METRICAL PROPERTIES OF CONVEX SETS

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CONTENTS

ABSTRACT

There have been many contributions of work in different fields of convexity giving various metrical properties of convex sets. In this thesis we shall consider some further ideas which seem interesting to study. A standard way of tackling certain types of problems is to prove the existence of an 'extremal' convex set with respect to the property in consideration and by a series of arguments determine its construction. Generally speaking the extremal set turns out to be regular in some sense with a correspondingly easy geometry.

In Chapters 1 and 2 we shall concern ourselves entirely with polytopes and we shall give some results on the metric properties of their faces. Following these results, we shall in Chapter 3 consider some continuity properties of the more general class of cell-complexes. In Chapters 4, 5 and 6, we shall confine ourselves to the plane. In Chapter 4, we shall consider sets which in certain senses correspond to the sets of constant width. This leads us in Chapter 5 to give some results concerning the minimal widths of triangles circumscribing convex sets. Finally, in Chapter 6 we consider the areas of certain subsets of a convex set which are determined by partitions of that set by three concurrent lines.

Papers which are relevant to the field of study in a particular chapter are mentioned briefly in an introduction to that chapter.

PREFACE

I should like to thank my supervisor, Professor H.G.Eggleston, for the advice he has given me in our many discussions on this work. Many of the areas for research which are investigated in this thesis were originally suggested by him. Finally, I should like to express my gratitude to the Science Research Council for the award of my studentship.

DEFINITIONS AND NOTATION

Let E^n denote n-dimensional Euclidean space. If $x \varepsilon E^n$ write $x = (\xi_1, \ldots \xi_n)$ where ξ_i is real for $i = 1, \ldots n$. If $y = (\gamma_1, \ldots \gamma_n) \varepsilon E^n$ define

$$x.y = \sum_{i=1}^{n} \xi_i \gamma_i$$

and

$$|x-y| = ((x-y) . (x-y))^{\frac{1}{2}}$$

$$= (\sum_{i=1}^{n} (\xi_i - \gamma_i)^2)^{\frac{1}{2}} .$$

we shall write xy to denote both the line through x and y and the line segment joining x to y. We shall call the point $o = (0, \ldots 0)$ the origin in E^n.

A set X will be said to be convex if whenever two points x and y belong to X all the points of the form $\lambda x + \mu y$ where $\lambda \geq 0, \mu \geq 0, \lambda + \mu = 1$ also belong to X. Define

$$\text{conv } X = \{ \lambda_1 x_1 + .. \lambda_m x_m \mid \sum_{i=1}^{m} \lambda_i = 1, x_i \varepsilon X \text{ and } 0 \leq \lambda_i \leq 1 \text{ for } i = 1, \ldots m \}$$

and

$$\text{aff } X = \{ \lambda_1 x_1 + .. \lambda_m x_m \mid \sum_{i=1}^{m} \lambda_i = 1 \text{ and } x_i \varepsilon X \text{ for } i = 1, \ldots m \} .$$

we shall say that conv X is the convex hull of X and that aff X is the affine hull of X or the affine subspace spanned by X.

If $\{ x_1, . . x_m \}$ is a finite set of m points, it is convenient to write,

conv $\{ x_1, . . x_m \}$ = conv $(x_1, . . x_m)$ and

aff $\{ x_1, . . x_m \}$ = aff $(x_1, . . x_m)$.

We shall say that a set of points $\{ x_1, . . x_m \}$ is affinely independant if a relation of the form

$$\lambda_1 x_1 + . \lambda_m x_m = o$$

$$\lambda_1 + . \lambda_m = 0$$

where λ_i is real for $i = 1, . . m$ implies that $\lambda_i = 0$ for $i = 1 . . m$.

The dimension of a convex set X is defined to be the dimension of the subspace aff X or equivalently to be one less than the maximum number of affinely independant points contained in X.

If X is an n-dimensional convex set in E^n, let int X and fr X denote the interior and frontier of X respectively. If X has dimension less than n, let rel int X and rel fr X denote the interior and frontier of X respectively relative to aff X.

An affine subspace R of n-1 dimensions in E^n will be called a hyperplane and can be written in the form

$$R = \{x \in E^n \mid x.a = \lambda \text{ where } a \text{ is a fixed vector and } \lambda \text{ is real}\}.$$

If X is an arbitrary set and a is a fixed vector, define the space spanned by X and a to be the set of points J given by

$$J = \{x + \lambda a \mid x \in X \text{ and } \lambda \text{ is real}\} .$$

If a is perpendicular to a hyperplane R, then $J \cap R$ is called the orthogonal projection of X on to R.

If X is a convex set and b is a point not contained in X, define the cone spanned by X and b to be the set of points C given by

$$C = \{ \lambda x + (1 - \lambda) b \mid x \in X \text{ and } \lambda \geq 0 \} .$$

The set of points lying on, or to one side of a hyperplane will be called a closed half-space; the set of points strictly to one side of a hyperplane will be called an open half-space.

A support hyperplane R to a convex set X is a hyperplane which intersects the closure of X and is such that X lies in one of the two half-spaces bounded by R. We shall say that a support hyperplane R to X supports X regularly or is a regular support hyperplane, if $R \cap X$ consists of a single point.

A polytope P is the convex hull of a finite number of points or equivalently a bounded intersection of half-spaces. If R is a supporting hyperplane to P then $F = R \cap P$ will be called a face of P. It is also convenient to call the empty set and P itself faces of P. If the dimension of F is r, we shall call F an r- face of P. We shall call the 0, 1 and n-1dimensional faces of an n-dimensional polytope P, vertices, edges and facets of P respectively. We note also that a polytope,which is two dimensional,will be called a polygon and a polytope, which is three dimensional, will be called a polyhedron.

If $\{ x_1, . . x_{m+1} \}$ is a set of affinely independant points and $T = \text{conv}\,(x_1, . . x_m)$ we shall say T is a simplex. If every edge of T has the same length then T will be called the regular simplex.

If P is any polytope define the path π in the graph of P from vertices x to y in P, to be the sub-graph of P having as vertices, a sequence of vertices $x = x_0, x_1, . . x_n = y$ of P and having as edges, the edges $x_0x_1, x_1x_2, . . x_{n-1}x_n$ of P. Two paths π_1 and π_2 in P will be called disjoint if they have only possibly end-points in common. It is well known[16] that the graph of an n-dimensional polytope is n-connected. In other words, for every pair of vertices x and y in P, there exist n pairwise disjoint paths in P having these vertices as end-points. Two paths π_1 and π_2 will be called edgewise disjoint if they have no edges in common. By the expression 'a path in P' we really mean 'a path in the graph of P'.

Define a cell-complex K to be the union of a finite family of polytopes $\{ P_i \}_{i=1}^{m}$ called cells such that

(a) Each face of any P_i is contained in the family

$\{ P_i \}_{i=1}^{m}$

(b) The intersection of any two members of the family

$\{ P_i \}_{i=1}^{m}$ is a face of both.

Let f_o (K) denote the number of vertices of K and we shall write σ_s (K) to denote the union of the s-dimensional faces of K.

Let $D^n [o, \lambda]$ denote the n-dimensional ball or solid sphere, centre o and radius λ in E^n.

$$D^n [o, \lambda] = \{ x \varepsilon E^n \mid |x| \leqslant \lambda \} .$$

If $n = 2$ we shall call $D^2 [o, \lambda]$ a disc.

Let $S^n [o, \lambda]$ denote the frontier of $D^n [o, \lambda]$ i.e; the sphere, centre o and radius λ.

$$S^n [o, \lambda] = \{ x \varepsilon E^n \mid |x| = \lambda \} .$$

If R is a hyperplane which meets $D^n [o, \lambda]$ and H is the open half-space bounded by R which does not contain o then the closure of $H \cap D^n [o, \lambda]$ is called a cap of $D^n [o, \lambda]$.

We shall say that a polytope P is inscribed in $S^n [o, \lambda]$ if each vertex of P lies on $S^n [o, \lambda]$. If a polytope P contains a convex set X and each facet of P supports X, then we shall say that P circumscribes X.

If $\{x_1, \ldots x_m\}$ are arbitrary points in E^n, define the centroid of these points to be $\frac{1}{m}(x_1 + \ldots x_m)$.

For each bounded set X define

$$\rho(X, x) = \inf_{y \in X} |x-y| ,$$

$$\rho(X, Y) = \inf_{\substack{x, \in X \\ y \in Y}} |x-y| \text{ and}$$

the diameter of X, D(X) to be

$$D(X) = \sup_{x, y \in X} |x-y| .$$

For each $\varepsilon > 0$ let

$$[X, \varepsilon] = \{ x \mid \rho(X, x) \leq \varepsilon \} .$$

we call this set the closed ε-neighbourhood of X.

If X and Y are compact let

$$\Delta(X, Y) = \delta_1 + \delta_2$$

where δ_1 and δ_2 are the smallest numbers for which $[X, \delta_1] \supset Y$ and $[Y, \delta_2] \supset X$.

Δ is called the Hausdorff metric function defined on the class of compact sets. We may say that a function f is continuous on this class if f is continuous with respect to this metric.

We say that a sequence of compact sets $\{ X_i \}_{i=1}^{\infty}$ is convergent if there is a compact set Y such that

$$\triangle(X_i, Y) \to 0 \quad \text{as} \quad i \to \infty$$

and we write

$$X_i \to Y \quad \text{as} \quad i \to \infty \ .$$

If Z is a compact set such that foreach $\varepsilon > 0$,

$$[X_i, \varepsilon] \supset Z$$

whenever $i \geq i_0(\varepsilon)$, then we write

$$\liminf X_i \supset Z.$$

If k is a given vector define the width of a convex set X in the direction k, $H(X; k)$ to be the perpendicular distance apart of two distinct parallel hyperplanes, each of which supports X and is perpendicular to k.

The minimal width of X, $H(X)$ is defined to be

$$H(X) = \inf_k H(X; k)$$

where the infimum is taken over all vectors k. If $H(X; k)$ is constant for all vectors k then X is a set of constant width.

An open 'rectangle' C is defined as follows

$$C = \{x = (\xi_1, \ldots \xi_n) \mid \gamma_i < \xi_i < \delta_i \quad \text{for } i = 1, 2 \ldots n\} \ .$$

write $T(C) = \prod_{i=1}^{n} (\delta_i - \gamma_i)$.

The n-dimensional Lebesgue measure of a set X in E^n which we shall denote by $\Lambda_n(X)$ is given by

$$\Lambda_n(X) = \inf_{\bigcup_{i=1}^{\infty} C_i \supset X} \sum_{i=1}^{\infty} \Upsilon(C_i)$$

where the infimum is taken over all coverings of X by open 'rectangles' $\{C_i\}_{i=1}^{\infty}$.

The m-dimensional Hausdorff measure of a set X in E^n with $m \leq n$ which we shall denote by $\mathcal{M}_m(X)$ is given by

$$\mathcal{M}_m(X) = \sup_{\delta > 0} \inf_{\substack{D(S_i) \leq \delta \\ \bigcup_{i=1}^{\infty} S_i \supset X}} \sum_{i=1}^{\infty} (D(S_i))^m$$

where for each $\delta > 0$, the infimum is taken over all coverings of X by sets $\{S_i\}_{i=1}^{\infty}$ with $D(S_i) \leq \delta$ for $i = 1, 2, \cdots$

For arbitrary sets X and Y let

$$X | Y = \{ x \mid x \in X, x \notin Y \} .$$

A set X in E^n is said to be Λ_n- measurable if, for all sets Y and Z in E^n with

$$Y \subset X , Z \subset E^n | X$$

we have

$$\Lambda_n (Y \cup Z) = \Lambda_n(Y) + \Lambda_n(Z).$$

A set X in E^n is said to be $\mathcal{M}_m$-measurable ($m \leq n$) if, for all sets Y and Z in E^n with

$$Y \subset X , \; Z \subset E^n \mid X$$

we have

$$\mathcal{M}_m (Y \cup Z) = \mathcal{M}_m(Y) + \mathcal{M}_m(Z) .$$

We shall call $\mathcal{M}_1$ (X) the linear measure of X or the 'length' of X.

The volume of an n-dimensional convex set X denoted by $\Phi_n(X)$ is defined to be its n-dimensional Lebesgue measure. Thus

$$\Phi_n (X) = \Lambda_n(X) .$$

The surface area of an n-dimensional convex set X denoted by $\Phi_{n-1}(X)$ is, see for example [7] page 88, given by

$$\Phi_{n-1}(X) = \lim_{\delta \to 0+} \frac{\Phi_n ([X, \delta]) - \Phi_n (X)}{\delta} .$$

Thus if P is a polytope then $\Phi_{n-1}(P)$ is the sum of the (n-1)-dimensional Lebesgue measures of the facets of P. Also by convention P is a face of itself and so $\Phi_n(P)$ is the n-dimensional

Lebesgue measure of its n-dimensional face. Thus if P is a polytope we extend the definition of Φ and write $\Phi_r(P)$ to denote the sum of the r-dimensional Lebesgue measures of the r-dimensional faces of P for $r = 1, 2 . . n$ if P has dimension n.

We shall call $\Phi_1(P)$ the total edge-length of P. Also if $n = 2$ and X is a 2-dimensional convex set, we shall sometimes write $P(X)$ and $A(X)$ to denote the perimeter and area of X respectively and so, in the above notation

$$P(X) = \Phi_1(X)$$

and
$$A(X) = \Phi_2(X) .$$

We shall say that a convex set X is central if there exists a point p such that X coincides with its reflexion in p. The point p is called the centre of X.

An affine transformation is a transformation $\sigma : E^n \to E^n$ of the form

$$\sigma(x) = Ax + b$$

where b is a fixed vector and A is a non-singular $n \times n$ matrix.

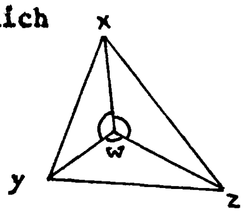
If a planar compact convex set X can be rotated continuously through 2π radians inside a regular polygon P so that each side of P is always supporting X, then X will be called a rotor for P.

For the remaining definitions we shall assume that X is a arbitrary set in the plane.

Let x, y and z be three non-collinear points. We shall let $x\hat{y}z$ denote the angle subtended at y by the line-segments $x\,y$ and $z\,y$.

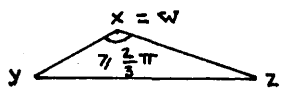
Let $C(x, y, z)$ denote the connected set of minimal length (linear measure) which contains x, y and z. Suppose $C(x, y, z)$ has length $l(x, y, z)$.

If each angle of triangle $x\,y\,z$ is less than $\frac{2}{3}\pi$ let w be the unique point in triangle $x\,y\,z$ for which



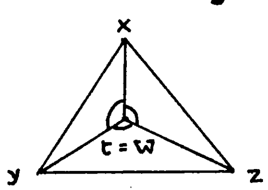
$$x\hat{w}y = y\hat{w}z = x\hat{w}z = \frac{2}{3}\pi,$$

and if the angle $y\hat{x}z$ say of triangle $x\,y\,z$ is greater than or equal to $\frac{2}{3}\pi$ let $w = x$.



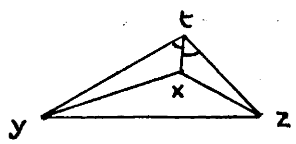
The point w will be called the centre of connection of triangle $x\,y\,z$.

We define a point t as follows. If each angle of triangle $x\,y\,z$ is less than or equal to $\frac{2}{3}\pi$ let $t = w$.



$(t = x$ if $y\hat{x}z = \frac{2}{3}\pi)$

If the angle $y\hat{x}z$ say of triangle $x\,y\,z$ is greater than $\frac{2}{3}\pi$ let t be the unique point which lies on the same side of the line yz as x and such that $y\hat{t}x = z\hat{t}x = \frac{\pi}{3}$.



The point t will be called the centre of revolution of triangle $x\,y\,z$. We shall let $D(x, y, z)$ denote the set union of the segments tx, ty and tz in all cases.

If each angle of triangle $x\,y\,z$ is less than or equal to $\frac{2}{3}\pi$ let

$$K(x, y, z) = |t-x| + |t-y| + |t-z| .$$

If the angle $y\hat{x}z$ is greater than $\frac{2}{3}\pi$ let

$$K(x, y, z) = -|t-x| + |t-y| + |t-z| .$$

It will in fact turn out that if each angle of triangle $x\,y\,z$ is less than or equal to $\frac{2}{3}\pi$ then

$$C(x, y, z) = D(x, y, z)$$

and

$$I(x, y, z) = K(x, y, z) .$$

If x, y and z are collinear choose sequences $\{x_i\}_{i=1}^{\infty}$, $\{y_i\}_{i=1}^{\infty}$ and $\{z_i\}_{i=1}^{\infty}$ which are convergent to x, y and z respectively such that x_i, y_i and z_i are the vertices of a triangle for $i = 1, 2 \ldots$

Then define

$$I(x, y, z) = \lim_{i \to \infty} I(x_i, y_i, z_i)$$

and

$$K(x, y, z) = \lim_{i \to \infty} K(x_i, y_i, z_i) .$$

For each point x, write

$$I(X; x) = \sup_{y,\, z \in X} I(x, y, z)$$

and

$$K(X; x) = \sup_{y, z \in X} K(x, y, z).$$

we shall call $I(X; x)$ and $K(X; x)$ the I-stretch of x with respect to X and the K-stretch of x with respect to X respectively.

Define

$$I(X) = \sup_{x, y, z \in X} I(x, y, z)$$

and

$$K(X) = \sup_{x, y, z \in X} K(x, y, z).$$

we shall call $I(X)$ and $K(X)$ the I-stretch and K-stretch of X respectively.

Finally we say that X is completely I-stretched if $x \notin X$ implies

$$I(X \cup \{x\}) > I(X)$$

and that X is completely K-stretched if $x \notin X$ implies

$$K(X \cup \{x\}) > K(X).$$

CHAPTER 1

INTRODUCTION

Problems concerning the length of a net to hold a sphere, the total edge-length of a crate (a frame formed by the edges of a convex polyhedron) to hold a sphere, and the total edge-length of a convex polyhedron to contain a sphere have been considered in [1], [2], [3] and [4].

In this chapter we shall prove some further results concerning the metric properties of polytopes which do not seem to have been included in the literature. In theorem 1 we shall give a lower bound for the total edge-length of a simplex which is inscribed in a sphere, and which contains the centre of the sphere.

Theorem 1

Let T be an n-dimensional simplex inscribed in the sphere $S^n [o, \lambda]$ in E^n and containing the centre o.

Then $\Phi_1 (T) > 2n\lambda$ for each $n \geq 2$.

Lemma 1

Let P be a polygon inscribed in the unit circle $S^2 [o, \lambda]$ in E^2 and containing the centre o. Then $\Phi_1 (P) > 4\lambda$.

Proof

Let P have N edges $E_1, \ldots E_N$. Let θ_i be the angle subtended by E_i at the centre o for $i = 1, \ldots N$. Then for each i with $1 \leq i \leq N$, $0 < \theta_i \leq \pi$ and for at least one j with $1 \leq j \leq N$, $0 < \theta_j < \pi$ since P has a non-empty interior. Now for $0 \leq \theta \leq \frac{\pi}{2}$, $\sin\theta \geq \frac{2}{\pi}\theta$ with equality if and only if $\theta = 0$ or $\theta = \frac{\pi}{2}$, and $\sin\theta$ is a concave function of θ in this range. Hence for all i with $1 \leq i \leq N$, $\sin\frac{\theta_i}{2} \geq \frac{\theta_i}{\pi}$ and $\sin\frac{\theta_j}{2} > \frac{\theta_j}{\pi}$ for at least one j with $1 \leq j \leq N$. Thus

$$\Phi_1(P) = \sum_{i=1}^{N} 2\lambda \sin\frac{\theta_i}{2}$$

$$> 2\lambda \sum_{i=1}^{N} \frac{\theta_i}{\pi} = 4\lambda .$$

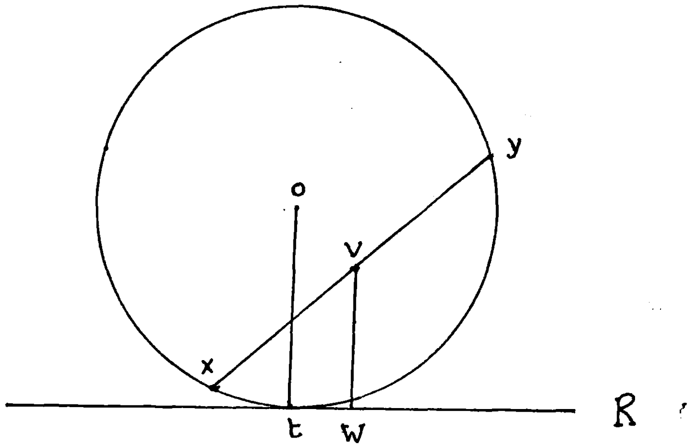
The lemma is proved.

Lemma 2

Let ot be a fixed radius of the circle $S^2[o, \lambda]$ in E^2 and $x\,y$ a chord of fixed length which meets the line ot between o and t. Let R be the tangent to $S^2[o, \lambda]$ at t and suppose that x is nearer to R than y is to R. Then, if v is the mid-point of $x\,y$ and w the foot of the perpendicular from v on to R.

$$|v - w| < |v - x| .$$

<u>Proof</u>



As the chord xy rotates in $S^2[o, \lambda]$, the locus of v is a circle centre o. Thus $|v - w|$ is maximal when $x = t$ and the result follows. The lemma is proved.

<u>Lemma 3</u>

Let P be a polytope and let o be any point in P. Then, either o belongs to some facet of P or there exists a facet F of P such that the line through o perpendicular to aff F meets F.

<u>Proof</u>

Let the facets of P be $F_1, \ldots F_M$. Let $G = \bigcup_{i=1}^{M} F_i$. Then G is compact and thus there exists a point $y \varepsilon G$ such that

$$|o - y| = \inf_{x \varepsilon G} |o - x| .$$

Then $y \varepsilon F_j$ for some j. If $o = y$ the lemma is proved. If $o \neq y$, suppose that oy is not perpendicular to aff F_j. Then there is a point $z \varepsilon$ aff F_j such that

$$|o - y| > |o - z| .$$

But this implies there is a point $w \varepsilon G$ between o and z, and so

$$|o - y| > |o - z| > |o - w| \text{ which is impossible.}$$

Thus oy is perpendicular to aff F_j and writing $F = F_j$ the lemma is proved.

Proof of Theorem 1

The proof of the result is by induction on the dimension n. By lemma 1 the result is true for $n = 2$. We assume inductively the result is true in each dimension k, $2 \leq k \leq n-1$.

Let the n-simplex T have vertices $x_1, \ldots x_{n+1}$. It is well known that $T = \text{conv}(x_1, \ldots x_{n+1})$. In view of lemma 3 we may assume that either $o \varepsilon \text{conv}(x_1, \ldots x_n)$ or the line through o perpendicular to aff $(x_1, \ldots x_n)$ meets the $(n-1)$ simplex conv $(x_1, \ldots x_n)$. Let $x_{n+1}o$ produced meet $S^n[o, \lambda]$ again in y_{n+1} and suppose R is the tangent hyperplane to the sphere $S^n[o,\lambda]$ at y_{n+1}. Let y_i be the foot of the perpendicular from x_i on to R for $i = 1, \ldots n$. We shall consider two cases.

Case I

Suppose $o \in$ conv $(x_1, \ldots x_n)$ (1)

Now let $(\xi_1, \ldots \xi_n)$ be a co-ordinate system whose origin $(0, \ldots 0)$ is the point y_{n+1} and whose ξ_n axis is perpendicular to the hyperplane R. Then R corresponds to the hyperplane $\xi_n = 0$ and thus the ξ_n co-ordinate of x_i is equal to $|x_i - y_i|$ for $i = 1, \ldots n$.

we shall assume without loss in generality that

$$|x_1 - y_1| \leqslant |x_2 - y_2| \leqslant \ldots \leqslant |x_n - y_n| . \qquad (2)$$

Now o, the centre of $S^n[o, \lambda]$ has co-ordinates $(0, \ldots 0, \lambda)$ relative to the above system and so by (1) it follows by convexity that there exist numbers $\lambda_i \geqslant 0$ such that

$$\lambda = \sum_{i=1}^{n} \lambda_i \left|x_i - y_i\right| , \qquad (3)$$

and

$$1 = \sum_{i=1}^{n} \lambda_i . \qquad (4)$$

Then (2), (3) and (4) imply

$$\lambda \geqslant |x_1 - y_1| . \qquad (5)$$

Let the line $x_1 o$ meet $S^n[o, \lambda]$ again in the point v_1.

Now x_1 is a vertex of the $(n - 1)$ simplex conv $(x_1, \ldots x_n)$ and so it follows by (1) that the line $x_1 o$ meets conv $(x_2, \ldots x_n)$ in some point v_2; i.e. $v_2 \in$ conv $(x_2, \ldots x_n)$. (6)

Let w_i be the foot of the perpendicular from v_i onto R, for $i = 1$ and 2. By (6) and by convexity it follows that there exist numbers $\mu_i \geq 0$ such that

$$|v_2 - w_2| = \sum_{i=2}^{n} \mu_i \; |x_i - y_i| , \qquad (7)$$

and

$$1 = \sum_{i=2}^{n} \mu_i . \qquad (8)$$

Then (2), (7) and (8) imply

$$|v_2 - w_2| \geq |x_2 - y_2| . \qquad (9)$$

Now clearly v_2 lies on the line $x_1 o$ between o and v_1 and thus by (5) and (9),

$$|v_1 - w_1| \geq |v_2 - w_2| \geq |x_2 - y_2| . \qquad (10)$$

Since the diameter of $S^n [o, \lambda]$ is 2λ, it follows that

$$|x_1 - y_1| + |v_1 - w_1| = 2\lambda \qquad (11)$$

and

$$\max \{ |x_2 - y_2| , |x_3 - y_3| , \ldots |x_n - y_n| \} \leq 2\lambda . \qquad (12)$$

Now by (10)

$$\sum_{i=1}^{n} |x_i - y_i| \leq |x_1 - y_1| + |v_1 - w_1| + \sum_{i=3}^{n} |x_i - y_i| \qquad (13)$$

$$\leq 2\lambda + 2\lambda \; (n-2) \text{ by (11) and (12)}$$

$$= 2\lambda \; (n-1) . \qquad (14)$$

By (1) and by the induction hypothesis it follows that

$$\Phi_1 (\text{conv } (x_1, \ldots x_n)) > 2\lambda (n - 1), \tag{15}$$

and thus by (14)

$$\Phi_1 (\text{conv } (x_1, \ldots x_n)) > \sum_{i=1}^{n} |x_i - y_i| . \tag{16}$$

$$\text{Then } \Phi_1 (T) = \sum_{i=1}^{n} |x_i - x_{n+1}| + \Phi_1 (\text{conv } (x_1, \ldots x_n))$$

$$> \sum_{i=1}^{n} |x_i - x_{n+1}| + \sum_{i=1}^{n} |x_i - y_i| \quad \text{by (16)}$$

$$= \sum_{i=1}^{n} (|x_{n+1} - x_i| + |x_i - y_i|)$$

$$\geq 2 n \lambda$$

since for each i, $|x_{n+1} - x_i| + |x_i - y_i| \geq 2\lambda$.

Thus case I is proved and we consider

Case II

The line through o perpendicular to aff $(x_1, \ldots x_n)$ meets the $(n - 1)$ simplex $V = \text{conv } (x_1, \ldots x_n)$ in a point u say.

Now aff $(x_1, \ldots x_n)$ meets $S^n [o, \lambda]$ in an $(n - 1)$ dimensional sphere $S^{n-1}[u, \mu]$, centre u and radius μ for some μ, $0 < \mu < \lambda$.

Let t be the foot of the perpendicular from u on to R and Q a two-dimensional plane containing o, t and u.

Applying lemma 2 (in the plane Q) it follows that

$$|t - u| < \mu. \tag{17}$$

Supposing the co-ordinate system $(\xi_1, \ldots \xi_n)$ is defined as in case I, we now translate V away from R in a direction parallel to the ξ_n axis by an amount $\mu - |t - u|$. write

$$u^1 = u + (0, 0, \ldots 0, \mu - |t - u|) = (0, 0, \ldots 0, \mu),$$

$$x_i^1 = x_i + (0, 0, \ldots 0, \mu - |t - u|) \text{ for } i = 1, 2 \ldots n.$$

Consider the sphere $S^{n-1}[u^1, \mu]$.

The $(n - 1)$ simplex conv $(x_1^1, \ldots x_n^1)$ is inscribed in $S^{n-1}[u^1, \mu]$ and contains u^1. Moreover the hyperplane R is tangent to $S^n[u^1, \mu]$ at y_{n+1}.

Thus by applying case I to the sphere $S^{n-1}[u^1, \mu]$ we have by (16),

$$\Phi_1 (\text{conv } (x_1^1, \ldots x_n^1)) > \sum_{i=1}^{n} |x_i^1 - y_i| . \tag{18}$$

But clearly

$$\Phi_1 (\text{conv } (x_1^1, \ldots x_n^1)) = \Phi_1 (\text{conv } (x_1, \ldots x_n))$$

and

$$\sum_{i=1}^{n} |x_i^1 - y_i| > \sum_{i=1}^{n} |x_i - y_i|$$

and thus by (18),

$$\Phi_1 (\text{conv } (x_1 \ldots x_n)) > \sum_{i=1}^{n} |x_i - y_i| .$$

The remainder of the proof follows exactly as in Case I from equation 16 and is omitted. The theorem is proved.

We show next in theorem 2 that the lower bound given in theorem 1 is the greatest lower bound, and that there are simplices inscribed in a sphere containing the centre for which the sums of the r-dimensional measures of the r-faces are arbitrarily small, for r = 2, . . n.

<u>Theorem 2</u>

There is a sequence of n-dimensional simplices $\{T_m\}_{m=1}^{\infty}$ in E^n inscribed in the sphere $S^n[o, \lambda]$, containing the centre o with the following properties

(i) $\Phi_1(T_m) \to 2n\lambda$ as $m \to \infty$,

(ii) $\Phi_r(T_m) \to 0$ as $m \to \infty$ for r = 2, 3 . . n.

<u>Proof</u>

Let x y be a diameter of $S^n[o, \lambda]$. Let y_m be the point on o y distant $\lambda(1 - \frac{1}{m^2})^{\frac{1}{2}}$ from o and suppose H_m is the hyperplane through y_m perpendicular to oy. Let V_m be any (n - 1) -simplex inscribed in the (n - 1).sphere $H_m \cap S^n[o, \lambda]$ of radius $\frac{\lambda}{m}$.

Let V_m have vertices $x_1, \ldots x_n$ and write $T_m = \text{conv}(x, x_1, \ldots x_n) = \text{conv}\{V_m \cup \{x\}\}$.

$$\text{Then} \quad \Phi_1(T_m) = \Phi_1(V_m) + \sum_{i=1}^{n} |x - x_i|$$

$$= 0\left(\frac{1}{m}\right) + \sum_{i=1}^{n} \left(\left(\frac{\lambda}{m}\right)^2 + \left(\lambda + \left(\lambda^2 - \left(\frac{\lambda}{m}\right)^2\right)^{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}}$$

$$\to 2n\lambda \quad \text{as} \quad m \to \infty \ .$$

$$\Phi_r(T_m) \leq \Phi_r(V_m) + \frac{2\lambda}{r} \Phi_{r-1}(V_m)$$

$$= 0\left(\frac{1}{m}\right)^{r-1}$$

$$\to 0 \quad \text{as} \quad m \to \infty \quad \text{for } r = 2, \ldots m.$$

The theorem is proved.

Next in theorem 3 we shall prove that among the simplices contained in a given sphere, the regular simplex which is inscribed in that sphere has maximal total edge-length. The proof is a direct extension of the method given for the tetrahedron which appears in [5].

Theorem 3

Let T be an n-dimensional simplex contained in the sphere $S^n[o, \lambda]$ in E^n with $n \geq 2$.

Then $\Phi_1(T) \leq \left(\frac{n}{2}\right)^{\frac{1}{2}} (n+1)^{\frac{3}{2}} \lambda$

with equality if and only if T is the regular simplex inscribed in $S^n[o, \lambda]$.

Proof

Let T have vertices $x_1, \dots x_{n+1}$ and centroid g.

Now $$\sum_{1 \leq i < k}^{n+1} |x_i - x_k|^2 = n \cdot \sum_{i=1}^{n+1} |x_i - g|^2 - 2 \sum_{1 \leq i < k}^{n+1} (x_i - g)\cdot(x_k - g)$$ (1)

and $$0 = \left(\sum_{i=1}^{n+1} (x_i - g)\right) \cdot \left(\sum_{i=1}^{n+1} (x_i - g)\right) = \sum_{i=1}^{n+1} |x_i - g|^2 +$$

$$2 \sum_{1 \leq i < k}^{n+1} (x_i - y) \cdot (x_k - g) .$$ (2)

Thus, adding (1) and (2) we have

$$\sum_{1 \leq i < k}^{n+1} |x_i - x_k|^2 = n + 1 \cdot \sum_{i=1}^{n+1} |x_i - g|^2 .$$ (3)

By a theorem of Steiner, see for example [5] page 56,

$$\sum_{i=1}^{n+1} |x_i - o|^2 = \sum_{i=1}^{n+1} |x_i - g|^2 + (n+1)\cdot|o - g|^2 .$$ (4)

By (3) and (4),

$$(n+1)\,\lambda^2 \geq \frac{1}{n+1} \cdot \sum_{1 \leq i < k}^{n+1} |x_i - x_k|^2 + (n+1)\cdot|o - g|^2$$

$$\geq \frac{1}{n+1} \cdot \frac{(n+1)\,n}{2} \cdot \left(\frac{\sum_{1 \leq i < k}^{n+1} |x_i - x_k|}{\frac{(n+1)\,n}{2}} \right)^2$$

$$= \frac{2}{n\,(n+1)^2} \left(\Phi_1\,(T) \right)^2$$

with equality only if $g = o$, $|x_i - o| = \lambda$ and $\dfrac{\Phi_1(T)}{n\frac{(n+1)}{2}} = |x_i - x_k|$

for all i and k, $1 \leqslant i < k \leqslant n + 1$.

Thus $\Phi_1(T) \leqslant \left(\frac{n}{2}\right)^{\frac{1}{2}} (n + 1)^{\frac{3}{2}} \lambda$

with equality only if T is the regular simplex inscribed in $S^n[o, \lambda]$.

The theorem is proved.

In view of theorem 1 we make the following conjecture.

<u>Conjecture</u>

Let P be an n-dimensional polytope inscribed in the sphere $S^n[o, \lambda]$ in E^n and containing the centre o.

Then $\Phi_1(P) > 2n\lambda$ for each $n \geqslant 2$.

We shall now establish this conjecture for the case $n = 3$ and prove that a polyhedron, which is inscribed in a sphere containing its centre, has total edge-length at least six times the radius of the sphere.

Theorem 4

If P is a polyhedron inscribed in the sphere $S^3 [o, \lambda]$ in E^3 and containing the centre o, then $\phi_1 (P) > 6\lambda$.

Proof

The idea of this proof is to choose four vertices x_1, x_2, x_3 and x_4 in P which are the vertices of a tetrahedron containing o and to prove the existence of paths σ_{ij} in P which join x_i to x_j for $i < j$, $(i, j) \in \{1, 2, 3, 4\}$ and which are edgewise disjoint in pairs. The theorem is then a consequence of theorem 1. We proceed as follows with a lemma.

Lemma 4

Let x_1, x_2, x_3 and x_4 be four points on the sphere $S^3 [o, \lambda]$ in E^3 with $o \in$ conv (x_1, x_2, x_3, x_4).
write $\phi(x_1, x_2, x_3, x_4) = \sum_{1 \leq i < j}^{4} |x_i - x_j|$.

Then $\phi(x_1, x_2, x_3, x_4) \geq 6\lambda$ with equality only if conv (x_1, x_2, x_3, x_4) is a diameter of $S^3 [o,\lambda]$.

Proof

If x_1, x_2, x_3, x_4 are the vertices of a tetrahedron the lemma is true by theorem 1. We may thus suppose that x_1, x_2, x_3, x_4 lie on the circle $S^2 [o, \lambda]$ in E^2. Now if conv (x_1, x_2, x_3, x_4) is a diameter of $S^2 [o, \lambda]$ then $\phi(x_1, x_2, x_3, x_4) = 6\lambda$ or 8λ.

Otherwise we may assume that $x_1, . . x_4$ lie in cyclic order around $S^2 [o, \lambda]$ and by lemma 1,

$$|x_1 - x_2| + |x_3 - x_2| + |x_4 - x_3| + |x_1 - x_4| > 4\lambda. \quad (1)$$

Let x be the point of intersection x_1x_3 and x_2x_4. we may assume that $o \epsilon$ conv (x, x_1, x_2) without loss in generality since $\bigcup_{i=1}^{4}$ conv (x, x_i, x_{i+1}) = conv (x_1, x_2, x_3, x_4) (we reduce indices mod 4).

Thus

$$\begin{aligned} 2\lambda &= |o - x_1| + |o - x_2| \\ &\leq |x - x_1| + |x - x_2| \qquad (2) \\ &\leq |x_3 - x_1| + |x_4 - x_2| . \end{aligned}$$

Adding equations (1) and (2) we obtain

$$\Phi(x_1, x_2, x_3, x_4) > 6\lambda. \quad (3)$$

The lemma is proved.

Now let v be any fixed vertex of P and let the line vo produced meet P again in u. Then u is contained in some facet F say with vertices $v_1, v_2, . . v_n$. Then, given any vertex v_p of F, there exist vertices v_q, v_{q+1} such that u is contained in the triangle $v_p v_q v_{q+1}$. The tetrahedron v, v_p, v_q, v_{q+1} then contains o and so by theorem 1 has total edge-length greater than 6λ. Now it is clear that the total edge-length of F is greater than or equal to that of the triangle $v_p v_q v_{q+1}$. Therefore, in order to prove the result, it is sufficient to prove that there existsthree paths in P joining v to v_p, v to v_q and v to v_{q+1} respectively which are pairwise edgewise disjoint and each of which has no edge in common with F.

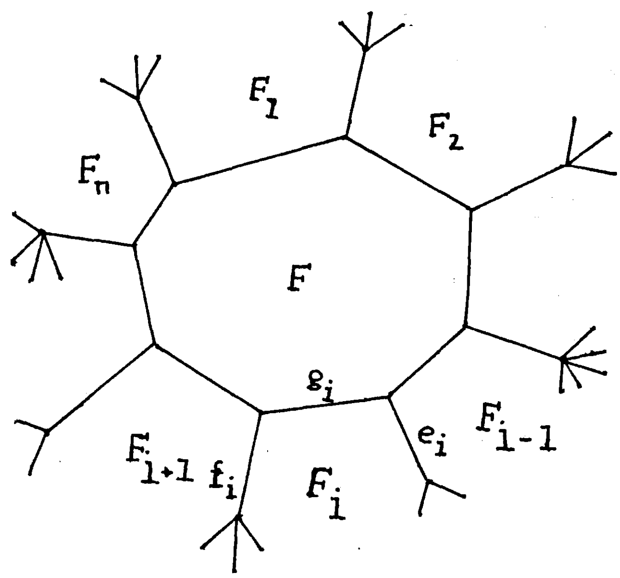
We prove next that we may suppose that every facet of P which meets F, actually meets F in a segment. Let H be a plane parallel to aff F and on the same side of aff F as P. Then, if H is sufficiently close to aff F, then H meets precisely those facets of P which meet F and no others. Let $F^1 = H \cap P$. Consider the polyhedron P^1 contained in P which is obtained by taking F^1 in place of F. Then P^1 has the property that every facet of P^1 which meets F^1 actually meets F^1 in a segment.

Next for the remainder of this proof we may assume that u is not a vertex of P for if this is the case then uv is a diameter and $|u - v| = 2\lambda$. Then since the graph of P is 3-connected, see for example [16] page 213, it is well knownthat there exist three pairwise disjoint paths in P from v to u and thus the total edge-length of P is greater than 6λ.

Let the line vo meet F^1 in u^1. Then it follows by continuity and by lemma 4, that all tetrahedra whose vertices consist of three vertices say v_p^1, v_q^1, v_{q+1}^1 of F^1, and v, and which contain u^1, have total edge-length greater than $(6 + \mu)\lambda$ for some positive number μ for all positions of H sufficiently close to aff F, since u is not a vertex of P. Hence, if paths in P^1 from v to v_p^1, v to v_q^1 and v to v_{q+1}^1 can be found of the type described for P it will follow that the total edge-length of P^1 is greater than $(6 + \mu)\lambda$. Then letting H approach aff F, we have that the total edge-length of P is greater than or equal to $(6 + \mu)\lambda$ and thus greater than 6λ.

For the remainder of the proof then we shall assume that each facet of P which meets F, meets F in a segment.

Let $F_1, F_2, \ldots F_n$ be a labelling in order of the facets of P which meet F excluding F itself. We may assume that each facet F_i has exactly one edge e_i in common with F_{i-1}, exactly one edge f_i in common with F_{i+1}, $e_i \nparallel f_i$, and that there is an edge g_i joining e_i and f_i which lies in $F \cap F_i$. Moreover the vertices corresponding to the edge g_i are the only vertices that F and F_i have in common.



We shall consider the graph of P which is 3-connected. Thus, if w is any vertex in F, then there exist three pairwise disjoint paths in P from v to w which consequently have only the end-points v and w in common.

Denote these paths by π_1, π_2 and π_3. Suppose the vertices of π_1 are $v = w_1, w_2, \ldots w_m = w$. Let $a \geq 1$ be the least integer for which w_a belongs to some facet of P which meets F. Let $y_1 = w_a$. Similarly define y_2 and y_3 for paths π_2 and π_3. By definition y_1, y_2 and y_3 are distinct or $y_1 = y_2 = y_3 = v$ and we may suppose the facets in which they lie which meet F are K_1, K_2 and K_3 respectively.

We essentially then have three cases to consider.

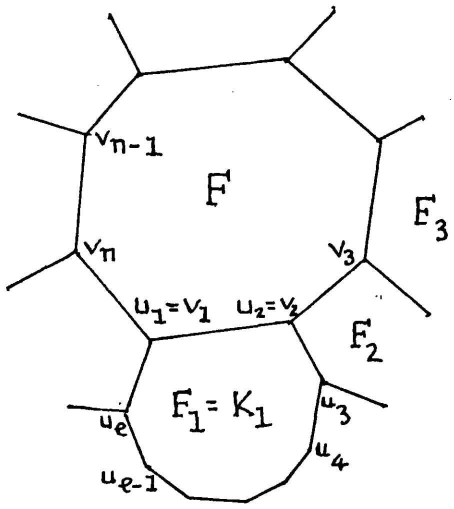
<u>Case I</u> $K_1 = K_2 = K_3$.

<u>Case II</u> $K_1 = K_2$, $K_1 \neq K_3$.

<u>Case III</u> $K_1 \neq K_2$, $K_2 \neq K_3$, $K_1 \neq K_3$.

We shall suppose without loss in generality that $K_1 = F_1$.

Let $u_1, u_2, \ldots u_e$ be a labelling of the vertices of F_1 in order around F_1. We may suppose that F_i and F have the vertices v_i and v_{i+1} in common, that $u_1 = v_1$ and $u_2 = v_2$ where we define $v_{n+1} = v_1$.



Before we consider the above cases we shall prove a lemma.

Lemma 5

In cases I and II, there exists another facet $K \neq K_1$, K meeting F, and paths τ_1, τ_2 and τ_3 in P with τ_1 joining v to u_3, τ_2 joining v to u_e and τ_3 joining v to some vertex u^{11} of K such that τ_i and τ_j are edgewise disjoint for $i \neq j$ and, τ_1, τ_2 and τ_3 have edges in common only with $K_1 = F_1$ of all the facets of P which meet F.

Proof

Consider first case II. We may suppose that $y_3 \notin K_1$.
Let $y_1 = u_{i_1}$ and $y_2 = u_{i_2}$ where $3 \leq i_1 \leq i_2 \leq e$.
Then let τ_1 be the sub-path of π_1 joining v to y_1 together with the edges $u_{i_1}u_{i_1-1}$, $u_{i_1-1}u_{i_1-2}$, . . u_4u_3, τ_2 be the sub-path of π_2 joining v to y_2 together with the edges $u_{i_2}u_{i_2+1}$, $u_{i_2+1}u_{i_2+2}$, . . $u_{e-1}u_e$, and τ_3 the sub-path of π_3 joining v to y_3. Define

$y_3 = u^{11}$ and $K = K_3$. The paths τ_1, τ_2 and τ_3 have the required properties of the lemma.

We now consider case I.

Let $y_1 = u_{i_1}$, $y_2 = u_{i_2}$ and $y_3 = u_{i_3}$ where we can assume $3 \leq i_1 \leq i_3 \leq i_2 \leq e$. Then π_3 must leave y_3 for w since π_1, π_2 and π_3 are pairwise disjoint. Let x_1 be the first vertex which π_3 meets after π_3 has left y_3 for w and which belongs to some F_i. Suppose $x_1 \in F_j$ for some $j \neq 1$. Let $K = F_j$, $u^{11} = x_1$ and τ_3 be the sub-path of π_3 joining v to x_1.

Suppose the sub-path of π_i joining v to y_i is ψ_i for $i = 1$ and 2. Extend ψ_1 and ψ_2 to give paths from v to u_3 and v to u_e respectively in the same manner as was used in case II, and denote the resulting paths by τ_1 and τ_2 respectively. The paths τ_1, τ_2 and τ_3 have the required property of the lemma.

On the other hand if $x_1 \in F_1$ and $x_1 \notin \bigcup_{i=2}^{n} F_i$, then x_1 must be one of the u_i with $3 < i < e$, $i \neq i_1$, $i \neq i_2$ and $i \neq i_3$. Let $x_1 = u_{j_1}$. Firstly if $i_1 < j_1 < i_2$ then as above π_3 must leave x_1 for w. Let x_2 be the first vertex which π_3 meets after π_3 has left x_1 for w and which belongs to some F_i. If $x_2 \in F_j$ for some $j \neq 1$, let $K = F_j$, $u^{11} = x_2$ and repeat the argument in the previous paragraph with x_2 in place of x_1. Again the result follows. Otherwise $x_2 \in F_1$ and $x_2 \notin \bigcup_{1=2}^{n} F_i$. Then as before

$x_2 = u_i$, where $3 < i < e$, $i \neq i_1$, $i \neq i_2$, $i \neq i_3$ and $i \neq j_1$. Suppose $x_2 = u_{j_2}$. Again if $i_1 < j_2 < i_2$ define x_3 in the same way as x_1 and x_2 and continue the arguments above.

Now since there exist only a finite number of integers i satisfying $i_1 < i < i_2$ there is a least integer $t \geq 1$ such that the first vertex $z_1 \in \bigcup_{i=1}^{n} F_i$, which π_3 meets after having left x_t for w is not equal to u_i for any i satisfying $i_1 \leq i \leq i_2$.

If $z_1 \in F_j$ for some $j \neq 1$ the lemma is proved as in case II.

Otherwise $z_1 \in F_1$ and $z_1 \notin \bigcup_{i=2}^{n} F_i$.

Thus $z_1 = u_{k_1}$ say where $3 \leq k_1 < i_1$ or $i_2 < k_1 \leq e$.

Note that if $i_1 = 3$ and $i_2 = e$ then this case could not occur and the lemma would have been proved at the previous stage. We shall suppose then without any real loss in generality that $3 \leq k_1 < i_1$.

Now $k_1 \neq 3$ for if $k_1 = 3$ then $z_1 = u_3 \in F_2$. We suppose then that $3 < k_1 < i_1 < i_2 \leq e$.

But now we must have that π_1 leaves $y_1 = u_{i_1}$ for w since π_1, π_2 and π_3 are pairwise disjoint. We now apply the whole argument again with π_1 in place of π_3 and either the lemma is proved or we can define $z_2 = u_{i_{k_2}}$ in a similar way as we defined z_1 where k_2 satisfies $3 < k_2 < k_1 < i_1$ or $i_2 < k_2 < e$. By continuing in this fashion we must have a strictly monotonic sub-sequence of the integer sequence $\{k_i\}$ tending either down to 3 or up to e if the lemma were false. But since $u_3 \in F_2$ and $u_e \in F_n$ there must

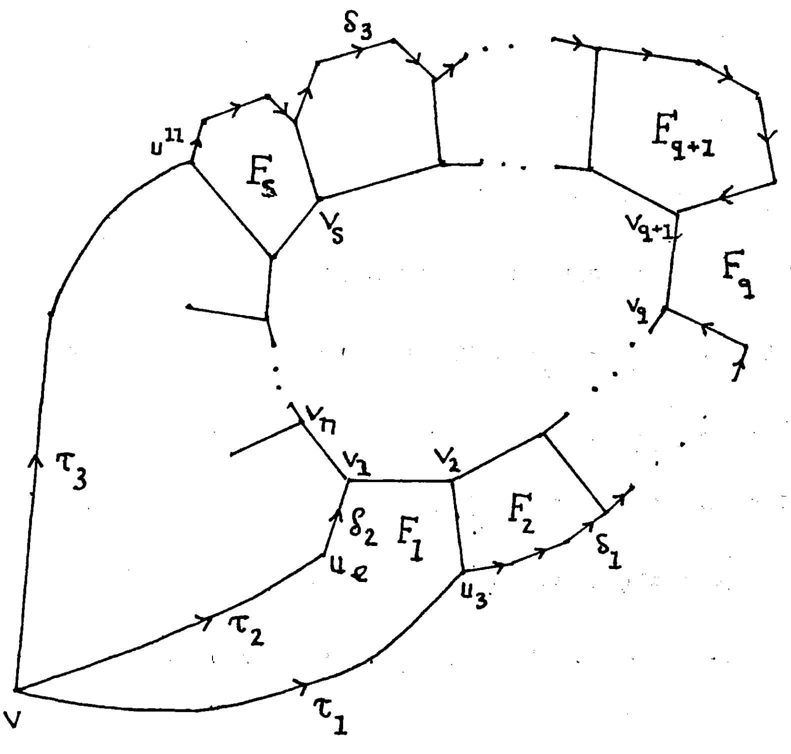
be a path π_{e_3} for some e_3 with $1 \leq e_3 \leq 3$ which joins v to some vertex u^{11} say where $u^{11} \varepsilon F_j$ for some $j \neq 1$.
Let τ_3 be the sub-path of π_{e_3} which joins v to u^{11}.
Suppose at this stage π_{e_1} and π_{e_2} with $\{e_1, e_2, e_3\} \subset \{1, 2, 3\}$ and $e_i \neq e_j$ for $i \neq j$ join v to $u_{k_{i_m}}$ say, and v to $u_{k_{i_n}}$ respectively where $3 \leq k_{i_m} \leq k_{i_{m-1}} \cdot\cdot \leq k_i < i_1$ and $i_2 \leq \cdot\cdot \leq k_{i_{n-1}} \leq k_{i_n} \leq e$.

Let ψ_1 be the sub-path of π_{e_1} joining v to $u_{k_{i_m}}$ and ψ_2 the sub-path joining v to $u_{k_{i_n}}$. Extend ψ_1 and ψ_2 as in case II to obtain τ_1 and τ_2. Then τ_1, τ_2 and τ_3 have the required properties and the proof of the lemma is complete.

We shall now prove case I and II simultaneously by means of the lemma 5. We shall suppose that $K = F_s$ for some s with $2 \leq s \leq n$ where we again define $v_{n+1} = v_1$. For the remainder of this theorem a path which has no edges in common with F will be called an α-path in P. We suppose then that τ_1, τ_2 and τ_3 are defined as in lemma 5. Suppose that u, the point of intersection of vo produced with F, lies in the polygon conv $(v_1, v_2, \cdot\cdot v_s)$. The case $u \varepsilon$ conv$(v_s, v_{s+1}, \cdot\cdot v_n, v_1)$ is similar and is omitted. We may suppose then that $u \varepsilon$ conv(v_1, v_q, v_{q+1}) where $2 \leq q < q+1 \leq s$.
Let δ_2 denote the edge $u_e u_1$ in F_1. Then there exist α-paths δ_1 and δ_3 in P with δ_1 from u_3 to v_q and with δ_3 from u^{11} to v_{q+1} which are edgewise

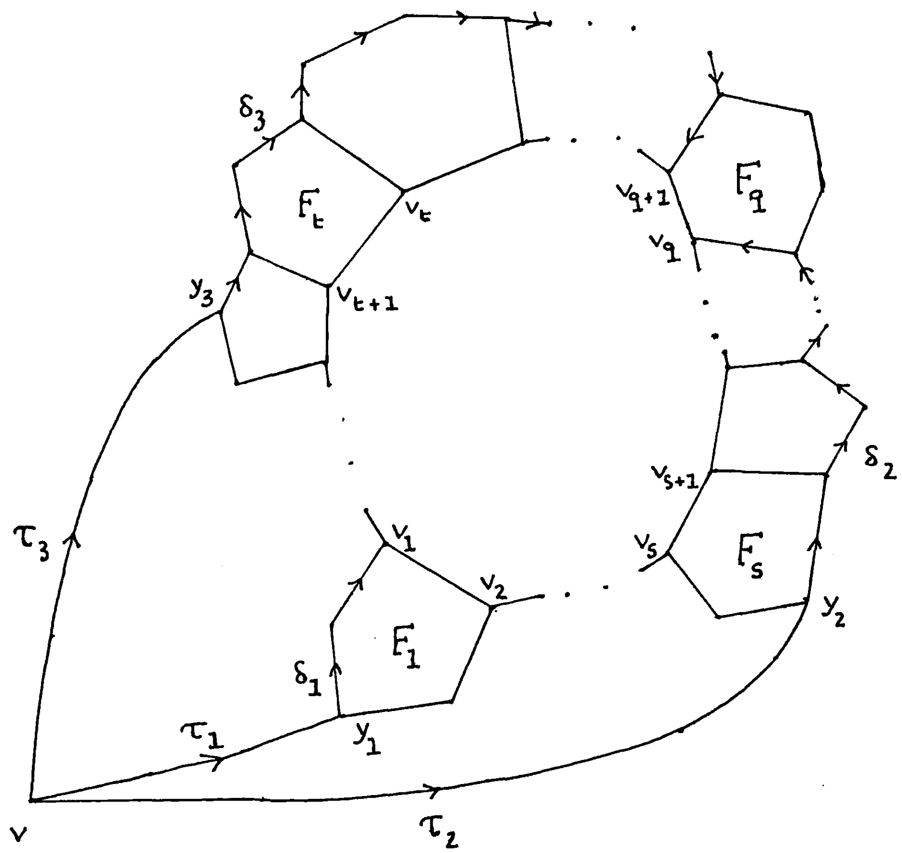
disjoint with δ_2, τ_1, τ_2 and τ_3, with δ_1 consisting of pairwise edgewise disjoint α-arcs of $F_2, F_3 \ldots F_q$ and with δ_3 consisting of pairwise edgewise disjoint α-arcs of $F_s, F_{s-1} \ldots F_{q+1}$.

Define $\theta_i = \tau_i \cup \delta_i$ for $i = 1$, 2 and 3. The θ_i are three pairwise edgewise disjoint α-paths in P from v to v_1, v to v_q and v to v_{q+1} respectively and each is edgewise disjoint with F. The theorem thus follows in cases I and II. We illustrate the situation.



It remains to prove case III.

We shall suppose $y_1 \in F_1$, $y_2 \in F_s$ and $y_3 \in F_t$ where $1 < s < t \leq n$. Let τ_i denote the sub-path of π_i joining v to y_i for $i = 1$, 2 and 3. Then the τ_i are pairwise edgewise disjoint α-paths in P which have no edges in common with any F_j for $j = 1, 2 .. n$. We suppose first that $u \varepsilon$ conv $(V_1, V_s, V_{s+1}, .. V_t)$ and so u lies in some triangle with vertices $V_1 V_q, V_{q+1}$ where $s \leq q < q+1 \leq t$. Then let δ_1 be the α-arc from y_1 to v_1 in F_1. Then there exist α-paths δ_2 and δ_3 in P with δ_2 from y_2 to v_q and with δ_3 from y_3 to v_{q+1} which are edgewise disjoint with δ_1, τ_1, τ_2 and τ_3, with δ_2 consisting of pairwise edgewise disjoint α-arcs of $F_s, F_{s+1}, .. F_q$ and with δ_3 consisting of pairwise edgewise disjoint α-arcs of $F_t, F_{t-1}, . . F_q$. Define $\theta_1 = \tau_i \cup \delta_i$ for $i = 1, 2$ and 3. The θ_i are three pairwise edgewise disjoint α-paths in P from v to v_1, v to v_q and v to v_{q+1} respectively and each is edgewise disjoint with F. The theorem is thus proved if $u \varepsilon$ conv $(v_1, v_s, v_{s+1}, .. v_t)$. However, if this is not the case then $u \varepsilon$ conv $(v_s, v_t, v_{t+1}, .. v_n, v_1)$ or $u \varepsilon$ conv $(v_t, v_1, v_2 .. v_s)$ and the proof is exactly similar to the above. Thus the theorem is completely proved. We illustrate the case $u \varepsilon$ conv $(v_1, v_s, v_{s+1}, .. v_t)$.



We now introduce a new function γ defined on a simplex which we show in theorem 5 takes a minimum on the regular simplex among the class of simplices of given volume.

<u>Theorem 5</u>

Let T be an n-dimensional simplex in E^n with $n \geq 2$ with vertices $x_1, x_2, \ldots x_{n+1}$.

$$\text{Define } \gamma(T) = \inf_{x \in E^n} \sum_{i=1}^{n+1} |x - x_i|.$$

Then

$$\frac{[\gamma(T)]^n}{\Phi_n(T)} \geq n^{\frac{n}{2}} n! (n+1)^{\frac{n-1}{2}}$$

with equality if and only if T is a regular simplex.

<u>Proof</u>

We first state two lemmas.

<u>Lemma 6</u>

Let $\xi_1, \ldots \xi_n$ and $\eta_1, \ldots \eta_n$ be $2n$ non-negative numbers.

Then

$$\sum_{i=1}^{n} (\xi_i^2 + \eta_i^2)^{\frac{1}{2}} \geq \left(\left(\sum_{i=1}^{n} \xi_i \right)^2 + \left(\sum_{i=1}^{n} \eta_i \right)^2 \right)^{\frac{1}{2}}$$

with equality if and only if $\xi_1 = .. \xi_n$ and $\eta_1 = \ldots \eta_n$.

<u>Proof</u>

This is the well known Minkowski inequality and the proof is omitted.

If T is an n-dimensional simplex we shall now write $T = T^n$ in order to emphasise the dimension of T.

<u>Lemma 7</u>

In E^n, $n \geq 2$, let S^n be the regular simplex with centroid g, and with vertices $x_1, \ldots x_{n+1}$.

Suppose $|g - x_i| = \delta_n$ for $i = 1, . . n + 1$.

Then $$\gamma(S^n) = \sum_{i=1}^{n+1} |g - x_i| = (n + 1)\, \delta_n .$$

If g^1 is not the centroid of S^n then

$$\sum_{i=1}^{n+1} |g^1 - x_i| > \gamma(S^n).$$

Proof

The proof is by induction on n.

When $n = 2$, the lemma is true by lemma 15.

Suppose the theorem true in each dimension k with $2 \leqslant k \leqslant n-1$.
By the continuity of $\sum_{i=1}^{n+1} |x - x_i|$ as a function of x it follows that there exists a point g^1 for which

$$\gamma\, (S^n) = \sum_{i=1}^{n+1} |g^1 - x_i| . \tag{1}$$

Suppose that g^1 is not the centroid g of S^n. Clearly $g^1 \varepsilon S^n$.
Then we can assume the notation is chosen so that

$$|g^1 - x_1| > |g^1 - x_2| . \tag{2}$$

Consider the regular $(n - 1)$-simplex $S^{n-1} = \text{conv}\,(x_1, . . x_n)$.
Let h and h^1 denote the orthogonal projections of g and g^1 respectively on to aff $(x_1, . . x_n)$.
Then h is the centroid of S^{n-1}.
Also $h^1 \neq h$ for, if $h^1 = h$ then $|g^1 - x_1| = |g^1 - x_2|$ which by (2) is false.

Thus by the induction hypothesis

$$\sum_{i=1}^{n} |h^1 - x_i| > \sum_{i=1}^{n} |h - x_i| \qquad (3)$$

$$= n\,\delta_{n-1}$$

where $\delta_{n-1} = |h - x_i|$ for $i = 1, 2 \ldots n$.

Now hx_{n+1} is perpendicular to aff $(x_1, \ldots x_n)$ and so if we choose a point g^{11} on hx_{n+1} such that

$$|h^1 - g^1| = |h - g^{11}| \qquad (4)$$

then $$|g^1 - x_{n+1}| \geqslant |g^{11} - x_{n+1}| . \qquad (5)$$

Then $$\gamma(S^n) = \sum_{i=1}^{n+1} |g^1 - x_i|$$

$$= \sum_{i=1}^{n} |g^1 - x_i| + |g^1 - x_{n+1}|$$

$$\geqslant \sum_{i=1}^{n} (|h^1 - x_i|^2 + |h^1 - g^1|^2)^{\frac{1}{2}} + |g^{11} - x_{n+1}| \text{ by (5)}$$

$$= \sum_{i=1}^{n} (|h^1 - x_i|^2 + |h - g^{11}|^2)^{\frac{1}{2}} + |g^{11} - x_{n+1}|$$

$$\geqslant n \left(\left(\frac{\sum_{i=1}^{n} |h^1 - x_i|}{n} \right)^2 + |h - g^{11}|^2 \right)^{\frac{1}{2}}$$

$$+ |g^{11} - x_{n+1}| \text{ by lemma 6}$$

$$> n\left(\left(\left(\frac{\sum_{i=1}^{n}|h - x_i|}{n}\right)^2 + |h - g^{11}|^2\right)^{\frac{1}{2}} + |g^{11} - x_{n+1}|\right.$$

$$= n\,(\delta_{n-1}^2 + |h - g^{11}|^2)^{\frac{1}{2}} + |g^{11} - x_{n+1}| \text{by} \qquad (3)$$

where $\delta_{n-1} = |h - x_i|$ for $i = 1, 2 \ldots n$

$$= \sum_{i=1}^{n+1} |g^{11} - x_i|$$

$$\geq \gamma(S^n) \quad \text{by definition.} \qquad (6)$$

But (6) is impossible and so the assumption that g^1 was not the centroid was false.

Thus lemma 7 is proved.

<u>Proof of theorem 5</u>

For each n write

$$\sigma_n = \inf_{T^n} \frac{\{\gamma(T^n)\}^n}{\phi_n(T^n)} . \qquad (7)$$

By the Blaschke selection theorem we may assume there exists a sequence of simplices $\{T_i^n\}_{i=1}^{\infty}$ convergent to a polytope S^n of at most $n + 1$ vertices of dimension less than or equal to n such that

$$\frac{\left(\gamma(T_i^n)\right)^n}{\phi_n(T_i^n)} \to \sigma_n \text{ as } i \to \infty . \qquad (8)$$

Since the ratio in consideration is invariant under similarity transformations we may suppose

$$\gamma\,(T_i^n) = 1 \quad \text{for } i = 1, 2 \ldots \quad . \tag{9}$$

Now $\sigma_n < \infty$ and so $\Phi_n\,(T_i^n) \not\to 0$ as $i \to \infty$.

Thus $\Phi_n\,(S^n) > 0$ and so S^n is a simplex.

Then clearly

$$\frac{\left(\gamma(S^n)\right)^n}{\Phi_n(S^n)} = \sigma_n \tag{10}$$

and so the inf imum in (7) is attained.

We show next that if T^n is a simplex for which

$$\frac{\left(\gamma(T^n)\right)^n}{\Phi_n(T^n)} = \sigma_n \tag{11}$$

then T^n can only be the regular simplex.

The proof is by induction on the dimension n.

<u>when $n = 2$</u>

Suppose $\dfrac{\left(\gamma(T^2)\right)^2}{\Phi_2\,(T^2)} = \sigma_2$ where T^2 is the (12)

triangle x y z.

Then by lemma 15 it is easy to see that $\gamma(T^2)$ is the length of the connected set of minimal length containing x, y and z.

Suppose now that T^2 is not the equilateral triangle and say

$$|x-y| > |x-z| \quad . \tag{13}$$

we consider two cases.

<u>Case I</u>

Each angle of triangle x y z is less than $\frac{2}{3}\pi$.

On yz erect the equilateral triangle u y z with u on the side of yz opposite to x. By lemma 15

$$\gamma (T^2) = |u - x|. \qquad (14)$$

Then there exists a point x^1 on the circle, centre u, radius $|u - x|$ on the same size of yz as x such that

$$\rho (yz, x) < \rho(yz, x^1). \qquad (15)$$

Let $T_1^{\ 2}$ be triangle x^1yz.

We may assume by taking x^1 sufficiently close to x that each angle of $T_1^{\ 2}$ is less than $\frac{2}{3}\pi$.

Then $\gamma (T^2) = \gamma (T_1^{\ 2})$ (16)

but $\Phi_2(T^2) < \Phi_2(T_1^{\ 2})$. (17)

Thus (16) and (17) combined contradict (12) and Case I is impossible. We consider

Case II

The angle $y \hat{x} z$, say of T^2 is greater than or equal to $\frac{2}{3}\pi$.

Then again by lemma 15

$$\gamma (T^2) = |x - y| + |x - z|. \qquad (18)$$

we now choose a point y^1 on the circle, centre x, radius $|x - y|$ such that

$$\rho(xz, y) < \rho(xz, y^1). \tag{19}$$

but $T_2^{\,2}$ be the triangle xy^1z.

Then $$\gamma\ (T^2) = |x - y^1| + |x - z|$$
$$\geqslant \gamma\ (T_2^{\,2}) \tag{20}$$

and $$\Phi_2\ (T^2) < \Phi_2(T_2^{\,2}). \tag{21}$$

Thus (20) and (21) combined contradict (12) and so case II is also impossible.

Thus T^2 must be equilateral and the statement is true for $n = 2$.

Suppose now the result is true in each dimension k with $2 \leq k \leq n-1$.

Suppose the result false in E^n and that there exists a simplex T^n which is not regular such that

$$\sigma_n = \frac{\{\gamma\ (T^n)\}^n}{\Phi_n(T^n)}. \tag{22}$$

Let T^n have vertices $x_1, \ldots x_{n+1}$ and suppose

$$|x_1 - x_2| > |x_1 - x_3|. \tag{23}$$

Let $T^{n-1} = \text{conv}\ (x_1, \ldots x_n)$.

Let $S(T^{n-1}) = \text{conv}\ (y_1, \ldots y_n)$ say be a regular simplex such that

$$\Phi_{n-1}\ (S(T^{n-1})) = \Phi_{n-1}\ (T^{n-1}). \tag{24}$$

Let L be a line through the centroid g of $S(T^{n-1})$ which is perpendicular to aff $S(T^{n-1})$.

Choose a point y_{n+1} on L such that

$$\rho(S(T^{n-1}), y_{n+1}) = \rho(T^{n-1}, x_{n+1}). \tag{25}$$

Then if $S(T^n) = \text{conv}(y_1, \ldots y_{n+1})$ it is clear that

$$\Phi_n(S(T^n)) = \Phi_n(T^n). \tag{26}$$

we may assume as in lemma 6 that

$$\gamma(T^n) = \sum_{i=1}^{n+1} |x - x_i| \tag{27}$$

for some $x \in T^n$.

Then if x^1 is the projection of x on to aff T^{n-1} it follows by the induction hypothesis and lemma 7 that

$$\sum_{i=1}^{n} |x^1 - x_i| \geq \gamma(T^{n-1}) > \gamma(S(T^{n-1}))$$

$$= n|g - y_i| \text{ for } i = 1, .. n. \tag{28}$$

This implies using a similar argument to that given in lemma 6 that

$$\gamma(T^n) > \gamma(S(T^n)). \tag{29}$$

But (26) and (29) contradict (22) and so T^n must be the regular simplex.

Theorem 5 now follows by calculation and the theorem is proved.

Next using theorem 5, we prove in theorems 6 and 7 respectively, that among the class of simplices of given volume the regular simplex has minimal total edge-length and surface area. It follows then by a result in [11] page 313 that among the simplices containing a given sphere, the regular simplex which circumscribes that sphere has minimal total edge-length and surface area.

Theorem 6

Let T be an n-dimensional simplex in E^n with $n \geq 2$.

Then
$$\frac{\left(\Phi_1(T)\right)^n}{\Phi_n(T)} \geq \frac{n^n \cdot n! \cdot (n+1)^{\frac{2n-1}{2}}}{2^{\frac{n}{2}}}$$

with equality if and only if T is the regular simplex.

Proof

If T is any n-dimensional simplex it is again convenient to write $T = T^n$.

Using the same arguments as in theorem 5 we may assume that there exists a simplex S^n for which

$$\frac{\left(\Phi_1(S^n)\right)^n}{\Phi_n(S^n)} = \alpha_n = \inf_{T^n} \frac{\left(\Phi_1(T^n)\right)^n}{\Phi_n(T^n)} . \qquad (1)$$

we show first that if T^n is a simplex for which

$$\frac{\left(\Phi_1(T^n)\right)^n}{\Phi_n(T^n)} = \alpha_n \qquad (2)$$

then T^n can only be the regular simplex.

The proof is by induction on n.

<u>when $n = 2$</u>

It is well known and easy to prove that of all triangles of given area only the equilateral triangle has minimal perimeter.

The statement then is true for $n = 2$.

Suppose now the statement is true in each dimension k with $2 \leq k \leq n-1$.

Suppose T^n is a simplex for which

$$\alpha_n = \frac{\left(\phi_1(T^n)\right)^n}{\phi_n(T^n)} \tag{3}$$

and that T^n is not regular.

In order to avoid repetition we shall assume the same convention and notation as that introduced in theorem 5 from equation (22) onwards.

Let x^1_{n+1} and y^1_{n+1} be the orthogonal projections of x_{n+1} and y_{n+1} on to aff T^{n-1} and aff $S(T^{n-1})$ respectively. Then y^1_{n+1} is the centroid of $S(T^{n-1})$.

By theorem 5,

$$\begin{aligned}\sum_{i=1}^{n} |x_i - x^1_{n+1}| &\geq \gamma(T^{n-1}) \\ &> \gamma(S(T^{n-1})) \\ &= n|y_i - y^1_{n+1}| \quad \text{for } i = 1, \ldots n \end{aligned} \tag{4}$$

This implies again using the arguments of theorem 5 that

$$\sum_{i=1}^{n} | x_i - x_{n+1} | > \sum_{i=1}^{n} | y_i - y_{n+1} | . \tag{5}$$

By induction hypothesis

$$\Phi_1 (T^{n-1}) > \Phi_1 (S(T^{n-1})) . \tag{6}$$

Thus by (5) and (6)

$$\begin{aligned} \Phi_1 (T^n) &= \Phi_1 (T^{n-1}) + \sum_{i=1}^{n} | x_i - x_{n+1} | \\ &> \Phi_1 (S(T^{n-1})) + \sum_{i=1}^{n} | y_i - y_{n+1} | \\ &= \Phi_1 (S(T^n)). \end{aligned} \tag{7}$$

But since as before

$$\Phi_n (T^n) = \Phi_n (S(T^n)), \tag{8}$$

(7) and (8) contradict (3) and it follows that T^n must be the regular simplex. Theorem 6 now follows by calculation.

<u>Theorem 7</u>

Let T be an n-dimensional simplex in E^n with $n \geq 2$.

Then $$\frac{\left(\Phi_{n-1}(T)\right)^n}{\left(\Phi_n(T)\right)^{n-1}} \geq \frac{(n+1)^{\frac{n+1}{2}} n^{\frac{3n-2}{2}}}{(n-1)!}$$

with equality if and only if T is the regular simplex.

Proof

If T is n-dimensional we again write $T = T^n$. Using the same arguments as in theorem 5 we may assume that there exists a simplex S^n for which

$$\frac{\left(\Phi_{n-1}(S^n)\right)^n}{\left(\Phi_n\ (S^n)\right)^{n-1}} = \beta_n = \inf_{T^n} \frac{\left(\Phi_{n-1}(T^n)\right)^n}{\left(\Phi_n\ (T^n)\right)^{n-1}} . \qquad (1)$$

we shall show that if T^n is a simplex for which

$$\frac{\left(\Phi_{n-1}(T^n)\right)^n}{\left(\Phi_n\ (T^n)\right)^{n-1}} = \beta_n \qquad (2)$$

then T^n can only be the regular simplex.

The proof is again by induction on n.

when $n = 2$ the statement follows by putting $n = 2$ in theorem 6.

Suppose now the statement is true in each dimension k, $2 \leq k \leq n-1$.

Let T^n be a simplex for which

$$\beta_n = \frac{\left(\Phi_{n-1}(T^n)\right)^n}{\left(\Phi_n\ (T^n)\right)^{n-1}} \qquad (3)$$

and suppose T^n is not regular.

Throughout the remainder of the proof of this theorem we shall assume the same convention and notation at that introduced in theorem 6 from equation(3) onwards.

Let T^{n-1} have (n-2) dimensional facets $F_1, \ldots F_n$ and write

$$\rho(F_i, x^1_{n+1}) = p_i \quad \text{for } i = 1, 2 \ldots n. \tag{4}$$

Suppose $S(T^{n-1})$ has n-2 dimensional facets $G_1, \ldots G_n$ and write

$$\rho(G_i, y^1_{n+1}) = p. \quad \text{for } i = 1, 2 \ldots n. \tag{5}$$

Also let

$$|x_{n+1} - x^1_{n+1}| = |y_{n+1} - y^1_{n+1}| = h. \tag{6}$$

Then

$$\Phi_{n-1}(T^n) = \Phi_{n-1}(T^{n-1}) + \frac{1}{(n-1)} \left(\sum_{i=1}^{n} \Phi_{n-2}(F_i) \, |h^2 + p_i^2|^{\frac{1}{2}} \right). \tag{7}$$

we note that by the induction hypothesis

$$\Phi_{n-2}(T^{n-1}) > \Phi_{n-2}(S(T^{n-1})) \tag{8}$$

and we also have

$$\Phi_{n-1}(T^{n-1}) = \Phi_{n-1}(S(T^{n-1})). \tag{9}$$

Thus by (7) and lemma 6,

$$\Phi_{n-1}(T^n) - \Phi_{n-1}(T^{n-1}) \geq \frac{1}{n-1} \left(\left(\sum_{i=1}^{n} \Phi_{n-2}(F_i) \cdot p_i \right)^2 + \left(\sum_{i=1}^{n} \Phi_{n-2}(F_i) h \right)^2 \right)^{\frac{1}{2}}$$

$$= \frac{1}{n-1} \left\{ \left((n-1)\Phi_{n-1}(T^{n-1}) \right)^2 + h^2 \left(\Phi_{n-2}(T^{n-1}) \right)^2 \right\}^{\frac{1}{2}}$$

$$> \frac{1}{n-1} \left(\left((n-1)\Phi_{n-1}(S(T^{n-1})) \right)^2 + h^2 \left(\Phi_{n-2}(S(T^{n-1})) \right)^2 \right)^{\frac{1}{2}}$$

by (8) and (9)

$$= \frac{1}{n-1}\left(\left(\sum_{i=1}^{n} \Phi_{n-2}(G_i)\cdot p\right)^2 + \left(\sum_{i=1}^{n} \Phi_{n-2}(G_i)\cdot h\right)^2\right)^{\frac{1}{2}}$$

$$= \frac{1}{n-1}\left(\sum_{i=1}^{n} \Phi_{n-2}(G_i)\,(h^2+p^2)^{\frac{1}{2}}\right)$$

$$= \Phi_{n-1}(S(T^n)) - \Phi_{n-1}(S(T^{n-1})) . \qquad (10)$$

Thus (9) and (10) imply

$$\Phi_{n-1}(T^n) > \Phi_{n-1}(S(T^n)). \qquad (11)$$

But since we have also

$$\Phi_n(T^n) = \Phi_n(S(T^n)) \qquad (12)$$

it is clear that (11) and (12) contradict (3)

and so it follows that T^n is the regular simplex.

The theorem is completed by calculation .

<u>Corollary to theorems 5, 6 and 7</u>

Let T be an n-dimensional simplex containing the sphere $S^n[o,\lambda]$.

Then

i) $\gamma(T) \geq n(n+1)\lambda$,

ii) $\Phi_1(T) \geq \left(\frac{n^3(n+1)^3}{2}\right)^{\frac{1}{2}}\lambda$,

iii) $\Phi_{n-1}(T) \geq \dfrac{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}\lambda^{n-1}}{(n-1)!}$.

Equality holds in (i), (ii) or (iii) if and only if T is regular

and T circumscribes $S^n[o,\lambda]$.

Proof

It is known, see for example [11] page 313 that if T is a simplex containing the sphere $S^n[o,\lambda]$ then

$$\Phi_n(T) \geq \frac{(n+1)^{\frac{n+1}{2}} n^{\frac{n}{2}} \lambda^n}{n!} \tag{1}$$

with equality if and only if T is regular and circumscribes $S^n[o,\lambda]$.

In view of (1) then the corollary is a trivial consequence of theorems 5, 6 and 7.

Finally we end this chapter by giving an elementary result concerning two polygons which circumscribe a circle. The result although very easy to prove does not seem to have appeared anywhere.

In [6] J.V.Uspensky proved that if two polygons were inscribed in a circle and the length of the largest edge of one polygon was less than the length of the smallest edge of the other then the perimeter and surface area of the former polygon was greater than the perimeter and surface area of the latter.

We shall prove in theorem 8 that if two polygons circumscribe a circle and the length of the largest side of one polygon is less than half the length of the smallest side of the other then the perimeter and surface area of the former polygon is less than the perimeter and surface area of the latter.

Theorem 8

(i) If P and P^1 are two polygons circumscribing $S^2[o, \lambda]$ and the length of the largest edge of P is less than half the length of the smallest edge of P^1

then $\Phi_i(P) < \Phi_i(P^1)$ for $i = 1$ and 2.

(ii) There exist polygons Q and Q^1 circumscribing $S^2[o, \lambda]$ such that all except two edges of Q have length less than half the length of the smallest edge of Q^1 and

$$\Phi_i(Q) > \Phi_i(Q^1) \qquad \text{for } i = 1 \text{ and } 2.$$

Proof

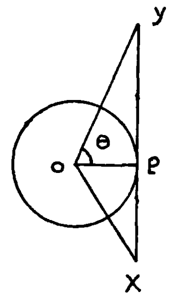
We shall assume throughout that $\lambda = 1$ without loss in generality.

Then $$\frac{1}{2}\Phi_1(P) = \Phi_2(P) \qquad (1)$$

and a similar equation holds for P^1.

Thus we need only prove the result for $i = 1$.

(i) We first prove the following. Let p be a fixed point on $S^2[o, \lambda]$. Suppose that a segment E of fixed length e and end-points x and y is moved from a position where the mid-point of E coincides with p to a position where x coincides with p and E is always tangent to $S^2[o, 1]$ at p. It is clear that the angle $\theta = y\hat{o}p$ increases from $\tan^{-1}\frac{e}{2}$ to $\tan^{-1} e$.



Let $y\hat{o}x = \phi$.

Then $\tan\theta + \tan(\phi-\theta) = e$

and thus

$$\sec^2\theta + \left(\frac{d\phi}{d\theta} - 1\right)\sec^2(\phi-\theta) = 0.$$

Hence $$\frac{d\phi}{d\theta} = 1 - \frac{\cos^2(\phi-\theta)}{\cos^2\theta}$$

$$< 0 \text{ if } \tan^{-1}\frac{e}{2} < \theta < \tan^{-1} e.$$

Thus $\left(\frac{e}{\phi}\right)$ is an increasing function of θ in the range $\tan^{-1}\frac{e}{2} < \theta < \tan^{-1} e$.

Now let P have edges $E_1, \ldots E_m$ of lengths $e_1, \ldots e_m$ respectively. Suppose that E_i subtends an angle ϕ_i at o for $i = 1, \ldots m$. Let P^1 have edges $E^1_1, \ldots E^1_{m^1}$ of lengths $e^1_1, \ldots e^1_{m^1}$ and suppose E^1_i subtends and angle ϕ^1_i at o. The above paragraph implies

$$\frac{e_i}{\phi_i} < \frac{e^1_j}{\phi^1_j} \qquad (2)$$

whenever $1 \leq i \leq m$ and $1 \leq j \leq m^1$.

Thus

$$\frac{\sum_{i=1}^{m} e_i}{2\pi} < \frac{\sum_{i=1}^{m^1} e_i^1}{2\pi} \qquad (3)$$

since $\sum_{i=1}^{m} \phi_i = 2\pi$ and $\sum_{i=1}^{m^1} \phi_i^1 = 2\pi$.

Thus

$$\phi_1 (P) < \phi_1 (P^1)$$

and part (i) is proved.

(ii) Let Q^1 be an equilateral triangle circumscribing $S^2 [o, 1]$. Let Q^{11} be an isosceles triangle $x\ y\ z$ circumscribing $S^2 [o, 1]$ which is not equilateral.

It is well known and easy to prove that

$$\phi_1 (Q^1) < \phi_1 (Q^{11}) .$$

Now choose points x^1, y^1 and z^1 on ox, oy and oz respectively which are close to x, y and z.

Let Q be the polygon which is bounded by the tangents to $S^2 [o, 1]$ from x^1, y^1 and z^1. It follows that if x^1, y^1 and z^1 are sufficiently close to x, y and z respectively then

$$\phi_1 (Q^1) < \phi_1 (Q)$$

and all except two edges of Q have length less than half the length of an edge of Q^1. The theorem is proved.

CHAPTER 2

INTRODUCTION

In this chapter we shall investigate some metric properties of polytopes, which are inscribed in a sphere, which contain its centre, and whose r-dimensional faces have small r-measure for different values of r.

We show first in theorem 9 that an n-dimensional polytope, inscribed in an n-sphere containing its centre and with its r-faces of small r-measure for $r = 1, . . n-1$ 'fills' most of the sphere in a sense described below. This of course implies that the volume and surface area of such a polytope differs by only a small amount from the volume and surface area of the solid-sphere or ball, a fact which is stated and proved in a corollary.

Theorem 9

Let λ and δ be given positive numbers with $\delta < \lambda$.
Let $P(\varepsilon)$ be an n-dimensional polytope inscribed in the sphere $S^n [o, \lambda]$ in E^n with $n \geq 2$ and containing the centre o, with the property that the r-dimensional faces of $P(\varepsilon)$ have Lebesque Λ_r - measure less than ε for $r = 1, 2 . . n - 1$. Let $\mathcal{P}(\varepsilon)$ denote the class of all such polytopes $P(\varepsilon)$ and write

$$\Delta(\varepsilon) = \sup_{P(\varepsilon) \in \mathcal{P}(\varepsilon)} \Delta(P(\varepsilon), D^n [o, \lambda]) .$$

Then $\Delta(\varepsilon) \leq \delta$ whenever $\varepsilon \leq \varepsilon_0 (n, \lambda, \delta)$.

Proof

We prove the result by induction on the dimension n. When $n = 2$ suppose λ and δ are given with $\delta < \lambda$ and let $\varepsilon_0(2, \lambda, \delta) = 2\,(2\lambda\delta - \delta^2)^{\frac{1}{2}}$.

Then for all $P(\varepsilon) \in \mathcal{P}(\varepsilon)$ with $\varepsilon \leq \varepsilon_0\,(2, \lambda, \delta)$,

$$\begin{aligned}\Delta\,(P(\varepsilon), D^2\,[o, \lambda]) &\leq \lambda - \left(\lambda^2 - \frac{\varepsilon_0^{\,2}}{4}\right)^{\frac{1}{2}}\\ &= \lambda - (\lambda^2 - 2\lambda\delta + \delta^2)^{\frac{1}{2}}\\ &= \delta\end{aligned}$$

and so

$$\Delta\,(\varepsilon) \leq \delta.$$

The theorem then is proved for the case $n = 2$.

we assume inductively that the result is true in each dimension k with $2 \leq k \leq n-1$.

We suppose then that δ and λ are given with $\delta < \lambda$.

Let $P(\varepsilon)$ be an n-dimensional polytope inscribed in $S^n\,[o, \lambda]$ with the property that

$$\Delta\,(P(\varepsilon), D^n\,[o, \lambda]) \geq \Delta\,(\varepsilon,\lambda) - \frac{\delta}{2}\quad . \qquad (1)$$

Let the $(n - 1)$ dimensional faces or facets of $P(\varepsilon)$ be $F_i(\varepsilon)$ for $i = 1, 2 \ldots m(\varepsilon)$. Let $o_i\,(\varepsilon)$ and $\mu_i(\varepsilon)$ denote the centre and radius respectively of the $(n - 1)$-sphere $S^{n-1}\,[o_i(\varepsilon), \mu_i(\varepsilon)]$ which is the intersection of aff $F_i\,(\varepsilon)$ and $S^n\,[o, \lambda]$.

Now since $\mu_i^2(\varepsilon) + |o - o_i(\varepsilon)|^2 = \lambda^2$ it follows by lemma 3 that if $\mu_j(\varepsilon) = \max_{1 \leq i \leq m(\varepsilon)} \mu_i(\varepsilon)$ then $o_j(\varepsilon)$ is contained in $F_j(\varepsilon)$.

Thus we may assume that for each $\varepsilon > 0$ there is a polytope $P(\varepsilon)$ satisfying (1) having a facet $F(\varepsilon)$ which is inscribed in an $(n - 1)$-sphere $S^{n-1}[o(\varepsilon), \mu(\varepsilon)]$ and containing $o(\varepsilon)$ with the property that each other facet of $P(\varepsilon)$ is inscribed in an $(n-1)$-sphere of radius less than or equal to $\mu(\varepsilon)$.

Suppose that there is an infinite sequence $\{\varepsilon_i\}_{i=1}^{\infty}$ tending to zero as i tends to infinity such that

$$\mu(\varepsilon_i) \geq \eta \quad \text{where} \quad \eta = \left(\lambda\delta - \frac{\delta^2}{4}\right)^{\frac{1}{2}}. \tag{2}$$

Let $G(\varepsilon_i)$ be a polytope similar to $F(\varepsilon_i)$ but reduced in the ratio $\eta : \mu(\varepsilon_i)$ about $o(\varepsilon_i)$ as centre of similitude.

Then by an appropriate translation we may assume that $G(\varepsilon_i)$ is inscribed in the $(n - 1)$-sphere $S^{n-1}[o, \eta]$ for $i = 1, 2, \ldots$.

Moreover the r-dimensional faces of $G(\varepsilon_i)$ have Λ_r-measure less than $\left(\frac{\eta}{\mu(\varepsilon_i)}\right)^r \varepsilon_i$ and thus less than or equal to ε_i for $r = 1, \ldots n - 2$ and $i = 1, 2, \ldots$.

Thus by the induction hypothesis,

$$\Delta\left(G(\varepsilon_i), D^{n-1}[o, \eta]\right) \to 0 \quad \text{as} \quad i \to \infty. \tag{3}$$

Hence for all $i \geq i_0$,

$$0 < \tfrac{1}{2}\, \Phi_{n-1}\,(D^{n-1}[o, \eta]) < \Phi_{n-1}(G(\varepsilon_i)) \leq \Phi_{n-1}(F(\varepsilon_i)) < \varepsilon_i. \qquad (4)$$

But (4) is contradictory since $\varepsilon_i \to 0$ as $i \to \infty$ and thus $\mu(\varepsilon) < \eta$ whenever $\varepsilon \leq \varepsilon_0\,(n, \delta, \lambda)$.

Thus if $\varepsilon \leq \varepsilon_0(n, \delta, \lambda)$ it follows that aff $F_i(\varepsilon)$ is distant at least $(\lambda^2 - \eta^2)^{\frac{1}{2}}$ from o for $i = 1, 2 \, . \, . \, m(\varepsilon)$. It follows that the solid sphere $D^n\,[o, (\lambda^2 - \eta^2)^{\frac{1}{2}}] \subset P(\varepsilon) \subset D^n\,[o, \lambda]$.

Hence $\Delta\,(P(\varepsilon), D^n\,[o,\lambda]\,) \leq \Delta(D^n\,[o, (\lambda^2 - \eta^2)^{\frac{1}{2}}], D^n\,[o, \lambda])$

$$= \lambda - (\lambda^2 - \eta^2)^{\frac{1}{2}}$$

$$= \lambda - (\lambda^2 - \lambda\delta + \frac{\delta^2}{4})^{\frac{1}{2}} \quad \text{by (2)}$$

$$= \frac{\delta}{2},$$

and thus by (1)

$$\Delta(\varepsilon, \lambda) \leq \delta.$$

The theorem is proved.

Corollary

In E^n and in the same notation as theorem 9 define

$$\Phi_i\,(\varepsilon) = \inf_{P(\varepsilon)\in\mathcal{P}(\varepsilon)} \Phi_i(P(\varepsilon)) \quad \text{for } i = n-1 \text{ and } i = n.$$

Then
$$\lim_{\varepsilon \to 0+} \phi_i(\varepsilon) = \phi_i(S^n[o,\lambda]) = \frac{2\pi^{\frac{1}{2}n}\lambda^{n-1}}{\Gamma(\frac{1}{2}n)} \quad \text{if } i = n-1,$$
$$= \frac{2\pi^{\frac{1}{2}n}\lambda^{n}}{n\Gamma(\frac{1}{2}n)} \quad \text{if } i = n.$$

Proof

Let $\delta > 0$ be given. For each $\varepsilon > 0$ choose n-dimensional polytopes $P_n(\varepsilon)$ and $P_{n-1}(\varepsilon)$ in $\mathcal{P}(\varepsilon)$ such that

$$\phi_i(\varepsilon) + \delta \geq \phi_i(P_i(\varepsilon)) \quad \text{for } i = n-1 \text{ and } i = n. \tag{1}$$

By continuity of volume and surface area, and also by theorem 9 it follows that

$$\phi_i(S^n[o, \lambda]) - \phi_i(P_i(\varepsilon)) \leq \delta \quad \text{for } i = n-1 \text{ and } i = n \tag{2}$$

whenever $\varepsilon \leq \varepsilon_0(n, \delta, \lambda)$.

Thus (1) and (2) imply

$$\phi_i(S^n[o, \lambda]) - 2\delta \leq \phi_i(\varepsilon) \quad \text{for } i = n-1 \text{ and } i = n \tag{3}$$

whenever $\varepsilon \leq \varepsilon_0(n, \delta, \lambda)$.

Thus (3) implies

$$\lim_{\varepsilon \to 0+} \phi_i(\varepsilon) \geq \phi_i(S^n[o, \lambda]) \quad \text{for } i = n-1 \text{ and } i = n. \tag{4}$$

Since trivially

$$\phi_i(P_i(\varepsilon)) \leq \phi_i(S^n[o, \lambda]) \text{ as } P_i(\varepsilon) \subset S^n[o, \lambda]$$

it follows that

it follows that

$$\lim_{\varepsilon \to 0+} \Phi_i(\varepsilon) \leq \Phi_i (S^n [o, \lambda]) \quad \text{for } i = n-1 \text{ and } i = n . \quad (5)$$

Hence (4) and (5) imply

$$\lim_{\varepsilon \to 0+} \Phi_i(\varepsilon) = \Phi_i(S^n [o, \lambda]) \quad \text{for } i = n-1 \text{ and } i = n$$

and the theorem is proved.

We now prove a theorem which shall require in the proof of theorem 11.

Theorem 10

Let X be a bounded set in E^n, $n \geq 2$ with positive Lebesgue Λ_n-measure. Then, if N is a given positive integer there exists $\varepsilon(N) > 0$ such that if $\{x_1, x_2, \ldots x_m\}$ is any set with the property that $\sup_{x \in X} \left(\min_{1 \leq i \leq m} |x - x_i| \right) < \varepsilon(N)$, then all arc-wise connected sets E containing $\{x_1, x_2, \ldots x_m\}$ have linear measure $\mathcal{M}_1(E) > N$.

Proof

Let C be a closed hypercube containing X. Let C have edge-length e. For each integer k divide C into $(3k)^n$ open disjoint equal hypercubes of side $\frac{e}{3k}$. Suppose exactly $m(k)$ of these open hypercubes contain at least one point of X. Then X is contained in the union of these $m(k)$ disjoint open hypercubes together with a closed set Y of Λ_n-measure zero.

Since each open hypercube in Λ_n-measurable and Y is Λ_n-measurable it follows that

$$m(k) \left(\frac{e}{3k} \right)^n \geq \Lambda_n(X). \qquad (1)$$

Consider the partition of C as an array of $(3k)^n$ open hypercubes $C(i_1, \ldots i_n)$ where $1 \leq i_1 \leq 3k, \ldots 1 \leq i_n \leq 3k$.

Let $$T(p_1, \ldots p_n) = \bigcup_{\substack{0 \leq r_1 \leq k-1 \\ \vdots \\ 0 \leq r_n \leq k-1}} C(3r_1 + p_1, \ldots, 3r_n + p_n)$$

where $1 \leq p_1 \leq 3 \; \ldots \; 1 \leq p_n \leq 3$.

Each $T(p_1, \ldots p_n)$ is a union of open hypercubes and each is distant $\frac{2e}{3k}$ from any other.

Moreover,

$$C = \bigcup_{\substack{1 \leq p_1 \leq 3 \\ \vdots \\ 1 \leq p_n \leq 3}} T(p_1, \ldots p_n) .$$

Suppose $m(p_1, \ldots p_n)$ hypercubes of $T(p_1, \ldots p_n)$ contain at least one point of X.

Then $$\sum_{p_1=1}^{3} \ldots \sum_{p_n=1}^{3} m(p_1, \ldots p_n) = m(k) \geq \left(\frac{3k}{e}\right)^n \Lambda_n(X) \qquad (2)$$

by (1).

Thus by (2) there exist integers $p_1, \ldots p_n$ with $1 \leqslant p_1 \leqslant 3, \ldots$ $1 \leqslant p_n \leqslant 3$ such that

$$m(p_1, \ldots p_n) \geqslant \frac{1}{3^n} \left(\frac{3k}{e}\right)^n \Lambda_n(X)$$
$$= \left(\frac{k}{e}\right)^n \Lambda_n(X). \qquad (3)$$

For these particular values of $p_1, \ldots p_n$ let

$$T(p_1 \ldots p_n) = T(p) \text{ and } m(p_1, \ldots p_n) = m(p).$$

we shall consider k so large that

$$\frac{\Lambda_n(X)}{6} \left(\frac{k}{e}\right)^{n-1} > N. \qquad (4)$$

We then choose $\varepsilon > 0$ such that $0 < \varepsilon < \frac{e}{6k}$.

The m(p) open hypercubes of $T(p)$ which contain a point of X will be denoted by $C_1, \ldots C_{m(p)}$.

Let C_i^1 and C_i^{11} be the hypercubes obtained from C_i by magnifications of C_i in the ratio $2:1$ and $3:1$ respectively about the centre of C_i as centre of similitude.

Now each C_i is distant at least $\frac{2e}{3k}$ from any C_j for $j \neq i$ and since C_i^1 and C_i^{11} are both open it follows that $C_i^1 \cap C_j^1 = \phi$ and $C_i^{11} \cap C_j^{11} = \phi$ for $i \neq j$.

Also each C_i contains a point of X and by the choice of ε it follows that any set $\{x_1, \ldots x_m\}$ satisfying the condition of the theorem must have points in common with C_i^1 for each i with $1 \leq i \leq m(p)$.

We suppose then $x_{k_1} \in C_1^1, x_{k_2} \in C_2^1, \ldots x_{k_{m(p)}} \in C_{m(p)}^1$.

Then since E is arc-wise connected it follows that there is an arc of E joining x_{k_i} to the boundary of C_i^{11} which must therefore have length at least $\frac{e}{6k}$ for $i = 1, \ldots m(p)$.

Thus $\Lambda_1(E) \geq m(p) \cdot \frac{e}{6k}$

$$\geq \frac{\Lambda_n(X)}{6} \cdot \left(\frac{k}{e}\right)^{n-1} \quad \text{by (3)}$$

$$> N . \quad \text{by (4)}$$

The theorem is proved.

We are now able to prove that an n-dimensional polytope, inscribed in an n-sphere containing its centre and with its r-faces of small r-measure for $r = 1, \ldots n-1$, has 'large' total edge-length.

Theorem 11

In E^n, $n \geq 3$ and in the same notation as theorem 9 define

$$\Phi_1(\varepsilon) = \inf_{P(\varepsilon) \in \mathcal{P}(\varepsilon)} \Phi_1(P(\varepsilon)).$$

Then $$\lim_{\varepsilon \to 0+} \Phi_1(\varepsilon) = \infty.$$

Proof

Let N be a given positive integer.

For each $\varepsilon > 0$ let $P(\varepsilon)$ be a polytope in $\mathcal{P}(\varepsilon)$ such that

$$\Phi_1(\varepsilon) + \frac{N}{2} \geq \Phi_1(P(\varepsilon)). \tag{1}$$

Let $P(\varepsilon)$ have vertices $x_1^\varepsilon, x_2^\varepsilon, \ldots x_{m(\varepsilon)}^\varepsilon$.

Since $S^n[o, \lambda]$ is compact we may assume that there exists $x(\varepsilon) \in S^n[o, \lambda]$ for which

$$\min_{1 \leq i \leq m(\varepsilon)} |x(\varepsilon) - x_i^\varepsilon| = \sup_{x \in S^n[o,\lambda]} \left(\min_{1 \leq i \leq m(\varepsilon)} |x - x_i^\varepsilon| \right). \tag{2}$$

Suppose if possible that there exists small $\delta > 0$ and a sequence $\{\varepsilon_j\}_{j=1}^\infty$ tending to zero as j tends to infinity for which

$$\min_{1 \leq i \leq m(\varepsilon_j)} |x(\varepsilon_j) - x_i^{\varepsilon_j}| \geq \delta > 0 \quad \text{for } j = 1, 2 \ldots \tag{3}$$

Then for each j, $x(\varepsilon_j)$ does not belong to a facet of $P(\varepsilon_j)$ and so there is a facet $F(\varepsilon_j)$ of $P(\varepsilon_j)$ for which

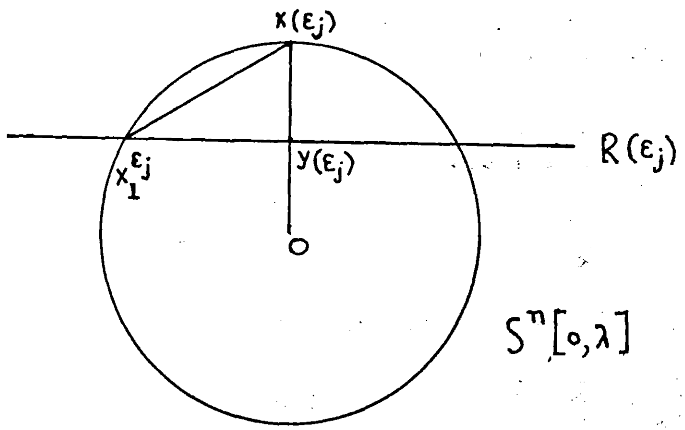
$$\rho(x(\varepsilon_j), F(\varepsilon_j)) = \rho(x(\varepsilon_j), P(\varepsilon_j)). \tag{4}$$

Suppose without loss in generality that $F(\varepsilon_j)$ has vertices $x_1^{\varepsilon_j}, \ldots x_{p(\varepsilon_j)}^{\varepsilon_j}$ where $n \leq p(\varepsilon_j) < m(\varepsilon_j)$, and that

$$|x(\varepsilon_j) - x_1^{\varepsilon_j}| = \min_{1 \leq i \leq p(\varepsilon_j)} |x(\varepsilon_j) - x_i^{\varepsilon_j}|. \tag{5}$$

Suppose the hyperplane $R(\varepsilon_j)$ which is perpendicular to $ox(\varepsilon_j)$ and passes through $x_1^{\varepsilon_j}$ meets $ox(\varepsilon_j)$ in $y(\varepsilon_j)$. Then it follows by (5) that

$$\begin{aligned} \rho\,(x(\varepsilon_j),\ F(\varepsilon_j)\ ) &\geq |x(\varepsilon_j) - y(\varepsilon_j)| \\ &= \frac{1}{2\lambda}\ |x(\varepsilon_j) - x_1^{\varepsilon_j}|^2 \qquad (6) \\ &> 0 \qquad \text{by (3).} \end{aligned}$$



It follows by (3), (4) and (6) that

$$\rho(x(\varepsilon_j),\ P(\varepsilon_j)\ ) \geq \eta > 0 \qquad \text{for } j = 1, 2 \ldots \quad (7)$$

where $\eta = \dfrac{\delta^2}{2\lambda}$.

But by theorem 9, if $j \geq j_0$, then

$$\Delta\ (P(\varepsilon_j),\ D^n\ [o,\ \lambda]) < \frac{\eta}{2}\ , \qquad (8)$$

and thus $\rho(x(\varepsilon_j),\ P(\varepsilon_j)\ ) < \dfrac{\eta}{2}$.

Thus (7) and (8) are contradictory and so (3) is impossible.

Thus for each $\varepsilon > 0$ we may assume

$$\min_{1 \leq i \leq m(\varepsilon)} |x(\varepsilon) - x_i^{\varepsilon}| < \delta(\varepsilon) \qquad (9)$$

where $\delta(\varepsilon)$ tends to zero as ε tends to zero.

We project E^n by orthogonal projection on to a hyperplane R. For any point x let x^1 denote the orthogonal projection of x on R. Then $S^n\ [o, \lambda]$ projects on to an $(n-1)$-dimensional solid sphere $D^{n-1}\ [u, \lambda]$ of radius λ and centre u and thus by (9)

$$\sup_{y \in D^{n-1}[u, \lambda]} \left(\min_{1 \leq i \leq m(\varepsilon)} |y - (x_i^{\varepsilon})^1| \right) < \delta(\varepsilon). \qquad (10)$$

Moreover the union of the 1-dimensional faces of $P(\varepsilon)$ projects on to an arc-wise connected set $E(\varepsilon)$ which contains $\{(x_1^{\varepsilon})^1, (x_2^{\varepsilon})^1, \ldots (x_{m(\varepsilon)}^{\varepsilon})^1\}$. But then by theorem 10, if $\varepsilon < \varepsilon(N)$ then

$$\mathcal{M}_1(E(\varepsilon)) > N. \qquad (11)$$

Thus since the distance between two vertices is decreased by orthogonal projection it follows that

$$\Phi_1(P(\varepsilon)) > N$$

and so by (1)

$$\Phi_1(\varepsilon) > \frac{N}{2}.$$

Hence $\lim_{\varepsilon \to 0+} \Phi_1(\varepsilon) > \frac{N}{2}$.

But N was arbitrary and so

$$\lim_{\varepsilon \to 0+} \Phi_1(\varepsilon) = \infty$$

and the theorem is proved.

For the remainder of this chapter we shall work in three dimensional Euclidean space E^3. We consider next polyhedra inscribed in a sphere containing its centre and which have small edge-lengths but which do not necessarily have facets of small area. In theorem 12 we look at the surface area and volume of such polyhedra and then in theorem 13 consider the total edge-lengths.

Theorem 12

Let $Q(\varepsilon)$ be a 3-dimensional polyhedron inscribed in the sphere $S^3[o, \lambda]$ in E^3 and containing the centre o with the property that each edge of $Q(\varepsilon)$ is of length less than ε.
Let $\mathcal{Z}(\varepsilon)$ denote the class of all such polyhedra $Q(\varepsilon)$ and define

$$\Psi_i(\varepsilon) = \inf_{Q(\varepsilon)\in\mathcal{Z}(\varepsilon)} \Phi_i(Q(\varepsilon)) \quad \text{for } i = 2 \text{ and } 3.$$

Then (i) $\lim_{\varepsilon \to 0+} \Psi_2(\varepsilon) = 2\pi\lambda^2,$

(ii) $\lim_{\varepsilon \to 0+} \Psi_3(\varepsilon) = 0$.

Lemma 8

For any integer N, the surface area σ of the set obtained by removing N disjoint caps from the solid sphere $D^3\,[o, \mu]$ is greater than or equal to $2\pi\mu^2$.

Proof

Let the caps be $C_1, C_2 \,.\,.\, C_N$ and suppose each cap C_i is distant λ_i from o for $i = 1, 2. \,.\, N$.

$$\sigma = 4\pi\lambda^2 - \sum_{i=1}^{N} 2\pi\lambda(\lambda - \lambda_i) + \sum_{i=1}^{N} \pi\,(\lambda^2 - \lambda_i^2) . \qquad (1)$$

If $\sum_{i=1}^{N} \pi\,(\lambda^2 - \lambda_i^2) \geq 2\pi\lambda^2$ then certainly

$$\sigma \geq 2\pi\lambda^2 .$$

If $\sum_{i=1}^{N} \pi\,(\lambda^2 - \lambda_i^2) < 2\pi\lambda^2,$ (2)

then
$$\begin{aligned}\sigma &= 4\pi\lambda^2 - \sum_{i=1}^{N} \pi\,(\lambda - \lambda_i)^2 \\ &\geq 4\pi\lambda^2 - \sum_{i=1}^{N} \pi\,(\lambda^2 - \lambda_i^2) \\ &> 2\pi\lambda^2 \qquad \text{by (2).}\end{aligned}$$

The lemma is proved.

<u>Proof of Theorem 12</u>

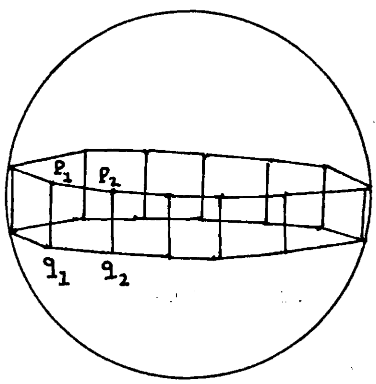
We prove first by an appropriate example that $\lim\limits_{\varepsilon \to 0+} \Psi_2(\varepsilon) \leq 2\pi\lambda^2$.

Let $p_1, p_2, \ldots p_n$ be the n vertices of a regular polygon which is inscribed in the sphere $S^3 [o, \lambda]$ and distant $\frac{\lambda}{n^2}$ from o.

Let q_i denote the reflection of p_i in the plane through o which is parallel to aff $(p_1, \ldots p_n)$.

Let $Q(n) = \text{conv}\,(p_1, \ldots p_n, q_1, \ldots q_n)$.

Then $Q(n)$ has two faces which are regular polygons of side $2\lambda(1 - \frac{1}{n^4})^{\frac{1}{2}} \sin \frac{\pi}{n}$ and n faces which are rectangles each of which has one side of length $2\lambda(1 - \frac{1}{n^4})^{\frac{1}{2}} \sin \frac{\pi}{n}$ and the other side of length $\frac{2\lambda}{n^2}$.



$S^3\,[o; \lambda]$

Thus for all $n \geq N(\varepsilon)$, $Q(n) \in \mathcal{Z}(\varepsilon)$

and so

$$\Psi_2(\varepsilon) \leq \lim_{n \to \infty} \Phi_2(Q(n))$$

$$= \lim_{n \to \infty} \left(2n \cdot \frac{\lambda^2}{2} \cdot \left(1 - \frac{1}{n^4}\right) \sin \frac{2\pi}{n} + n . \frac{2\lambda}{n^2}\left(2\lambda\left(1 - \frac{1}{n^4}\right)^{\frac{1}{2}} \sin \frac{\pi}{n}\right)\right)$$

$$= \lim_{n \to \infty} \left(2\pi\lambda^2\left(1 - \frac{1}{n^4}\right) . \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}} + \frac{2}{n}\left(2\lambda^2\left(1 - \frac{1}{n^4}\right)^{\frac{1}{2}} \sin\frac{\pi}{n}\right)\right)$$

$$= 2\pi\lambda^2.$$

Thus $\lim\limits_{\varepsilon \to 0+} \Psi_2(\varepsilon) \leqslant 2\pi\lambda^2$.

It remains to show in order to prove (i) that

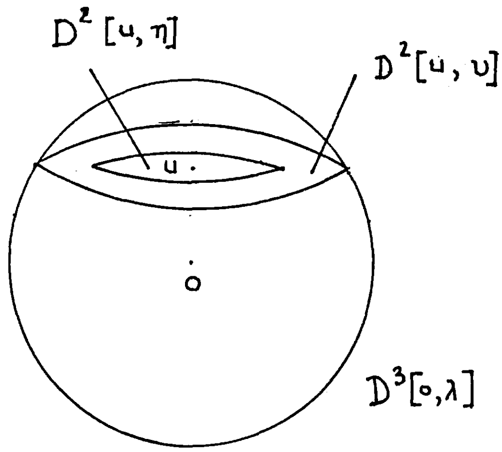
$$\lim_{\varepsilon \to 0+} \Psi_2(\varepsilon) \geqslant 2\pi\lambda^2.$$

Choose δ such that $\lambda>\delta>0$ and then consider any $\varepsilon > 0$ with

$0 < \frac{\varepsilon^2}{4} \leqslant 2\delta\lambda-\delta^2$. (3)

Let $Q(\varepsilon)$ be any polyhedron in $\mathcal{L}(\varepsilon)$.

Now suppose there is a facet F of $Q(\varepsilon)$ such that aff F meets $D^3[o,\lambda - \delta]$ in a disc $D^2[u, \eta]$ centre u and radius $\eta > 0$. Then if aff F meets $D^3[o, \lambda]$ in the disc $D^2[u, \upsilon]$, $\upsilon > \eta$ it follows that each edge of F of length ζ is distant $(\upsilon^2 - \frac{\zeta^2}{4})^{\frac{1}{2}}$ from u.



Also
$$\eta = ((\lambda - \delta)^2 - (\lambda^2 - \upsilon^2))^{\frac{1}{2}}$$
$$= (\upsilon^2 - 2\delta\lambda + \delta^2)^{\frac{1}{2}}$$
$$\leq (\upsilon^2 - \frac{\varepsilon^2}{4})^{\frac{1}{2}} \quad \text{by (3)}$$
$$< (\upsilon^2 - \frac{\zeta^2}{4})^{\frac{1}{2}}$$

since each edge of $Q(\varepsilon)$ and in particular F has length less than ε.

Thus each facet F of $Q(\varepsilon)$ satisfies one of the following;

(a) F does not meet $D^3[o, \lambda - \delta]$

(b) F meets $D^3[o, \lambda - \delta]$ in at most a point or

(c) F meets $D^3[o, \lambda - \delta]$ in a disc.

Thus the set $Q(\varepsilon) \cap D^3[o, \lambda - \delta]$ can be obtained by removing $N(\varepsilon,\delta)$ disjoint caps from the solid sphere $D^3[o, \lambda - \delta]$ and thus by lemma 8 has surface area

$$\sigma(\varepsilon,\delta) \geq 2\pi(\lambda - \delta)^2.$$

Thus $\Phi_2 (Q(\varepsilon)) \geq 2\pi (\lambda - \delta)^2$

and so $\Psi_2 (\varepsilon) = \inf\limits_{Q(\varepsilon)\varepsilon\mathcal{L}(\varepsilon)} \Phi_2 (Q(\varepsilon)) \geq 2\pi (\lambda - \delta)^2$

whenever $0 < \dfrac{\varepsilon^2}{4} \leq 2\delta\lambda - \delta^2.$

Thus $\lim\limits_{\varepsilon \to 0+} \Psi_2(\varepsilon) \geq 2\pi (\lambda - \delta)^2$

and since the choice of δ was arbitrary it follows that

$$\lim_{\varepsilon \to 0+} \Psi_2 (\varepsilon) \geq 2\pi\lambda^2$$

and part (i) of the theorem is proved.

It is obvious that

$$0 \leq \Psi_3 (\varepsilon) \leq \lim_{n \to \infty} \Phi_3 (Q(n)) = 0$$

and so

$$\lim_{\varepsilon \to 0+} \Psi_3 (\varepsilon) = 0$$

and so the theorem is proved.

Theorem 13

In E^3 and in the same notation as theorem 12 define

$$\Psi_1 (\varepsilon) = \inf_{Q(\varepsilon)\varepsilon\mathcal{L}(\varepsilon)} \Phi_1 (Q(\varepsilon)) .$$

Then $\lim\limits_{\varepsilon \to 0+} \Psi_1 (\varepsilon) = 4\pi\lambda.$

Lemma 9

Let $\{C_i\}_{i=1}^{\infty}$ be a collection of caps of $D^3\,[o, \lambda]$ such that $C_i \cap C_j$ has no interior points. Let σ_i denote the surface area of the portion of the frontier of C_i which is common to $S^3\,[o, \lambda]$ for $i = 1, 2 \ldots$.

Then, if $\sum_{i=1}^{\infty} \sigma_i = 4\pi\lambda^2$, the total sum $\mathcal{L}$ of the lengths of the circumferences of the discs which form the bases of the caps C_i is greater than $4\pi\lambda$.

Note we allow finite collection $\{C_i\}_{i=1}^{N}$ by defining $C_i = \phi$ for $i > N$.

Proof

Given $\sum_{i=1}^{\infty} \sigma_i = 4\pi\lambda^2$. Thus if each cap C_i is distant λ_i from o for $i = 1, 2 \ldots$ then

$$\sum_{i=1}^{\infty} 2\pi\lambda\,(\lambda - \lambda_i) = 4\pi\lambda^2 \tag{1}$$

and so

$$\begin{aligned}
\mathcal{L} &= \sum_{i=1}^{\infty} 2\pi\,(\lambda^2 - \lambda_i^2)^{\frac{1}{2}} \\
&= \sum_{i=1}^{\infty} 2\pi\,(\lambda - \lambda_i)\,\left(\frac{\lambda + \lambda_i}{\lambda - \lambda_i}\right)^{\frac{1}{2}} \\
&> \sum_{i=1}^{\infty} 2\pi\,(\lambda - \lambda_i) \\
&= 4\pi\lambda \qquad \text{by (1).}
\end{aligned}$$

The lemma is proved.

Lemma 10

Let X be a subset of $S^3[o, \lambda]$ of positive $\mathcal{M}_2$-measure. Let R be any fixed plane and let X^1 denote the orthogonal projection of X on to R. Then X^1 has positive $\mathcal{M}_2$-measure.

Proof

For any set Y in E^3 let Y^1 denote the orthogonal projection of Y on R.

Let R_1 and R_2 be two planes $R_1 \neq R_2$ each of which is the same distance η from o and parallel to R.

Let G denote the closed subset of $S^3[o, \lambda]$ which lies between R_1 and R_2. We shall suppose that η is so small that

$$0 < \mathcal{M}_2(G) < \tfrac{1}{2}\mathcal{M}_2(X) \,. \tag{2}$$

Now G is $\mathcal{M}_2$-measurable and so

$$\mathcal{M}_2(X) = \mathcal{M}_2(G \cap X) + \mathcal{M}_2(X|G). \tag{3}$$

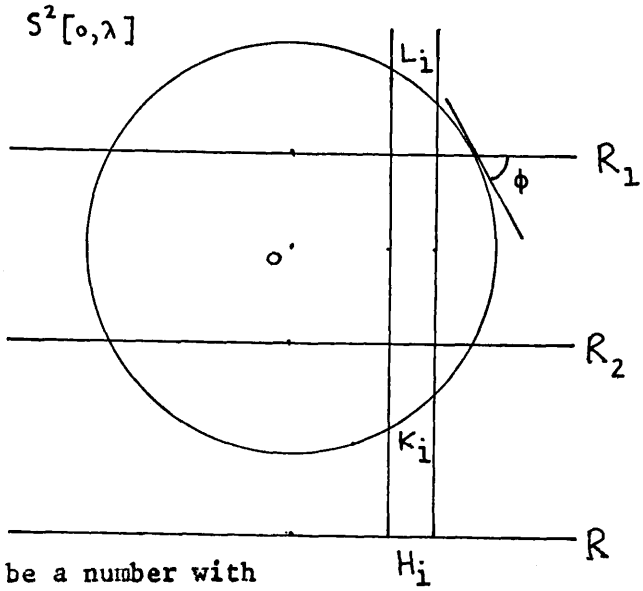
Hence by (2) and (3)

$$\mathcal{M}_2(X|G) > \tfrac{1}{2}\mathcal{M}_2(X) > 0. \tag{4}$$

We shall suppose $\mathcal{M}_2(X^1) = 0$ and obtain a contadiction. If

$$\mathcal{M}_2(X^1) = 0 \text{ then certainly } \mathcal{M}_2((X|G)^1) = 0. \tag{5}$$

Let $S^2[o, \lambda]$ be a great circle of $S^3[o, \lambda]$ which lies in a plane perpendicular to R and let the tangents to $S^2[o, \lambda]$ make an acute angle ϕ with R_1 and R_2 at the points of intersection of $S^2[o, \lambda]$ with R_1 and R_2 respectively.



Let ε be a number with

$$0 < \varepsilon < \tfrac{1}{2}\mathcal{M}_2(X). \tag{6}$$

Now equation (5) implies that for each $\delta > 0$ there is a sequence of sets $\{H_i\}_{i=1}^{\infty}$ such that

$$H_i \subset R,$$

$$\bigcup_{i=1}^{\infty} H_i \supset (X|G)^1,$$

$$D(H_i) \leq \delta \cos\phi,$$

and $$\sum_{i=1}^{\infty} \left(D(H_i)\right)^2 < \frac{\varepsilon}{2} \cos^2\phi. \tag{7}$$

Let J_i be the space spanned by H_i and the normal to R and define K_i and L_i to be the intersections of J_i with $S^3[o, \lambda]$.

Then $X|G \subset \bigcup_{i=1}^{\infty} (K_i \cup L_i)$

and also

$$\max (D(K_i), D(L_i)) \leq \sec \phi \; D(H_i) \leq \delta. \qquad (8)$$

Hence $$\inf_{\substack{S_i \subset E^3 \\ D(S_i) \leq \delta \\ \bigcup_{i=1}^{\infty} S_i \supset X|G}} \left(D(S_i)\right)^2 \leq \sum_{i=1}^{\infty} \left(D(K_i)\right)^2 + \sum_{i=1}^{\infty} \left(D(L_i)\right)^2$$

$$\leq 2 \sec^2 \phi \sum_{i=1}^{\infty} \left(D(H_i)\right)^2$$

$$< \varepsilon \qquad \text{by (7).} \qquad (9)$$

But this implies

$\mathcal{M}_2 (X|G) \leq \varepsilon < \frac{1}{2}\mathcal{M}_2(X) < \mathcal{M}_2 (X|G)$ by (4) and (9) which is impossible.

Thus $\mathcal{M}_2 (X^1) > 0$ and the lemma is proved.

Proof of Theorem 13

We prove first that $\lim_{\varepsilon \to 0+} \Psi_1(\varepsilon) \leq 4\pi\lambda$. For each integer n, consider the polyhedron $Q(n)$ defined in theorem 12. Then for all $n \geq N(\varepsilon)$, $Q(n) \varepsilon \mathcal{Z}(\varepsilon)$ and so

$$\Psi_1(\varepsilon) \leq \lim_{n \to \infty} \Phi_1 (Q(n))$$

$$= \lim_{n \to \infty} \left(2.n.2 \sin\frac{\pi}{n} \cdot \lambda\left(1 - \frac{1}{n^4}\right)^{\frac{1}{2}} + \frac{2\lambda}{n^2} \cdot n\right)$$

$$= \lim_{n \to \infty} \left(4\pi\lambda \left(1 - \frac{1}{n^4}\right)^{\frac{1}{2}} \left(\frac{\sin \frac{\pi}{n}}{\frac{\pi}{n}}\right)\right)$$

$$= 4\pi\lambda.$$

Thus $\lim_{\varepsilon \to 0+} \Psi_1(\varepsilon) \leq 4\pi\lambda.$

It remains to show

$$\lim_{\varepsilon \to 0+} \Psi_1(\varepsilon) \geq 4\pi\lambda.$$

Let $\delta > 0$ be given. For each $\varepsilon > 0$ choose a polyhedron $Q(\varepsilon)$ in the class $2(\varepsilon)$ such that

$$\Phi_1(Q(\varepsilon)) \leq \Psi_1(\varepsilon) + \delta. \qquad (10)$$

Let $\{\varepsilon_i\}_{i=1}^{\infty}$ be a sequence tending to zero as i tends to infinity.

Since $\Phi_1(Q(\varepsilon_i)) \leq 4\pi\lambda + \delta$ (11)

and the sequence of polyhedra $\{Q(\varepsilon_i)\}_{i=1}^{\infty}$ is uniformly bounded we may assume by the Blaschke selection theorem that there is a subsequence $\{\varepsilon_{i_j}\}_{j=1}^{\infty}$ of $\{\varepsilon_i\}_{i=1}^{\infty}$ for which $\Phi_1(Q(\varepsilon_{i_j}))$ tends to some number Φ_1 and $Q(\varepsilon_{i_j})$ tends to a convex set Q contained in $D^3[o, \lambda]$ as j tends to infinity. Thus by omitting appropriate terms we may assume to simplify the notation that

$$\Phi_1(Q(\varepsilon_i)) \to \Phi_1 \quad \text{as} \quad i \to \infty \qquad (12)$$

and

$$Q(\varepsilon_i) \to Q \quad \text{as } i \to \infty . \tag{13}$$

Now let $\eta > 0$ be given with $\eta < \lambda$. Then it was shown in theorem 12 that either

(a) $D^3 [o, \lambda - \eta] \subset Q(\varepsilon_i)$ or

(b) $D^3 [o, \lambda - \eta]$ meets the frontier fr $Q(\varepsilon_i)$ of $Q(\varepsilon_i)$ in closed discs, whenever i is so large that $\frac{\varepsilon_i^2}{4} \leq 2\eta\lambda - \eta^2$.

Hence taking the limit as i tends to infinity it follows that either

(a) $D^3 [o, \lambda - \eta] \subset Q$ or

(b) $D^3 [o, \lambda - \eta]$ meets the frontier fr Q of Q in closed discs.

We consider two cases.

Case I

$D^3 [o, \lambda - \eta] \subset Q$ for all η with $0 < \eta < \lambda$

and

Case II

There exists $\eta_0 > 0$ for which $D^3 [o, \lambda - \eta_0]$ meets the fr Q in at least one closed disc of positive radius.

We show first that case I is impossible for, if were true then Q would be the solid sphere $D^3 [o, \lambda]$. But this implies using the same arguments as in the proof of theorem 11 that

$\lim_{i \to \infty} (Q(\varepsilon_i)) = \infty$ which contradicts (11).

Thus case II must apply. We prove next the following lemma.

Lemma 11

For each $\eta > 0$, $D^3 [o, \lambda - \eta]$ meets fr Q in a finite number of discs of positive radius.

Proof

Let η be given with $0 < \eta < \lambda$.

If $i \geq i_0(\eta)$, then $\frac{\varepsilon_i^2}{4} \leq 2\eta\lambda - \eta^2$ and it follows as in the proof of theorem 12 that fr $Q(\varepsilon_i)$ meets $D^3 [o, \lambda - \eta]$ in m_i discs where possibly $m_i = 0$.

Now each facet of $Q(\varepsilon_i)$ which meets $D^3 [o, \lambda - \eta]$ is inscribed in a circle, centre u_i and radius σ_i, where

$$\sigma_i \geq (\lambda - (\lambda - \eta)^2)^{\frac{1}{2}}$$
$$= (2\eta\lambda - \eta^2)^{\frac{1}{2}}.$$

Moreover each such facet contains u_i.

Thus given ζ with $0 < \zeta < (2\eta\lambda - \eta^2)^{\frac{1}{2}}$ it follows by theorem 9, that all facets of $Q(\varepsilon_i)$ which meet $D^3 [o, \lambda - \eta]$ contain a disc of radius $(2\eta\lambda - \eta^2)^{\frac{1}{2}} - \zeta$ whenever $i \geq i_0(\zeta, \eta) \geq i_0(\eta)$.

Since $Q(\varepsilon_i) \subset D^3 [o, \lambda]$ for all i it follows that

$$4\pi\lambda^2 \geq m_i \pi((2\eta\lambda - \eta^2)^{\frac{1}{2}} - \zeta)^2$$

for all $i \geq i_0(\zeta, \eta)$.

The lemma is proved.

Now let $\{\eta_i\}_{i=1}^{\infty}$ be a monotonic sequence with $\eta_i \geq \eta_{i+1}$ and η_i tending to zero as i tends to infinity. By lemma 11 we may assume that $D^3 [o, \lambda-\eta_i]$ meets fr Q in e_i discs for $i = 1, 2 \ldots$ We shall ignore discs of zero radius or points and any disc mentioned will necessarily have positive radius.

Suppose $D^3 [o, \lambda-\eta_1]$ meets fr Q in the discs $D^1_1 \ldots D^1_{e_1}$ which have radii $\lambda^1_1, \ldots \lambda^1_{e_1}$ and centres $o^1_1, \ldots o^1_{e_1}$ respectively. Let $C(o, D^1_k)$ denote the cone subtended by D^1_k at o, the centre of $D^3 [o, \lambda]$ for $k = 1, \ldots e_1$.

Then the frontier fr $C(o, D^1_k)$ of $C(o, D^1_k)$ meets $S^3 [o, \lambda]$ in a circle of radius μ^1_k say where $\mu^1_k \geq \lambda^1_k$ for $k = 1, \ldots e_1$.

We next consider $D^3 [o, \lambda - \eta_2]$.

Now $D^3 [o, \lambda - \eta_2] \supset D^3 [o, \lambda - \eta_1]$ so certainly $D^3 [o, \lambda - \eta_2]$ meets fr Q in discs $D^2_1, \ldots D^2_{e_1}$ of radii $\lambda^2_1, \ldots \lambda^2_{e_1}$ and centres $o^1_1, \ldots o^1_{e_1}$ respectively where $\lambda^2_k \geq \lambda^1_k$ for $k = 1, \ldots e_1$. Let the remainder of the discs (if any) formed by $D^3 [o, \lambda - \eta_2] \cap$ fr Q be $D^2_{e_1+1}, \ldots D^2_{e_2}$ with radii $\lambda^2_{e_1+1}, \ldots \lambda^2_{e_2}$ and centres $o^2_{e_1+1}, \ldots o^2_{e_2}$ respectively.

So summarising we have that $D^3 [o, \lambda-\eta_2]$ meets fr Q in discs $D^2_1, \ldots D^2_{e_2}$ with radii $\lambda^2_1, \ldots \lambda^2_{e_2}$ and centres $o^2_1, \ldots o^2_{e_2}$ respectively where $o^1_k = o^2_k$ and $\lambda^1_k \leq \lambda^2_k$ for $k = 1, 2, \ldots e_1$.

Then as before the frontier fr $C(o, D^2_k)$ of $C(o, D^2_k)$, the cone subtended by D_k^2 at o, meets S^3 [o, λ] in a circle of radius μ_k^2 say where $\mu_k^2 \geq \mu_k^1$ for $k = 1, \ldots e_1$.

By continuing this process inductively we must have that the sequence $\{e_i\}_{i=1}^{\infty}$ is monotonically increasing and for each i, $D^3 [o, \lambda-\eta_i]$ meets fr Q in e_i discs of radius $\lambda_1^i, \ldots \lambda_{e_i}^i$, centres $o_1^i, \ldots o_{e_i}^i$ and the frontier fr C (o, D_k^i) of $C(o, D_k^i)$, the cone subtended by D_k^i at o, meets S^3 [o, λ] in a circle for radius μ_k^i for $k = 1, o. \ldots e_i$.

Lemma 12

If $e = \lim_{i \to \infty} e_i$ and $\mu_k = \lim_{i \to \infty} \mu_k^i$ for each k with $1 \leqslant k \leqslant e$ then

$$2\pi \sum_{k=1}^{e} \mu_k \leqslant \Phi_1.$$

Proof

We now consider fixed $\eta_i \leqslant \eta_0$ so that $D^3 [o, \lambda-\eta_i]$ meets fr Q in at least one disc.

Thus for all j sufficiently large $D^3 [o, \lambda-\eta_i]$ will meet fr $Q(\varepsilon_j)$ in at least one disc.

Let $Q_1^j, \ldots Q_{p_j}^j$ be the p_j polygons in fr $Q(\varepsilon_j)$ which meet $D^3 [o, \lambda-\eta_i]$ in discs.

These polygons are facets of $Q(\varepsilon_j)$.

Let $\alpha(j)$ denote the sum of lengths of the edges of these polygons, $Q_1^j, \ldots Q_{p_j}^j$.

Let $\beta(j, k)$ be the length of the perimeter of Q_k^j for $k = 1, \ldots p_j$ and write

$$\beta(j) = \sum_{k=1}^{p_j} \beta(j, k). \qquad (14)$$

Now $Q^j_{k_1}$ and $Q^j_{k_2}$ have at most one side in common for $k_1 \neq k_2$ and thus

$$\beta(j) \leq \alpha(j) + (p_j^{\ p_j})\varepsilon_j. \qquad (15)$$

Let $\gamma(j)$ be the sum of the circumferences of the discs formed by the intersection of $D^3 [o, \lambda-\eta_i]$ and fr $Q(\varepsilon_j)$. Let γ denote the corresponding sum for $D^3 [o, \lambda-\eta_i] \cap$ fr Q.

Since $Q(\varepsilon_j)$ tends to Q as j tends to infinity it follows that $D^3 [o, \lambda-\eta_i] \cap Q(\varepsilon_j)$ tends to $D^3 [o, \lambda-\eta_i] \cap Q$ as j tends to infinity.

Thus

$$\gamma = \lim_{j \to \infty} \gamma_j. \qquad (16)$$

Now if j is sufficiently large then

$$\gamma(j) \leq \beta(j) \qquad (17)$$

and so by (15)

$$\gamma(j) \leq \alpha(j) + (p_j^{\ p_j})\varepsilon j. \qquad (18)$$

Now by lemma 11, the numbers p_j are bounded for all j and thus letting j tend to infinity we have

$$\Upsilon \leq \lim_{j \to \infty} \phi_1(Q(\varepsilon_j)) = \phi_1 \text{ by (16) and (18) since obviously}$$

$\alpha(j) \leq \phi_1(Q(\varepsilon_j))$ for $j = 1, 2 \ldots$.

Since we were considering fixed i, Υ is really a function of i so we now write $\Upsilon = \Upsilon_i$.

Thus we have proved

$$\Upsilon_i = 2\pi \sum_{k=1}^{e_i} \lambda_k^i \leq \phi_1$$

and so $2\pi \sum_{k=1}^{e_i} \mu_k^i \dfrac{(\lambda - \eta_i)}{\lambda} \leq \phi_1$ for $i = 1, 2 \ldots$.

It follows that in the limit as i tends to infinity that

$$2\pi \sum_{k}^{e} \mu_k \leq \phi_1 \qquad (19)$$

and the lemma is proved.

Now for each $\eta_i > 0$, $D^3[o, \lambda - \eta_i]$ meets fr Q in e_i disjoint discs of radii $\lambda_1^i, \ldots \lambda_{e_i}^i$ and so fr $(D^3[o, \lambda-\eta_i] \cap Q)$ consists of these discs together with an open subset $W(\eta_i)$ of $S^3[o, \lambda-\eta_i]$. Project $W(\eta_i)$ on to $S^3[o, \lambda\ \ ]$ from o and let the set obtained be denoted by $V(\eta_i)$.

Hence for each integer i,

$$4\pi\lambda^2 = \frac{\pi}{4}\mathcal{M}_2\,(V(\eta_i)\,) + 2\pi \sum_{k=1}^{e_i} \lambda(\lambda - (\lambda - (\mu_k^i)^2)^{\frac{1}{2}}). \qquad (20)\ \dagger$$

Now for each i, $V(\eta_i)$ is $\mathcal{M}_2$-measurable and

$$V(\eta_i) \supset V(\eta_{i+1})$$

and so

$$\mathcal{M}_2(\bigcap_{i=1}^{\infty} V(\eta_i)\,) = \lim_{i\to\infty}\mathcal{M}_2(V(\eta_i)\,). \qquad (21)$$

Thus taking the limit as i tends to infinity in (20) and applying (21) it follows that

$$4\pi\lambda^2 = \frac{\pi}{4}\mathcal{M}_2(\bigcap_{i=1}^{\infty} V(\eta_i)\,) + 2\pi \sum_{k=1}^{e} \lambda(\lambda - (\lambda^2 - \mu_k^2)^{\frac{1}{2}}).$$

we suppose first that $\mathcal{M}_2(\bigcap_{i=1}^{\infty} V(\eta_i)\,) > 0$.

Let $Q(\varepsilon_j)$ have vertices $v_1^j, \ldots v_{n_j}^j$. We show next that

$\sup\limits_{w\in \bigcap_{i=1}^{\infty} V(\eta_i)} (\min\limits_{1\le k\le n_j} |w-v_k^j|)$ tends to zero as j tends to infinity.

For if this is not the case then there exists $\zeta > 0$ and $w_j \varepsilon \bigcap_{i=1}^{\infty} V(\eta_i)$ such that

† See note at the end of the theorem.

$$\min_{1 \leq k \leq n_j} |w_j - v_k^j| \geq \zeta \quad \text{for } j = 1, 2 \ldots, \tag{22}$$

and we may assume that w_j tends to a point w as j tends to infinity. Thus, a similar argument to that used in theorem 11 implies that $\rho(w_j, Q(\varepsilon_j))$ is bounded away from 0 as j tends to infinity and thus

$$\rho(w, Q) > 0. \tag{23}$$

This implies that $\rho(w, V(\eta_i)) > 0$ for all i sufficiently large and thus since $V(\eta_i) \supset V(\eta_{i+1})$ it follows that $\rho(w, \bigcap_{i=1}^{\infty} V(\eta_i)) > 0$.
Thus for all j sufficiently large

$$\rho(w_j, \bigcap_{i=1}^{\infty} V(\eta_i)) > 0$$

which is impossible by definition of w_j.

Thus $\sup_{w \in \bigcap_{i=1}^{\infty} V(\eta_i)} (\min_{1 \leq k \leq n_j} |w - v_k^j|)$ tends to zero as i tends to infinity.

We now project E^3 by orthogonal projection on to a plane R. By lemma 10, the set $\bigcap_{i=1}^{\infty} V(\eta_i)$ projects down into a set of positive $\mathcal{M}_2$-measure and thus positive Λ_2-measure.
Thus using again the argument of theorem 11 it follows that $\Phi_1(Q(\varepsilon_i))$ tends to infinity as i tends to infinity. But again by (11) this is impossible.

Thus we must have

$$\mathcal{M}_2\left(\bigcap_{i=1}^{\infty} V(\eta_i) \right) = 0.$$

Hence $4\pi\lambda^2 = 2\pi \sum_{k=1}^{e} \lambda(\lambda - (\lambda^2 - \mu_k^2)^{\frac{1}{2}})$

and so by lemma 9 it follows that

$$4\pi\lambda \leq 2\pi \sum_{k=1}^{e} \mu_k.$$

Thus by lemma 12

$$4\pi\lambda \leq \Phi_1 = \lim_{i \to \infty} \Phi_1(Q(\varepsilon_i)) \text{ by (12).}$$

and so by (10)

$$4\pi\lambda \leq \lim_{i \to \infty} \Psi(\varepsilon_i) + \delta$$

where δ was arbitrary.

Hence $4\pi\lambda \leq \lim_{i \to \infty} \Psi(\varepsilon_i)$

and the theorem is proved.

Note

The reader will see that we have used the fact that the area of the curved surface of a cap is equal to $\frac{4}{\pi}$ times its Hausdorff 2-dimensional measure. This follows from a result in [8] page 54.

We note that for small ε the lower bounds of the total edge-lengths, surface areas and volumes of polyhedra taken over the class $\mathcal{L}(\varepsilon)$ defined in theorem 12 are considerably lower than the corresponding quantities taken over the class $\mathcal{P}(\varepsilon)$ defined in theorem 9.

Further theorems 12 and 13 show that the lower bounds for the total edge-lengths and surface areas of polyhedra taken over the whole class of polyhedra inscribed in a sphere and containing the centre given in theorem 2 are not best possible if we restrict ourselves to polyhedra in the class $\mathcal{L}(\varepsilon)$ for small ε where of course we are insisting that the edge-lengths are small. We note however that the lower bounds of the volumes of polyhedra taken over these two classes are both equal to zero.

We finally complete the three dimensional case and consider polyhedra which are inscribed in a sphere containing the centre which have facets of small area but which have no restrictions placed on their edge-lengths.

Theorem 11

Let $R(\varepsilon)$ be a 3-dimensional polyhedron inscribed in the sphere $S^3\,[o, \lambda]$ in E^3 and containing the centre o with the property that each facet of $R(\varepsilon)$ is of area less than ε. Let $\mathcal{R}(\varepsilon)$ denote the class of all such polyhedra $R(\varepsilon)$ and define

$$\chi_i(\varepsilon) = \inf_{R(\varepsilon)\in\mathcal{R}(\varepsilon)} \phi_i(R(\varepsilon)). \qquad i = 1, 2 \text{ and } 3.$$

Then (i) $\lim_{\varepsilon \to 0+} \chi_1(\varepsilon) = 6\lambda$

(ii) $\lim_{\varepsilon \to 0+} \chi_i(\varepsilon) = 0 \qquad i = 2 \text{ and } 3.$

Proof

By theorem 2 it follows that

$$\lim_{\varepsilon \to 0+} \chi_1(\varepsilon) \leqslant 6\lambda \tag{1}$$

and

$$\lim_{\varepsilon \to 0+} \chi_i(\varepsilon) = 0 \qquad \text{for } i = 2 \text{ and } 3. \tag{2}$$

It follows by theorem 4 that

$$\phi_1(R(\varepsilon)) > 6\lambda \tag{3}$$

for all polyhedra $R(\varepsilon) \in \mathcal{R}(\varepsilon)$,

and so

$$\lim_{\varepsilon \to 0+} \chi_i(\varepsilon) \geqslant 6\lambda.$$

The theorem is proved.

We now see that for small ε the lower bounds of the total edge-lengths and surface areas of polyhedra taken over the class

$\mathcal{R}(\varepsilon)$ are in fact lower than the corresponding quantities taken over the class $\mathcal{Q}(\varepsilon)$ and thus still lower than the corresponding quantities taken over the class $\mathcal{P}(\varepsilon)$. We note however that the lower bounds of the volumes of polyhedra taken over the classes $\mathcal{L}(\varepsilon)$ and $\mathcal{R}(\varepsilon)$ are both equal to zero for any positive number ε.

Also theorem 14 shows us that the lower bounds for the total edge-lengths, surface areas and volumes of polyhedra taken over the whole class of polyhedra inscribed in a sphere, containing the centre, cannot be improved by restricting ourselves to the class $\mathcal{R}(\varepsilon)$ for any positive number ε.

It is clear that we have now fully investigated the three dimensional case. However, there are of course many open problems in higher dimensions associated with this chapter.

CHAPTER 3

INTRODUCTION

In this chapter we shall give three theorems concerning the behaviour of the 'higher' dimensional polytopes or cells of arbitrary convergent sequences of cell-complexes. The ideas comes from a study of a paper by Eggleston, Grünbaum and Klee [9] . In theorem 3.1 in [9] they essentially give a condition on a cell-complex K to ensure that if a sequence $\{K_i\}_{i=1}^{\infty}$, convergent to K (in the usual Hausdorff metric), is such that the number of vertices of K_i is uniformly bounded as i tends to infinity, then for each integer s and each $\varepsilon > 0$, the cells of K of dimension s are contained in the ε-neighbourhoods of the cells of K_i of dimension s, whenever i is sufficiently large.

In theorem 15 we shall prove a corresponding result for arbitrary cell-complexes, concerning the cells of dimension greater than or equal to s. We shall also show that our theorem is false if the number of vertices of K_i is not uniformly bounded as i tends to infinity.

Theorem 15

In E^n let $\{K_i\}_{i=1}^{\infty}$ be a sequence of cell-complexes tending to the cell-complex K as i tends to infinity with $f_o(K_i) < \infty$. Then $\liminf \bigcup_{t=s}^{n} \sigma_t(K_i) \supset \bigcup_{t=s}^{n} \sigma_t(K)$ for each s with $0 \leq s \leq n$.

Moreover if $s \neq 0$ the condition $f_o(K_i) < \infty$ is necessary.

Lemma 13

If K_i is a sequence of compact sets and K_i tends to sets L and M in the Hausdorff metric then $L = M$ provided L and M are both compact.

Proof

The proof is well-known and is omitted.

Proof of Theorem 15

Suppose that the result is false. Then there is a subsequence $\{L_i\}_{i=1}^{\infty}$ of $\{K_i\}_{i=1}^{\infty}$ such that for some $\varepsilon > 0$, and for some s with $0 \leq s \leq n$

$$\bigcup_{t=s}^{n} \sigma_t(K) \not\subset \left[\bigcup_{t=s}^{n} \sigma_t(L_i), \varepsilon\right].$$

we shall assume $\bigcup_{t=s}^{n} \sigma_t(K) \neq \phi$ for otherwise the result is trivial.

Since $f_o(K_i) < \infty$ we may assume, extracting suitable subsequence of $\{L_i\}_{i=1}^{\infty}$ if necessary, that the following conditions are satisfied.

i) There are non-negative integers $m_0, \ldots m_n$ such that $f_t(L_i) = m_t$ for each i with $1 \leqslant i < \infty$ and for each t with $0 \leqslant t \leqslant n$.

ii) For each t with $0 \leqslant t \leqslant n$ and for each i with $1 \leqslant i < \infty, L_i$ has exactly m_t t-faces $P_i^{t,1}, P_i^{t,2}, \ldots P_i^{t,m_t}$ where $P_i^{t,h}$, for each h with $1 \leqslant h \leqslant m_t$, is convergent as i tends to infinity to a compact convex set $P^{t,h}$ of dimension less than or equal to t.

Since $\bigcup_{t=s}^{n} \sigma_t(K)$ is a finite union of cells we may further assume that $Q^t \not\subset \left[\bigcup_{t=s}^{n} \sigma_t(L_i), \varepsilon\right]$ for some t-dimensional cell Q^t of K with $s \leqslant t \leqslant n$ and thus that there is a sequence $\{y_i\}_{i=1}^{\infty} \subset Q^t$, convergent to a point y in Q^t such that

$$y_i \notin \left[\bigcup_{t=s}^{n} \sigma_t(L_i), \varepsilon\right]. \tag{1}$$

Now suppose if possible that

$$y \in \bigcup_{t=s}^{n} \qquad \bigcup_{h=1}^{m_t} P^{t,h} \tag{2}$$

This implies that if $i \geqslant i_0(\varepsilon)$ then

$$\rho(y, y_i) < \frac{\varepsilon}{4}, \tag{3}$$

and $$\rho(y, x_i) < \frac{\varepsilon}{4} \text{ for some } x_i \in \bigcup_{t=s}^{n} \sigma_t(L_i). \tag{4}$$

Then (3) and (4) imply

$$\rho(y_i, x_i) < \frac{\varepsilon}{2} \text{ whenever } i \geq i_0(\varepsilon).$$

This is impossible by (1) and so

$$\rho(y, \bigcup_{t=s}^{n} \bigcup_{h=1}^{m_t} P^{t,h}) > 0. \tag{5}$$

Thus there is a t-dimensional ball $B^t \subset Q^t$ with $s \leq t \leq n$ such that

$$B^t \cap \left(\bigcup_{t=s}^{n} \bigcup_{h=1}^{m_t} P^{t,h} \right) = \phi. \tag{6}$$

By lemma 13, $\bigcup_{t=0}^{n} \bigcup_{h=1}^{m_t} P^{t,h} = K$ and so (6) implies that $B^t \subset \bigcup_{t=0}^{s-1} \bigcup_{h=1}^{n} P^{t,h}$ which is a finite union of convex sets each of which has dimension less than or equal to s-1. This is impossible since $s \leq t \leq n$ and so there is a contradiction and the theorem is proved.

We show finally that if $s \neq 0$ then the condition $f_o(K_i) < \infty$ is necessary.

Let P be an s-dimensional cell of K with $s \neq 0$. Let Q be a cell of maximal dimension that contains P.

Then $Q \subset \bigcup_{t=s}^{n} \sigma_t(K)$. Also no cell of K can meet rel int Q for suppose that R is one such cell. Then $R \cap Q$ is a face of Q and so $R \cap Q = Q$. Thus $Q \subset R$ and this implies that R has dimension greater than that of Q. This is contrary to the definition of Q and so Q has the stated property.

Now for each integer i divide Q into p_i disjoint sub-cells $Q_1^i, \ldots Q_{p_i}^i$ each of which has dimension that of Q and diameter less than $\frac{1}{i}$. Take a point $x_r^i \varepsilon$ rel int Q_r^i for each r with $1 \leq r \leq p_i$. Let S_i denote the set union of $\{x_r^i\}_{r=1}^{p_i}$ for $i = 1, 2, \ldots$ Let K_i be the set union of S_i and all the cells of K apart from Q. Then K_i is a cell-complex for $i = 1, 2 \ldots$

Moreover $K_i \to K$ as $i \to \infty$ but $\liminf \bigcup_{t=s}^{n} \sigma_t(K_i) \not\supset \bigcup_{t=s}^{n} \sigma_t(K)$.

The proof of the theorem is complete.

We next give a condition on a cell-complex K to ensure that if $\{K_i\}_{i=1}^{\infty}$ is any sequence of cell-complexes convergent to K, then for given integer s the union of the cells of K_i of dimension greater than or equal to s tends to the union of the cells of K of dimension greater than or equal to s as i tends to infinity.

Theorem 16

In E^n, let s be a given integer with $0 \leq s \leq n$. Let $\mathcal{A}(K)$ denote the class of sequences of cell-complexes $\{K_i\}_{i=1}^{\infty}$ which satisfy the following properties;

(a) $K_i \to K$ as $i \to \infty$ and

(b) $f_o(K_i) < \infty$ for each $\{K_i\}_{i=1}^{\infty} \in \mathcal{A}(K)$.

Then $\bigcup_{t=s}^{n} \sigma_t(K_i) \to \bigcup_{t=s}^{n} \sigma_t(K)$ as $i \to \infty$ for each $\{K_i\}_{i=1}^{\infty} \in \mathcal{A}(K)$ if and only if every t-dimensional cell of K with $0 \leqslant t \leqslant s$ is contained in a cell of dimension s.

Proof

Suppose first that every t-dimensional cell of K with $0 \leqslant t \leqslant s$ is contained in a cell of dimension s. In view of theorem 15 we have $\liminf \bigcup_{t=s}^{n} \sigma_t(K_i) \supset \bigcup_{t=s}^{n} \sigma_t(K)$ for all sequences $\{K_i\}_{i=1}^{\infty} \in \mathcal{A}(K)$. Thus, if the result is false, then for some sequence $\{K_i\}_{i=1}^{\infty} \in \mathcal{A}(K)$ there exists $\varepsilon > 0$ and a subsequence $\{L_i\}_{i=1}^{\infty}$ of $\{K_i\}_{i=1}^{\infty}$ for which

$$\bigcup_{t=s}^{n} \sigma_t(L_i) \not\subset \left[\bigcup_{t=s}^{n} \sigma_t(K), \varepsilon\right] \quad \text{for } i = 1, 2 \ldots \quad (1)$$

We may assume that $\{L_i\}_{i=1}^{\infty}$ satisfies conditions (1) and (11) of theorem 15 and thus in the same notation as theorem 15 there exists t with $s \leqslant t \leqslant n$ and h with $1 \leqslant h \leqslant m_t$ for which

$$P_i^{t,h} \not\subset \left[\bigcup_{t=s}^{n} \sigma_t(K), \varepsilon\right].$$

Thus there is a point $y_i \in P_i^{t,h} \Big| [\bigcup_{t=s}^{n} \sigma_t(K), \varepsilon]$ (3)

for $i = 1, 2, \ldots$

and since, by extracting a suitable subsequence, we may assume that y_i tends to a point y as i tends to infinity it follows that then

$$y \in P^{t,h} \Big| [\bigcup_{t=s}^{n} \sigma_t(K), \frac{\varepsilon}{2}]. \qquad (4)$$

Hence $$y \notin \bigcup_{t=s}^{n} \sigma_t(K). \qquad (5)$$

But $y \in P^{t,h} \subset K$ and so since every t-dimensional cell with $0 \leq t \leq s-1$ is contained in a cell of dimension s it follows that

$$y \in \bigcup_{t=s}^{n} \sigma_t(K). \qquad (6)$$

But (5) and (6) are contradictory and so that theorem is proved in the one direction.

Suppose next that a t-dimensional cell P of K with $0 \leq t < s$ is not contained in a cell of dimension s. Then certainly P is not contained in a cell of dimension greater than s.

If P has dimension 0, then for each i with $1 \leq i < \infty$ there is an s-dimensional cube C_i of diameter $\frac{1}{i}$ which contains P and which does not meet any other cell of K for all i sufficiently large. Let K_i be the set union of C_i and each other cell of K apart from P.

Then for all i sufficiently large K_i is a cell-complex. Also $\{K_i\}_{i=1}^{\infty} \in \mathcal{A}(K)$, but $\bigcup_{t=s}^{n} K_i \not\to \bigcup_{t=s}^{n} K_i$ as $i \to \infty$.

We may assume that P is a maximal in that P is not contained in a cell of dimension greater than t. Then as in the proof of theorem 15 it follows that no other cells of K meet rel int P. Choose $x \in$ rel int P. We assume now P has dimension greater than 0.

Let $e_1, e_2, \ldots e_s$ be a set of pairwise mutually orthogonal lines which meet at x with $e_1, \ldots e_t$ in the affine hull of P. For each integer i with $1 \leq i < \infty$ and each j with $t+1 \leq j \leq s$, choose a point y_i^j on e_j which is distant $\frac{1}{i}$ from x. Then the polytope $P_i = \mathrm{conv}\,(P, y_i^{t+1}, y_i^{t+2}, \ldots y_i^s)$ is s-dimensional and $P_i \to P$ as $i \to \infty$. Also for all i sufficiently large no cell of K meets P_i in its relative interior. Let K_i denote the set union of P_i and the cells of K for $i = 1, 2$. Then for all i sufficiently large K_i is a cell-complex. Also $\{K_i\}_{i=1}^{\infty} \in \mathcal{A}(K)$, but $\bigcup_{t=s}^{n} \sigma_t(K_i) \not\to \bigcup_{t=s}^{n} \sigma_t(K)$ as $i \to \infty$. Thus the theorem is proved.

We next state a lemma which will be used in the following theorem.

<u>Lemma 14</u>

If an s-dimensional compact convex set S is the limit of a sequence of polytopes $\{P_i\}_{i=1}^{\infty}$ then $\{\sigma_s(P_i)\}_{i=1}^{\infty}$ is also convergent to S.

Proof

The lemma is stated and proved in the paper 'Some semicontinuity theorems for convex polytopes and cell complexes' [9] under the assumption that S is a convex polytope. This is however not required in the proof and so the lemma is proved.

Finally we finish this chapter by giving a theorem which tells us that if each cell of a cell-complex K has dimension less than or equal to m for some integer m, and $\{K_i\}_{i=1}^{\infty}$ is a sequence of cell-complexes convergent to K, then the union of the cells of K_i of dimension less than or equal to m also tends to K as i tends to infinity. This enables us to sharpen theorems 15 and 16.

Theorem 17

In E^n let $\{K_i\}_{i=1}^{\infty}$ be a sequence of cell-complexes tending to a cell-complex K as i tends to infinity. Suppose that each cell of K has dimension less than or equal to m for some integer m with $m \leq n$. Then $\bigcup_{t=0}^{m} \sigma_t(K_i) \to K$ as $i \to \infty$.

Proof

Now since $K_i \to K$ as $i \to \infty$ it follows that for each $\varepsilon > 0$

there exists $i_0(\varepsilon) > 0$ such that

$$\bigcup_{t=0}^{m} \sigma_t(K_i) \subset K_i \subset [K,\varepsilon] \text{ whenever } i \geq i_0(\varepsilon).$$

Thus if the result is false then there exists $\varepsilon > 0$ and a subsequence $\{L_i\}_{i=1}^{\infty}$ of $\{K_i\}_{i=1}^{\infty}$ for which

$$K \not\subset [\bigcup_{t=0}^{m} \sigma_t(L_i), \varepsilon]. \tag{1}$$

For each integer i with $1 \leq i < \infty$ choose a point y_i with

$$y_i \in K \,\Big|\, [\bigcup_{t=0}^{m} \sigma_t(L_i), \varepsilon]. \tag{2}$$

By extracting suitable subsequences if necessary we may assume that there is a point y and compact sets H and L such that

$$y_i \to y, \tag{3}$$

$$\bigcup_{t=0}^{m} \sigma_t(L_i) \to H, \tag{4}$$

and

$$\bigcup_{t=m+1}^{n} \sigma_t(L_i) \to L \quad \text{as } i \to \infty. \tag{5}$$

Now by lemma 13 it follows that $K = H \cup L$ and by (2)
$y \in K \,\Big|\, [H, \frac{\varepsilon}{2}]$.

Thus $y \in L$ and $P(H, y) \geq \frac{\varepsilon}{2} > 0.$ (6)

Let $\{\varepsilon_i\}_{i=1}^{\infty}$ be a sequence tending to zero as i tends to infinity such that

$$[L_i, \varepsilon_i] \supset K \quad \text{for } i = 1, 2, \quad . \tag{7}$$

Since $\varepsilon_i < \varepsilon$ for all $i \geq i_0(\varepsilon)$ it follows by (2) that

$$y_i \in K \Big| \Big[\bigcup_{t=0}^{m} \sigma_t(L_i), \varepsilon_i \Big]. \tag{8}$$

Thus by (7)

$y_i \in \Big[\bigcup_{t=m+1}^{n} \sigma_t(L_i), \varepsilon_i \Big]$ for $i \geq i_0(\varepsilon)$. Hence, by extracting a suitable subsequence if necessary, we may assume that there is a cell P_i of $\bigcup_{t=m+1}^{n} \sigma_t(L_i)$ which is of fixed dimension r with $m+1 \leq r \leq n$, and which is such that

$$y_i \in [P_i, \varepsilon_i] \quad \text{for } i \geq i_0(\varepsilon). \tag{9}$$

we may assume further that the sequence $\{P_i\}_{i=1}^{\infty}$ is convergent to a compact convex set P in K which thus has dimension at most m. But then by lemma 14 it follows that

$$\sigma_m(P_i) \to P \quad \text{as} \quad i \to \infty.$$

Also (9) implies that $y \in P$ and so if i is sufficiently large then

$$y \in [\sigma_m(P_i), \frac{\varepsilon}{4}] \subset [\bigcup_{t=0}^{m} \sigma_t(L_i), \frac{\varepsilon}{4}].$$

Thus in the limit as i tends to infinity we have

$$y \in [H, \frac{\varepsilon}{4}].$$

This is contrary to (6) and so there is a contradiction. The theorem is proved.

Corollary

If, in the notation of theorems 15 and 16, each cell of K of dimension less than or equal to m where $0 \leq m \leq n$, then we may replace $\bigcup_{t=s}^{n} \sigma_t(K_i)$ by $\bigcup_{t=s}^{m} \sigma_t(K_i)$ in the statement of each theorem.

Proof

In order to prove theorem 15 we assume the result is false and then simply replace

$$\bigcup_{t=s}^{n} \sigma_t(K_i) \text{ by } \bigcup_{t=s}^{m} \sigma_t(K_i) \text{ throughout the proof.}$$

Using the fact that $\bigcup_{t=0}^{m} \sigma_t(K_i) \to K$ as $i \to \infty$ we obtain a contradiction by similar arguments as before.

In order to prove theorem 16 we note that by lemma 16, if $\varepsilon > 0$, then there exists $i_0(\varepsilon) > 0$ such that for all $i \geq i_0(\varepsilon)$

$$\left[\bigcup_{t=s}^{m}\sigma_t(K),\varepsilon\right]\supset\bigcup_{t=s}^{n}\sigma_t(K_i)\supset\bigcup_{t=s}^{m}\sigma_t(K_i),$$

the second inclusion being obvious since $m \leqslant n$.

The corollary then is proved.

CHAPTER 4

INTRODUCTION

The main purpose of this chapter is to give some characterisations of those sets which are completely I-stretched or completely K-stretched. We shall see in theorem 18, corollary 2 and theorem 21, corollary 2 that any such sets are necessarily compact and convex. By the nature of their definition we might expect them to possess properties analogous to the properties of the sets of constant width and indeed this turns out to be the case.

In [12] Fujiwara and Kakeya give an analytic characterisation of the rotors for an equilateral triangle. Later on in this chapter we shall be able to give a geometric characterisation of such sets.

We shall of course be working throughout in the plane. We start with a lemma which is fundamental.

Lemma 15

Let $x\,y\,z$ be a triangle. On $y\,z$ erect the triangle $u_1\,y\,z$ which is equilateral and such that u_1 lies on the side of $y\,z$ opposite to x. Similarly define u_2 and u_3 for the sides $x\,z$ and $x\,y$ respectively.

(a) If each angle of triangle $x\,y\,z$ is less than $\frac{2}{3}\pi$, let w be the unique point such that $x\hat{w}y = y\hat{w}z = x\hat{w}z = \frac{2}{3}\pi$.

Then (i) each connected set containing x, y and z has length greater than or equal to $|x-w| + |y-w| + |z-w|$;

(ii) the lines $u_1 x$, $u_2 y$ and $u_3 z$ contain wx, wy and wz respectively. Moreover

$$|x-w| + |y-w| + |z-w| = |u_1-x| = |u_2-y| = |u_3-z|.$$

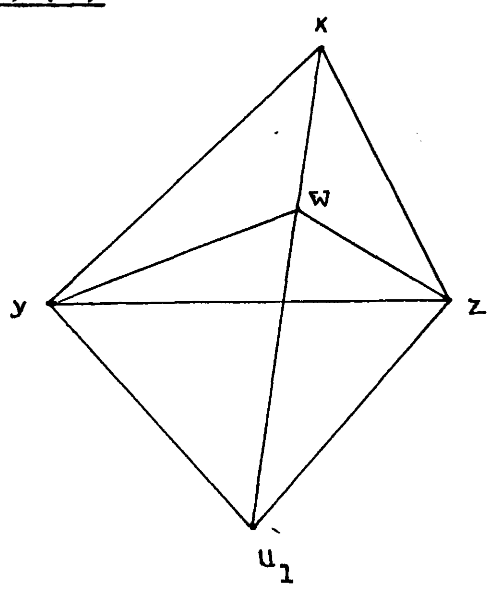
(b) If one angle of triangle x y z, say $y \hat{x} z$ is greater than or equal to $\frac{2}{3}\pi$, let $t = x$ if $y \hat{x} z = \frac{2}{3}\pi$, and let t be the unique point which lies on the same side of the line yz as x and such that $y\hat{t}x = z\hat{t}x = \frac{\pi}{3}$, if $y\hat{x}z > \frac{2}{3}\pi$.

Then (i) each connected set containing x, y and z has length greater than or equal to $|y-x| + |z-x|$;

(ii) the lines u_1x, u_2y and u_3z contain tx, ty and tz respectively. Moreover

$$- |x-t| + |y-t| + |z-t| = |u_1-x| = |u_2-y| = |u_3-z|.$$

Proof (a), (i), (ii)



(a) parts (i) and (ii) are proved in a lemma by H.G.Eggleston in the paper 'on the projection of a plane set of finite linear measure' [10] on page 63.

we consider

(b)(i).

It is clear using the same argument as that given for the proof of (a), (i) that any connected set containing x, y and z has length greater than or equal to the minimum of the function $|x-s| + |y-s| + |z-s|$ taken over all points $s \varepsilon E^2$.

we may assume that this minimum is attained at some point $s = s_0$.

Moreover it was shown in the proof of (a), (i) that either s_0 is a vertex of triangle $x\ y\ z$ or $y\hat{s}_0 x = z\hat{s}_0 x = z\hat{s}_0 y = \frac{2\pi}{3}$.

Now since $y\hat{x}z$ is greater than or equal to $\frac{2\pi}{3}$ the latter case is impossible.

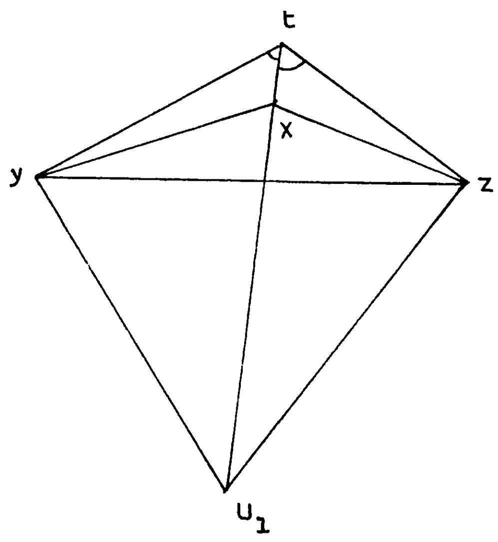
Hence $|x-s_0| + |y-s_0| + |z-s_0|$ is equal to

$$\min(|x-y| + |x-z|,\ |y-z| + |y-x|,\ |z-x| + |z-y|)$$

$$= |x-y| + |x-z|.$$

Thus (b), (i) is proved. we now consider (b), (ii) and show first that the line u_1x contains tx and

$$|u_1-x| = -|x-t| + |y-t| + |z-t|.$$



In order to prove that the line u_1x contains tx it is sufficient, by the definition of t, to prove that $y\hat{t}u_1 = z\hat{t}u_1$.

Since $z\hat{t}y + y\hat{u}_1z = \pi$, yu_1zt is a cyclic quadrilateral whence

$$y\hat{t}u_1 = y\hat{z}u_1 = \frac{\pi}{3} \quad \text{and}$$

$$z\hat{t}u_1 = z\hat{y}u_1 = \frac{\pi}{3} .$$

Thus $y\hat{t}u_1 = z\hat{t}u_1$ and the line u_1x contains tx.

But now from case (a) applied to triangle tyz we have (with $t = w$)

$$|u_1 - t| = |y-t| + |z-t|$$

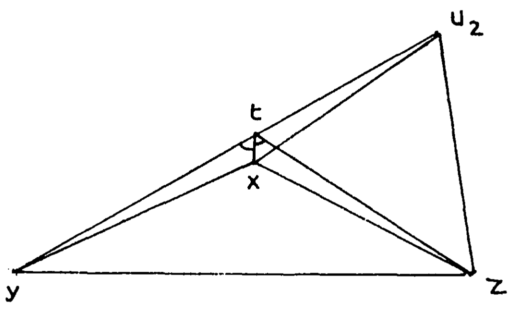
and so

$$|u_1 - x| = -|x-t| + |y-t| + |z-t| .$$

In order to complete the proof of the lemma we show finally that the line u_2y contains ty and

$$|u_2 - y| = -|x-t| + |y-t| + |z-t| .$$

(The argument with 'y' replaced by 'z' is similar and is omitted.).



In order to prove the line u_2y contains ty it is sufficient to prove that $u_2\hat{t}z = \frac{\pi}{3}$.

Since $x\hat{t}z = x\hat{u}_2z = \frac{\pi}{3}$, xtu_2z is a cyclic quadrilateral whence

$$u_2\hat{t}z = u_2\hat{x}z = \frac{\pi}{3}.$$

Thus u_2y contains ty.

But now we apply case (a) to triangle txu_2 and so (since $t = w$)

$$|x-t| + |u_2-t| = |z-t|.$$

Thus

$$|u_2-y| = -|x-t| + |y-t| + |z-t|$$

and the lemma is proved.

<u>Note</u>

In the notation of lemma 15 we now see that if each angle of triangle $x\,y\,z$ is less than $\frac{2}{3}\pi$, then $C(x, y, z)$ is the union of the segments wx, wy and wz where w is the centre of connection of triangle x y z and so

$$I(x, y, z) = |x-w| + |y-w| + |z-w|$$
$$= |u_1-x|$$
$$= |u_2-y|$$
$$= |u_3-z| .$$

Also if the angle $y\hat{x}z$ say of triangle x y z is greater than or equal to $\frac{2}{3}\pi$ then C(x, y, z) is the union of the segments xy and xz and

$$I(x, y, z) = |y-x| + |z-x|,$$

where x = w is the centre of connection of triangle x y z.

We note further that if each angle of triangle x y z is less than or equal to $\frac{2}{3}\pi$, the sets C(x, y, z) and D(x, y, z) coincide, the centre of connection of triangle x y z is equal to the centre of revolution of triangle x y z and

$$I(x, y, z) = K(x, y, z)$$
$$= |u_1-x|$$
$$= |u_2-y|$$
$$= |u_3-z|.$$

Also in the notation of lemma 15 we see that for any triangle x y z,

$$K(x, y, z) = |u_1 - x| = |u_2 - y| = |u_3 - z|.$$

We next state two corollarys to lemma 15.

<u>Corollary 1</u>

Let X be a compact set. Suppose $I(X) = I(x, y, z)$ for three points x, y and z in X. Then x, y and z are collinear if and only if int (conv X) $= \phi$.

<u>Proof</u>

If int (conv X) $= \phi$ then X is a subset of a line and then trivially x, y and z are collinear.

On the other hand if $I(X) = I(x, y, z)$ where x, y and z are collinear then

$$I(X) = \max (|x-y|, |x-z|, |y-z|) = |y-z| \text{ say.}$$

Now if int (conv X) $\neq \phi$ then $y \neq z$ and there exists a point $x^1 \varepsilon X$ which does not lie on the line yz. But then it is a direct consequence of lemma 15 that

$$I(X) = I(x, y, z) = |y-z| < I(x^1, y, z) \leq I(X)$$

which is impossible.

The corollary is proved.

Corollary 2

The set functions I and K are continuous on the class of compact sets in the plane.

Proof

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of compact sets convergent to a compact set Y.

By compactness arguments we may assume there exist sequences of points $\{x_i\}_{i=1}^{\infty}$, $\{y_i\}_{i=1}^{\infty}$ and $\{z_i\}_{i=1}^{\infty}$ which are convergent to x, y and z in Y respectively such that

$$I(X_i) = I(x_i, y_i, z_i) \qquad \text{for } i = 1, 2 \ldots \quad . \tag{1}$$

Then clearly by definition of I (x, y, z)

$$\begin{aligned} \lim_{i\to\infty} I(X_i) &= \lim_{i\to\infty} I(x_i, y_i, z_i) \\ &= I(x, y, z) \\ &\leq I(Y) \ . \end{aligned} \tag{2}$$

Conversely we choose points x, y and z in Y for which

$$I(Y) = I(x, y, z) \ . \tag{3}$$

Then there exist points x_i, y_i and z_i contained in X_i such that

$$x_i \to x, \quad y_i \to y, \quad \text{and } z_i \to z \quad \text{as} \quad i \to \infty \ .$$

Hence
$$\begin{aligned} \lim_{i\to\infty} I(X_i) &\geq \lim_{i\to\infty} I(x_i, y_i, z_i) \\ &= I\,(x, y, z) \\ &= I\,(Y). \end{aligned} \tag{4}$$

Equations (2) and (4) imply

$$\lim_{i \to \infty} I(X_i) = I(Y)$$

and the corollary is proved since we apply the same arguments to prove

$$\lim_{i \to \infty} K(X_i) = K(Y).$$

We now prove a lemma which will be useful in the sequel.

<u>Lemma 16</u>

(i) Of all equilateral triangles circumscribing a segment yz there is a maximal one of height $|y-z|$ which is unique up to reflection in the perpendicular bisector of yz.

Let $x\,y\,z$ be a triangle with $\alpha = y\hat{x}z \geq y\hat{z}x \geq x\hat{y}z$. Then

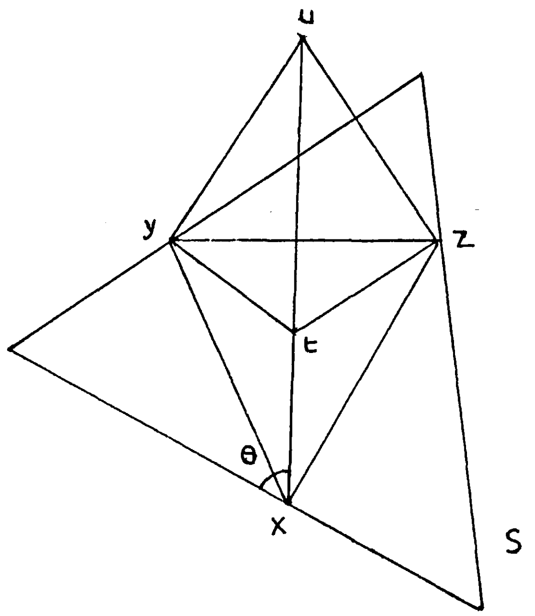
(ii) Of all equilateral triangles S which have the property that x, y and z each belong to an edge of S there is one triangle T of maximal height. Then T is unique, has height $K(x, y, z)$ and does not contain any side of triangle $x\,y\,z$.

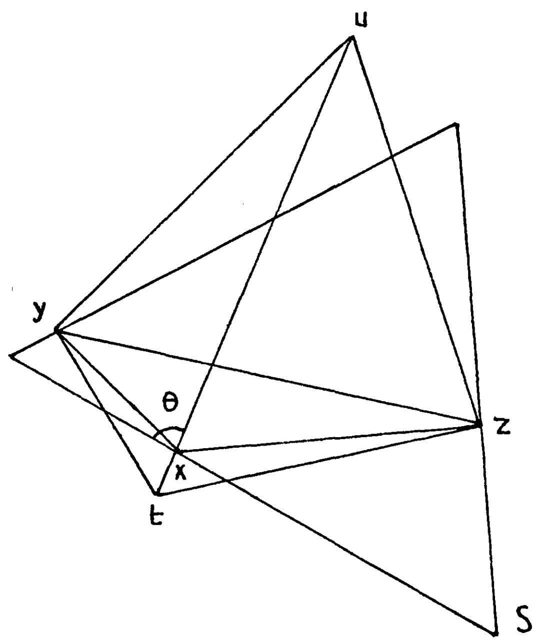
(iii) $K(x, y, z) \geq |y-z|$ if $\alpha \leq \frac{5\pi}{6}$ with equality only if $= \frac{5\pi}{6}$, and $K(x, y, z) < |y-z|$ if $\alpha > \frac{5\pi}{6}$.

Proof

The proof of (i) is obvious and is omitted. We prove (ii). We consider the set $D(x, y, z)$ which has centre of revolution t and length $K(x, y, z)$. On yz erect the equilateral triangle $u\,y\,z$ with u on the side of yz opposite to x. It is a consequence of lemma 15 that $u\hat{x}y$ and $u\hat{x}z$ are both positive. Then by lemma 15 any equilateral triangle S with the required property has height $|u-x|\sin\theta$ where θ is the angle that the edge of S which contains x makes with ux.
It is then clear that T has height $|u-x| = K(x, y, z)$ and that the edge of T which contains x is perpendicular to ux. We illustrate the two cases,

(a) $\alpha < \frac{2\pi}{3}$ and (b) $\alpha \geq \frac{2\pi}{3}$.

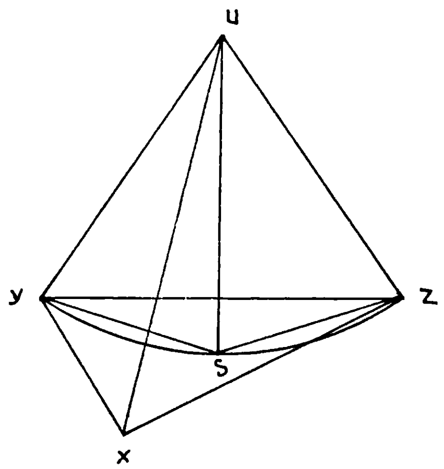




Thus (ii) is proved.

(iii)

On yz erect the equilateral triangle uyz with u on the side of yz opposite to x. Draw the arc Υ of the circle centre u, radius $|y-z|$ which lies on the same side of the line yz as x. Let s be the point on Υ such that $|y-s| = |z-s|$.



Now $s\hat{u}z = \frac{\pi}{6}$.

Thus $u\hat{s}z = u\hat{s}y = \frac{1}{2}(\pi - \frac{\pi}{6})$

and so $y\hat{s}z = \frac{5\pi}{6}$.

Thus the chord yz of Υ subtends an angle $\frac{5\pi}{6}$ on Υ. Now $K(x, y, z) = |u-x| = |y-z|$ if and only if $\alpha = \frac{5\pi}{6}$.

Thus the lemma is proved.

Corollary 1

Let X be a compact set and suppose $K(X) = K(x, y, z)$ for some points x, y and z in X. Then

(i) Each angle of triangle $x\ y\ z$ is less than or equal to $\frac{5\pi}{6}$ or

(ii) The points, x, y, and z are not all distinct.

Proof

If x, y and z are distinct and collinear with x between y and z, then $K(x, y, z) = |u-x|$ where u is the third vertex of the equilateral triangle uyz and so $K(x, y, z) < |y-z|$.

If x, y and z are the vertices of a triangle and $\alpha > \frac{5\pi}{6}$ then by lemma 16 part (iii) $K(x,y,z) < |y-z|$.

Thus in all cases we have

$$K(X) = K(x, y, z) < |y-z| = K(y,y,z) \leq K(X)$$

which is impossible. The corollary is proved.

Our first theorem shows that if three points attain the I-stretch of a compact convex set X then these points possess similar kinds of properties as two points which attain the diameter of X.

Theorem 18

Let X be a compact convex set with int $X \neq \phi$, and suppose $I(X) = I(x, y, z)$ for some x, y and z in X. Let $C(x, y, z)$ have centre of connection w, and v be any vertex of triangle $x\,y\,z$. Then if $w \neq v$ the line through v perpendicular to wv supports X regularly and if $w = v$ (e.g. if $C(x, y, z)$ has two segments) then the external bisector of the obtuse angle between the two lines of the triangle incident at w supports X regularly.

Proof

We consider two cases.we note that by the corollary 1 to lemma 15 x, y and z are the vertices of a triangle.

Case I

$C(x, y, z)$ has three segments.

Suppose without loss in generality that $v = x$. Let u be the third vertex of the equilateral triangle $u\,y\,z$ which lies on the side of yz opposite to x and suppose that the line L through x perpendicular to wx does not support X regularly. Then there is a point $x^1 \neq x$, $x^1 \in X \cap L$, such that each angle of triangle x^1yz is less than $\frac{2}{3}\pi$ and so by lemma 15,

$$I(X) \geq I(x^1, y, z) = |x^1 - u| > |x-u| = I(x, y, z) = I(X). \tag{1}$$

But (1) is impossible and so L supports X regularly.

Case II

If C(x, y, z) has two segments.

We suppose throughout the remainder of this theorem that $x = w$. Let M be the line through y perpendicular to wy and suppose that M does not support X regularly. Then if $y\hat{x}z > \frac{2}{3}\pi$ there is a point $y^1 \neq y$, $y^1 \in M \cap X$, with $y^1\hat{x}z > \frac{2}{3}\pi$ such that $|x-y^1| > |x-y|$ which implies

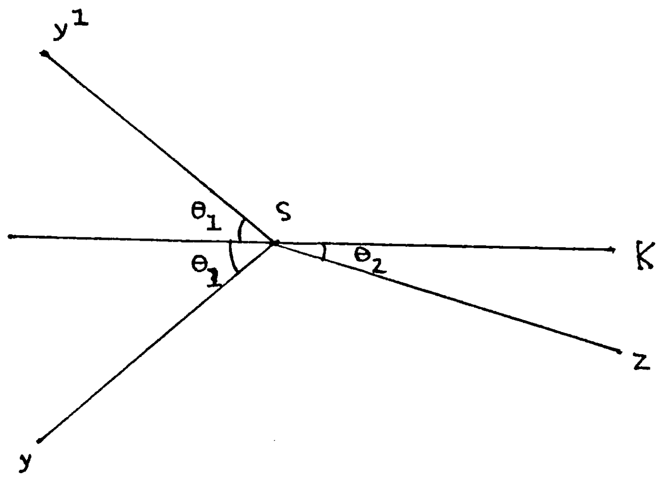
$$\begin{aligned} I(X) \geq I(x,y^1,z) &= |x-y^1| + |x-z| \\ &> |x-y| + |x-z| \\ &= I(X). \qquad (2) \end{aligned}$$

This is impossible, so $y\hat{x}z = \frac{2}{3}\pi$ and we may assume that there is a point $y^1 \neq y$, $y^1 \in M \cap X$ such that either each angle of triangle xy^1z is less than $\frac{2}{3}\pi$ or that $y^1\hat{x}z > \frac{2}{3}\pi$. The case $y^1\hat{x}z > \frac{2}{3}\pi$ is impossible as in the above paragraph and so we may assume that each angle of triangle xy^1z is less than $\frac{2}{3}\pi$. But then by lemma 15,

$$\begin{aligned} I(X) \geq I(x, y^1, z) &= |y^1-u| > |y-u| \\ &= |x-y| + |x-z| \\ &= I(X) \qquad (3) \end{aligned}$$

where u is the vertex of the equilateral triangle uxz which lies on the side of the line xz opposite to y. Thus (3) is impossible and so M is a regular support line to X at y. Similarly the line through z perpendicular to xz supports X regularly.

Now let K be the external bisector of $y\hat{x}z$. We show first that of all positions of the point s on K, $|s-y| + |s-z|$ takes a minimum when $s = x$.



Suppose the lines ys and zs make acute angles θ_1 and θ_2 respectively with K. Let y^1 be the reflection of y in K. It is then clear that $|s-y| + |s-z| = |s-y^1| + |s-z|$ takes a minimum when $\theta_1 = \theta_2$, i.e. when $s=x$.

Then if K does not support X regularly then there is a point $x^1 \neq x$, with $x^1 \in X \cap K$. Then if $y\hat{x}z > \frac{2}{3}\pi$ we can assume $y\hat{x^1}z > \frac{2}{3}\pi$ and in view of the statement in the previous paragraph we have

$$\begin{aligned} I(X) \geq I(x^1, y, z) &= |x^1-y| + |x^1-z| \\ &> |x-y| + |x-z| \\ &= I(X). \qquad (4) \end{aligned}$$

But (4) is impossible and so $y\hat{x}z = \frac{2}{3}\pi$.

We may assume $y\hat{x}^1z \geq \frac{2}{3}\pi$ or each angle of triangle x^1yz is less than $\frac{2}{3}\pi$.

The former case again by (4) is impossible. In the latter case, erect the equilateral triangle uyz on yz with u on the side of yz opposite to x. Then ux is perpendicular to K and so by lemma 15,

$$\begin{aligned} I(X) \geq I(x^1, y, z) &= |u-x^1| \\ &> |u-x| \\ &= I(X). \qquad (5) \end{aligned}$$

Thus finally (5) is impossible and the theorem is proved.

<u>Corollary 1</u>

If L is a line segment in the frontier of a compact convex set X and int $X \neq \phi$ then

$$I(X;x) < I(X) \quad \text{for each } x \in \text{rel int } L.$$

<u>Proof</u>

If $x \in$ rel int L and $I(X;x) = I(X)$ then by theorem 18 there would exist a regular support line to X at x. But this is not so and thus $I(X;x) < I(X)$.

Corollary 2

If X is compact then $I(X) = I(\text{conv } X)$.

Proof

If int (conv X) $= \phi$ then this is trivial and we suppose int (conv X) $\neq \phi$. If the result is false then $I(\text{conv } X) > I(X)$. Thus there exist points x, y and z in conv X such that $I(x, y, z) = I(\text{conv } X)$ since, as X is compact it follows that conv X is compact. Then at least one of x, y or z does not belong to X. Suppose $x \in (\text{conv } X) | X$. Now by theorem 18 there exists a regular support line M to conv X. Thus $M \cap \text{conv } X = \{x\}$ and so $M \cap X = \phi$. But X is compact so $\rho(X,x) > 0$ and this implies $M \cap \text{conv } X = \phi$ which is false. Thus the assumption $I(\text{conv } X) > I(X)$ was false and since trivially $I(X) \leq I(\text{conv } X)$ the corollary is proved.

Note

We are now able to see that if a set X is completely I-stretched then X is necessarily compact and convex. For if X is unbounded then $I(X) = I(X \cup \{x\}) = \infty$ for any point x. Thus X is bounded. Also if X is not closed, we choose any point x in the closure of X which is not in X and then it is easy to see by lemma 15 that $I(X) = I(X \cup \{x\})$. Thus X is compact and by corollary 2 above it follows that X is convex.

Corollary 3

Let X be a compact convex set with int $X \neq \phi$. In the same notation as theorem 18 there exists a triangle containing X of minimal width $I(X)$ whose three edges support X at x, y and z respectively, which is equilateral if each angle of triangle $x\ y\ z$ is less than or equal to $\frac{2}{3}\pi$ and which is isosceles with its shortest edge supporting X at x if $y\hat{x}z > \frac{2}{3}\pi$.

Proof

By drawing the lines, described in theorem 18, through x, y and z it is clear that the proof is immediate.

Corollary 4

For any points x, y or z,

$$I(x, y, z) = I(\text{conv } (x, y, z)).$$

Proof

If all the points are not distinct or collinear then the result is obvious. Suppose then conv (x, y, z) is a triangle. By Corollary (2) of theorem 18,

$$I(\text{conv}(x, y, z)) = I(\{x\}\cup\{y\}\cup\{z\}). \tag{6}$$

The result then clearly follows from the corollary 1 to lemma 15 since (6) implies

$$I(\mathrm{conv}(x, y, z)) = \max(|x-y|, |x-z|, |y-z|, I(x,y,z)) = I(x,y,z).$$

The corollary is proved.

Our next theorem shows that a compact convex set X with a non-empty interior is completely I-stretched if and only if the I-stretch of x with respect to X is constant for each x in the frontier of X. This is a direct analogue of the fact (see for example [7] page 122) that a compact convex set X has constant width if and only if X is complete.

Theorem 19

Let X be a compact convex set with int $X \neq \phi$. Then $I(X;x) = \lambda$ for each $x \in \mathrm{fr}\, X$ if and only if X is completely I-stretched with $I(X) = \lambda$.

Proof

Suppose first that $I(X; x) = \lambda$ for each $x \in \mathrm{fr}\, X$. By the compactness of X, $I(X) = I(x, y, z)$ for three points x, y and z in X and it is obvious by theorem 18 that x, y and z each belong to fr X.

Hence $I(X) = I(X; x) = \lambda$.

Now let $x_1 \notin X$. Let x_2 be that point of X which is nearest x_1. Let y_2 and z_2 be points in fr X for which $I(X; x_2) = \lambda$ $I(x_2, y_2, z_2) = \lambda$. By the corollary 1 to lemma 15, x_2, y_2 and z_2 are the vertices of a triangle. Let L_1 and L_2 be lines through x_2 perpendicular to y_2x_2 and z_2x_2 respectively. Then, since x_2 is the point of X nearest to x_1, it follows that x_1 lies on the side of L_1 opposite to y_2 and also on the side of L_2 opposite to z_2. Let H_1 and H_2 denote respectively the close half-space bounded by L_1 which does not contain y_2 and the closed half-space bounded by L_2 which does not contain z_2. Then $x_1 \in H_1 \cap H_2$ and $x_1 \neq x_2$. Also if w is the centre of connection of $C(x_2, y_2, z_2)$ let M be the line through x_2 perpendicular to wx_2 if $w \neq x_2$, and the external bisector of $y_2 \hat{x}_2 z_2$ if $w = x_2$. Then M does not meet int $(H_1 \cap H_2)$ since wx_2 lies in triangle $x_2 y_2 z_2$ and, by theorem 18 supports X regularly. Since $x_1 \neq x_2$ and x_1 lies on the side of M opposite to X, we may assume one of the lines x_1y_2, x_1z_2 meets M in a point $x_3 \neq x_2$.

This implies using similar arguments as in the proof of theorem 18 that as M is a regular support line to X then

$$I(x_3, y_2, z_2) > I(x_2, y_2, z_2) = I(X). \qquad (1)$$

But since the triangle $x_3 y_2 z_2$ is contained in the triangle $x_1 y_2 z_2$ it follows that

$$I(\text{conv}(x_3, y_2, z_2)) \leqslant I(\text{conv}(x_1, y_2, z_2)). \qquad (2)$$

By the corollary 4 to theorem 18

$$I(\text{conv}(x_3, y_2, z_2)) = I(x_3, y_2, z_2)$$

and $$I(\text{conv}(x_1, y_2, z_2)) = I(x_1, y_2, z_2).$$

Thus by (1) and (2)

$$I(x_1, y_2, z_2) > I(X)$$

whence

$$I(X \cup \{x_1\}) > I(X).$$

Thus the first part of the theorem is proved.

Suppose now that $x \notin X$ implies $I(X \cup \{x\}) > \lambda$.

If the result is false then there exists $x_0 \varepsilon$ fr X such that

$$I(X;\, x_0) < I(X). \qquad (3)$$

Again by compactness there exist y_0 and z_0 both in fr X such that

$$I(X; x_0) = I(x_0, y_0, z_0). \qquad (4)$$

We show next that if $\delta > 0$ is sufficiently small, then $I(x, y, z) < I(X)$ for all x, y and z with x in the closed disc $D^2\,[x_0, \delta]$ and y and z in $D^2[x_0,\delta] \cup X$. If this is false, then there exists a sequence $\{\delta_i\}_{i=1}^{\infty}$ which tends to zero as i tends to infinity and points $x_i \;\varepsilon\; D^2[x_0, \delta_i]$, together with points y_i and z_i in

$D^2[x_0, \delta_i] \cup X$, such that

$$I(x_i, y_i, z_i) \geq I(X) \quad \text{for } i = 1, 2 \ldots \quad . \qquad (5)$$

Then $x_i \to x_0$ as $i \to \infty$ and by extracting suitable subsequences it follows that there exist points y_0 and z_0 in X such that $y_i \to y_0$ and $z_i \to z_0$ as $i \to \infty$.

$$\begin{aligned} \text{Hence} \quad I(X;x_0) &\geq I(x_0, y_0, z_0) \\ &= \lim_{i \to \infty} I(x_i, y_i, z_i) \\ &\geq I(X) \quad \text{by (5)} \\ &> I(X; x_0) \quad \text{by (3)} \quad . \qquad (6) \end{aligned}$$

But (6) is contradictory and there exists $\delta > 0$ with $I(x, y, z) < I(X)$ for all x, y and z with $x \in D^2[x_0, \delta]$ and y and z in $D^2[x_0, \delta] \cup X$.

Since $x_0 \in \text{fr } X$ there exists a point $x^1 \in D^2[x_0, \delta] \,|\, X$ and so $I(X \cup \{x^1\}) \leq I(X)$ by the previous line. But this is a contradiction since X is completely I-stretched. Thus the theorem is proved.

We shall now show that a compact set X of I-stretch equal to λ is contained in a compact convex set Y which is completely I-stretched with I-stretch equal to λ. This is a direct analogue of the well known fact that a set of diameter λ is contained in a compact convex set of constant width λ.

Theorem 20

Let X be a compact set with $I(X) = \lambda$. Then X is contained in a compact convex set Y which is completely I-stretched with $I(Y) = \lambda$.

Proof

For any compact set Z define the following sets,

$$V(Z) = \{z \mid I(Z \cup \{z\}) = I(Z)\} ,$$

$$\rho(Z) = \sup_{z \in V(Z)} \rho(Z, z),$$

$$W(Z) = \{z \mid z \in V(Z) \text{ and } \rho(Z, z) = \rho(Z)\}.$$

Since Z is compact, $V(Z)$ is compact and so $\rho(Z)$ is attained. Thus $W(Z) \neq \phi$ since clearly $V(Z) \neq \phi$.

Now let X be a given compact set with $I(X) = \lambda$. Write $X_1 = X$. Select a point $x_1 \in W(X_1)$ and define $X_2 = \text{conv}(X_1, x_1)$. Inductively define $X_{i+1} = \text{conv}\ (X_i, x_i)$ with x_i selected from $W(X_i)$ for $i = 1, 2 \ldots$.

Now for each integer i,

$$I(X_i \cup \{x_i\}) = I(X_i) \tag{1}$$

and so by corollary 2 to theorem 18

$$I(X_{i+1}) = I(\text{conv}(X_i, x_i)\ )$$

$$= I(X_i \cup \{x_i\})$$

$$= I(X_i) \text{ in view of (1).} \tag{2}$$

Thus for all integers i and j,

$$I(X_i) = I(X_j) \;. \tag{3}$$

Also $X_1 \subset X_2 \ldots \subset X_i \subset X_{i+1}$.

These sets converge in the Hausdorff metric to a compact convex set $Y \supset \bigcup_{i=1}^{\infty} X_i$.

Suppose $I(Y) = I(x, y, z)$ (4)

Since $X_i \to Y$ as $i \to \infty$ there exist points x_i^1, y_i^1 and z_i^1 in X_i such that $x_i^1 \to x$, $y_i^1 \to y$ and $z_i^1 \to z$ as $i \to \infty$. Thus for fixed j, it follows by (3) and (4) that

$$\begin{aligned} I(X_j) &= \lim_{i \to \infty} I(X_i) \geq \lim_{i \to \infty} I(x_i^1, y_i^1, z_i^1) \\ &= I(x, y, z) \\ &= I(Y). \end{aligned} \tag{5}$$

Trivially for each integer j,

$$I(Xj) \leq I(Y) \tag{6}$$

and thus by (5) and (6)

$$I(X) = I(X_j) = I(Y). \tag{7}$$

Suppose finally that Y is not completely I-stretched and choose a point $y \notin Y$ such that

$$I(Y \cup \{y\}) = I(Y). \tag{8}$$

Let $\rho(Y, y) = \delta > 0$.

Now for any pair x_i, x_j $(j > i)$,

$x_i \subset X_{i+1} \subset X_j$ and so

$|x_i - x_j| \geq \rho(X_j, x_j) = \rho(X_j)$ by definition of x_j since

$$x_j \in W(X_j). \tag{9}$$

Also $|y - x| \geq \delta$ for each $x \in X_j$ for $j = 1, 2 \ldots$ (10)

since $Y \supset X_j$.

But $I(X_j) \leq I(X_j \cup \{y\})$

$\leq I(Y \cup \{y\})$

$= I(Y)$ by (8)

$= I(X_j)$ by (7)

and so we have

$$I(X_j \cup \{y\}) = I(X_j) \quad \text{for } j = 1, 2 \ldots .$$

Thus $y \in V(X_j)$ for $j = 1, 2 \ldots$,

and hence

$$\rho(X_j) \geq (X_j, y) \geq \delta \quad \text{by (10)}. \tag{11}$$

But (11) and (9) then imply

$$|x_i - x_j| \geq \delta. \tag{12}$$

But the sequence $\{x_i\}_{i=1}^{\infty} \subset Y$ which is bounded. Thus (12) is impossible and so Y is completely I-stretched.

The theorem is proved.

Theorems 18, 19 and 20 show the resemblance between the completely I-stretched sets and the sets of constant width. This is hardly surprising for if we consider the length of the connected set of minimal length containing just two points x and y (i.e. the length of the segment xy), in place of the length of the connected set of minimal length containing three points x, y and z then our corresponding sets are precisely the sets of constant width.

In our next theorem we look at three points which attain the K-stretch of a compact convex set X.

<u>Theorem 21</u>

Let X be a compact convex set and suppose $K(X) = K(x, y, z)$ for some x, y and z in X.

(i) If x, y and z are the vertices of a triangle, let $D(x, y, z)$ have centre of revolution t, and let v be any vertex of triangle $x\,y\,z$. Then if $t \neq v$ the line through v perpendicular to tv supports X regularly and if $t = v$ then the external bisector of the obtuse angle between the two lines of the triangle incident at v supports X regularly.

(ii) If x, y and z are not all distinct we may assume $x = y \neq z$. Then all lines through y or z which make an acute angle of $\frac{\pi}{6}$ with yz support X regularly.

<u>Proof</u>

(i) Let x, y and z be the vertices of a triangle for which $K(X) = K(x, y, z)$.

Let L be the line through x perpendicular to tx if $t \neq x$, and the external bisector of $y\hat{x}z$ if $t = x$.

Suppose L does not support X regularly.

Then there is a point $x^1 \neq x$ with $x^1 \in X \cap L$ which lies on the same side of the line yz as x.

On yz erect the equilateral triangle uyz such that u lies on the side of yz opposite to x. By lemma 15, ux contains tx and ux is perpendicular to L. But then

$$\begin{aligned} K(X) \geq K(x^1, y, z) &= |u-x^1| \\ &> |u-x| \\ &= K(X), \end{aligned}$$

which is impossible and so there is a contradiction. Thus L is a regular support line to X at x and part (i) of the theorem is proved.

We now consider part (ii) with $x = y \neq z$. It is sufficient to only consider the point y. Let L be a line through y making an acute angle of $\frac{\pi}{6}$ with yz and suppose that L does not support X regularly. Then there is a point y^1 in $X \cap L$ and $y^1 \neq y$. If $y^1\hat{y}z = \frac{5}{6}\pi$, then this implies by lemma 16 part (iii) that

$$\begin{aligned} K(X) \geq K(x, y^1, z) &= |y^1 - z| \\ &> |y - z| \\ &= K(x, y, z) \\ &= K(X) \end{aligned}$$

which is impossible.

If $y^1\hat{y}z = \frac{\pi}{6}$, erect the equilateral triangle uyy^1 with u on the side of L opposite z. This implies

$$\begin{aligned} K(X) \geq K(x, y^1, z) &= |u-z| \\ &> |y-z| \\ &= K(x, y, z) \\ &= K(X) \end{aligned}$$

which is impossible and so there is a contradiction. Thus L is a regular support line to X at x and the theorem is proved.

Corollary 1

If L is a line segment in the frontier of a compact convex set X and $\operatorname{int} X \neq \phi$ then

$$K(X; x) < K(X) \quad \text{for each} \quad x \in \operatorname{rel\,int} L.$$

Corollary 2

If X is compact then $K(X) = K(\text{conv } X)$.

Proofs

The proofs of corollary 1 and corollary 2 are the same as the proofs of corollary 1 and corollary 2 of theorem 18 and are omitted.

Note

We are now able to see that if a set X is completely K-stretched then X is necessarily compact and convex. The proof is exactly the same as the corresponding proof for 'I' given in the note in corollary 2 of theorem 18.

Corollary 3

Let X be a compact convex set. In the same notation as theorem 21 there exists an equilateral triangle containing X of minimal width $K(X)$ whose three edges support X at x,y and z respectively.

Proof

By drawing the lines described in theorem 21, through x, y and z it is clear that the proof is immediate.

Corollary 4

For any three points x, y and z

$K(x, y, z) = K(\text{conv}(x, y, z))$ if and only if x,y and z are not all distinct or x, y, z form the vertices of triangle, each of whose angles is less than or equal to $\frac{5\pi}{6}$. Otherwise

$$K(\text{conv}(x, y, z)) = \max(|x-y|, |x-z|, |y-z|).$$

Proof

By corollary 2 of theorem 21

$$K(\text{conv}(x, y, z)) = K(\{x\} \cup \{y\} \cup \{z\}).$$

Thus

$$K(\text{conv}(x, y, z)) = \max(|x-y|, |x-z|, |y-z|, K(x, y, z)).$$

The result is now a direct consequence of lemma 16 part (iii).

We show in our next theorem, that if the K-stretch of x with respect to a compact convex set X with a non-empty interior is constant and equal to λ for each x in the frontier of X, then X is a rotor for an equilateral triangle of height λ.

Theorem 22

Let X be a compact convex set with int $X \neq \phi$.
Then $K(X; x) = \lambda$ for each $x \in \text{fr}\, X$ if and only if X is a rotor for an equilateral triangle of height λ.

Proof

Suppose that X is a rotor for an equilateral triangle T

of height λ. Then suppose there exists $x_0 \in \mathrm{fr}\, X$ such that

$$K(X;\, x_0) < \lambda. \tag{1}$$

It is known [12] that the normals to the edges of T at the points of intersection of X with the edges of T are concurrent in a point t. Then the sum of the distances of t from the three edges of T is equal to λ, with the convention that if an edge separates t from X, then the distance from t to that side is negative. So if y_0 and z_0 are points of X such that x_0, y_0 and z_0 lie one on each edge of T then

$$K(X;\, x_0) \geq K(x_0,\, y_0,\, z_0) = \lambda. \tag{2}$$

This contradicts (1) and so

$$K(X;\, x) = \lambda \text{ for each } x \in \mathrm{fr}\, X.$$

We now suppose $K(X;\, x) = \lambda$ for each $x \in \mathrm{fr}\, X$. Again by the compactness of X, $K(X) = K(x,\, y,\, z)$ for three points x, y and z and it is obvious by theorem 21 that x, y and z each belong to $\mathrm{fr}\, X$.

Thus $K(X) = \lambda$ and so by theorem 21, corollary 3, it follows that X is circumscribed by at least one equilateral triangle T of height λ. Now there does not exist an equilateral triangle T^1 of height $\mu > \lambda$ which circumscribes X for suppose this is the case. Let x^1, y^1 and z^1 be three points of X, one or each edge of T^1.

Then by lemma 16 parts (i) and (ii),

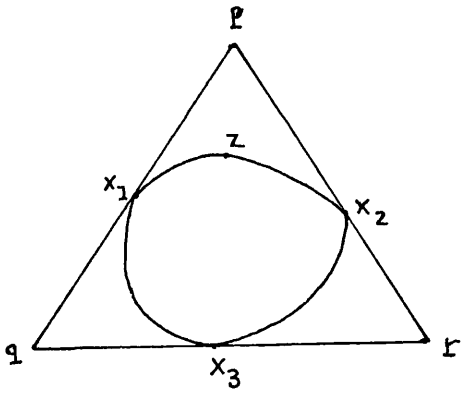
$$K(X) \geq K(x^1, y^1, z^1) \geq \mu > \lambda = K(X). \qquad (3)$$

But (3) is impossible and so we can assume that T can be moved continuously round X so as always to contain X. Moreover T circumscribes X in at least one orientation.

Let T have vertices p, q and r and suppose that qr makes an angle θ with $0 \leq \theta < 2\pi$, with some fixed line. We shall suppose p, q and r are labelled in anti-clockwise order round T with respect to an interior point of X and that θ increases as T is moved in a clockwise sense. We shall call θ, the orientation of T. Now suppose that X is not a rotor of T.

Then there exist orientations θ_0, θ_1 of T such that T circumscribes X when $\theta = \theta_0$ and $\theta = \theta_1$ but does not circumscribe X for any θ in the interval $\theta_1 > \theta > \theta_0$. Let pq, pr and qr support X at points x_1, x_2 and x_3 respectively when $\theta = \theta_0$. It is clearly impossible that $x_1 = x_2 = x_3$ and so we can assume without any real loss in generality that $x_1 \neq x_2$.

By corollary 1 to theorem 21, fr X does not contain a segment. Thus there exists a point $z \in$ fr X strictly on the same side of the line x_1x_2 as p, with $z \neq x_1$, $z \neq x_2$, and such that no support line to X through z can contain either x_1 or x_2.



Let $\theta_2 > \theta_0$ be the least value of θ for which pq is a support line to X at z, if T were moved with pq always supporting X.

Let $\theta_3 = \min(\theta_2, \theta_1)$.

Then $0 < |\theta_3 - \theta_0| < \frac{2\pi}{3}$ since $\theta_0 \neq \theta_1$ and zx_1 and zx_2 are not support lines to X.

We suppose that pq, pr and qr support X at points y_1, y_2 and y_3 respectively when $\theta = \theta_1$ and consider two cases.

Case I

$\theta_3 = \theta_1$.

Thus $0 < |\theta_1 - \theta_0| < \frac{2\pi}{3}$

Now it is not possible that $y_i = x_i$ for each $i = 1$, 2 and 3 for if this were the case we should have, by considering the length $K(x_1, x_2, x_3)$ of $D(x_1, x_2, x_3)$, that $K(x_1, x_2, x_3) = K(X) = \lambda$

and, by lemma 16 parts (i) and (ii) this implies that $|\theta_0 - \theta_1| \geq \frac{2}{3}\pi$. This is not so.

We may assume $y_1 \neq x_1$ for the proof of the other cases is similar.

We choose $x^1 \in \text{fr } X$ with $x^1 \neq x_1$ and $x^1 \neq y_1$.

But then if θ is the angle of orientation of T when the edge pq supports X at x^1 (assuming that T is moved with pq always supporting X), then θ satisfies

$$\theta_0 < \theta < \theta_1.$$

Thus, since $K(X; x^1) = \lambda = K(X)$, it follows by theorem 21 corollary 3 that there is an orientation θ^1 with

$$\theta_0 < \theta^1 < \theta_1.$$

for which T circumscribes X. But this is a contradiction with the definition of θ_1.

Thus we consider,

<u>Case II</u>

$\theta_3 = \theta_2$. We now choose a point $x^1 \in \text{fr } X$ with $x^1 \neq x_1$ and $x^1 \neq z$. But again using the same arguments as in case I, we have there is an orientation θ^1 with

$$\theta_0 < \theta^1 < \theta_2 \leq \theta_1$$

for which T circumscribes X. This is a contradiction as before and so the theorem is proved.

We now prove in theorem 23 that a compact convex set X is completely K-stretched with K-stretch equal to λ if and only if X is a rotor for an equilateral triangle of height λ. Following theorem 23, we show finally in this chapter that if a compact set X has K-stretch equal to λ, then X is contained in a rotor for an equilateral triangle of height λ.

Theorem 23

Let X be a compact convex set.
Then X is completely K-stretched with $K(X) = \lambda$ if and only if X is a rotor for an equilateral triangle of height λ.

Proof

(a) Suppose X is a rotor for an equilateral triangle T of height λ.
Let $x_1 \notin X$ and consider the point $x_2 \in X$ which is nearest to x_1. Then there is an orientation of T such that one edge passes thoough x_2 and that edge is perpendicular to x_1x_2. Suppose the other edges of T support X in the points y_2 and z_2 in this orientation. By theorem 22, $K(X) = \lambda$ and so

$$
\begin{aligned}
K(X) &= K(x_2, y_2, z_2) \\
&= K(x_1, y_2, z_2) - |x_2 - x_1| \\
&< K(x_1, y_2, z_2) \\
&\leq K(\{x_1\} \cup X).
\end{aligned}
$$

Thus X is completely K-stretched.

(b) It is evident from lemma 16 part (iii) that if X is completely K-stretched then int $X \neq \phi$.

The proof that X completely K-stretched with $K(X) = \lambda$ implies that $K(X; x) = \lambda$ for each $x \in$ fr X is exactly the same as the proof that X completely I-stretched with $I(X) = \lambda$ implies that $I(X; x) = \lambda$ for each $x \in$ fr X, which is given in theorem 19. Hence X completely K-stretched with $K(X) = \lambda$ implies that X is a rotor for an equilateral triangle of height λ by theorem 22, and so the theorem is proved.

Theorem 24

Let X be a compact set with $K(X) = \lambda$.
Then X is contained in a compact convex set Y which is a rotor for an equilateral triangle of height λ.

Proof

The proof that X is contained in a compact convex set Y which is completely K-stretched is obtained from the proof of theorem 20 by replacing 'I' by 'K' and references to theorem 18 by references to theorem 21 in the argument. Theorem 24 then follows from theorem 23.

In view of theorems 22, 23 and 24 it is easy to see that we now have theorems 19 and 20 with the function I replaced by the function K.

It is evident by a consideration of the results obtained in this chapter, that the completely K-stretched sets not only possess properties analogous to the 'completeness' properties of the sets of constant width, but also are rotors for an equilateral triangle. Since the class of sets of constant width λ coincides with the class of rotors of the square of side λ, we might say that of the class of completely K-stretched sets and the class of completely I-stretched sets, the former is more analogous to the class of sets of constant width. This is possibly rather surprising whan we look at the nature of the corresponding constructions of the completely I-stretched sets and the completely K-stretched sets.

Finally we note that theorems 20 and 24 are also true for arbitrary bounded sets X since clearly the I-stretch (K-stretch) of a bounded set X is equal to the I-stretch (K-stretch) of the closure of X. We then work with the closure of X in place of X.

CHAPTER 5

INTRODUCTION

Our main purpose in this chapter will be to give some bounds over various classes of sets for the I-stretch and K-stretch of such sets in terms of other well known set functions. We shall also reveal the precise geometric meaning of the I-stretch and K-stretch of a compact convex set.

However in our first theorem in this chapter we give a property of the 'maximal' equilateral triangle which circumscribes a compact set X. Of course the existance of such a triangle is guaranteed by the Blaschke selection theorem and the compactness of X.

Theorem 25

Let $T(X,\theta)$ be the smallest equilateral triangle which contains a compact set X and which is such that one of its edges makes an angle θ with a given fixed direction. Let $T(X)$ denote the largest such triangle $T(X,\theta)$.

Then there exists a compact convex set Y which contains X and which is a rotor for $T(X)$.

Proof

Since for each θ it is clear that $T(X, \theta) = T(\text{conv } X, \theta)$ we may assume X is convex. By definition of $T(X, \theta)$ it follows that $T(X, \theta)$ circumscribes X for all θ.

Let $W(X, \theta)$ and $W(X)$ denote the widths of $T(X, \theta)$ and $T(X)$ respectively. We next prove a lemma.

Lemma 17

If for some θ_0, $W(X, \theta_0) < W(X)$ then there exists $z \notin X$ such that

$$W(\text{conv}(X, z), \theta_0) > W(X, \theta_0) \text{ but}$$

$$W(\text{conv}(X, z)) = W(X).$$

Proof

Suppose that the edges of $T(X, \theta_0)$ meet X in the points x_0, y_0 and z_0 respectively. We may assume $x_0 \neq y_0$ and $x_0 \neq z_0$. By continuity and by lemma 16 we may assume that if θ is measured in an appropriate sense, then there exists a point $x_1 \in \text{fr } X$, $x_1 \neq x_0$ and an angle $\theta_1 > \theta_0$, such that one edge $L(X, \theta_1)$ of $T(X, \theta_1)$ supports X at x_1 and does not contain x_0 and also

$$W(X, \theta) < W(X) \quad \text{for all } \theta \text{ with } \theta_0 \leq \theta \leq \theta_1 . \qquad (1)$$

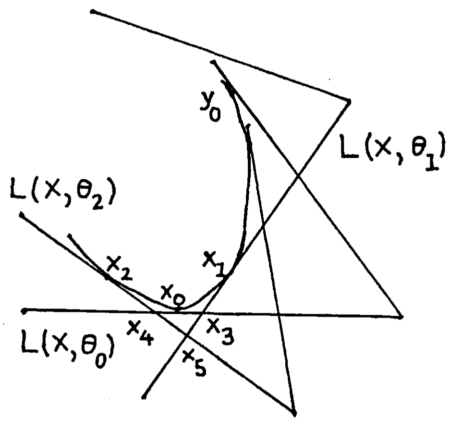
(Note if X is a segment we can assume X has end-points x_0 and y_0 and then take $x_1 = y_0$).

Again by continuity we may assume that there exists a point $x_2 \in \text{fr } X$, and an angle $\theta_2 < \theta_0$ such that one edge $L(X, \theta_2)$ of $T(X, \theta_2)$

supports X at x_2 and

$$w(X, \theta) < w(X) \text{ for all } \theta \text{ with } \theta_2 \leq \theta \leq \theta_0. \qquad (2)$$

Let $L(X, \theta_0)$ be the edge of $T(X, \theta_0)$ which supports X at x_0. Let $L(X, \theta_0)$ meet $L(X, \theta_1)$ in $x_3 \neq x_0$, let $L(X, \theta_2)$ meet $L(X, \theta_0)$ in x_4 and let $L(X, \theta_1)$ meet $L(X, \theta_2)$ in x_5.



we may assume that $\theta_1 - \theta_2 < \frac{2}{3}\pi$ by lemma 16.

Then if z is chosen in triangle $x_3\, x_4\, x_5$ exterior to X but sufficiently close to x_0 it follows by continuity and by (1) and (2) that

$$w(\text{conv}(X, z), \theta) < w(X), \text{ for all } \theta \text{ with } \theta_2 \leq \theta \leq \theta_1.$$

It is also clear that

$$w(\text{conv}(X, z), \theta_0) > w(X, \theta_0).$$

These results imply the lemma as stated.

Proof of Theorem 25

Let $\{\theta_j\}_{j=1}^{\infty}$ be a sequence dense in $[o, \frac{2}{3}\pi]$. For each compact set Z and each integer $j \geq 1$ define the following sets,

$$V(Z) = \{z \mid W(\mathrm{conv}(Z, z)) = W(Z)\} ,$$

$$\rho_j(Z) = \sup_{z \varepsilon V(Z)} W(\mathrm{conv}(Z, z), \theta_j),$$

$$U_j(Z) = \{z \varepsilon V(Z) \mid W(\mathrm{conv}(Z, z), \theta_j) = \rho_j(Z)\} .$$

Since Z is compact, $V(Z)$ is compact and so $\rho_j(Z)$ is attained. Thus $U_j(Z) \neq \phi$ for each j with $1 \leq j < \infty$.

We shall define a sequence $\{Y_j\}_{j=0}^{\infty}$ of compact convex sets inductively in the following fashion.

Let $Y_1 = X$. When Y_j, $j \geq 1$ has been defined take $y_j^1 \varepsilon U_j(Y_j)$ and let $Y_j^1 = \mathrm{conv}(Y_j, y_j^1)$. In general when Y_j^i has been defined select $y_j^{i+1} \varepsilon U_j(Y_j^i)$ and define $Y_j^{i+1} = \mathrm{conv}(Y_j^i, y_j^{i+1})$. For fixed j the sequence $\{Y_j^i\}_{i=1}^{\infty}$ is a uniformly bounded monotonic increasing sequence of compact convex sets, and so $\{Y_j^i\}_{i=1}^{\infty}$ converges in the Hausdorff metric to a compact convex set, which we define to be Y_{j+1}.

By construction

$$Y_{j+1} \supset Y_j^{i+1} \supset Y_j^i \supset Y_j^1 \tag{3}$$

and by continuity

$$W(Y_{j+1}) = W(Y_j^{i+1}) = W(Y_j^i) = W(Y_j) \tag{4}$$

for all integers i and j.

Thus (3) and (4) imply that for each j with $1 \leq j < \infty$

$$Y_j \supset X \tag{5}$$

and

$$W(Y_j) = W(X) \quad . \tag{6}$$

Moreover for each j with $1 \leq j < \infty$

$$W(Y_{j+1}, \theta_j) = W(Y_{j+1})$$

for suppose this is not the case and

$$W(Y_{j+1}, \theta_j) < W(Y_{j+1}) \quad \text{for some } j. \tag{7}$$

Then by lemma 17 there exists $\delta > 0$ and $z \notin Y_{j+1}$ such that

$$W(\text{conv}(Y_{j+1}, z), \theta_j) > W(Y_{j+1}, \theta_j) + \delta \tag{8}$$

and

$$W(\text{conv}(Y_{j+1}, z)) \doteq W(Y_{j+1}). \tag{9}$$

Now $\text{conv}(Y_j^i, y_j^{i+1}) \subset Y_{j+1}$ for each integer i so

$$W(Y_{j+1}, \theta_j) \geq W(\text{conv}(Y_j^i, y_j^{i+1}), \theta_j) \quad . \tag{10}$$

Since $Y_j^i \to Y_{j+1}$ as $i \to \infty$, for all $i \geq i_0(\delta)$

we have

$$W(\text{conv}(Y_{j+1}, z), \theta_j) - W(\text{conv}(Y_j^i, z), \theta_j) < \frac{\delta}{2} \quad . \tag{11}$$

Then by (8) and (11) we have

$$W(\text{conv}(Y_j^i, z), \theta_j) > W(Y_{j+1}, \theta_j) + \frac{\delta}{2} \tag{12}$$

and so by (10)

$$W(\mathrm{conv}(Y_j^i, z), \theta_j) > W(\mathrm{conv}(Y_j^i, y_j^{i+1}), \theta_j) + \frac{\delta}{2} \qquad (13)$$

for all $i \geq i_0(\delta)$.

We show next that $z \in V(Y_j^i)$ for each i with $1 \leq i < \infty$, for

$$\begin{aligned} W(Y_j^i) &\leq W(\mathrm{conv}(Y_j^i, z)) \\ &\leq W(\mathrm{conv}(Y_{j+1}, z)) \\ &= W(Y_{j+1}) \quad \text{by (9)} \\ &= W(Y_j) \quad \text{by (4)} \\ &\leq W(Y_j^i) \end{aligned}$$

for all i with $1 \leq i < \infty$.

Thus $W(Y_j^i) = W(\mathrm{conv}(Y_j^i, z))$ and so $z \in V(Y_j^i)$.

But $y_j^{i+1} \in U_j(Y_j^i)$ and so

$$W(\mathrm{conv}(Y_j^i, y_j^{i+1}), \theta_j) \geq W(\mathrm{conv}(Y_j^i, z), \theta_j) \qquad (14)$$

for each i with $1 \leq i < \infty$.

But (13) and (14) are contradictory and so

$$W(Y_{j+1}, \theta_j) = W(Y_{j+1}) \text{ for each } j \text{ with } 1 \leq j < \infty. \qquad (15)$$

Again the sequence $\{Y_j\}_{j=1}^{\infty}$ is a uniformly bounded monotonic increasing sequence of compact convex sets and so converges in the Hausdorff metric to a compact convex set Y, which by (5) contains X.

Also for each k with $1 \leq k < \infty$,

$$\begin{aligned} W(Y, \theta_k) &\geq W(Y_{k+1}, \theta_k) \\ &= W(Y_{k+1}) \quad \text{by (15)}, \\ &= W(Y) \quad \text{by continuity}, \\ &\geq W(Y, \theta_k). \end{aligned}$$

Since the sequence $\{\theta_k\}_{k=1}^{\infty}$ was chosen to be dense in $[o, \frac{2}{3}\pi])$ it follows that Y is a rotor of T(X). The theorem is proved.

In our next theorem the geometric meanings of the I-stretch and K-stretch of a compact convex set X are brought to light. We shall see that the I-stretch of X, I(X) is the maximum of the minimum widths taken over all triangles which circumscribe X, and that the K-stretch of X, K(X) is the maximum of the widths of all equilateral triangles which circumscribe X.

Theorem 26

Let X be a compact convex set.

(i) Of all equilateral triangles which circumscribe X there is at least one T(X) whose minimal width W(X) is maximal. Then K(X) = W(X).

(ii) Of all triangles which circumscribe X there is at least one S(X) whose minimal width V(X) is maximal. Then S(X) is isosceles with its two longer sides equal and I(X) = V(X).

(iii) I(X) $\geqslant$ K(X) with equality if and only if there exists an equilateral triangle of minimal width V(X) which circumscribes X.

Proof

(i) By compactness there exist points x, y and z in X for which $K(X) = K(x, y, z)$. Then by theorem 21 corollary 3 we have

$$W(X) \geq K(X) . \tag{1}$$

Suppose the three edges of $T(X)$ support X in the points x, y and z respectively. Then by lemma 16,

$$K(X) \geq W(X) . \tag{2}$$

Thus (1) and (2) imply $K(X) = W(X)$ and part (i) is proved.

(ii) By compactness there exist points x, y and z in X for which $I(X) = I(x, y, z)$.

Now if $\text{int } X = \phi$ then we can suppose that X is the segment xy and it is easy to see that

$I(X) = |x - y| = V(X)$ whence part (ii) is proved.

Thus we may suppose $\text{int } X \neq \phi$.

Then by theorem 18 corollary 3 it follows that

$$V(X) \geq I(X), \tag{3}$$

and part (ii) is proved in the one direction.

Next suppose that $S(X)$ has a pair of parallel sides which support X in the points x and y respectively. Then clearly

$$I(X) \geq |x-y| \geq V(X). \tag{4}$$

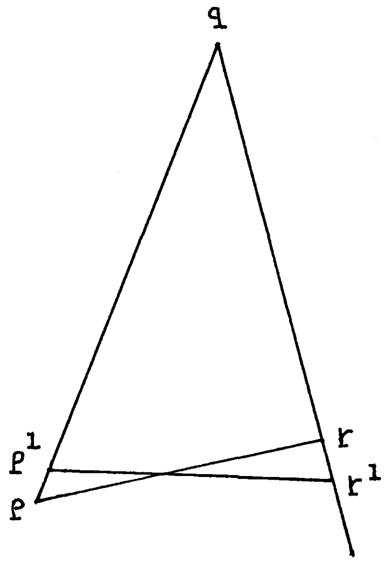
Thus by (3) and (4) part (ii) is proved if S(X) has a pair of parallel sides. We suppose then that S(X) has vertices pqr. Now I(X) = D(X) if and only if X is a segment and we are assuming this is not the case.

Thus by (3)

$$V(X) > D(X), \tag{5}$$

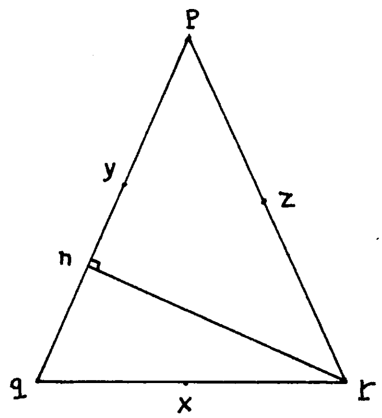
and so no vertex of S(X) belongs to X. We show next that we may assume S(X) is isosceles with its two equal sides at least as long as its third side. For if this were not the case we could choose notation so that

$$|p-q| > |q-r| \geq |p-r|. \tag{6}$$



Let r^1 be a point on qr on the side of pr opposite to q. Let p^1 be the point on pq such that p^1r^1 is the support line to X which is different from qr.

Now since there is a point of X on pq it follows by (5) that $r \notin X$. Thus there is a point of X on pr which is not r. It follows then from (6) that if r^1 is taken sufficiently close to r then the minimal width of triangle p^1qr^1 is greater than the minimal width of triangle pqr or S(X). This is contrary to the definition of S(X). Hence we can assume that S(X) is isosceles with its two equal sides pq and pr meeting X in y and z respectively and with its third side qr which is no longer than pq and pr meeting X in x. Let e, m and n be the feet of the perpendiculars from p on to qr, from q on to pr and from r on to pq respectively. Suppose S(X) has area μ. We consider different cases.



(a) Suppose C(x, y, z) has three segments and centre of connection w.

Then

$$2\mu = |p-q|\ |n-r| \leq |p-q|\ |w-y| + |p-r||w-z| + |q-r||w-x|$$
$$\leq |p-q|(|w-y| + |w-z| + |w-x|).$$

But $V(X) = |n-r|$ and $I(X) \geq |w-x| + |w-y| + |w-z|$ and so

$V(X) \leq I(X).$ (7)

(b) Suppose $C(x, y, z)$ has two segments and centre of connection $w = x$. Then

$$2\mu = |p-q||n-r| \leq |p-q||x-y| + |p-r||x-z|$$

and again

$V(X) \leq I(X)$ since $I(X) \geq |x-y| + |x-z|$. (8)

(c) Suppose that $C(x, y, z)$ has two segments and centre of connection $w = y$ or $w = z$.

We suppose $w = y$ without loss in generality and then

$$2\mu = |p-q||n-r| \leq |q-r||x-y| + |p-r||y-z|$$
$$\leq |p-r|\ (|x-y| + |y-z|).$$

Thus $V(X) \leq I(X)$ since in this case (9)

$I(X) \geq |x-y| + |y-z|.$

Thus all cases have been considered and so by (3)

$V(X) = I(X)$ and part (ii) is proved.

(iii) This is an immediate consequence of parts (i) and (ii). The theorem is then proved.

Thus in view of theorem 26 it is easy to see that theorems 24 and 25 are equivalent.

In our next theorem we give an inequality concerning the minimal width H(X) and diameter D(X) of a compact convex set X which is completely I-stretched with I-stretch λ. This enables us to give a characterisation of the rotors of a regular hexagon.

Theorem 27

Let X be a compact convex set.

(i) If X is completely I-stretched with $I(X) = \lambda$ then

$$0 \leq H(X) \leq \frac{2}{3}\lambda \leq D(X) \leq \lambda \ .$$

(ii) X is a rotor for a regular hexagon of side $\frac{2\lambda}{3\sqrt{3}}$ if and only if X is completely I-stretched with $I(X) = \lambda$ and either $H(X) = \frac{2}{3}\lambda$ or $D(X) = \frac{2}{3}\lambda$.

Proof

Suppose that $I(X) = I(X; x) = \lambda$ for each $x \in \text{fr } X$. (If X is a segment then (i) is trivially true and so we shall suppose $\text{int } X \neq \phi$.)

We shall suppose first that for some x in fr X,

$I(X; x) = I(x, y, z)$ say, where $C(x, y, z)$ has two segments, and

assume that $y\hat{x}z \geq \frac{2}{3}\pi$.

Then $\lambda = |x-y| + |x-z|$ and (1)

$$(D(X))^2 \geq |y-z|^2 = |x-y|^2 + |x-z|^2 - 2|x-y||x-z| \cos y\hat{x}z$$
$$\geq |x-y|^2 + |x-z|^2 - |x-y||x-z|$$
$$= \frac{3}{4}(|x-y| + |x-z|)^2 + \frac{1}{4}(|x-y| - |x-z|)^2.$$

Thus by (1)

$$D(X) \geq \frac{\sqrt{3}}{2}\lambda > \frac{2}{3}\lambda. \qquad (2)$$

Now (2) implies that X contains a segment uv of length $\frac{\sqrt{3}}{2}\lambda$.

Let Υ_1 be the small arc of a circle which contains u and v and is such that uv subtends an angle $\frac{2}{3}\pi$ on Υ_1. Let Υ_2 be the reflection of Υ_1 in uv. Let L and M be support lines to Υ_1 and Υ_2 respectively which are parallel to uv and are on either side of uv.

It is then clear by lemma 13 that X lies in the strip bounded by L and M which contains uv.

Now $\rho(L, M) = \frac{\lambda}{2}$ and thus

$$H(X) \leq \frac{\lambda}{2} < \frac{2}{3}\lambda. \qquad (3)$$

Thus part (i) of the theorem is proved if there exists $x \in$ fr X for which $I(X; x) = I(x, y, z)$ say, where $C(x, y, z)$ has two segments.

We suppose now that this is not the case, and for each $x \in \text{fr } X$ there exist y and z in $\text{fr } X$ such that $I(X; x) = I(x, y, z)$, where $C(x, y, z)$ has three segments.

It then follows by theorem 26 part (iii) that

$$K(X; x) = I(X; x) = \lambda \quad \text{for each} \quad x \in \text{fr } X,$$

and so by theorem 22 X is a rotor for an equilateral triangle of height λ.

Thus by [12] the perimeter of X is equal to $\frac{2}{3}\pi\lambda$ and so

$$\frac{2}{3}\pi\lambda = \frac{1}{2}\int_0^{2\pi} H(X, \theta)\, d\theta \qquad (4)$$

where $H(X, \theta)$ is the width of X in the direction θ.

Hence,

$$\begin{aligned} \pi D(X) &= \pi \sup_{0 \leqslant \theta \leqslant 2\pi} H(X, \theta) \\ &\geqslant \frac{2}{3}\pi\lambda \\ &\geqslant \pi \inf_{0 \leqslant \theta \leqslant 2\pi} H(X, \theta) \\ &= \pi H(X). \end{aligned} \qquad (5)$$

Hence (2), (3) and (5) imply

$$H(X) \leqslant \frac{2}{3}\lambda \leqslant D(X).$$

Let $\{\theta_i\}_{i=1}^{\infty}$ be a sequence of angles convergent to π as $i \to \infty$.

Let T_i be the isosceles triangle whose two equal sides are of length $\frac{\lambda}{2}$ and contain the angle θ_i for $i = 1, 2 \ldots$.

We assume $\theta_i \geq \frac{2}{3}\pi$ for each i.

Then T_i, by theorem 20, is contained in a completely I-stretched set X_i with $I(X_i) = \lambda$.

It follows by theorem 18 that

$$H(X_i) \to 0$$

and $$D(X_i) \to \lambda \qquad \text{as } i \to \infty.$$

Thus part (i) is proved and the last paragraph shows that there exist sets completely I-stretched with non-empty interiors with arbitary small area and minimal width. We now prove (ii).

Suppose X is completely I-stretched with $I(X = \lambda$, and $H(X) = \frac{2}{3}\lambda$ or $D(X) = \frac{2}{3}\lambda$. Then certainly int $X \neq \phi$. Also for each $x \in$ fr X there exist y and z in fr X such that $I(X; x) = I(x, y, z)$, where $C(x, y, z)$ has three segments, for otherwise by (2) and (3) $D(X) \geq \frac{\sqrt{3}}{2}\lambda$ and $H(X) \leq \frac{\lambda}{2}$. This is not so.

Hence as before X is a rotor for an equilateral triangle of height λ.

Thus equality holds in (5) and so

$$H(X) = \frac{2}{3}\lambda = D(X),$$

whence X has constant width $\frac{2}{3}\lambda$.

Thus X is a rotor for a regular hexagon of side $\frac{2\lambda}{3\sqrt{3}}$.

We suppose finally that X is a rotor for a regular hexagon of side $\frac{2\lambda}{3\sqrt{3}}$. Then X has constant width $\frac{2\lambda}{3}$. We show next that it is not possible for I(X; x) to equal I(x, y, z) say, where C(x, y, z) has two segments. For suppose this is the case. Then it is clear, since X has diameter $\frac{2}{3}\lambda$, that I(X; x) is smaller than or equal to the sum of the two equal sides of an isosceles triangle with base $\frac{2}{3}\lambda$ and base angles of $\frac{\pi}{6}$.

$$\text{i.e. } I(X;\, x) \leq \frac{4\lambda}{3\sqrt{3}} \,. \tag{6}$$

But by [7] page 125, the radius of the incircle of X is at least $(1 - \frac{1}{\sqrt{3}}) \cdot \frac{2\lambda}{3}$.

Then if o is the centre of this circle choose points y^1 and z^1 in fr X such that $y^1 \hat{o} x = z^1 \hat{o} x = y^1 \hat{o} x = \frac{2}{3}\pi$.

Then

$$\begin{aligned} I(X;\, x) &\geq I(x,\, y^1,\, z^1) \\ &\geq 3(1 - \frac{1}{\sqrt{3}}) \frac{2\lambda}{3} \\ &> \frac{4\lambda}{3\sqrt{3}} \\ &\geq I(X;\, x) \qquad \text{by (6)} \,. \end{aligned} \tag{7}$$

But (7) is contradictory and the statement is proved.

Thus I(X; x) = K(X; x) = λ for each $x \in \text{fr}X$ by theorem 19 since it follows, as X is a rotor for a regular hexagon of side $\frac{2\lambda}{3\sqrt{3}}$, that in particular X is a rotor for an equilateral triangle of height λ.

The theorem is proved.

Since a compact convex set which is completely K-stretched is a rotor for an equilateral triangle, it is clear that we may replace 'I' for 'K' in theorem 27 part (ii) and the theorem remains true. The proof is then however rather trivial and is omitted.

In our next theorem we give an upper bound for the perimeter P(X) of a completely I-stretched set X. Again, since the completely K-stretched sets are rotors for an equilateral triangle, it follows (see for example [12] page 106) that all such sets have the same perimeter. However, this is not so for the completely I-stretched sets as we shall now see.

Theorem 28

Let X be a compact convex set which is completely I-stretched with $I(X) = \lambda$. Then

(i) $P(X) \leq \frac{2}{3}\pi\lambda$ with equality if $\frac{2}{3}\lambda \leq D(X) \leq \frac{\sqrt{3}}{2}\lambda$.

(ii) Given any number μ with $\frac{\sqrt{3}}{2}\lambda < \mu < \lambda$, there exists a compact convex set Y, of diameter $D(Y) = \mu$ and perimeter $P(Y) < \frac{2}{3}\pi\lambda$, which is completely I-stretched with $I(Y) = \lambda$.

Proof

(i) By theorem 26, the equilateral triangle T of maximal minimal width which circumscribes X has minimal width K(X) less than or equal to λ. Then by theorem 24 or theorem 25, X is contained in a rotor Y of T, which consequently has perimeter P(Y) less than or equal to $\frac{2}{3}\pi\lambda$. Thus $P(X) \leq \frac{2}{3}\pi\lambda$.

Suppose now $\frac{2}{3}\lambda \leq D(X) \leq \frac{\sqrt{3}}{2}\lambda$. Then certainly int $X \neq \phi$. Now there does not exist $x \in \text{fr } X$ for which $I(X) = I(X; x) = I(x, y, z)$ say, where $C(x, y, z)$ has two segments and where one of the angles of triangle xyz, say $y\hat{x}z > \frac{2}{3}\pi$. For if this were the case, then this implies by equation (i) of theorem 27 that

$$(D(X))^2 > \frac{3}{4}\lambda^2 + \frac{1}{4}(|x-y| - |x-z|)^2$$

and so $D(X) > \frac{\sqrt{3}}{2}\lambda$ which is not the case.

Hence for each $x \in \text{fr}X$, there exists y and z in fr X such that either $C(x, y, z)$ has three segments or $C(x, y, z)$ has two segments where one of the angles of triangle x y z is equal to $\frac{2}{3}\pi$.

This implies by theorem 26 that

$$K(X; x) = I(X; x) = \lambda \quad \text{for each } x \in \text{fr } X,$$

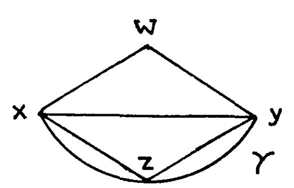
and so by theorem 22 X is a rotor for an equilateral triangle of height λ and thus has perimeter

$$P(X) = \frac{2}{3}\pi\lambda.$$

we next prove (ii).

Let μ be given with $\frac{\sqrt{3}}{2}\lambda < \mu < \lambda$ Let xy be a segment of length μ. Let w, z be points, one on either side xy with

$$|x-w| = |y-w| = |x-z| = |y-z| = \frac{\lambda}{2}.$$



Let γ be the arc of the circle $x\,y\,z$. Let X be the set bounded by γ and the segments xw and yw.

Now $\sin \frac{1}{2}(x \hat{w} y) = \frac{\mu}{\lambda} > \frac{\sqrt{3}}{2}$.

Hence $x\hat{w}y > \frac{2\pi}{3}$.

$$\text{Also} \quad |w-z| = 2 \cdot \frac{\lambda}{2} \cos \tfrac{1}{2} (x\hat{w}y)$$
$$< \frac{\lambda}{2}. \qquad (1)$$

Now suppose $I(X) = I(x^1, y^1, z^1)$.

By theorem 18 corollary 1, none of the points x^1, y^1 or z^1 can lie in relative interior of xw or wy. Thus we consider two distinct cases.

<u>Case I</u>

One of the points x^1, y^1 or z^1 say z^1 is equal to w.

Then by (1)

$$\begin{aligned} I(X) = I(x^1, y^1, z^1) &\leq |w-x^1| + |w-y^1| \\ &\leq |w-x\ | + |w-y\ | \\ &\leq I(X). \qquad (2) \end{aligned}$$

For equality to hold in (2) we must necessarily have $x = x^1$ and $y = y^1$. Thus $I(X) = \lambda$.

Case II

Each of the points x^1, y^1 and z^1 lies on γ.

But then if z^1 lies between x^1 and y^1 it is easy to see that

$$\begin{aligned} I(X) = I(x^1, y^1, z^1) &\leq |z^1 - x| + |z^1 - y| \\ &\leq |z - x| + |z - y| \\ &\leq I(X). \end{aligned} \quad (3)$$

For equality to hold in (3) we must necessarily have

$x = x^1$, $y = y^1$ and $z = z^1$.

Thus $I(x^1, y^1, z^1) = \lambda$ in both cases so

$I(X) = \lambda$.

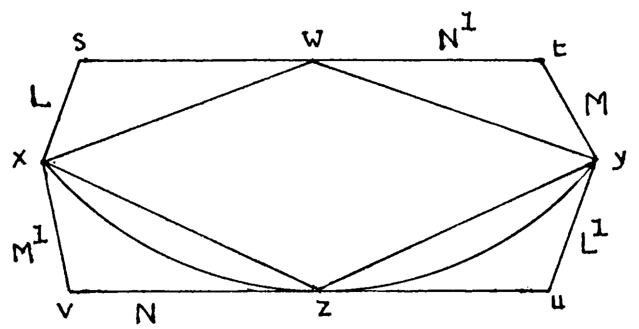
Then by theorem 20 there exists a set Y, which is completely I-stretched with $I(Y) = \lambda$, and which contains X.

Since $I(x, y, z) = I(Y)$ it follows by theorem 18 that there exist support lines L, M and N to Y with L passing through x perpendicular to xz, M passing through y perpendicular to yz and N passing through z parallel to xy. Since N is the unique support line to X at z it follows that N is the unique support line to Y at z.

Since also $I(x, y, w) = I(Y)$ we can define corresponding lines L^1, M^1 and N^1 through y, x and w respectively, which are parallel to L, M and N respectively.

Let N meet L^1 and M^1 in u and v respectively and let N^1 meet L and M in s and t respectively.

Then Y is contained in the hexagon $x\ s\ t\ y\ u\ v$.

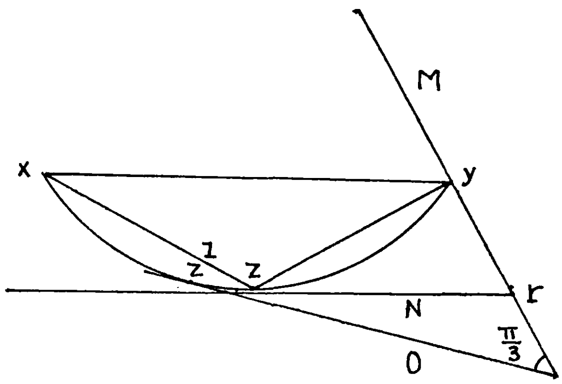


By considering the parallelogram $x\ v\ y\ t$ it is clear that $|x-y|>|t-v|$ since yz produced meets xv produced. Thus the diameter of the hexagon $x\ s\ t\ y\ u\ v$ is equal to $|x-y| = \mu$. Since Y is contained in this hexagon and Y contains X it follows that Y has diameter $D(Y) = \mu$.

Now by theorem 26 the maximal equilateral triangle which circumscribes Y has height $K(Y) = \eta \leqslant \lambda$, and so by theorem 24 or theorem 25 Y is contained in a rotor Z of an equilateral triangle of height η. Hence Y has perimeter $\frac{2}{3}\pi\lambda$ if and only if Y is a rotor for an equilateral triangle of height λ. We shall show that this is not the case and then part (ii) of this theorem follows from part (i).

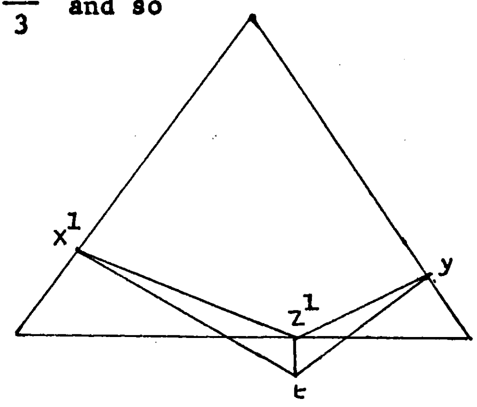
Suppose that Y is a rotor for an equilateral triangle T of height λ. Consider the orientation of T when one side of T

coincides with the line M. Suppose that the lines M and N meet in r. Since $x\hat{z}y > \frac{2\pi}{3}$ it follows that $y\hat{r}z > \frac{\pi}{3}$. Let O be the line which supports Y, which makes an angle $\frac{\pi}{3}$ with M, and which meets M on the side of r opposite to y. Then O contains one side of T when T has the above orientation. Also O does not contain z since N is the unique support line to Y at z.



Now since we are assuming that Y is a rotor of T, the normals to the three edges of T at the points of contact of T with Y are coincident in a point t say.

Thus in the above orientation of T it follows that t lies on yz produced and thus exterior to T. Let the three points of contact of the edges of T with Y be x^1, y, z^1 in this orientation. Then clearly $x^1\hat{z}^1y > \frac{2\pi}{3}$ and so



$$I(Y) \geq |x^1 - z^1| + |y - z^1|$$
$$> |x^1 - t| + |y - t| - |z^1 - t|$$
$$= \lambda$$
$$= I(Y).$$

But this is impossible and so Y is not a rotor for T.

The theorem is proved.

We shall now give a characterisation of discs among the rotors for an equilateral triangle.

The proof given here is due to my supervisor and is considerably shorter than my original proof.

If X is a compact convex set which is completely I-stretched, and we consider 'w the centre of connection of C(x, y, z)' in place of 't the centre of revolution of D(x, y, z)', together with 'I' in place of 'K' in the statement of this theorem, it will be clear from the argument that the theorem still remains true.

Theorem 29

Let X be a compact convex set which is rotor for an equilateral triangle T of height λ. If there is a fixed point t that is the centre of revolution of D(x, y, z) for each $x \in$ fr X, where y, z $\in$ fr X such that

$$K(X;\ x) = K(x,\ y,\ z) = \lambda$$

then X is a disc, centre t and radius $\frac{\lambda}{3}$.

<u>Proof</u>

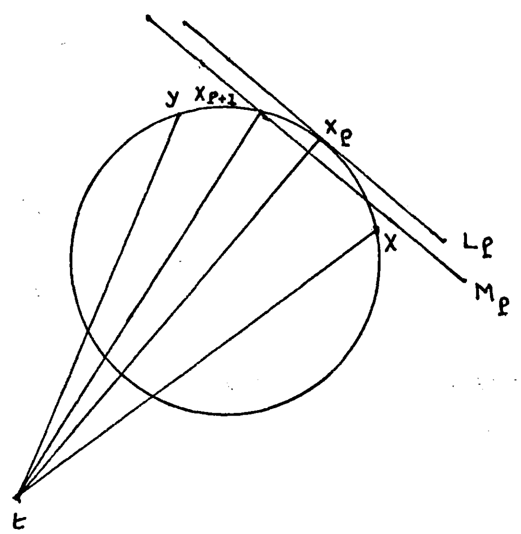
Let x and y be any two distinct points in fr X such that $x \hat{t} y = \theta$ where $0 < \theta < \frac{\pi}{2}$.

Now for each integer n choose points $x_0, x_1, \ldots x_n$ in order on the arc x y of fr X which lies on the side of the line x y opposite to t such that $x = x_0$, $y = x_n$ and $x_p \hat{t} x_0 = \theta_p$ where

$$\theta_p = \frac{p}{n} . \theta \quad \text{for } p = 1, 2 \ldots n.$$

By theorem 21, the line L_p through x_p perpendicular to tx_p supports X and a similar statement is true for x_{p+1} for $p = 0, \ldots n-1$.

Thus the line M_p through x_{p+1} which is parallel to L_p cuts the line $x_p t$ between t and x_p for $p = 0, \ldots n-1$.



Then clearly, if $r_p = |t-x_p|$ for $p = 0, . . n$, then

$$r_{p+1} \cos(\theta_{p+1}-\theta_p) \leqslant r_p \tag{1}$$

for $p = 0, . . n-1$.

Thus $r_{p+1} \cos \frac{\theta}{n} \leqslant r_p$ and (2)

Similarly $r_p \cos \frac{\theta}{n} \leqslant r_{p+1}$ for $p = 0, . . n-1$. (3)

But $$1 - \frac{(\frac{\theta}{n})^2}{2} \leqslant \cos \frac{\theta}{n} \leqslant 1 \tag{4}$$

and so by (2) and (3)

$$r_{p+1} \left(1 - \frac{(\frac{\theta}{n})^2}{2}\right) \leqslant r_p \tag{5}$$

and $$r_p \left(1 - \frac{(\frac{\theta}{n})^2}{2}\right) \leqslant r_{p+1} \quad \text{for } p = 0, . . n-1. \tag{6}$$

Then (5) and (6) imply

$$- r_{p+1} \frac{(\frac{\theta}{n})^2}{2} \leqslant r_p - r_{p+1} \leqslant r_p \frac{(\frac{\theta}{n})^2}{2} \tag{7}$$

for $p = 0, . . n-1$.

Thus by (7)

$$|r_p - r_{p+1}| \leqslant (\frac{\theta}{n})^2 . \tfrac{1}{2} . \max (r_p, r_{p+1})$$
$$\leqslant A (\frac{\theta}{n})^2 \tag{8}$$

where $A = \frac{1}{2} \sup_{x \in \text{fr } X} |t-x| < \infty$ for $p = 0, . . n-1$.

Hence $$\Big| |t-x| - |t-y| \Big| = |r_0 - r_n|$$

$$= \Big| \sum_{p=0}^{n-1} r_{p+1} - r_p \Big|$$

$$\leqslant \sum_{p=0}^{n-1} |r_{p+1} - r_p|$$

$$\leqslant \frac{A\theta^2}{n} \quad \text{by (8).}$$

But this is true for all n and thus

$$|t-x| = |t-y|.$$

Since the choice of x and y in fr X was arbitrary subject to $x\hat{t}y = \theta$ where $0 < \theta < \frac{\pi}{2}$ it is clear that the theorem follows.

Note

We can use the above argument to show that if X is an arbitrary convex set which is not a disc, and t is a given point, then there exists a point $x \neq t$ in fr X for which the line through x perpendicular to tx does not support X.

In tuitively one might expect the triangle of maximal minimal width which circumscribes a compact convex set X to be equilateral. However as we have already seen in theorem 26 this is not always the case.

In our next theorem we shall give an upper bound for the ratio of the minimal width of the triangle of maximal minimal width which circumscribes X to the width of the maximal equilateral triangle which circumscribes X. We shall also consider the ratio of these quantities to the perimeter of X. At the end of this theorem we shall give some results concerning other set functions.

Finally in theorem 31 we shall show that the upper bounds obtained in theorem 30 can be reduced if we consider either the class of sets of constant width or the class of rotors for an equilateral triangle. The lower bounds however remain the same.

Theorem 30

Let X be a compact convex set. Then

(i) $1 \leq \frac{I(X)}{K(X)} \leq \frac{2}{(2+\sqrt{3})^{\frac{1}{2}}}$,

(ii) $\frac{3}{2\pi} \leq \frac{K(X)}{P(X)} \leq \frac{I(X)}{P(X)} \leq \frac{1}{\sqrt{3}}$.

$I(X) = \frac{3}{2\pi} P(X)$ only if X is a rotor for an equilateral triangle.

$K(X) = \frac{3}{2\pi} P(X)$ if and only if X is a rotor for an equilateral triangle.

$K(X) = \frac{1}{\sqrt{3}} P(X)$ if and only if X is an equilateral triangle.

Proof

Since all the set functions mentioned in the above inequalities are continuous on the class of compact convex sets, we may assume, by the Blaschke selection theorem that in each case considered, there is a corresponding compact convex extremal set Y for which the upper or lower bound in question is attained.

We first prove (i).

By theorem 26 part (iii) $\frac{I(X)}{K(X)} \geq 1$.

Now let Y be a compact convex set for which

$$\frac{I(Y)}{K(Y)} = \sup_X \frac{I(X)}{K(X)} .$$

We choose points x, y and z in Y such that

$$I(Y) = I(x, y, z).$$

Since conv (x, y, z) $\subset$ Y we must have

$$K(\text{conv}(x, y, z)) \leq K(Y).$$

Thus we may assume that Y = conv (x, y, z).

Let Y^1 be an isosceles triangles which is such that the two equal sides of Y^1 contain the angle $\frac{5\pi}{6}$.

Suppose Y^1 has vertices x^1, y^1 and z^1 where $y^1\hat{x}^1z^1 = \frac{5\pi}{6}$.

By theorem 18 corollary 4

$$I(\text{conv}(x^1, y^1, z^1)) = I(x^1, y^1, z^1)$$

and so

$$I(Y^1) = |x^1-y^1| + |x^1-z^1|. \qquad (1)$$

By theorem 21 corollary 4

$$K(\text{conv}(x^1, y^1, z^1)) = K(Y^1) = |y^1 - z^1|. \qquad (2)$$

Then $\dfrac{I(Y^1)}{K(Y^1)} = \sec\dfrac{\pi}{12} = \dfrac{2}{(2+\sqrt{3})^{\frac{1}{2}}}$ by (1) and (2).

Hence $\dfrac{I(Y)}{K(Y)} \geq \dfrac{2}{(2+\sqrt{3})^{\frac{1}{2}}}$. (3)

Thus we may assume that Y is a triangle $x\,y\,z$ for, if Y was a segment, then this would imply that

$$\frac{I(Y)}{K(Y)} = 1.$$

Now no angle of Y can be greater than $\frac{5\pi}{6}$ for if, say $y\hat{x}z > \frac{5\pi}{6}$ then by theorem 21 corollary 4 and lemma 16 part (iii) it follows that

$$K(Y) = |y-z| > \max(|x-z|, |y-x|, K(x, y, z)). \qquad (4)$$

we now choose a point x_1 near x on the side of the line xy opposite to z and on the side of the line xz opposite to y such that $y\hat{x}_1 z > \frac{5\pi}{6}$.

Then again, if $Y^{11} = \text{conv}(x_1, y, z)$

$$K(Y^{11}) = |y-z| > \max(|x_1-z|, |x_1-y|, K(x_1, y, z)) \quad (5)$$

but by theorem 18 corollary 4, it follows that

$$\begin{aligned} I(Y^{11}) &= |x_1-y| + |x_1-z| \\ &> |x-y| + |x-z| \qquad (6) \\ &= I(Y). \end{aligned}$$

But (5) and (6) imply

$$\frac{I(Y^{11})}{K(Y^{11})} > \frac{I(Y)}{K(Y)}$$

which is contrary to the definition of Y.

Thus each angle of Y is less than or equal to $\frac{5\pi}{6}$.

Also each angle of Y is not less than or equal to $\frac{2}{3}\pi$ for if this were the case then by lemma 15

$$I(x, y, z) = K(x, y, z). \qquad (7)$$

But then also in view of lemma 16 part (iii)

$$K(x, y, z) = K(Y)$$

and we know

$$I(x, y, z) = I(Y).$$

Thus by (7) we have $\frac{K(Y)}{I(Y)} = 1$ which is impossible.

Hence we may assume that Y is such that $y\hat{x}z = \alpha$ where $\frac{2}{3}\pi < \alpha \leq \frac{5\pi}{6}$.

If $\alpha = \frac{5\pi}{6}$ then we must have $|x-y| = |x-z|$ for otherwise we could choose the point x_1 such that $|x_1-y| = |x_1-z|$ and $y\hat{x}_1z = \frac{5\pi}{6}$. This implies $K(conv(x_1, y, z)) = K(Y)$ and

$$\begin{aligned} I(conv(x_1, y, z)) &= |x_1-y| + |x_1-z| \\ &> |x-y| + |x-z| \\ &= I(Y). \end{aligned}$$

As before this contradicts the extremal property of Y and so if $\alpha = \frac{5\pi}{6}$ then the result is proved.

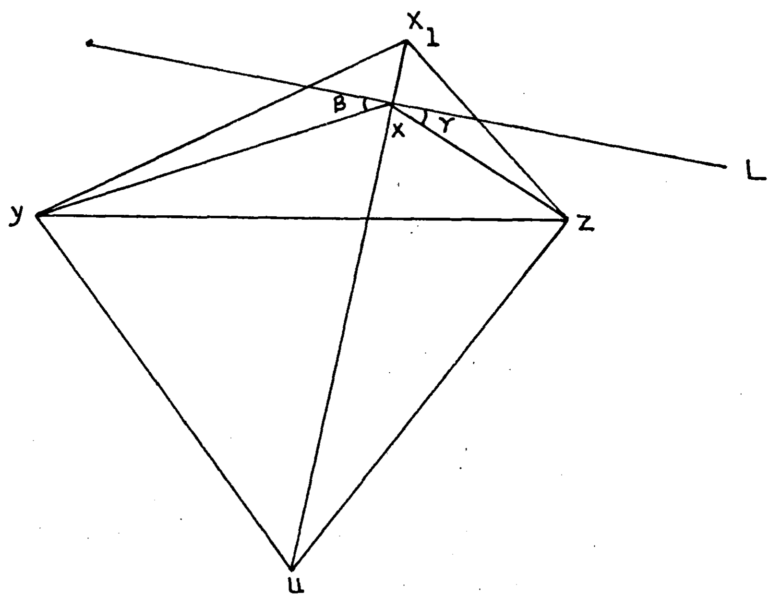
Suppose now $\frac{2}{3}\pi < \alpha < \frac{5\pi}{6}$. We shall finally show that this is impossible. On yz erect the equilateral triangle uyz with u on the side of yz opposite to x.

Let x_1 be the point on ux produced such that $y\hat{x}_1z = \frac{2}{3}\pi$.

Then using the same arguments as in the proof of lemma 15 it follows that

$$y\hat{x}_1x = z\hat{x}_1x = \frac{\pi}{3}.$$

Let L be the line through x perpendicular to xu which makes acute angles β and γ with yx and zx respectively.

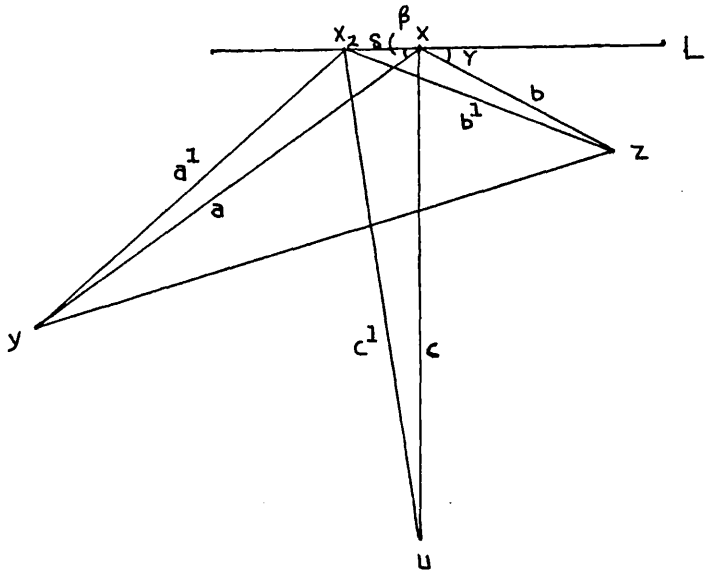


Now $|x-y| = |x-z| \Leftrightarrow \frac{|x-x_1|}{\sin x_1\hat{y}x} = \frac{|x-x_1|}{\sin x_1\hat{z}x}$

$\Leftrightarrow \beta = \gamma.$

We shall suppose $|x-y| \neq |x-z|$ and obtain a contradiction. We suppose then without loss in generality that $\beta > \gamma$. Let x_2 be a point on L near x on the same side of ux_1 as y.

Let $|x-y| = a$, $|x-z| = b$, $|u-x| = c$, $|x_2-y| = a^1$, $|x_2-z| = b^1$, $|u-x_2| = c^1$ and $|x-x_2| = \delta$.



Then $(c^1)^2 = c^2 + \delta^2$. (8)

$$\therefore \quad c^1 = c\left(1 + \frac{\delta^2}{c^2}\right)^{\frac{1}{2}}$$

$$(a^1)^2 = a^2 + \delta^2 - 2a\delta \cos \beta \qquad (9)$$

$$\therefore \quad a^1 = a\left(1 + \frac{\delta^2 - 2a\delta \cos \beta}{a^2}\right)^{\frac{1}{2}}$$

$$= a - \delta \cos \beta + O(\delta^2). \qquad (10)$$

Similarly

$$b^1 = b + \delta \cos \gamma + O(\delta^2). \qquad (11)$$

$\therefore$ From (8), (10) and (11),

$$\frac{a^1 + b^1}{c^1} = \frac{1}{c} (a + b + \delta(\cos \gamma - \cos \beta)) (1 + \frac{\delta^2}{c^2})^{-\frac{1}{2}} + O(\delta^2).$$

$$\therefore \frac{a^1 + b^1}{c^1} = \frac{a + b}{c} + \frac{\delta}{c} (\cos \gamma - \cos \beta) + O(\delta^2) .$$

This implies that if x_2 is sufficiently close to x, then δ is sufficiently small and

$$\frac{a^1 + b^1}{c^1} > \frac{a + b}{c} . \qquad (12)$$

But also if x_2 is sufficiently close to x then

$$\frac{2}{3}\pi < y\hat{x}_2 z < \frac{5\pi}{6}$$

and so by (12), theorem 18 corollary 4 and theorem 21 corollary 4 it follows that

$$\frac{I(\mathrm{conv}(x_2, y, z))}{K(\mathrm{conv}(x_2, y, z))} = \frac{|x_2-y| + |x_2-z|}{|u-x_2|} > \frac{I(Y)}{K(Y)} . \qquad (13)$$

But (13) is impossible and so

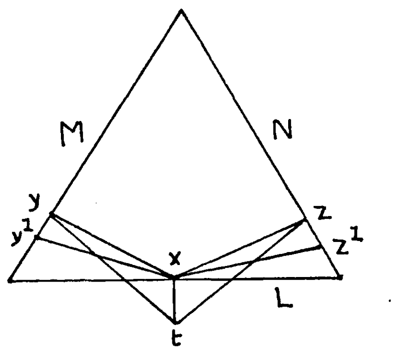
$$|x-y| = |x-z|.$$

We finally show $\alpha < \frac{5\pi}{6}$ is impossible.

Suppose that $D(x, y, z)$ has centre of revolution t.

The equilateral triangle whose three edges L, M and N pass through x, y and z and which are perpendicular to tx, ty and tz respectively has height K(Y).

Let y^1 and z^1 be points on M and N respectively such that $y^1\hat{x}z^1 = \frac{5\pi}{6}$ and such that the edge L externally bisetts $y^1\hat{x}z^1$.



Then clearly

$$I(\text{conv}(x, y^1, z^1)) = |x-y^1| + |x-z^1|$$

$$> |x-y| + |x-z| \qquad (14)$$

$$= I(Y) \ ,$$

but

$$K(\text{conv}(x, y^1, z^1)) = K(x, y, z)$$

$$= K(Y). \qquad (15)$$

The equations (14) and (15) again imply a contradiction and so part (i) of the theorem is proved.

(ii) By theorems 24 or 25 it is easy to see that

$\frac{K(X)}{P(X)} \geq \frac{3}{2}$ with equality if and only if X is a rotor for an equilateral triangle .

Thus $\frac{I(X)}{P(X)} \geq \frac{3}{2\pi}$ and there is equality only if X is a rotor for an equilateral triangle.

Note

The converse is false for let xy be a chord of an arc γ such that xy subtends an angle $\frac{5\pi}{6}$ on γ. Let γ^1 denote the reflection of γ in xy. Suppose X^1 is the set bounded by γ and γ^1. It is known [12] that X^1 is a rotor for an equilateral triangle of height $|x-y|$. Thus $\frac{K(X^1)}{P(X^1)} = \frac{3}{2\pi}$. (16)

Let z be the point on γ such that $|x-z| = |y-z|$.

Then $\frac{I(X^1)}{K(X^1)} \geq \frac{|x-z| + |y-z|}{|x-y|} \geq \frac{2}{(2+\sqrt{3})^{\frac{1}{2}}}$ (17)

and so by part (i) of this theorem there is equality in (17).

Thus by (16) $\frac{I(X^1)}{P(X^1)} = \frac{3}{2\pi} \cdot \frac{2}{(2+\sqrt{3})^{\frac{1}{2}}} > \frac{3}{2\pi}$.

We prove finally that $I(X) \leq \frac{1}{\sqrt{3}} P(X)$ with equality if and only if X is an equilateral triangle. This is sufficient in order to complete the proof (of (ii) for if this is known then $K(X) \leq \frac{1}{\sqrt{3}} P(X)$, and equality could only possibly occur if X was an equilateral triangle. But if X is the equilateral triangle then the triangle which circumscribes X and which has sides parallel to those of X but is oriented in the opposite sense has height K(X) and

$K(X) = \frac{1}{\sqrt{3}} P(X)$.

Hence the bound is attained.

Now let Y be a compact convex set for which

$$\frac{I(Y)}{P(Y)} = \sup_X \frac{I(X)}{P(X)} .$$

Suppose $I(Y) = I(x, y, z)$ for some x, y and z in Y.

Then clearly we must have $Y = \text{conv}(x, y, z)$.

Now if Y^1 is the equilateral triangle then

$$\frac{I(Y^1)}{P(Y^1)} = \frac{1}{\sqrt{3}} .$$

Thus $$\frac{I(Y)}{P(Y)} \geqslant \frac{1}{\sqrt{3}} . \qquad (18)$$

Now x, y and z cannot be collinear for this would imply

$$\frac{I(Y)}{P(Y)} = \tfrac{1}{4} < \frac{1}{\sqrt{3}} ,$$

which is impossible.

(We define the perimeter of a segment to be twice its length so that P is continuous on the class of compact convex sets in the plane).

We show next that each angle of triangle $x\,y\,z$ is less than $\frac{2\pi}{3}$. For suppose this is not the case and $y\hat{x}z = \alpha \geqslant \frac{2\pi}{3}$. Let x_1 and x_2 be two points such that $y\hat{x}_1z = \alpha$, $y\hat{x}_2z = \frac{2\pi}{3}$,

$|x_1-y| = |x_1-z|$ and $|x_2-y| = |x_2-z|$. Then clearly

$$\frac{I(Y)}{P(Y)} = \frac{I(x, y, z)}{P(Y)} = \frac{|x-y| + |x-z|}{|x-y| + |x-z| + |y-z|}$$

$$\leq \frac{|x_1-y| + |x_1-z|}{|x_1-y| + |x_1-z| + |y-z|}$$

$$\leq \frac{|x_2-y| + |x_2-z|}{|x_2-y| + |x_2-z| + |y-z|}$$

$$= \frac{2}{2+\sqrt{3}} < \frac{1}{\sqrt{3}}$$

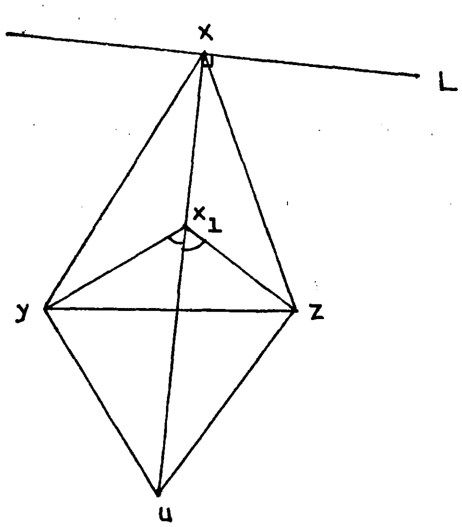
which by (18) is impossible.

Thus we may assume each angle of Y is less than $\frac{2}{3}\pi$.

Now on yz erect the equilateral triangle uyz with u on the side of yz opposite to x.

Let x_1 be the point on ux such that $y\hat{x}_1z = \frac{2}{3}\pi$.

(The existance of x_1 is guaranteed by lemma 15). Let L be the line through x perpendicular to ux.



By lemma 15, $|u-x| = I(x, y, z)$.

Now if $|x-y| \neq |x-z|$ we could, by the same sort of arguments as those given in part (i) of this theorem, choose a point x_2 on L such that each angle of triangle x_2yz is less than $\frac{2}{3}\pi$ and

$$\frac{I(Y)}{P(Y)} = \frac{|u-x|}{|x-y| + |x-z| + |y-z|}$$

$$< \frac{|u-x_2|}{|x_2-y| + |x_2-z| + |y-z|}$$

$$= \frac{I(\text{conv } x_2, y, z)}{P(\text{conv}(x_2, y, z)} \qquad (20)$$

But this would contrary to the definition of Y and so $|x-y| = |x-z|$.

Similarly

$|x-y| = |y-z|$.

Thus Y must be equilateral so

$$\frac{I(Y)}{P(Y)} = \frac{1}{\sqrt{3}} \text{ and } \frac{I(X)}{P(X)} \leq \frac{1}{\sqrt{3}} .$$

Moreover it is easy to see, by using similar sorts of arguments as those in the proof above that equality can hold only if X is an equilateral triangle. The theorem then is proved.

Corollary

(i) $1 \leq \frac{K(X)}{D(X)} \leq \frac{I(X)}{D(X)} \leq \sqrt{3}$.

(ii) $\frac{3}{2} \leq \frac{K(X)}{H(X)} \leq \frac{I(X)}{H(X)} \leq \infty$.

$\frac{I(X)}{H(X)} = \frac{3}{2}$ if and only if X is a rotor for a regular hexagon.

(iii) $\frac{9}{\pi} \leq \frac{K(X)^2}{A(X)} \leq \frac{I(X)^2}{A(X)} \leq \infty$.

$\frac{I(X)^2}{A(X)} = \frac{9}{\pi}$ if and only if X is a disc.

Proof

(i) Let xy be a diameter of X. Then

$$D(X) = |x-y| = K(x, x, y) \leq K(X).$$

Thus $D(X) \leq I(X)$ and equality is attained when X is a segment.

Let Y be a compact convex set for which

$$\frac{I(Y)}{D(Y)} = \sup_X \frac{I(X)}{D(X)} .$$

Suppose $I(Y) = I(x, y, z)$ for x, y and z in Y.

Then clearly we may suppose $Y = \text{conv}(x, y, z)$.

Thus $D(Y) \geq \frac{1}{3} P(Y)$. (21)

By theorem 30

$$P(Y) \geq \sqrt{3}\ I(Y) \qquad (22)$$

and so $D(Y) \geq \frac{1}{\sqrt{3}} I(Y)$ by (21) and (22).

Thus $\frac{K(X)}{D(X)} \leq \frac{I(X)}{D(X)} \leq \sqrt{3}$.

Obviously equality occurs when X is an equilateral triangle and so the bound is best possible.

(ii) Clearly $\frac{K(X)}{H(X)} = \frac{I(X)}{H(X)} = \infty$ when X is a segment.

Now $P(X) \geq \pi H(X)$ with equality if and only if X has constant width.

But by theorem 30 part (ii), $K(X) \geq \frac{3}{2\pi} P(X)$ with equality if and only if X is a rotor for an equilateral triangle.

Thus $K(X) \geq \frac{3}{2} H(X)$ with equality if and only if X is both a rotor for an equilateral triangle and X has constant width.

In other words X is a rotor for a regular hexagon.

Hence $I(X) \geq \frac{3}{2} H(X)$ with equality only possibly when X is a rotor for a regular hexagon.

On the other hand if X is a rotor for a regular hexagon then by theorem 27, $I(X) = \frac{3}{2} H(X)$. Thus the bound is attained. Finally we prove,

(iii) Clearly $\frac{K(X)^2}{A(X)} = \frac{I(X)^2}{A(X)} = \infty$ when X is a segment.

Now let Y be a compact convex set for which

$$\frac{K(Y)^2}{A(Y)} = \inf_X \frac{K(X)^2}{A(X)} .$$

Then by theorems 24 or 25 Y is contained in a rotor for an equilateral triangle T of height $K(Y)$ and by the extremal property of Y it follows that Y must be a rotor for T. It follows then by the iso-perimetric inequality that Y must be a disc.

Hence it is easy to see that

$$\frac{9}{\pi} \leq \frac{K(X)^2}{A(X)}$$

with equality if and only if X is a disc.

Thus $\frac{9}{\pi} \leq \frac{I(X)^2}{A(X)}$ and it can be seen that equality is only attained when X is a disc.

The corollary then is proved.

Theorem 31

(i) Let X be a compact set of constant width. Then

$$\text{(a)}\quad \frac{K(X)}{H(X)} = \frac{I(X)}{H(X)} \leq \sqrt{3} \quad \text{and (b)}\quad \frac{K(X)^2}{A(X)} = \frac{I(X)^2}{A(X)} \leq \frac{6}{\pi-\sqrt{3}}$$

with equality if and only if X is the Reuleaux triangle.

(ii) Let X be a rotor for an equilateral triangle. Then

$$\text{(a)}\quad \frac{K(X)}{H(X)} \leq 2+\sqrt{3} \quad \text{and}\quad \frac{I(X)}{H(X)} \leq 2(2+\sqrt{3})^{\frac{1}{2}},$$

$$\text{(b)}\quad \frac{K(X)^2}{A(X)} \leq \frac{6}{2\pi-3\sqrt{3}} \quad \text{and}\quad \frac{I(X)^2}{A(X)} \leq \frac{24}{(2+\sqrt{3})(2\pi-3\sqrt{3})}.$$

Equality holds in any expression if and only if X is the set bounded by the closed convex curve consisting of two circular arcs whose radii are equal to the height of the equilateral triangle, and for which the distance between the two angular points is equal to the height of the equilateral triangle. We shall denote this set by X_T.

Proof

(i) Now if X has constant width it was shown in theorem 27 that $I(X; x) = I(x, y, z)$, say, where $C(x, y, z)$ has three segments for each $x \in \text{fr } X$. Hence $I(X) = K(X)$ as before by theorem 26 part (iii). Also it is well known, see for example [7] page 128, that

$$\frac{H(X)^2}{A(X)} \leq \frac{2}{\pi - \sqrt{3}}$$ with equality if and only if X is the Reuleaux triangle. Hence (i) (b) is a consequence of (i) (a) and we only prove (i) (a).

We show next, that if T is an equilateral triangle pqr of height λ, and x, y and z are points on pq, pr and qr respectively, then

$$f(x, y, z) = |x-y| + |y-z| + |x-z| \geq \sqrt{3}\lambda \qquad (1)$$

with equality if and only if x,y and z are the mid-points of pq, pr and qr respectively.

By theorem 30 part (ii), if $S = \text{conv}(x, y, z)$, then

$$\frac{\lambda}{f(x, y, z)} \leq \frac{K(S)}{f(x, y, z)} \leq \frac{1}{\sqrt{3}}$$

and, it is easy to see that if x,y and z are the midpoints of pq, pr and qr respectively, then $f(x, y, z) = \sqrt{3}\lambda$.

Thus (1) follows and implies

$$g(x, y, z) = \max(|x-y|, |y-z|, |x-z|) \geq \frac{\lambda}{\sqrt{3}} \qquad (2)$$

with equality if and only if x, y and z are the mid-points of pq, pr and qr respectively.

It is easily seen that if x, y and z are the mid-points of pq, pr and qr respectively then the Reuleaux triangle of diameter $\frac{\lambda}{\sqrt{3}}$ containing x, y and z is inscribed in T.

Thus if X is a Reuleaux triangle then

$$\frac{K(X)}{H(X)} \geq \sqrt{3}.$$

Now suppose X is any set of constant width which is circumscribed by an equilateral triangle $T(X)$ of height $K(X)$. Suppose the three sides of $T(X)$ meet X in the points x, y and z respectively.

Then

$$\frac{K(X)}{H(X)} \leq \frac{K(X)}{\max(|x-y|, |x-z|, |y-z|)} \leq \sqrt{3} \text{ by (2)} \qquad (3)$$

and by the above paragraph equality can be attained.

Also if there is equality in (3) then by (2) x, y and z are the mid-points of the sides of $T(X)$. But this implies that X is the Reuleaux triangle, for if not, then by lemma 1 in [15], the Reuleaux triangle Y of diameter $\frac{K(X)}{\sqrt{3}}$ which contains x, y and z is strictly contained in X and so $H(Y) < H(X)$. Also $K(X) = K(Y)$ and so this implies

$$\sqrt{3} = \frac{K(X)}{H(X)} < \frac{K(Y)}{H(Y)} \text{ which is impossible by (3).}$$

Thus part (i) is proved.

We now prove part (ii). Suppose X is a rotor for an equilateral triangle.

Now it is known, see for example [12] page 106, that

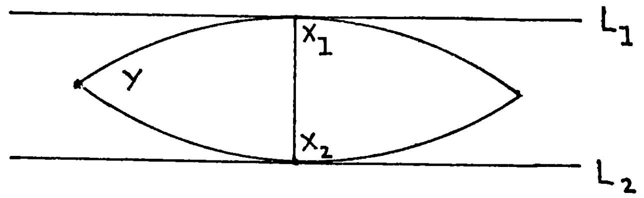
$$\frac{K(X)^2}{A(X)} \leq \frac{6}{2\pi - 3\sqrt{3}} \text{ with equality if and only if } X = X_T.$$

Thus in order to prove (ii), it is sufficient to show

$$\frac{K(X)}{H(X)} \leq 2 + \sqrt{3} \quad \text{and} \quad \frac{I(X)}{K(X)} \leq \frac{2}{(2+\sqrt{3})^{\frac{1}{2}}}$$ with equality if and only if $X = X_T$.

Let x_1 and x_2 be points in X for which $H(X) = |x_1 - x_2|$. Then the lines L_1 and L_2 through x_1 and x_2 respectively perpendicular to x_1x_2 support X.

Let D_1 and D_2 be discs of radius $K(X)$, with centres on x_1x_2 produced and x_2x_1 produced respectively, and whose frontiers pass through x_1 and x_2 respectively.



We shall show that the set $Y = D_1 \cap D_2$ contains X. For suppose this is not the case and say $x_1^1 \in X$ is exterior to D_1. Since X is a rotor for an equilateral triangle of height $K(X)$, there are points y_1 and z_1 in X such that

$$K(X) = K(x_1, y_1, z_1) \tag{4}$$

and such that the line u_1x_1 contains x_1x_2, where u_1 is the vertex of the equilateral triangle $u_1y_1z_1$ on the side of y_1z_1 opposite to x_1.

But then (4) implies

$$\begin{aligned} K(X) = K(x_1, y_1, z_1) &= |u_1-x_1| \\ &< |u_1-x_1^1| \\ &= K(x_1^1, y_1, z_1) \\ &\leqslant K(X) \end{aligned}$$

which is impossible. Thus Y contains X and so

$$P(Y) \geqslant P(X) \text{ with equality if and only if } Y = X. \tag{5}$$

Now, if $\frac{K(X)}{H(X)} > 2 + \sqrt{3}$ it is easy to prove that

$$\frac{P(Y)}{K(X)} < \frac{2}{3}\pi, \tag{6}$$

and (5) and (6) imply that X is not a rotor for an equilateral triangle. Thus

$$\frac{K(X)}{H(X)} \leqslant 2 + \sqrt{3} .$$

If there is equality then

$$\frac{P(X)}{K(X)} \leq \frac{P(Y)}{K(X)} = \frac{2}{3}\pi \qquad (7)$$

and so equality can hold in (7) if and only if $X = Y$. Thus if X is a rotor for an equilateral triangle and $\frac{K(X)}{H(X)} = 2+\sqrt{3}$ then $X = X_T$.

We show finally that

$\frac{I(X)}{K(X)} \leq \frac{2}{(2+\sqrt{3})^{\frac{1}{2}}}$ with equality if and only if $X = X_T$.

We first note that no three points x, y and z in the frontier of a rotor X of an equilateral triangle T of height λ can lie on a circle of radius greater than λ. For suppose this is the case and y, x and z lie in order on an arc, which we may suppose subtends an angle less than $\frac{\pi}{3}$ at the centre of the circle. Suppose that $\lambda = K(X) = K(x, y^1, z^1)$ say, and let u^1 be the vertex of the equilateral triangle $u^1y^1z^1$ on the side y^1z^1 opposite to x. Then it is easy to prove that $\max(|u^1-y|, |u^1-z|) > |u-x|$.

In other words either

$$K(y, y^1, z^1) > K(X) \text{ or } K(z, y^1, z^1) > K(X).$$

But this is impossible and so the statement is proved. The statement implies that if x and y are points in fr X, and γ is a circle of radius λ containing x and y, then the small arc of γ lies in X and the larger arc, apart from the points x and y lies exterior to X.

We have already shown in theorem 30 that

$$\frac{I(X)}{K(X)} \leq \frac{2}{(2 + \sqrt{3})^{\frac{1}{2}}}$$

for all sets X, and so we investigate the case of equality when X is a rotor.

Suppose $I(X) = I(x, y, z)$ for x, y and z in X.

Then

$$\frac{I(X)}{K(X)} \leq \frac{I(\text{conv}(x, y, z))}{K(\text{conv}(x, y, z))} \leq \frac{2}{(2 + \sqrt{3})^{\frac{1}{2}}} . \qquad (8)$$

Now it was shown in the proof of theorem 30 that

$$\frac{I(\text{conv}(x, y, z))}{K(\text{conv}(x, y, z))} = \frac{2}{(2 + \sqrt{3})^{\frac{1}{2}}}$$ if and only if conv(x, y, z)

is an isosceles triangle with its two equal sides containing an angle of $\frac{5\pi}{6}$.

Thus if equality holds throughout (8), then X must contain an isosceles triangle with its two equal sides containing an angle of $\frac{5\pi}{6}$, and which has diameter K(X), for otherwise $K(\text{conv}(x, y, z)) < K(X)$.

Since every arc of radius K(X) containing two points of X is contained in X, it follows that $X \supset X_T$ where $K(X_T) = K(X)$. Thus $X = X_T$ and the theorem is proved.

CHAPTER 6

INTRODUCTION

The ideas in this chapter come from a study of [14] in which Grunbaum gives a measure of asymmetry of plane convex sets, a notion introduced by Besicovitch in [13].

In [14], Grunbaum defines a set functional which is a ratio of the areas of certain 'portions' of a compact convex set X with a non-empty interior, which are determined by a partition of X by three non-concurrent lines.

In this chapter we shall look at some corresponding ratios of areas of different 'portions' of X which are determined by three concurrent lines.

We shall assume that all sets considered lie in a plane.

Theorem 32

Let X be a plane compact convex set X with int X $\neq \phi$. Let $o \in$ int X and suppose L_1, L_2 and L_3 are three distinct lines through o. Let L_1 meet fr X in x_1 and y_2, L_2 meet fr X in x_2 and y_3 and L_3 meet fr X in x_3 and y_1, where the points lie in order $x_1, y_1, x_2, y_2, x_3, y_3$ round fr X. Suppose L_1, L_2 and L_3 divide X into six regions $X_1, X_2, X_3, Y_1, Y_2, Y_3$ as shown below in the diagram.

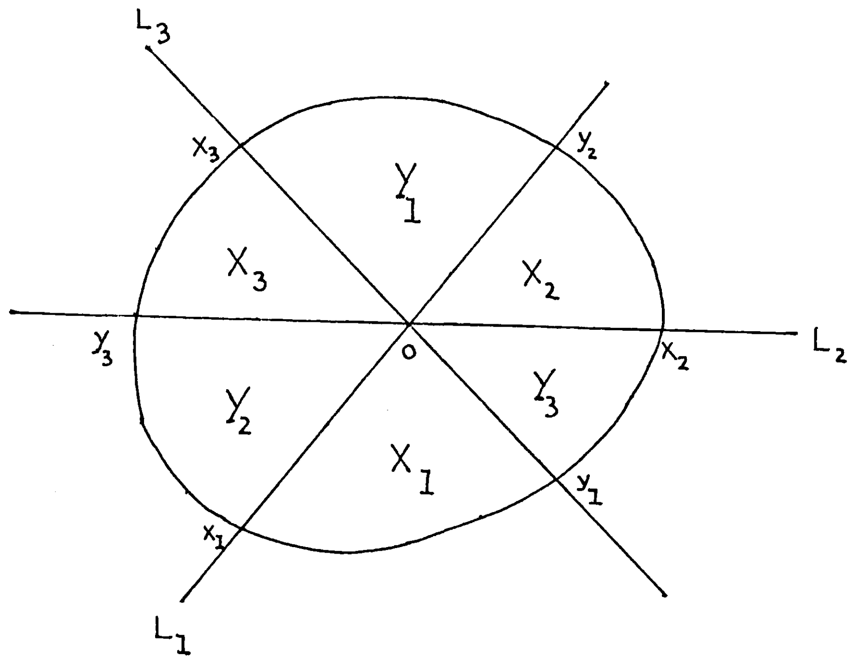


Figure 1.

Define $f(X; L_1, L_2, L_3) = \max\limits_{1\leq i\leq 3} \dfrac{A(X_1) + A(X_2) + A(X_3)}{A(Y_i)}$.

Then $f(X; L_1, L_2, L_3) \geq \frac{3}{2}$ with equality if and only if the lines M_1, M_2 and M_3 through x_1y_1, x_2y_2 and x_3y_3 respectively are parallel to L_2, L_3 and L_1 respectively, the resulting triangle $T(X)$ bounded by M_1, M_2 and M_3 has centroid o, and $T(X) = X$.

Lemma 18

In the notation of theorem 32, let $T(X)$ be the convex set bounded by the lines M_1, M_2 and M_3 through x_1y_1, x_2y_2 and x_3y_3 respectively which contains o. Then

$$f(X; L_1, L_2, L_3) \geq f(T(X); L_1, L_2, L_3),$$

and if $T(X)$ is unbounded there is a triangle $T^1(X)$ such that

$$f(T(X); L_1, L_2, L_3) > f(T^1(X); L_1, L_2, L_3).$$

Proof

Suppose that L_1, L_2 and L_3 divide $T(X)$ into six regions X_1^1, X_2^1, X_3^1, Y_1^1, Y_2^1 and Y_3^1 where $X_i^1 \subset X_i$ and $Y_i^1 \supset Y_i$ for $i = 1, 2$ and 3.

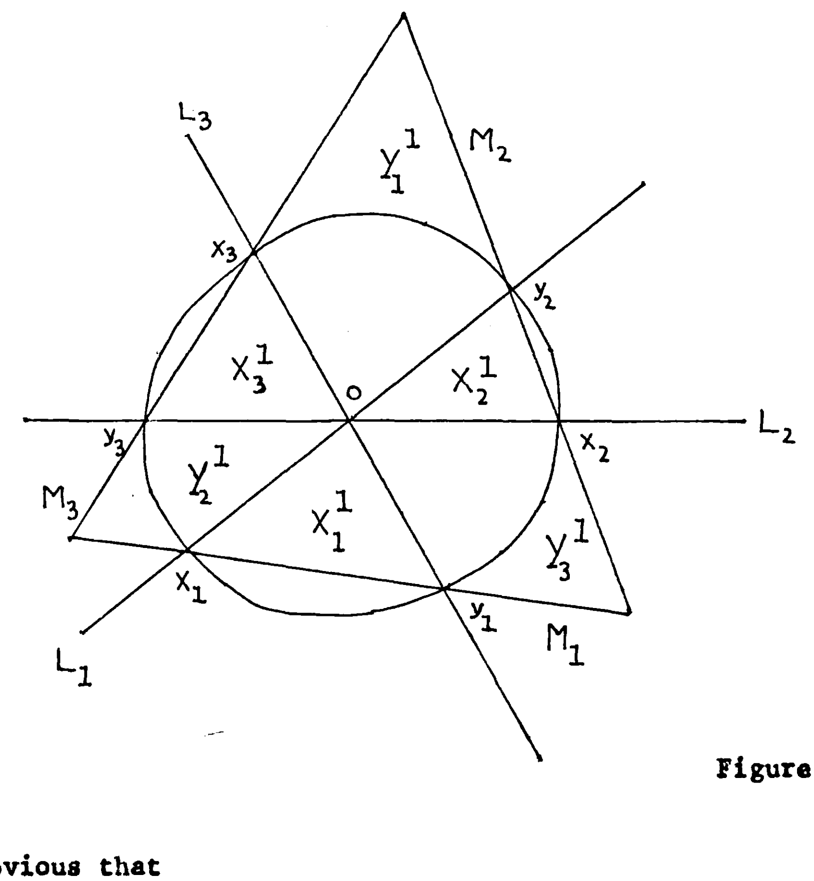


Figure 2.

It is obvious that

$$f(X; L_1, L_2, L_3) \geq f(T(X); L_1, L_2, L_3).$$

Now if $T(X)$ is not a triangle then one of the regions Y_i^1 is unbounded, say for example Y_1^1.

we may suppose without loss in generality that M_1 is parallel to L_2 or M_1 meets L_2 on the same side of L_2 as y_3. We now consider varying M_2 in such a way that the area of the region Y_3^1 tends to increase, but the area of the triangular region bounded by L_1, L_2 and M_2 remains fixed and equal to $A(X_2^1)$. We vary M_2 in this manner until the region bounded by L_1, M_2, M_3 and L_3 has finite area greater than $\min_{1 \leq i \leq 3} A(Y_i^1)$ and the lines M_1, M_2 and M_3 bound a triangle $T^*(X)$. Suppose M_1 and M_3 meet in p. We may assume $p \neq x_1$.* We can then vary M_1 a little, keeping the area of the triangular region bounded by L_1, L_3 and M_1 fixed and equal to $A(X_1^1)$ in such a direction as to increase $A(Y_2^1)$. If the variation of M_1 is sufficiently small then we obtain a triangle and $T^1(X)$ and

$$f(T^1(X); L_1, L_2, L_3) < f(T(X); L_1, L_2, L_3).$$

*(Note. If $p = x_1$, then $p \neq y_3$, and we would then vary M_3 in place of M_1).

The lemma is then proved.

<u>Lemma 19</u>

In the notation of theorem 32 define for each compact convex set X,

$$f(X) = \inf_{L_1, L_2, L_3} f(X; L_1, L_2, L_3)$$

where the inf imum is taken over all points $o \in \text{int } X$ and lines

L_1, L_2 and L_3 through o.

Then, given any triangle T, there exists a point $o^* \varepsilon$ int T and lines L_1^*, L_2^* and L_3^* distinct and meeting in the point o^*, for which the regions X_1^*, X_2^* and X_3^* corresponding to L_1^*, L_2^* and L_3^* are triangular and

$$f(T; L_1^*, L_2^*, L_3^*) = \inf_X f(X) ,$$

where the infimum is taken over all compact convex sets X with int $X \neq \phi$.

<u>Proof</u>

We first note that for any compact convex set X there is a point o in X and lines L_1, L_2 and L_3 through o for which

$$f(X) = f(X; L_1, L_2, L_3).$$

We may assume that $o \varepsilon$ int X and the lines L_1, L_2 and L_3 are all distinct, for if this were not the case then $f(X)$ would be infinite.

Thus we can choose a sequence of compact convex sets $\{X_i\}_{i=1}^{\infty}$, sequences of lines $\{L_{1i}\}_{i=1}^{\infty}$, $\{L_{2i}\}_{i=1}^{\infty}$ and $\{L_{3i}\}_{i=1}^{\infty}$, and a sequence of points $\{o_i\}_{i=1}^{\infty}$ such that $o_i \varepsilon$ int X_i, L_{1i}, L_{2i} and L_{3i} meet in o_i,

$$f(X_i; L_{1i}, L_{2i}, L_{3i}) = f(X_i) \quad \text{for } i = 1, 2 \ldots, \quad (1)$$

and

$$f(X_i) \to \inf_X f(X) \quad \text{as} \quad i \to \infty \quad . \tag{2}$$

By lemma 18 we may assume that there is a triangle S_i for which the regions X_{1i}, X_{2i} and X_{3i} in S_i corresponding to L_{1i}, L_{2i} and L_{3i} are triangular and

$$f(X_i;\, L_{1i},\, L_{2i},\, L_{3i}) \geq f(S_i;\, L_{1i},\, L_{2i},\, L_{3i}) \tag{3}$$

for $i = 1, 2 \ldots$.

Now if T is a given triangle then there exists an affine transformation $\sigma_i : E^2 \to E^2$ for which

$$\sigma_i(S_i) = T \qquad \text{for} \quad i = 1, 2 \ldots \quad . \tag{4}$$

Then

$$f(S_i;L_{1i},L_{2i},L_{3i}) = f(T;\sigma_i(L_{1i}),\, \sigma_i(L_{2i}),\, \sigma_i(L_{3i})) \tag{5}$$

and the regions $\sigma_i(X_{1i})$, $\sigma_i(X_{2i})$ and $\sigma_i(X_{3i})$ in T corresponding to $\sigma_i(L_{1i})$, $\sigma_i(L_{2i})$ and $\sigma_i(L_{3i})$ are triangular.

By extracting a suitable subsequence we may assume

$$\sigma_i(L_{ji}) \to L_j^* \quad \text{for } j = 1, 2 \text{ and } 3 \text{ and } o_i \to o^* \text{ as } i \to \infty. \tag{6}$$

Thus in the limit as $i \to \infty$ it follows from (1), (2), (3), (4) and (5) that

$$\inf_X f(X) \geq f(T;\, L_1^*,\, L_2^*,\, L_3^*)$$

whence there is equality.

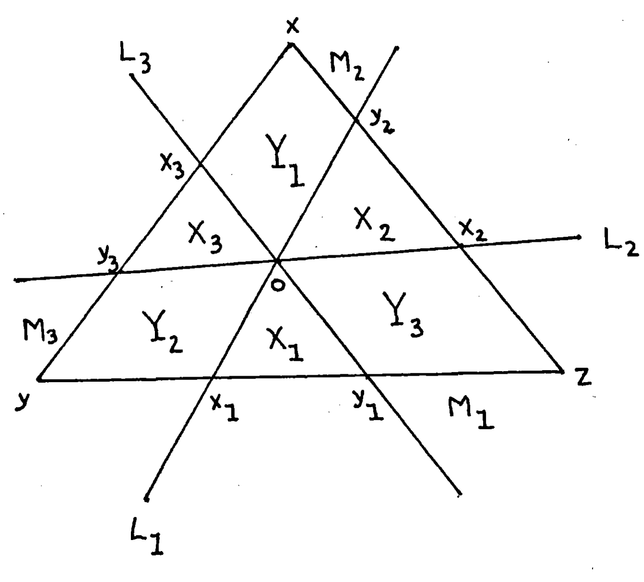
By the first paragraph in this lemma $o^* \varepsilon$ int T and L_1^*, L_2^* and L_3^* are distinct. It also follows from (5) that the regions X_1^*, X_2^* and X_3^* corresponding to L_1^*, L_2^* and L_3^* are triangular. Hence the lemma is proved.

Proof of theorem 32

Let T be the triangle x y z. By lemma 19 we may choose a point $o \varepsilon$ int T, distinct lines L_1, L_2 and L_3 containing o such that xy meets L_3 and L_2 in x_3 and y_3 respectively, xz meets L_2 and L_1 in x_2 and y_2 respectively, yz meets L_1 and L_3 in x_1 and y_1 respectively and in the notation of theorem 32, Figure 1, such that

$$\inf_X f(X) = f(T; L_1, L_2, L_3) = \max_{1 \leq i \leq 3} \frac{A(X_1)+A(X_2)+A(X_3)}{A(Y_i)} . \tag{7}$$

Again let M_i be the lines containing $x_i y_i$ for $i = 1, 2$ and 3.



In order to prove theorem 32, it is sufficient, in view of lemmas 18 and 19, to prove that L_1, L_2 and L_3 are parallel to xy, yz and xz respectively and o is the centroid of T.

We may assume by taking an appropriate affine transformation of the plane that the lines L_1, L_2 and L_3 make acute angles of $\frac{\pi}{3}$ with each other.

We show first that

$$A(Y_1) = A(Y_2) = A(Y_3).$$

For suppose this is not the case and say $A(Y_1) > A(Y_3)$. We rotate M_2 through an angle δ, keeping the area of the triangular region bounded by L_1, L_2 and M_2 fixed and equal to $A(X_2)$ in such a direction as to increase $A(Y_3)$. We then rotate M_3 through an angle η keeping the area of the triangular region bounded by L_2, L_3 and M_3 fixed and equal to $A(X_3)$ in such a direction as to increase $A(Y_2)$.

If δ and η are chosen appropriately and sufficiently small, the resulting new triangle T^1 bounded by M_1, M_2 and M_3 is such that

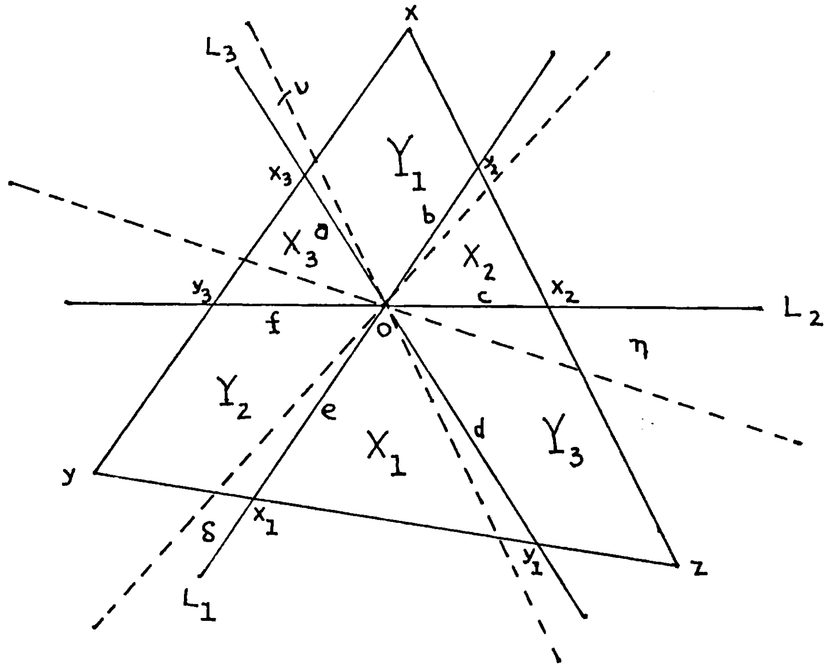
$$f(T^1; L_1, L_2, L_3) < f(T; L_1, L_2, L_3).$$

But by (7) this is impossible. Hence

$$A(Y_1) = A(Y_2) = A(Y_3). \tag{8}$$

We shall suppose that x, y and z are labelled in anti-clockwise order with respect to o.

Let $|o-x_3| = a$, $|o-y_2| = b$, $|o-x_2| = c$, $|o-y_1| = d$, $|o-x_1| = e$ and $|o-y_3| = f$.



Suppose the lines L_1, L_2 and L_3 are given rotations in a clockwise direction through small angles δ, η and υ respectively. Let L_1^1, L_2^1 and L_3^1 denote the new positions of L_1, L_2 and L_3 respectively and let X_1^1, X_2^1, X_3^1, Y_1^1, Y_2^1 and Y_3^1 be the 'new' regions corresponding to X_1, X_2, X_3, Y_1, Y_2 and Y_3.

Suppose $bdf > ace$. (9)

Then $$1 > \frac{a^2}{b^2} \cdot \frac{c^2}{d^2} \cdot \frac{f^2}{e^2} . \qquad (10)$$

we choose $\varepsilon > 0$ so that

$$1 = \left(\frac{a^2}{b^2} + \varepsilon\right)\left(\frac{c^2}{d^2} + \varepsilon\right)\left(\frac{f^2}{e^2} + \varepsilon\right) . \qquad (11)$$

We shall suppose δ is given and then define

$$\eta = \delta\left(\frac{e^2}{f^2} + \varepsilon\right), \tag{12}$$

$$\upsilon = \delta\left(\frac{c^2}{d^2} + \varepsilon\right)\left(\frac{e^2}{f^2} + \varepsilon\right). \tag{13}$$

Then by (11) and (13),

$$\frac{\delta}{\upsilon} = \frac{a^2}{b^2} + \varepsilon \tag{14}$$

by (12) and (13),

$$\frac{\upsilon}{\eta} = \frac{c^2}{d^2} + \varepsilon \tag{15}$$

and by (12)

$$\frac{\eta}{\delta} = \frac{e^2}{f^2} + \varepsilon . \tag{16}$$

Then by (13) and (14),

$$b^2 \delta - a^2 \upsilon = b^2 \varepsilon \left(\frac{c^2}{d^2} + \varepsilon\right)\left(\frac{e^2}{f^2} + \varepsilon\right) \delta \tag{17}$$

by (16),

$$f^2\eta - e^2\delta = f^2 \varepsilon \delta \tag{18}$$

and by (12) and (15)

$$d^2\upsilon - c^2\eta = d^2\varepsilon \left(\frac{e^2}{f^2} + \varepsilon\right) \delta . \tag{19}$$

Now in view of (12) and (13),

$$A(Y_1^1) - A(Y_1) = \tfrac{1}{2}(b^2\delta - a^2\upsilon) + O(\delta^2), \qquad (20)$$

$$A(Y_2^1) - A(Y_2) = \tfrac{1}{2}(f^2\eta - e^2\delta) + O(\delta^2), \qquad (21)$$

$$A(Y_3^1) - A(Y_3) = \tfrac{1}{2}(d^2\upsilon - c^2\eta) + O(\delta^2). \qquad (22)$$

Thus in view of (17), (18) and (19), if δ is sufficiently small then

$$A(Y_i^1) > A(Y_i) \quad \text{for } i = 1, 2 \text{ and } 3 \qquad (23)$$

and so

$$A(X_1^1) + A(X_2^1) + A(X_3^1) < A(X_1) + A(X_2) + A(X_3). \qquad (24)$$

But (23) and (24) imply

$$f(T;\, L_1^1, L_2^1, L_3^1) < f(T;\, L_1, L_2, L_3)$$

which by (7) is impossible.

Thus the assumption that $b\,d\,f > a\,c\,e$ was false and so

$b\,d\,f \leqslant a\,c\,e$. (25)

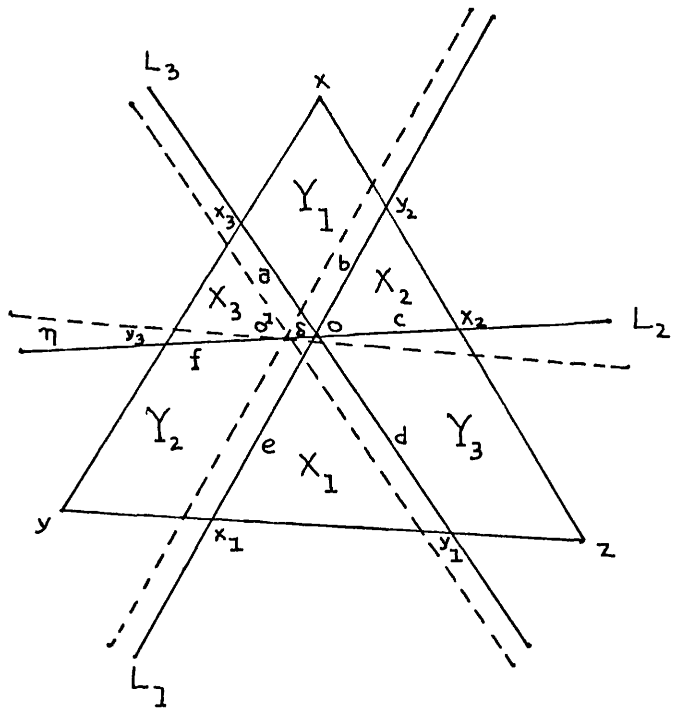
By taking similar small notations in an anti-clockwise direction the roles of a and b, c and d, e and f are each reversed and so we have

$$a\,c\,e \geqslant b\,d\,f \,. \qquad (26)$$

Thus by (25) and (26)

$$a\,c\,e = b\,d\,f \,. \qquad (27)$$

Suppose now that $a \neq b$.



We suppose without loss in generality that $a > b$. (28)

Then suppose the lines L_1 and L_3 are translated by a small amount δ in a direction parallel to L_2 so as to increase $A(Y_1)$. Let L_1^1 and L_3^1 denote the 'new' positions of L_1 and L_3 and suppose L_1^1, L_2 and L_3^1 intersect in o^1. Then suppose the line L_2 is given a rotation about o^1 through a small angle η in a direction so as to decrease $A(X_3)$. Let L_2^1 denote the 'new' position of L_2 and as before suppose the regions X_1^1, X_2^1, X_3^1, Y_1^1, Y_2^1 and Y_3^1 correspond to the regions X_1, X_2, X_3, Y_1, Y_2 and Y_3.

Suppose if possible that

$$df^2 > ec^2 \,. \tag{29}$$

we choose $\varepsilon > 0$ so that

$$1 = \left(\frac{e\sqrt{3}}{f^2} + \varepsilon\right)\left(\frac{c^2}{\sqrt{3}d} + \varepsilon\right) . \tag{30}$$

By (29) this is possible.

We shall again suppose δ is given and define

$$\eta = \delta(\frac{e\sqrt{3}}{f^2} + \varepsilon) \quad . \tag{31}$$

Then by (31)

$$\frac{\eta}{\delta} = \frac{e\sqrt{3}}{f^2} + \varepsilon \tag{32}$$

and by (30) and (31)

$$\frac{\delta}{\eta} = \frac{c^2}{\sqrt{3}d} + \varepsilon \, . \tag{33}$$

Then by (32)

$$f^2\eta - e\sqrt{3}\delta = \varepsilon f^2\delta \tag{34}$$

and by (33) and (31)

$$\sqrt{3}d\delta - c^2\eta = \sqrt{3}\,\varepsilon\, d\, (\frac{e\sqrt{3}}{f^2} + \varepsilon)\delta \quad . \tag{35}$$

Now in view of (31),

$$A(Y_1^1) - A(Y_1) = \frac{\sqrt{3}}{2}(a-b)\delta + O(\delta^2) \quad , \tag{36}$$

$$A(Y_2^1) - A(Y_2) = \tfrac{1}{2}(f^2\eta - \sqrt{3}e\delta) + O(\delta^2) \quad , \tag{37}$$

$$A(Y_3^1) - A(Y_3) = \tfrac{1}{2}(\sqrt{3}d - c^2\eta) + O(\delta^2) \quad . \tag{38}$$

Thus if δ is sufficiently small then by (28), (34) and (35) we have

$$A(Y_i^1) > A(Y_i) \quad \text{for } i = 1, 2 \text{ and } 3. \tag{39}$$

Again this implies

$$A(X_1^1) + A(X_2^1) + A(X_3^1) < A(X_1) + A(X_2) + A(X_3) \tag{40}$$

and (39) and (40) imply

$$f(T;\ L_1^1,\ L_2^1,\ L_3) < f(T;\ L_1,\ L_2,\ L_3)$$

which by (7) is impossible.

Thus the assumption (29) that $df^2 > ec^2$ was false and so

$$df^2 \leqslant ec^2 \ . \qquad (41)$$

Now suppose

$$c \leqslant f \ . \qquad (42)$$

Then $\dfrac{c^2}{f^2} \leqslant \dfrac{c}{f}$

and so by (41)

$$df \leqslant ec \ . \qquad (43)$$

Then (27) and (43) imply

$$a \leqslant b \ .$$

This by (28) is impossible.

Thus (42) is false and

$$c > f \ . \qquad (44)$$

Also if $e \geqslant d$, (45)

then $\dfrac{e^2}{d^2} \geqslant \dfrac{e}{d}$

and so by (41)

$$d^2f^2 \leqslant e^2c^2$$

and $df \leqslant ec$

which as before is impossible.

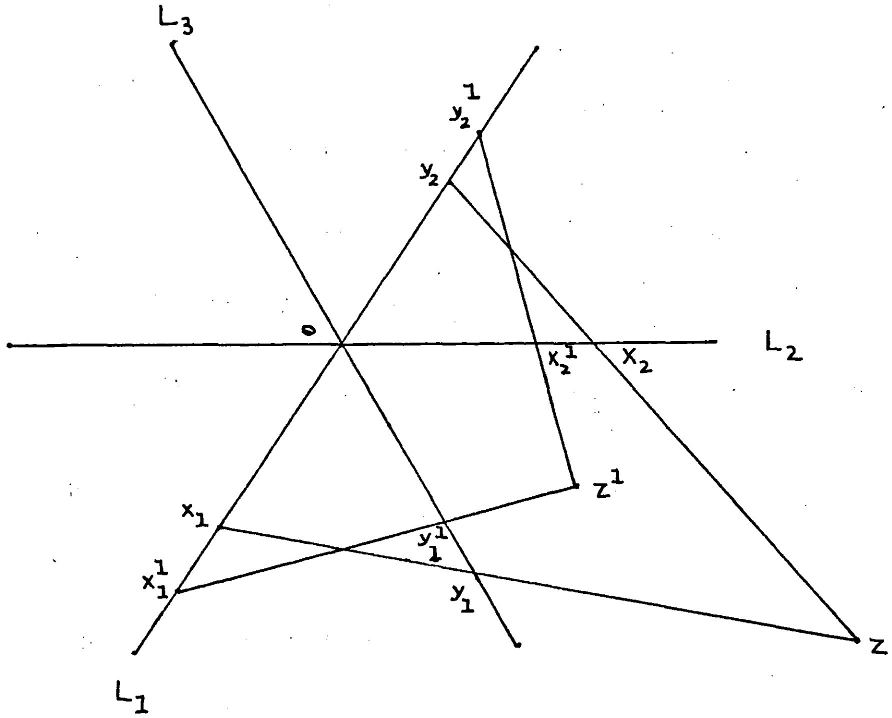
Thus $e < d$. (46)

Thus we have so far by (28), (44) and (46) that

$$a > b, \quad c > f \quad \text{and} \quad d > e.$$

We shall show this is impossible.

Reflect the triangle $y_1 y x_3$ in L_3 and rotate the triangle so obtained in a clockwise direction through an angle of $\frac{\pi}{3}$.
Suppose the resulting triangle has vertices x_1^1, y_2^1 and z^1 where x_1^1 lies on L_1 and on the same side of o as x_1 and y_2^1 lies on L_1 and on the same side of o as y_2.



Suppose $y_2^1 z^1$ meets L_2 in x_2^1 and $x_1^1 z^1$ meets L_3 in y_1^1.
Then

$$|o-y_2^1| = a > b = |o-y_2| \quad ,$$

$$|o-x_1^1| = d > e = |o-x_1| \quad ,$$

$$|o-y_1^1| = e < d = |o-y_1|$$

and

$$|o-x_2^1| = f < c = |o-x_2| .$$

Hence $\mathrm{conv}(o, x_2^1, z^1, y_1^1)$ is strictly contained in $\mathrm{conv}\ (o, x_2, z, y_1)$. This implies that

$$\begin{aligned} A(Y_2) &= A(\mathrm{conv}(o, x_2^1, z^1, y_1^1)) \\ &< A(\mathrm{conv}(o, x_2, z, y_1)) \\ &= A(Y_3) . \end{aligned} \tag{47}$$

But (47) is impossible by (8).

Thus the assumption $a > b$ was false and so $a = b$. Also clearly we can apply similar arguments to prove $c = d$ and $e = f$.

Now there exists a function $g(\xi_1, \xi_2, \xi_3)$ of the three real variables ξ_1, ξ_2 and ξ_3, $\Uparrow \xi_1$ for fixed ξ_2 and ξ_3, $\Downarrow \xi_2$ for fixed ξ_1 and ξ_3, $\Downarrow \xi_3$ for fixed ξ_1 and ξ_2 and for which

$$\begin{aligned} A(Y_1) &= g(a, e, c) , \\ A(Y_2) &= g(c, a, e) , \\ A(Y_3) &= g(e, c, a) . \end{aligned}$$

Now (8) implies

$$g(a, e, c) = g(c, a, e) = g(e, c, a). \tag{48}$$

If we suppose without loss in generality that $a \leqslant c \leqslant e$ and in fact $a < e$ then this implies

$g(e, c, a) > g(a, e, c)$ which by (48) is impossible.

Thus $a = c = e$ and so

$a = b = c = d = e = f.$

The theorem is then completed by calculation.

Corollary 1

In the same notation as theorem 32 define

$$g(X; L_1, L_2, L_3) = \max_{\substack{1 \leq i \leq 3 \\ 1 \leq j \leq 3}} \frac{A(X_j)}{A(Y_i)} .$$

Then $g(X; L_1, L_2, L_3) \geq \frac{1}{2}$ with equality if and only if

$f(X; L_1, L_2, L_3) = \frac{3}{2}$.

Proof

The proof is trivial since

$$3g(X; L_1, L_2, L_3) \geq f(X; L_1, L_2, L_3).$$

Corollary 2

In the same notation as theorem 32 define

$$h(X;L_1,L_2,L_3) = \max_{1 \leq i \leq 3} \left(\frac{A(X_1)+ A(X_2)}{A(Y_i)} , \frac{A(X_2)+ A(X_3)}{A(Y_i)} , \frac{A(X_1)+ A(X_3)}{A(Y_i)} \right) .$$

Then $h(X; L_1, L_2, L_3) \geq 1$ with equality if and only if

$f(X; L_1, L_2, L_3) = \frac{3}{2}$.

Proof

The proof is immediate since

$$3h(X;L_1,L_2,L_3) \geq \max_{1\leq i\leq 3} \frac{(A(X_1)+A(X_2))+(A(X_2)+A(X_3))+(A(X_1)+A(X_3)}{A(Y_i)}$$

$$= 2\, f\,(X;\, L_1,\, L_2,\, L_3).$$

Corollary 3

In the same notation as theorem 32 define

$$k(X;\, L_1,\, L_2,\, L_3) = \max_{1\leq i\leq 3} \frac{A(X)}{A(Y_i)}\,.$$

Then $k(X;\, L_1,\, L_2,\, L_3) \geq \frac{9}{2}$ with equality if and only if $f(X;\, L_1,\, L_2,\, L_3) = \frac{3}{2}$.

Proof

The proof is again immediate since

$$k(X;L_1,L_2,L_3) = \max_{1\leq i\leq 3} \frac{A(X_1)+A(X_2)+A(X_3)+A(Y_1)+A(Y_2)+A(Y_3)}{A(Y_i)}$$

$$\geq \max_{1\leq i\leq 3} \frac{A(X_1)+A(X_2)+A(X_3)}{A(Y_i)} + 3$$

$$\geq f(X;\, L_1,\, L_2,\, L_3) + 3.$$

We now consider a different functional which we see as in the previous theorem takes its lower bound on a triangle. The lower bound given in theorem 33 is not best possible. I would conjecture that the correct answer is $\frac{3}{2}$ although I have been unable to prove this.

Theorem 33

In the same notation as theorem 32, Figure 1 define

$$e(X: L_1, L_2, L_3) = \frac{A(X_1)}{A(Y_1)} + \frac{A(X_2)}{A(Y_2)} + \frac{A(X_3)}{A(Y_3)} \quad .$$

Then $e(X; L_1, L_2, L_3) \geq \frac{9}{8}$.

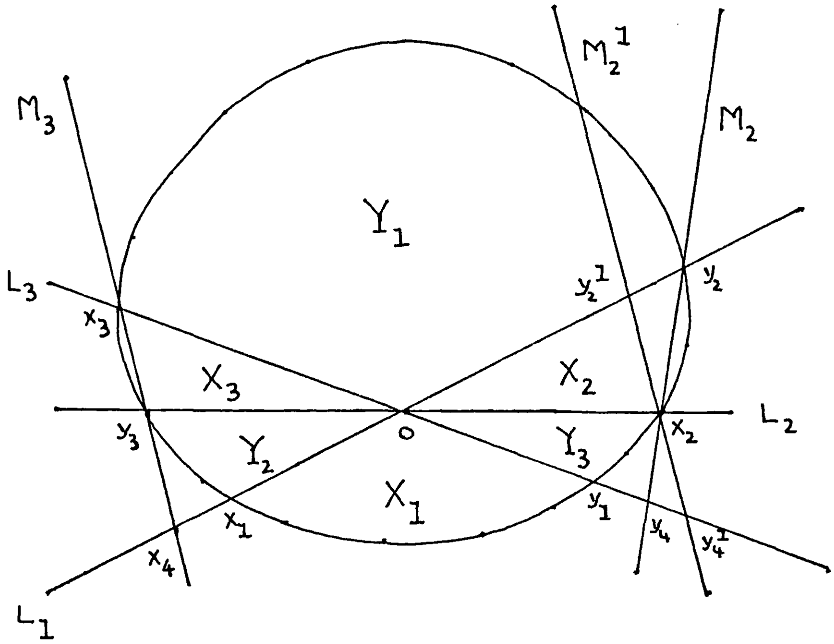
Proof

Throughout this theorem we shall use the same notation as in theorem 32, Figures 1 and 2. It is clear that if $T(X)$ is defined as in lemma 18 then

$$e(X; L_1, L_2, L_3) \geq e(T(X); L_1, L_2, L_3). \qquad (1)$$

Thus it is sufficient to prove the result for $T(X)$ in place of X. We suppose first that $T(X)$ is unbounded. We assume then that X_1^1, X_2^1, X_3^1, Y_1^1, Y_2^1 and Y_3^1 are defined as in lemma 18 and Y_1^1 is unbounded. Suppose M_2 meets L_3 in y_4 and M_3 meets L_1 in x_4.

Let M_2^1 be the line through x_2 parallel to M_3 and suppose M_2^1 meets L_1 in y_2^1 and L_3 in y_4^1.



write $|o-x_2| = \mu$ and $|o-y_3| = \eta$.

Now clearly,

$$\frac{A(X_1^1)}{A(Y_1^1)} + \frac{A(X_2^1)}{A(Y_2^1)} + \frac{A(X_3^1)}{A(Y_3^1)} \geq \frac{A(\text{conv}(o,\ x_2,\ y_2))}{A(\text{conv}(o,\ x_4,\ y_3))} + \frac{A(\text{conv}(o,\ x_3,\ y_3))}{A(\text{conv}(o,\ x_2,\ y_4))}$$

$$\geq \frac{A(\text{conv}(o,\ x_2,\ y_2^1))}{A(\text{conv}(o,\ x_4,\ y_3))} + \frac{A(\text{conv}(o,\ x_3,\ y_3))}{A(\text{conv}(o,\ x_2,\ y_4^1))}$$

$$= \frac{\mu^2}{\eta^2} + \frac{\eta^2}{\mu^2}$$

$$\geq 2 \cdot \frac{\mu\eta}{\eta\mu}$$

$$= 2$$

$$> \frac{9}{8}. \qquad (2)$$

Thus if $T(X)$ is unbounded then the theorem is true and so we now suppose $T(X)$ is bounded.

We may suppose taking an appropriate affine transformation of the plane that $T(X)$ is equilateral with unit height. For simplicity we shall now write $T = T(X)$,

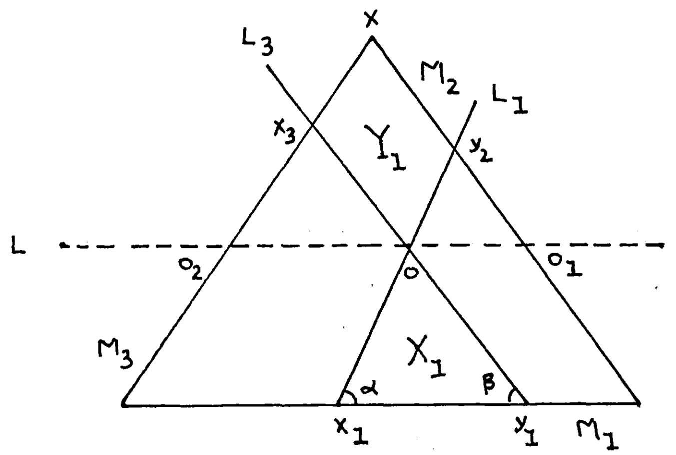
$$X_i = X_i^1 \text{ and } Y_i = Y_i^1 \text{ for } i = 1, 2 \text{ and } 3.$$

Suppose that o is distant h_i from M_i and that the segment $x_i y_i$ subtends an angle θ_i at o for $i = 1, 2$ and 3. Then

$$h_1 + h_2 + h_3 = 1, \tag{3}$$

$$\theta_1 + \theta_2 + \theta_3 = \pi. \tag{4}$$

Now we consider for the moment, just the ratio $\dfrac{A(X_1)}{A(Y_1)}$.



Let L be the line through o parallel to M_1 which meets M_2 in o_1 and M_3 in o_2. Let x be the point of intersection of M_2 and M_3. Let $y_2\hat{x}_1y_1 = \alpha$ and $x_3\hat{y}_1x_1 = \beta$.

We now suppose that L_1 and L_3 are varied such that L_1 and L_3 make fixed angles α and β respectively with M_1 and o the point of intersection of L_1 and L_3 lies on o_1o_2. Clearly $A(\text{conv}(o, x_1, y_1))$ remains invariant and equal to $A(X_1)$ and there is a position of o, say $o = o^1$ where $A(\text{conv}(x, x_3, o, y_2))$ is maximal. When $o = o^1$, write $x_1 = x_1^1$, $y_1 = y_1^1$, $x_3 = x_3^1$ and $y_2 = y_2^1$.

Then

$$\frac{A(X_1)}{A(Y_1)} \geqslant \frac{A(\text{conv}(o^1, x_1^1, y_1^1))}{A(\text{conv}(x, x_3^1, o^1, y_2^1))} . \tag{5}$$

Also clearly $o^1 \varepsilon$ rel int o_1o_2 and $x_1^1 \hat{o}^1 y_1^1 = \theta_1$.

We show next that $x_3^1y_2^1$ is parallel to $x_1^1y_1^1$.

Suppose L_1 and L_3 are translated in the manner described from any position o by a small amount δ.

Then the change in $A(\text{conv}(x, x_3, o, y_2))$ is given by

$$\pm \delta(|o-x_3| \sin \beta - |o-y_2| \sin \alpha) + O(\delta^2). \tag{6}$$

Thus if $o = o^1$ we must have

$$|o^1-x_3^1| \sin \beta = |o^1-y_2^1| \sin \alpha. \tag{7}$$

But then

$$\frac{|o^1-x_1^1|}{|o^1-y_1^1|} = \frac{\sin \beta}{\sin \alpha} = \frac{|o^1-y_2^1|}{|o^1-x_3^1|} . \tag{8}$$

Thus the triangles $o^1x_1^1y_1^1$ and $o^1x_3^1y_2^1$ are similar and $x_3^1y_2^1$ is parallel to $x_1^1y_1^1$.

Next let o_3 be the midpoint of o_1o_2. Suppose the lines $x_3^1o_3$

produced and $y_2^1 o_3$ produced meet M_1 in y_4^1 and x_4^1 respectively.

Let $\Psi_1 = x_4^1 \hat{o}_3 y_4^1$.

Then $x_3^1 \hat{o}_3 y_2^1 = \Psi_1$ and by considering the circumcircle of triangle $x_3^1 o^1 y_2^1$ it can be seen that

$$\Psi_1 \geq \theta_1. \tag{9}$$

Also $A(\text{conv}(o^1, x_3^1, y_2^1)) = A(\text{conv}(o_3, x_3^1, y_2^1))$. (10)

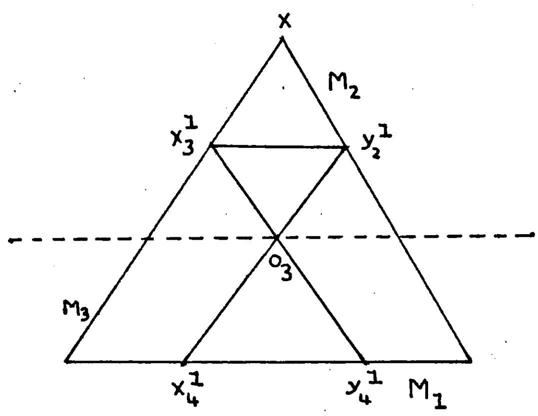
Thus $A(\text{conv}(o_3, x_4^1 y_4^1) = A(\text{conv}(o^1 x_1^1 y_1^1))$ (11)

and

$$A(\text{conv}(o_3, x_3^1, x, y_2^1)) = A(\text{conv}(o^1, x_3^1, x, y_2^1)). \tag{12}$$

Thus by (5), (11) and (12)

$$\frac{A(X_1)}{A(Y_1)} \geq \frac{A(\text{conv}(o_3, x_4^1, y_4^1))}{A(\text{conv}(o_3, x_3^1, x, y_2^1))} . \tag{13}$$



By considering triangle $o_3\ x\ x_3^1$ we have

$$\frac{|o_3-x_3^1|}{\sin\frac{\pi}{6}} = \frac{|x-x_3^1|}{\sin\frac{\Psi_1}{2}} = \frac{1-h_1}{\sin(\frac{\pi}{6}+\frac{\Psi_1}{2})} \quad . \tag{14}$$

Thus

$$|o_3-x_3^1| = \frac{(1-h_1)}{2\sin(\frac{\pi}{6}+\frac{\Psi_1}{2})} \tag{15}$$

and $$|x-x_3^1| = \sin\frac{\Psi_1}{2}\cdot\frac{(1-h_1)}{\sin(\frac{\pi}{6}+\frac{\Psi_1}{2})} \quad . \tag{16}$$

write $\xi_1 = \dfrac{\Psi_1}{2}$.

Then

$$\frac{A(\mathrm{conv}(o_3,\ x_4^1,\ y_4^1))}{A(\mathrm{conv}(o_3,\ x_3^1,\ x,\ y_2^1))}$$

$$= \frac{A(\mathrm{conv}(o_3,\ x_4^1,\ y_4^1))}{A(\mathrm{conv}(o_3,\ x_3^1\ y_2^1)) + A(\mathrm{conv}(x,\ x_3^1,\ y_2^1))}$$

$$= \frac{\frac{1}{2}.\ h_1^2\ \sec^2\xi_1\ \sin 2\xi_1}{\left(\frac{1}{2}\cdot\frac{(1-h_1)^2\sin 2\xi_1}{4\sin^2(\frac{\pi}{6}+\xi_1)} + \frac{1}{2}\cdot\frac{(1-h_1)^2\sin^2\xi_1\cdot\sqrt{3}}{\sin^2(\frac{\pi}{6}+\xi_1)\ .\ 2}\right)}$$

$$= \left(\frac{h_1}{1-h_1}\right)^2 \sin^2\left(\frac{\pi}{6}+\xi_1\right)\left(\frac{2\sin\xi_1\cos\xi_1\sec^2\xi_1}{\frac{1}{2}\sin\xi_1\cos\xi_1+\frac{\sqrt{3}}{2}\sin^2\xi_1}\right)$$

$$= \frac{2\left(\frac{h_1}{1-h_1}\right)^2 \sin^2\left(\frac{\pi}{6} + \xi_1\right)}{\left(\frac{1}{2}\cos \xi_1 + \frac{\sqrt{3}}{2}\sin \xi_1\right)\cos \xi_1}$$

$$= 2\left(\frac{h_1}{1-h_1}\right)^2 \sin\left(\frac{\pi}{6} + \xi_1\right) \cdot \frac{1}{\cos \xi_1}$$

$$= \left(\frac{h_1}{1-h_1}\right)^2 (1 + \sqrt{3}\tan \xi_1) . \tag{17}$$

Now write $\phi_1 = \frac{\theta_1}{2}$.

Then since by (9) $\Psi_1 \geq \theta_1$, we have

$$\left(\frac{h_1}{1-h_1}\right)^2 (1 + \sqrt{3}\tan \xi_1) \geq \left(\frac{h_1}{1-h_1}\right)^2 (1 + \sqrt{3}\tan \phi_1) \tag{18}$$

since $0 < \phi_1 \leq \xi_1 < \frac{\pi}{2}$.

Thus by (13), (17) and (18)

$$\frac{A(X_1)}{A(Y_1)} \geq \left(\frac{h_1}{1-h_1}\right)^2 (1 + \sqrt{3}\tan \phi_1). \tag{19}$$

we have similar inequalities for $\frac{A(X_2)}{A(Y_2)}$ and $\frac{A(X_3)}{A(Y_3)}$ and so

$$\sum_{i=1}^{3} \frac{A(X_i)}{A(Y_i)} \geq \sum_{i=1}^{3} \left(\frac{h_i}{1-h_i}\right)^2 (1 + \sqrt{3}\tan \phi_i) \tag{20}$$

where $\sum_{i=1}^{3} h_i = 1$, $\sum_{i=1}^{3} \phi_i = \frac{\pi}{2}$, $0 \leq h_i \leq 1$, and $0 \leq \phi_i \leq \frac{\pi}{2}$,

for $i = 1, 2$ and 3.

The remainder of the proof is concerned with obtaining a lower bound to the function $f(h_1, h_2, h_3, \delta_1, \delta_2, \delta_3) = \sum_{i=1}^{3} \left(\frac{h_i}{1-h_i}\right)^2 (1+\sqrt{3}\tan\phi_i)$ of the six variables $h_1, h_2, h_3, \phi_1, \phi_2$ and ϕ_3 subject to the conditions $\sum_{i=1}^{3} h_i = 1$, $\sum_{i=1}^{3} \phi_i = \frac{\pi}{2}$, $0 \leq h_i \leq 1$ and $0 \leq \phi_i \leq \frac{\pi}{2}$ for $i = 1, 2$ and 3.

First consider the function

$g = g(h_1, h_2, h_3) = \sum_{i=1}^{3} \frac{h_i}{1-h_i}$ subject to

$\sum_{i=1}^{3} h_i = 1$ and $0 \leq h_i \leq 1$.

$$g = \sum_{i=1}^{3} \frac{h_i}{1-h_i} + \lambda (1 - h_i - h_2 - h_3) \qquad (21)$$

for some multiplier λ.

$$\frac{\partial g}{\partial h_i} = \frac{1}{(1-h_i)^2} - \lambda \qquad \text{for } i = 1, 2 \text{ and } 3 \qquad (22)$$

$\therefore$ At stationary points,

$$\frac{\partial g}{\partial h_i} = \frac{\partial g}{\partial h_j} \quad (i \neq j) \qquad (23)$$

and this implies there is an unique stationary point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Also $\frac{\partial^2 g}{\partial h_i^2} > 0$ and $\frac{\partial^2 g}{\partial h_i \partial h_j} = 0$ for $i \neq j$. (24)

Thus $g(h_1, h_2, h_3)$ takes a minimum at $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

and $g(h_1, h_2, h_3) \geq g(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \frac{3}{2}$. (25)

We now suppose ϕ_1, ϕ_2, ϕ_3 are fixed and define

$$a_i = 1 + \sqrt{3}\tan\phi_i \quad \text{and} \quad k_i = \frac{h_i}{1-h_i} \tag{26}$$

for $i = 1, 2$ and 3.

Then by (25), if $0 \leq h_i \leq 1$ for $i = 1, 2$ and 3 and $\sum_{i=1}^{3} h_i = 1$ then

$$\sum_{i=1}^{3} k_i = \delta \geq \frac{3}{2}. \tag{27}$$

Now consider the function

$$h = h(k_1, k_2, k_3) = \sum_{i=1}^{3} a_i k_i^2 \text{ subject to } \sum_{i=1}^{3} k_i = \delta.$$

$$\hat{h} = \sum_{i=1}^{3} a_i k_i^2 + \mu(\delta - k_1 - k_2 - k_3) \tag{28}$$

for some multiplier μ.

$$\frac{\partial h}{\partial k_i} = 2a_i k_i - \mu. \tag{29}$$

Thus there is an unique stationary point (k_1, k_2, k_3) where

$$k_1 a_1 = k_2 a_2 = k_3 a_3 \tag{30}$$

and $$k_1 + k_2 + k_3 = \delta. \tag{31}$$

Thus (30) and (31) imply

$$k_1\left(1 + \frac{a_1}{a_2} + \frac{a_1}{a_3}\right) = \delta.$$

Thus

$$k_1 = \frac{a_2 a_3 \delta}{a_1 a_2 + a_1 a_3 + a_2 a_3}, \tag{32}$$

$$k_2 = \frac{a_1 a_3 \delta}{a_1 a_2 + a_1 a_3 + a_2 a_3}, \tag{33}$$

$$k_3 = \frac{a_1 a_2 \delta}{a_1 a_2 + a_1 a_3 + a_2 a_3}. \tag{34}$$

Now $\dfrac{\partial^2 h}{\partial k_i^2} = 2a_i > 0$ and $\dfrac{\partial^2 h}{\partial k_i \partial k_j} = 0$ for $i \neq j$. (35)

Thus the stationary point is a minimum.

Thus

$$h(k_1, k_2, k_3) \geq \delta^2 \frac{(a_1(a_2a_3)^2 + a_2(a_1a_3)^2 + a_3(a_1a_2)^2}{(a_1a_2 + a_1a_3 + a_2a_3)^2}$$

$$= \frac{\delta^2 a_1 a_2 a_3}{a_1a_2 + a_1a_3 + a_2a_3}$$

$$\geq \frac{9}{4\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right)} \quad \text{by (27).} \tag{36}$$

Thus for any given ϕ_1, ϕ_2 and ϕ_3 with $0 \leq \phi_i \leq \frac{\pi}{2}$ for $i = 1, 2$ and 3, we have

$$f(h_1, h_2, h_3, \phi_1, \phi_2, \phi_3) \geq \frac{q}{4 \sum_{i=1}^{3} \left(\frac{1}{1+\sqrt{3}\tan\phi_i}\right)} . \qquad (37)$$

we finally consider the function

$$k = k(\phi_1, \phi_2, \phi_3) = \sum_{i=1}^{3} \frac{1}{1+\sqrt{3}\ \tan\phi_i} \quad \text{subject to}$$

$\sum_{i=1}^{3} \phi_i = \frac{\pi}{2}$ and $0 \leq \phi_i \leq \frac{\pi}{2}$ for $i = 1, 2$ and 3.

$$k = \sum_{i=1}^{3} \frac{1}{1+\sqrt{3}\ \tan\phi_i} + \eta\left(\frac{\pi}{2} - \phi_1 - \phi_2 - \phi_3\right) \qquad (38)$$

for some multiplier η.

$$\frac{\partial k}{\partial \phi_i} = \frac{-\sqrt{3}\ \sec^2\phi_i}{(1+\sqrt{3}\ \tan\phi_i)^2} - \eta \qquad (39)$$

$$= -\frac{\sqrt{3}}{4}\ \operatorname{cosec}^2\left(\frac{\pi}{6}+\phi_i\right) - \eta.$$

At stationary points,

$$\operatorname{cosec}^2\left(\frac{\pi}{6}+\phi_i\right) = \operatorname{cosec}^2\left(\frac{\pi}{6}+\phi_j\right) . \qquad (40)$$

Thus $\phi_i = \phi_j$ and there is an unique stationary point $\left(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}\right)$.

Now $$\frac{\partial k}{\partial \phi_i} = -\frac{\sqrt{3}}{4}\ \operatorname{cosec}^2\left(\frac{\pi}{6}+\phi_i\right) - \eta.$$

$$\therefore \quad \frac{\partial^2 k}{\partial \phi_i^2} = \frac{\sqrt{3}}{2}\ \operatorname{cosec}^2\left(\frac{\pi}{6}+\phi_i\right)\ \cot\phi_i$$

$$> 0 \text{ when } \phi_i = \frac{\pi}{6} \text{ for } i = 1, 2 \text{ and } 3. \qquad (41)$$

Also $\dfrac{\partial^2 k}{\partial\phi_i \partial\phi_j} = 0$ for $i \neq j$.

Thus the function $k(\phi_1, \phi_2, \phi_3)$ takes a minimum at $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$.

Now $k(\phi_1, \phi_2, \phi_3)$ is a continuous function defined on the closed and bounded convex domain D given by the intersection of the cube $\{\phi_1, \phi_2, \phi_3 \mid 0 \leq \phi_i \leq \frac{\pi}{2}\}$ with the plane $\phi_1 + \phi_2 + \phi_3 = \frac{\pi}{2}$.

Thus $k(\phi_1, \phi_2, \phi_3)$ attains a bounded maximum over D, which from the above must be attained on the frontier of D.

Thus we may assume

$$\phi_1 = 0 \quad \text{or} \quad \phi_1 = \frac{\pi}{2}.$$

If $\phi_1 = 0$, then

$$k(\phi_1, \phi_2, \phi_3) \leq \max_{\phi_2+\phi_3=\frac{\pi}{2}} k(0, \phi_2, \phi_3). \tag{43}$$

Now since we may apply the arguments above to a function of two variables it follows that $k(0, \phi_2, \phi_3)$ takes a maximum subject to $\phi_2 + \phi_3 = \frac{\pi}{2}$ when either $\phi_2 = 0$ or $\phi_2 = \frac{\pi}{2}$.

Thus $k(\phi_1, \phi_2, \phi_3) \leq k(0, 0, \frac{\pi}{2}) = 2.$ (44)

If $\phi_1 = \frac{\pi}{2}$, then we have immediately that

$$k(\phi_1, \phi_2, \phi_3) \leq k(\frac{\pi}{2}, 0, 0) = 2. \tag{45}$$

Thus in all cases $k(\phi_1, \phi_2, \phi_3) \leq 2$.

Thus by (20), (37) and(46) we have

$$\sum_{i=1}^{3} \frac{A(X_i)}{A(Y_i)} \geq f(h_1, h_2, h_3, \phi_1, \phi_2, \phi_3) \geq \frac{9}{8}.$$

The theorem then is proved.

In our final theorem we look at a function with particular reference to central sets.

Theorem 34

In the same notation as theorem 32, Figure 1, define

$$m(X; L_1, L_2, L_3) = \max_{1 \leq i \leq 3} \frac{A(X_i)}{A(Y_i)} .$$

(i) If X is central with centre c then $m(X; L_1, L_2, L_3) \geq 1$ with equality if $o = c$. If $m(X; L_1, L_2, L_3) = 1$ and $o \neq c$, then $\text{fr } X$ contains a pair a parallel line segments which are parallel to oc.

(ii) If X is not central then there exist lines L_1^*, L_2^* and L_3^* which are coincident in a point $o^* \in X$ for which $m(X; L_1^*, L_2^*, L_3^*) < 1$.

Proof

(i) In the proof of this theorem we shall use the notation of theorem 32, Figure 1. Let M_1, M_2 and M_3 denote respectively the

the lines x_1y_1, x_2y_2 and x_3y_3.

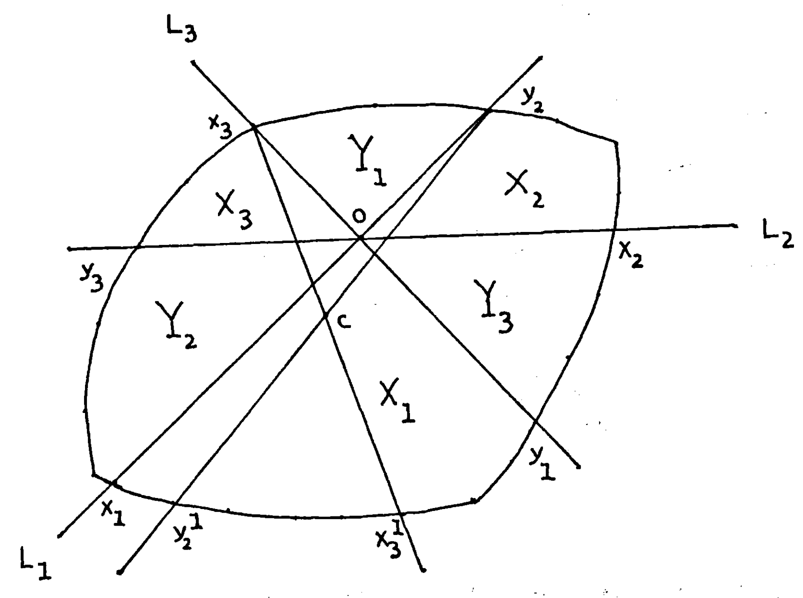
Suppose that X is central with centre c and that there exist lines L_1, L_2 and L_3 coincident in a point o contained in int X for which

$$m(X;\ L_1,\ L_2,\ L_3) < 1. \qquad (1)$$

Then certainly $o \neq c$.

We show first that c does not lie in any region X_i for any i with $1 \leq i \leq 3$.

For suppose this is the case and say $c \in X_1$.



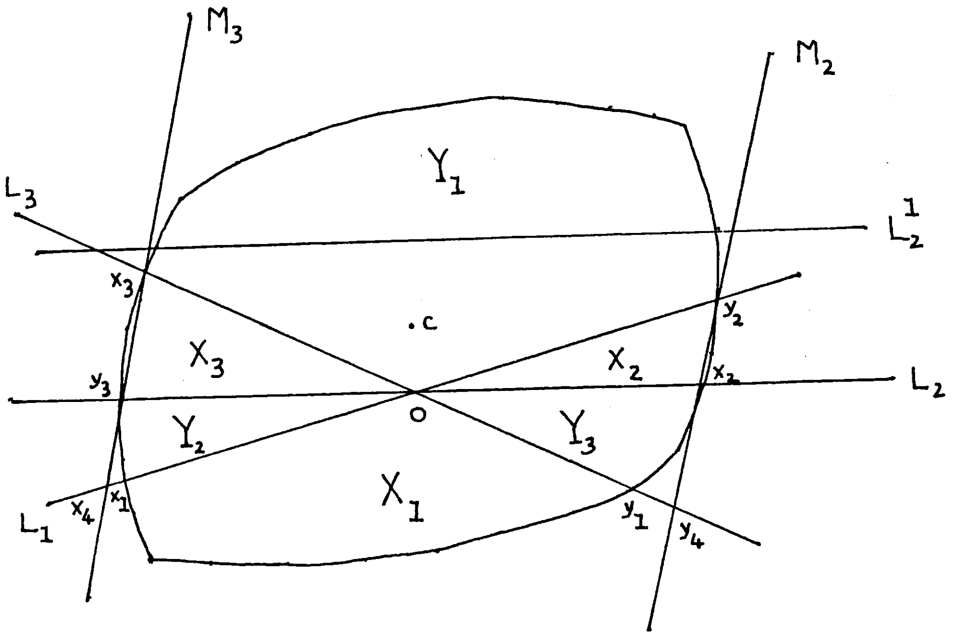
Let the lines x_3c produced and y_2c produced meet fr X in x_3^1 and y_2^1 respectively.

Clearly Y_1 is contained in the set Y_1^1 bounded by cx_3, cy_2 and fr X, and X_1 contains the set X_1^1 bounded by cx_3^1, cy_2^1 and fr X.

But this implies

$$A(Y_1) \leq A(Y_1^1) = A(X_1^1) \leq A(X_1). \tag{2}$$

But (2) is impossible by (1), and so c does not lie in any region X_i.
We may suppose then without loss in generality that $c \in Y_1$.
Let L_2^1 denote the reflection of L_2 in c. We show next that not both x_3 and y_2 lie in the strip bounded by L_2 and L_2^1. For suppose this is the case.



Then M_2 and M_3 meet on the side of L_2 opposite to L_2^1. Let M_2 meet L_3 in y_4 and M_3 meet L_1 in x_4. But then using the same arguments as those given in the beginning of the proof of theorem 33 we have

$$\frac{A(X_2)}{A(Y_2)} + \frac{A(X_3)}{A(Y_3)} \geqslant \frac{A(\mathrm{conv}(o,\ x_2,\ y_2))}{A(\mathrm{conv}(o,\ x_4,\ y_3))} + \frac{A(\mathrm{conv}(o,\ x_3,\ y_3))}{A(\mathrm{conv}(o,\ x_2,\ y_4))} \qquad (3)$$

$$\geqslant 2. \qquad (3)$$

Thus $\max\left(\frac{A(X_2)}{A(Y_2)}, \frac{A(X_3)}{A(Y_3)}\right) \geqslant 1.$ (4)

But by (1), (4) is impossible.

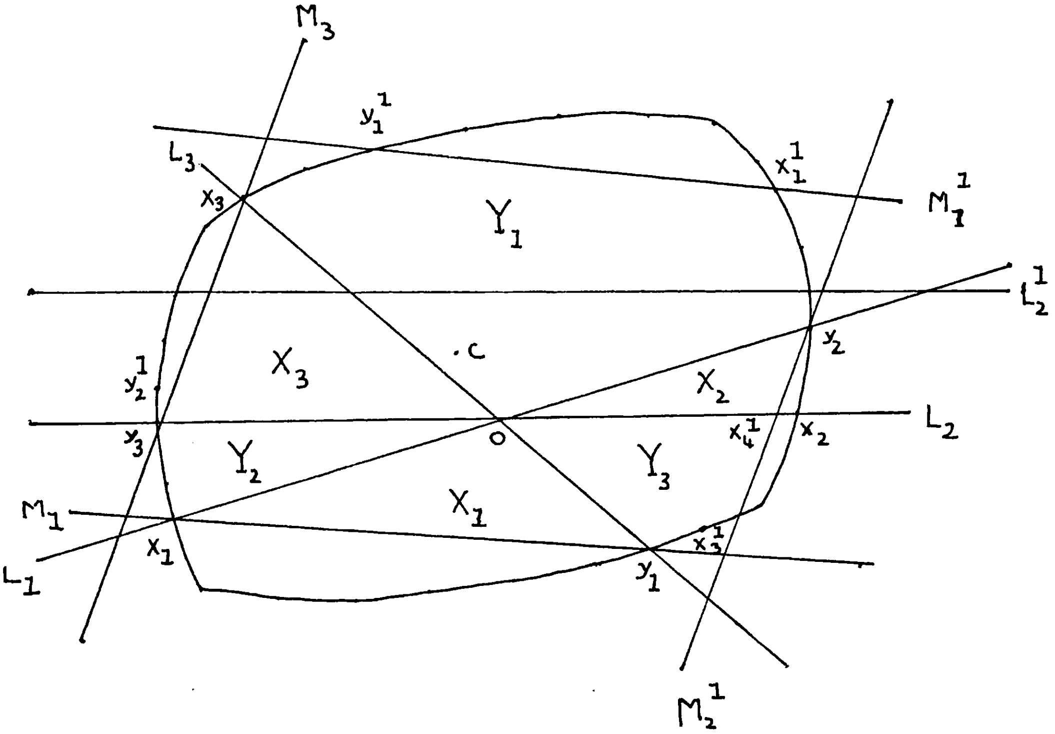
Thus we may assume x_3 lies exterior to the strip, bounded by L_2 and L_2^1. Now let x_3^1 and y_2^1 denote the reflections in c of the points x_3 and y_2 respectively.

Let x_1^1 and y_1^1 denote the reflections in c of the points x_1 and y_1.

We next construct a parallelogram Y for which $m(Y; L_1,L_2,L_3) < 1$. Let H denote the closed strip bounded by L_2 and L_2^1. We consider two cases.

<u>Case I</u>

$x_3 \notin H$ and $y_2 \in H$.



We first note that the lines $x_3y_2^1$ and $y_2x_3^1$ are parallel. Thus the line M_3 through x_3 and y_3 is either parallel to $y_2x_3^1$ or meets the line $y_2x_3^1$ on the side of L_2 opposite to L_2^1. Now the lines M_3 and M_2 must meet on the same side of L_2 as L_2^1, for otherwise we would have as before that

$$\frac{A(X_2)}{A(Y_2)} + \frac{A(X_3)}{A(Y_3)} \geqslant 2. \qquad (5)$$

which is impossible.

Thus there exists a line M_2^1 through y_2 which is parallel to M_3 and which separates x_2 and x_3^1.

Let M_1^1 be the line $x_1^1y_1^1$.

Let Y be the parallelogram bounded by M_1^1, M_3, M_1 and M_2^1. We shall show Y has the required properties.

Let $X_1^1 = conv(o, x_1, y_1)$ and let Y_1^1 be the set bounded by M_1^1, M_3, x_3o, oy_2 and M_2^1.

Since $A(X_1) < A(Y_1)$ it follows that

$$A(X_1^1) < A(Y_1^1). \qquad (6)$$

Let M_2^1 meet L_2 in x_4^1.

Let $X_2^1 = conv(o, y_2, x_4^1)$. Then $X_2^1 \subset X_2$,

Let Y_2^1 be the set bounded by oy_3, M_3, M_1 and x_1o.

Then $Y_2^1 \supset Y_2$.

Thus $A(X_2^1) \leqslant A(X_2) < A(Y_2) \leqslant A(Y_2^1)$. (7)

Now the subset X_3^{11} of X_3 which lies on the side of $x_3y_2^1$ opposite to o has the same area as the subset Y_3^{11} of $X_2 \cup Y_3$ which lies on the side of the line $y_2x_3^1$ opposite to o.

Write $X_3^1 = conv(o, x_3, y_3)$ and let Y_3^1 be the set bounded by ox_4^1, M_2^1, M_1 and y_1o.

We have

$$\begin{aligned} A(X_3^1) &\leqslant A(X_3) - A(X_3^{11}) \\ &< A(Y_3) - A(Y_3^{11}) \\ &\leqslant A(Y_3^1). \end{aligned} \qquad (8)$$

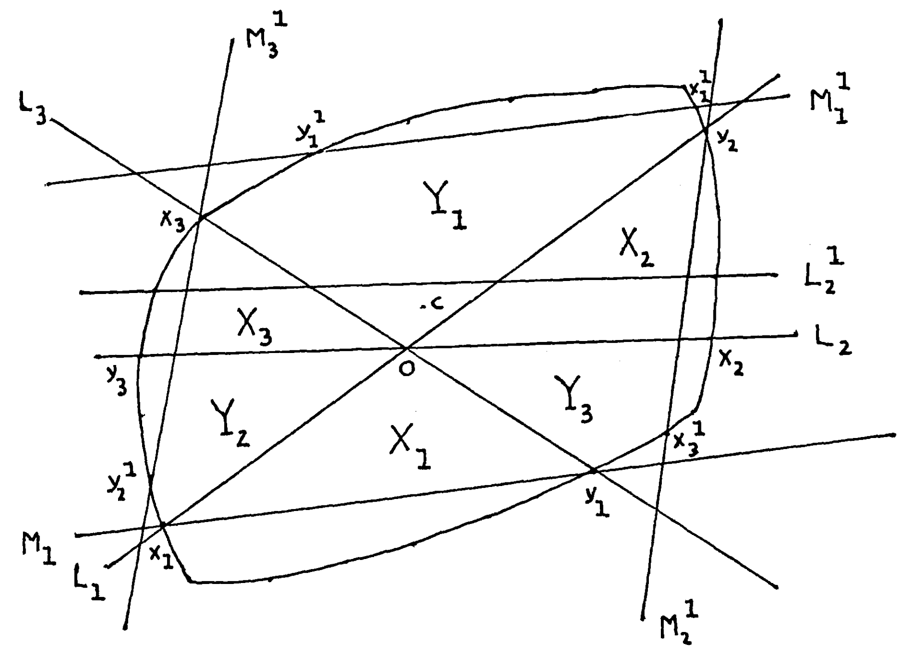
Hence by (6), (7) and (8)

$$m(Y;\ L_1,\ L_2,\ L_3) = \max_{1\le i\le 3} \frac{A(X_i^1)}{A(Y_i^1)} < 1. \tag{9}$$

In order to make the notation consistent with Case II we shall now write $M_3 = M_3^1$ and so Y is the parallelogram bounded by M_1^1, M_3^1, M_1 and M_2^1.

<u>Case II</u>

$x_3 \notin H$ and $y_2 \notin H$.



Let M_1^1, M_3^1, M_1 and M_2^1 be the lines through $x_1^1 y_1^1$, $x_3 y_2^1$, $x_1 y_1$ and $y_2 x_3^1$ respectively.

Let Y be the parallelogram bounded by M_1^1, M_3^1, M_1 and M_2^1.

From the figure above it is easy to see

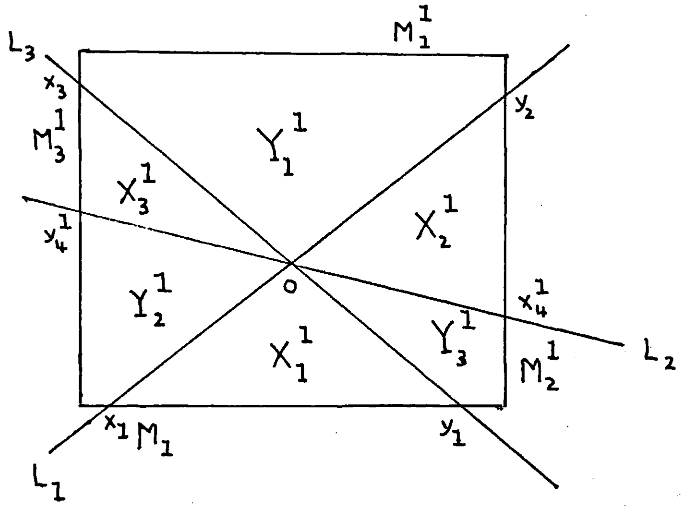
$$m(Y;\ L_1,\ L_2,\ L_3) < 1 \quad \text{if} \quad m(X;\ L_1,\ L_2,\ L_3) < 1. \tag{10}$$

By taking an appropriate affine transformation of the plane we may suppose, in cases I and II, that Y is a rectangle with sides bounded by M_1^1, M_3^1, M_1 and M_2^1 where M_1^1 is parallel to M_1 and M_2^1 is parallel to M_3^1. In viww of the construction of Y we may assume

M_1 meets L_1 and L_3 in x_1 and y_1 respectively,

M_3^1 meets L_3 and L_2 in x_3 and y_4^1 respectively and

M_2^1 meets L_2 and L_1 in x_4^1 and y_2 respectively.



If $X_1^1 = \text{conv}\ (o, x_1, y_1)$, $X_2^1 = \text{conv}(o, y_2, x_4^1)$, $X_3^1 = \text{conv}(o, x_3, y_4^1)$ and Y_i^1 is the region in Y 'opposite' to X_i^1 for $i = 1$, 2 and 3, then we have proved

$$\max_{1 \leq i \leq 3} \frac{A(X_i^1)}{A(Y_i^1)} < 1. \qquad (11)$$

we shall show that (11) is impossible.

Let the distance of o from M_2^1 be μ and the distance of o from M_3^1 be η.

Let Y_3^{11} be the triangle bounded by L_2, M_2^1 and oy_1 produced.

Then $$\frac{A(X_3^1)}{A(Y_3^1)} \geq \frac{A(X_3^1)}{A(Y_3^{11})}$$

$$= \frac{\eta^2}{\mu^2} .$$

Similarly

$$\frac{A(X_2^1)}{A(Y_2^1)} \geq \frac{\mu^2}{\eta^2} .$$

Thus $$\max \left(\frac{A(X_2^1)}{A(Y_2^1)} , \frac{A(X_3^1)}{A(Y_3^1)} \right) \geq 1 \qquad (12)$$

which contradicts (11). Thus the original assumption (1) that there exist lines L_1, L_2 and L_3 for which $m(X; L_1, L_2, L_3) < 1$ was false. Thus if X is central then $m(X; L_1, L_2, L_3) \geq 1$ for all lines L_1, L_2 and L_3 and the first part of (i) is proved.

Now suppose there exists a point $o \in \text{int } X$ which is central with centre c for which

$$m(X; L_1, L_2, L_3) = 1 \qquad (13)$$

and $o \neq c$.

Now as before $c \notin X_i$ for any i and so we may assume $c \in Y_1$. Thus

$$A(X_1) < A(Y_1). \qquad (14)$$

Since $m(X; L_1, L_2, L_3) = 1$ it follows that either $\frac{A(X_2)}{A(Y_2)} = 1$ or $\frac{A(X_3)}{A(Y_3)} = 1$.

We suppose that $\frac{A(X_2)}{A(Y_2)} = 1$. (15)

We show first that this implies $\frac{A(X_3)}{A(Y_3)} = 1$

for suppose $\frac{A(X_3)}{A(Y_3)} < 1$. (16)

We choose a point $o^1 \varepsilon X_2$ and lines L_1^1, L_2^1 and L_3^1 through o^1 parallel to L_1, L_2 and L_3 respectively such that o^1 is near o. But by (14) and (16) it follows that if o^1 is sufficiently close to o then

$$m(X; L_1^1, L_2^1, L_3^1) < m(X; L_1, L_2, L_3) = 1 \qquad (17)$$

which is impossible.

Thus $\frac{A(X_3)}{A(Y_3)} = 1$. (18)

we show next in the notation of theorem 32, Figure 1 that

$$|o-x_1| = |o-y_2|$$

and $|o-x_3| = |o-y_1|$.

For suppose this is not the case and $|o-x_3| \neq |o-y_1|$.

Suppose $|o-x_3| < |o-y_1|$. (19)

We may rotate L_3 through a small angle θ to a new position L_3^1 in the direction so as to increase X_3.

Suppose the regions X_1^1, X_3^1, Y_1^1 and Y_3^1 for L_3^1 correspond to the regions X_1, X_3, Y_1 and Y_3 for L_3.

Then

$$A(X_3^1) = A(X_3) + \tfrac{1}{2}|o-x_3|^2\ \theta + O(\theta^2) \tag{20}$$

and

$$A(Y_3^1) = A(Y_3) + \tfrac{1}{2}|o-y_1|^2\ \theta + O(\theta^2)\ . \tag{21}$$

If θ is sufficiently small then (14), (18), (19), (20) and (21) imply

$$A(X_1^1) < A(Y_1^1) \tag{22}$$

and

$$A(X_3^1) < A(Y_3^1). \tag{23}$$

Thus $1 = m(X;\ L_1,\ L_2,\ L_3^1) = \dfrac{A(X_2)}{A(Y_2)}$

$$> \max\left(\frac{A(X_1^1)}{A(Y_1^1)},\ \frac{A(X_3^1)}{A(Y_3^1)}\right)\ . \tag{24}$$

But by repeating the arguments from equations (15) to (17) we see that (24) is impossible. Thus (19) is impossible.

If $|o-x_3|>|o-y_1|$, then we rotate L_3 in the opposite direction and again obtain a contradiction.

Thus $|o-x_3| = |o-y_1|$, (25)

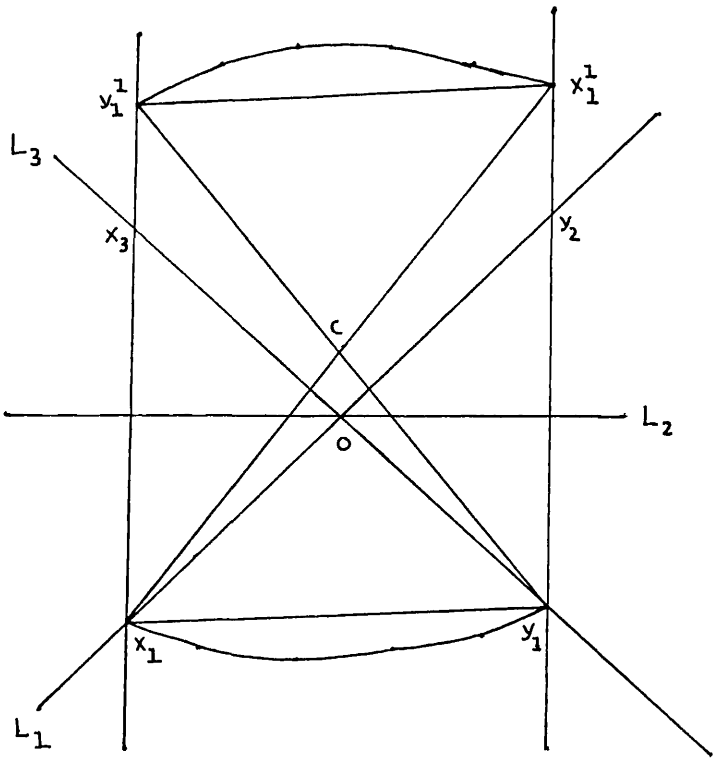
and similarly

$$|o-x_1| = |o-y_2|. \tag{26}$$

Now let x_1^1 and y_1^1 be the reflection of x_1 and y_1 in c.
Let H be the closed strip bounded by the lines x_1x_3 and y_1y_2.
Now (25) and (26) imply that the lines x_1y_1, x_3y_2, $x_1^1y_1^1$ are parallel and

$$|x_1-y_1| = |x_3-y_2| = |x_1^1-y_1^1|. \tag{27}$$

Moreover the line x_3y_2 separates the lines x_1y_1 and $x_1^1y_1^1$. Then convexity implies that the line segment $x_1^1y_1^1$ is contained in H.
Also convexity again implies that $X \subset H$.
Then if $m(X; L_1, L_2, L_3) = 1$ we must have L_2 is parallel to x_1y_1 and also oc is parallel to the line segments $x_1y_1^1$ and $y_1x_1^1$. Thus part (i) is proved.



We finally prove part (ii).

It is well known that X is central if and only if all the lines which bisect the area of X are concurrent. Thus if X is not central then there exist lines K_1, K_2 and K_3 which are not concurrent and which bisect the area of X. Suppose these lines bound a triangle T with vertices x, y and z, where K_1 contains xy, K_2 contains xz and K_3 contains yz.

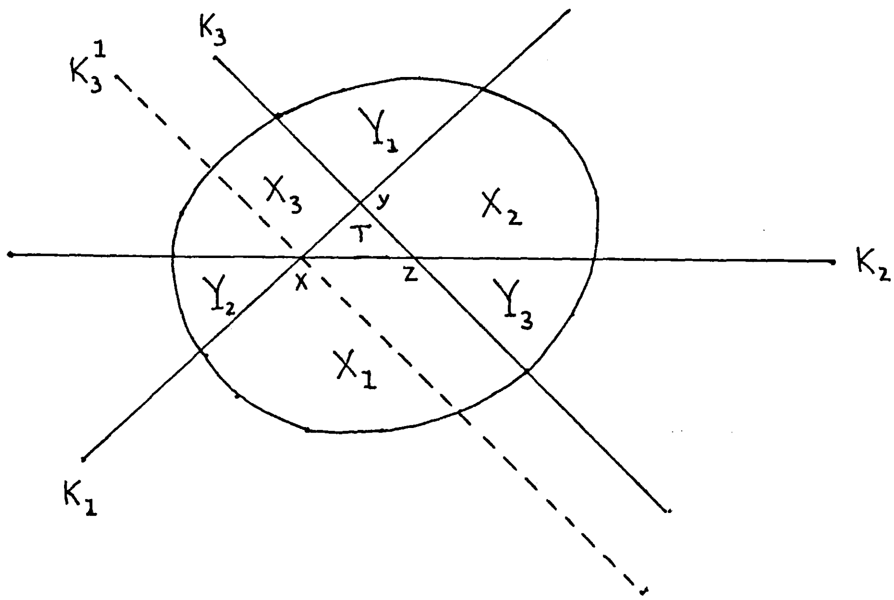
Let Y_1 be the region bounded by K_1, K_3 and fr X, Y_2 be the region bounded by K_1, K_2 and fr X and

Y_3 be the region bounded by K_2, K_3 and fr X.

Suppose $X_i \cup T$ is the region 'opposite' Y_i for i = 1, 2 and 3.

Then

$$A(T) + A(X_i) = A(Y_i) \quad \text{for } i = 1, 2 \text{ and } 3. \tag{28}$$

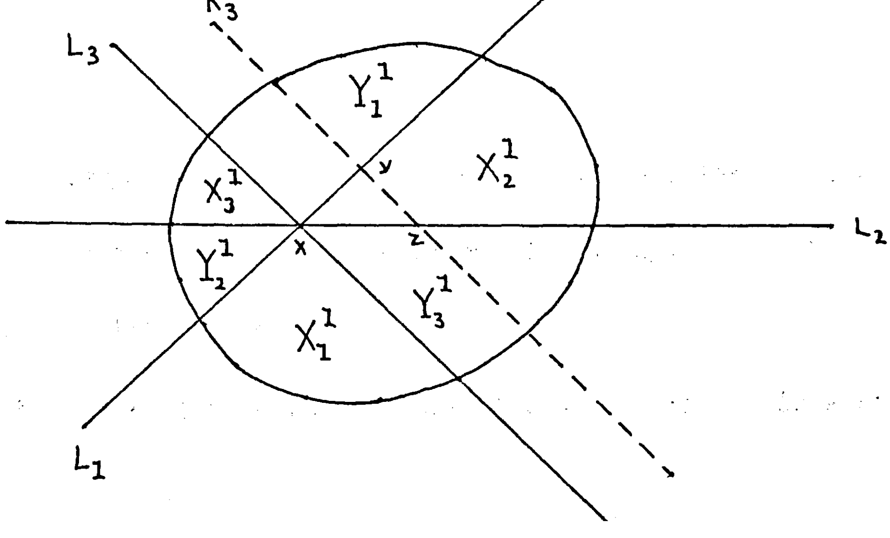


Let K_3^1 be the line through x parallel to K_3.

Suppose K_1, K_2 and K_3^1 divide X into the regions X_1^1, X_2^1, X_3^1 Y_1^1, Y_2^1, Y_3^1 where

$$X_1^1 \subset X_1,\ X_2^1 = T \cup X_2,\ X_3^1 \subset X_3,\ Y_1^1 \supset Y_1,\ Y_2^1 = Y_2 \text{ and } Y_3^1 \supset Y_3.$$

We now write $K_1 = L_1$, $K_2 = L_2$ and $K_3^1 = L_3$.



Clearly then $1 = m(X; L_1, L_2, L_3) = \max_{1 \leq i \leq 3} \dfrac{A(X_i^1)}{A(Y_i^1)}$

$$= \frac{A(X_2^1)}{A(Y_2^1)} \quad \text{by (28)}$$

$$> \max \left(\frac{A(X_2^1)}{A(Y_2^1)}, \frac{A(X_3^1)}{A(Y_3^1)} \right).$$

But now we again apply the argument from equations (15) to (17) and obtain lines L_1^*, L_2^* and L_3^* for which

$$m(X; L_1^*, L_2^*, L_3^*) < 1.$$

The theorem then is proved.

We finish this thesis with a conjecture that seems interesting.

<u>Conjecture</u>

In the notation of theorem 34 define

$$m(X) = \inf_{L_1, L_2, L_3} m(X; L_1, L_2, L_3)$$

for each compact convex set X with a non-empty interior, where the inf imum is taken over all concurrent lines L_1, L_2 and L_3 which meet in the interior of X.

Then $m(X) \geq \frac{1}{2}$ with equality if and only if X is a triangle.

Note

We have shown in theorem 34 that $m(X) = 1$ if and only if X is central. It is not difficult to show that $m(X)$ takes its lower bound when X is a triangle and so the functional m is a measure of symmetry. It seems to be surprisingly difficult however to determine this bound even for a triangle. It follows of course from theorem 33 that $m(X) \geq \frac{3}{8}$.

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