{12}(1) && \text{by 1.} \end{aligned}$$

$\therefore$ $E(0)^2 = -I$ by 1.

$$\begin{aligned} \text{Thus } E(x)E(0)E(y) &= B_{12}(-x)E(0)^2B_{12}(-y)E(0) && \text{by (b)} \\ &= -B_{12}(-x-y)E(0) && \text{by 1.} \\ &= -E(x+y) && \text{by (b).} \end{aligned}$$

From 2. we get a similar relation

6. $B_{ij}(1-\alpha)B_{ji}(-1) = D_{ij}(\alpha)B_{ji}(-\alpha)B_{ij}(\alpha^{-1}-1)$

(write $[-1]_iB_{ij}(1-\alpha)B_{ji}(-1) = B_{ij}(\alpha-1)B_{ji}(1)[-1]_i$ (by 4.) and use 2.)

$$\begin{aligned} \text{Then } E(\alpha) &= B_{12}(1-\alpha)B_{21}(-1)B_{12}(1) \\ &= D_{12}(\alpha)B_{21}(-\alpha)B_{12}(\alpha^{-1}) && \text{by 6. and 1.} \end{aligned}$$

So $E(\alpha)E(\alpha^{-1})E(\alpha)$

$= D_{12}(\alpha)B_{21}(-\alpha)B_{12}(\alpha^{-1})D_{12}(\alpha^{-1})B_{21}(-\alpha^{-1})B_{12}(\alpha)$
$\quad .D_{12}(\alpha)B_{21}(-\alpha)B_{12}(\alpha^{-1})$

$= D_{12}(\alpha)\{B_{21}(-\alpha)B_{12}(\alpha^{-1})\}^3$ by 4. and 5.

Now $\{B_{21}(-1)B_{12}(1)\}^3$

$= B_{21}(-1)B_{12}(1)B_{21}(-1)B_{12}(-1)B_{21}(1)$
$\quad .B_{21}(-1)B_{12}(2)B_{21}(-1)B_{12}(1)$ by 1.

$= B_{12}(1)B_{21}(-1)D_{12}(-1)B_{21}(1)B_{12}(-2)B_{12}(1)$ by 3. and 6.

$= -I$ by 1. and 4.

So $\{B_{21}(-\alpha)B_{12}(\alpha^{-1})\}^3 = [\alpha]_1\{B_{21}(-1)B_{12}(1)\}^3[\alpha^{-1}]_1$ by 4,5.

$= [\alpha]_1(-I)[\alpha^{-1}]_1 = -I$

$\therefore$ $E(\alpha)E(\alpha^{-1})E(\alpha) = -D_{12}(\alpha)$ is a consequence of 1.-5.

Then $B_{12}(\alpha)B_{21}(-\alpha^{-1})\{B_{12}(\alpha-1)B_{21}(1)\}B_{12}(-1)$

$= D_{12}(\alpha)B_{12}(\alpha^{-1})B_{21}(-\alpha)B_{21}(\alpha)B_{12}(-\alpha^{-1})$ by 2,5.

$= D_{12}(\alpha)$ by 1.

Replace α by $\alpha^{-1}\beta$:

$\{B_{12}(\alpha^{-1}\beta)B_{21}(-\beta^{-1}\alpha)B_{12}(\alpha^{-1}\beta)\}\{B_{12}(-1)B_{21}(1)B_{12}(-1)\}$

$= D_{12}(\alpha^{-1}\beta) = [\alpha,\beta]^{-1}[\beta,\alpha]$ by 1,5.

$\therefore$ $[\alpha,\beta]\{B_{12}(\alpha^{-1}\beta)B_{21}(-\beta^{-1}\alpha)B_{12}(\alpha^{-1}\beta)\}$

$= [\beta,\alpha]B_{12}(1)B_{21}(-1)B_{12}(1)$ by 1.

or $B_{12}(1)B_{21}(-1)B_{12}(1)[\alpha,\beta] = [\beta,\alpha]B_{12}(1)B_{21}(-1)B_{12}(1)$ by 4.

i.e. $E(0)[\alpha,\beta] = [\beta,\alpha]E(0)$

So $E(x)[\alpha,\beta] = B_{12}(-x)E(0)[\alpha,\beta]$

$= B_{12}(-x)[\beta,\alpha]E(0)$

$= [\beta,\alpha]B_{12}(-\beta^{-1}x\alpha)E(0)$ by 4.

$= [\beta,\alpha]E(\beta^{-1}x\alpha)$

It remains to shew that the relations implicit in (a), (b) and (c) are consequences of 1.-5. These relations are:

$B_{12}(x) = E(-x)E(0)^{-1} = B_{12}(1+x)B_{21}(-1)B_{12}(1)$
$\quad .\{B_{12}(1)B_{21}(-1)B_{12}(1)\}^{-1}$

$B_{21}(x) = E(0)^{-1}E(x) = \{B_{12}(1)B_{21}(-1)B_{12}(1)\}^{-1}$
$\quad .B_{12}(1-x)B_{21}(-1)B_{12}(1)$

The first of these, by 1, is equivalent to

$I = B_{12}(1)B_{21}(-1)B_{21}(1)B_{12}(-1)$

which is a consequence of 1. The second is, by 1,

$$B_{21}(x) = B_{12}(-1)B_{21}(1)B_{12}(-x)B_{21}(-1)B_{12}(1)$$

which is just 3. So we have a presentation.

Converse: suppose $GE_2(R)$ has the presentation

$$\{B_{ij}(x),\ [\alpha,\beta]\,|\,1.-5.\}$$

We must shew R is universal for GE_2. First we shew that the universal relations imply 1.-5. (using (a), (b) and (c)). From the universal relations, we have $E(0)^2=-I$: in fact, we can assume all the relations proved in [1], Theorem 2.2.

So $B_{12}(x)B_{12}(y) = E(-x)E(0)^{-1}E(-y)E(0)^{-1}$

$= -E(-x)E(0)E(-y)E(0)^{-1}$

$= E(-x-y)E(0)^{-1} = B_{12}(x+y)$

Similarly $B_{21}(x)B_{21}(y) = E(0)^{-1}E(x)E(0)^{-1}E(y)$

$= E(0)^{-1}E(x+y) = B_{21}(x+y)$

Then $B_{12}(\alpha-1)B_{21}(1)B_{12}(\alpha^{-1}-1)B_{21}(-\alpha)$

$= E(1-\alpha)E(0)^{-2}E(1)E(1-\alpha^{-1})E(0)^{-2}E(-\alpha)$

$= E(-\alpha)E(0)E(1)^3E(0)E(-\alpha^{-1})E(-\alpha)$

$= E(-\alpha)E(-\alpha^{-1})E(-\alpha)$

$= -D_{12}(-\alpha) = D_{12}(\alpha)$: using 1, 2, follows (case ij=12).

Similarly $B_{21}(\alpha-1)B_{12}(1)B_{21}(\alpha^{-1}-1)B_{12}(-\alpha)$

$= E(0)^{-1}E(\alpha-1)E(-1)E(0)^{-2}E(\alpha^{-1}-1)E(\alpha)E(0)^{-1}$

$= -E(0)^{-1}E(\alpha)E(0)E(-1)^3E(0)E(\alpha^{-1})E(\alpha)E(0)^{-1}$

$= E(0)^{-1}E(\alpha)E(\alpha^{-1})E(\alpha)E(0)^{-1}$

$= -E(0)^{-1}D_{12}(\alpha)E(0)^{-1} = D_{21}(\alpha)$. Use 1. as before.

Then $B_{12}(-1)B_{21}(1)B_{12}(-x)B_{21}(-1)B_{12}(1)$

$= E(1)E(0)^{-2}E(1)E(x)E(0)^{-2}E(-1)^2E(0)^{-1}$

$= E(1)^2E(x)E(-1)^2E(0)^{-1}$

$= E(1)^2E(x)E(-1)^{-1}E(0)^{-1}$

$= E(1)^2E(x+1)E(0)^{-2}$ by [1], Theorem 2.2, equation 2.7.

$= E(1)^3E(0)E(x) = E(0)^{-1}E(x) = B_{21}(x)$

And $B_{21}(-1)B_{12}(1)B_{21}(-x)B_{12}(-1)B_{21}(1)$

$= E(0)^{-1}E(-1)^2E(0)^{-2}E(-x)E(1)E(0)^{-2}E(1)$

$= E(0)^{-1}E(-1)^2E(-x)E(1)^2$

$= -E(0)^{-1}E(-1)^2E(-x)E(1)^{-1}$

$= -E(0)^{-1}E(-1)^2E(-x-1)E(0)^{-1}$ by [1], Thm 2.2, eqn 2.7.

$$= E(0)^{-1}E(-1)^3E(0)E(-x)E(0)^{-1}$$
$$= E(-x)E(0)^{-1} = B_{12}(x)$$

Then
$$\begin{aligned} B_{12}(x)[\alpha,\beta] &= E(-x)E(0)^{-1}[\alpha,\beta] \\ &= E(-x)[\beta,\alpha]E(0)^{-1} \\ &= [\alpha,\beta]E(-\alpha^{-1}x\beta)E(0)^{-1} \\ &= [\alpha,\beta]B_{12}(\alpha^{-1}x\beta) \end{aligned}$$

and
$$\begin{aligned} B_{21}(x)[\alpha,\beta] &= E(0)^{-1}E(x)[\alpha,\beta] \\ &= E(0)^{-1}[\beta,\alpha]E(\beta^{-1}x\alpha) \\ &= [\alpha,\beta]E(0)^{-1}E(\beta^{-1}x\alpha) \\ &= [\alpha,\beta]B_{21}(\beta^{-1}x\alpha) \end{aligned}$$

Finally we must check that the relation implicit in (a), (b) and (c) is a consequence of the universal relations. This relation is:

$$\begin{aligned} E(x) &= B_{12}(1-x)B_{21}(-1)B_{12}(1) \\ &= E(x-1)E(0)^{-2}E(-1)^2E(0)^{-1} \end{aligned}$$

and this does indeed follow from the universal relations. □

3. Universal rings.

In defining 'universal for GE_n', we could generalize the definition of 'universal for GE_2' by taking $E_{ij}(x)$ and $[\alpha]_k$ as generators, where

$$E_{ij}(x) = B_{ij}(1-x)B_{ji}(-1)B_{ij}(1)$$

but this seems a little awkward; it is much easier to work directly with the elementary matrices $B_{ij}(x)$. Accordingly, we make the following definition:

A ring R is <u>universal for GE_n</u> if $GE_n(R)$ has the presentation:

Generators: $B_{ij}(x)$, $[\alpha_1, .. \alpha_n]$ $(x \in R,\ \alpha_k \in U(R),\ 1 \leq i,j,k \leq n,\ i \neq j)$

Relations:

1. $B_{ij}(x)B_{ij}(y) = B_{ij}(x+y)$
2. $B_{ij}(x)B_{km}(y) = B_{km}(y)B_{ij}(x)$ $\qquad (i \neq m,\ j \neq k)$
3. $B_{ij}(x)B_{jk}(y) = B_{jk}(y)B_{ij}(x)B_{ik}(xy)$ $\qquad (i \neq k)$
4. $B_{ij}(\alpha-1)B_{ji}(1) = D_{ij}(\alpha)B_{ji}(\alpha)B_{ij}(1-\alpha^{-1})$
5. $B_{ij}(x) = B_{ji}(1)B_{ij}(-1)B_{ji}(-x)B_{ij}(1)B_{ji}(-1)$
6. $B_{ij}(x)[\alpha_1, .. \alpha_n] = [\alpha_1, .. \alpha_n]B_{ij}(\alpha_i^{-1}x\alpha_j)$
7. $[\alpha_1, .. \alpha_n][\beta_1, .. \beta_n] = [\alpha_1\beta_1, .. \alpha_n\beta_n]$

All these relations hold in $GE_n(R)$, for any ring R. Note that 2. and 3. are vacuous for n=2, so by (2.9) the definition coincides with the previous one in this case. The definition is justified by (3.7) and (3.13).

We already know from [1;(4.1)] that every local ring is universal for GE_2, but we give a direct proof here in terms of the above definition, in the belief that familiarity with the argument for this case will make the argument for general n easier to follow.

<u>Lemma (3.1)</u>. (Normal form for $GL_2(R)$, R local.) Put $B_1 = B_{21}(1)$ and $B_2 = I_2$. Then if $A \in GL_2(R)$ (R local) there is a unique expression

$$A = B_r B_{12}(x)B_{21}(y)[\alpha,\beta]_{12}$$

where $x,y \in R$, $\alpha,\beta \in U(R)$, r=1 or 2, and $1+x \notin U(R)$ if r=1.

<u>Proof</u>. Let $A = (a_{ij})$. One of a_{12}, a_{22} must be a unit. If $a_{22} \in U(R)$, put r=2. Otherwise put r=1. In either case

$$A = B_r \begin{bmatrix} * & * \\ * & \beta \end{bmatrix} = B_r B_{12}(x) \begin{bmatrix} \alpha & 0 \\ * & \beta \end{bmatrix} \qquad \text{(i)}$$

$$= B_r B_{12}(x)B_{21}(y)[\alpha,\beta]_{12} \qquad \text{(ii)}$$

If r=1, the last column of A is $\begin{bmatrix} x\beta \\ (1+x)\beta \end{bmatrix}$ which shews that

$1+x$ is a non-unit. Thus r, β and x are unique. From (i), α is unique. From (ii), y is unique. □

If R is any ring and $1+xy \in U(R)$, then $1+yx \in U(R)$; indeed

$$(1+yx)^{-1} = 1-y(1+xy)^{-1}x.$$

Alternatively we may see this by noting that, for any $x,y \in R$,

$$B_{ij}(x)B_{ji}(y)[1+yx]_j = [1+xy]_i B_{ji}(y)B_{ij}(x).$$

This relation will assume great importance in chapter 4, when we define 'quasi-universal' rings.

Lemma (3.2). If R is a local ring, the relation

$$B_{ij}(x)B_{ji}(y)[1+yx]_j = [1+xy]_i B_{ji}(y)B_{ij}(x) \qquad (1+xy \in U(R))$$

is a consequence of the universal relations 1, 4, 6 and 7.

Proof. (i) Suppose $y \in U(R)$.

Then $B_{ij}(x)B_{ji}(y)[1+yx]_j$

$= [y]_j B_{ij}(xy)B_{ji}(1)[y^{-1}]_j[1+yx]_j$ by 6, 7.

$= [y]_j B_{ij}((1+xy)-1)B_{ji}(1)[y^{-1}]_j[1+yx]_j$

$= [y]_j D_{ij}(1+xy)B_{ji}(1+xy)B_{ij}(1-(1+xy)^{-1})[y^{-1}]_j[1+yx]_j$ by 4.

$= [y]_j[1+xy]_i B_{ji}(1)B_{ij}(xy)[1+xy]_j^{-1}[y^{-1}]_j[1+yx]_j$ by 6, 7.

$= [y]_j[1+xy]_i B_{ji}(1)B_{ij}(xy)[y^{-1}]_j$ by 7.

$= [1+xy]_i B_{ji}(y)B_{ij}(x)$ by 6, 7.

(ii) Suppose $x \in U(R)$. By (i),

$$B_{ji}(y)B_{ij}(x)[1+xy]_i = [1+yx]_j B_{ij}(x)B_{ji}(y)$$

is a consequence of 1, 4, 6 and 7.

$\therefore$ $B_{ij}(x)B_{ji}(y)[1+yx]_j$

$= [1+yx]_j^{-1}B_{ji}(y)B_{ij}(x)[1+xy]_i[1+yx]_j$

$= [1+xy]_i B_{ji}((1+yx)^{-1}y(1+xy))B_{ij}((1+xy)^{-1}x(1+yx))$ by 6, 7.

$= [1+xy]_i B_{ji}(y)B_{ij}(x)$

(iii) Suppose x,y are both non-units.

$B_{ij}(x)B_{ji}(y)[1+yx]_j$

$= B_{ij}(x)B_{ji}(1)[1+x]_j[1+x]_j^{-1}B_{ji}(y-1)[1+yx]_j$ by 1, 7.

$= [1+x]_i B_{ji}(1)B_{ij}(x)[1+x]_j^{-1}B_{ji}(y-1)[1+yx]_j$ by (i).

$= [1+x]_i[1+x]_j^{-1}B_{ji}(1+x)B_{ij}(x(1+x)^{-1})B_{ji}(y-1)$

$\quad .[1+(y-1)x(1+x)^{-1}]_j[1+(y-1)x(1+x)^{-1}]_j^{-1}[1+yx]_j$ by 6,7.

$= [1+x]_i[1+x]_j^{-1}B_{ji}(1+x)[1+x(1+x)^{-1}(y-1)]_i B_{ji}(y-1)$

$\quad .B_{ij}(x(1+x)^{-1})[1+(y-1)x(1+x)^{-1}]_j^{-1}[1+yx]_j$ by (i).

$= [1+xy]_i[1+x]_j^{-1}B_{ji}(1+xy)B_{ji}(y-1)B_{ij}(x(1+x)^{-1})$

$\quad .[1+(y-1)x(1+x)^{-1}]_j^{-1}[1+yx]_j$ by 6,7.

$= [1+xy]_i [1+x]_j^{-1} B_{ji}((1+x)y) B_{ij}(x(1+x)^{-1})$

$\cdot [1+(y-1)x(1+x)^{-1}]_j^{-1} [1+yx]_j$ by 1.

$= [1+xy]_i B_{ji}(y) B_{ij}(x) [(1+x)^{-1}\{1+(y-1)x(1+x)^{-1}\}^{-1}(1+yx)]_j$ by 6,7.

$= [1+xy]_i B_{ji}(y) B_{ij}(x)$ by 7. □

We are now in a position to prove that a local ring is universal for GE_2. The proof here is longer than that in [1], though the difference is less great than would appear at first sight, since [1;(4.1)] uses results from [1; section 2]. The point is that this proof (3.3) provides a relatively simple illustration of the method that will be used to prove that a local ring is universal for GE_n.

Proposition (3.3). (P.M.Cohn) Every local ring is universal for GE_2.

Proof. Let $A = B_r B_{12}(x) B_{21}(y) [\alpha,\beta]_{12}$ be in normal form.

Then $A[\alpha',\beta'] = B_r B_{12}(x) B_{21}(y) [\alpha\alpha',\beta\beta']$ by 7.

and $AB_{21}(y') = B_r B_{12}(x) B_{21}(y+\beta y'\alpha^{-1}) [\alpha,\beta]$ by 1,6.

So it remains to shew that $A\cdot B_{12}(w)$ can be put in normal form using only 1.-7.

$A\cdot B_{12}(w) = B_r B_{12}(x) B_{21}(y) B_{12}(\alpha w \beta^{-1}) [\alpha,\beta]$ by 6.

so it is sufficient to prove that

$A_0 = B_r B_{12}(x) B_{21}(y) B_{12}(z)$ can be put in normal form.

(i) $z \notin U(R)$. From (3.2)

$B_{21}(y) B_{12}(z) [1+zy]_1 = [1+yz]_2 B_{12}(z) B_{21}(y)$

so $B_{21}(y) B_{12}(z) = B_{12}(z(1+yz)^{-1}) B_{21}((1+yz)y) [(1+zy)^{-1}, 1+yz]$ by 6,7.

$A_0 = B_r B_{12}(x+z(1+yz)^{-1}) B_{21}((1+yz)y) [(1+zy)^{-1}, 1+yz]$ by 1.

This is now in normal form, for $1+x \in U(R) \iff 1+x+z(1+yz)^{-1} \in U(R)$.

(ii) $z \in U(R)$. First suppose $r = 2$.

Then $A_0 = B_{12}(x) B_{21}(y) B_{12}(z)$

If $1+yz \in U(R)$, by (3.2) and 7. we have

$A_0 = B_{12}(x) [1+yz]_2 B_{12}(z) B_{21}(y) [1+zy]_1^{-1}$

$= B_{12}(x+z(1+yz)^{-1}) B_{21}((1+yz)y) [(1+zy)^{-1}, 1+yz]$ by 6,7.

and this is in normal form.

If $1+yz \notin U(R)$, then $y \in U(R)$, and

$A_0 = [y]_2 B_{12}(xy) B_{21}(1) B_{12}(zy) [y]_2^{-1}$

$= [y]_2 B_{21}(1) B_{12}(-1) \cdot B_{12}(1) B_{21}(-1) B_{12}(xy) B_{21}(1) B_{12}(-1)$

$\cdot B_{12}(1+zy) [y]_2^{-1}$ by 1.

$= [y]_2 B_{21}(1)B_{12}(-1)B_{21}(-xy)B_{12}(1+zy)[y]_2^{-1}$ by 5.

$= B_{21}(y)B_{12}(-y^{-1})[y]_2 B_{21}(-xy)B_{12}(1+zy)[y]_2^{-1}$ by 6.

$= B_{21}(1)B_{21}(y-1)B_{12}(-y^{-1})[1+(-y^{-1})(y-1)]_1[y,y]$
$\cdot B_{21}(-xy)B_{12}(1+zy)[y]_2^{-1}$ by 1,7.

$= B_{21}(1)[y^{-1}]_2 B_{12}(-y^{-1})B_{21}(y-1)[y,y]$
$\cdot B_{21}(-xy)B_{12}(1+zy)[y]_2^{-1}$ by (3.2)

$= B_{21}(1)[y^{-1}]_2 B_{12}(-y^{-1})B_{21}(y-1-yx)B_{12}(y(1+zy)y^{-1})[y]_1$
by 1,6,7.

$= B_{21}(1)[y^{-1}]_2 B_{12}(-y^{-1})B_{21}(y-1-yx)B_{12}(1+yz)[y]_1$

$= B_{21}(1)[y^{-1}]_2 B_{12}(-y^{-1})[1+(y-1-yx)(1+yz)]_2 B_{12}(1+yz)$
$\cdot B_{21}(y-1-yx)[1+(1+yz)(y-1-yx)]_1^{-1}[y]_1$
by (3.2) and 7.

$= B_1 \cdot B_{12}(-1+(1+yz)y^{-1}(1+(y-1-yx)))$
$\cdot B_{21}((1+(y-1-yx)(1+yz))^{-1}y(y-1-yx))$
$\cdot [(1+(1+yz)(y-1-yx))^{-1}y \ , \ y^{-1}(1+(y-1-yx)(1+yz)]$
by 1,6,7.

and this is in normal form, since

$1+(-1+(1+yz)y^{-1}(1+(y-1-yx)(1+yz)))$

is a multiple of $1+yz$, and so is a non-unit.

Now suppose $r=1$. Thus $A_0 = B_{21}(1)B_{12}(x)B_{21}(y)B_{12}(z)$, with $z \in U(R)$ and $1+x \notin U(R)$ (and so $x \in U(R)$).

$A_0 = B_{21}(1)B_{12}(-1)B_{12}(1+x)B_{21}(y)B_{12}(z)$ by 1.

$= B_{21}(1)B_{12}(-1)B_{21}(y')B_{12}(z')[\alpha,\beta]$ by (3.2),1,6,7.
(suitable y',z',α,β)

$= B_{12}(-y')B_{21}(1)B_{12}(z'-1)[\alpha,\beta]$ by 1,5.

and so now we can use the argument as for $r=2$, for $A_{00} = B_{12}(-y')B_{21}(1)$ is in normal form with $r=2$. □

There is a natural embedding of $GL_{n-1}(R)$ in $GL_n(R)$ by the map

$$\tau : A' \mapsto \begin{pmatrix} & A' & & 0 \\ & & & \vdots \\ & & & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \qquad (A' \in GL_{n-1}(R))$$

With this in mind, if $A \in GL_n(R)$, the statement $A \in GL_{n-1}(R)$ will be used to mean $\exists\, A' \in GL_{n-1}(R)$ such that

$$A = \tau A'$$

Lemma (3.4). (Normal form for $GL_n(R)$, R local.) Put $B_n=I_n$ and $B_r=B_{nr}(1)$, $1\leq r<n$. Then if $A\in GL_n(R)$ (R local) there is a unique expression

$$A = B_r B_{1n}(x_1)\ldots B_{n-1n}(x_{n-1})B_{n1}(y_1)\ldots B_{nn-1}(y_{n-1})[\alpha]_n A_1$$

where $\alpha\in U(R)$, $A_1\in GL_{n-1}(R)$, and if $r<n$, x_r+1 and x_s, $r<s<n$, are non-units. So by induction we have a normal form for A, expressed as a product of $\Sigma_1^n(2m-1) = n^2$ matrices (the last one diagonal, the others of type $B_{ij}(x)$).

[N.B. It is well known that a field, and even a local ring, is a GE-ring, and so some such expression for A exists. It is the particular form of the expression and its uniqueness which are new here.]

Proof. Every matrix in $GL_n(R)$ has a unit in every row and column (note that this property actually characterizes local rings). So we can choose r maximal such that $B_r^{-1}A$ has a unit α in the n,n position.

Then $\exists\ x_i$ $(i=1\ldots n-1)$ $\in R$ such that

$$A = B_r B_{1n}(x_1)\ldots B_{n-1n}(x_{n-1}) \cdot \begin{pmatrix} C & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ a_1\ldots a_{n-1} & \alpha \end{pmatrix} \qquad (1)$$

where $C\in GL_{n-1}(R)$, $a_i\in R$.

Put $(y_1\ldots y_{n-1}) = (a_1\ldots a_{n-1})C^{-1}$ and we have the required form, with $A_1 = \tau C$.

If $r<n$, the last column of A is $\begin{pmatrix} x_1\alpha \\ \vdots \\ x_{n-1}\alpha \\ (1+x_r)\alpha \end{pmatrix}$

By choice of r, $1+x_r$ and x_s $(r<s<n)$ are non-units.

This also shews that r, x_i $(1\leq i<n)$ and α are unique. The uniqueness of A_1 and $y_1\ldots y_{n-1}$ follows. □

Lemma (3.5). For any ring R, the following relations in $GE_n(R)$ are consequences of the universal relations:

8. $B_{ij}(x)B_{jk}(y)=B_{jk}(y)B_{ij}(x)B_{ik}(xy)$
$=B_{jk}(y)B_{ik}(xy)B_{ij}(x)=B_{ik}(xy)B_{jk}(y)B_{ij}(x)$
$B_{jk}(y)B_{ij}(x)=B_{ij}(x)B_{jk}(y)B_{ik}(-xy)$
$=B_{ij}(x)B_{ik}(-xy)B_{jk}(y)=B_{ik}(-xy)B_{ij}(x)B_{jk}(y)$

9. $B_{ij}(1-\alpha)B_{ji}(-1) = D_{ij}(\alpha)B_{ji}(-\alpha)B_{ij}(\alpha^{-1}-1)$

10. $\{B_{ij}(\alpha)B_{ji}(-\alpha^{-1})\}^3 = D_{ij}(-1)$

11. $\{B_{ij}(1)B_{ik}(1)B_{ji}(-1)B_{ki}(-1)\}^2 = D_{ij}(-1)B_{jk}(1)B_{kj}(-1)B_{jk}(1)$

12. $[\alpha_1,.. \alpha_n]B_{ij}(x) = B_{ij}(\alpha_i x\alpha_j^{-1})[\alpha_1,.. \alpha_n]$

Proof. 8. consists of various ways of writing 3, all equivalent, by 1 and 2. 12 is just another way of writing 6.
Then $[-1]_i B_{ij}(1-\alpha)B_{ji}(-1) = B_{ij}(\alpha-1)B_{ji}(1)[-1]_i$ by 6
$= D_{ij}(\alpha)B_{ji}(\alpha)B_{ij}(1-\alpha^{-1})[-1]_i$ by 4
$= [-1]_i D_{ij}(\alpha)B_{ji}(-\alpha)B_{ij}(\alpha^{-1}-1)$ by 6,7

whence 9 follows.

We next prove 10 with $\alpha=1$:

$\{B_{ij}(1)B_{ji}(-1)\}^3$
$= \{B_{ij}(1)B_{ji}(-1)B_{ij}(1)B_{ji}(1)B_{ij}(-1)\}\{B_{ij}(1)B_{ji}(-2) \cdot B_{ij}(1)B_{ji}(-1)\}$ by 1
$= B_{ji}(-1)B_{ij}(1)D_{ji}(-1)B_{ij}(-1)B_{ji}(2)B_{ji}(-1)$ by 4,5
$= B_{ji}(-1)B_{ij}(1)B_{ij}(-1)B_{ji}(1)D_{ij}(-1)$ by 1,6
$= D_{ij}(-1)$ by 1

Then $\{B_{ij}(\alpha)B_{ji}(-\alpha^{-1})\}^3 = [\alpha]_i\{B_{ij}(1)B_{ji}(-1)\}^3[\alpha^{-1}]_i$ by 6,7
$= [\alpha]_i D_{ij}(-1)[\alpha^{-1}]_i$ by the above
$= D_{ij}(-1)$ by 7.

11: $\{B_{ij}(1)B_{ik}(1)B_{ji}(-1)B_{ki}(-1)\}^2$
$= B_{ik}(1)B_{ij}(1)B_{ji}(-1)B_{ij}(1)B_{ki}(-1)B_{kj}(-1)B_{ik}(1)B_{ji}(-1)B_{ki}(-1)$ by 2,3
$= B_{ik}(1)\{B_{ij}(1)B_{ji}(-1)\}^2B_{ji}(1)B_{ki}(-1)B_{kj}(-1)B_{ik}(1) \cdot B_{ji}(-1)B_{ki}(-1)$ by 1
$= B_{ik}(1)\{B_{ij}(1)B_{ji}(-1)\}^2B_{ki}(-1)B_{kj}(-1)B_{ki}(1)B_{ik}(1) \cdot B_{jk}(1)B_{ki}(-1)$ by 1,2,3
$= B_{ik}(1)\{B_{ij}(1)B_{ji}(-1)\}^2B_{kj}(-1)B_{ik}(1)B_{jk}(1)B_{ki}(-1)$ by 1,2
$= B_{ik}(1)\{B_{ij}(1)B_{ji}(-1)\}^2B_{ik}(1)B_{ij}(1)B_{kj}(-1) \cdot B_{jk}(1)B_{ki}(-1)$ by 2,3
$= B_{ik}(1)\{B_{ij}(1)B_{ji}(-1)\}^3B_{ji}(1)B_{ik}(1)B_{kj}(-1) \cdot B_{jk}(1)B_{ki}(-1)$ by 1,2
$= B_{ik}(1)D_{ij}(-1)B_{ik}(1)B_{jk}(1)B_{ji}(1)B_{kj}(-1)B_{jk}(1)B_{ki}(-1)$ by 3,10
$= D_{ij}(-1)B_{jk}(1)B_{kj}(-1)B_{ji}(1)B_{ki}(1)B_{jk}(1)B_{ki}(-1)$ by 1,2,3,6

$= D_{ij}(-1)B_{jk}(1)B_{kj}(-1)B_{ji}(1)B_{ji}(-1)B_{jk}(1)$ by 1,2,3

$= D_{ij}(-1)B_{jk}(1)B_{kj}(-1)B_{jk}(1)$ by 1. □

We introduce the following notation: if $A,B \in GE_n(R)$ are expressions in the generators $B_{ij}(x)$, $[\alpha]_k$, then $A \underset{n}{\to} B$ will mean $\exists$ C in $GE_{n-1}(R)$, i.e. an expression in $B_{ij}(x)$, $[\alpha]_k$ with $i,j,k<n$, such that $A = BC$, and furthermore that this relation is a consequence of the universal relations. Clearly $\underset{n}{\to}$ is an equivalence relation; the arrow symbol is chosen as its use will be in a stepwise reduction to normal form. Normally we shall write $\to$ for $\underset{n}{\to}$ where the value of n is clear from the context.

Lemma (3.6). Let R be a local ring. The following hold, the R.H.S. being in normal form in each case:

(i) $\prod_{i<n} B_{ni}(y_i)\prod_{i<n} B_{in}(w_i) \to \prod_{i<n} B_{in}(w_i\alpha^{-1})\prod_{i<n} B_{ni}(\alpha y_i)[\alpha]_n$

where $1 + \Sigma y_i w_i = \alpha \in U(R)$

(ii) $\prod_{i<n} B_{in}(x_i)\prod_{i<n} B_{ni}(y_i)\prod_{i<n} B_{in}(w_i)$

$\to \prod_{i<n} B_{in}(x_i+w_i\alpha^{-1})\prod_{i<n} B_{ni}(\alpha y_i)[\alpha]_n$

where $1 + \Sigma y_i w_i = \alpha \in U(R)$

(iii) $\prod_{i<n} B_{ni}(y_i)\prod_{i<n} B_{in}(w_i)$

$\to B_s\prod_{i<n} B_{in}(w_i\beta^{-1})\prod_{\substack{i<n\\ i\neq s}} B_{ni}(\beta y_i)\cdot B_{ns}(\beta(y_s-1))[\beta]_n$

where $1+\Sigma y_i w_i = z \notin U(R)$ and s is maximal such that

$z-w_s = \beta \in U(R)$

(iv) $\prod_{i<n} B_{in}(x_i)\prod_{i<n} B_{ni}(y_i)\prod_{i<n} B_{in}(w_i)$

$\to B_s\prod_{i<n} B_{in}((x_i z+w_i)\alpha^{-1})\prod_{\substack{i<n\\ i\neq s}} B_{ni}(\alpha y_i)\cdot B_{ns}(\alpha(y_s-z'))[\alpha]_n$

with conditions as in (iii), and also $\alpha = \beta-x_s z$

and $z' = 1+\Sigma y_i x_i$

(v) If $B_s\prod_{i<n} B_{in}(x_i)\prod_{i<n} B_{ni}(y_i)$ is in normal form, then

$B_r\cdot B_s\prod_{i<n} B_{in}(x_i)\prod_{i<n} B_{ni}(y_i) \to A$

where $A \in GE_n(R)$ is expressed in normal form.

[N.B. By relation 2. the order of the terms in the products $\prod_{i<n} B_{ni}(*)$, $\prod_{i<n} B_{in}(*)$ is immaterial.]

<u>Proof</u>. (i) This holds for n=2, by (3.2). Now consider the case n=3. Suppose $1+y_1w_1 = \beta\in U(R)$.

$$B_{31}(y_1)B_{32}(y_2)B_{13}(w_1)B_{23}(w_2)$$
$$= B_{32}(y_2)B_{23}(w_2\beta^{-1})B_{31}(y_1)B_{21}(-w_2\beta^{-1}y_1)B_{13}(w_1)B_{23}(w_2(1-\beta^{-1})) \quad \text{by } 1,2,3$$
$$= B_{32}(y_2)B_{23}(w_2\beta^{-1})B_{31}(y_1)B_{13}(w_1)B_{23}(w_2(1-\beta^{-1}-\beta^{-1}y_1w_1)) \cdot B_{21}(-w_2\beta^{-1}y_1) \quad \text{by } 1,2,3$$
$$\underset{s}{\to} B_{32}(y_2)B_{23}(w_2\beta^{-1})B_{31}(y_1)B_{13}(w_1) \quad \text{by } 1$$
$$\to B_{32}(y_2)B_{23}(w_2\beta^{-1})B_{13}(w_1\beta^{-1})B_{31}(\beta y_1)[\beta]_3 \quad \text{by } (3.2),\ 6$$
$$\to B_{23}(w_2\alpha^{-1})B_{32}(\alpha\beta^{-1}y_2)B_{13}(w_1\alpha^{-1})B_{31}(\alpha y_1)[\alpha]_3 \quad \text{by } (3.2),6,7$$
$$\to B_{13}(w_1\alpha^{-1})B_{23}(w_2\alpha^{-1})B_{32}(\alpha\beta^{-1}y_2)B_{12}(-w_1\beta^{-1}y_2)B_{31}(\alpha y_1)[\alpha]_3 \quad \text{by } 2,3$$
$$\to B_{13}(w_1\alpha^{-1})B_{23}(w_2\alpha^{-1})B_{31}(\alpha y_1)B_{32}(\alpha\beta^{-1}y_2+\alpha y_1w_1\beta^{-1}y_2)[\alpha]_3 \quad \text{by } 1,2,3$$
$$= B_{13}(w_1\alpha^{-1})B_{23}(w_2\alpha^{-1})B_{31}(\alpha y_1)B_{32}(\alpha y_2)[\alpha]_3$$

If $1+y_2w_2 \in U(R)$, a similar calculation gives the result. In the remaining case, $1+y_1w_1$ and $1+y_2w_2$ are both non-units, so y_1, y_2, w_1, w_2 are all units.

Then $B_{31}(y_1)B_{32}(y_2)B_{13}(w_1)B_{23}(w_2)$
$$= [y_1^{-1},y_2^{-1}]_{12}B_{31}(1)B_{32}(1)B_{13}(-1)B_{23}(-1)\cdot M_0 \quad \text{by } 1,2,6,7$$
where $M_0= B_{13}(1+y_1w_1)B_{23}(1+y_2w_2)[y_1,y_2]_{12}$

Now use 11. We have $B_{31}(y_1)B_{32}(y_2)B_{13}(w_1)B_{23}(w_2)$
$$\underset{s}{\to} [y_1^{-1},y_2^{-1}]_{12}B_{13}(1)B_{23}(1)B_{31}(-1)B_{32}(-1)\cdot M$$
where $M = D_{13}(-1)B_{12}(1)B_{21}(-1)B_{12}(1)\cdot M_0$
$$\to B_{13}(1+y_1w_1)D_{13}(-1)B_{12}(1)B_{21}(-1)B_{23}(-1-y_1w_1)B_{12}(1) \cdot B_{23}(1+y_2w_2) \quad \text{by } 2,3,6$$
$$\to B_{13}(1+y_1w_1)D_{13}(-1)B_{12}(1)B_{21}(-1)B_{23}(y_2w_2-y_1w_1) \cdot B_{13}(1+y_2w_2) \quad \text{by } 1,2,3$$
$$\to B_{13}(1+y_1w_1)B_{23}(y_1w_1-y_2w_2)D_{13}(-1)B_{12}(1) \cdot B_{13}(y_2w_2-y_1w_1)B_{21}(-1)B_{13}(1+y_2w_2) \quad \text{by } 2,3,6$$
$$\to B_{13}(1+y_2w_2)B_{23}(y_1w_1-y_2w_2)D_{13}(-1)B_{13}(1+y_2w_2) \cdot B_{12}(1)B_{23}(-1-y_2w_2) \quad \text{by } 1,2,3,6$$
$$\to B_{13}(1+y_2w_2)B_{23}(y_1w_1-y_2w_2)D_{13}(-1)B_{13}(1+y_2w_2) \cdot B_{23}(-1-y_2w_2)B_{13}(-1-y_2w_2) \quad \text{by } 2,3$$

$\rightarrow B_{13}(1+y_2w_2)B_{23}(1+y_1w_1)[-1]_3$ by 1,2,6,7

So $B_{31}(y_1)B_{32}(y_2)B_{13}(w_1)B_{23}(w_2)$

$\rightarrow [y_1^{-1},y_2^{-1}]_{12}B_{13}(1)B_{23}(1)\{B_{31}(-1)B_{32}(-1)$
$\cdot B_{13}(1+y_2w_2)B_{23}(1+y_1w_1)\}[-1]_3$

$\rightarrow [y_1^{-1},y_2^{-1}]_{12}B_{13}(1)B_{23}(1)\{B_{13}((1+y_2w_2)(-\alpha^{-1}))$
$\cdot B_{23}((1+y_1w_1)(-\alpha^{-1}))B_{31}(\alpha)B_{32}(\alpha)\}[\alpha]_3$
by previous case

$\rightarrow [y_1^{-1},y_2^{-1}]_{12}B_{13}(y_1w_1\alpha^{-1})B_{23}(y_2w_2\alpha^{-1})B_{31}(\alpha)B_{32}(\alpha)[\alpha]_3$ by 1,2

$\rightarrow B_{13}(w_1\alpha^{-1})B_{23}(w_2\alpha^{-1})B_{31}(\alpha y_1)B_{32}(\alpha y_2)[\alpha]_3$ by 6,7

So (i) holds for $n = 2,3$. We now use induction.

We have
$$\alpha = 1 + \sum_1^{n-1} y_i w_i \in U(R).$$

Suppose first that $1 + \sum_2^{n-1} y_i w_i = \beta \in U(R)$.

Write Σ_k for $\sum_k^{n-1}$. Then $\alpha=1+\Sigma_1 y_i w_i$, $\beta=1+\Sigma_2 y_i w_i$.

Similarly, write Π_k for $\prod_k^{n-1}$.

Then $\Pi_1 B_{ni}(y_i)\Pi_1 B_{in}(w_i)$

$= B_{n1}(y_1)B_{1n}(w_1\beta^{-1})\Pi_2\{B_{ni}(y_i)B_{1i}(-w_1\beta^{-1}y_i)\}$
$\cdot\Pi_2 B_{in}(w_i)B_{1n}(w_1(1-\beta^{-1}))$ by 1,2,3

$\underset{n}{\rightarrow} B_{n1}(y_1)B_{1n}(w_1\beta^{-1})\Pi_2 B_{ni}(y_i)\Pi_2 B_{in}(w_i)$
$\cdot B_{1n}(w_1(1-\beta^{-1}-\beta^{-1}\Sigma_2 y_i w_i))$ by 1,2,3

$\rightarrow B_{n1}(y_1)B_{1n}(w_1\beta^{-1})\Pi_2 B_{ni}(y_i)\Pi_2 B_{in}(w_i)$ by 1

$\rightarrow B_{n1}(y_1)B_{1n}(w_1\beta^{-1})\Pi_2 B_{in}(w_i\beta^{-1})\Pi_2 B_{ni}(\beta y_i)[\beta]_n$ by induction

$\rightarrow B_{1n}(w_1\alpha^{-1})B_{n1}(\alpha\beta^{-1}y_1)\Pi_2 B_{in}(w_i\alpha^{-1})\Pi_2 B_{ni}(\alpha y_i)[\alpha]_n$ by (3.2), 6,7

$\rightarrow \Pi_1 B_{in}(w_i\alpha^{-1})\Pi_2 B_{i1}(-w_i\beta^{-1}y_1)B_{n1}(\alpha\beta^{-1}y_1)\Pi_2 B_{ni}(\alpha y_i)[\alpha]_n$ by 2,3

$\rightarrow \Pi_1 B_{in}(w_i\alpha^{-1})B_{n1}(\alpha\beta^{-1}y_1+\alpha\Sigma_2 y_i w_i\beta^{-1}y_1)\Pi_2 B_{ni}(\alpha y_i)[\alpha]_n$ by 1,2,3

$\rightarrow \Pi_1 B_{in}(w_i\alpha^{-1})\Pi_1 B_{ni}(\alpha y_i)[\alpha]_n$

Now suppose $\exists$ r, $1\le r<n$ such that $1 + \sum_{\substack{i<n\\ i\ne r}} y_i w_i \in U(R)$.

Then a proof similar to the above applies.

The remaining case is when $1 + \sum_{\substack{i<n\\ i\ne r}} y_i w_i = z_r$ is a non-unit, $1\le r<n$.

Then $z_r + y_r w_r = \alpha \in U(R)$, so y_r, w_r are units, $1\le r<n$.

Then $1+\Sigma_3 y_i w_i = z_1 - y_2 w_2 \in U(R)$: put $\beta = 1+\Sigma_3 y_i w_i$.

$\Pi_1 B_{ni}(y_i)\Pi_1 B_{in}(w_i)$

$= B_{n1}(y_1)B_{n2}(y_2)B_{1n}(w_1\beta^{-1})B_{2n}(w_2\beta^{-1})\Pi_3\{B_{ni}(y_i)B_{1i}(-w_1\beta^{-1}y_i)$
$\cdot B_{2i}(-w_2\beta^{-1}y_i)\}\Pi_3 B_{in}(w_i)B_{1n}(w_1(1-\beta^{-1}))B_{2n}(w_2(1-\beta^{-1}))$ by 1,2,3

$\to B_{n1}(y_1)B_{n2}(y_2)B_{1n}(w_1\beta^{-1})B_{2n}(w_2\beta^{-1})\Pi_3 B_{ni}(y_i)\Pi_3 B_{in}(w_i)$
$\cdot B_{1n}(w_1(1-\beta^{-1}-\beta^{-1}\Sigma_3 y_i w_i))B_{2n}(w_2(1-\beta^{-1}-\beta^{-1}\Sigma_3 y_i w_i))$ by 1,2,3

$\to B_{n1}(y_1)B_{n2}(y_2)B_{1n}(w_1\beta^{-1})B_{2n}(w_2\beta^{-1})\Pi_3 B_{ni}(y_i)\Pi_3 B_{in}(w_i)$ by 1

$\to B_{n1}(y_1)B_{n2}(y_2)B_{1n}(w_1\beta^{-1})B_{2n}(w_2\beta^{-1})$
$\cdot\Pi_3 B_{in}(w_i\beta^{-1})\Pi_3 B_{ni}(\beta y_i)[\beta]_n$ by induction

$\to B_{1n}(w_1\alpha^{-1})B_{2n}(w_2\alpha^{-1})B_{n1}(\alpha\beta^{-1}y_1)B_{n2}(\alpha\beta^{-1}y_2)$
$\cdot\Pi_3 B_{in}(w_i\alpha^{-1})\Pi_3 B_{ni}(\alpha y_i)[\alpha]_n$ by case n=3, and 6,7.

$\to \Pi_1 B_{in}(w_i\alpha^{-1})\Pi_3\{B_{i1}(-w_i\beta^{-1}y_1)B_{i2}(-w_i\beta^{-1}y_2)\}$
$\cdot B_{n1}(\alpha\beta^{-1}y_1)B_{n2}(\alpha\beta^{-1}y_2)\Pi_3 B_{ni}(\alpha y_i)[\alpha]_n$ by 2,3

$\to \Pi_1 B_{in}(w_i\alpha^{-1})B_{n1}(\alpha\beta^{-1}y_1+\alpha\Sigma_3 y_i w_i\beta^{-1}y_1)$
$\cdot B_{n2}(\alpha\beta^{-1}y_2+\alpha\Sigma_3 y_i w_i\beta^{-1}y_2)\Pi_3 B_{ni}(\alpha y_i)[\alpha]_n$ by 1,2,3

$\to \Pi_1 B_{in}(w_i\alpha^{-1})\Pi_1 B_{ni}(\alpha y_i)[\alpha]_n$

Thus (i) is proved. (ii) is now immediate, by 1,2.

(iii) $\Pi_1 B_{ni}(y_i)\Pi_1 B_{in}(w_i)$

$= B_s\cdot\prod_{\substack{i<n\\ i\neq s}} B_{ni}(y_i)B_{ns}(y_s-1)\Pi_1 B_{in}(w_i)$ by 1,2

$\to B_s\cdot\Pi_1 B_{in}(w_i\beta^{-1})\prod_{\substack{i<n\\ i\neq s}} B_{ni}(\beta y_i)B_{ns}(\beta(y_s-1))[\beta]_n$ by (i)

Note that $1+w_s\beta^{-1} = (\beta+w_s)\beta^{-1} = z\beta^{-1}$ is a non-unit, and so is $w_r\beta^{-1}$, s<r<n, so we have normal form.

(iv) Put $z'' = z'-y_s x_s$. Write Π_1^s for $\prod_{\substack{i<n\\ i\neq s}}$ and Σ_1^s for $\sum_{\substack{i<n\\ i\neq s}}$.

$\Pi_1 B_{in}(x_i)\Pi_1 B_{ni}(y_i)\Pi_1 B_{in}(w_i)$

$\to \Pi_1 B_{in}(x_i)B_{ns}(1)\Pi_1 B_{in}(w_i\beta^{-1})\Pi_1^s B_{ni}(\beta y_i)B_{ns}(\beta(y_s-1))[\beta]_n$ by (iii)

$\to B_{sn}(x_s)B_{ns}(1)\Pi_1^s\{B_{in}(x_i)B_{is}(x_i)\}\Pi_1 B_{in}(w_i\beta^{-1})$
$\cdot\Pi_1^s B_{ni}(\beta y_i)B_{ns}(\beta(y_s-1))[\beta]_n$ by 2,3

$\rightarrow B_{sn}(x_s)B_{ns}(1)B_{sn}(w_s\beta^{-1})\Pi_1^s B_{in}(x_i+w_i\beta^{-1}+x_iw_s\beta^{-1})$
$\cdot\Pi_1^s B_{ni}(\beta y_i)B_{ns}(\beta(y_s-z''))[\beta]_n$ by 1,2,3

$\rightarrow B_{sn}(x_s)B_{ns}(1)B_{sn}(w_s\beta^{-1})\Pi_1^s B_{in}((x_iz+w_i)\beta^{-1})$
$\cdot\Pi_1^s B_{ni}(\beta y_i)B_{ns}(\beta(y_s-z''))[\beta]_n$

Now $B_{21}(1)B_{12}((x_sz+w_s)\alpha^{-1})B_{21}(-x_s\delta^{-1})[\delta,\alpha\beta^{-1}]_{12}$ $(\delta=1+z\alpha^{-1}x_s)$

$$= \begin{pmatrix} \delta-(x_sz+w_s)\alpha^{-1}x_s & (x_sz+w_s)\beta^{-1} \\ \delta-x_s-(x_sz+w_s)\alpha^{-1}x_s & (\alpha+x_sz+w_s)\beta^{-1} \end{pmatrix}$$

Then $\delta-x_s-(x_sz+w_s)\alpha^{-1}x_s = 1+(z-\alpha-x_sz-w_s)\alpha^{-1}x_s$
$= 1+(\beta-\alpha-x_sz)\alpha^{-1}x_s$
$= 1$

So $\delta-(x_sz+w_s)\alpha^{-1}x_s = 1+x_s$

Then $(x_sz+w_s)\beta^{-1} = (x_s(w_s+\beta)+w_s)\beta^{-1} = x_s+w_s\beta^{-1}+x_sw_s\beta^{-1}$

Also $(\alpha+x_sz+w_s)\beta^{-1} = (\alpha+x_sz)\beta^{-1}+w_s\beta^{-1} = 1+w_s\beta^{-1}$

But $B_{12}(x_s)B_{21}(1)B_{12}(w_s\beta^{-1})$

$$= \begin{pmatrix} 1+x_s & x_s+w_s\beta^{-1}+x_sw_s\beta^{-1} \\ 1 & 1+w_s\beta^{-1} \end{pmatrix}$$

Thus $B_{sn}(x_s)B_{ns}(1)B_{sn}(w_s\beta^{-1})$
$= B_{ns}(1)B_{sn}((x_sz+w_s)\alpha^{-1})B_{ns}(-x_s\delta^{-1})[\delta,\alpha\beta^{-1}]_{sn}$

This relation only involves the two indices s,n and so by (3.3) it is a consequence of the universal relations.

So $\Pi_1 B_{in}(x_i)\Pi_1 B_{ni}(y_i)\Pi_1 B_{in}(w_i)$

$\rightarrow B_s B_{sn}((x_sz+w_s)\alpha^{-1})B_{ns}(-x_s\delta^{-1})\Pi_1^s B_{in}((x_iz+w_i)\alpha^{-1})$
$\cdot\Pi_1^s B_{ni}(\alpha y_i)B_{ns}(\alpha(y_s-z'')\delta^{-1})[\alpha]_n$ by the above and 6,7

$\rightarrow B_s\Pi_1 B_{in}((x_iz+w_i)\alpha^{-1})\Pi_1^s B_{is}((x_iz+w_i)\alpha^{-1}x_s\delta^{-1})$
$\cdot\Pi_1^s B_{ni}(\alpha y_i)B_{ns}((\alpha y_s-\alpha z''-x_s)\delta^{-1})[\alpha]_n$ by 1,2,3

$\rightarrow B_s\Pi_1 B_{in}((x_iz+w_i)\alpha^{-1})\Pi_1^s B_{ni}(\alpha y_i)$
$\cdot B_{ns}((\alpha y_s-\alpha z''-x_s-\Sigma_1^s\alpha(y_ix_iz+y_iw_i)\alpha^{-1}x_s)\delta^{-1})[\alpha]_n$
by 1,2,3

Now $(\alpha y_s-\alpha z''-x_s-\Sigma_1^s\alpha(y_ix_iz+y_iw_i)\alpha^{-1}x_s)\delta^{-1}$
$= (\alpha y_s-\alpha z''-\alpha((z''-1)z+z-1-y_sw_s)\alpha^{-1}x_s)\delta^{-1}$
$= \alpha(y_s-z''+(y_sw_s-z''z)\alpha^{-1}x_s)\delta^{-1}$
$= \alpha(y_s-z''+y_s(z-\alpha-x_sz)\alpha^{-1}x_s-z''z\alpha^{-1}x_s)\delta^{-1}$
$(\because w_s = z-\beta = z-\alpha-x_sz)$

$= \alpha(y_s - z'' - y_s x_s)(1 + z\alpha^{-1} x_s)\delta^{-1}$

$= \alpha(y_s - z'' - y_s x_s)$

$= \alpha(y_s - z')$. Substitution back gives (iv).

Note that for $i>s$, w_i is a non-unit; so is z, and hence so is $(x_i z + w_i)\alpha^{-1}$. Then $1+(x_s z + w_s)\alpha^{-1}$

$$= (\alpha + x_s z + w_s)\alpha^{-1} = z\alpha^{-1} \text{ is a non-unit,}$$

so we have normal form.

(v) We have $r<n$, otherwise the result is trivial.

Firstly suppose $s=n$. There are three cases:

(a) $1+x_r$ and x_i $(r<i<n)$ all non-units: then we have normal form already.

(b) $1+x_r \notin U(R)$ and $x_t \in U(R)$ some $t>r$, t maximal.

Then put $\beta = 1+x_r - x_t \in U(R)$.

$B_r \Pi_1 B_{in}(x_i)\Pi_1 B_{ni}(y_i)$

$= B_t\{B_{nr}(1)B_{nt}(-1)B_{rn}(x_r)B_{tn}(x_t)\}\Pi_1^{r,t} B_{in}(x_i)\Pi_1 B_{ni}(y_i)$ by 1,2

(where $\Pi_1^{r,t}$ stands for $\prod_{\substack{i<n \\ i\neq r,t}}$)

$\rightarrow B_t B_{rn}(x_r\beta^{-1})B_{tn}(x_t\beta^{-1})B_{nr}(\beta)B_{nt}(-\beta)$

$\cdot \Pi_1^{r,t} B_{in}(x_i\beta^{-1})\Pi_1 B_{ni}(\beta y_i')[\beta]_n$

(suitable $y_i' \in R$), by (i),1,2,3,6,7

$\rightarrow B_t \Pi_1 B_{in}(x_i\beta^{-1})\Pi_1^{r,t}\{B_{ir}(-x_i)B_{it}(x_i)\}\Pi_1 B_{ni}(y_i'')[\beta]_n$

(suitable $y_i'' \in R$), by 1,2,3

$\rightarrow B_t \Pi_1 B_{in}(x_i\beta^{-1})\Pi_1 B_{ni}(\hat{y}_i)[\beta]_n$ $(\hat{y}_i \in R)$ by 1,2,3.

Note that $x_i\beta^{-1}$ $(t<i<n)$ is a non-unit, and so is $1+x_t\beta^{-1} = (\beta+x_t)\beta^{-1} = (1+x_r)\beta^{-1}$. So we have normal form.

(c) $1+x_r = \alpha \in U(R)$.

$B_r \Pi_1 B_{in}(x_i)\Pi_1 B_{ni}(y_i)$

$= B_{nr}(1)B_{rn}(x_r)\Pi_1^r B_{in}(x_i)\Pi_1 B_{ni}(y_i)$ by 2

$\rightarrow B_{rn}(x_r\alpha^{-1})B_{nr}(\alpha)\Pi_1^r B_{in}(x_i\alpha^{-1})\Pi_1 B_{ni}(y_i')[\alpha]_n$ (suitable $y_i' \in R$) by (3.2),6

$\rightarrow \Pi_1 B_{in}(x_i\alpha^{-1})\Pi_1^r B_{ir}(-x_r)\Pi_1 B_{ni}(y_i'')[\alpha]_n$ (suitable $y \in R$) by 1,2,3

$\rightarrow \Pi_1 B_{in}(x_i\alpha^{-1})\Pi_1 B_{ni}(\hat{y}_i)[\alpha]_n$ (suitable $\hat{y} \in R$) by 1,2,3

This is in normal form.

Now suppose $s=r<n$. Then $1+x_r$ is a non-unit, and so $1+2x_r = \alpha \in U(R)$.

$B_r B_s \Pi_1 B_{in}(x_i)\Pi_1 B_{ni}(y_i)$

$= B_{nr}(2)B_{rn}(x_r)\Pi_1^r B_{in}(x_i)\Pi_1 B_{ni}(y_i)$ by 1,2

$\to B_{rn}(x_r\alpha^{-1})B_{nr}(2\alpha)\Pi_1^r B_{in}(x_i\alpha^{-1})\Pi_1 B_{ni}(y_i')[\alpha]_n$ $(y_i' \in R)$ by (3.2), 6

$\to \Pi_1 B_{in}(x_i\alpha^{-1})\Pi_1^r B_{ir}(-2x_i)\Pi_1 B_{ni}(y_i'')[\alpha]_n$ $(y_i'' \in R)$ by 1,2,3

$\to \Pi_1 B_{in}(x_i\alpha^{-1})\Pi_1 B_{ni}(\hat{y}_i)[\alpha]_n$ $(\hat{y} \in R)$ by 1,2,3.

This is now in normal form.

Finally suppose s,r,n are all distinct. There are two cases:

(a) $x_r \in U(R)$. We know $1+x_s \notin U(R)$, so $1+x_s+x_r = \alpha \in U(R)$.

Then $B_r B_s \Pi_1 B_{in}(x_i)\Pi_1 B_{ni}(y_i)$

$= B_{nr}(1)B_{ns}(1)B_{rn}(x_r)B_{sn}(x_s)\Pi_1^{r\,s} B_{in}(x_i)\Pi_1 B_{ni}(y_i)$ by 2

$\to B_{rn}(x_r\alpha^{-1})B_{sn}(x_s\alpha^{-1})B_{nr}(\alpha)B_{ns}(\alpha)$
$\cdot\Pi_1^{r\,s} B_{in}(x_i\alpha^{-1})\Pi_1 B_{ni}(y_i')[\alpha]_n$ $(y_i' \in R)$ by (i), 1,2,3,6,7

$\to \Pi_1 B_{in}(x_i\alpha^{-1})\Pi_1^{r\,s}\{B_{ir}(x_i)B_{is}(x_i)\}\Pi_1 B_{ni}(y_i'')[\alpha]_n$ $(y_i'' \in R)$ by 1,2,3

$\to \Pi_1 B_{in}(x_i\alpha^{-1})\Pi_1 B_{ni}(\hat{y}_i)[\alpha]_n$ $(\hat{y}_i \in R)$ by 1,2,3

This is in normal form.

(b) $x_r \notin U(R)$. Then $1+x_r \in U(R)$, so by a previous case,

$B_r \Pi_1 B_{in}(x_i)\Pi_1 B_{ni}(y_i)$

$\to \Pi_1 B_{in}(x_i')\Pi_1 B_{ni}(y_i')[\alpha]_n$ (suitable x_i', y_i', α)

So $B_r B_s \Pi_1 B_{in}(x_i)\Pi_1 B_{ni}(y_i)$

$\to B_s \Pi_1 B_{in}(x_i')\Pi_1 B_{ni}(y_i')[\alpha]_n$ by 2.

$\to$ normal form, by a previous case. □

<u>Theorem (3.7)</u>. Every local ring is universal for GE_n, all n.

<u>Proof</u>. Let R be local. By (3.3) the theorem holds for n=2.

We use induction on n.

Let $A = B_r \Pi_1 B_{in}(x_i)\Pi_1 B_{ni}(y_i)[\alpha]_n A_0$ $(A_0 \in GE_{n-1}(R))$ be in normal form. Let B be a generator of $GE_n(R)$.

Then by induction it is sufficient to shew AB→normal form.

Firstly suppose $B \in GE_{n-1}(R)$.

Then $AB = B_r \Pi_1 B_{in}(x_i)\Pi_1 B_{ni}(y_i)[\alpha]_n A_0 B$

$\to B_r \Pi_1 B_{in}(x_i)\Pi_1 B_{ni}(y_i)[\alpha]_n$, and this is in normal form.

Next suppose $B = [\beta_1 \dots \beta_n]$

Then $AB \to B_r \Pi_1 B_{in}(x_i) \Pi_1 B_{ni}(y_i)[\alpha\beta_n]_n$, by 6,7, and this is normal form.

Next suppose $B = B_{nj}(w)$.

Then $AB \to B_r \Pi_1 B_{in}(x_i) \Pi_1 B_{ni}(y_i) \Pi_1 B_{ni}(w_i)[\alpha]_n$ (suitable $w_i \in R$) by 1,2,3,6

$\to B_r \Pi_1 B_{in}(x_i) \Pi_1 B_{ni}(y_i + w_i)[\alpha]_n$ by 1,2, and this is in normal form.

Finally suppose $B = B_{jn}(w)$.

Then $AB \to B_r \Pi_1 B_{in}(x_i) \Pi_1 B_{ni}(y_i) \Pi_1 B_{in}(w_i)[\alpha]_n$ (suitable $w_i \in R$) by 1,2,3,6

Now $\Pi_1 B_{in}(x_i) \Pi_1 B_{ni}(y_i) \Pi_1 B_{in}(w_i)[\alpha]_n$

$\to B_s \Pi_1 B_{in}(x_i') \Pi_1 B_{ni}(y_i')[\alpha']_n$ (suitable x_i', y_i', α')

where this is in normal form (using (ii) or (iii) of (3.6) as appropriate). So

$AB \to B_r B_s \Pi_1 B_{in}(x_i') \Pi_1 B_{ni}(y_i')[\alpha']_n$

$\to$ normal form, by (v) of (3.6). □

We shall prove later (chapter 3) that if $R/J(R)$ is universal for GE_n and R is universal for GE_2 then R is universal for GE_n. Thus (3.7) follows from the special case that all fields are universal for GE_n, all n. However, this fact is non-trivial; indeed, the proof is scarcely shorter in the classical case than that given in (3.7).

In [1;(5.2)] it was shewn that any discretely normed ring is universal for GE_2. In particular, the ring Z of rational integers is universal for GE_2. With the help of a result in [4], we now shew that Z is universal for GE_3.

<u>Theorem (3.8)</u>. The ring Z of rational integers is universal for GE_3.

<u>Proof</u>. In [4; section 2] the following is proved (a sketch of the proof is given at the end of this proof) :

Let $P_{ik} = B_{ik}(1)B_{ki}(-1)B_{ik}(1)[-1]_i = P_{ki}$

(e.g. $P_{12} = \begin{pmatrix} 0\,1 & 0 \\ 1\,0 & \\ 0 & I \end{pmatrix}$, $P_{1n} = \begin{pmatrix} 0 & & & 1 \\ & 1 & & \\ & & \ddots & \\ & & & 1 & \\ 1 & & & 0 \end{pmatrix}$)

and $O_i = [-1]_i$, $(ik) = B_{ik}(1)$.

Then $GE_3(Z)$ has the presentation:

Generators: P_{ik}, O_i, (ik) $(1 \leq i,k \leq 3,\ i \neq k)$

Relations:

$$A \begin{cases} (1) & P_{ik}^2 = I \\ (2) & P_{km}^{-1} P_{ik} P_{km} = P_{im} \\ (3) & O_i^2 = I \\ (4) & O_i O_k = O_k O_i \\ (5) & P_{ik}^{-1} O_i P_{ik} = O_k \\ (6) & P_{km}^{-1} O_i P_{km} = O_i \end{cases}$$

$$B \begin{cases} (7) & P_{ik}^{-1}(ik)P_{ik} = (ki) \\ (8) & P_{im}^{-1}(ik)P_{im} = (mk) \\ (9) & P_{km}^{-1}(ik)P_{km} = (im) \\ (10) & O_m^{-1}(ik)O_m = (ik) \\ (11) & O_i^{-1}(ik)O_i = (ik)^{-1} \\ (12) & O_k^{-1}(ik)O_k = (ik)^{-1} \end{cases}$$

$$C \begin{cases} (13) & O_i P_{ik}(ik)(ki)^{-1}(ik) = I \\ (14) & (ik)(im) = (im)(ik) \\ (15) & (ik)(mk) = (mk)(ik) \\ (16) & (ik)(km)(ik)^{-1}(km)^{-1}(im)^{-1} = I \end{cases}$$

The generators $B_{ik}(n)$ and $[\alpha,\beta,\delta]$ of $GE_3(R)$ are defined in terms of the above generators by

$$B_{ik}(n) = (ik)^n$$

$$[\alpha,\beta,\delta] = O_1^{\epsilon(\alpha)} O_2^{\epsilon(\beta)} O_3^{\epsilon(\delta)} \quad \text{where } \epsilon(\lambda) = \begin{cases} 0 & \text{if } \lambda = 1 \\ 1 & \text{if } \lambda = -1 \end{cases}$$

The relations implicit in the definitions of the two sets of generators are:

$$B_{ik}(1)B_{ki}(-1)B_{ik}(1)[-1]_i = B_{ki}(1)B_{ik}(-1)B_{ki}(1)[-1]_k$$

$$[\alpha,\beta,\delta] = [\alpha]_1[\beta]_2[\delta]_3$$

$$B_{ik}(n) = B_{ik}(1)^n$$

The second and third of these follow immediately from universal relations 7 and 1.

Then $B_{ik}(1)B_{ki}(-1)B_{ik}(1)[-1]_i$

$= B_{ik}(1)B_{ki}(-1)B_{ik}(1)B_{ki}(1)B_{ik}(-1)\cdot B_{ik}(1)B_{ki}(-1)[-1]_i$ by 1

$= B_{ki}(-1)B_{ik}(1)B_{ki}(-1)[-1]_i$ by 5

$= B_{ki}(1)B_{ik}(-1)B_{ki}(1)\cdot B_{ki}(-1)B_{ik}(1)B_{ki}(-2)B_{ik}(1)$
$\cdot B_{ki}(-1)[-1]_i$ by 1

$= B_{ki}(1)B_{ik}(-1)B_{ki}(1)\cdot B_{ki}(-1)B_{ik}(1)$
$\cdot D_{ik}(-1)B_{ik}(-1)B_{ki}(2)B_{ki}(-1)[-1]_i$ by 4

$= B_{ki}(1)B_{ik}(-1)B_{ki}(1)\cdot D_{ik}(-1)[-1]_i$ by 1,6

$= B_{ki}(1)B_{ik}(-1)B_{ki}(1)[-1]_k$ by 7

So it remains to shew that the relations (1)-(16) are consequences of 1.-7.

(1) $P_{ik}^2 = B_{ik}(1)B_{ki}(-1)B_{ik}(1)[-1]_iB_{ik}(1)B_{ki}(-1)B_{ik}(1)[-1]_i$
$= B_{ik}(1)B_{ki}(-1)B_{ik}(1)B_{ik}(-1)B_{ki}(1)B_{ik}(-1)$ by 6,7
$= I$ by 1

So we may now replace P_{ik}^{-1} by P_{ik} where convenient.

(2) $P_{km}^{-1}P_{ik}P_{km}$
$= B_{mk}(1)B_{km}(-1)B_{mk}(1)[-1]_mB_{ik}(1)B_{ki}(-1)B_{ik}(1)[-1]_i$
$\cdot B_{mk}(1)B_{km}(-1)B_{mk}(1)[-1]_m$

$= B_{mk}(1)B_{km}(-1)B_{mk}(1)B_{ik}(1)B_{ki}(-1)B_{ik}(1)$
$\cdot B_{mk}(-1)B_{km}(1)B_{mk}(-1)[-1]_i$ by 6,7

$= B_{mk}(1)B_{km}(-1)B_{ik}(1)B_{ki}(-1)B_{mi}(-1)B_{ik}(1)$
$\cdot B_{km}(1)B_{mk}(-1)[-1]_i$ by 1,2,3

$= B_{mk}(1)B_{ik}(1)B_{im}(1)B_{ki}(-1)B_{mi}(-1)B_{ki}(1)B_{ik}(1)$
$\cdot B_{im}(1)B_{mk}(-1)[-1]_i$ by 1,2,3

$= B_{mk}(1)B_{ik}(1)B_{im}(1)B_{mi}(-1)B_{ik}(1)B_{im}(1)B_{mk}(-1)[-1]_i$ by 1,2

$= B_{ik}(1)B_{im}(1)B_{ik}(-1)B_{mi}(-1)B_{ik}(1)B_{im}(1)B_{ik}(-1)[-1]_i$ by 1,2,3

$= B_{im}(1)B_{mi}(-1)B_{im}(1)[-1]_i$ by 1,2

$= P_{im}$

(3) $O_i^2 = [-1]_i^2 = I$ by 7

(4) $O_iO_k = [-1]_i[-1]_k = [-1]_k[-1]_i = O_kO_i$ by 7

(5) $P_{ik}^{-1}O_iP_{ik}$
$= B_{ik}(1)B_{ki}(-1)B_{ik}(1)[-1]_i^2B_{ik}(1)B_{ki}(-1)B_{ik}(1)[-1]_i$
$= B_{ik}(1)B_{ki}(-1)B_{ik}(2)B_{ki}(-1)B_{ik}(1)[-1]_i$ by 1,7
$= B_{ik}(1)B_{ki}(-1)D_{ik}(-1)B_{ki}(1)B_{ik}(-2)B_{ik}(1)[-1]_i$ by 9 (3.5)
$= D_{ik}(-1)[-1]_i$ by 1,6
$= [-1]_k$ by 7
$= O_k$

(6) $P_{km}^{-1} O_i P_{km}$

$= [-1]_k B_{km}(-1) B_{mk}(1) B_{km}(-1) [-1]_i B_{km}(1) B_{mk}(-1) B_{km}(1) [-1]_k$

$= [-1]_k B_{km}(-1) B_{mk}(1) B_{km}(-1) B_{km}(1) B_{mk}(-1) B_{km}(1) [-1]_k [-1]_i$ by 6,7

$= [-1]_i$ by 1,7

$= O_i$

(7) $P_{ik}^{-1}(ik) P_{ik}$

$= B_{ik}(1) B_{ki}(-1) B_{ik}(1) [-1]_i B_{ik}(1) B_{ik}(1) B_{ki}(-1) B_{ik}(1) [-1]_i$

$= B_{ik}(1) B_{ki}(-1) B_{ik}(-1) B_{ki}(1) B_{ik}(-1)$ by 1,6,7

$= B_{ki}(1)$ by 5

$= (ki)$

(8) $P_{im}^{-1}(ik) P_{im}$

$= B_{im}(1) B_{mi}(-1) B_{im}(1) [-1]_i B_{ik}(1) B_{im}(1) B_{mi}(-1) B_{im}(1) [-1]_i$

$= B_{im}(1) B_{mi}(-1) B_{im}(1) B_{ik}(-1) B_{im}(-1) B_{mi}(1) B_{im}(-1)$ by 6,7

$= B_{im}(1) B_{mi}(-1) B_{ik}(-1) B_{mi}(1) B_{im}(-1)$ by 1,2

$= B_{im}(1) B_{ik}(-1) B_{mk}(1) B_{im}(-1)$ by 1,2,3

$= B_{ik}(-1) B_{mk}(1) B_{ik}(1)$ by 1,2,3

$= B_{mk}(1)$ by 1,2

$= (mk)$

(9) $P_{km}^{-1}(ik) P_{km}$

$= B_{km}(1) B_{mk}(-1) B_{km}(1) [-1]_k B_{ik}(1) B_{km}(1) B_{mk}(-1) B_{km}(1) [-1]_k$

$= B_{km}(1) B_{mk}(-1) B_{km}(1) B_{ik}(-1) B_{km}(-1) B_{mk}(1) B_{km}(-1)$ by 6,7

$= B_{km}(1) B_{mk}(-1) B_{ik}(-1) B_{im}(1) B_{mk}(1) B_{km}(-1)$ by 1,2,3

$= B_{km}(1) B_{ik}(-1) B_{im}(1) B_{ik}(1) B_{km}(-1)$ by 1,2,3

$= B_{km}(1) B_{im}(1) B_{km}(-1)$ by 1,2

$= B_{im}(1)$ by 1,2

$= (im)$

(10) $O_m^{-1}(ik) O_m = [-1]_m^{-1} B_{ik}(1) [-1]_m$

$= B_{ik}(1)$ by 6

$= (ik)$

(11) $O_i^{-1}(ik) O_i = [-1]_i^{-1} B_{ik}(1) [-1]_i$

$= B_{ik}(-1)$ by 6

$= B_{ik}(1)^{-1}$ by 1

$= (ik)^{-1}$

(12) $O_k^{-1}(ik) O_k = [-1]_k^{-1} B_{ik}(1) [-1]_k$

$= B_{ik}(-1)$ by 6

$= B_{ik}(1)^{-1}$ by 1

$= (ik)^{-1}$

(13) $O_i P_{ik}(ik)(ki)^{-1}(ik)$

$= [-1]_i B_{ik}(1) B_{ki}(-1) B_{ik}(1) [-1]_i B_{ik}(1) B_{ki}(1)^{-1} B_{ik}(1)$

$= B_{ik}(-1) B_{ki}(1) B_{ik}(-1) B_{ik}(1) B_{ki}(-1) B_{ik}(1)$ by 6,7,1

$= I$ by 1

(14) $(ik)(im) = B_{ik}(1) B_{im}(1) = B_{im}(1) B_{ik}(1)$ by 2

$= (im)(ik)$

(15) $(ik)(mk) = B_{ik}(1) B_{mk}(1) = B_{mk}(1) B_{ik}(1)$ by 2

$= (mk)(ik)$

(16) $(ik)(km)(ik)^{-1}(km)^{-1}(im)^{-1}$

$= B_{ik}(1) B_{km}(1) B_{ik}(1)^{-1} B_{km}(1)^{-1} B_{im}(1)^{-1}$

$= B_{km}(1) B_{ik}(1) B_{im}(1) B_{ik}(1)^{-1} B_{km}(1)^{-1} B_{im}(1)^{-1}$ by 3

$= I$ by 2. □

It may be helpful here to give a brief sketch of Nielsen's proof that (1)-(16) are a set of defining relations for $GE_3(Z)$.

Let Ω be the subgroup of $GE_3(Z)$ generated by the P_{ik} and O_i. This is just the orthogonal group, or the matrices with exactly one entry of ±1 in each row and column, and zero elsewhere. Then a simple order calculation shews that the relations (A) (page 29) present Ω. The relations (B) enable any matrix in $GE_3(Z)$ to be written in the form

$$\omega \cdot \Pi(ik)$$

where $\omega \in \Omega$. If $M \in GE_3(Z)$, $M = (e_{ik})$, put $\sigma(M) = \sum_{i,k} e_{ik}^2$

Then a straightforward calculation shews $\sigma(M) \geqslant 3$, with equality iff $M \in \Omega$. Further, $\sigma(M) = \sigma(M\omega) = \sigma(\omega M)$, any $\omega \in \Omega$. Now suppose $M = F_1 F_2 \ldots F_r$ where $F_j = P_{ik}$ or O_i or (ik). Define

$$\begin{aligned} \sigma_1 &= \sigma(F_1 F_2 \ldots F_r) \\ \sigma_2 &= \sigma(F_2 F_3 \ldots F_r) \\ &\vdots \\ \sigma_r &= \sigma(F_r) \\ \sigma_{r+1} &= \sigma(I) = 3 \end{aligned}$$

The numbers $\sigma_1, \sigma_2, \ldots \sigma_{r+1}$ are called the diagram of M. Then by an inductive argument, Nielsen shews that for any such M, using only (1)-(16), we can obtain $M = M'$, where $M' = F_1' F_2' \ldots F_s'$ has monotone diagram, $\sigma_1' \geqslant \sigma_2' \geqslant \ldots \geqslant \sigma_{s+1}' = 3$. Thus if $M=I$, we must have $\sigma_1' = \sigma_2' = \ldots = \sigma_{s+1}' = 3$ (since $\sigma_1' = \sigma(M) = \sigma(I) = 3$) and so $F_i' \in \Omega$, and the relation $M' = I$ is a relation of Ω and so is a consequence of (A); further, $M = M'$ is a consequence of (A,B,C) and so $M=I$ is a consequence of (A,B,C), i.e., of (1)-(16).

In [3], Magnus uses Nielsen's result to get a presentation of $GE_n(Z)$, $n \geq 3$. We can generalize his method to prove theorems that hold for Z or $k[x]$ and indeed for a class of rings (see (3.11)) which includes any Euclidean ring; essentially our results shew that to see whether such rings are universal for GE_n, $n \geq 3$, it is sufficient to look at the case $n = 3$.

Let $A_n(R)$ be the subgroup of $GE_n(R)$ generated by $GE_{n-1}(R)$ and all $B_{in}(x)$, $i<n$. Every matrix in $A_n(R)$ has bottom row $(0,0,\ldots,0,1)$, but unless R is a GE_{n-1}-ring, the converse need not hold. For example, let $R = k[x,y]$, and put

$$A = \begin{pmatrix} 1+xy & x^2 \\ -y^2 & 1-xy \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} A & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0\ 0 & 1 \end{pmatrix}$$

Then $B \in GE_3(R)$, but $B \notin A_3(R)$ since $A \notin GE_2(R)$ (see [1; Proposition (7.3)]).

Lemma (3.9). If $A \in A_n(R)$ (any ring R) then there is a unique normal form

(a) $\quad A = \prod_{i<n} B_{in}(x_i)\cdot A_0 \qquad (x_i \in R,\ A_0 \in GE_{n-1}(R))$

Further, A can be brought to this form using only the universal relations, i.e. if A is a product of $B_{ij}(x)$ $(1\leq i<n,\ 1\leq j\leq n)$ and $[\alpha_1,\ldots\alpha_{n-1},1]$ then $\exists\ x_i$ $(1\leq i<n)$ such that

$$A \to \prod_{i<n} B_{in}(x_i)$$

Proof. We have $(i<n)$

$$B_{ij}(x)B_{kn}(y) = \begin{cases} B_{kn}(y)B_{ij}(x) & \text{by 2, if } j\neq k \\ B_{kn}(y)B_{in}(xy)B_{ij}(x) & \text{by 2,3, if } j=k \end{cases}$$

and

$$[\alpha_1,\ldots\alpha_{n-1},1]B_{kn}(y) \doteq B_{kn}(\alpha_k y)[\alpha_1,\ldots\alpha_{n-1},1] \qquad \text{by 6.}$$

Thus, by an inductive argument, if $A_1 \in GE_{n-1}(R)$

$$A_1\cdot\prod_{i<n} B_{in}(x_i) = \prod_{i<n} B_{in}(y_i)\cdot A_1 \qquad \text{by 2,3,6}$$

where

$$A_1\cdot\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ 0 \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \\ 0 \end{pmatrix}$$

So $A_1\cdot\prod_{i<n} B_{in}(x_i) \to \prod_{i<n} B_{in}(y_i)$

Then

$$\prod_{i<n} B_{in}(y_i)\prod_{i<n} B_{in}(y_i') = \prod_{i<n} B_{in}(y_i+y_i') \qquad \text{by 1,2}$$

Thus if $A \in A_n(R)$, $\exists\, x_i \in R$ such that

$$A \to \prod_{i<n} B_{in}(x_i)$$

So $\exists\, A_0 \in GE_{n-1}(R)$ such that $A = \prod_{i<n} B_{in}(x_i)\cdot A_0$

Suppose also $(B_0 \in GE_{n-1}(R))$ $A = \prod_{i<n} B_{in}(y_i)\cdot B_0$

Then $\prod_{i<n} B_{in}(x_i - y_i) = B_0 A_0^{-1}$, whence $x_i = y_i$, $A_0 = B_0$, and we have uniqueness. □

Lemma (3.10). $A_n(R)$ (any ring R) has the following presentation:

Generators: $B_{ij}(x)$ $(i<n)$ and $[\alpha_1, \ldots \alpha_{n-1}, 1]$

Relations: The universal relations 1.-7., where applicable, together with the relations of $GE_{n-1}(R)$.

Proof. By (3.9), all that remains to be shewn is that $A\cdot B$ can be put in normal form (where each of A,B is in normal form) using only the prescribed relations.

$$\begin{aligned} A\cdot B &= \prod_{i<n} B_{in}(x_i)A_0 \cdot \prod_{i<n} B_{in}(y_i)B_0 \\ &= \prod_{i<n} B_{in}(x_i + y_i')\cdot A_0 B_0 \quad (y_i' \in R) \text{ by the same argument as in (3.9)} \\ &= \prod_{i<n} B_{in}(x_i + y_i')\cdot C_0 \quad (C_0 = A_0 B_0 \in GE_{n-1}(R)). \quad \square \end{aligned}$$

We note in passing that a similar proof shews that the group of upper triangular invertible matrices, and the group of unitriangular matrices, each have presentations consisting of the obvious generators together with the applicable universal relations.

Definition: R is a strong GE_n-ring if given $a_1, \ldots a_k$ $(k \leq n)$ and $b_1, \ldots b_k$ in R, b_i not all zero, such that

$$a_1 b_1 + \ldots + a_k b_k = 0$$

there exists $P \in GE_k(R)$ such that $(a_1, \ldots a_k)\cdot P$ has at least one zero entry.

Note that a strong GE_n-ring is a GE_n-ring; it is also a strong GE_m-ring for all $m \leq n$. A strong GE_1-ring is just an integral domain (not necessarily commutative).

Definition: R is a right Ore-ring if for all a_1, $a_2 \in R$ $\exists\, b_1$, $b_2 \in R$, not both zero, such that $a_1 b_1 + a_2 b_2 = 0$.

Now suppose R is both a right Ore-ring and a strong GE_2-ring (e.g. R = any Euclidean ring); then given $a,b \in R$ $\exists\, P \in GE_2(R)$, $c \in R$, such that $(a,b)\cdot P = (c,0)$.
Conversely, suppose R has this property; then it is a right Ore-ring, and it is a strong GE_2-ring iff it is an integral domain.

Definition: A ring R is <u>GE_2-reducible</u> if for all $a,b \in R$ $\exists\, P \in GE_2(R)$, $c \in R$, such that $(a,b)\cdot P = (c,0)$.

<u>Theorem (3.11)</u>. (cf. Magnus [3]) If R is GE_2-reducible, then $GE_n(R)$, $n \geqslant 3$, has the following presentation:

Generators: $B_{ij}(x)$, $[\alpha_1,\ldots\alpha_n]$

Relations: The universal relations 1.-7., together with the relations of $GE_3(R)$ (i.e. relations involving just three subscripts).

<u>Proof</u>. The theorem is trivial for $n = 3$, so assume $n > 3$ and use induction.

Let M_{ij} denote any product of $B_{ij}(x)$, $B_{ji}(y)$, $[\alpha,\beta]_{ij}$, i.e. (since R is certainly a GE_2-ring) any matrix in $GE_n(R)$ which differs from the identity matrix only at the intersections of the i,j rows and columns. We stress that M_{ij} will denote <u>any</u> matrix of the appropriate form, so we shall write $M_{ji} = M_{ij}$, $M_{ij}M_{ij} = M_{ij}$, $B_{ij}(x)M_{ij} = M_{ij}$, etc.

Let $A \in GE_n(R)$,

$$A = \begin{bmatrix} \vdots & & \\ \vdots & & \\ a & b & \cdots \end{bmatrix}$$

Now $\exists\, M \in GE_2(R)$ such that $(a,b)\cdot M = (c,0)$ i.e. $\exists\, M_{12}$ such that

$$A = \begin{bmatrix} \vdots & & \\ \vdots & & \\ 0 & c & \cdots \end{bmatrix} M_{12} = \begin{bmatrix} \vdots & & & \\ \vdots & & & \\ 0 & 0 & d & \cdots \end{bmatrix} M_{32}M_{21}$$

$$= \ldots = A_1 M_{n\,n-1} \cdots M_{21}, \text{ where } A_1 \in A_n(R).$$

(This reduction of A is essentially due (in the case R = Z) to Magnus, op. cit.)

Then $A_1 = \prod_1^{n-1} B_{in}(x_i)\cdot A_0 \quad (A_0 \in GE_{n-1}(R))$

Suppose $\prod_1^{n-1} B_{in}(x_i)A_0M_{nn-1}\cdots M_{21} = I$ (*)

Then $M_{n-1n-2}\cdots M_{21}\prod_1^{n-1} B_{in}(x_i)A_0M_{nn-1} = I$ by 1,7

By relations of $A_n(R)$ we can write this as

$$\prod_1^{n-1} B_{in}(x_i')A_{00}M_{nn-1} = I$$

so $$\prod_1^{n-1} B_{in}(x_i')A_{00} = M_{nn-1} \qquad (**)$$

and thus $M_{nn-1} \in A_n(R)$, and so (**) is a relation of $A_n(R)$. By (3.10), (*) is thus a consequence of the universal relations and the relations of $GE_{n-1}(R)$.

So it remains to shew that

$$A = \prod_1^{n-1} B_{in}(x_i)A_0M_{nn-1}\cdots M_{21}\cdot\prod_1^{n-1} B_{in}(y_i)A_1M_{nn-1}\cdots M_{21}$$

can be expressed in the form

$$\prod_1^{n-1} B_{in}(x_i')A_2M_{nn-1}\cdots M_{21}$$

(A_0, A_1, $A_2 \in GE_{n-1}(R)$) using only the universal relations and the relations of $GE_3(R)$; by induction we can use relations of $GE_{n-1}(R)$, and hence by (3.10) we can use relations of $A_n(R)$.

For the rest of this proof, ' $\rightarrow$ ' will mean ' =, using only relations of $GE_3(R)$, $\overline{GE}_{n-1}(R)$, $A_n(R)$, and the universal relations.'

By (3.9), $M_{n-1n-2}\cdots M_{21}\prod_1^{n-1} B_{in}(y_i)A_1 \rightarrow \prod_1^{n-1} B_{in}(y_i')A_2$ ($A_2 \in GE_{n-1}(R)$)

So $A \rightarrow \prod_1^{n-1} B_{in}(x_i)A_0M_{nn-1}\prod_1^{n-1} B_{in}(y_i')A_2M_{nn-1}\cdots M_{21}$

$\rightarrow \prod_1^{n-1} B_{in}(x_i)A_0M_{nn-1}\prod_1^{n-2} B_{in}(y_i')A_2M_{nn-1}\cdots M_{21}$

Then $M_{nn-1}\prod_1^{n-2} B_{in}(y_i') \rightarrow \prod_1^{n-2} B_{in-1}(y_i'')\prod_1^{n-2} B_{in}(\hat{y}_i)M_{nn-1}$ by 1,2,3,6

$$A_0\prod_1^{n-2} B_{in-1}(y_i'') = A_3 \in GE_{n-1}(R)$$

and $\prod_1^{n-1} B_{in}(x_i)A_3\prod_1^{n-2} B_{in}(\hat{y}_i) \rightarrow \prod_1^{n-1} B_{in}(x_i')A_3$

So $A \rightarrow \prod_1^{n-1} B_{in}(x_i')A_3M_{nn-1}A_2M_{nn-1}\cdots M_{21}$

Now $A_2 = \prod_1^{n-2} B_{in-1}(z_i)\hat{A}_2M_{n-1n-2}\cdots M_{21}$ ($\hat{A}_2 \in GE_{n-2}(R)$)

Then $M_{n\,n-1}A_2 = M_{n\,n-1}\prod_1^{n-2} B_{i\,n-1}(z_i)\hat{A}_2 M_{n-1\,n-2}\cdots M_{21}$

$\rightarrow \prod_1^{n-1} B_{i\,n-1}(z_i')\prod_1^{n-2} B_{i\,n}(z_i'')M_{n\,n-1}\hat{A}_2 M_{n-1\,n-2}\cdots M_{21}$ by 1,2,3,6

$\rightarrow \prod_1^{n-1} B_{i\,n-1}(z_i')\prod_1^{n-2} B_{i\,n}(z_i'')\hat{A}_2 M_{n\,n-1}M_{n-1\,n-2}\cdots M_{21}$ by 2,6

$\rightarrow \prod_1^{n-2} B_{i\,n}(z_i'')A_4 M_{n\,n-1}\cdots M_{21}$ by 2 ($A_4 \in GE_{n-1}(R)$).

So $A \rightarrow \prod_1^{n-1} B_{i\,n}(x_i')A_3\prod_1^{n-2} B_{i\,n}(z_i'')A_4 M_{n\,n-1}\cdots M_{21}M_{n\,n-1}\cdots M_{21}$

$\rightarrow \prod_1^{n-1} B_{i\,n}(x_i'')A_5 M_{n\,n-1}\cdots M_{21}M_{n\,n-1}\cdots M_{21}$ ($A_5 \in GE_{n-1}(R)$)

Then $M_{n-2\,n-3}\cdots M_{21}M_{n\,n-1} \rightarrow M_{n\,n-1}M_{n-2\,n-3}\cdots M_{21}$ by 2,6

and $M_{n\,n-1}M_{n-1\,n-2}M_{n\,n-1} \rightarrow B_{n-2\,n}(w_1)B_{n-1\,n}(w_2)M_{n-1\,n-2}\cdot M_{n\,n-1}M_{n-1\,n-2}$ by relations of GE (R)

So $A \rightarrow \prod_1^{n-1} B_{i\,n}(x_i'')A_5 B_{n-2\,n}(w_1)B_{n-1\,n}(w_2)M_{n-1\,n-2}M_{n\,n-1}M_{n-1\,n-2}\cdots M_{21}M_{n-1\,n-2}\cdots M_{21}$

$\rightarrow \prod_1^{n-1} B_{i\,n}(\hat{x}_i)A_5 M_{n-1\,n-2}M_{n\,n-1}M_{n-1\,n-2}\cdots M_{21}M_{n-1\,n-2}\cdots M_{21}$

$\rightarrow \prod_1^{n-1} B_{i\,n}(\hat{x}_i)A_6 M_{n\,n-1}\cdots M_{21}M_{n-1\,n-2}\cdots M_{21}$ ($A_6 \in GE_{n-1}(R)$)

Then $A_7 = M_{n-1\,n-2}\cdots M_{21}M_{n-1\,n-2}\cdots M_{21} \in GE_{n-1}(R)$

and as before, $M_{n\,n-1}A_7 \rightarrow \prod_1^{n-2} B_{i\,n}(\hat{z}_i)A_8 M_{n\,n-1}\cdots M_{21}$ ($A_8 \in GE_{n-1}(R)$)

So $A \rightarrow \prod_1^{n-1} B_{i\,n}(\hat{x}_i)A_6\prod_1^{n\ 2} B_{i\,n}(\hat{z}_i)A_8 M_{n\,n-1}\cdots M_{21}$

$\rightarrow \prod_1^{n-1} B_{i\,n}(\hat{\hat{x}}_i)A_9 M_{n\,n-1}\cdots M_{21}$ ($A_9 \in GE_{n-1}(R)$). □

Thus we have immediately:

Corollary (3.12). If R is GE_2-reducible and universal for GE_3, it is universal for GE_n, all $n \geqslant 3$. □

Corollary (3.13). The ring Z of rational integers is universal for GE_n, all n.

Proof. The case n=2 is covered by [1;(5.2)], and the case n=3 by (3.8). (3.12) now gives the result. □

We conclude this chapter with some remarks on the interdependence of certain of the universal relations.

<u>Proposition (3.14)</u>. In $GE_n(R)$ (any ring R, $n \geqslant 3$) the universal relation 5. is a consequence of the other universal relations.

<u>Proof</u>. $B_{ji}(1)B_{ij}(-1)B_{ji}(-x)B_{ij}(1)B_{ji}(-1)$

$$= B_{ji}(1)B_{ij}(-1)B_{ki}(-1)B_{jk}(-x)B_{ki}(1)B_{jk}(x)B_{ij}(1)B_{ji}(-1)$$
by 1,3 ($k \neq i,j$)

$$= B_{ji}(1)B_{ki}(-1)B_{kj}(-1)B_{jk}(-x)B_{ik}(x)B_{ki}(1)B_{kj}(1) \cdot B_{jk}(x)B_{ik}(-x)B_{ji}(-1)$$
by 1,3

$$= B_{ki}(-1)B_{kj}(-1)B_{ki}(1)B_{jk}(-x)B_{ik}(x)B_{jk}(x) \cdot B_{ki}(1)B_{kj}(1)B_{ki}(-1)B_{jk}(x)B_{ik}(-x)B_{jk}(-x)$$
by 1,2,3

$$= B_{kj}(-1)B_{ik}(x)B_{kj}(1)B_{ik}(-x)$$
by 1,2

$$= B_{ij}(x)$$
by 1,2,3. □

For n=2, this need not be the case:

<u>Proposition (3.15)</u>. In $GE_2(Z)$, the universal relation 5. is independent of the other universal relations.

<u>Proof</u>. Consider the group $G = \{\pm 1, \pm\epsilon\}$ where $\epsilon^2 = 1$, and the map $GE_2(Z) \to G$ given by

$$B_{12}(n) \to \epsilon^n \quad (n \in Z)$$
$$B_{21}(n) \to 1 \quad (n \in Z)$$
$$[\alpha,\beta] \to \alpha\beta \quad (\alpha,\beta = \pm 1)$$

Then it is clear that the map is consistent with the relations 1,4,6,7 (2,3 are vacuous in $GE_2(R)$) but not with 5. □

For certain values of the element x occuring in 5, however, 5. is a consequence of 1,4,6,7:

<u>Proposition (3.16)</u>. (Any R) The relation

$$B_{ij}(1-\alpha) = B_{ji}(1)B_{ij}(-1)B_{ji}(\alpha-1)B_{ij}(1)B_{ji}(-1) \quad (\alpha \in U(R))$$

is a consequence of 1,4,6,7 in $GE_2(R)$.

<u>Proof</u>. We have

$$B_{ij}(\alpha^{-1}-1)B_{ji}(1)B_{ij}(\alpha-1)B_{ji}(-\alpha^{-1}) = D_{ij}(\alpha^{-1}) \quad \text{by } 1,4$$

and

$$B_{ji}(\alpha^{-1}-1)B_{ij}(1)B_{ji}(\alpha-1)B_{ij}(-\alpha^{-1}) = D_{ji}(\alpha^{-1}) \quad \text{by } 1,4$$

By 7, $D_{ij}(\alpha^{-1})D_{ji}(\alpha^{-1}) = I$.

So, using 1,

$B_{ij}(\alpha^{-1}-1)B_{ji}(1)B_{ij}(\alpha-1)B_{ji}(-1)B_{ij}(1)B_{ji}(\alpha-1)B_{ij}(-\alpha^{-1}) = I$

Using 1 again,

$B_{ij}(-1)B_{ji}(1)B_{ij}(\alpha-1)B_{ji}(-1)B_{ij}(1)B_{ji}(\alpha-1) = I$

and the result follows. □

Corollary (3.17). If R is a local ring, and if $|R/J| > 2$ (J=Jacobson radical) then 5. is a consequence of 1,4,6,7 in $GE_2(R)$.

Proof. If $x \notin U(R)$ then $x = \alpha-1$, where $\alpha = 1+x \in U(R)$.
If $x \in U(R)$ and $1+x = \alpha \in U(R)$ then $x = \alpha-1$ as before.
Other case: $x \in U(R)$, $1+x \notin U(R)$.
Then $\exists\ \alpha,\beta \in U(R)$ with $\alpha+\beta = 1$.
So $-1 = (\alpha-1) + (\beta-1)$
and $x = (\alpha-1) + (\delta-1)$ where $\delta = \beta+x+1 \in U(R)$.
By 1, $B_{ij}(x) = B_{ij}(\alpha-1)B_{ij}(\delta-1)$ and the result now follows. □

Note that in the excluded cases of (3.17) we can use an argument similar to (3.15) to shew that 5. is independent; if $\epsilon^2 = 1$, just map

$$\begin{aligned} B_{12}(\alpha) &\mapsto \epsilon \quad (\alpha\in U(R)) \\ B_{12}(x) &\mapsto 1 \quad (x\notin U(R)) \\ B_{21}(y) &\mapsto 1 \quad (y\in R) \\ [\alpha,\beta] &\mapsto 1 \quad (\alpha,\beta\in U(R)). \end{aligned}$$

4. Quasi-universal rings.

We already know from (3.7) that skew fields are universal for GE_n, all n. The Wedderburn-Artin structure theorem states that every semi-simple ring with the minimum condition on right ideals is a finite direct product of full matrix rings over skew fields. Now if R is any ring,

$$(R_n)_m \cong R_{nm}$$

and if R,S are rings,

$$GE_n(R\times S) \cong GE_n(R)\times GE_n(S)$$

which prompts us to ask whether the property of being universal for GE_n is preserved under formation of direct products and of matrix rings; counter-examples to these hypotheses are given in (4.1) and (4.7). However, if R,S are universal for GE_n, we can shew (see (4.2)) that $GE_n(R\times S)$ has a presentation consisting of the universal relations, together with

$$B_{ij}(x)B_{ji}(y) = B_{ji}(y)B_{ij}(x) \quad \text{whenever } xy=0=yx.$$

Then if R is a GE_n-ring, universal for GE_{nm}, we can shew (see (4.9)) that $GE_m(R_n)$ has a presentation consisting of the universal relations together with

$$B_{ij}(x)B_{ji}(y)[1+yx]_j = [1+xy]_i B_{ji}(y)B_{ij}(x) \quad \text{whenever } 1+xy \in U(R_n).$$

Thus we make the following definition:

Definition: A ring R is <u>quasi-universal for GE_n</u> if $GE_n(R)$ has the following presentation:

Generators: $B_{ij}(x)$, $[\alpha_1,\dots\alpha_n]$ $(x\in R, \alpha_k \in U(R),\ 1\leq i,j,k\leq n,\ i\neq j)$

Relations: The universal relations (page 15) with the following in place of 4:

4'. $B_{ij}(x)B_{ji}(y)[1+yx]_j = [1+xy]_i B_{ji}(y)B_{ij}(x)$ whenever $1+xy \in U(R)$.

(Recall that $1+xy \in U(R) \Rightarrow 1+yx \in U(R)$; indeed, this follows from 4'.)

Since 4 is a special case of 4' (just put $x=\alpha-1$, $y=1$ and use 6), a ring which is universal for GE_n is quasi-universal for GE_n. Note also that a ring which is quasi-universal for GE_n and universal for GE_2 is universal for GE_n.

<u>Proposition (4.1)</u>. Let R be the field of two elements. Then $R\times R$ is not universal for GE_2.

Proof. $R\times R = \{(n,m) | n,m = 0,1\}$ where $1+1 = 0$

$\therefore U(R\times R) = 1$

Let $S = \{0, 1, x, 1+x\}$ where $1+1 = 0$ and $x^2 = 1$.
The map $\theta: R\times R \to S$ determined additively by $(1,1)^\theta = 1$ and $(1,0)^\theta = x$ is a U-homomorphism. But

$$(1,0)\cdot(0,1) = 0 = (0,1)\cdot(1,0)$$

whereas

$$(1,0)^\theta(0,1)^\theta = x(1+x) = 1+x \neq 0.$$

Thus by (2.2), $R\times R$ is not universal for GE_2. □
However, $R\times R$ is quasi-universal for GE_2, by (4.2)(ii).

Theorem (4.2). (i) If R, S are universal for GE_n, $GE_n(R\times S)$ has a presentation consisting of the usual generators, and the universal relations together with

(*) $\quad B_{ij}(x)B_{ji}(y) = B_{ji}(y)B_{ij}(x)$ whenever $xy=0=yx$.

(ii) If R, S are quasi-universal for GE_n, so is $R\times S$.

Proof. Clearly $(R\times S)_n \cong R_n\times S_n$

and $GE_n(R\times S) \cong GE_n(R)\times GE_n(S)$.

Thus $GE_n(R\times S)$ has a presentation consisting of presentations of $GE_n(R)$ and $GE_n(S)$, together with relations ensuring that these two subgroups commute with each other elementwise.

If $(x,y)\in R\times S$, we write $B_{ij}(x,y)$ for $B_{ij}((x,y))$.
Then

$B_{ij}(x,0)B_{ij}(0,y) = B_{ij}(0,y)B_{ij}(x,0)$ by 1

$B_{ij}(x,0)B_{ji}(0,y) = B_{ji}(0,y)B_{ij}(x,0)$ by (*)

$B_{ij}(x,0)B_{jk}(0,y) = B_{jk}(0,y)B_{ij}(x,0)$ by 3,1

$B_{ij}(x,0)B_{ki}(0,y) = B_{ki}(0,y)B_{ij}(x,0)$ by 3,1

$B_{ij}(x,0)B_{kr}(0,y) = B_{kr}(0,y)B_{ij}(x,0)$ by 2

Now $U(R\times S) = U(R)\times U(S)$. Let $\alpha_k\in U(R)$, $\beta_k\in U(S)$.

$B_{ij}(x,0)[(1,\beta_1),....(1,\beta_n)] = [(1,\beta_1),....(1,\beta_n)]B_{ij}(x,0)$ by 6

$[(\alpha_1,1),....(\alpha_n,1)]B_{ij}(0,y) = B_{ij}(0,y)[(\alpha_1,1),....(\alpha_n,1)]$ by 6

$[(\alpha_1,1),....(\alpha_n,1)][(1,\beta_1),....(1,\beta_n)]$

$= [(1,\beta_1),....(1,\beta_n)][(\alpha_1,1),....(\alpha_n,1)]$ by 7.

Thus the universal relations for $R\times S$, together with (*), are sufficient to ensure that $GE_n(R)$ and $GE_n(S)$ (as subgroups of $GE_n(R\times S)$) commute elementwise.

It remains to shew that the universal relations for $GE_n(R)$ and $GE_n(S)$ follow from 1.-7. and (*).
Now 1,2,3,6,7 for $GE_n(R)$ are just special cases of the corresponding relations for $GE_n(R\times S)$. Suppose $\alpha \in U(R)$.

$$B_{ij}(\alpha-1,0)B_{ji}(1,0)$$
$$= B_{ij}((\alpha,1)-(1,1))B_{ji}(1,1)B_{ji}(0,-1) \quad \text{by } 1$$
$$= D_{ij}(\alpha,1)B_{ji}(\alpha,1)B_{ij}((1,1)-(\alpha^{-1},1))B_{ji}(0,-1) \quad \text{by } 4$$
$$= D_{ij}(\alpha,1)B_{ji}(\alpha,0)B_{ij}(1-\alpha^{-1},0) \quad \text{by } 1,(*)$$

which is the form taken by 4 for $GE_n(R)$ in $GE_n(R\times S)$.
If $x \in R$, $B_{ij}(x,0)$

$$= B_{ji}(1,1)B_{ij}(-1,-1)B_{ji}(-x,0)B_{ij}(1,1)B_{ji}(-1,-1) \quad \text{by } 5$$
$$= B_{ji}(1,0)B_{ij}(-1,0)B_{ji}(-x,0)B_{ij}(1,0)B_{ji}(-1,0) \quad \text{by } 1,(*)$$

which is the form taken by 5 for $GE_n(R)$ in $GE_n(R\times S)$.
Similarly for $GE_n(S)$. This completes (i).
(ii): Since (*) is a special case of 4′, we have only to shew that 4′ for $GE_n(R)$ and $GE_n(S)$ is a consequence of the quasi-universal relations for $GE_n(R\times S)$.
If $x,y \in R$ and $1+xy \in U(R)$, then $(1,1)+(x,0)(y,0) \in U(R\times S)$.
So $B_{ij}(x,0)B_{ji}(y,0)[(1+yx,1)]_j$

$$= [(1+xy,1)]_i B_{ji}(y,0)B_{ij}(x,0) \quad \text{by } 4'$$

which is the form taken by 4′ for $GE_n(R)$ in $GE_n(R\times S)$.
Similarly for $GE_n(S)$. □

<u>Corollary (4.3)</u>. Let $S = \prod_{\lambda\in\Lambda} R_\lambda$, $|\Lambda| < \infty$

(i) If R_λ is universal for GE_n, all $\lambda\in\Lambda$, $GE_n(S)$ has a presentation consisting of the universal relations and (*).
(ii) If R_λ is quasi-universal for GE_n, all $\lambda\in\Lambda$, so is S.
<u>Proof</u>. The proof is a straightforward generalization of (4.2). □

We note in passing that there does not seem to be any reason why the above should hold for an infinite direct product of rings, except in some special cases (e.g. when R_λ is a local ring, all $\lambda\in\Lambda$) when the proof that any relation of a given length in $GE_n(R_\lambda)$ follows from the universal relations is a standard process whose form and length are independent of λ. It seems unlikely that the direct product of infinitely many copies of Z is a GE_n-ring (for n=2, it isn't; see [1; page 11]) and whether it is quasi-universal

for GE_n does not appear to be a trivial question.
Definitions: Let $E_n(R)$ be the subgroup of $GE_n(R)$ generated by all $B_{ij}(x)$, $x \in R$, $1 \leq i,j \leq n$, $i \neq j$. Let $D_n(R)$ be the subgroup of $GE_n(R)$ generated by all $[\alpha]_k$, $\alpha \in U(R)$, $1 \leq k \leq n$.

Lemma (4.4). If R is universal for GE_n, $E_n(R) \cap D_n(R)$ is generated by all $D_{ij}(\alpha)$, $\alpha \in U(R)$, $1 \leq i,j \leq n$, $i \neq j$.
(cf. [1;(9.1), Corollary 1])

Proof. $D_{ij}(\alpha) \in E_n(R)$ by 4 and 1.
Now if $[\alpha_1, \ldots \alpha_n] = \prod_{i,j,x} \{B_{ij}(x)\}$, this relation must follow from 1.-7. No diagonal matrices are introduced by any of these relations except 4, and it follows that $[\alpha_1, \ldots \alpha_n]$ is a product of $D_{ij}(\alpha)$, suitable i,j,α. □

Note that (4.4) need not hold for quasi-universal R. Let $[\alpha]_1 = \prod_{ijk} D_{ij}(\alpha_{ijk})$ in some order (α's $\in U(R)$). Then, as in the proof of (4.5) below, it follows that $\alpha \in U(R)'$ (the derived group of U(R)). Now put K = the field of two elements, and put $R = K_2$.

$$I_2 + \begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix} = I_2$$

$$I_2 + \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix} = \begin{pmatrix}1&1\\0&1\end{pmatrix}$$

So by relation 4′, $[\begin{pmatrix}1&1\\0&1\end{pmatrix}]_1 \in E_n(R) \cap D_n(R)$. But U(R) is the dihedral group of order 6, generated by $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and $\begin{pmatrix}1&1\\1&0\end{pmatrix}$. So $U(R)'$ is cyclic of order 3, generated by $\begin{pmatrix}1&1\\1&0\end{pmatrix}$. Thus $\begin{pmatrix}1&1\\0&1\end{pmatrix} \notin U(R)'$, and so $E_n(R) \cap D_n(R)$ is not generated by all $D_{ij}(\alpha)$, $\alpha \in U(R)$. It follows from (4.4) that R is not universal for GE_n, any n : for a second proof of this, see (4.7). But, as we shall see in (4.11), R is quasi-universal for GE_n, all n.

Proposition (4.5). If R is universal for GE_n and $[\alpha]_1 \in E_n(R)$, then $\alpha \in U(R)'$.

Proof. If

(*) $\quad [\alpha]_1 = \prod_{ijk} D_{ij}(\alpha_{ijk})$ in some order, we have the relation $D_{ij}(\beta) = D_{1j}(\beta)D_{1i}(\beta^{-1})$, and so

(**) $[\alpha]_1 = \prod_{ij} D_{1i}(\beta_{ij})$

Then $\alpha = \theta_0\delta_1\theta_1\delta_2\theta_2\ldots\ldots\delta_r\theta_r$

where δ_i are the arguments of the D_{1n} in (**): thus

$$\delta_1^{-1}\delta_2^{-1}\ldots\ldots\delta_r^{-1} = 1$$

$$\therefore \quad \alpha \equiv \theta_0\theta_1\ldots\ldots\theta_r \mod U(R)'$$

Now $\theta_0\theta_1\ldots\ldots\theta_r = \phi_0\psi_1\phi_1\psi_2\phi_2\ldots\ldots\psi_s\phi_s$

where ψ_i are the arguments of the D_{1n-1} in (**): repeat the argument to get

$$\alpha \equiv \phi_0\phi_1\ldots\ldots\phi_s \mod U(R)'$$

After n-2 such steps we have

$$\alpha \equiv \lambda_1\lambda_2\ldots\ldots\lambda_t \mod U(R)'$$

where λ_i are the arguments of the D_{12} in (**). Finally we get, $\because \lambda_1^{-1}\lambda_2^{-1}\ldots\ldots\lambda_t^{-1} = 1$,

$$\alpha \equiv 1 \mod U(R)'$$

Note that we have not yet used the fact that R is universal for GE_n.

If R is universal for GE_n and $[\alpha]_1 \in E_n(R)$, then (*) follows by (4.4), whence the result. □

Now note that for any ring R, $GE_2'(R) \subseteq E_2(R)$ (see [1;(9.1), Corollary 3] or (5.4)). Then we have:

<u>Corollary (4.6)</u>. Let R be a GE_2-ring, and $S = R_2$.

If $GE_2'(R) \neq E_2(R)$, then S is not universal for GE_n, any n.

<u>Proof</u>. Given $x \in R$ and i,j = 1,2 or 2,1 we have (in $GE_{2n}(R)$, n>1)

$$B_{ij}(x) = B_{i3}(-x)B_{3j}(-1)B_{i3}(x)B_{3j}(1)$$

As an equation in $GE_n(S)$, this reads

$$[B_{ij}(x)]_1 = B_{12}(-xe_{i1})B_{21}(-e_{1j})B_{12}(xe_{i1})B_{21}(e_{1j}) \quad \in E_n(S)$$

If S is universal for GE_n, we have by (4.5)

$$B_{ij}(x) \in U(S)' = GE_2'(R), \text{ since R is a } GE_2\text{-ring.}$$

$$\therefore \quad E_2(R) \subseteq GE_2'(R).$$

Now $[\alpha\beta]_i = D_{ji}(\beta)[\beta\alpha]_i D_{ij}(\beta)$, so since $D_{km}(\beta) \in E_2(R)$, it follows that $GE_2(R)/E_2(R)$ is abelian, and so

$$GE_2'(R) \subseteq E_2(R)$$

Thus $GE_2'(R) = E_2(R)$ and we have a contradiction. □

Corollary (4.7). Let R be the field of two elements, and $S = R_2$. Then S is not universal for GE_n, any n.
Proof. By (4.6) we have only to shew $GE_2'(R) \neq E_2(R)$. Since $U(R) = 1$, $E_2(R) = GE_2(R)$.
Then $|GE_2(R)| = 6$. Since $GE_2(R)$ is not abelian, it is the dihedral group of order 6, which is not a perfect group. □

We now introduce some more notation.

Write $E_{ij}(x) = xe_{ij}$ where e_{ij} are the usual 'matrix units'
= the matrix with x in the i,j position and 0 elsewhere.

Write $\underset{n}{B}{}^{ij}_{kr}(x) = B_{kn-n+i,\,rn-n+j}(x)$

Where there is no ambiguity, we shall write $B^{ij}_{kr}(x)$ for $\underset{n}{B}{}^{ij}_{kr}(x)$.

Write $\underset{n}{[\alpha]^i_k} = [\alpha]_{kn-n+i}$. Again, we shall generally write $[\alpha]^i_k$ for $\underset{n}{[\alpha]^i_k}$. Put $D^{ij}_{kr}(\alpha) = [\alpha]^i_k[\alpha^{-1}]^j_r$.

Now let R be a ring and $S = R_n$. Then $R_{nm} \cong S_m$ in a natural way: specifically, if $A = (a_{ij}) \in R_{nm}$ and $B = (b_{ij}) \in S_m$, then since $b_{ij} \in S = R_n$, $b_{ij} = (c^{ij}_{kr})$. We identify A and B if $c^{ij}_{kr} = a_{in-n+k,\,jn-n+r}$, all i,j,k,r

This isomorphism induces an isomorphism

$$U(R_{nm}) \cong U(S_m)$$

i.e. $\quad GL_{nm}(R) \cong GL_m(S)$.

Proposition (4.8). If R is a ring and $S = R_n$, there is a natural isomorphism $\theta: R_{nm} \to S_m$. Assume $m \geq 2$. θ induces an isomorphism between $GL_{nm}(R)$ and $GL_m(S)$, and an isomorphism between $E_{nm}(R)$ and $E_m(S)$. It induces an embedding of $GE_{nm}(R)$ in $GE_m(S)$, and for this to be an isomorphism it is sufficient that R should be a GE_n-ring.

Proof. We already have that θ and its restriction $GL_{nm}(R) \to GL_m(S)$ are isomorphisms. Then

$$\theta: \underset{n}{B}{}^{ij}_{kr}(x) \mapsto B_{kr}(E_{ij}(x)) \qquad (k \neq r)$$

$$\theta: B^{ij}_{kk}(x) \mapsto [B_{ij}(x)]_k \qquad (i \neq j)$$

$$\theta: [\alpha_{11}, \ldots\ldots \alpha_{nm}] \mapsto [[\alpha_{11}, \ldots \alpha_{n1}], \ldots\ldots [\alpha_{1m}, \ldots \alpha_{nm}]]$$

which shews that θ maps $GE_{nm}(R)$ into $GE_m(S)$. Then

$B_{kk}^{ij}(x) = B_{kr}^{ij}(-x)B_{rk}^{jj}(-1)B_{kr}^{ij}(x)B_{rk}^{jj}(1)$ where $r \neq k$

and so, since $m \geqslant 2$, $E_{nm}(R)$ is generated by all $B_{kr}^{ij}(x)$ ($x \in R$, $k \neq r$). Thus θ maps $E_{nm}(R)$ into $E_m(S)$.

If $A \in S_m$, $A = (a_{ij})$ then

$$B_{kr}(A) = \prod_{i,j} B_{kr}(E_{ij}(a_{ij}))$$
$$= \prod_{i,j} B_{kr}^{ij}(a_{ij})^{\theta}$$

so θ maps $E_{nm}(R)$ onto $E_m(S)$.

Let R be a GE_n-ring. To shew that θ maps $GE_{nm}(R)$ onto $GE_m(S)$ it is sufficient to shew that $\alpha \in U(S) \Rightarrow [\alpha]_k \in GE_{nm}(R)^{\theta}$. But $U(S) \cong GE_n(R)$, so θ: $\prod_i C_i \to [\alpha]_k$ for suitable C_i of the form $B_{kk}^{ij}(x)$ or $\prod_i [\beta_i]_k^i$. □

<u>Theorem (4.9)</u>. Let R be a GE_n-ring. Put $S = R_n$.

(I) If $n \geqslant 3$, $m \geqslant 2$ and R is universal for GE_{nm}, then S is universal for GE_m.

(II) If $n, m \geqslant 2$ and R is quasi-universal for GE_{nm}, then S is quasi-universal for GE_m.

<u>Proof</u>. We have to shew that 1.-7. for S imply 1.-7. for R, with 4′ in place of 4 in case (II).

R1(i): $i \neq j$:
$$B_{kk}^{ij}(x)B_{kk}^{ij}(y) = [B_{ij}(x)]_k[B_{ij}(y)]_k$$
$$= [B_{ij}(x)B_{ij}(y)]_k \quad \text{by S7}$$
$$= [B_{ij}(x+y)]_k$$
$$= B_{kk}^{ij}(x+y)$$

(ii) $k \neq r$:
$$B_{kr}^{ij}(x)B_{kr}^{ij}(y) = B_{kr}(E_{ij}(x))B_{kr}(E_{ij}(y))$$
$$= B_{kr}(E_{ij}(x)+E_{ij}(y)) \quad \text{by S1}$$
$$= B_{kr}(E_{ij}(x+y))$$
$$= B_{kr}^{ij}(x+y)$$

This completes R1.

R2(i) $i \neq s$, $j \neq r$:
$$B_{kk}^{ij}(x)B_{kk}^{rs}(y) = [B_{ij}(x)]_k[B_{rs}(y)]_k$$
$$= [B_{ij}(x)B_{rs}(y)]_k \quad \text{by S7}$$
$$= [B_{rs}(y)B_{ij}(x)]_k$$
$$= [B_{rs}(y)]_k[B_{ij}(x)]_k \quad \text{by S7}$$
$$= B_{kk}^{rs}(y)B_{kk}^{ij}(x)$$

(ii) $k \neq p$: $B^{ij}_{kk}(x)B^{rs}_{pp}(y) = [B_{ij}(x)]_k[B_{rs}(y)]_p$

$= [B_{rs}(y)]_p[B_{ij}(x)]_k$ by S7

$= B^{rs}_{pp}(y)B^{ij}_{kk}(x)$

(iii) $k \neq p,q$, $p \neq q$: $B^{ij}_{kk}(x)B^{rs}_{pq}(y) = [B_{ij}(x)]_k B_{pq}(E_{rs}(y))$

$= B_{pq}(E_{rs}(y))[B_{ij}(x)]_k$ by S6

$= B^{rs}_{pq}(y)B^{ij}(x)$

(iv) $k \neq p$, $i \neq s$: $B^{ij}_{kk}(x)B^{rs}_{pk}(y) = [B_{ij}(x)]_k B_{pk}(E_{rs}(y))$

$= B_{pk}(E_{rs}(y)B_{ij}(-x))[B_{ij}(x)]_k$ by S6

$= B_{pk}(E_{rs}(y))[B_{ij}(x)]_k$

$= B^{rs}_{pk}(y)B^{ij}_{kk}(x)$

(v) $k \neq p$, $j \neq r$: $B^{ij}_{kk}(x)B^{rs}_{kp}(y) = [B_{ij}(x)]_k B_{kp}(E_{rs}(y))$

$= B_{kp}(B_{ij}(x)E_{rs}(y))[B_{ij}(x)]_k$ by S6

$= B_{kp}(E_{rs}(y))[B_{ij}(x)]_k$

$= B^{rs}_{kp}(y)B^{ij}_{kk}(x)$

(Vi) $k \neq q$, $t \neq p$; $k \neq t$, $p \neq q$: $B^{ij}_{kt}(x)B^{rs}_{pq}(y) = B_{kt}(E_{ij}(x))B_{pq}(E_{rs}(y))$

$= B_{pq}(E_{rs}(y))B_{kt}(E_{ij}(x))$ by S2

$= B^{rs}_{pq}(y)B^{ij}(x)$

(vii) $k \neq q$, $j \neq r$, $k \neq t$, $t \neq q$: $B^{ij}_{kt}(x)B^{rs}_{tq}(y) = B_{kt}(E_{ij}(x))B_{tq}(E_{rs}(y))$

$= B_{tq}(E_{rs}(y))B_{kt}(E_{ij}(x))B_{kq}(E_{ij}(x)E_{rs}(y))$ by S3

$= B_{tq}(E_{rs}(y))B_{kt}(E_{ij}(x))$ by S1

$= B^{rs}_{tq}(y)B^{ij}_{kt}(x)$

(viii) $k \neq t$, $j \neq r$, $i \neq s$, $i \neq j$, $r \neq s$:

$B^{ij}_{kt}(x)B^{rs}_{tk}(y) = B_{kt}(E_{ij}(x))B_{tk}(E_{rs}(y))$

$= B_{kt}(B_{ij}(x)-I)B_{tk}(I)B_{tk}(-B_{rs}(-y))$ by S1

$= D_{kt}(B_{ij}(x))B_{tk}(B_{ij}(x))B_{kt}(I-B_{ij}(-x))B_{tk}(-B_{rs}(-y))$ by S4

$$= D_{kt}(B_{ij}(x))[-B_{rs}(y)]_t B_{tk}(-B_{rs}(y)B_{ij}(x))$$

$$\cdot B_{kt}(B_{ij}(-x)B_{rs}(-y)-B_{rs}(-y))B_{tk}(I)[-B_{rs}(-y)]_t \quad \text{by S6,7}$$

$$= [B_{ij}(x),\ -B_{ij}(-x)B_{rs}(y)]_{kt}B_{tk}(-B_{rs}(y)B_{ij}(x))$$

$$\cdot B_{kt}(B_{ij}(-x)-I)B_{tk}(I)[-B_{rs}(-y)]_t \quad \text{by S7}$$

$$= [B_{ij}(x),\ -B_{ij}(-x)B_{rs}(y)]_{kt}B_{tk}(-B_{rs}(y)B_{ij}(x))$$

$$\cdot D_{kt}(B_{ij}(-x))B_{tk}(B_{ij}(-x))B_{kt}(E_{ij}(-x))[-B_{rs}(-y)]_t \quad \text{by S4}$$

$$= [-B_{rs}(y)]_t B_{tk}(-B_{rs}(y)B_{ij}(-x)+B_{ij}(-x))$$

$$\cdot B_{kt}(E_{ij}(-x))[-B_{rs}(-y)]_t \quad \text{by S1,6,7}$$

$$= B_{tk}(B_{ij}(-x)-B_{rs}(-y)B_{ij}(-x))B_{kt}(E_{ij}(x)B_{rs}(-y)) \quad \text{by S6,7}$$

$$= B_{tk}(E_{rs}(y))B_{kt}(E_{ij}(x))$$

$$= B^{rs}_{tk}(y)B^{ij}_{kt}(x)$$

The remaining cases of R2, i.e. as in (viii) but with $i=j$ or $r=s$ or both, will be dealt with after R3 (i)-(v).

R3(i) $i\neq q$: $B^{ij}_{kk}(x)B^{jq}_{kk}(y) = [B_{ij}(x)]_k[B_{jq}(y)]_k$

$$= [B_{ij}(x)B_{jq}(y)]_k \quad \text{by S7}$$

$$= [B_{jq}(y)B_{ij}(x)B_{iq}(xy)]_k$$

$$= [B_{jq}(y)]_k[B_{ij}(x)]_k[B_{iq}(xy)]_k \quad \text{by S7}$$

$$= B^{jq}_{kk}(y)B^{ij}_{kk}(x)B^{iq}_{kk}(xy)$$

(ii) $k\neq p$: $B^{ij}_{kk}(x)B^{jr}_{kp}(y) = [B_{ij}(x)]_k B_{kp}(E_{jr}(y))$

$$= B_{kp}(B_{ij}(x)E_{jr}(y))[B_{ij}(x)]_k \quad \text{by S6}$$

$$= B_{kp}(E_{jr}(y)+E_{ir}(xy))[B_{ij}(x)]_k$$

$$= B_{kp}(E_{jr}(y))B_{kp}(E_{ir}(xy))[B_{ij}(x)]_k \quad \text{by S1}$$

$$= B^{jr}_{kp}(y)B^{ij}_{kk}(x)B^{ir}_{kp}(xy) \quad \text{by R2(v)}$$

(iii) $k\neq p$: $B^{ij}_{kp}(x)B^{jr}_{pp}(y) = B_{kp}(E_{ij}(x))[B_{jr}(y)]_p$

$$= [B_{jr}(y)]_p B_{kp}(E_{ij}(x)B_{jr}(y)) \quad \text{by S6}$$

$$= [B_{jr}(y)]_p B_{kp}(E_{ij}(x)+E_{jr}(xy))$$

$= [B_{jr}(y)]_p B_{kp}(E_{ij}(x))B_{kp}(E_{ir}(xy))$ by S1

$= B^{jr}_{pp}(y)B^{ij}_{\ p}(x)B^{ir}_{\ p}(xy)$

(iv) $k\neq t$, $t\neq p$, $k\neq p$: $B^{ij}_{kt}(x)B^{jr}_{tp}(y) = B_{kt}(E_{ij}(x))B_{tp}(E_{jr}(y))$

$= B_{tp}(E_{jr}(y))B_{kt}(E_{ij}(x))B_{kp}(E_{ij}(x)E_{jr}(y))$ by S3

$= B_{tp}(E_{jr}(y))B_{kt}(E_{ij}(x))B_{kp}(E_{ir}(xy))$

$= B^{jr}_{tp}(y)B^{ij}_{kt}(x)B^{ir}_{kp}(xy)$

(v) $k\neq t$, $i\neq j$, $j\neq r$, $i\neq r$:

$B^{ij}_{kt}(x)B^{jr}_{tk}(y) = B_{kt}(E_{ij}(x))B_{tk}(E_{jr}(y))$

$= B_{kt}(B_{ij}(x)-I)B_{tk}(I)B_{tk}(-B_{jr}(-y))$ by S1

$= D_{kt}(B_{ij}(x))B_{tk}(B_{ij}(x))B_{kt}(E_{ij}(x))B_{tk}(-B_{jr}(-y))$ by S4

$= D_{kt}(B_{ij}(x))[-B_{jr}(-y)]_t B_{tk}(-B_{jr}(y)B_{ij}(x))$

$\cdot B_{kt}(E_{ij}(-x)B_{jr}(-y))B_{tk}(I)[-B_{jr}(y)]_t$ by S6,7

$= [B_{ij}(x), -B_{ij}(-x)B_{jr}(-y)]_{kt}B_{tk}(-B_{jr}(y)B_{ij}(x))$

$\cdot B_{kt}(B_{ij}(-x)B_{ir}(xy)-I)B_{tk}(I)[-B_{jr}(y)]_t$ by S7

$= [B_{ij}(x), -B_{ij}(-x)B_{jr}(-y)]_{kt}B_{tk}(-B_{jr}(y)B_{ij}(x))$

$\cdot D_{kt}(B_{ij}(-x)B_{ir}(xy))B_{tk}(B_{ij}(-x)B_{ir}(xy))$

$\cdot B_{kt}(E_{ij}(-x)+E_{ir}(xy))[-B_{jr}(y)]_t$ by S4

$= [B_{ir}(xy), -B_{jr}(-y)]_{kt}B_{tk}(\{-B_{ij}(-x)B_{ir}(xy)B_{jr}(y)B_{ij}(x)$

$\cdot B_{ij}(-x)B_{ir}(xy)+B_{ij}(-x)B_{ir}(xy)\})$

$\cdot B_{kt}(E_{ij}(-x)+E_{ir}(xy))[-B_{jr}(y)]_t$ by S1,6,7

$= [B_{ir}(xy)]_k B_{tk}(\{B_{jr}(-y)B_{ij}(-x)B_{ir}(xy)\{B_{jr}(y)B_{ir}(xy)-I\}\})$

$\cdot B_{kt}(-E_{ir}(xy)+E_{ij}(x)B_{jr}(y))$ by S6,7

$= [B_{ir}(xy)]_k B_{tk}(\{B_{ij}(-x)\{B_{ir}(xy)-B_{jr}(-y)\}\})B_{kt}(E_{ij}(x))$

$= [B_{ir}(xy)]_k B_{tk}(E_{jr}(y))B_{kt}(E_{ij}(x))$

$= B^{ir}_{kk}(xy)B^{jr}_{tk}(y)B^{ij}_{kt}(x)$

$= B^{jr}_{tk}(y)B^{ij}_{kt}(x)B^{ir}_{kk}(xy)$ by R2(iv),(v)

The remaining cases of R3, i.e. as in (v) but with either $i=j$ or $j=r$, will be dealt with after R2(ix),(x).
We are now in a position to complete R2:

R2(ix) $k\neq t$, $i\neq j$, $j\neq r$, $i\neq r$:

$$B^{ij}_{kt}(x)B^{rr}_{tk}(y) = B^{ij}_{kt}(x)B^{rj}_{tk}(-y)B^{jr}_{kk}(-1)B^{rj}_{tk}(y)B^{jr}_{kk}(1) \quad \text{by R3(iii), R1}$$

$$= B^{rj}_{tk}(-y)B^{jr}_{kk}(-1)B^{rj}_{tk}(y)B^{jr}_{kk}(1)B^{ij}_{kt}(x) \quad \text{by R2(v),(viii)}$$

$$= B^{rr}_{tk}(y)B^{ij}_{kt}(x) \quad \text{by R3(iii), R1}$$

(x), case (I): $k\neq t$, $i\neq j$. Since $n\geqslant 3$, choose $r\neq i,j$.
Then $B^{ii}_{kt}(x)B^{jj}_{tk}(y)$

$$= B^{ir}_{kt}(-x)B^{ri}_{tt}(-1)B^{ir}_{kt}(x)B^{ri}_{tt}(1)B^{jj}_{tk}(y) \quad \text{by R3(iii), R1}$$

$$= B^{jj}_{tk}(y)B^{ir}_{kt}(-x)B^{ri}_{tt}(-1)B^{ir}_{kt}(x)B^{ri}_{tt}(1) \quad \text{by R2(v), (ix)}$$

$$= B^{jj}_{tk}(y)B^{ii}_{kt}(x) \quad \text{by R3(iii), R1}$$

Case (II): Here we may have $n=2$. Suppose $k\neq t$, $i\neq j$:

$$B^{ii}_{kt}(x)B^{jj}_{tk}(y) = B_{kt}(E_{ii}(x))B_{tk}(E_{jj}(y))$$

$$= B_{tk}(E_{jj}(y))B_{kt}(E_{ii}(x)) \quad \text{by S4}', 7$$

$$= B^{jj}_{tk}(y)B^{ii}_{kt}(x)$$

This completes R2; we now complete R3:

R3(vi)(I): $k\neq t$, $i\neq j$. Choose $s\neq i,j$.

$$B^{ii}_{kt}(x)B^{ij}_{tk}(y)$$

$$= B^{is}_{kt}(-x)B^{si}_{tt}(-1)B^{is}_{kt}(x)B^{si}_{tt}(1)B^{ij}_{tk}(y) \quad \text{by R3(iii), R1}$$

$$= B^{ij}_{tk}(y)B^{is}_{kt}(-x)B^{si}_{tt}(-1)B^{sj}_{tk}(-y)B^{is}_{kt}(x)B^{si}_{tt}(1)B^{sj}_{tk}(y) \quad \text{by R3(ii), R2}$$

$$= B^{ij}_{tk}(y)B^{is}_{kt}(-x)B^{si}_{tt}(-1)B^{is}_{kt}(x)B^{sj}_{tk}(-y)B^{ij}_{kk}(xy)B^{si}_{tt}(1)B^{sj}_{tk}(y) \quad \text{by R3(v), R1}$$

$$= B^{ij}_{tk}(y)B^{is}_{kt}(-x)B^{si}_{tt}(-1)B^{is}_{kt}(x)B^{si}_{tt}(1)B^{ij}_{kk}(xy) \quad \text{by R1,2}$$

$$= B^{ij}_{tk}(y)B^{ii}_{kt}(x)B^{ij}_{kk}(xy) \quad \text{by R3(iii), R1}$$

(II): $B^{ii}_{kt}(x)B^{ij}_{tk}(y) = B_{kt}(E_{ii}(x))B_{tk}(E_{ij}(y))$

$$= [B_{ij}(xy)]_k B_{tk}(E_{ij}(y))B_{kt}(E_{ii}(x)) \quad \text{by S4}'$$

$= B^{ij}_{kk}(xy)B^{ii}_{tk}(y)B^{ij}_{kt}(x)$

$= B^{ij}_{tk}(y)B^{ii}_{kt}(x)B^{ij}_{kk}(xy)$ by R2

(vii)(I): $k\neq t$, $i\neq j$. Choose $r \neq i,j$.

$B^{ij}_{kt}(x)B^{jj}_{tk}(y)$

$= B^{ij}_{kt}(x)B^{jr}_{tk}(-y)B^{rj}_{kk}(-1)B^{jr}_{tk}(y)B^{rj}_{kk}(1)$ by R3(iii), R1

$= B^{jr}_{tk}(-y)B^{ir}_{kk}(-xy)B^{rj}_{kk}(-1)B^{jr}_{tk}(y)B^{ir}_{kk}(xy)B^{rj}_{kk}(1)B^{ij}_{kt}(x)$ by R3(v), R2

$= B^{jr}_{tk}(-y)B^{rj}_{kk}(-1)B^{ij}_{kk}(xy)B^{jr}_{tk}(y)B^{rj}_{kk}(1)B^{ij}_{kt}(x)$ by R3(i), R2, R1

$= B^{jr}_{tk}(-y)B^{rj}_{kk}(-1)B^{jr}_{tk}(y)B^{rj}_{kk}(1)B^{ij}_{kt}(x)B^{ij}_{kk}(xy)$ by R2

$= B^{jj}_{tk}(y)B^{ii}_{kt}(x)B^{ij}_{kk}(xy)$ by R3(iii), R1

(II): $k\neq t$, $i\neq j$. $B^{ij}_{kt}(x)B^{jj}_{tk}(y)$

$= B_{kt}(E_{ij}(x))B_{tk}(E_{jj}(y))$

$= [B_{ij}(xy)]_k B_{tk}(E_{jj}(y))B_{kt}(E_{ij}(x))$ by S4′

$= B^{ij}_{kk}(xy)B^{jj}_{tk}(y)B^{ij}_{kt}(x)$

$= B^{jj}_{tk}(y)B^{ij}_{kt}(x)B^{ij}_{kk}(xy)$ by R2

This completes R3.

(I) R4(i) $i\neq j$: $B^{ij}_{kk}(\alpha-1)B^{ji}_{kk}(1) = [B_{ij}(\alpha-1)]_k[B_{ji}(1)]_k$

$= [B_{ij}(\alpha-1)B_{ji}(1)]_k$ by S7

$= [D_{ij}(\alpha)B_{ji}(\alpha)B_{ij}(1-\alpha^{-1})]_k$

$= [D_{ij}(\alpha)]_k[B_{ji}(\alpha)]_k[B_{ij}(1-\alpha^{-1})]_k$ by S7

$= D^{ij}_{kk}(\alpha)B^{ji}_{kk}(\alpha)B^{ij}_{kk}(1-\alpha^{-1})$

(ii) $k\neq t$, $i\neq j$: $B^{ij}_{kt}(\alpha-1)B^{ji}_{tk}(1) = B_{kt}(E_{ij}(\alpha-1))B_{tk}(E_{ji}(1))$

$= B_{kt}(B_{ij}(\alpha-1)-I)B_{tk}(I)B_{tk}(-B_{ji}(1))$ by S1

$= D_{kt}(B_{ij}(\alpha-1))B_{tk}(B_{ij}(\alpha-1))B_{kt}(I-B_{ij}(1-\alpha))B_{tk}(-B_{ji}(-1))$ by S4

$= D_{kt}(B_{ij}(\alpha-1))[-B_{ji}(-1)]_t B_{tk}(-B_{ji}(1)B_{ij}(\alpha-1))$

$\cdot B_{kt}(B_{ij}(1-\alpha)B_{ji}(-1)-B_{ji}(-1))B_{tk}(I)[-B_{ji}(1)]_t$ by S6,7

$= [B_{ij}(\alpha-1),\ -B_{ij}(1-\alpha)B_{ji}(-1)]_{kt}B_{tk}(-B_{ji}(1)B_{ij}(\alpha-1))$

$\quad\cdot B_{kt}(B_{ij}(1-\alpha)[\alpha]_i-I)B_{tk}(I)[-B_{ji}(1)]_t$ by S7

$= [B_{ij}(\alpha-1),\ -B_{ij}(1-\alpha)B_{ji}(-1)]_{kt}B_{tk}(-B_{ji}(1)B_{ij}(\alpha-1))$

$\quad\cdot D_{kt}(B_{ij}(1-\alpha)[\alpha]_i)B_{tk}(B_{ij}(1-\alpha)[\alpha]_i)$

$\quad\cdot B_{kt}(I-[\alpha^{-1}]_iB_{ij}(\alpha-1))[-B_{ji}(1)]_t$ by S4

$= [[\alpha]_i,\ -B_{ij}(1-\alpha)B_{ji}(-1)[\alpha^{-1}]_iB_{ij}(\alpha-1)]_{kt}$

$\cdot B_{tk}(\{-B_{ij}(1-\alpha)[\alpha]_iB_{ji}(1)B_{ij}(\alpha-1)B_{ij}(1-\alpha)[\alpha]_i+B_{ij}(1-\alpha)[\alpha]_i\})$

$\cdot B_{kt}(I-[\alpha^{-1}]_iB_{ij}(\alpha-1))[-B_{ji}(1)]_t$ by S1,6,7

$= [[\alpha]_i,\ -[\alpha^{-1}]_jB_{ji}(-1)]_{kt}B_{tk}(\{B_{ij}(1-\alpha)[\alpha]_i-[\alpha]_jB_{ji}(\alpha)$
$\cdot B_{ij}(1-\alpha)[\alpha]_i\})$

$\quad\cdot B_{kt}(I-[\alpha^{-1}]_iB_{ij}(\alpha-1))[-B_{ji}(1)]_t$

$= [[\alpha]_i,[\alpha^{-1}]_j]_{kt}B_{tk}(\{[\alpha]_jB_{ij}(1-\alpha)[\alpha]_i-B_{ji}(-1)B_{ij}(1-\alpha)[\alpha]_i\})$

$\quad\cdot B_{kt}(\{[\alpha^{-1}]_iB_{ij}(\alpha-1)B_{ji}(1)-B_{ji}(1)\})$ by S6,7

$= [[\alpha]_i,[\alpha]_j^{-1}]_{kt}B_{tk}(E_{ji}(\alpha))B_{kt}(E_{ij}(1-\alpha^{-1}))$

$= D_{kt}^{ij}(\alpha)B_{tk}^{ji}(\alpha)B_{kt}^{ij}(1-\alpha^{-1})$

(iii) k≠t: $B_{kt}^{ii}(\alpha-1)B_{tk}^{ii}(1) = B_{kt}(E_{ii}(\alpha-1))B_{tk}(E_{ii}(1))$

$\quad = B_{kt}([\alpha]-I)B_{tk}(I)B_{tk}(E_{ii}(1)-I)$ by S1

$= D_{kt}([\alpha]_i)B_{tk}([\alpha]_i)B_{kt}(I-[\alpha^{-1}]_i)B_{tk}(E_{ii}(1)-I)$ by S4

$= D_{kt}([\alpha]_i)B_{tk}([\alpha]_i)B_{kt}(E_{ii}(1-\alpha^{-1}))\{\prod_{j\neq i}B_{tk}(E_{jj}(-1))\}$ by S1

$= D_{kt}([\alpha]_i)B_{tk}([\alpha]_i)B_{kt}^{ii}(1-\alpha^{-1})\{\prod_{j\neq i}B_{tk}^{jj}(-1)\}$

$= D_{kt}([\alpha]_i)B_{tk}([\alpha]_i)\{\prod_{j\neq i}B_{tk}^{jj}(-1)\}B_{kt}^{ii}(1-\alpha^{-1})$ by R2

$= D_{kt}([\alpha]_i)B_{tk}([\alpha]_i+\sum_{j\neq i}E_{jj}(-1))B_{kt}^{ii}(1-\alpha^{-1})$ by S1

$= D_{kt}([\alpha]_i)B_{tk}(E_{ii}(\alpha))B_{kt}^{ii}(1-\alpha^{-1})$

$= D_{kt}^{ii}(\alpha)B_{tk}^{ii}(\alpha)B_{kt}^{ii}(1-\alpha^{-1})$

(II) R4′ (i) $i\neq j$, $1+xy\in U(R)$:

$B^{ij}_{kk}(x)B^{ji}_{kk}(y)[1+yx]^{j}_{k} = [B_{ij}(x)]_k[B_{ji}(y)]_k[[1+yx]_j]_k$

$= [B_{ij}(x)B_{ji}(y)[1+yx]_j]_k$ by S7

$= [[1+xy]_i B_{ji}(y)B_{ij}(x)]_k$

$= [[1+xy]_i]_k[B_{ji}(y)]_k[B_{ij}(x)]_k$ by S7

$= [1+xy]^{i}_{k}B^{ji}_{kk}(y)B^{ij}_{kk}(x)$

(ii) $k \neq t$ (we do not insist that $i \neq j$), and $1+xy \in U(R)$:

$B^{ij}_{kt}(x)B^{ji}_{tk}(y)[1+yx]^{j}_{t} = B_{kt}(E_{ij}(x))B_{tk}(E_{ji}(y)[[1+yx]_j]_t$

$= B_{kt}(E_{ij}(x))B_{tk}(E_{ji}(y))[I+E_{ji}(y)E_{ij}(x)]_t$

$= [I+E_{ij}(x)E_{ji}(y)]_k B_{tk}(E_{ji}(y))B_{kt}(E_{ij}(x))$ by S4′

$= [[1+xy]_i]_k B_{tk}(E_{ji}(y))B_{kt}(E_{ij}(x))$

$= [1+xy]^{i}_{k}B^{ji}_{tk}(y)B^{ij}_{kt}(x)$

This completes R4 and R4′ in cases (I) and (II) respectively.

R5(i) If $i \neq j$, $B^{ij}_{kk}(x) = [B_{ij}(x)]_k$

$= [B_{ji}(1)B_{ij}(-1)B_{ji}(-x)B_{ij}(1)B_{ji}(-1)]_k$

$= [B_{ji}(1)]_k[B_{ij}(-1)]_k[B_{ji}(-x)]_k[B_{ij}(1)]_k[B_{ji}(-1)]_k$ by S7

$= B^{ji}_{kk}(1)B^{ij}_{kk}(-1)B^{ji}_{kk}(-x)B^{ij}_{kk}(1)B^{ji}_{kk}(-1)$

(ii) As a consequence of 1,2,3 we have

$\{B_{ij}(x)\}^{B_{jk}(1)B_{kj}(-1)B_{jk}(1)}$

$= B_{jk}(-1)B_{kj}(1)B_{jk}(-1)B_{ij}(x)B_{jk}(1)B_{kj}(-1)B_{jk}(1)$

$= B_{jk}(-1)B_{kj}(1)B_{ij}(x)B_{ik}(+x)B_{kj}(-1)B_{jk}(1)$

$= B_{jk}(-1)B_{ik}(x)B_{jk}(1)$

$= B_{ik}(x)$

and $\{B_{ji}(x)\}^{B_{jk}(1)B_{kj}(-1)B_{jk}(1)}$

$= B_{jk}(-1)B_{kj}(1)B_{jk}(-1)B_{ji}(x)B_{jk}(1)B_{kj}(-1)B_{jk}(1)$

$= B_{jk}(-1)B_{kj}(1)B_{ji}(x)B_{kj}(-1)B_{jk}(1)$

$= B_{jk}(-1)B_{ji}(x)B_{ki}(x)B_{jk}(1)$

$= B_{ki}(x)$

So we can find P such that, if $k \neq t$,

$B_{kt}^{ij}(x) = \{B_{kk}^{ir}(x)\}^P$ (where $r \neq i$) by R1,2,3

$= \{B_{kk}^{ri}(1)B_{kk}^{ir}(-1)B_{kk}^{ri}(-x)B_{kk}^{ir}(1)B_{kk}^{ri}(-1)\}^P$ by R5(i)

$= B_{tk}^{ji}(1)B_{kt}^{ij}(-1)B_{tk}^{ji}(-x)B_{kt}^{ij}(1)B_{tk}^{ji}(-1)$ by R1,2,3

Note that we do not insist that $i \neq j$.

This completes R5.

R6(i) $i \neq j$: $B_{kk}^{ij}(x)[\alpha_{11},\ldots\alpha_{nm}]$

$= [B_{ij}(x)]_k[[\alpha_{11},\ldots\alpha_{n1}],\ldots[\alpha_{1m},\ldots\alpha_{nm}]]$

$= [[\alpha_{11},\ldots\alpha_{n1}],\ldots[B_{ij}(x)[\alpha_{1k},\ldots\alpha_{nk}]],\ldots[\alpha_{1m},\ldots\alpha_{nm}]]$ by S7

$= [[\alpha_{11},\ldots\alpha_{n1}],\ldots[[\alpha_{1k},\ldots\alpha_{nk}]B_{ij}(\alpha_{ik}^{-1}x\alpha_{jk})],\ldots[\alpha_{1m},\ldots\alpha_{nm}]]$

$= [[\alpha_{11},\ldots\alpha_{n1}],\ldots[\alpha_{1m},\ldots\alpha_{nm}]][B_{ij}(\alpha_{ik}^{-1}x\alpha_{jk})]_k$ by S7

$= [\alpha_{11},\ldots\alpha_{nm}]B_{kk}^{ij}(\alpha_{ik}^{-1}x\alpha_{jk})$

(ii) $k \neq t$ (we do not insist that $i \neq j$):

$B_{kt}^{ij}(x)[\alpha_{11},\ldots\alpha_{nm}]$

$= B_{kt}(E_{ij}(x))[[\alpha_{11},\ldots\alpha_{n1}],\ldots[\alpha_{1m},\ldots\alpha_{nm}]]$

$= [[\alpha_{11},\ldots\alpha_{n1}],\ldots[\alpha_{1m},\ldots\alpha_{nm}]]B_{kt}([\alpha_{1k},\ldots\alpha_{nk}]^{-1}E_{ij}(x)$
$\cdot[\alpha_{1t},\ldots\alpha_{nt}])$ by S6

$= [[\alpha_{11},\ldots\alpha_{n1}],\ldots[\alpha_{1m},\ldots\alpha_{nm}]]B_{kt}(E_{ij}(\alpha_{ik}^{-1}x\alpha_{jt}))$

$= [\alpha_{11},\ldots\alpha_{nm}]B_{kt}^{ij}(\alpha_{ik}^{-1}x\alpha_{jt})$

This completes R6.

R7: $[\alpha_{11},\ldots\alpha_{nm}][\beta_{11},\ldots\beta_{nm}]$

$= [[\alpha_{11}\ldots\alpha_{n1}],\ldots[\alpha_{1m},\ldots\alpha_{nm}]][[\beta_{11},\ldots\beta_{n1}],\ldots[\beta_{1m},\ldots\beta_{nm}]]$

$= [[\alpha_{11},\ldots\alpha_{n1}][\beta_{11},\ldots\beta_{n1}],\ldots[\alpha_{1m},\ldots\alpha_{nm}][\beta_{1m},\ldots\beta_{nm}]]$ by S7

$= [[\alpha_{11}\beta_{11},\ldots\alpha_{n1}\beta_{n1}],\ldots[\alpha_{1m}\beta_{1m},\ldots\alpha_{nm}\beta_{nm}]]$

$= [\alpha_{11}\beta_{11}, \ldots \alpha_{nm}\beta_{nm}]$

This completes the proof of the theorem. □

Theorem (4.10). Every semi-simple Artin ring is quasi-universal for GE_n, all n.

Proof. If R is semi-simple and has the minimum condition on right ideals, then by the Wedderburn-Artin structure theorem,

$$R \cong \prod_{i=1}^{r} K_{m_i}^{(i)}$$

where m_i is a positive integer and $K^{(i)}$ is a skew field, each i.

By (3.7), $K^{(i)}$ is universal for GE_n (∵ a skew field is a local ring) and hence is quasi-universal for GE_n, all i,n.

By (4.9), $K_{m_i}^{(i)}$ is quasi-universal for GE_n, all i,n.

By (4.3), R is quasi-universal for GE_n, all n. □

Note that by (4.1) and (4.7) we cannot hope to replace 'quasi-universal' by 'universal' in (4.10). Indeed we may now restate (4.1) and (4.7) as:

Corollary (4.11). If K is the field of two elements, then K_2 and K×K are quasi-universal for GE_n, all n; K_2 is not universal for GE_n, all n, and K×K is not universal for GE_2. □

We now shew that, if we restrict attention to skew fields R, the above example (4.7) is the only case in which the restriction $n \geqslant 3$ in (4.9)(I) is needed.

Proposition (4.12). Let R be a skew field containing more than two elements, and put $S = R_2$. Then S is universal for GE_m, all m.

Proof. In (4.9)(I), the only use of the condition $n \geqslant 3$ was in R2(x) and R3(vi),(vii). So it is sufficient to find alternative arguments for these cases when n = 2 and R is a skew field with $|R| \geqslant 3$.

R2(x) $k\neq t$, $i\neq j$. Assume first that $x \neq -1$ and $y \neq 1$.

$B_{kt}^{ii}(x)B_{tk}^{jj}(y) = B_{kt}(E_{ii}(x))B_{tk}(E_{jj}(y))$

$= B_{kt}([\alpha]_i - I)B_{tk}(I)B_{tk}(-[\beta]_j)$ by S1 $(\alpha=1+x,\ \beta=1-y)$

$= D_{kt}([\alpha]_i)B_{tk}([\alpha]_i)B_{kt}(I-[\alpha]_i^{-1})B_{tk}(-[\beta]_j)$ by S4

$= D_{kt}([\alpha]_i)[-[\beta]_j]_t B_{tk}(-[\alpha,\beta^{-1}]_{ij})B_{kt}(-I+[\alpha]_i^{-1})$

$\cdot B_{tk}(I)[-[\beta]_j^{-1}]_t$ by S6,7

$= D_{kt}([\alpha]_i)[-[\beta]_j]_t B_{tk}(-[\alpha,\beta^{-1}]_{ij})D_{kt}([\alpha]_i^{-1})B_{tk}([\alpha]_i^{-1})$

$\cdot B_{kt}(I-[\alpha]_i)[-[\beta]_j^{-1}]_t$ by S4

$= [-[\beta]_j]_t B_{tk}(-[\alpha,\beta]_{ij}^{-1}+[\alpha]_i^{-1})B_{kt}(E_{ii}(-x))[-[\beta]_j^{-1}]_t$ by S1,6,7

$= B_{tk}([\alpha]_i^{-1}-[\alpha^{-1},\beta]_{ij})B_{kt}(E_{ii}(x))$ by S6,7

$= B_{tk}(E_{jj}(y))B_{kt}(E_{ii}(x))$

$= B_{tk}^{jj}(y)B_{kt}^{ii}(x)$

If $x = -1$ or $y = 1$ or both, we can choose $\gamma,\delta \in U(R)$ (=R^*, the non-zero elements of R) such that $\gamma^{-1}x \neq -1$, $y\delta \neq 1$.

$B_{kt}^{ii}(x)B_{tk}^{jj}(y) = [\gamma]_k^i[\delta]_k^j B_{kt}^{ii}(\gamma^{-1}x)B_{tk}^{jj}(y\delta)[\gamma^{-1}]_k^i[\delta^{-1}]_k^j$ by R6,7

$= [\gamma]_k^i[\delta]_k^j B_{tk}^{jj}(y\delta)B_{kt}^{ii}(\gamma^{-1}x)[\gamma^{-1}]_k^i[\delta^{-1}]_k^j$ by the above

$= B_{tk}^{jj}(y)B_{kt}^{ii}(x)$ by R6,7

R3(vi) $k\neq t$, $i\neq j$. First suppose $x \neq -1$; put $\alpha=1+x$, $z=1-\alpha^{-1}$.

$B_{kt}^{ii}(x)B_{tk}^{ij}(y) = B_{kt}(E_{ii}(x))B_{tk}(E_{ij}(y))$

$= B_{kt}([\alpha]_i - I)B_{tk}(I)B_{tk}(-B_{ij}(-y))$ by S1

$= D_{kt}([\alpha]_i)B_{tk}([\alpha]_i)B_{kt}(I-[\alpha]_i^{-1})B_{tk}(-B_{ij}(-y))$ by S4

$= D_{kt}([\alpha]_i)B_{tk}([\alpha]_i)B_{kt}(E_{ii}(z))B_{tk}(-B_{ij}(-y))$

$= D_{kt}([\alpha]_i)[-B_{ij}(-y)]_t B_{tk}(-B_{ij}(y)[\alpha]_i)$

$\cdot B_{kt}(-E_{ii}(z)+E_{ij}(zy))B_{tk}(I)[-B_{ij}(y)]_t$ by S6,7

$= D_{kt}([\alpha]_i)[-B_{ij}(-y)]_t B_{tk}(-B_{ij}(y)[\alpha]_i)$

$\cdot B_{kt}(B_{ij}(zy)[\alpha]_i^{-1}-I)B_{tk}(I)[-B_{ij}(y)]_t$

$= D_{kt}([\alpha]_i)[-B_{ij}(-y)]_t B_{tk}(-B_{ij}(y)[\alpha]_i)$

$\cdot D_{kt}(B_{ij}(zy)[\alpha]_i^{-1})B_{tk}(B_{ij}(zy)[\alpha]_i^{-1})$

$\cdot B_{kt}(I-[\alpha]_i B_{ij}(-zy))[-B_{ij}(y)]_t$ by S4

$= [[\alpha]_i B_{ij}(zy)[\alpha]_i^{-1}, -[\alpha]_i^{-1}B_{ij}(-y)[\alpha]_i B_{ij}(-zy)]$

$\cdot B_{tk}(\{-B_{ij}(zy)[\alpha]_i^{-1}B_{ij}(y)[\alpha]_i B_{ij}(zy)[\alpha]_i^{-1}+B_{ij}(zy)[\alpha]_i^{-1}\})$

$\cdot B_{kt}(E_{ii}(-x)+E_{ij}(xy))[-B_{ij}(y)]_t$ by S1,6,7

$= [B_{ij}(xy)]_k B_{tk}(B_{ij}(-y+zy+\alpha^{-1}y+zy)[\alpha]_i^{-1}+B_{ij}(-y+zy)[\alpha]_i^{-1})$

$\cdot B_{tk}(E_{ii}(x)+E_{ij}(-xy)+E_{ij}(xy))$ by S6,7

$= [B_{ij}(xy)]_k B_{tk}((B_{ij}(zy)-B_{ij}(zy-y))[\alpha]_i^{-1})B_{kt}(E_{ii}(x))$

$= [B_{ij}(xy)]_k B_{tk}(E_{ij}(y)[\alpha]_i^{-1})B_{kt}(E_{ii}(x))$

$= [B_{ij}(xy)]_k B_{tk}(E_{ij}(y))B_{kt}(E_{ii}(x))$

$= B_{kk}^{ij}(xy)B_{tk}^{ij}(y)B_{kt}^{ii}(x)$

$= B_{tk}^{ij}(y)B_{kt}^{ii}(x)B_{kk}^{ij}(xy)$ by R2

In the case $x = -1$, choose $\gamma \neq 0,1$; so $\gamma^{-1}x \neq -1$.

$B_{kt}^{ii}(x)B_{tk}^{ij}(y) = [\gamma]_k^i B_{kt}^{ii}(\gamma^{-1}x)B_{tk}^{ij}(y)[\gamma^{-1}]_k^i$ by R6,7

$= [\gamma]_k^i B_{tk}^{ij}(y)B_{kt}^{ii}(\gamma^{-1}x)B_{kk}^{ij}(\gamma^{-1}xy)[\gamma^{-1}]_k^i$ by the above

$= B_{tk}^{ij}(y)B_{kt}^{ii}(x)B_{kk}^{ij}(xy)$ by R6,7

(vii) $k \neq t$, $i \neq j$. Assume $y \neq 1$; put $\beta = 1-y$.

$B_{kt}^{ij}(x)B_{tk}^{jj}(y) = B_{kt}(E_{ij}(x))B_{tk}(E_{jj}(y))$

$= B_{kt}(B_{ij}(x)-I)B_{tk}(I)B_{tk}(-[\beta]_j)$ by S1

$= D_{kt}(B_{ij}(x))B_{tk}(B_{ij}(x))B_{kt}(I-B_{ij}(-x))B_{tk}(-[\beta]_j)$ by S4

$= D_{kt}(B_{ij}(x))[-[\beta]_j]_t B_{tk}(-[\beta]_j^{-1}B_{ij}(x))$

$\cdot B_{kt}(B_{ij}(-x\beta)-I)B_{tk}(I)[-[\beta]_j^{-1}]_t$ by S6,7

$= D_{kt}(B_{ij}(x))[-[\beta]_j]_t B_{tk}(-[\beta]_j^{-1}B_{ij}(x))$

$\cdot D_{kt}(B_{ij}(-x\beta))B_{tk}(B_{ij}(-x\beta))B_{kt}(I-B_{ij}(+x\beta))[-[\beta]_j^{-1}]_t$ by S4

$= [B_{ij}(x)B_{ij}(-x\beta),\ -B_{ij}(-x)[\beta]_j B_{ij}(x\beta)]_{kt}$

$\cdot B_{tk}(\{-B_{ij}(-x\beta)[\beta]_j^{-1}B_{ij}(xy)+B_{ij}(-x\beta)\})$

$\cdot B_{kt}(E_{ij}(-x\beta))[-[\beta]_j^{-1}]_t$ by S1,6,7

$= [B_{ij}(xy)]_k B_{tk}(B_{ij}(-x\beta)-[\beta]_j B_{ij}(-x\beta))B_{kt}(E_{ij}(x\beta)[\beta]_j^{-1})$ by S6,7

$= [B_{ij}(xy)]_k B_{tk}(E_{jj}(y))B_{kt}(E_{ij}(x))$

$= B^{ij}_{kk}(xy)B^{jj}_{tk}(y)B^{ij}_{kt}(x)$

$= B^{jj}_{tk}(y)B^{ij}_{kt}(x)B^{ij}_{kk}(xy)$ by R2

In the case $y = 1$, choose $\delta \neq 0,1$; so $y\delta \neq 1$.

$B^{ij}_{kt}(x)B^{jj}_{tk}(y) = [\delta]^j_k B^{ij}_{kt}(x)B^{jj}_{tk}(y\delta)[\delta^{-1}]^j_k$ by R6,7

$= [\delta]^j_k B^{jj}_{tk}(y\delta)B^{ij}_{kt}(x)B^{ij}_{kk}(xy\delta)[\delta^{-1}]^j_k$ by the above

$= B^{jj}_{tk}(y)B^{ij}_{kt}(x)B^{ij}_{kk}(xy)$ by R6,7. □

Now by the Wedderburn-Artin theorem, the rings K_m (K= sfield, m=1,2,..) are the 'building blocks' for semi-simple Artin rings. We now know that all such K_m are universal for GE_n, all n, except K_2 when $|K| = 2$. We now investigate further to see which semi-simple Artin rings are universal for GE_n.

Proposition (4.13). Let $R = K_m$ where K is a sfield and m a positive integer. Then, provided $|R| > 2$, for all $x \in R$ $\exists\ \alpha,\beta \in U(R)$ such that $x = \alpha+\beta$.

Proof. Let J_r be the matrix with r 1's in leading positions on the diagonal, and 0's elsewhere, $0 \le r \le m$. So in particular $J_0 = 0$, $J_1 = e_{11}$, $J_m = 1_R$.

Then given $x \in R$, $\exists\ \theta,\phi \in U(R) = GL_m(K)$ such that

$\theta x \phi = J_r$, some r (= rank of x)

If $|K| > 2$, choose $k \in K$, $k \neq 0,-1$. Then $1 = (1+k)-k$, and both $1+k$ and $-k$ are units of K.

Then $J_r = [1+k,1+k,....1+k,1,1,....1]+[-k,-k,....-k,-1,-1,....-1]$

$= \alpha + \beta \quad (\alpha,\beta \in U(R))$

and so $x = \theta^{-1}\alpha\phi^{-1} + \theta^{-1}\beta\phi^{-1}$ as required.

Now suppose $|K| = 2$ and $m \geq 2$.

Put $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

and $D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $E = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Then $J_0 = 1_R + 1_R$

$$J_1 = \begin{bmatrix} A & 0 \\ 0 & I_{m-2} \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & I_{m-2} \end{bmatrix}$$

$$J_2 = \begin{bmatrix} A & 0 \\ 0 & I_{m-2} \end{bmatrix} + \begin{bmatrix} C & 0 \\ 0 & I_{m-2} \end{bmatrix}$$

$$J_3 = \begin{bmatrix} D & 0 \\ 0 & I_{m-3} \end{bmatrix} + \begin{bmatrix} E & 0 \\ 0 & I_{m-3} \end{bmatrix}$$

and for r even,

$$J_r = \begin{bmatrix} A & & & & 0 \\ & A & & & \\ & & \ddots & & \\ & & & A & \\ 0 & & & & I_{m-r} \end{bmatrix} + \begin{bmatrix} C & & & & 0 \\ & C & & & \\ & & \ddots & & \\ & & & C & \\ 0 & & & & I_{m-r} \end{bmatrix}$$

and for r odd,

$$J_r = \begin{bmatrix} A & & & & 0 \\ & \ddots & & & \\ & & A & & \\ & & & D & \\ 0 & & & & I_{m-r} \end{bmatrix} + \begin{bmatrix} C & & & & 0 \\ & \ddots & & & \\ & & C & & \\ & & & E & \\ 0 & & & & I_{m-r} \end{bmatrix}$$

So in every case we have $J_r = \alpha+\beta$, $\alpha,\beta \in U(R)$ and so $x = \theta^{-1}\alpha\phi^{-1} + \theta^{-1}\beta\phi^{-1}$. □

<u>Theorem (4.14)</u>. If R,S are rings, both universal for GE_n, and if for all $x \in R$, $y \in S$ we have $x = \alpha+\beta$, $y = \gamma+\delta$ for some $\alpha,\beta \in U(R)$, $\gamma,\delta \in U(S)$, then $R \times S$ is universal for GE_n.

<u>Proof</u>. By (4.2) it is sufficient to shew that

(*) $B_{ij}(x)B_{ji}(y) = B_{ji}(y)B_{ij}(x)$, where $xy = 0 = yx$

is a consequence of the universal relations. Examining the proof of (4.2), we see that in fact we need only consider the case of (*) where $x \in R$, $y \in S$.

Write $R \times S$ as the set of all pairs (x,y) $(x \in R, y \in S)$.

Then $B_{ij}(x,0)B_{ji}(0,y)$

$= B_{ij}(\alpha+\beta,0)B_{ji}(0,\gamma+\delta)$ (suitable $\alpha,\beta \in U(R)$, $\gamma,\delta \in U(S)$)

$= [(-\beta,\delta^{-1})]_i B_{ij}(\theta-1,0)B_{ji}(0,\phi+1)[(-\beta^{-1},\delta)]_i$ by 6,7

where $\theta = -\beta^{-1}\alpha$, $\phi = \gamma\delta^{-1}$

$= [(-\beta,\delta^{-1})]_i B_{ij}((\theta,1)-(1,1))B_{ji}(1,1)B_{ji}(-1,\phi)[(-\beta^{-1},\delta)]_i$ by 1

$= [(-\beta,\delta^{-1})]_i D_{ij}(\theta,1)B_{ji}(\theta,1)B_{ij}(1-\theta^{-1},0)$
$\cdot B_{ji}(-1,\phi)[(-\beta^{-1},\delta)]_i$ by 4

$= [(\alpha,\delta^{-1}),(-\theta^{-1},\phi)]_{ij}B_{ji}(-\theta,\phi^{-1})$
$\cdot B_{ij}((\theta^{-1},1)-(1,1))B_{ji}(1,1)[(-\beta^{-1},\delta),(-1,\phi^{-1})]_{ij}$ by 6,7

$= [(\alpha,\delta^{-1}),(-\theta^{-1},\phi)]_{ij}B_{ji}(-\theta,\phi^{-1})D_{ij}(\theta^{-1},1)B_{ji}(\theta^{-1},1)$
$\cdot B_{ij}(1-\theta,0)[(-\beta^{-1},\delta),(-1,\phi^{-1})]_{ij}$ by 4

$= [(-\beta,\delta^{-1}),(-1,\phi)]_{ij}B_{ji}((-\theta^{-1},\phi^{-1})+(\theta^{-1},1))$
$\cdot B_{ij}(1-\theta,0)[(-\beta^{-1},\delta),(-1,\phi^{-1})]_{ij}$ by 1,6,7

$= B_{ji}(0,\phi(\phi^{-1}+1)\delta)B_{ij}(\beta(1-\theta),0)$ by 6,7

$= B_{ji}(0,\phi\delta+\delta)B_{ij}(-\beta\theta+\beta,0)$

$= B_{ji}(0,\gamma+\delta)B_{ij}(\alpha+\beta,0)$

$= B_{ji}(0,y)B_{ij}(x,0)$. □

<u>Corollary (4.15)</u>. Let K be the field of two elements. Then a sufficient condition for a semi-simple Artin ring R to be universal for GE_n, all n, is that R should not contain K or K_2 as direct factors.

<u>Proof</u>.

$$R \cong \prod_{i=1}^{r} K^{(i)}_{m_i}$$ where $K^{(i)}$ is a sfield, and

$|K^{(i)}| = 2 \Rightarrow m_i \geqslant 3$. Each $K^{(i)}_{m_i}$ is universal for GE_n, by (3.7), (4.9), and (4.12). Then R is universal for GE_n (all n) by (4.13) and (4.14). □

The proof of (3.7) depended on the fact that the set of non-units in a local ring is an ideal. We now try to generalize this. Let J(R) be the Jacobson radical of R. If R/J(R) is universal for GE_n, what about R?

Let R be semi-primitive and let M be an R-bimodule. Define $S = R \times M$ with addition componentwise, and multiplication given by

$$(r,m)(r',m') = (rr',rm'+mr')$$

(S is the split null extension of R by M.) Identify R,M with (R,0), (0,M) respectively. M is an ideal of S and $M^2 = 0$, so $M \subseteq J(S)$. Further, $S/M \cong R$ is semi-primitive, so $J(S) = M$.

Assume that $U(R)' = 1$, and that $\alpha m = m\alpha$ for all $\alpha \in U(R)$, $m \in M$. Then $U(S) =$ all $\alpha+m$, $\alpha \in U(R)$, $m \in M$. Further, $U(S)' = 1$, for

$$\begin{aligned}(\alpha+m)(\alpha'+m') &= \alpha\alpha'+\alpha m'+m\alpha' \\ &= \alpha'\alpha+m'\alpha+\alpha' m \\ &= (\alpha'+m')(\alpha+m)\end{aligned}$$

Now suppose $\exists\, x \in R$, $y \in M$ with $xy \neq yx$. Then $1 \neq 1+xy-yx \in U(S)$, and

$$\begin{aligned}[1+xy-yx]_1 &= [1+xy]_1[1+yx]_1^{-1} \\ &= [1+xy]_1[1+yx]_2^{-1}D_{21}(1+yx) \\ &\in E_n(S) \text{ by } 4,\ 4' \text{ and } 7.\end{aligned}$$

But $U(S)' = 1$, so $1+xy-yx \notin U(S)'$. $\therefore$ S is not universal for GE_n, any n, by (4.5).

As an example, we can take $R = k[x]$ and $M = k\langle x,y\rangle$ (the free associative algebra over the field k on the free generators x,y). M is an R-bimodule in a natural way. R is semi-primitive, $U(R)' = 1$ and $U(R) = k^*$ commutes elementwise with M. Further, $x \in R$, $y \in M$ and $xy \neq yx$. So if we construct S as above, S is not universal for GE_n, any n. But $S/J(S) \cong k[x]$ is universal for GE_2 (see [1;(5.2)]).

Note: For $n > 2$ we do not know whether $k[x]$ is universal for GE_n; it seems reasonable to conjecture that it is. We do in fact obtain a presentation for $GE_n(k[x])$ in (6.4).

In spite of the above, we can give an easy sufficient condition for R to be universal if R/J is universal; and as before, the property of being quasi-universal is better behaved: R/J quasi-universal implies R quasi-universal, without extra conditions (see (4.17)).

Write $GE_n(R,J)$ for the subgroup of $GE_n(R)$ generated by all $B_{ij}(x)$, $x \in J(R)$, and all $[\alpha_1,\dots,\alpha_n]$ where $\alpha_i = 1+x_i$, $x_i \in J$.

Proposition (4.16). For any ring R, $GE_n(R,J)$ has the presentation:

Generators: $B_{ij}(x)$, $[1+x_1,\dots,1+x_n]$ (x's $\in J(R)$)

Relations: The quasi-universal relations (1,2,3,4′,5,6,7) where applicable.

Proof. $A \in GE_n(R,J) \Rightarrow A = I_n + (z_{ij})$ where $z_{ij} \in J(R)$.

So

$$A = \prod_{i<n} B_{in}(x_i) \begin{bmatrix} & & 0 \\ & A' & \vdots \\ & & 0 \\ y'_1 \dots y'_{n-1} & & \alpha \end{bmatrix}$$

$$= \prod_{i<n} B_{in}(x_i) \prod_{i<n} B_{ni}(y_i) \begin{bmatrix} & & 0 \\ & A' & \vdots \\ & & 0 \\ 0 \dots 0 & & \alpha \end{bmatrix}$$

for suitable $A' \in GE_{n-1}(R,J)$, $\alpha = 1+z_{nn}$ and x's, y's $\in J(R)$. Furthermore, this expression for A is unique.

Applying the same reduction to A', and continuing inductively, we get a normal form for A:

$$A = \prod_{m=n,..2} \{ \prod_{i<m} B_{im}(x_{im}) \prod_{i<m} B_{mi}(y_{mi}) \} [\alpha_1, \dots \alpha_n]$$

where x_{ij}, $y_{ij} \in J$ and $\alpha_i \equiv 1 \bmod J$.

Clearly $A \cdot [\beta_1, \dots \beta_n]$ can be put in normal form, by 7.

It remains to shew that $A \cdot B_{ij}(z)$ $(z \in J)$ can be put in normal form using only the prescribed relations. Suppose n = 2.

$B_{12}(x)B_{21}(y)[\alpha,\beta]B_{21}(z)$

$= B_{12}(x)B_{21}(y+\beta z\alpha^{-1})[\alpha,\beta]$ by 1,6

Also $B_{12}(x)B_{21}(y)[\alpha,\beta]B_{12}(z)$

$= B_{12}(x)B_{21}(y)B_{12}(\alpha z\beta^{-1})[\alpha,\beta]$ by 6

$= B_{12}(x)[1+y\alpha z\beta^{-1}]_2 B_{12}(\alpha z\beta^{-1})B_{21}(y)[1+\alpha z\beta^{-1}y]_1^{-1}[\alpha,\beta]$ by 4'

$= B_{12}(x+\alpha z\beta^{-1}(1+y\alpha z\beta^{-1})^{-1})B_{21}((1+y\alpha z\beta^{-1})y)$

$\cdot[(1+\alpha z\beta^{-1}y)^{-1}\alpha, \beta+y\alpha z]$ by 1,6,7

So the proposition holds for n = 2. Assume n > 2 and use induction.

If i,j < n we can put $A \cdot B_{ij}(z)$ in normal form, by the induction hypothesis.

If i,j < n,

$B_{ij}(x)B_{nr}(z) = B_{nr}(z)B_{ij}(x)$ $(r \neq i)$ by 2

$B_{ij}(x)B_{ni}(z) = B_{ni}(z)B_{nj}(-zx)B_{ij}(x)$ by 2,3

$[\alpha_1, \dots \alpha_n]B_{nr}(z) = B_{nr}(\alpha_n z\alpha_r^{-1})[\alpha_1, \dots \alpha_n]$ by 6

So using only 1,2,3 and 6 we have

$A \cdot B_{nr}(z) = \prod_i B_{in}(x_i) \prod_i B_{ni}(y_i) A'[\alpha]_n B_{nr}(z)$ $(A' \in GE_{n-1}(R,J))$

$= \prod_i B_{in}(x_i) \prod_i B_{ni}(y_i) \prod_i B_{ni}(z_i) A'[\alpha]_n$ suitable $z_i \in J$

$= \prod_i B_{in}(x_i)\prod_i B_{ni}(y_i+z_i)A'[\alpha]_n$ by 1,2

Now if $i,j < n$,

$B_{ij}(x)B_{rn}(z) = B_{rn}(z)B_{ij}(x)$ $(r\neq j)$ by 2

$B_{ij}(x)B_{jn}(z) = B_{jn}(z)B_{in}(xz)B_{ij}(x)$ by 2,3

$[\alpha_1,\dots\alpha_n]B_{rn}(z) = B_{rn}(\alpha_r z\alpha_n^{-1})[\alpha_1,\dots\alpha_n]$ by 6

So using only 1,2,3 and 6 we have

$$A\cdot B_{rn}(z) = \prod_i B_{in}(x_i)\prod_i B_{ni}(y_i)A'[\alpha']_n B_{rn}(z) \quad (A'\in GE_{n-1}(R,J))$$

$$= \prod_i B_{in}(x_i)\prod_i B_{ni}(y_i)\prod_i B_{in}(z_i)A'[\alpha']_n \text{ suitable } z_i\in J.$$

Now it is sufficient to prove that the following is a consequence of the quasi-universal relations:

$$(**)\ \prod_i B_{ni}(y_i)\prod_i B_{in}(z_i) = \prod_i B_{in}(z_i\alpha^{-1})\prod_i B_{ni}(\alpha y_i)[\alpha]_n A''$$

where $\alpha = 1 + \Sigma y_i z_i$ and $A''\in GE_{n-1}(R,J)$

For then, substituting back,

$$A\cdot B_{rn}(z) = \prod_i B_{in}(x_i+z_i\alpha^{-1})\prod_i B_{ni}(\alpha y_i)A''A'[\alpha\alpha']_n \text{ by } 1,2,6,7$$

and by induction this can now be put in normal form.

For $n = 2$, (**) reads

$$B_{21}(y)B_{12}(z) = [\alpha]_2 B_{12}(z)B_{21}(y)A'' \quad (A''=[1+zy]_1^{-1},\ \alpha=1+yz)$$ by 4'

$$= B_{12}(z\alpha^{-1})B_{21}(\alpha y)[\alpha]_2 A'' \text{ by } 6.$$

So assume (**) is true for $n-1$: put

$$1 + \sum_1^{n-1} y_i z_i = \alpha, \quad 1 + \sum_2^{n-1} y_i z_i = \beta.$$

We write $B \to C$ when $B = CD$, some $D \in GE_{n-1}(R,J)$

Also we write Π_k, Σ_k for $\prod_k^{n-1}$, $\sum_k^{n-1}$.

$\Pi_1 B_{ni}(y_i)\Pi_1 B_{in}(z_i)$

$= B_{n1}(y_1)B_{1n}(z_1\beta^{-1})\Pi_2\{B_{ni}(y_i)B_{1i}(-z_i\beta^{-1}y_i)\}\Pi_2 B_{in}(z_i)$
$\quad\cdot B_{1n}(z_1(1-\beta^{-1}))$ by 1,2,3

$\to B_{n1}(y_1)B_{1n}(z_1\beta^{-1})\Pi_2 B_{ni}(y_i)\Pi_2 B_{in}(z_i)$
$\quad\cdot B_{1n}(z_1(1-\beta^{-1}-\beta^{-1}\Sigma_2 y_i z_i))$ by 1,2,3

$= B_{n1}(y_1)B_{1n}(z_1\beta^{-1})\Pi_2 B_{ni}(y_i)\Pi_2 B_{in}(z_i)$ by 1

$\to B_{n1}(y_1)B_{1n}(z_1\beta^{-1})\Pi_2 B_{in}(z_i\beta^{-1})\Pi_2 B_{ni}(\beta y_i)[\beta]_n$ by induction

$= B_{1n}(z_1\alpha^{-1})B_{n1}(\alpha\beta^{-1}y_1)[1+z_1\beta^{-1}y_1]_1^{-1}[\alpha\beta^{-1}]_n$
$\quad\cdot\Pi_2 B_{in}(z_i\beta^{-1})\Pi_2 B_{ni}(\beta y_i)[\beta]_n$ by case $n=2$

$\to B_{1n}(z_1\alpha^{-1})B_{n1}(\alpha\beta^{-1}y_1)[\alpha\beta^{-1}]_n\Pi_2B_{in}(z_i\beta^{-1})\Pi_2B_{ni}(\beta y_i)[\beta]_n$ by 6,7

$= B_{1n}(z_1\alpha^{-1})B_{n1}(\alpha\beta^{-1}y_1)\Pi_2B_{in}(z_i\alpha^{-1})\Pi_2B_{ni}(\alpha y_i)[\alpha]_n$ by 6,7

$= B_{1n}(z_1\alpha^{-1})\Pi_2\{B_{in}(z_i\alpha^{-1})B_{i1}(-z_i\beta^{-1}y_1)\}$
$\cdot B_{n1}(\alpha\beta^{-1}y_1)\Pi_2B_{ni}(\alpha y_i)[\alpha]_n$ by 2,3

$\to \Pi_1B_{in}(z_i\alpha^{-1})B_{n1}(\alpha\beta^{-1}y_1+\alpha\Sigma_2y_iz_i\beta^{-1}y_1)\Pi_2B_{ni}(\alpha y_i)[\alpha]_n$ by 1,2,3,6

$= \Pi_1B_{in}(z_i\alpha^{-1})\Pi_1B_{ni}(\alpha y_i)[\alpha]_n$

This proves (**), and hence the proposition. □

<u>Theorem (4.17)</u>. (I) Let R (or, equivalently, R/J(R)) be generated as a ring by its units. Then if R/J(R) is universal for GE_n, so is R.

(II) Any R: If R/J(R) is quasi-universal for GE_n, so is R.

<u>Proof</u>. (II): If $x\in J$, $y\notin J$, $\alpha_i\in 1+J$, $\beta_j\in U(R)-(1+J)$ then

(i) $B_{ij}(x)B_{kl}(y) = B_{kl}(y)B_{ij}(x)$ by 2

(ii) $B_{ij}(x)B_{jk}(y) = B_{jk}(y)B_{ij}(x)B_{ik}(xy)$ by 3

(iii) $B_{ij}(x)B_{ki}(y) = B_{ki}(y)B_{ij}(x)B_{kj}(-yx)$ by 1,3

(iv) $B_{ij}(x)B_{ji}(y) = B_{ji}(y\alpha^{-1})B_{ij}(\alpha x)[1+xy,(1+yx)^{-1}]_{ij}$ by 4′, 6,7 $(\alpha=1+xy)$

(v) $[\alpha_1,\dots\alpha_n]B_{ij}(y) = B_{ij}(\alpha_iy\alpha_j^{-1})[\alpha_1,\dots\alpha_n]$ by 6

(vi) $[\alpha_1,\dots\alpha_n][\beta_1,\dots\beta_n] = [\alpha_1\beta_1\alpha_1^{-1},\dots\alpha_n\beta_n\alpha_n^{-1}][\alpha_1,\dots\alpha_n]$ by 7

So from these, if $C\in GE_n(R)$ is some product of $B_{ij}(z)$'s and $[\gamma_1,\dots\gamma_n]$'s , we can write

$C = A\cdot B$ by 1,2,3,4′,6,7 only

where A is a product (possibly empty) of elementary and diagonal matrices each incongruent to I_n mod J, and B is in $GE_n(R,J)$; furthermore, if $r\mapsto\overline{r}$ is the natural map $R\to R/J$ then $\overline{B} = I_n$, and $\overline{C},\overline{A}$ are formally identical, once $[1]_i$ and $B_{ij}(0)$ have been dropped from $\overline{C}$. (Just note that in each of (i)-(vi) the second term on the LHS is congruent mod J to the first term on the RHS, and all other terms are in $GE_n(R,J)$.)

Now suppose $C_0 = I_n$ is a relation of $GE_n(R)$. Then

$\overline{C}_0 = I_n$ is a relation of $GE_n(R/J)$, and since R/J is quasi-universal for GE_n, we have

$$\overline{C}_0 = \overline{C}_1 = \dots = \overline{C}_m$$

where $\overline{C}_m$ is just I_n, and the step $\overline{C}_i = \overline{C}_{i+1}$ involves one application of one of 1.-7. or 4′, or putting $B_{ij}(0)$ or $[1,\dots 1]$ equal to 1.

Now $C_0 = A_0 \cdot B_0$ (notation as above for C) and $\overline{C}_0$, $\overline{A}_0$ are formally identical. If the step $\overline{C}_0 = \overline{C}_1$ involves putting $B_{ij}(0)=I$ or $[1,\dots 1]=I$, on lifting to A_0B_0 we obtain matrices of $GE_n(R,J)$ which we pass through to the right as before, to get $A_1B_1 = C_1$, with $\overline{A}_1, \overline{C}_1$ formally identical.
If the step $\overline{C}_0 = \overline{C}_1$ involves an application of 1,2,3,5,6 or 7, this lifts to the same application to A_0B_0, giving $A_1B_1 = C_1$ as before.
An application of 4′ arises from terms in A_0:

$B_{ij}(x)B_{ji}(y)[1+yx+z]_j$ where $1+yx \in U(R)$ and $z \in J$.
$\exists\, z' \in J$ such that $(1+yx+z)(1+z')^{-1} = 1+yx$. So

$B_{ij}(x)B_{ji}(y)[1+yx+z]_j = B_{ij}(x)B_{ji}(y)[1+yx]_j[1+z']_j$ by 7
and now we can apply 4′ and also pull $[1+z']_j$ through to the right, to get $A_1B_1 = C_1$ as before.

Repeating the above process, we have as a consequence of 1.-7. and 4′,

$$C_0 = A_mB_m$$

where $\overline{A}_m$ and $\overline{C}_m$ are formally identical; but $\overline{C}_m$ is just I, so A_m is the empty product, and $C_0 = B_m$ is a consequence of 1.-7. and 4′. So since $C_0 = I$, we have $B_m = I$, and $B_m \in GE_n(R,J)$, so $B_m = I$ is a consequence of 1.-7. and 4′, by (4.16).

$\therefore C_0 = I$ is a consequence of 1.-7. and 4′, as required.

(I): We have merely to shew that, with the given conditions on R, use of 4′ in (II) above can be replaced by use of 1.-7.
If $x \in J$ and $\alpha \in U(R)$,

$B_{ij}(x)B_{ji}(\alpha) = [\alpha]_j B_{ij}(x\alpha)B_{ji}(1)[\alpha^{-1}]_j$ by 6,7

$= [\alpha]_j B_{ij}(\beta-1)B_{ji}(1)[\alpha^{-1}]_j$ where $\beta = 1+x$ $(\in U(R))$

$= [\alpha]_j D_{ij}(\beta)B_{ji}(\beta)B_{ij}(1-\beta^{-1})[\alpha^{-1}]_j$ by 4

$= [\beta]_i B_{ji}(\alpha)B_{ij}(x)[\alpha\beta^{-1}\alpha^{-1}]_j$ by 6,7

Using 7, $B_{ij}(x)B_{ji}(\alpha)[1+\alpha x]_j = [1+x\alpha]_i B_{ji}(\alpha)B_{ij}(x)$.

Now suppose $y = \alpha_1+\dots+\alpha_r$: apply the above r times to obtain (for $x\in J$)

$B_{ij}(x)B_{ji}(y)[1+yx]_j = [1+xy]_i B_{ji}(y)B_{ij}(x)$ by 1.-7.

Inductive step:

$B_{ij}(x)B_{ji}(y+\alpha)[1+(y+\alpha)x]_j$

$= B_{ij}(x)B_{ji}(y)[1+yx]_j[1+yx]_j^{-1}B_{ji}(\alpha)[1+(y+\alpha)x]_j$ by 1,7

$= [1+xy]_i B_{ji}(y)B_{ij}(x)B_{ji}(\beta^{-1}\alpha)[1+\beta^{-1}\alpha x]_j[(1+\beta^{-1}\alpha x)^{-1} \cdot\beta^{-1}(\beta+\alpha x)]_j$

by the inductive hypothesis and 6,7 ($\beta=1+yx$)

$= [1+xy]_i B_{ji}(y)[1+x\beta^{-1}\alpha]_i B_{ji}(\beta^{-1}\alpha)B_{ij}(x)$ by the above case (r=1)

$= [(1+xy)(1+x\beta^{-1}\alpha)]_i B_{ji}(y+yx\beta^{-1}\alpha+\beta^{-1}\alpha)B_{ij}(x)$

$= [1+x(y+\alpha)]_i B_{ji}(y+\alpha)B_{ij}(x)$

So (iv) on page 64 and 4′ as used in (4.16) are consequences of 1.-7.

The only other use of 4′ was in the steps $\overline{C}_k = \overline{C}_{k+1}$.

Use of 4 arises from terms $B_{ij}(1-\alpha)B_{ji}(1+z)$ in A_k $(z\in J)$.

(Note that $1-\alpha+z' = 1-\alpha'$ where $\alpha' = \alpha-z' \in U(R)$ $(z'\in J)$)

We have $B_{ij}(1-\alpha)B_{ji}(1+z) = B_{ij}(1-\alpha)B_{ji}(1)B_{ji}(z)$ by 1 and now we can use 4 and pull the last term through to the right. □

Recall that a semi-local ring is a ring R such that R/J(R) has the minimum condition on right ideals; in particular, of course, any Artin ring is also semi-local.

Corollary (4.18). Every semi-local ring R (and in particular, every Artin ring) is quasi-universal for GE_n, all n. A sufficient condition for such R to be universal for GE_n is that R/J(R) should not contain K or K_2 as direct factor, where $|K| = 2$.

Proof. The first part is immediate from (4.10) and (4.17).

The second part follows from (4.15) and (4.17), once we note that every Artin ring is generated as a ring by its units. □

We now shew that not every ring is quasi-universal for GE_2. The question of whether every ring is quasi-universal for GE_n $(n>2)$ is undecided; it seems unlikely.

Proposition (4.19). The ring R of integers in $Q(\sqrt{-11})$ (Q = rationals) is not quasi-universal for GE_2.

Proof. $U(R) = \{\pm 1\}$, so $1+xy \in U(R) \Rightarrow xy=0$ (so $x=0$ or $y=0$) or $xy = -2$ (so $x,y = \pm 1, \pm 2$ in some order).

So in $GE_2(R)$, 4′ is a consequence of 4.

But R is not universal for GE_2, by [2; page 163], so it cannot be quasi-universal for GE_2. □

In [2] it is also shewn that the rings of integers in $Q(\sqrt{-2})$ and in $Q(\sqrt{-7})$ are not universal for GE_2; but in these rings the equation $xy = -2$ has solutions other than $x,y = \pm 1, \pm 2$, so the above proof breaks down.

Note that if R is a local ring, $R/J(R)$ is a skew field, which is generated as a ring by its units; by (4.17) we could have deduced (3.7) from the weaker statement that every skew field is universal for GE_n, but the proof of this statement is scarcely shorter than the proof of (3.7).

5. The commutator quotient structure of $GE_n(R)$ and $E_n(R)$.

In this chapter we generalize some of the results of [1;paragraph 9]. We have already seen in (4.4) that [1;(9.1)Cor.1] has an immediate generalization. We now generalize [1;(9.1)].

Proposition (5.1). (Any R) If $A \in GE_n(R)$, then

$$A \equiv [\alpha]_1 \bmod E_n(R), \text{ some } \alpha \in U(R).$$

Proof. By 7, $A \equiv [\alpha_1,\dots\alpha_n] \bmod E_n(R)$

$$= [\alpha_1\alpha_2\dots\alpha_n]_1 \prod_{i=2}^{n} D_{i1}(\alpha_i) \quad \text{by 7}$$

and now note that by 4, $D_{ij}(\beta) \in E_n(R)$. □

We may ask: to what extent is α (in (5.1)) determined by A? This is equivalent to determining the subgroup $W \triangleleft U(R)$ in the following:

Corollary (5.2). For any R, $GE_n(R)/E_n(R) \cong U(R)/W$, some $W\ (=W(n)) \triangleleft U(R)$.

Proof. Immediate from (5.1), once we note that the LHS is well-defined, since $E_n(R) \triangleleft GE_n(R)$, by 6. □

Now for $n = 1$, $GE_n(R) = U(R)$ and $E_n(R) = 1$, so $W = 1$.

For $n > 1$, we have

Proposition (5.3). For any R, and $n > 1$,

$$GE_n(R)/E_n(R) \cong U(R)/W, \text{ some } U(R)' \leq W \leq U(R).$$

Proof. $[\alpha]_1[\beta]_1 = D_{21}(\beta)[\beta]_1[\alpha]_1D_{12}(\beta)$

Thus, since $D_{ij}(\beta) \in E_n(R)$, $GE_n(R)/E_n(R)$ is abelian. The result now follows from (5.2). □

Corollary (5.4). For $n \geq 2$, $GE_n(R)' \subseteq E_n(R)$, any R. □
(cf. [1;(9.1)Cor.3])

For $n \geq 3$, we can improve on this:

Proposition (5.5). For $n \geq 3$, $GE_n(R)' = E_n(R)$, any R.
(cf. [1;(9.2)])

Proof. We already have LHS $\subseteq$ RHS by (5.4). The reverse relationship is immediate from

$$B_{ij}(x) = B_{ik}(-x)B_{kj}(-1)B_{ik}(x)B_{kj}(1) \quad \text{where } k \neq i,j. \quad \square$$

Notation: for any group G, write $G^a = G/G'$.

Proposition (5.6). If R is universal for GE_n $(n \geq 2)$,

$$GE_n(R)/E_n(R) \cong U(R)^a \qquad \text{(cf. [1;(9.1)])}$$

Proof. Define a map $f: GE_n(R) \to U(R)^a$ by

$$f: B_{ij}(x) \mapsto 1$$
$$f: [\alpha_1, \alpha_2, \ldots \alpha_n] \mapsto (\alpha_1 \alpha_2 \ldots \alpha_n)^a$$

Since R is universal for GE_n, and f is compatible with 1.-7, we have a well-defined homomorphism. Clearly $\mathrm{im}(f) = U(R)^a$ and $E_n(R) \subseteq \ker(f)$.

Then if $A \in \ker(f)$, by (5.1) $A \equiv [\alpha]_1 \bmod E_n(R)$.

So it is sufficient to prove that $\alpha \in U(R)' \Rightarrow [\alpha]_1 \in E_n(R)$ (i.e. the converse of (4.5)).

But if $\beta, \gamma \in U(R)$,

$$[\beta^{-1}\gamma^{-1}\beta\gamma]_1 = D_{21}(\beta)D_{21}(\gamma)D_{12}(\beta\gamma)$$
$$\in E_n(R), \text{ since } D_{ij}(\delta) \in E_n(R), \text{ by 4.} \quad \square$$

Corollary (5.7). If R is universal for GE_n, and $1+xy \in U(R)$, then $1+xy \equiv 1+yx \bmod U(R)'$.

Proof. 4': $B_{12}(x)B_{21}(y)[1+yx]_2 = [1+xy]_1 B_{21}(y)B_{12}(x)$

The result now follows from (5.6) and the fact that $D_{12}(1+yx) \in E_n(R)$. $\square$

This gives us another way of constructing rings which are not universal for GE_n, any n. Let K be a (commutative) field. Put $R = K\langle x,y\rangle/(xy)$.

R is the set of all finite K-linear sums of monomials $y^r x^s$ $(r,s \geq 0)$ with multiplication defined by

$$(y^r x^s)(y^{r'} x^{s'}) = 0 \quad \text{if } s > 0,\ r' > 0$$
$$(y^r x^s)x^{s'} = y^r x^{s+s'}$$
$$y^r(y^{r'} x^{s'}) = y^{r+r'} x^{s'}$$

Suppose $(\Sigma a_{rs} y^r x^s)(\Sigma b_{r's'} y^{r'} x^{s'}) = 1$

Consider the homomorphism $R \to K[x]$ formed by mapping $y \mapsto 0$.

$$(\Sigma a_{0s} x^s)(\Sigma b_{0s'} x^{s'}) = 1$$

whence $a_{00}b_{00} = 1$ and $a_{0s} = 0$, $b_{0s'} = 0$ all $s,s' > 0$.

Similarly, $a_{r0} = 0$, $b_{r'0} = 0$ all $r,r' > 0$.

Conversely, if $\alpha \in K^*$ $(=K-\{0\})$ and $f \in R$, then $\alpha+yfx \in U(R)$ $((\alpha+yfx)^{-1} = \alpha^{-1}-\alpha^{-2}yfx)$. So $U(R)$ = all $\alpha+yfx$ $(\alpha \in K^*, f \in R)$.

Then $(\alpha+yfx)(\beta+ygx) = \alpha\beta+y(\alpha g+\beta f)x$

$= (\beta+ygx)(\alpha+yfx)$. Thus $U(R)' = 1$.

But $1+xy = 1$ and $1+yx \neq 1$, so by (5.7) R is not universal for GE_n, any n.

The following example of an integral domain which behaves in a similar way has been found by P.M.Cohn:
Let K be a commutative field, and put $R_0 = K\langle x,y \rangle$.
Consider a monomial

$$\lambda x^{r_1}y^{s_1}\dots x^{r_n}y^{s_n}$$

where $\lambda \in K$ and $r_i, s_i > 0$ except possibly $r_1 = 0$ or $s_n = 0$ or both. Define its height h to be

$2n$ if all $r_i, s_i > 0$

$2n-1$ if $r_1 = 0$ or $s_n = 0$ but not both

$2n-2$ if $r_1 = 0 = s_n$

Put R = all power series $f = \sum_0^\infty f_n$ where f_n is the sum of a finite number of monomials of height n.

Put H_m = all $\alpha + \sum_m^\infty f_n$ $(\alpha \in K^*$, f_n as above).

We claim $U(R) \subseteq H_2$ and $U(R)' \subseteq H_4$. For suppose $f = \sum_0^\infty f_n \in U(R)$ and without loss of generality we may suppose $f_0 = 1$.

Let $f_1 = p(x)+q(y)$ where $p(0) = 0$, $q(0) = 0$.

Suppose $f^{-1} = \sum_0^\infty g_n$ where $g_0 = 1$ and $g_1 = r(x)+s(y)$, $r(0) = 0$, $s(0) = 0$.

Looking at the terms of height 1 in $(\Sigma f_n)(\Sigma g_n) = 1$ we have

$$p(x)+q(y)+r(x)+s(y)+p(x)r(x)+q(y)s(y) = 0$$

Thus $p(x)+r(x)+p(x)r(x) = 0$. $\therefore p = r = 0$.

Similarly $q = s = 0$. $\therefore U(R) \subseteq H_2$.

Now each H_m is multiplicatively closed. Suppose $\alpha+a \in H_n$, $\beta+b \in H_m$ (where a,b contain only terms of height $\geq$ n,m respectively). Then

$$(\alpha+a)^{-1} = (1-\alpha^{-1}a+(\alpha^{-1}a)^2-\dots)\alpha^{-1}$$

and similarly for $\beta+b$. So

$$(\alpha+a)^{-1}(\beta+b)^{-1}(\alpha+a)(\beta+b)$$
$$= (\alpha^{-1}-\alpha^{-2}a+\dots)(\beta^{-1}-\beta^{-2}b+\dots)(\alpha+a)(\beta+b)$$
$$= 1+\beta^{-1}b+\alpha^{-1}a-\beta^{-1}b-\alpha^{-1}a + \text{terms of height} \geqslant m+n$$
$$\in H_{m+n}.$$

So since $U(R) \subseteq H_2$, we have $U(R)' \subseteq H_4$.
Then $1+yx \in U(R)$ $((1+yx)^{-1} = 1-yx+yxyx-\dots)$ and $1+xy \in U(R)$,
but $(1+xy)(1+yx)^{-1} = 1+(xy-yx) + \text{terms of height} \geqslant 4$
$$\in H_2-H_4$$
$\therefore$ $1+xy \not\equiv 1+yx \bmod U(R)'$. By (5.7) R is not universal for GE_n, any n. Further, R is clearly an integral domain.(skew).

We note that, in view of (5.5) and (5.6) we have:
<u>Corollary (5.8)</u>. If R is universal for GE_n, some $n \geqslant 3$, then
$$GE_n(R)^a \cong U(R)^a. \qquad \square$$

We now generalize (5.6) to quasi-universal rings. (5.6) was proved, essentially, by observing that, from the universal relations, if R is universal for GE_n, then $GE_n(R)/E_n(R)$ has a presentation:

Generators: $[\alpha_1,\dots\alpha_n]$ $(\alpha_i \in U(R))$
Relations: $[\alpha_1,\dots\alpha_n][\beta_1,\dots\beta_n] = [\alpha_1\beta_1,\dots\alpha_n\beta_n]$
$D_{ij}(\alpha) = 1$

and this is then seen to be equivalent to the presentation:

Generators: $[\alpha]$ $(\alpha \in U(R))$
Relations: $[\alpha][\beta] = [\alpha\beta] = [\beta][\alpha]$

Similarly, if R is quasi-universal for GE_n, $GE_n(R)/E_n(R)$ has the presentation:

Generators: $[\alpha_1,\dots\alpha_n]$ $(\alpha_i \in U(R))$
Relations: $[\alpha_1,\dots\alpha_n][\beta_1,\dots\beta_n] = [\alpha_1\beta_1,\dots\alpha_n\beta_n]$
$[1+xy]_i = [1+yx]_j$ whenever $1+xy \in U(R)$

and this is equivalent to the presentation:

Generators: $[\alpha]$ $(\alpha \in U(R))$
Relations: $[\alpha][\beta] = [\alpha\beta]$
$[1+xy] = [1+yx]$ whenever $1+xy \in U(R)$.

So let $U_2(R)$ be the subgroup of $U(R)$ generated by all expressions $(1+xy)(1+yx)^{-1}$ $(1+xy \in U(R))$, and let $U_1(R) = U(R)'$. Now if $\alpha,\beta \in U(R)$,

$$\begin{aligned}\alpha\beta &= \beta + (\alpha-1)\beta \\ &= \beta(1 + \beta^{-1}(\alpha-1)\beta) \\ &\equiv \beta(1 + (\alpha-1)) \mod U_2(R) \\ &= \beta\alpha\end{aligned}$$

So $U_1(R) \subseteq U_2(R)$. We have proved:

Proposition (5.9). Let R be either universal or quasi-universal for GE_n, any n, and put m = 1,2 respectively. Then $GE_n(R)/E_n(R) \cong U(R)/U_m(R)$. □

Corollary (5.10). If R, m are as in (5.9), and $[\alpha]_1 \in E_n(R)$, then $\alpha \in U_m(R)$. □

This result provides, potentially, a way of constructing rings which are not quasi-universal for GE_n, for any n. We return to this question in the next chapter.

Finally,

Proposition (5.11). (Any R) If $n \geq 3$, $E_n(R)^a = 1$. (cf.[1;(9.3)])

Proof. Given $x \in R$, and $i \neq j$, choose $k \neq i,j$. Then

$$B_{ij}(x) = B_{ik}(-x)B_{kj}(-1)B_{ik}(x)B_{kj}(1).$$ □

6. The general case.

Here we give a presentation, albeit a clumsy one, for $GE_n(R)$, that holds for any ring R; we shew how it takes a specially simple form when R is GE_2-reducible.

We start by working in as general a context as possible. Let R,S be rings; let M be an (R,S)-bimodule and N an (S,R)-bimodule. Suppose we have balanced maps

$$M \times N \to R \qquad ((x,y) \mapsto (x;y))$$
$$N \times M \to S \qquad ((y,x) \mapsto (y;x))$$

satisfying the additional conditions

$$(x;y)x' = x(y;x') \qquad (x,x' \in M,\ y \in N)$$
$$(y;x)y' = y(x;y') \qquad (x \in M,\ y,y' \in N)$$

Then we can define a ring A consisting of all $\begin{pmatrix} r & x \\ y & s \end{pmatrix}$ $(r \in R,\ s \in S,\ x \in M,\ y \in N)$ with ordinary matrix multiplication and addition, once we agree to write xy for (x;y) etc.
If $x \in M$, $y \in N$, $\alpha \in U(R)$, $\beta \in U(S)$, write

$$B_{12}(x) = \begin{pmatrix} 1_R & x \\ 0 & 1_S \end{pmatrix},\quad B_{21}(y) = \begin{pmatrix} 1_R & 0 \\ y & 1_S \end{pmatrix},$$

$$[\alpha,\beta] = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad ; \text{ all these are in } U(A).$$

Let F,G be subgroups of U(R), U(S) respectively.
Let H be the subgroup of U(A) generated by all $B_{12}(x)$ $(x \in M)$, all $B_{21}(y)$ $(y \in N)$, and all $[\alpha,\beta]$ $(\alpha \in F,\ \beta \in G)$.
The following universal relations hold:

(i) $B_{12}(x)B_{12}(x') = B_{12}(x+x')$ $(x,x' \in M)$
(ii) $B_{21}(y)B_{21}(y') = B_{21}(y+y')$ $(y,y' \in N)$
(iii) $B_{12}(x)[\alpha,\beta] = [\alpha,\beta]B_{12}(\alpha^{-1}x\beta)$ $(x \in M,\ \alpha \in F,\ \beta \in G)$
(iv) $B_{21}(y)[\alpha,\beta] = [\alpha,\beta]B_{21}(\beta^{-1}y\alpha)$ $(y \in N,\ \alpha \in F,\ \beta \in G)$
(v) $[\alpha,\beta][\alpha',\beta'] = [\alpha\alpha',\beta\beta']$ $(\alpha,\alpha' \in F,\ \beta,\beta' \in G)$

Now suppose we have some relation

(*) $\qquad C = 1$

where C is a product of the generators of H. Using (i)-(v), we can reduce (*) to the form:

(vi) $\prod_1^m \{B_{12}(x_i)B_{21}(y_i)\} = [\alpha,\beta] \quad (x_i\in M,\ y_i\in N,\ \alpha\in F,\ \beta\in H)$

It follows that (i)-(v), and (vi) when it holds, are a complete set of defining relations for H.

In order to see when (vi) holds, we define a generalized form of the continuant polynomials of [1; section 8].

Define $p_1(x) = x$ $(x\in M)$ and $p_1(y) = y$ $(y\in N)$.

Define $p_2(x,y) = 1_R+xy$ and $p_2(y,x) = 1_S+yx$ $(x\in M,\ y\in N)$.

Then inductively:

$$p_n(t_1,\dots t_n) = p_{n-1}(t_1,\dots t_{n-1})t_n + p_{n-2}(t_1,\dots t_{n-2})$$

where $t_i\in M$, i odd, and $t_i\in N$, i even; or vice versa.

p_{-1} will mean 0_M or 0_N and $p_0 = p_{-2}$ will mean 1_R or 1_S; exactly which will be clear from the context.

For n>0 we shall sometimes write $p(t_1,\dots t_n)$ for $p_n(t_1,\dots t_n)$.

Then if $x\in M$, $y\in N$, we have $p(x_1,y_1,\dots x_r,y_r)\in R$, $p(x_1,y_1,\dots x_r)\in M$, $p(y_1,x_2,\dots x_r,y_r)\in N$, $p(y_1,x_2,\dots x_r)\in S$.

We claim

$$\prod_1^m \{B_{12}(x_i)B_{21}(y_i)\} = \begin{pmatrix} p_{2m}(x_1,y_1\dots x_m y_m) & p_{2m-1}(x_1,y_1\dots x_m) \\ p_{2m-1}(y_1\dots x_m,y_m) & p_{2m-2}(y_1,x_2\dots x_m) \end{pmatrix}$$

This is certainly true for m = 0, when we have

$$1_H = \begin{pmatrix} p_0 & p_{-1} \\ p_{-1} & p_{-2} \end{pmatrix}$$

and for m = 1 we have

$$B_{12}(x)B_{21}(y) = \begin{pmatrix} 1_R & x \\ 0 & 1_S \end{pmatrix}\begin{pmatrix} 1_R & 0 \\ y & 1_S \end{pmatrix}$$

$$= \begin{pmatrix} 1_R + xy & x \\ y & 1_S \end{pmatrix}$$

$$= \begin{pmatrix} p_2(x,y) & p_1(x) \\ p_1(y) & p_0 \end{pmatrix}$$

Then $\prod_1^m \{B_{12}(x_i)B_{21}(y_i)\}$

$$= \begin{pmatrix} p_{2m-2}(x_1\dots y_{m-1}) & p_{2m-3}(x_1\dots x_{m-1}) \\ p_{2m-3}(y_1\dots y_{m-1}) & p_{2m-4}(y_1\dots x_{m-1}) \end{pmatrix}\begin{pmatrix} 1_R & x_m \\ 0 & 1_S \end{pmatrix}\begin{pmatrix} 1_R & 0 \\ y_m & 1_S \end{pmatrix}$$

$$= \begin{pmatrix} p_{2m-2}(x_1 \dots y_{m-1}) & p_{2m-1}(x_1 \dots x_m) \\ p_{2m-3}(y_1 \dots y_{m-1}) & p_{2m-2}(y_1 \dots x_m) \end{pmatrix} \begin{pmatrix} 1_R & 0 \\ y_m & 1_S \end{pmatrix}$$

$$= \begin{pmatrix} p_{2m}(x_1 \dots y_m) & p_{2m-1}(x_1 \dots x_m) \\ p_{2m-1}(y_1 \dots y_m) & p_{2m-2}(y_1 \dots x_m) \end{pmatrix}$$

So now we can give the following conditions for (vi) to hold:

$p_{2m}(x_1 \dots y_m) = \alpha \in F$

$p_{2m-2}(y_1 \dots x_m) = \beta \in G$

$p_{2m-1}(x_1 \dots x_m) = 0_M$

$p_{2m-1}(y_1 \dots y_m) = 0_N$

We now turn to a special case. Let K be any ring; put $R = K_r$, $S = K_s$, where $r+s = n$. Let M,N be, respectively, all $r \times s$, all $s \times r$ matrices over K. The bimodule structure and the balanced maps (see page 73) are given by matrix multiplication, and then $A \cong K_n$. H is a subgroup of $GL_n(K)$; the actual subgroup will depend on the choice of F and G. We put $F = GE_r(K)$ and $G = GE_s(K)$: so $H = GE_n(K)$.

Then $[\alpha,\beta]$ is a product of $B_{ij}(z)$ $(z \in K,\ 1 \leq i,j \leq r$ or $r < i,j \leq n)$ and $[\alpha_1, \dots \alpha_n]$ $(\alpha_i \in U(K))$.

Further, if $x = (x_{ij}) \in M$ and $y = (y_{ji}) \in N$ $(1 \leq i \leq r,\ 1 \leq j \leq s)$ we have

$$B_{12}(x) = \prod_{i,j} B_{i\ r+j}(x_{ij}) \ ; \ B_{21}(y) = \prod_{i,j} B_{r+j\ i}(y_{ji}),$$

and by universal relation 2, the order of these products is immaterial. Then (i) and (ii) follow from 1 and 2; (iii) and (iv) follow from 2, 3 and 6. Finally 2, 6 and 7 ensure that $[\alpha,1]$ and $[1,\beta]$ $(\alpha \in F,\ \beta \in G)$ commute, and now (v) follows by relations of $GE_r(K)$ and $GE_s(K)$.

Our results may now be stated as

Theorem (6.1). Let K be any ring, and $r,s \geq 1$ with $r+s = n$. Then $GE_n(K)$ has as defining relations:

(a) 1,2,3,6,7

(b) The relations of $GE_t(K)$ $(t = \max(r,s))$

(c) $\prod_1^m \{B_{12}(X_i)B_{21}(Y_i)\} = [A,B]$ where $m \geq 1$ and

X_i, Y_i (i=1....m) are respectively $r\times s$, $s\times r$ matrices over K and

$$p_{2m}(X_1,Y_1,\dots X_m,Y_m) = A \in GE_r(K)$$
$$p_{2m-1}(X_1,Y_1,\dots X_m) = 0$$
$$p_{2m-1}(Y_1,X_2,\dots X_m,Y_m) = 0$$
$$p_{2m-2}(Y_1,X_2,\dots X_m) = B \in GE_s(K). \quad \square$$

N.B. By (6.1)(b) we mean any relation of $GE_n(K)$ not involving more than t distinct suffices.

As a special case of (6.1) we take $r = n-1$, $s = 1$, and then use induction on n. Note that the relations of $GE_1(K)$ are covered by 7. We have

<u>Theorem (6.2)</u>. For any K, any n, $GE_n(K)$ has as defining relations:

(a) 1,2,3,6,7

(b) For $1\leq k\leq n-1$, and $m \geq 1$,

$$(*) \qquad \prod_1^m \{B_{12}(X_i)B_{21}(Y_i)\} = A[\beta]_{k+1} \qquad \text{where}$$

X_i, Y_i (i=1....m) are respectively $k\times 1$, $1\times k$ matrices over K and

$$p_{2m}(X_1,Y_1,\dots X_m,Y_m) = A' \in GE_k(K)$$
$$p_{2m-1}(X_1,\dots X_m) = 0$$
$$p_{2m-1}(Y_1,\dots Y_m) = 0$$
$$p_{2m-2}(Y_1,\dots X_m) = \beta \in U(K)$$

and

$$A = \begin{pmatrix} A' & 0 \\ 0 & I_{n-k} \end{pmatrix}. \quad \square$$

We note that we could add universal relation 4 to the list in (6.2)(a) and then insist in (b) that $\beta = 1$. We also note that universal relation 5 is a special case of (*).

Now we have already seen in (3.11) that if R is GE_2-reducible, the relations of $GE_n(R)$ ($n \geq 3$) are just the universal relations together with the relations of $GE_3(R)$; i.e. in (6.2) just (a),and (b) for $1\leq k\leq 2$. We shew now that for such a ring it is sufficient to take (a), and (b) for k=1, all m, and for k=2, $m\leq 4$.

<u>Proposition (6.3)</u>. Let R be GE_2-reducible. Then if $A \in GE_3(R)$, there is an expression

$$A = B_{13}(x_1)B_{23}(x_2)B_{31}(y_1)B_{32}(y_2)$$
$$\cdot B_{13}(x_3)B_{23}(x_4)B_{31}(y_3)B_{32}(y_4)\cdot M$$

where x_i, $y_i \in R$ and $M \in GE_2(R)$.

<u>Proof</u>. Write 'A → B' for 'A = BM, some $M \in GE_2(R)$'.

$$\text{Then } A \to \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & 0 & z \end{pmatrix} \to \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & z & z \end{pmatrix} \cdot B_{32}(-1)$$

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & z & z \end{pmatrix} \to \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 1 & z \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & 1 \end{pmatrix} \cdot B_{32}(y)B_{23}(x) \quad \text{some } x,y$$

$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & 1 \end{pmatrix} \to \quad B_{13}(x_1)B_{23}(x_2) \quad \text{some } x_1,\ x_2$$

(Each dot stands for an unspecified element of R.)

Putting this together, we see

$$A = B_{13}(x_1)B_{23}(x_2)M_1B_{32}(y)B_{23}(x)M_2B_{32}(-1)M_3 \quad (M_i \in GE_2(R))$$
$$= B_{13}(x_1)B_{23}(x_2)B_{31}(y_1)B_{32}(y_2)$$
$$\cdot B_{12}(x_3)B_{23}(x_4)B_{31}(y_1)B_{32}(y_4)\cdot M$$

where $(y_1,y_2) = (0,y)M_1^{-1}$, $\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = M_1\begin{pmatrix} 0 \\ x \end{pmatrix}$,

$(y_3,y_4) = (0,-1)M_2^{-1}M_1^{-1}$ and $M = M_1M_2M_3 \in GE_2(R)$. □

We may restate (6.3) as: Every $A \in GE_3(R)$ (R GE_2-reducible) has an expression

$$A = \prod_1^2 \{B_{12}(X_i)B_{21}(Y_i)\}\cdot M$$

where $X_i = \begin{pmatrix} x_{i1} \\ x_{i2} \end{pmatrix}$, $Y_i = (y_{i1}, y_{i2})$, $x_{ij}, y_{ij} \in R$ and $M \in GE_2(R)$.

Now suppose m>4 and

$$\prod_1^m \{B_{12}(X_i)B_{21}(Y_i)\} = A\cdot[\beta]_3$$

Then $\prod_1^m \{B_{12}(X_i)B_{21}(Y_i)\}$

$$= B_{12}(X_1)\{B_{21}(Y_1)B_{12}(X_2)B_{21}(Y_2)B_{12}(X_3)\}B_{21}(Y_3)$$
$$\cdot \prod_4^m \{B_{12}(X_i)B_{21}(Y_i)\}$$

$= B_{12}(X_1)\prod_1^2\{B_{12}(X_i')B_{21}(Y_i')\}M\cdot B_{21}(Y_3)\prod_4^m\{B_{12}(X_i)B_{21}(Y_i)\}$ by (6.3)

$= \prod_1^{m-2}\{B_{12}(X_i'')B_{21}(Y_i'')\}$ suitable X_i', Y_i', X_i'', Y_i''.

This gives the induction: we can discard all the relations for $m > 4$, leaving just

$$B_{21}(Y_1)B_{12}(X_2)B_{21}(Y_2)B_{12}(X_3) = \prod_1^2\{B_{12}(X_i')B_{21}(Y_i')\}$$

which in effect is just (6.2)(b) with $k = 2$, $m = 4$ and $\beta = 1$. Thus we have proved:

Theorem (6.4). Let R be GE_2-reducible. Then $GE_n(R)$ (any n) has as defining relations:

(a) 2,3,6,7

(b) The relations of $GE_2(R)$

(c) $\prod_1^4\{B_{12}(X_i)B_{21}(Y_i)\} = A$

where $X_i = \begin{pmatrix} x_{i1} \\ x_{i2}\end{pmatrix}$, $Y_i = (y_{i1}, y_{i2})$, $x_{ij}, y_{ij} \in R$

and (i) $p_8(X_1 \ldots Y_4) = A \in GE_2(R)$

(ii) $p_7(X_1 \ldots X_4) = \begin{pmatrix} 0 \\ 0\end{pmatrix}$

(iii) $p_7(Y_1 \ldots Y_4) = (0,0)$

(iv) $p_6(Y_1 \ldots X_4) = 1$. □

Note that, since a GE_2-reducible ring is always a GE-ring, condition (i) of (6.4)(c) is a consequence of (ii),(iii), and (iv): the force of it is that for each set of X_i, Y_i satisfying (ii),(iii) and (iv) we must pick an expression for A in terms of the generatoɒs of $GE_2(R)$, and write down the corresponding relation (c).

We return now to the general case, and prove some identities for the continuant polynomials.

Lemma (6.5).

(a) $p(x_1, x_2, \ldots x_m)p(x_{m-1}, x_{m-2}, \ldots x_1) = p(x_1, x_2, \ldots x_{m-1})\cdot p(x_m, x_{m-1}, \ldots x_1)$

(b) $p(x_1 \ldots x_m)p(x_{m-1} \ldots x_2) - p(x_1 \ldots x_{m-1})p(x_m \ldots x_2) = (-1)^m \quad (m \geq 2)$

(c) $p(x_2 \ldots x_{m-1})p(x_m \ldots x_1) - p(x_2 \ldots x_m)p(x_{m-1} \ldots x_1) = (-1)^m \quad (m \geq 2)$.

Proof. (a) $m=1$: $p(x_1)p_0 = p_0 p(x_1)$

$m > 1$: $p(x_1 \dots x_m)p(x_{m-1} \dots x_1)$

$= p(x_1 \dots x_{m-1})x_m p(x_{m-1} \dots x_1) + p(x_1 \dots x_{m-2})p(x_{m-1} \dots x_1)$

$= p(x_1 \dots x_{m-1})x_m p(x_{m-1} \dots x_1) + p(x_1 \dots x_{m-1})p(x_{m-2} \dots x_1)$ by induction

$= p(x_1 \dots x_{m-1})p(x_m \dots x_1)$

(b) $m=2$: $p(x_1, x_2)p_0 - p(x_1)p(x_2) = 1+x_1x_2-x_1x_2 = 1$

$m=3$: $p(x_1, x_2, x_3)p(x_2) - p(x_1, x_2)p(x_3, x_2)$

$= (x_1+x_3+x_1x_2x_3)x_2 - (1+x_1x_2)(1+x_3x_2) = -1$

$m > 3$: $p(x_1 \dots x_m)p(x_{m-1} \dots x_2) - p(x_1 \dots x_{m-1})p(x_m \dots x_2)$

$= x_1 p(x_2 \dots x_m)p(x_{m-1} \dots x_2) + p(x_3 \dots x_m)p(x_{m-1} \dots x_2)$

$\quad - x_1 p(x_2 \dots x_{m-1})p(x_m \dots x_2) - p(x_3 \dots x_{m-1})p(x_m \dots x_2)$

$= p(x_3 \dots x_m)p(x_{m-1} \dots x_2) - p(x_3 \dots x_{m-1})p(x_m \dots x_2)$ by (a)

$= p(x_3 \dots x_m)p(x_{m-1} \dots x_3)x_2 + p(x_3 \dots x_m)p(x_{m-1} \dots x_4)$

$\quad - p(x_3 \dots x_{m-1})p(x_m \dots x_3)x_2 - p(x_3 \dots x_{m-1})p(x_m \dots x_4)$

$= p(x_3 \dots x_m)p(x_{m-1} \dots x_4) - p(x_3 \dots x_{m-1})p(x_m \dots x_4)$ by (a)

$= (-1)^{m-2}$ by induction

$= (-1)^m$

(c) similarly. □

Now we have (using the notation of pp. 73-75; $X_i \in M$, $Y_i \in N$)

$$\prod_1^m \{B_{12}(X_i)B_{21}(Y_i)\}[p_{2m}(Y_m, X_m, \dots X_1)]_2$$

$$= \begin{pmatrix} p_{2m}(X_1 \dots Y_m) & p_{2m-1}(X_1 \dots X_m) \\ p_{2m-1}(Y_1 \dots Y_m) & p_{2m-2}(Y_1 \dots X_m) \end{pmatrix} \cdot [p_{2m}(Y_m \dots X_1)]_2$$

$$= \begin{pmatrix} p_{2m}(X_1 \dots Y_m) & p_{2m-1}(X_1 \dots X_m)p_{2m}(Y_m \dots X_1) \\ p_{2m-1}(Y_1 \dots Y_m) & p_{2m-2}(Y_1 \dots X_m)p_{2m}(Y_m \dots X_1) \end{pmatrix}$$

$$= \begin{pmatrix} p_{2m}(X_1 \dots Y_m) & p_{2m}(X_1 \dots Y_m)p_{2m-1}(X_m \dots X_1) \\ p_{2m-1}(Y_1 \dots Y_m) & 1 + p_{2m-1}(Y_1 \dots Y_m)p_{2m-1}(X_m \dots X_1) \end{pmatrix} \text{ by (6.5)}$$

$$= [p_{2m}(X_1 \dots Y_m)]_1 B_{21}(p_{2m-1}(Y_1 \dots Y_m))B_{12}(p_{2m-1}(X_m \dots X_1))$$

So we have $p_{2m}(X_1 \dots Y_m) \in U(R) \iff p_{2m}(Y_m \dots X_1) \in U(S)$.

Now any relation of H was reducible, by (i)-(v) (page 73), to the form $\prod_{1}^{m} \{B_{12}(x_i)B_{21}(y_i)\} = [\alpha,\beta]$ — (iv)

By (i),(ii) $\prod_{1}^{m-1} \{B_{12}(x_i)B_{21}(y_i)\}$

$= [\alpha,\beta]B_{21}(-y_m)B_{12}(-x_m)$

$= [\alpha]_1 B_{21}(-\beta y_m)B_{12}(-x_m\beta^{-1})[\beta]_2$ by (iii),(iv),(v)

By (v), $\prod_{1}^{m-1} \{B_{12}(x_i)B_{21}(y_i)\}[\beta^{-1}]_2 = [\alpha]_1 B_{21}(-\beta y_m)B_{12}(-x_m\beta^{-1})$

From (vi), $\alpha = p_{2m}(x_1 \ldots y_m) = p_{2m-2}(x_1 \ldots y_{m-1})$, $\because p_{2m-1}(x_1 \ldots x_m) = 0$.

By (i),(ii),(v),

$\prod_{1}^{m} \{B_{21}(-y_i)B_{12}(-x_i)\} = [\alpha^{-1},\beta^{-1}]$

So $\beta^{-1} = p_{2m}(-y_m \ldots -x_1)$

$= p_{2m-2}(-y_{m-1} \ldots -x_1)$ $\because$ $p_{2m-1}(-x_m \ldots -x_1) = 0$

$= p_{2m-2}(y_{m-1} \ldots x_1)$.

Thus if we write E for the subgroup of H generated by all $B_{12}(x)$, $B_{21}(y)$ $(x \in M,\ y \in N)$, we have $E \triangleleft H$ (by (iii),(iv)), and H/E has the presentation:

Generators: $[\alpha,\beta]$ where $\alpha = \alpha' p(x_1 \ldots y_m) \in U(R)$, $\alpha' \in F$
and $\beta = \beta' p(y_m \ldots x_1)^{-1} \in U(S)$, $\beta' \in G$

Relations; $[\alpha_1,\beta_1][\alpha_2,\beta_2] = [\alpha_1\alpha_2,\beta_1\beta_2]$
$[p(x_1 \ldots y_m), p(y_m \ldots x_1)^{-1}] = 1$ whenever $p(x_1 \ldots y_m) \in U(R)$.

Now consider the case $R = S = M = N$, $F = G = U(R)$.
So $H = GE_2(R)$, $E = E_2(R)$, and we have proved that for any ring R, $GE_2(R)/E_2(R)$ has the presentation:

Generators: $[\alpha]$ $(\alpha \in U(R))$

Relations: $[\alpha][\beta] = [\alpha\beta]$
$[p(x_1 \ldots y_m)] = [p(y_m \ldots x_1)]$ whenever $p(x_1 \ldots y_m) \in U(R)$.

Now note that $p(1, x_1-1, x_2, \ldots x_n) = p(x_1-1, x_2, \ldots x_n) + p(x_2 \ldots x_n)$

$= (x_1-1)p(x_2 \ldots x_n) + p(x_3 \ldots x_n) + p(x_2 \ldots x_n)$

$= x_1 p(x_2 \ldots x_n) + p(x_3 \ldots x_n)$

$= p(x_1, x_2, \ldots x_n)$

and similarly $p(x_1, x_2, \ldots x_{n-1}, x_n-1, 1) = p(x_1, x_2, \ldots x_n)$

So $(m \geqslant 2)$ we define $U_m(R)$ to be the subgroup of $U(R)$ generated by all expressions

$$p_m(x_1 \dots x_m)p_m(x_m \dots x_1)^{-1} \quad (p(x_1 \dots x_m) \in U(R)).$$

We have $U_m(R) \subseteq U_{m+1}(R)$. Put $W_1(R) = 1$ and $W_2(R) = \bigcup_1^\infty U((R)$

We have proved

Lemma (6.6). For any R, and n = 1 or 2,

$$GE_n(R)/E_n(R) \cong U(R)/W_n(R). \qquad \square$$

Now use induction: suppose $n \geqslant 2$, and we have defined $W_n(R)$ such that $GE_n(R)/E_n(R) \cong U(R)/W_n(R)$. So we have a function $\phi_n : GE_n(R) \to U(R)$ given by $\theta_n : GE_n(R) \to U(R)/W_n(R)$ followed by a choice of coset representative in $U(R)$.

Let $W_{n+1}(R)$ be the subgroup of $U(R)$ generated by $W_n(R)$ and all expressions

$$p(X_1 \dots Y_m)^{\phi_n} p(Y_m \dots X_1)^{-1}$$

where X_i, Y_i are respectively $n \times 1$, $1 \times n$ matrices over R,

$$p(X_1 \dots Y_m) \in GE_n(R) \quad (m = 1, 2, 3, \dots)$$

$$\text{(and hence } p(Y_m \dots X_1) \in U(R)).$$

Then we have

Theorem (6.7). For all n, all R,

$$GE_n(R)/E_n(R) \cong U(R)/W_n(R). \qquad \square$$

If R is commutative, $W_n(R) = 1$, all n.

If R is universal for GE_n, $W_n(R) = U_1(R)$.

If R is quasi-universal for GE_n, $W_n(R) = U_2(R)$.

We recall that in (4.19) we shewed that a certain commutative ring was not quasi-universal for GE_2. We would like to find an example of a ring which is not quasi-universal for GE_n, for all n. Clearly it would be sufficient to find a ring R such that $U_2(R) \underset{\neq}{\subset} W_2(R)$, i.e. such that for some $m > 2$, $U_2(R) \subsetneq U_m(R)$, but the present author is unable to say whether such a ring exists.

REFERENCES.

[1] P.M.Cohn, On the structure of the GL_2 of a ring. (Pub. Math. I.H.E.S. No. 30, 1966.)

[2] P.M.Cohn, A presentation of SL_2 for Euclidean imaginary quadratic number fields. (Mathematika 15, 1968.)

[3] W.Magnus, Über n-dimensionale Gittertransformationen. (Acta Math., 1935.)

[4] J.Nielsen, Die Gruppe der dreidimensionalen Gittertransformationen. (D.K.Danske Vid. Sel., Math-fys. Med.V, 1924.)

[5] O.T.O'Meara, On the finite generation of linear groups over Hasse domains. (J. reine angew. Math. 217, 1956.)

[6] O.T.O'Meara, The automorphisms of the linear groups over any integral domain. (J. reine angew. Math. 223, 1966.)

[7] Yan Shi-Jian, Linear groups over a ring. (Chinese Mathematics, vol. 7 No. 2, 1965.)