SOME APPLICATIONS OF SET THEORY

TO ALGEBRA

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ABSTRACT

This thesis deals with two topics.

In Part I it is shown that if ZFC is consistent, then so is ZF + the order extension principle + there is an abelian group without a divisible hull. The proof is by forcing.

In Part II a technique is developed which, in many varieties of algebras, enables the construction for each positive integer n of a non-free $\aleph_{\alpha+n}$-free algebra of cardinality $\aleph_{\alpha+n}$ from a suitable non-free $\aleph_{\alpha}$-free algebra, when $\aleph_{\alpha}$ is regular. The algebras constructed turn out to be elementarily equivalent in the language $L_{\infty\aleph_{\alpha+n}}$ to free algebras in the variety.

As applications of the technique, it is shown that for any positive integer n there are $2^{\aleph_n}$ $\aleph_n$-free algebras which are generated by $\aleph_n$ elements, cannot be generated by fewer than this number and are $L_{\infty\aleph_n}$-equivalent to free algebras in each of the following varieties: any torsion-free variety of groups, all rings with a 1, all commutative rings with a 1, all K-algebras (with K a not-necessarily commutative integral domain), all Lie algebras over a given field.

By a different analysis it is shown too that in any variety of nilpotent groups, a λ-free group of uncountable cardinality λ is free (respectively, equivalent in $L_{\infty\lambda}$ to a free group) if and only if its abelianisation is, in the abelian part of the variety.

Finally, sufficient conditions are given for a λ-free group in a

variety of groups to be also parafree in the variety. The results imply that in the varieties of all groups soluble of length at most k and of all groups polynilpotent of given class, if λ is singular or weakly compact, then a λ-free group of cardinality λ is parafree, while if λ is strongly compact, then a λ-free group of any cardinality is parafree.

für Inge,

der ich mehr schulde, als ich hoffen kann, je gut zu machen

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INTRODUCTION

This thesis is about set theory and algebra.

In Part I a question which has a "logical" flavour is considered. It is an instance of the more general question: how effective are the common constructions of algebra? The construction considered here is that of the divisible hull of an abelian group. It had been shown (in /Hodges1/) that ZF alone is not sufficient to establish the existence of divisible hulls. That proof involved the construction by forcing of a model of ZF in which choice failed badly and there was an abelian group without a divisible hull. By a similar construction, it is shown in Part I that there is a model of ZF in which there is an abelian group without a divisible hull and the order extension principle holds. This means a "moderately strong" axiom of choice can hold while there is an abelian group without a divisible hull. The natural question of whether the Boolean prime ideal theorem holds in the model constructed is not settled. My guess is that it does hold; it is in any case a notoriously difficult thing to prove in models in which choice fails.

In Part II the methods used and the questions considered are more algebraic in flavour. For the purposes of this introduction, let us say that an algebra in a variety is almost free if it has the property that every subalgebra of cardinality less than that of the given algebra is contained in a free subalgebra. (This definition will be widened later.) The main theme in Part II is: when are almost free algebras free?

Historically, this question was first considered mainly for the variety of all groups and the variety of all abelian groups, and there are constructions by Higman and Hill of non-free almost free groups in these varieties. The methods used were principally algebraic. It was in /Eklof1/ that it was recognised that methods from logic were applicable here and in /Eklof2/, using the concept of stationary set , it was shown that if κ is a regular cardinal and there is an almost free but non-free abelian group of cardinality κ, then there is one of cardinality κ^+. Since the additive group of the rationals is almost free but not free, this showed that there are almost free but not free abelian groups in all infinite cardinalities up to the first limit cardinal. In his PhD thesis, Mekler extended this to the variety of all groups, and there have also been extensions to modules.

In a different direction, in /Shelah1/ it was shown that in any variety of algebras with the property that every subalgebra of a free algebra is free (called Schreier varieties), any almost free algebra of singular cardinality is free. Again the methods used are more set-theoretic than algebraic (and in fact a much more general theorem is proved - see also /Hodges2/).

The results of Part II are concerned with non-Schreier varieties. It is shown in §3 that, with some restrictions on the variety (principally that factoring out by a subalgebra should make sense), if there is a sufficiently nice almost free non-free algebra of regular cardinality κ, then there is one of cardinality κ^+. This is applied in §§4,5 to an arbitrary torsion-free variety of groups and also to varieties of rings, K-algebras and Lie algebras. The results suggest that the algebraic content of the proofs is not very great and that there is a model-theoretic concept still to be found which subsumes the notion of freeness and for which the proofs could be carried out using only set-theoretic machinery.

On the question of whether Shelah's theorem on singular cardinals holds for non-Schreier varieties, the results of §§6,7 combine to show that a counterexample cannot be found in any nilpotent variety of groups. (This was known already (/Hodges2/), but we give a different proof.) More than this is shown: in §6 it is proved that these varieties behave exactly as their abelian parts do with respect to questions of almost-freeness. The methods of §6 rely heavily on the concept of purity, which although it originated in abelian group theory has recently found a more suitable home in model theory.

The final section centres on the concept of parafreeness in a variety of groups. Here it is shown that with certain conditions on λ and the variety, a λ-free group of cardinality λ must be parafree.

All of the theorems in Parts I and II which are not attributed to someone else are original work of the author. In some places it is the ideas, rather than the results, which should be so attributed. I have tried to make clear in the text where I have used someone else's ideas.

There is a third part of the thesis that has not been mentioned yet: Part 0 is a review of notation and terminology as well as some topics the reader is assumed to be familiar with. None of the theorems in Part 0 is due to the author.

As the reader will already have noticed, references to other works are marked off by /. The end of a proof is indicated by //. The empty set is denoted by 0.

Internal references to lemmas, theorems and corollaries are given as z, y.z or x.y.z. Just z means the zth of the current section, y.z means the zth of §y of the current Part and x.y.z means the zth of §y of Part x. References to another section of the same Part are written §y, while a reference to §y of Part x is written §x.y.

It is said elsewhere, but will bear repetition here, that the axiom of choice is used in Part 0 only in those sections whose title is followed by (AC); in Part I, only in §2, except where indicated in the text; and in Part II, throughout, without mention.

Finally, there are many people I would like to thank, but I shall only mention a few: my supervisor, Wilfrid Hodges, for his help and guidance, and directing me towards the problems considered; Mark Roberts, without whose incredulity this thesis would have been much longer (and falser); Harold Simmons, for making his time so generously available to me; and my wife, Inge, without whose support this would never have existed.

Despite the generous and helpful criticisms of the above people, such faults as remain are of course entirely due to me.

PART 0

PRELIMINARIES

In this Part, some notation is explained and an outline is given of what is assumed. (AC) indicates Choice is assumed.

§1 Set theory

Our main source of information on set theory is the book /Jech2/. Concepts and notations in this area that we do not define may be found there. The set theory we shall use is ZF or ZFC. A ZF-formula is a formula of the language of set theory (/Jech2/, pp2-3).

§2 Functions

Let $f: X \longrightarrow Y$ and $Z \subseteq X$. Generally we shall follow algebraic practice and write $f(Z)$ for

$$\{ y \in Y : \exists z \in Z \ \ y = f(z)\} ,$$

but we shall also from time to time use the notation f"Z for this set, to avoid ambiguity.

The restriction of f to Z is written $f|Z$.

If $A \subseteq Y$, the inverse image of A under f is written $f^{-1}(A)$, that is,

$$f^{-1}(A) = \{ x \in X : \exists y \in A \ \ y = f(x)\} .$$

If $A = \{a\}$, then we write $f^{-1}(a)$ instead of $f^{-1}(\{a\})$.

If f is onto Y, we write $f: X \twoheadrightarrow Y$, and if f is 1-1, $f: X \rightarrowtail Y$.

The notation ${}^{X}Y$ always represents the set of all functions from X into Y, but we shall sometimes write Y^X for this set, in circumstances where this is more usual and no ambiguity will occur as a result.

§3 Powers and sequences

The power set of X is written PS(X).

A <u>sequence</u> of members of X is a function s from an ordinal, called the <u>length</u> of s (written length(s)), to X. The set of all finite sequences of members of X is written Seq(X), and also sometimes ${}^{<\omega}X$. Sequences are written with round brackets, thus: $(x_0, x_1, \ldots)$. The notation $\vec{s}$ indicates that $\vec{s}$ is a sequence. If s and t are sequences, then their concatenation is written $s^\frown t$.

§4 Relative constructibility

Let X be a set and T' its transitive closure (/Jech2/, p71). Put $T = \{X\} \cup T'$. By L(X) we mean L(T), obtained by the following inductive definition ((15.13), p131 of /Jech2/):

$$L_0(T) = 0;$$

$$L_\delta(T) = \bigcup_{\alpha<\delta} L_\alpha(T) \qquad \text{if } \delta \text{ is a limit ordinal;}$$

$$L_{\alpha+1}(T) = \bigcup_{A\in T} \mathrm{def}_A(L_\alpha(T));$$

$$L(T) = \bigcup_{\alpha\in \mathrm{Ord}} L_\alpha(T).$$

Here, $\mathrm{def}_A(U)$ is the set of all subsets of U definable over the model $(U; \in \cap U^2, A)$, and "definable over" means what it does in /Jech2/, p82. (See also ex 15.1 on p128 of /Jech2/.) In line three of this definition one takes of course $U = \quad L_\alpha (T)$.

Then L(X) is a model of ZF and is the least transitive model containing X as a subset and all the ordinals.

If X = 0 in this definition, it is omitted, and the <u>constructible universe</u>, L, is obtained. It is well-known that L is a model of <u>strong choice</u>, that is, that there is a definable class which is a bijection between L and Ord. Since L(X) need not be a model of the axiom of choice, we cannot hope for this to hold for L(X), but as we indicate below, L(X) does share some of the properties of L.

It is well-known that there a class definable from T which is a surjection of $\mathrm{Seq}(T) \times \mathrm{Ord}$ onto L(X). We shall consider only L(X) and for this X we shall be able to get a slight improvement in the properties of the surjection. The point is that it will turn out to be definable from X. Here, and in the following theorem, "definable from X" means definable with X as a parameter, as in /Jech2/, p3.

<u>Theorem 1</u>: Let X be a set of ordered triples of members of $PS(\omega)$. Then there is a surjection

$$F: \mathrm{Seq}(X) \times \mathrm{Ord} \twoheadrightarrow L(X)$$

which is definable from X and absolute for transitive models.

Proof: (in outline): We sketch first the construction of such an F which is definable from the transitive closure, T, of X.

There is a definable bijection between ω and the set of all ZF-formulas (a Gödel numbering) and there is too a definable bijection between κ and Seq(κ) for each infinite cardinal κ.

The definition of F is by transfinite induction. Put $F(0,0) = 0$, and let α be an ordinal. Suppose that for each $\beta < \alpha$, for all $x \in L_\beta(T)$ there is $(s_x, \gamma_x) \in Seq(T) \times Ord$ such that $(s_x, \gamma_x) \longmapsto x$ defines a function. If α is a limit ordinal, it is clear how to proceed, so suppose $\alpha = \beta + 1$. If $x \in L_\beta(T)$, then there is a formula $\theta(v, x_1, x_2, \ldots, x_n)$ with just v free and $x_1, x_2, \ldots, x_n \in L_\beta(T)$, and there is $A \in T$ such that

$$x = \{y \in A : (A; \varepsilon \cap A^2, L_\beta(T) \cap A) \models \theta[y, x_1, x_2, \ldots, x_n]\}.$$

Now $x_1, x_2, \ldots, x_n$ are by induction $F(s_1, \alpha_1), F(s_2, \alpha_2), \ldots, F(s_n, \alpha_n)$ for some $s_1, s_2, \ldots, s_n$ in Seq(T) and some ordinals $\alpha_1, \alpha_2, \ldots, \alpha_n$. Form the new sequences

$$s = (A)^\frown s_1{}^\frown s_2{}^\frown \cdots {}^\frown s_n,$$

and $$\vec{\gamma} = (f, \alpha_1, \alpha_2, \ldots, \alpha_n),$$

where f is the Gödel number of θ.

Let γ correspond to $\vec{\gamma}$ under the bijection $\kappa \longleftrightarrow Seq(\kappa)$, where $\kappa = |\max(\omega, \alpha_1, \alpha_2, \ldots, \alpha_n)|^+$, and put $F(s, \gamma) = x$.

Since we have used bijections, it is clear that x is indeed recoverable from (s, γ), and so this does define a function. We can expand the domain if necessary to all of $Seq(T) \times Ord$ by putting $F(s, \gamma) = 0$ if it has not been given a value by the above process.

So what we have done so far is to produce an F satisfying the conclusion of the theorem with T in place of X. Since ω is definable and $X \subseteq PS(\omega)^3$, T has a particularly simple form, and turns out to be definable from X. Hence Seq(T) is definable from X and it is easy to see that the above construction can be modified to produce F of the required form. //

§5 Forcing

We assume familiarity with all the relevant parts of /Jech2/, especially §§16-19.

§6 The axiom of choice

The following abbreviations will be used:

AC: the axiom of choice

BPI: the Boolean prime ideal theorem: every Boolean algebra contains a proper prime ideal

OEP: the order extension principle: every partial order can be extended to a linear order

OP: the order principle: every set can be linearly ordered

ACF: every set of finite sets has a choice function.

The main theorem is:

Theorem 1: In ZF, each member of the above list implies those below it, and none of the implications can be reversed.

Proof: The implications are all proved in /Jech1/, as are all the non-implications except OEP $\not\longrightarrow$ BPI, which is to be found in /Felgner/. //

The reason for introducing this theorem is that it gives a scale against which to measure how badly choice fails in a model of ZF. Of course, it is a crude scale.

Each of the axioms listed above (AC,BPI, etc) has the form:

$$\forall x \exists y \quad \Phi(x) \to \Psi(x,y).$$

The global form of such an axiom is the assertion:

there is a definable proper class C which is a function such that $\forall x \quad \Phi(x) \to \Psi(x,C(x))$.

Thus C allows a uniform choice to be made. Strong choice is the global form of AC and, as remarked before, holds in L.

§7 Stationary sets (AC)

Let δ be an ordinal of regular uncountable cardinality κ. The set C is closed in δ iff the supremum of any subset of C is either δ or a member of C; C is unbounded in δ iff δ is the supremum of C. We say that C is a club in δ iff C is closed and unbounded in δ.

The set of clubs in δ is a filter in the Boolean algebra PS(δ) and this filter is closed under intersections of fewer than κ members (that is, is κ-complete). The ideal dual to this filter is called the ideal of non-stationary sets (in δ), denoted by NS_δ. A subset S of δ is said to be stationary (in δ) iff $S \notin NS_\delta$. Thus S is stationary in δ iff S has non-empty intersection with every club in δ.

If X,Y are subsets of δ, then we write $X \equiv Y \bmod NS_\delta$ iff the symmetric difference of X and Y is a member of NS_δ. This is an equivalence relation on $PS(\delta)$ and we denote the equivalence class of X by $\tilde{X}$. It follows easily from the definitions that $X \equiv Y \bmod NS_\delta$ iff there is a club C in δ such that $C \cap X = C \cap Y$.

§8 Some large cardinals (AC)

We give here a few definitions and point out a few well-known consequences. The survey /Kanamori & Magidor/ contains proofs and much more besides.

A cardinal κ is <u>weakly compact</u> iff whenever S is a set of sentences in the language $L_{\kappa\kappa}$ involving at most κ non-logical constants and every subset of S of cardinality less than κ has a model, S too has a model. A weakly compact cardinal is a regular limit cardinal.

A cardinal κ is <u>strongly compact</u> iff κ satisfies the above definition with the restriction on the number of non-logical constants occurring in S deleted.

Let $j:V \to M$ be an elementary embedding of the universe of all s sets into a standard model M. (We are assuming AC.) If j is not the identity then j must move some ordinal, and the least such is called the critical point of j. If $\kappa < \lambda$ are cardinals, we say κ is <u>λ-compact</u> iff there is an elementary embedding j of the universe V into a standard model M such that κ is the critical point of j and for all $X \subseteq M$ if $|X| < \lambda$, then there is $Y \in M$ so that $X \subseteq Y$ and $M \models |Y| < j(\kappa)$.

Then κ is strongly compact iff κ is λ-compact for all $\lambda > \kappa$.

We note finally that these are all large cardinals and if ZFC is consistent then the existence of cardinals of these types cannot be proved (or even proved to be consistent) in ZFC.

§9 Abelian groups

Groups will appear from time to time in what follows. In Part I, these groups will all be abelian and additive notation will be used.

The additive groups of the integers and the rationals will be written $\mathbb{Z}$ and $\mathbb{Q}$ respectively, while $\mathbb{Z}_n$ denotes the cyclic group of order n, for n a positive integer; $\mathbb{Z}(p^\infty)$, for p a prime, denotes the abelian group generated by $\{x_i : i<\omega\}$ subject to the relations $\{px_{i+1} = x_i : i < \omega\}$. The group $\mathbb{Z}(p^\infty)$ is divisible and directly indecomposable.

The standard reference /Fuchs/ is our source for most of the information about abelian groups that we shall require and unexplained terminology is to be found there.

With the axiom of choice it is well-known that any finitely-generated abelian group is a direct sum of cyclic groups. In fact, AC is not necessary for this:

<u>Theorem 1</u>: The following is a theorem of ZF:

each finitely-generated abelian group is the direct sum of finitely many cyclic groups.

Proof: The two proofs given on pp78-9 of /Fuchs/ do not use the axiom of choice. //

In Part I, if A is an abelian group and X is a subset of A, the notation $\langle X \rangle$ always means the subgroup of A generated by X, but we sometimes prefer gp(X) or $gp_A(X)$ for this, to avoid confusion with the common set-theoretic practice of writing, for example, $\langle x,y \rangle$ for the ordered pair of x and y (which we write (x,y)).

§10 Some universal algebra (AC)

In Part II we shall deal with varieties of algebras. For us an algebra is an algebra in the sense of /Cohn1/, that is, a pair (A, Ω) where A is a set and Ω is a sequence of finitary operations on A. We refer to Ω as the similarity type of the algebra (A, Ω), but since this is almost always understood in the context, we usually just write A for the algebra (A, Ω). A variety of algebras is a class of algebras of the same similarity type which is closed under taking homomorphic images, subalgebras and direct products. A class of algebras of the same similarity type is a variety iff there is a set of identities (called the laws of the variety) such that an algebra is in the class iff it satisfies all the identities. (See /Cohn1/, ch IV.)

If $\underline{C}$ is a class of algebras of the same similarity type and X is a subset of the algebra A in $\underline{C}$ such that for every B in $\underline{C}$ and every map $f: X \longrightarrow B$ there is a unique homomorphism $f^*: A \longrightarrow B$ extending f, then we say X is a basis of A, or X is a free generating set for A. We say A is free in $\underline{C}$ iff A has a basis.

If X is a basis of A, then we say A is free on X. Note that the concept of freeness and basis depend on $\underline{C}$.

A subalgebra B of the algebra A in $\underline{C}$ is said to be a free factor of A iff A has a basis X such that there is a subset Y of X with the property that B is the subalgebra of A generated by Y. Again this depends on $\underline{C}$. Note that both A and B must be free if B is a free factor of A, and that if X, Y are as above, $Y = B \cap X$.

We shall be concerned with questions related to freeness. We shall however treat only varieties. The reason for this is that an algebra is free in some class $\underline{C}$ of algebras iff it is free in the variety generated by $\underline{C}$. (Compare /Cohn1/,IV.3.7, 3.11.) Let us note too that in any variety of algebras, there is a free algebra on any set. (See /Cohn1/, III.5.3.) Finally, let us note that a subset X of an algebra A in a variety $\underline{V}$ is a basis of A iff X generates A and every relation between the members of X is a law in $\underline{V}$. (See /Cohn1/, ch IV.)

§11 Purity

If A is an abelian group, then the subgroup B of A is pure in A iff for all integers n and all $b \in B$ if there is $a \in A$ such that $na = b$, there is $b' \in B$ such that $nb' = b$. Thus any finite system of equations with parameters from B which can be solved in A can also be solved in B. (See /Fuchs/, ch V.) This notion can be extended to arbitrary algebras, as follows.

Suppose B is a subalgebra of the algebra A. Then B is pure in

A iff every finite system of equations with parameters from B which can be solved in A can also be solved in B. We note that the union of a chain of pure subalgebras is a pure subalgebra, provided only that it is an algebra.

§1.2 Varieties of groups

The class of all groups is a variety; we denote it by <u>Gps</u>. If U and V are subvarieties of <u>Gps</u>, then $U \cap V$ denotes the class of all groups which belong to both U and V; it is a variety. The following names will be used:

- <u>Abgps</u>: the variety of all abelian groups (abbreviated to <u>Ab</u> in §II.6);
- $\underline{B}(p^n)$: the variety of all groups in which $x^{p^n} = 1$ is a law, where p is a prime and n a positive integer (B for Burnside);
- $\underline{Ab}(p^n)$: $\underline{Abgps} \cap \underline{B}(p^n)$;
- $\underline{N}_c$: the variety of all groups nilpotent of class at most c, a positive integer; defining law:

  $$[x_0, [x_1, [\ \dots [x_{c-1}, x_c] \dots]\ ]\ ] = 1,$$

  where $[x,y] = x^{-1}y^{-1}xy$.

A variety of groups has <u>exponent 0</u> iff $x^n = 1$ is not a law for any positive integer n, and <u>exponent $n > 0$</u> iff n is the least positive integer such that $x^n = 1$ is a law. So $\underline{B}(p^n)$ has exponent p^n. We also say a variety is <u>torsion-free</u> if it has exponent 0; a variety is torsion-free iff its free groups are.

Subvarieties of <u>Abgps</u> are referred to as <u>abelian</u> varieties; subvarieties of $\underline{N}_c$ are referred to as <u>nilpotent</u> varieties.

§13 The lower central series of a group (AC)

If G is a group and H is a subgroup of G, then $[G,H]$ is the subgroup of G generated by all the <u>commutators</u>

$$[g,h] = g^{-1}h^{-1}gh \quad \text{with } g \in G,\ h \in H.$$

The <u>lower central series</u> of G is the sequence $G^{(i)}$, i a non-negative integer, defined by:

$$G^{(0)} = G, \quad G^{(i+1)} = [G, G^{(i)}] .$$

The subgroup $G^{(1)}$ is called the <u>derived</u> subgroup of G, and we usually write G' for it. It will play an important role in §II.6. Of course G/G' is abelian, and is called the <u>abelianisation</u> of G, written G^{ab}. Abelianisation is a functor and we have:

<u>Lemma 1</u>: If H is a pure subgroup of the group G, then H^{ab} is naturally embeddable as a pure subgroup in G^{ab}.

Proof: Clearly $H' \subseteq G' \cap H$. Suppose $h \in G' \cap H$. Then there are $g_1, g_2, \ldots, g_{2n}$ in G such that

$$h = [g_1,g_2]\ [g_3,g_4]\ \cdots\ [g_{2n-1},g_{2n}] .$$

Now, this is an equation with the parameter $h \in H$, soluble (by g_1, g_2, etc) in G. Since H is pure in G, there are $h_1, h_2, \ldots, h_{2n}$ in H such that

$$h = [h_1,h_2]\ [h_3,h_4]\ \ldots\ [h_{2n-1},h_{2n}] ,$$

and hence $h \in H'$. Thus $G' \cap H \subseteq H'$ and $H' = G' \cap H$.

Hence $H/H' = H/G' \cap H$, and since this is naturally isomorphic to the subgroup HG'/G' of G/G', we are done if we show that HG'/G' is a pure subgroup of G/G'.

Suppose $g^nG' = hG'$ with $g \in G$, $h \in H$ and $n \in \mathbb{Z}$. Then there are $g_1, g_2, \ldots, g_{2n}$ in G such that

$$g^n = h\,[g_1,g_2]\,[g_3,g_4]\,\ldots\,[g_{2n-1},g_{2n}],$$

and as above there are $h_0, h_1, \ldots, h_{2n}$ in H such that

$$h_0{}^n = h\,[h_1,h_2]\,[h_3,h_4]\,\ldots\,[h_{2n-1},h_{2n}].$$

Hence

$$h_0{}^nG' = hG',$$

and HG'/G' is pure in G/G'. //

§14 Nilpotent and residually nilpotent groups (AC)

If the lower central series of the group G terminates, that is, if $G^{(i)} = (1)$ for some i, we say that G is <u>nilpotent</u>; G is <u>nilpotent of class</u> c iff G is nilpotent and c is least such that $G^{(c)} = (1)$. We say G is <u>residually nilpotent</u> iff $\bigcap_{i<\omega} G^{(i)} = (1)$. The (absolutely) free groups are residually nilpotent.

The following lemma is well-known. It follows from 31.23 and 31.24 of /Neumann/.

<u>Lemma 1</u>: If G is a nilpotent group and H a subgroup such that $HG' = G$, then $H = G$. //

Putting together 32.21, the proof of 32.22 and 42.35 of /Neumann/ gives:

<u>Theorem 2</u>: Suppose V is a nilpotent torsion-free variety of groups, and let F be free in V. If S is a subset of F which generates freely modulo F' a free abelian subgroup of F^{ab}, then S generates freely a V-free subgroup of F. //

In the same vein, the following appears as 42.31 of /Neumann/:

Theorem 3: Let V be a variety of groups whose free groups are residually nilpotent. If S is a subset of the free (in V) group F that generates ($\underline{\text{Abgps}} \cap V$)-freely modulo F' a direct factor of F/F', then S generates freely a free (in V) subgroup of F. //

Finally we record the following, taken from /Baumslag3/, p2:

Theorem 4: Let G be a nilpotent group such that there is a positive integer n such that $x^n = 1$ for all x in G. Let m be the least value of n for which this holds and suppose $p_1, p_2, \ldots, p_k$ are the distinct primes dividing m. Then for each i there is a unique p_i-Sylow subgroup S_i and, furthermore,

$$G = S_1 \times S_2 \times \ldots \times S_k. \qquad //$$

§15 Infinitary equivalence (AC)

We assume the reader is familiar with the language $L_{\infty\kappa}$ and knows what $L_{\infty\kappa}$-elementary equivalent structures are, where κ is an infinite cardinal. If not, the following may be taken as a definition.

If A and B are algebras of the same similarity type and J is a family of isomorphisms of subalgebras of A onto subalgebras of B, then J has the <u>$<\kappa$-back-and-forth property</u> iff whenever $f \in J$ and X (respectively, Y) is a subset of A (respectively, B) of cardinality less than κ, there is $g \in J$ such that g extends f and $X \subseteq \text{domain}(g)$ (respectively, $Y \subseteq \text{range}(g)$).

A proof of the next theorem may be found in /Kueker1/, p53, for example.

<u>Theorem 1</u>: If A and B are algebras of the same similarity type, then A and B are $L_{\infty\kappa}$-elementary equivalent iff there is a family of isomorphisms of subalgebras of A onto subalgebras of B which has the $<\kappa$-back-and-forth property. //

PART I

DIVISIBLE HULLS

AND

THE ORDER EXTENSION PRINCIPLE

§0 Introduction

In this Part a model of set theory is constructed in which there is an abelian group with no divisible hull, and the order extension principle holds.

The axiom of choice is not used in this Part except where this is made explicit. All groups mentioned in this Part are abelian.

In §1 the background from algebra is given, and in §2 the algebra which will be needed in the subsequent forcing construction is presented. The axiom of choice is assumed in §2. The forcing construction is carried out in §3. The main points of §3 are to be found in theorems 6,7 and 8.

In §4 it is shown that the order extension principle holds in the model of §3 and in §5 it is shown that there is a group in the model which does not have a divisible hull.

The whole Part may be summed up by saying that it is a proof that without the axiom of choice the order extension principle is not strong enough to construct divisible hulls.

§1 The algebraic background

In this section it is explained what a divisible hull is and how this is connected with the axiom of choice. Recall the convention that all groups in Part I are abelian. In this section the axiom of choice is used only where indicated by a vertical line in the left margin.

The group A is <u>divisible</u> iff for all $a \in A$, $n \in \mathbb{Z}$ there is $x \in A$ such that $nx = a$.

We sketch the classical development (that is, with AC) of divisible hulls.

The following propositions are proved in /Fuchs/, ch IV:

(1) a divisible subgroup of a group is a direct summand;

(2) if B is a subgroup of A and D is a divisible subgroup of A with $B \cap D = (0)$, then a complement C of D may be chosen so that $B \leqslant C$ (and $A = D \oplus C$);

(3) every group can be embedded in a divisible group;

(4) every divisible group containing the group A contains a group which is minimal with respect to the property of being divisible containing A;

(5) any two minimal divisible groups containing the group A are isomorphic over A.

Of course the axiom of choice is used in /Fuchs/, so these are theorems of ZFC. Let us consider briefly how (4) is obtained from (1)-(3). Let D be divisible and contain A. (Such exists by (3).) The set of subgroups of D that are divisible and have trivial intersection with A is inductive and hence, by AC, has a maximal member, M, say. By (2), there is $E \leqslant A$ such that $D = M \oplus E$. Clearly E is divisible, and by (1) and the maximality of M, E contains no proper direct summand. Thus E is minimal divisible containing A.

Before sketching the proof of (5), let us introduce a convenient concept. The subgroup A of the group B is <u>essential</u> iff for all $x \in B$ if $x \neq 0$, then $\langle x \rangle \cap A \neq (0)$. We shall sometimes say B is an <u>essential</u> <u>extension</u> of A or B is <u>essential</u> <u>over</u> A if A is an essential subgroup of B.

<u>Lemma 1</u>: Let A be a subgroup of the group B. Then A is essential iff whenever $f: B \longrightarrow G$ is a homomorhism with $f|A$ monic, f is monic too.

Proof: If A is essential in B and $f|A$ is monic, then $(0) = \ker(f|A) = (\ker f) \cap A$ and hence $\ker f = (0)$. If A is not essential and $\langle x \rangle \cap A = (0)$, with $x \neq 0$, then the canonical homomorphism $f: B \twoheadrightarrow B/\langle x \rangle$ is not monic although $f|A$ is. //

The axiom of choice is not used in this proof.

To prove (5), /Fuchs/ establishes first:

(6) a divisible group E containing the group A is minimal divisible containing A iff E is essential over A.

To deduce (5), suppose E_1, E_2 are two divisible groups containing A as an essential subgroup. Then E_1, E_2 are minimal divisible extensions of A, by (6). Embed E_1, E_2 in some group B (for example by forming the pushout of

$$\begin{array}{l} E_1 \\ \uparrow \\ A \hookrightarrow E_2 \end{array}$$).

By (1), E_2 is a direct summand of B, so there is an epimorphism $p: B \twoheadrightarrow E_2$ such that $p|E_2$ is the identity. Since $p(E_1)$ is divisible and contains A, $p|E_1$ is onto E_2. By lemma 1, since $p|A$ is monic, so is $p|E_1$. Thus $p|E_1$ is the required isomorphism, fixing A pointwise.

Thus we see that if the axiom of choice holds, there is for each group A a minimal divisible group D containing A, and that this group D is unique up to an isomorphism over A. The group D is known as the divisible hull of A.

The property of divisible groups expressed in (1) is often phrased:

(1') a divisible group is injective,

and since from (3) any injective group is embeddable in a divisible group, it follows that injective groups are divisible. Thus the group D of the last paragraph is often referred to as the injective hull of A.

Let us now identify some of the places at which AC is used in the proof outlined above that divisible hulls exist.

It is proved in /Blass/ that in ZF (1) implies the axiom of choice, so the axiom of choice is necessary for (1). Since (2) implies (1), it is also necessary for (2). We shall see in the theorem 3 below that choice is not necessary for (3). It is shown in /Hodges1/ that (4) cannot be proved in ZF alone. Whether (5) and (6) can be established in ZF is not clear (to the author); this question is discussed below after lemma 5.

Let us abandon AC now and establish something positive. Our first goal is to show that (3) is a theorem of ZF. It will be convenient to observe here that pushouts do not need AC. A word of warning though: the next lemma only claims the existence of the pushout object and does not claim that it has the universal property required for "real" pushouts.

<u>Lemma 2</u>: Suppose $\alpha : C \to A$ and $\beta : C \to B$ are homomorphisms, with A,B and C groups. Then there is a group G and two homomorphisms $\gamma : A \to G$ and $\delta : B \to G$ such that

(i) the diagram

$$\begin{tikzcd} A \arrow[r, "\gamma"] & G \\ C \arrow[u, "\alpha"] \arrow[r, "\beta"'] & B \arrow[u, "\delta"'] \end{tikzcd}$$

is commutative;

and (ii) if α is monic, then so is δ, while if α is epic, then so is δ.

Proof: Define G as the quotient of $A \oplus B$ by the subgroup

$$H = \{(\alpha c, -\beta c) : c \in C\},$$

and let γ, δ be defined by

$$\gamma : a \longmapsto (a,0) + H$$

$$\delta : b \longmapsto (0,b) + H.$$

Clearly we have not so far used choice, and (i) holds. To see that (ii) holds, proceed as in /Fuchs/, pp52-3. //

Now, as promised, we can establish (3):

<u>Theorem 3</u>: Each group can be embedded in a divisible group.

Proof: Let A be a group and let $F = \bigoplus_{a \in A} \mathbb{Z}$ be free on the underlying set of A. Let $f : F \twoheadrightarrow A$ be the obvious epimorphism. Embed F in the divisible group $Q = \bigoplus_{a \in A} \mathbb{Q}$ in the obvious way and form the pushout as in lemma 2:

$$\begin{tikzcd}
Q \arrow[r, two heads, "g"] & D \\
F \arrow[u, tail] \arrow[r, two heads, "f"'] & A \arrow[u, tail, "e"']
\end{tikzcd}$$

Then g is epic and e is monic and so D is divisible and contains a copy of A. //

If the reader is worried that there may be a hidden application of AC in this proof, let us observe that the construction of the proof is actually functorial (that is, $A \longmapsto D$ is the object function of a functor) and that the construction preserves ω-directed limits. The construction is thus concrete and hence does not involve AC. (For an explanation of these remarks, see /Hodges1/, from which theorem 3 is taken.)

Now we prove part of (6):

Lemma 4: Suppose D is a divisible essential extension of the group A. Then D is minimal divisible containing A.

Proof: Suppose E is a divisible group, $A \leqslant E \leqslant D$. Let $d \in D-E$. Since E is essential in D, there is a least positive integer n such that $nd \in E$ and is not 0. Say $nd = e \neq 0$.

Since E is divisible, there is $x \in E$ such that $nx = e$. Hence $n(x-d) = 0$ and $x-d \neq 0$, which means that the fact that $\langle x-d \rangle \cap E = (0)$ is a contradiction to the essentialness of E in D. //

Again, we have not used AC in this proof.

It is well-known (and proved in /Sharpe & Vámos/, p43, for example) that a maximal essential extension of a group A is a divisible hull of A (in the presence of AC of course), and since AC clearly enables us to obtain a maximal essential extension of A, we can also obtain a divisible hull this way.

Without AC, we still have:

Lemma 5: Let A,D be groups, $A \leqslant D$. Then D is a divisible essential extension of A iff D is a maximal essential extension of A.

Proof: Suppose D is divisible and A is an essential subgroup of D and let $E \geqslant D$ be essential over A. If $E \neq D$, then there is $x \in E - D$ and so a least positive integer n such that $nx \neq 0$ and $nx \in D$, say $nx = d$. Since D is divisible, there is $y \in D$

such that $ny = d$. Hence $\langle x-y\rangle \cap D = (0)$ while $x-y \neq 0$, a contradiction. So D is a maximal essential extension of A.

For the converse, suppose D is a maximal essential extension of A. By theorem 3, there is a divisible group E containing D. Suppose D is not divisible. Then there exist a prime p and $d \in D$ such that

$$\text{for all } x \in D \quad px \neq d. \qquad \ldots(*)$$

Let $y \in E$ be such that $py = d$. Since D is maximal essential over A, $D + \langle y\rangle$ is not essential over A, and hence not over D. Thus there is $d' \in D$ and an integer k relatively prime to p such that $ky + d' \neq 0$ and $\langle ky + d'\rangle \cap D = (0)$. Since $kpy + pd'$ is in $\langle ky + d'\rangle \cap D$, $kd + pd' = kpy + pd' = 0$.

Since k and p are relatively prime, there are integers u and v such that $uk + pv = 1$. Hence

$$\begin{aligned} d &= (uk + pv)d \\ &= u(-pd') + pvd \\ &= p(vd - ud'). \end{aligned}$$

Since $vd - ud' \in D$, this contradicts (*), and D is a divisible essential extension of A. //

As we have seen, with AC, the converse of lemma 4 is true. It is not clear to the author whether the converse is true in ZF. Let us examine the difficulty. Suppose D is a minimal divisible extension of the group A. We try to show that D is essential over A. Suppose not. Then there is $x \in D$ such that $\langle x\rangle \cap D = (0)$ but $x \neq 0$. The natural way to proceed is to observe that the order of x can be assumed, without loss of generality, to be either a prime, p, or ∞. Then embed $\langle x\rangle$ in B, a copy of $\mathbb{Z}(p^\infty)$

or $\mathbb{Q}$, as appropriate, inside D, disjoint from A, and factor out by B to get a "smaller" divisible group containing a copy of A. We have to lift back to D however to get a contradiction, and this involves choosing coset representatives. Alternatively, we might try to prove that B is a direct summand of D. But this leads to the question: can we even get B? Suppose D is in fact a p-group, p a prime. The natural way to get B is to choose x_1 such that $px_1 = x$, choose x_2 so that $px_2 = x_1$, etc, and let B be the subgroup of D generated by the x_is. Except in trivial cases there is more than one solution y of $py = x_i$, and so it seems that we are obliged to make infinitely many choices. If D is torsion-free, then there is only one solution of $py = x$ and we shall see in theorem 6 below that this can be exploited.

Of course, if D as constructed in theorem 3 were minimal divisible containing e(A), then it would clearly also be essential over e(A), since if $\langle x \rangle$ is disjoint from e(A) then there is a direct summand of Q mapping onto a subgroup of D which contains e(A) and is disjoint from $\langle x \rangle$. This would clearly contradict the minimality.

We have discussed above three possible ways in which D might be regarded as a divisible hull of A:

(a) D is a divisible essential extension of A;

(b) D is a maximal essential extension of A;

(c) D is a minimal divisible extension of A.

With AC, these are of course equivalent conditions on D; without AC, (a) and (b) are equivalent and imply (c), as we have seen. We shall adopt (a) as our definition: the group D is a <u>divisible hull</u> of the group A iff D is divisible and an essential extension of A.

In this definition, D is referred to as "a" divisible hull of A. Without AC, need it be unique? That is, if D_1 and D_2 are two divisible hulls of A, is there an isomorphism of D_1 onto D_2 which fixes A? If we could find a homomorphism from D_1 into D_2 fixing A, we could argue as we did above in deducing (5) and (6) from lemma 1. In that argument we obtained a homomorphism by an appeal to (1), which we know is equivalent to AC. So it seems that we may well need AC for uniqueness of divisible hulls. Of course, it has not yet been shown that we can have divisible hulls without choice, let alone unique ones. In the next theorem we show that torsion-free groups have divisible hulls without using AC. It appears in /Hodges1/.

<u>Theorem 6</u>: Each torsion-free group has a divisible hull.

Proof: Let A be a torsion-free group and let D be the group constructed as in the proof of theorem 3. If T is the torsion part of D then D/T is torsion-free and contains a copy of A, since A is torsion-free and A is embedded in D. Now D/T is certainly divisible and thus contains a solution x of nx = a for each integer n $\neq$ 0 and a in A. Since D/T is torsion-free, each such equation has exactly one solution. If we let H be the set of all solutions as n runs through the non-zero integers and a runs through A, we see easily that H is a divisible hull of A. //

It is because of this theorem that we deal only with torsion groups in the rest of Part I.

§2 The algebra for the forcing construction

In §3 we shall construct by forcing a model of set theory in which there is a group without a divisible hull. To do this we need to be able to produce suitable automorphisms of the notion of forcing. These will be obtained from automorphisms of a particular group in the ground model, in which choice holds. Hence we <u>assume AC in this section</u>.

We shall be concerned exclusively with 2-groups, and in fact with groups all of whose non-zero members have order 2 or 4. Recall that the cyclic group of order n is denoted by $\mathbb{Z}_n$.

<u>Lemma 1</u>: Suppose G is a direct sum of copies of $\mathbb{Z}_4$, and let B be a subgroup of G. If $C \leqslant G$ is a B-high subgroup of G, then C is a direct summand of G.

Proof: By 27.1 of /Fuchs/, it is sufficient to show that C is pure in G.

Suppose there is x in G such that $2x = c$, where $c \in C$. If $x \in C$, then there is nothing to prove; so suppose $x \notin C$. Then because C is B-high, $\langle x, C\rangle$ contains a non-zero member of B, say

$$0 \neq b = c_1 + kx,$$

where k is 1,2 or 3. In fact we cannot have $k = 2$, since then $kx = c$ and $b \in C$, contradicting $B \cap C = (0)$. So

$$2b = 2c_1 + k(2x) = 2c_1 + kc \in C \cap B = (0).$$

Hence $2c_1 = c$ and C is pure in G. //

<u>Lemma 2</u>: Suppose G is a direct sum of copies of $\mathbb{Z}_4$ and B is a

subgroup of G. If C is B-high in G, and $B_1 \geqslant B$ is C-high in G, then $G = B_1 \oplus C$, and B is an essential subgroup of B_1.

Proof: By lemma 1, B_1 and C are direct summands of G, so if they do not generate G, there is a direct summand of G disjoint from both of them. This contradicts the C-highness of B_1 and so establishes the first part.

For the second, suppose x is a non-zero member of B_1. Then $x \notin C$, so $\langle x, C\rangle \cap B \neq (0)$. Hence there is non-zero b in B such that for some integer k and c in C $b = c + kx$. But then $c \in B_1$ and so $c = 0$ and $b = kx$, which means B is essential in B_1. //

We shall be concerned with essential subgroups of torsion groups; the following concept is useful in this context. The <u>socle</u> of the group G, written soc(G), is the set of elements of G that have square-free order. In the case of 2-groups, it is the set of elements of order 2, together with 0. The socle is always a subgroup. The following lemma is easily proved:

<u>Lemma 3</u>: Suppose G is a torsion group and B is a subgroup of G. Then B is an essential subgroup of G iff $\mathrm{soc}(G) \leqslant B$. //

We need one last technical lemma before getting to the point.

<u>Lemma 4</u>: Suppose G and H are finite groups, and both direct sums of copies of $\mathbb{Z}_4$. Let S,T be essential subgroups of G,H respectively and suppose that there is an isomorphism f of S onto T. Then there is an isomorphism g of G onto H extending f.

Proof: We note that G and H are isomorphic iff they have the same number of direct summands in a decomposition by theorem 0.9.1. Since the socles of G and H are isomorphic (by lemma 3), we see that G and H are isomorphic. To see that the isomorphism can be taken to extend f, write $S = S_0 \oplus S_1$, where S_0 is a direct summand of G, maximal with respect to being a subgroup of S. Let $G_1 \geqslant S_1$ be S_0-high in G.

Since, by lemma 2, $G = S_0 \oplus G_1$, and S_1 is essential in G_1, we see that $S_1 = soc(G_1)$ and hence S_1 and G_1 have the same number of direct summands in a decomposition by theorem 0.9.1. If we put $T_0 = f(S_0)$, $T_1 = f(S_1)$, and hence $T = T_0 \oplus T_1$, we see that for any T_0-high $H_1 \geqslant T_1$, G_1 and H_1 have the same number of direct summands and are hence isomorphic, indeed, by an isomorphism extending $f|S_1$. Clearly T_0 is a direct summand of H and so we obtain g as required. //

For the rest of this section A denotes a fixed group, isomorphic to the direct sum of a countable number of copies of $\mathbb{Z}_4$.

<u>Theorem 5</u>: Suppose C is a finite group and the non-zero members of C all have order 2 or 4. Let B be a subgroup of C and suppose that $e:B \rightarrowtail A$ is an embedding. Then there is an embedding $f:C \rightarrowtail A$ extending e.

Proof: Let B_0 be maximal with respect to the property: $B_0 \leqslant B$ and $e(B_0)$ is a direct summand of A. Then B_0 is a direct summand of both B and C, and we can write

$$B = B_0 \oplus B_1 \quad \text{and } C = B_0 \oplus C_1,$$

where $C_1 \geqslant B_1$.

Since $e(B_1)$ contains no proper direct summand of A, $e(B_1)$ is contained in soc(A) and it is now easy to see how to embed C_1 in A so that the embedding extends $e|B_1$. Combining this with $e|B_0 \rightarrowtail A$ gives us f. //

Corollary 6: Any finite group C, all of whose non-zero members have order 2 or 4, can be embedded in A. //

The property of A expressed by this corollary is referred to as the universality of A.

We have to examine some automorphisms of A for the purposes of the following sections.

Theorem 7: Suppose C and D are finite subgroups of A with intersection B, and that there is an isomorphism f of C onto D which fixes B pointwise. Then f can be extended to an automorphism of A.

Proof: In this proof we repeatedly use the fact that given an automorphism of a direct summand of A and any complementary direct summand, one can find an automorphism of A extending the given automorphism and fixing the complement pointwise. The question turns on how B sits inside C and D. We progressively eliminate cases until we are left with the case where C and D are direct summands of A and B is the socle of C (and D).

First, let B_0 be maximal among the subgroups of B which are direct summands of A. Then we may write

$$B = B_0 \oplus B_1, \text{ where } B_1 \leqslant \mathrm{soc}(A).$$

Now we can write

$$C = B_0 \oplus C_0 \text{ and and } \quad D = B_0 \oplus D_0,$$

where $B_1 \leqslant C_0$ and $B_1 \leqslant D_0$.

If $b = c + d$ for some $b \in B_0$, $c \in C_0$, $d \in D_0$, then $b - c = d \in C \cap D = B$ and $c \in C_0 \cap B = B_1$. Similarly $d \in B_1$ and hence $b \in B_0 \cap B_1 = (0)$. Thus $B_0 \cap (C_0 + D_0) = (0)$, and we can find $A_0 \geqslant C_0 + D_0$ such that $A = B_0 \oplus A_0$. Hence any automorphism of A_0 extending $e|C_0$ can be extended to an automorphism of A extending e. Since $A \simeq A_0$, it is sufficient to prove the theorem for A_0. So we assume $B_0 = (0)$, that is:

$$B \text{ is a direct sum of copies of } \mathbb{Z}_2.$$

Now let A_1 be a B-high subgroup of A and let $A_2 \geqslant B$ be A_1-high. Then by lemma 2, B is essential in A_2 and hence also in $C_2 = A_2 \cap C$. Since A_2 is a direct summand of A, C_2 is a direct summand of C and so there is C_1 such that $C = C_1 \oplus C_2$, and we may put $D = D_1 \oplus D_2$, where $D_1 = e(C_1)$, $D_2 = e(C_2)$. Again B is essential in D_2, and we also have $C_2 \cap D_2 = B$.

Of course $C_1 \cap D_1 = (0)$, but we also have

$$(C_1 \oplus D_1) \cap (C_2 + D_2) = (0).$$

For, if $c_1 + d_1 = c_2 + d_2$ with $c_1 \in C_1$, $c_2 \in C_2$, $d_1 \in D_1$ and $d_2 \in D_2$, then $c_1 - c_2 = d_2 - d_1 \in C \cap D = B \leqslant C_2$, and hence $c_1 = c_2 + d_2 - d_1 \in C_1 \cap C_2 = (0)$. A similar argument shows $d_1 = 0$, too, and so $c_1 + d_1 = 0$, as claimed.

Thus we can find successively subgroups A_3, C_3, D_3 of A such that

$A_3 \geq C_2 + D_2$ and is $(C_1 \oplus D_1)$-high,

$C_3 \geqslant C_1$ and is $(D_1 \oplus A_3)$-high, and $\quad D_3 \geqslant D_1 \quad$ and is $(C_3 \oplus A_3)$-high.

By lemma 2, $A = C_3 \oplus D_3 \oplus A_3$. Since lemma 2 also tells us that C_1 and D_1 are essential in C_3 and D_3 respectively, and since by assumption $e|C_1$ is an isomorphism onto D_1, then by lemma 4 there is an isomorphism of C_3 onto D_3 extending $e|C_1$. It is clear that this isomorphism extends to an automorphism of A fixing A_3 pointwise. Thus it is sufficient to prove the theorem for the case when $C_1 = D_1 = (0)$. That is, we assume:

B is a direct sum of copies of $\mathbb{Z}_2$ and B is essential in C. Of course B is then also essential in D, and we may write

$$C = C_4 \oplus B_2,$$

where C_4 is a direct summand of A and

$$B = (2C_4) \oplus B_2.$$

Hence $D = D_4 \oplus B_2$, where $D_4 = e(C_4)$. It follows that $2C_4 = 2D_4 = C_4 \cap D_4$. Then $B_2 \cap (C_4 + D_4) = (0)$. For, if $b = c + d$ with $b \in B_2$, $c \in C_4$, $d \in D_4$, then $c = b - d \in C \cap D = B$ and hence $c \in 2C_4$. By the same argument, $d \in 2C_4$ and $b \in B_2 \cap (2C_4) = (0)$.

Thus there is $A_4 \geqslant B_2$ such that A_4 is $(C_4 + D_4)$-high in A. Now $C_4 + A_4$ is clearly essential in A (by lemma 1 and the fact that C_4 is a direct summand of A). It follows that

$$A = C_4 \oplus A_4,$$

and by symmetry,

$$A = D_4 \oplus A_4.$$

It is now clear that $e|C_4$ extends to an automorphism of A which fixes A_4 pointwise. Since $C \cap A_4 = B_2 = D \cap A_4$, this automorphism extends e. //

The property of A expressed by this theorem is referred to as the <u>homogeneity</u> of A. Finally we have:

<u>Theorem 8</u>: Suppose C and D are finite subgroups of A and that B is a subgroup of $C \cap D$. Then there is an automrphism π of A such that π fixes B pointwise and $\pi(C) \cap D = B$.

Proof: Let C_0 be another copy of C and form the push-out

$$\begin{tikzcd}
C_0 \arrow[r, tail] & E \\
B \arrow[u, "e"] \arrow[r, hook] & C + D \arrow[u]
\end{tikzcd}$$

where e is the embedding of B into C_0 induced by $C \simeq C_0$, and $C + D$ is the subgroup of A generated by C and D.

Since E is a quotient of the direct sum of C_0 and $C + D$, it follows that every non-zero element of E has order 2 or 4. If we identify C_0 and $C + D$ with their images in E, then theorem 5 gives us an embedding of E into A, over $C + D$. Let C' be the image of C_0 under this embedding. Then $C' \cap (C + D) = B$, and hence $C' \cap C = B = C' \cap D$. Since there is an isomorphism of C onto C' fixing B pointwise, there is by theorem 7 an automorphism π of A extending this isomorphism. Then $\pi(C) = C'$ and $\pi(C) \cap D = B$.

//

§3 Construction of the model

In this section we treat three transitive models of set theory which will be dentoted by M, M[G] and N. M will be the ground model, M[G] a generic extension of M and N will be between the other two. AC will hold in M and M[G] and fail in N.

Fix a transitive model M of ZF + V = L and a group A in M which is in M the direct sum of a countable number of copies of $\mathbb{Z}_4$. Thus the results of §2 hold in M, since choice holds in M.

Let P be the set of all functions p such that

(i) dom(p) is a finite subset of $A\times\omega$;

and (ii) range(p) $\subseteq \{0,1\}$.

Define p $<$ q iff p,q are in P and p $\supset$ q.

Then $(P,\leq)$ is a notion of forcing in M. Let G be P-generic over M and form M[G] as in /Jech2/. We are going to find a "generic copy" of A in M[G].

For each a ε A, let $g_a = \{n \in \omega : \exists p \in G \;\; p(a,n) = 1\}$, and put $g_A = \{g_a : a \in A\}$ and $g_+ = \{(g_a,g_b,g_c) : a,b,c \in A,\ a+b = c\}$. Now we define names for these sets: $\underline{g}_a, \underline{g}_A, \underline{g}_+$ are to be names for g_a, g_A, g_+, respectively:

$$\mathrm{dom}(\underline{g}_a) = \{\check{n} : n \in \omega\}$$

$$\underline{g}_a(\check{n}) = 1 \quad \text{for } n \in \omega$$

$$\mathrm{dom}(\underline{g}_A) = \{\underline{g}_a : a \in A\}$$

$$\underline{g}_A(\underline{g}_a) = 1 \quad \text{for } a \in A$$

$$\mathrm{dom}(\underline{g}_+) = \{(\underline{g}_a,\underline{g}_b,\underline{g}_c) : a,b,c \in A \text{ and } a+b = c\}$$

$$\underline{g}_+(\underline{g}_a,\underline{g}_b,\underline{g}_c) = 1 \quad \text{for } a,b,c \in A \text{ with } a+b = c.$$

The reader may wish to compare this with p185 of /Jech2/. The idea is that g_A is in $M[G]$ a generic set of reals and that g_+ is the addition table for a group structure on g_A. The following is easy:

Lemma 1: $(P,\leqslant)$ is a separative notion of forcing. //

We summarise what we have done so far:

Lemma 2: In $M[G]$ the following hold:

(i) for each $a \in A$, $g_a \in PS(\omega)$;

(ii) $(g_A;g_+)$ is a group, and $g:a \longmapsto g_a$ is an isomorphism of A onto $(g_A;g_+)$.

Proof: (i)

$$||\underset{\sim}{g}_a \subseteq \check{\omega}||$$

$$= \prod_{t \in \operatorname{dom}(\underset{\sim}{g}_a)} \sum \{p \in P : p.\underset{\sim}{g}_a(t) = 0 \text{ or } p \leqslant ||t \in \check{\omega}||\}$$

$$= \prod_{t \in \omega} \sum \{p \in P : p.\Sigma\{q \in P : q(a,t) = 1\} = 0 \text{ or } p \leqslant ||\check{t} \in \check{\omega}||\}.$$

Since for $t \in \omega$, $||\check{t} \in \check{\omega}|| = 1$, the expression under $\sum$ has value 1 for all $t \in \omega$ and hence in $M[G]$, $g_a \in PS(\omega)$.

(ii) It is clearly enough to show that $g:A \longrightarrow g_A$ is an isomorphism in $M[G]$. To see this, it is enough to find a name for g. Define $\underset{\sim}{g}$ by

$$\operatorname{dom}(\underset{\sim}{g}) = \{(\check{a},\underset{\sim}{g}_a) : a \in A\}$$

$$\underset{\sim}{g}(a,\underset{\sim}{g}_a) = 1 \quad \text{for } a \in A.$$

A calculation similar to that in (i) shows that $||\underset{\sim}{g} \subseteq A\times\underset{\sim}{g}_A|| = 1$. To see that $||\underset{\sim}{g}$ is a function$|| = 1$, we calculate as follows:

$$\| \forall x \in g_A\ (\check{a},x) \in g \to x = g_a \|$$
$$= \prod_{t \in A} \| (\check{a},g_t) \in g \to g_t = g_a \|$$
$$= \prod_{t \in A} (- \|(\check{a},g_t) \in g\| + \| g_t = g_a \|)$$
$$= 1,$$
since $\|(\check{a},g_t) \in g\| = 0$ if $t \neq a$, and $\| g_a = g_a \| = 1$.

It follows that g is the name of a function. That this function is 1-1 is proved exactly as lemma 19.11, p185 of /Jech2/ is proved. Now it is easy to see that g names g and g is an isomorphism. //

We shall use automorphisms of P to fail choice in N. We define these now. Suppose $\pi \in M$ is an automorphism of A. Then π extends to an automorphism, which we also denote by π, of P as follows, and this automorphism is again in M:

$$\mathrm{dom}(\pi p) = \{(\pi x,n) : (x,n) \in \mathrm{dom}(p)\}$$
$$(\pi p)(\pi x,n) = p(x,n) \text{ for } (x,n) \in \mathrm{dom}(p).$$

As on p184 of /Jech2/, π extends to an automorphism of the Boolean-valued universe corresponding to P.

<u>Lemma 3</u>: For any π as above and any $a \in A$:

(i) $\pi(g_a) = g_{\pi a}$;

(ii) $\pi(g_A) = g_A$;

(iii) $\pi(g_+) = g_+$.

Proof: $\pi(\mathrm{dom}(g_a)) = \pi(\{\check{n} : n \in \omega\}) = \{\check{n} : n \in \omega\}$.

$$(\pi g_a)(\check{n}) = \pi(g_a(\check{n}))$$
$$= \pi \Sigma\{p \in P : p(a,n) = 1\}$$
$$= \Sigma\{p \in P : (\pi p)(\pi a,n) = 1\}$$
$$= \Sigma\{p \in P : p(\pi a,n) = 1\}$$

$$= g_{\pi a}(\check{n}),$$

which proves (i), and (ii) now follows easily.

For (iii),

$$\begin{aligned}
\pi(\mathrm{dom}\ g_+) &= \pi(\{(g_a, g_b, g_c) : a+b = c \text{ in } A\}) \\
&= \mathrm{dom}(g_+), \text{ since } \pi \text{ is an automorphism of } A. \\
(\pi g_+)(g_a, g_b, g_c) &= \pi(g_+(g_a, g_b, g_c)) \\
&= \pi 1 \qquad \text{for } a+b = c \text{ in } A \\
&= 1 \qquad \text{for } a+b = c \text{ in } A.
\end{aligned}$$

//

For notational convenience, let us define for $p \in P$, $X \subseteq A$ and $a \in A$:

$$\mathrm{dom}_1(p) = \{a \in A : \exists n \in \omega\ (a,n) \in \mathrm{dom}(p)\},$$

$$p|X = \{(a,n,i) \in p : a \in X\}$$

and $$p(a) = \{(n,i) : (a,n,i) \in p\}.$$

Recall that if $X \subseteq A$ or g_A, $gp(X)$ denotes the subgroup of A or g_A, respectively, generated by X.

<u>Lemma 4</u>: Suppose $p \in P$, $x \in M$, ϕ is a ZF-formula in n+2 free variables and

$$p \Vdash \phi(g_+, g_{a_1}, g_{a_2}, \ldots, g_{a_n}, \check{x}).$$

Then

$$p|gp(a_1, a_2, \ldots, a_n) \Vdash \phi(g_+, g_{a_1}, g_{a_2}, \ldots, g_{a_n}, \check{x}).$$

Proof: We show that $\{q \in P : q \Vdash \phi\}$ is dense below $p|gp(a_1, \ldots, a_n)$. So suppose $r \leq p|gp(a_1, \ldots, a_n)$.

By theorem 2.8, there is an automorphism $\pi \in M$ of P which fixes $gp(a_1, \ldots, a_n)$ pointwise and is such that $gp(\mathrm{dom}_1(r)) \cap gp(\mathrm{dom}_1(p)) = gp(a_1, \ldots, a_n)$. Then $r \cup \pi p \leq p$ and by the permutation lemma

((19.16) on p184 of /Jech2/), $\pi p \Vdash \phi(\pi g_+, \pi g_{a_1}, \ldots, \pi g_{a_n}, \pi \check{x})$, that is, $\pi p \Vdash \phi(g_+, g_{a_1}, \ldots, g_{a_n}, \check{x})$. //

Suppose that $b_1, \ldots, b_m$ lists all the non-zero members of the finite subgroup B of ~~A~~ or g_A. Then there is a formula Γ of the language of groups such that Γ has just $v_1, \ldots, v_m$ free and for any group C, $C \models \Gamma(c_1, \ldots, c_m)$ iff $b_i \mapsto c_i$ is an isomorphism of B onto C. We note that such a Γ is definable and hence belongs to M. We say that $\Gamma(b_1, \ldots, b_m)$ is a <u>type</u> for B if Γ is as above.

<u>Lemma 5</u>: Suppose ϕ is a ZF-formula, $x \in M$ and $a_1, \ldots, a_n, b_1, \ldots, b_m$ lists without repetition the non-zero members of the finite subgroup B of A in such a way that $\{0\} \cup \{a_1, \ldots, a_n\}$ is also a subgroup of A. Let $p \in P$ be such that $\mathrm{dom}_1(p) \subseteq B$ and suppose

$$p \Vdash \phi(g_+, g_{a_1}, \ldots, g_{a_n}, g_{b_1}, \ldots, g_{b_m}, \check{x}).$$

Let $\Gamma(a_1, \ldots, a_n, b_1, \ldots, b_m)$ be a type for B, and put

$$h_i = p(b_i) \text{ for } i = 1, \ldots, m.$$

Then $p \Vdash$

$$(\forall r_1, \ldots, r_m \in g_A)((r_1 \supseteq \check{h}_1 \wedge \ldots \wedge r_m \supseteq \check{h}_m \wedge \Gamma(g_{a_1}, \ldots, g_{a_n}, r_1, \ldots, r_m)) \rightarrow \phi(g_+, g_{a_1}, \ldots, g_{a_n}, r_1, \ldots, r_m, \check{x})).$$

Proof: We have to show a statement of the form:

$$p \Vdash \forall \vec{r}\ (\theta(\vec{r}) \rightarrow \psi(\vec{r})),$$

which means we must show that for all names $\vec{\underline{r}}$,

$$\forall q \leqslant p\ ((q \Vdash \theta(\vec{\underline{r}})) \rightarrow \exists q' \leqslant q\ (q' \Vdash \psi(\vec{\underline{r}}))).$$

So suppose $q \leqslant p$ and q forces all of: $\underline{r}_1, \ldots, \underline{r}_m \in g_A$, $\Gamma(g_{a_1}, \ldots, g_{a_n}, \underline{r}_1, \ldots, \underline{r}_m)$ and $\underline{r}_i \supseteq \check{h}_i$ for $i = 1, \ldots, m$.

Now, there is $q' \leqslant q$ such that for $i = 1, \ldots, m$ $q' \Vdash g_{c_i} = \underline{r}_i$, for some c_i in A. It follows that q' also forces

$\Gamma(\underline{g}_{a_1},\dots,\underline{g}_{a_n},\underline{g}_{c_1},\dots,\underline{g}_{c_m})$ and hence for any G generic over P with $q' \in G$

$$M[G] \models \Gamma(g_{a_1},\dots,g_{a_n},g_{c_1},\dots,g_{c_m}),$$

and hence

$$M[G] \models \Gamma(a_1,\dots,a_n,c_1,\dots,c_m),$$

and by absoluteness this holds in M too. So by choice of Γ and theorem 2.7 there is an automorphism π of P which fixes $\underline{g}_+$ and each $\underline{g}_i$ and sends $\underline{g}_{b_j}$ to $\underline{g}_{c_j}$ for $j = 1,\dots,m$.

Thus, by the permutation lemma,

$$\pi p \Vdash \phi(\underline{g}_+,\underline{g}_{a_1},\dots,\underline{g}_{a_n},\underline{g}_{c_1},\dots,\underline{g}_{c_m},\check{x}).$$

Now we show that $\pi p \cup p \subseteq q'$. This we do by showing first that $\pi p \cup p$ is a condition, and then that $(\pi p \cup p)(d) \subseteq q'(d)$ for all $d \in \mathrm{dom}_1(\pi p \cup p)$. Now, $\pi p \cup p$ will be a condition provided that p and πp agree on the intersection of $\mathrm{dom}_1(p)$ and $\mathrm{dom}_1(\pi p)$. But this set is just the set of d which are fixed by π. If d is fixed by π, then $p(d) = \pi p(\pi d) = \pi p(d)$, and p and πp agree. So $p \cup \pi p$ is a condition. If $d \in \mathrm{dom}_1(p)$, then $p(d) \subseteq q'(d)$, since $p \subseteq q'$. If $d \in \mathrm{dom}_1(\pi p) - \mathrm{dom}_1(p)$, then d is c_i for some i. Now $q' \Vdash \underline{g}_{c_i} \supseteq \check{h}_i$, and hence, for any G generic over P with $q' \in G$, $M[G] \models g_{c_i} \supseteq h_i$, from which it follows, using the definition of g_{c_i} and the fact that $q' \in G$, that $q'(c_i) = h_i$. Now, $h_i = p(b_i) = \pi p(\pi b_i) = \pi p(c_i)$ and hence $q'(c_i) \supseteq \pi p(c_i)$. Thus $\pi p \cup p$ is a condition extended by q', as claimed.

It follows that

$$q' \Vdash \phi(\underline{g}_+,\underline{g}_{a_1},\dots,\underline{g}_{a_n},\dots,\underline{g}_{c_m},\check{x}).$$

Since

$$q' \Vdash \underline{g}_{c_1} = \underline{r}_1 \wedge \dots \wedge \underline{g}_{c_m} = \underline{r}_m,$$

it follows that

$$q' \Vdash \phi(\underline{g}_+, \underline{g}_{a_1}, \ldots, \underline{g}_{a_n}, \underline{r}_1, \ldots, \underline{r}_m),$$

as required. //

Now PS(ω) can be identified in a definable way with ${}^{\omega}2$. This set has a definable total order (the lexicographic). In this order there are intervals I_h defined as follows for $h \in {}^{<\omega}2$:

$$I_h = \{f \in {}^{\omega}2 : f \supseteq h\}.$$

We call such an I_h a <u>basic interval</u>.

<u>Theorem 6</u>: Suppose ϕ is a ZF-formula and $x \in M$. Then the following holds in $M[G]$:

Suppose $s_1, \ldots, s_n, r_1, \ldots, r_m$ lists without repetition all the non-zero members of the finite subgroup R of g_A and that $\{g_0, s_1, \ldots, s_n\}$ is a subgroup too. Let $\Gamma(s_1, \ldots, s_n, r_1, \ldots, r_m)$ be a type for R and suppose $\phi(g_+, s_1, \ldots, s_n, r_1, \ldots, r_m, x)$ holds.

Then there are basic intervals I_{h_i} $(i = 1, \ldots, m)$ such that $r_i \in I_{h_i}$ $(i = 1, \ldots, m)$ and for all $t_1, \ldots, t_m \in g_A$ with $t_i \in I_{h_i}$ $(i = 1, \ldots, m)$, if $\Gamma(s_1, \ldots, s_n, t_1, \ldots, t_m)$ holds, then so does $\phi(g_+, s_1, \ldots, s_n, t_1, \ldots, t_m, x)$.

Proof: Suppose the hypotheses hold in $M[G]$. Then there is $p \in G$ and there are $a_1, \ldots, a_n, b_1, \ldots, b_m \in A$ such that $s_i = g_{a_i}$ and $r_i = g_{b_i}$ for all the appropriate i, and, by the same sort of argument as we used on the last page, $\Gamma(a_1, \ldots, a_n, b_1, \ldots, b_m)$ holds, and

$$p \Vdash \phi(\underline{g}_+, \underline{g}_{a_1}, \ldots, \underline{g}_{a_n}, \underline{g}_{b_1}, \ldots, \underline{g}_{b_m}, \check{x}) \wedge \underline{g}_{b_1} \supseteq \check{h}_1 \ldots \wedge \underline{g}_{b_m} \supseteq \check{h}_m,$$

where $h_i = p(b_i)$ for $i = 1, \ldots, m$

and $\mathrm{dom}_1(p) \subseteq gp(a_1, \ldots, a_n, b_1, \ldots, b_m)$.

By lemma 5, p then forces

$$(\forall t_1,\dots,t_m \in g_A)\, \Gamma(g_{a_1},\dots,g_{a_n},t_1,\dots,t_m) \wedge$$
$$\wedge\, t_1 \supseteq \check{h}_1 \wedge \dots \wedge t_m \supseteq \check{h}_m \rightarrow \phi(g_+, g_{a_1},\dots,g_{a_n},t_1,\dots,t_m,\check{x}).$$

Since $p \in G$, this holds in $M[G]$ too, and the theorem now follows. //

In case the reader wonders what happens with g_0, let us note that g_0 is definable from g_+ and hence any formula involving g_0 is equivalent to one involving g_+ but not g_0, so there is no loss of generality here. Of course we cannot move g_0 around using automorphisms.

This theorem will be our main tool for showing that there can be a group without a divisible hull. Before we can use it, however, we had better check that there actually are $t_1,\dots,t_m$ such that the conclusion holds, different from $r_1,\dots,r_m$. It is here that we see the importance of universality and homogeneity of A, and why we described g_A as a generic copy of A.

<u>Theorem 7</u>: The following holds in $M[G]$:

Suppose $s_1,\dots,s_n,r_1,\dots,r_m$ lists without repetition the non-zero members of the finite subgroup R of g_A and that $\{g_0,s_1,\dots,s_n\}$ is a subgroup too. Let $\Gamma(s_1,\dots,s_n,r_1,\dots,r_m)$ be a type for R, and let h_i $(i = 1,\dots,m)$ be members of ${}^{<\omega}2$. Then there are $t_1,\dots,t_m \in g_A$ such that $t_i \in I_{h_i}$ for $i = 1,\dots,m$ and $\Gamma(s_1,\dots,s_n,t_1,\dots,t_m)$ holds.

Proof: Let $D = \{p \in P : \exists c_1,\dots,c_m \in A\ \Gamma(a_1,\dots,a_n,c_1,\dots,c_m)$ holds and $p(c_1) \supseteq h_1,\dots, p(c_m) \supseteq h_m\}$. Now, if $p \in D$, and

$c_1,\ldots,c_m$ witness this, then by the same argument as in the proof of lemma 5, $p \Vdash g_{c_i} \supseteq \check{h}_i$ for $i = 1,\ldots,m$. If $\Gamma(a_1,\ldots,a_n,c_1,\ldots,c_m)$ holds, then for any generic G, since

$$M[G] \models g \text{ is an isomorphism,}$$

$$M[G] \models \Gamma(g_{a_1},\ldots,g_{a_n},g_{c_1},\ldots,g_{c_m}),$$

and thus for any $p \in P$

$$p \Vdash \Gamma(g_{a_1},\ldots,g_{a_n},g_{c_1},\ldots,g_{c_m}).$$

Now, in the definiton of D take $a_1,\ldots,a_n$ in A such that $g_{a_1} = s_1, \ldots, g_{a_n} = s_n$. It will be enough to show that D is dense in P, for then there is $p \in G \cap D$ and for this p there are $c_1,\ldots,c_m$ such that

$$p \Vdash \Gamma(g_{a_1},\ldots,g_{a_n},g_{c_1},\ldots,g_{c_m}),\ g_{c_1} \supseteq \check{h}_1,\ldots,\ g_{c_m} \supseteq \check{h}_m,$$

and hence the conclusion holds in $M[G]$.

So, to show that D is dense, let $q \in P$ be any condition. Since $\mathrm{dom}_1(q)$ is finite, the universality of A means that there are $c_1,\ldots,c_m$ in A such that $c_1,\ldots,c_m \in gp(\mathrm{dom}_1(q))$ and $\Gamma(a_1,\ldots,a_n,c_1,\ldots,c_m)$ holds. Put

$$r = q \cup \bigcup_{i=1}^{m} \{(c_i,j,k) : (j,k) \in h_i\}.$$

Then $r \leqslant q$ and $r \in D$, so D is dense. //

Since each basic interval can clearly be partitioned into infinitely many subintervals, we see that this means that we can find infinitely many sequences $(t_1,\ldots,t_m)$ lying in the intervals $I_{h_1},\ldots,I_{h_m}$ of theorem 6 such that $\phi(g_+,s_1,\ldots,s_n,t_1,\ldots,t_m,x)$ as in that theorem holds.

Finally we can define our model N. We put $N = L(g_+)^{M[G]}$.

We could have followed /Jech2/ and taken $\mathrm{HOD}(g_+)^{M[G]}$ as our N. It makes no difference really (compare the remark at the bottom of p204 of /Jech2/), but since the definition of $L(g_+)$ is absolute, we do not need to worry about whether we are talking in $M[G]$ or the real world in what follows.

Note first that $g_A \in N$, and hence $g_A \subseteq N$.

Let F denote the function from $\mathrm{Seq}(g_+)\times\mathrm{Ord}$ onto N obtained from theorem 0.4.1. If $s \in \mathrm{Seq}(g_+)$, let

$$S = \{x \in g_A : \exists y,z \in g_A \text{ such that one of } (x,y,z),\ (y,x,z),\ (y,z,x) \text{ is a member of range}(s)\}.$$

For such s,S, if there is $\alpha \in \mathrm{Ord}$ such that $F(s,\alpha) = x$, then we say S is a <u>support</u> for x. The class of sets which have S as a support is written ∇S. Since ∇S is clearly closed under definability from g_+, $s \in \nabla S$ and $\nabla S = \nabla gp(S)$, we may if we wish restrict attention to supports which are subgroups of g_A. Because S is definable from s and g_+, it follows that there is a function

$$F': \mathrm{Seq}(g_A)\times\mathrm{Ord} \twoheadrightarrow N$$

which is definable from g_+ and F, and hence from g_+ alone.

If S is a support, then $S \subseteq {}^{\omega}2$, and is well-ordered (since it is finite) by the lexicographic order on ${}^{\omega}2$. It follows, using F', that ∇S has a well-order, definable from g_+.

We summarise these remarks in

<u>Theorem 8</u>:

(a) For each finite $S \subseteq g_A$, $x \in \nabla S$ iff there is $\alpha \in \mathrm{Ord}$ such that $F'(S,\alpha) = x$.

(b) For each finite $S \subseteq g_A$, ∇S is closed under definability from g_+ and in particular $S \in \nabla S$, $S \subseteq \nabla S$.

(c) If S and T are finite, $S \subseteq T \subseteq g_A$, then $\nabla S \subseteq \nabla T$.

(d) For each finite $S \subseteq g_A$, ∇S has a well-order definable from g_+.

(e) $M \subseteq \nabla 0$, and $I_h \in \nabla 0$ for each $h \in {}^{<\omega}2$.

(f) $x \in N$ iff there is a finite $S \subseteq g_A$ such that $x \in \nabla S$. //

§4 The order extension principle in N

In this section it is shown that the order extension principle, in fact in its global form, holds in N. The argument is a refinement of that in /Monro/, and is similar to that in /Felgner/.

The following lemma is not vital, but is useful since it enables us to simplify our notation by considering only those partial orders in N which have support 0. It appears in /Felgner/.

<u>Lemma 1</u>: For every partial order $(X, \leqslant)$, there is a set Y such that $(X, \leqslant)$ and $(Y, \subseteq)$ are isomorphic.

Proof: If $x \in X$, let $x^* = \{y \in X : y \leqslant x\}$ and put $Y = \{x^* : x \in X\}$. Then $x \mapsto x^*$ will do. //

In /Szpilrajn/ it is shown that if $(X, \leqslant)$ is a partial order and a and b are two incomparable elements, then there is a partial order $\leqslant'$ extending $\leqslant$ such that $a \leqslant' b$. It follows from this that a maximal partial order is total and so AC $\to$ OEP. The following lemma, which appears in both /Monro/ and /Felgner/, is a modification of Szpilrajn's lemma.

<u>Lemma 2</u>: Suppose $(X, \leqslant)$ is a partial order and $A, B \subseteq X$ are such that $a \in A$, $b \in B$ implies a,b are unrelated by $\leqslant$. Then there is a partial order $\leqslant'$ on X which extends $\leqslant$ and is such that if $a \in A$ and $b \in B$, then $a \leqslant' b$.

Proof: Put $x \leqslant' y$ iff $x \leqslant y$ or there are $a \in A$, $b \in B$ such that

$x \leqslant a$ and $b \leqslant y$. //

We do not actually use this lemma, but the idea behind it is employed several times in the proof of theorem 7 below.

We do not have the axiom of choice in N, but we do have:

<u>Lemma 3</u>: Suppose $(X,\leqslant)$ is a partial order which is an element of $\nabla 0$. Let Ω be the set of all partial orders on X which extend $\leqslant$ and are in $\nabla 0$. Then Ω has a maximal member with respect to $\subseteq$ and this is in $\nabla 0$.

Proof: Ω is inductive and $\nabla 0$ has a well-order. //

The rest of this section is devoted to showing that such a maximal partial order must be total.

The notation in what follows will be complicated enough, so we make some simplifying conventions. We shall write $\vec{s}$ for a sequence and s for the range of $\vec{s}$. All sequences we mention are 1-1. We shall write $\vec{st}$ for the concatenation of $\vec{s}$ and $\vec{t}$. The ith member of $\vec{s}$ is written s_i. If $\vec{a}$ is a sequence of members of A, we shall write $g(\vec{a})$ for the sequence whose ith member is g_{a_i}, where $i <$ length(a). Then g"a will denote the range of $g(\vec{a})$, by our other conventions. If $\vec{r}$ is a sequence of length n of members of g_A, and $\vec{J}$ is a sequence of basic intervals of ${}^{<\omega}2$, then we shall say $\vec{J}$ <u>distinguishes</u> $\vec{r}$ iff length$(\vec{J}) = n$, the members of J are pairwise disjoint and $r_i \in J_i$ for all $i < n$.

We shall be using theorem 3.6 and it will be convenient to rephrase it now to illustrate the terminology just introduced. If we suppress the mention of g_+ and $x \in M$ in its statement, it says that given $\Gamma, \vec{s}, \vec{r}$, such that $\phi(\vec{s}, \vec{r})$ holds there is a sequence $\vec{J}$ of basic intervals which distinguishes $\vec{r}$ and is such that for any $\vec{t}$ distinguished by $\vec{J}$ if $\Gamma(\vec{st})$ holds, so does $\phi(\vec{st})$. Thus theorem 3.6 says that certain sequences are indistinguishable by ϕ. We shall need to sharpen this. Since the precise statement is complicated, let us first examine a special case.

Consider two non-zero elements b,c of A and suppose b,c have order 2. Let $d = b+c$, and suppose we are given three pairwise disjoint intervals J,K,L, such that $g_b \in J, g_c \in K, g_d \in L$. Now let b',c' and d' also be of order 2 and suppose $g_{b'} \in J, g_{c'} \in K$, $g_{d'} = g_{b'} + g_{c'} \in L$. It is clear that $b \mapsto b'$ induces an isomorphism of gp(b,c) onto gp(b',c). The question is: where is $g_{d*} = g_{b'} + g_{c'}$? To answer this, suppose $p \in P$ is a condition with $\mathrm{dom}_1(p) = \{b,c,d\}$ and that $J = I_{p(b)}$, $K = I_{p(c)}$, $L = I_{p(d)}$ and

$$p \Vdash \underset{\sim}{g}_b \in \check{J},\ \underset{\sim}{g}_c \in \check{K},\ \underset{\sim}{g}_d \in \check{L} \qquad \ldots(1)$$

and

$$p \Vdash \underset{\sim}{g}_{b'} \in \check{J},\ \underset{\sim}{g}_{c'} \in \check{K},\ \underset{\sim}{g}_{d'} \in \check{L} \qquad \ldots(2).$$

We shall see that for some $q \leqslant p$, $q \Vdash \underset{\sim}{g}_{b'} + \underset{\sim}{g}_c \in \check{L}$. Let π be an automorphism of P induced by $b \mapsto b'$, fixing c. Then

$$p \Vdash \underset{\sim}{g}_{b'} \in \check{J},\ \underset{\sim}{g}_c \in \check{K},\ \underset{\sim}{g}_{\,d} \in \check{L},$$

by the permutation lemma and (1). Now if $x \in \mathrm{dom}_1(p) \cap \mathrm{dom}_1(\pi p)$, then $x = c$ (except in trivial cases) and $p(x) = \pi p(x)$, so $\pi p \cup p$ is a condition. Put $q = \pi p \cup p$. Then

$$q \Vdash \underset{\sim}{g}_{\,d} \in \check{L} \text{ and } \underset{\sim}{g}_{\,d} = \underset{\sim}{g}_{b'} + \underset{\sim}{g}_c,$$

since $d = b' + c$.

Now, in fact we could have found q as above extending any given $q' \leqslant p$ such that q' forced (1) and (2). This means that p forces the statement "for all r,s,t such that $r \in J$, $s \in K$, $t \in L$, if $t = s + r$ and s,r have order 2, then $r + g_c \in L$". Given this, it is clear that $M[G]$ will also satisfy this statement provided we started with $p \in G$. Thus the answer to the question is: $g_{d*} \in L$.

We shall have to deal with more complicated versions of the above, so we introduce still more notation. Suppose $\vec{a}$, $\overrightarrow{ab}$, $\overrightarrow{ac}$ list the non-zero members of finite subgroups of A or g_A and that $\overrightarrow{abcd}$ lists all the nonzero members of the subgroup D generated by $a \cup b \cup c$ (= abc, by our earlier conventions). In this case, we say $\overrightarrow{abcd}$ is a <u>tidy listing</u> of D (or sometimes abcd is a tidy listing of D). Note that D is then the direct sum of $\{0\} \cup ab$ and $\{0\} \cup ac$ with amalgamated subgroup $\{0\} \cup a$.

If $\vec{r}, \vec{s}, \vec{t}$ are sequences of members of g_A, then we shall say that $(\vec{J}, \vec{K}, \vec{L})$ is a <u>layout</u> for $(\vec{r}, \vec{s}, \vec{t})$ iff $\vec{J}, \vec{K}, \vec{L}$ are sequences of basic intervals of ${}^{<\omega}2$ and $\vec{J}, \vec{K}, \vec{L}, \overrightarrow{JKL}$ distinguish $\vec{r}, \vec{s}, \vec{t}, \overrightarrow{rst}$ respectively. (It follows from our conventions that $\{r,s,t\}$ and $\{J,K,L\}$ are each pairwise disjoint collections and that $\vec{r}$ and $\vec{J}$ have the same length (and $\vec{s}, \vec{K}$ and $\vec{t}, \vec{L}$).) If $\vec{b}, \vec{c}, \vec{d}$ are sequences of members of A, then we say $(\vec{J}, \vec{K}, \vec{L})$ is a layout for $(\vec{b}, \vec{c}, \vec{d})$ iff $(\vec{J}, \vec{K}, \vec{L})$ is a layout for $(g(\vec{b}), g(\vec{c}), g(\vec{d}))$. We note that "x is a layout for y" can be formalised in the forcing language; we shall sometimes say "y is laid out according to x" instead.

Lemma 4: Suppose $\overrightarrow{abcd}$ is a tidy listing of the finite subgroup D of A, and let $\Gamma(\overrightarrow{abcd})$ be a type for D. Let $z \in {}^{<\omega}2$ and let $\vec{h},\vec{j},\vec{k},\vec{l}$ be sequences of members of ${}^{<\omega}2$ of the same lengths as $\vec{a},\vec{b},\vec{c},\vec{d}$ respectively. Suppose the members of $\{z\} \cup h \cup j \cup k \cup l$ are pairwise incompatible and define sequences $\vec{J},\vec{K},\vec{L}$ of basic intervals of ${}^{\omega}2$, of the same length as $\vec{b},\vec{c},\vec{d}$ respectively, as follows:

$$J_i = I_{j_i}, \quad K_i = I_{k_i}, \quad L_i = I_{l_i}.$$

Define $p \in \mathbb{P}$ by:

$$\mathrm{dom}_1(p) = D, \; p(0) = z, \; p(a_i) = h_i,$$
$$p(b_i) = j_i, \; p(c_i) = k_i, \; p(d_i) = l_i.$$

Then

$$p \Vdash \Gamma(g(\overrightarrow{abcd})) \text{ and } "(\vec{J},\vec{K},\vec{L}) \text{ is a layout for } (g(\vec{b}),g(\vec{c}),g(\vec{d}))"$$

and

$p \Vdash$ " for all $\vec{r},\vec{s},\vec{t},\vec{u}$ in g_A, if $(\vec{r},\vec{s},\vec{t})$ is laid out according to $(\vec{J},\vec{K},\vec{L})$ and $(g(\vec{a})\overrightarrow{rst})$ and $\Gamma(g(\vec{a})\vec{r}g(\vec{c})\vec{u})$ hold, then $\vec{L}$ distinguishes $\vec{u}$".

Proof: As we have seen before (in the proof of lemma 3.5),

$$p \Vdash g_0 \supseteq \check{z}, \; g_{a_i} \supseteq \check{h}_i, \text{ etc}$$

and

$$p \Vdash \Gamma(g(\overrightarrow{abcd})),$$

so the first claim is established.

For the second, we have to show a statement of the form

$$p \Vdash \forall\vec{x}\,(\theta(\vec{x}) \rightarrow \psi(\vec{x})),$$

which means we must show that for all names $\vec{\underline{x}}$,

$$\forall q \leqslant p \; ((q \Vdash \theta(\vec{\underline{x}})) \rightarrow \exists q' \leqslant q \; (q' \Vdash \psi(\underline{x}))).$$

So suppose $q \leqslant p$ and

$q \Vdash$ "$\vec{\underline{r}},\vec{\underline{s}},\vec{\underline{t}},\vec{\underline{u}}$ are sequences from g_A, $(\vec{\underline{r}},\vec{\underline{s}},\vec{\underline{t}})$ is laid out according to $(\vec{J},\vec{K},\vec{L})$, and $\Gamma(\underline{g}(\vec{a})\overline{\underline{rst}})$ and $\Gamma(\underline{g}(\vec{a})\vec{\underline{r}}\underline{g}(\vec{c})\vec{\underline{u}})$ hold".

Now we can find $q' \leqslant q$ and sequences $\vec{b}',\vec{c}',\vec{d}',\vec{d}^*$ from A such that

$$q' \Vdash \ \vec{\underline{r}} = \underline{g}(\vec{b'}),\ \vec{\underline{s}} = \underline{g}(\vec{c'}),\ \vec{\underline{t}} = \underline{g}(\vec{d'}),\ \vec{\underline{u}} = \underline{g}(\vec{d}^*).$$

Now,

$$q' \Vdash \ \Gamma(\underline{g}(\overrightarrow{ab'c'd'})),$$

and as we have seen before, this means that $\Gamma(\overrightarrow{ab'c'd'})$ holds.

Let $B = \{0\} \cup a \cup b$, $C = \{0\} \cup a \cup c$. Since $\overrightarrow{abcd}$ is tidy, this means B and C are subgroups of D. Since $\Gamma(\overrightarrow{abcd})$ and $\Gamma(\overrightarrow{ab'c'd'})$ both hold, there is in M an isomorhism f of B onto $gp(a \cup b')$, sending b_i to b_i' and fixing a pointwise. If $y \in D$, then $y = b^{\$} + c^{\$}$ for some $b^{\$} \in B$, $c^{\$} \in C$. Define $e(y) = f(b^{\$}) + c^{\$}$ in $gp(a \cup b' \cup c)$. It is not hard to see that e is a homomorphism, provided it is well-defined. Suppose

$$b^{\$} + c^{\$} = b^{\$\$} + c^{\$\$}, \text{ where } b^{\$},\ b^{\$\$} \in B,\ c^{\$},\ c^{\$\$} \in C.$$

Then

$$b^{\$} - b^{\$\$} = c^{\$\$} - c^{\$} \in B \cap C = a,$$

and so

$$f(b^{\$} - b^{\$\$}) = b^{\$} - b^{\$\$}$$

and

$$f(b^{\$}) - f(b^{\$\$}) = c^{\$\$} - c^{\$},$$

that is,

$$e(b^{\$} + c^{\$}) = e(b^{\$\$} + c^{\$\$}).$$

Thus e is well-defined, and is a homomorphism from D to $gp(a \cup b' \cup c)$.

In fact e is an isomorphism onto. It is clearly onto. To see that it is 1-1, suppose

$$f(b^{\$}) + c^{\$} = 0.$$

Now $b^\$$ is either b_i or a_i for some i, and $-c^\$$ is c_m for some m. If $b^\$$ is a_i, then $f(b^\$) = a_i = c_m$, a contradiction. If $b^\$ = b_i$, then $b_i' = c_m$. Since $q' \Vdash g_{b_i'} \supseteq \check{j}_i \wedge g_{c_m} \supseteq \check{k}_m$, then for any generic G with $q' \in G$,

$$M[G] \models g_{b_i'} \neq g_{c_m},$$

while if $b_i' = c_m$, then for any generic G

$$M[G] \models g_{b_i'} = g_{c_m},$$

again a contradiction.

Thus e is an isomorphism in M and by the homogeneity of A, there is an automorphism $\pi \in M$ of the notion of forcing, induced by e.

Thus, by the permutation lemma,

$$p \Vdash \Gamma(\pi g(\overrightarrow{abcd})) \text{ and } "(\vec{J},\vec{K},\vec{L}) \text{ is a layout for } (\overrightarrow{\pi b},\overrightarrow{\pi c},\overrightarrow{\pi d})",$$

that is,

$$p \Vdash \Gamma(g(\overrightarrow{ab'c\pi d})) \text{ and } "(\vec{J},\vec{K},\vec{L}) \text{ is a layout for } (\vec{b'},\vec{c},\overrightarrow{\pi d})".$$

Suppose $x \in \mathrm{dom}_1(\pi p) \cap \mathrm{dom}_1(p)$. Then x is fixed by π and $p(x) = \pi p(\pi x) = \pi p(x)$. So $\pi p \cup p$ is a condition. Moreover, $\pi p \cup p \subseteq q'$. For, if $x \in \mathrm{dom}_1(p)$, then $(\pi p \cup p)(x) = p(x) \subseteq q'(x)$, since $p \subseteq q'$. If $x \in \mathrm{dom}_1(\pi p) - \mathrm{dom}_1(p)$, then x is b_i' or d_i. In the first case, since $(\pi p)(b_i') = p(b_i)$ and $q' \Vdash g_{b_i'} \supseteq \check{j}_i = \widecheck{p(b_i)}$, and by the sort of argument we have seen before, $q'(b_i') \supseteq p(b_i) = p(b_i')$. The second case is done similarly. Thus $q' \leq \pi p \cup p$.

Hence

$$q' \Vdash \Gamma(g(\overrightarrow{ab'c\,d})) \text{ and } "\vec{L} \text{ distinguishes } g(\pi \vec{d})".$$

Now since Γ is a type for D and $a \cup b \cup c$ generates D, it follows that

$$(\Gamma(\overrightarrow{xuvw}) \wedge \Gamma(\overrightarrow{xuvw'})) \rightarrow w = w'$$

is a theorem of ZF, from which it follows that

$$q' \Vdash \underline{g}(\pi\vec{d}) = \underline{g}(\vec{d}*).$$

Hence

$$q' \Vdash \text{"}\vec{L} \text{ distinguishes } \underline{g}(\vec{d}*)\text{"}$$

and, since $q' \Vdash \underline{g}(\vec{d}*) = \vec{\underline{u}}$,

$$q' \Vdash \text{"}\vec{L} \text{ distinguishes } \vec{\underline{u}}\text{"},$$

as required. //

Now we have the following improvement of theorem 3.6:

<u>Theorem 5</u>: Let ϕ be a ZF-formula and $x \in M$. Then the following holds in $M[G]$:

Suppose Δ is a finite subgroup of g_A and that $\overrightarrow{urst}$ is a tidy listing of Δ, and let $\Gamma(\overrightarrow{urst})$ be a type for Δ. Suppose that $\phi(g_+, x, \overrightarrow{urst})$ holds.

Then there are sequences $\vec{J}, \vec{K}, \vec{L}$ of basic intervals of ${}^{\omega}2$ such that no member of $\{g_0\} \cup u$ belongs to an interval in $J \cup K \cup L$, $(\vec{r}, \vec{s}, \vec{t})$ is laid out according to $(\vec{J}, \vec{K}, \vec{L})$, and for all $(\vec{r}', \vec{s}', \vec{t}')$ laid out according to $(\vec{J}, \vec{K}, \vec{L})$, if $\Gamma(\overrightarrow{ur's't'})$ holds, then so does $\phi(g_+, x, \overrightarrow{ur's't'})$.

Further, if $(\vec{r}', \vec{s}', \vec{t}')$ is laid out according to $(\vec{J}, \vec{K}, \vec{L})$ and $\Gamma(\overrightarrow{ur's't'})$ holds, then there is a unique $\vec{t}*$ such that $\Gamma(\overrightarrow{ur'st*})$ holds; this $\vec{t}*$ is distinguished by $\vec{L}$, and $\phi(g_+, x, \overrightarrow{ur'st*})$ holds.

Proof: Suppose the hypotheses hold in $M[G]$. Then there is $p \in G$ and there are sequences $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ of members of A such that

$$p \Vdash \vec{\underline{u}} = \underline{g}(\vec{a}),\ \vec{\underline{r}} = \underline{g}(\vec{b}),\ \vec{\underline{s}} = \underline{g}(\vec{c}),\ \vec{\underline{t}} = \underline{g}(\vec{d}),$$

$$\text{"}\underline{g}(\overrightarrow{abcd}) \text{ is a tidy listing of } \Delta\text{"}$$

$$\text{and } \phi(\underline{g}_+, \check{x}, \underline{g}(\overrightarrow{abcd})),$$

and $\mathrm{dom}_1(p) \subseteq D = \{0\} \cup a \cup b \cup c \cup d$.

Now part of the statement above forced by p is "the members of Δ are distinct". Suppose x and y are distinct members of D. Then

$$p \Vdash g_x \neq g_y$$

and so for all generic G with $p \in G$

$$M[G] \models g_x \neq g_y.$$

It follows that there is an extension p' of p with $p' \in G$ and $p'(x) \neq p'(y)$. This means, since D is finite, that we can extend p if necessary to a condition $r \in G$ such that for all distinct x and y in D $r(x) \neq r(y)$. So we may as well assume p has this property already. Thus p satisfies the hypotheses of lemma 4. Define $(\vec{J},\vec{K},\vec{L})$ as in that lemma. Now we see that with this definition we are in the situation of the proof of lemma 3.5 and so

$p \Vdash$ "for all $\vec{r}',\vec{s}',\vec{t}'$, if $(\vec{r}',\vec{s}',\vec{t}')$ is laid out according to $(\vec{J},\vec{K},\vec{L})$ and $\Gamma(g(\vec{a})\overrightarrow{r's't'})$ holds, then $\phi(g_+,x,g(\vec{a})\overrightarrow{r's't'})$ holds" .

Since $p \in G$, this proves the first claim of the theorem.

By lemma 4, p also forces

"for all $\vec{r}',\vec{s}',\vec{t}',\vec{t}^*$, if $(\vec{r}',\vec{s}',\vec{t}')$ is laid out according to $(\vec{J},\vec{K},\vec{L})$ and $\Gamma(g(\vec{a})\overrightarrow{r's't'})$ and $\Gamma(g(\vec{a})\vec{r}'g(\vec{c})\vec{t}^*)$ hold, then $\vec{L}$ distinguishes $\vec{t}^*$".

Since $p \in G$, this statement holds in $M[G]$. Since there always is such a $\vec{t}^*$ and it must be unique, the second claim of the theorem follows now by an application of the first. //

With this behind us, we come to the main weapon of this section.

Corollary 6: The sequence $\vec{J}$ of theorem 5 can be taken so that, in addition, the following holds in $M[G]$:

if $(\vec{r}', \vec{s}', \vec{t}')$ and $(\vec{r}^2, \vec{s}^2, \vec{t}^2)$ are laid out according to $(\vec{J}, \vec{K}, \vec{L})$ and $\Gamma(\overrightarrow{ur's't'})$ and $\Gamma(\overrightarrow{ur^2s^2t^2})$ hold, then there is a unique $\vec{t^*}$ such that $\Gamma(\overrightarrow{ur's^2t^*})$ hold; this $\vec{t^*}$ is distinguished by L, and $\Phi(g_+, x, \overrightarrow{ur's^2t^*})$ holds.

Proof: The conclusion of the theorem is of the form $\psi(g_+, x, \overrightarrow{us})$ and so by theorem 3.6. if $\Gamma_0(\overrightarrow{us})$ is a type for $gp(u \cup s)$, there is a sequence $\vec{H}$ of basic intervals of ${}^{\omega}2$ distinguishing $\vec{s}$ and such that if $\vec{H}$ distinguishes $\vec{s}^2$ and $\Gamma_0(\overrightarrow{us^2})$ holds, then so does $\Phi(g_+, x, \overrightarrow{us^2})$. Now put $J_i' = J_i \cap H_i$ for $i <$ length$(\vec{s})$, where $\vec{J}$ is as in the theorem. Noting that $\Gamma(\overrightarrow{ur^2s^2t^2}) \rightarrow \Gamma_0(\overrightarrow{us^2})$, we see that if $(\vec{r}^2, \vec{s}^2, \vec{t}^2)$ is laid out according to $(\vec{J}', \vec{K}, \vec{L})$ and $\Gamma(\overrightarrow{ur^2s^2t^2})$ holds, then so does $\psi(g_+, x, \overrightarrow{us^2})$. This is just the claim of the lemma with $\vec{J}'$ for $\vec{J}$. //

And now we can show that the OEP holds in N.

Theorem 7: The order extension principle holds in N.

Proof: By lemma 1, it is sufficient to show that the order $\subseteq$ on any set Y can be extended to a total order. Since any set Y is a subset of some V^N (/Jech2/, p71), and since $(V, \subseteq)^N$ has support 0, it is sufficient to show that any partial order in $\nabla 0$ can be extended to a total order.

So let $(X, \preccurlyeq)$ be a partial order in $\nabla 0$ and let $\leqslant$ be a partial order maximal amongst all those partial orders extending $\preccurlyeq$ which are in $\nabla 0$. Such exists by lemma 3. We show that $\leqslant$ is a total order on X.

Suppose $\leqslant$ is not total. Then there are x,y in X such that x and y are unrelated by $\leqslant$. We write $x{\downarrow}y$ in this case. The argument is by induction on the cardinality of the intersection of the supports of x and y. We show for each $n \in \omega$ that, if n is least such that there exist x and y in X with supports whose intersection has cardinality n and $x{\downarrow}y$ holds, then $\leqslant$ is not maximal, a contradiction. We assume in this proof that supports are subgroups of g_A.

So suppose x,y in X and n in ω are chosen so that $x{\downarrow}y$ and n is least such that the supports of x and y intersect in a set of cardinality n.

Case 1: $n = 1$: Of course this is the least possible value of n. Let $\vec{b},\vec{c}$ list all the non-zero members of subgroups supporting x and y respectively such that $b \cap c = 0$. Let $\vec{d}$ list the remaining non-zero members of $gp(b \cup c)$. For convenience, we shall write $x = x(\vec{b})$ and when, later, we move $\vec{b}$ to $\vec{b}'$ say, we shall write $x(\vec{b}')$ for the effect of this on x. More accurately, we have $x = F'(\vec{b},\alpha)$ for some α in Ord and we write $x(\vec{b}')$ for $F'(\vec{b}',\alpha)$, where F' is the function of theorem 3.8. We use similar notations $y(\vec{c}),y(\vec{c}')$ etc.

Let $\Gamma(\overrightarrow{bcd})$ be a type for $gp(b \cup c)$. Then, since $\overrightarrow{bcd}$ is tidy (with empty a), and the statement "$x(\vec{b}),\ y(\vec{c}) \in X$ and $x(\vec{b}){\downarrow}y(\vec{c})$" is of the form $\phi(g_+,x,\overrightarrow{bcd})$, we may apply theorem 5 and corollary 6 to obtain: sequences $\vec{J},\vec{K},\vec{L}$ such that $(\vec{J},\vec{K},\vec{L})$ is a layout for $(\vec{b},\vec{c},\vec{d})$ and sets $\mathcal{X}$ and $\mathcal{Y}$ such that if $x(\vec{b}^1) \in \mathcal{X}$ and $y(\vec{c}^2) \in \mathcal{Y}$, then $x(\overrightarrow{b^1}) \nmid y(\overrightarrow{c^2})$, where

$$X = \{x(\overrightarrow{b^1}) : \exists \overrightarrow{c^1 d^1} \ (\overrightarrow{b^1}, \overrightarrow{c^1}, \overrightarrow{d^1}) \text{ is laid out according to } (\vec{J}, \vec{K}, \vec{L}) \text{ and } \Gamma(\overrightarrow{b^1 c^1 d^1}) \text{ holds}\}$$

and

$$Y = \{y(\overrightarrow{c^2}) : \exists \overrightarrow{b^2 d^2} \ (\overrightarrow{b^2}, \overrightarrow{c^2}, \overrightarrow{d^2}) \text{ is laid out according to } (\vec{J}, \vec{K}, \vec{L}) \text{ and } \Gamma(\overrightarrow{b^2 c^2 d^2}) \text{ holds}\}.$$

Note that both X and Y are elements of ∇O.

From the proof of lemma 2 we see that there is a partial order definable from $\leqslant$ and extending $\leqslant$ properly, a contradiction.

Case 2: $n > 1$:

Let $\overrightarrow{ab}$, $\overrightarrow{ac}$ list the non-zero members of the supports of x and y respectively so that $b \cap c = 0$, and a has cardinality $n-1$. Let $\vec{d}$ list the remaining non-zero members of $gp(a \cup b \cup c)$, and let $\Gamma(\overrightarrow{abcd})$ be a type for this group. As in case 1, we can apply theorem 5 and corollary 6 to obtain $\vec{J}, \vec{K}, \vec{L}$ such that $(\vec{b}, \vec{c}, \vec{d})$ is laid out according to $(\vec{J}, \vec{K}, \vec{L})$ and .

if $(\overrightarrow{b^1}, \overrightarrow{c^1}, \overrightarrow{d^1})$ and $(\overrightarrow{b^2}, \overrightarrow{c^2}, \overrightarrow{d^2})$ are laid out according to $(\vec{J}, \vec{K}, \vec{L})$ and $\Gamma(\overrightarrow{ab^1 c^1 d^1})$ and $\Gamma(\overrightarrow{ab^2 c^2 d^2})$ hold, then
$x(\overrightarrow{ab^1}) \nmid y(\overrightarrow{ac^2})$ ...(3).

Now we can apply theorem 3.6 to (3) to obtain a sequence $\vec{H}$ distinguishing $\vec{a}$ and such that

if $\overrightarrow{a^0}$ is distinguished by $\vec{H}$ and $(\overrightarrow{b^1}, \overrightarrow{c^1}, \overrightarrow{d^1})$ and $(\overrightarrow{b^2}, \overrightarrow{c^2}, \overrightarrow{d^2})$ are laid out according to $(\vec{J}, \vec{K}, \vec{L})$ and $\Gamma(\overrightarrow{a^0 b^1 c^1 d^1})$ and $\Gamma(\overrightarrow{a^0 b^2 c^2 d^2})$ both hold, then $x(\overrightarrow{a^0 b^1}) \nmid y(\overrightarrow{a^0 c^2})$ ...(4).

We put

$$X = \{x(\overrightarrow{a^1b^1}) : \exists \overrightarrow{c^1d^1}\ \vec{H} \text{ distinguishes } \vec{a^1},\ (\vec{b^1},\vec{c^1},\vec{d^1}) \text{ is laid out according to } (\vec{J},\vec{K},\vec{L}) \text{ and } \Gamma(\overrightarrow{a^1b^1c^1d^1}) \text{ holds}\}$$

and

$$Y = \{y(\overrightarrow{a^2c^2}) : \exists \overrightarrow{b^2d^2}\ \vec{H} \text{ distinguishes } \vec{a^2},\ (\vec{b^2},\vec{c^2},\vec{d^2}) \text{ is laid out according to } (\vec{J},\vec{K},\vec{L}) \text{ and } \Gamma(\overrightarrow{a^2b^2c^2d^2}) \text{ holds}\}.$$

As before, X and Y are members of $\nabla 0$.

From (4), we have:

(A) if $x(\overrightarrow{a^1b^1}) \in X$ and $y(\overrightarrow{a^1c^2}) \in Y$, then $x(\overrightarrow{a^1b^1}) \nleq y(\overrightarrow{a^1c^2})$.

To go along with (A), we prove three other propositions:

(B) if $\vec{a^1} \neq \vec{a^2}$ and $x(\overrightarrow{a^1b^1}) \in X$, $y(\overrightarrow{a^2c^2}) \in Y$, then $x(\overrightarrow{a^1b^1})$ and $y(\overrightarrow{a^2c^2})$ are comparable by $\leqslant$.

For, since $\overrightarrow{HJKL}$ distinguishes $\overrightarrow{a^1b^1c^1d^1}$ and $\overrightarrow{a^2b^2c^2d^2}$, $(a^1 \cup b^1) \cap (a^2 \cup c^2) \subseteq a^1 \cap a^2$, which has cardinality less than n-1, which makes the two elements comparable by choice of n.

(C) if $\vec{a^1} \neq \vec{a^2}$ and $c^1 \cap c^2 = 0$, and $y(\overrightarrow{a^1c^1})$, $y(\overrightarrow{a^2c^2}) \in Y$, then $y(\overrightarrow{a^1c^1})$ and $y(\overrightarrow{a^2c^2})$ are comparable by $\leqslant$.

For, otherwise $(a^1 \cup c^1) \cap (a^2 \cup c^2) \subseteq a^1 \cap a^2$, and again this would contradict the choice of n.

(D) there do not exist $\vec{a^1},\vec{a^2},\vec{b^1},\vec{b^2},\vec{c^1},\vec{c^2}$ such that

$x(\overrightarrow{a^1b^1}), x(\overrightarrow{a^2b^2}) \in X$

$y(\overrightarrow{a^1c^1}), y(\overrightarrow{a^2c^2}) \in Y$

and

$$y(\overrightarrow{a^1c^1}) \leqslant x(\overrightarrow{a^2b^2})$$
$$y(\overrightarrow{a^2c^2}) < x(\overrightarrow{a^1b^1})$$
$$y(\overrightarrow{a^1c^1}) \not\leqslant x(\overrightarrow{a^1b^1})$$
$$y(\overrightarrow{a^2c^2}) \not\leqslant x(\overrightarrow{a^2b^2}).$$

For, suppose we have an instance of this, in the above notation. By (A), since $x(\overrightarrow{a^2b^2}) \geqslant y(\overrightarrow{a^1c^1})$, $\vec{a}^1 \neq \vec{a}^2$. If $c^1 \cap c^2 = 0$, then by (C) $y(\overrightarrow{a^1c^1})$ and $y(\overrightarrow{a^2c^2})$ are comparable, which is a contradiction. Thus $c^1 \cap c^2 \neq 0$. So we can assume that we have arranged our sequences so that we can write:

$$\vec{a}^1 = \overrightarrow{\alpha^1\alpha} \qquad \vec{a}^2 = \overrightarrow{\alpha^2\alpha}$$
$$\vec{b}^1 = \overrightarrow{\beta^1\beta} \qquad \vec{b}^2 = \overrightarrow{\beta^2\beta}$$
$$\vec{c}^1 = \overrightarrow{\gamma^1\gamma} \qquad \vec{c}^2 = \overrightarrow{\gamma^2\gamma}$$
$$\vec{d}^1 = \overrightarrow{\delta^1\delta} \qquad \vec{d}^2 = \overrightarrow{\delta^2\delta}$$

where $\alpha = a^1 \cap a^2$, $\beta = b^1 \cap b^2$, $\gamma = c^1 \cap c^2$, $\delta = d^1 \cap d^2$. Here, α, β, δ may be empty but γ is not.

Let $\vec{s}_0$ denote $\overrightarrow{\alpha^1\alpha^2\alpha\beta^1\beta^2\beta\gamma^1\gamma^2\gamma\delta^1\delta^2\delta}$ and let $\overrightarrow{\alpha^1}(s_0) = \overrightarrow{\alpha^1}$, etc, and for s a sequence of the same length as s_0, let $\overrightarrow{\alpha^1}(\vec{s})$ be the "$\overrightarrow{\alpha^1}$-part of $\vec{s}$" in the obvious way and so on for other subsequences and subsets of s and s respectively. Let $\vec{e}$ list the remaining non-zero members of $gp(s_0)$ and let $\Gamma'(\overrightarrow{s_0e})$ be a type for this group.

By theorem 3.6, there is a sequence $\vec{Q}$ of basic intervals such that $\vec{Q}$ distinguishes $\overrightarrow{s_0e}$ and whenever $\overrightarrow{se'}$ is distinguished by $\vec{Q}$ and $\Gamma'(\overrightarrow{se'})$ holds, then

$$x(\overrightarrow{\alpha^1\alpha\beta^1\beta}\,(\vec{s})),\ x(\overrightarrow{\alpha^2\alpha\beta^2\beta}\,(\vec{s})) \in X,$$
$$y(\overrightarrow{\alpha^1\alpha\gamma^1\gamma}(\vec{s})),\ y(\overrightarrow{\alpha^2\alpha\gamma^2\gamma}(\vec{s})) \in Y,$$

and

$$y(\overrightarrow{\alpha^1\alpha\gamma^1\gamma(\vec{s})}) \leqslant x(\overrightarrow{\alpha^2\alpha\beta^2\beta(\vec{s})})$$
$$y(\overrightarrow{\alpha^2\alpha\gamma^2\gamma(\vec{s})}) \leqslant x(\overrightarrow{\alpha^1\alpha\beta^1\beta(\vec{s})})$$
$$y(\overrightarrow{\alpha^1\alpha\gamma^1\gamma(\vec{s})}) \nleqslant x(\overrightarrow{\alpha^1\alpha\beta^1\beta(\vec{s})})$$
$$y(\overrightarrow{\alpha^2\alpha\gamma^2\gamma(\vec{s})}) \nleqslant x(\overrightarrow{\alpha^2\alpha\beta^2\beta(\vec{s})}).$$

By theorem 3.7, there is a sequence $\overrightarrow{se'}$ distinguished by $\vec{Q}$ such that $\Gamma'(\overrightarrow{se'})$ holds, $\Upsilon(\vec{s}) \cap \Upsilon(\vec{s_0}) = 0$, and
$\overrightarrow{\alpha^1\alpha^2\alpha\beta^1\beta^2\beta\gamma^1\gamma^2(\vec{s})} = \overrightarrow{\alpha^1\alpha^2\alpha\beta^1\beta^2\beta\gamma^1\gamma^2(\vec{s_0})}$.

By (C), either

$$y(\overrightarrow{\alpha^1\alpha\gamma^1\gamma(\vec{s_0})}) < y(\overrightarrow{\alpha^2\alpha\gamma^2\gamma(\vec{s})})$$
$$\text{or } y(\overrightarrow{\alpha^2\alpha\gamma^2\gamma(\vec{s})}) \leqslant y(\overrightarrow{\alpha^1\alpha\gamma^1\gamma(\vec{s_0})}).$$

In the first case we have $y(\overrightarrow{\alpha^2\alpha\gamma^2\gamma(\vec{s})}) \leqslant x(\overrightarrow{\alpha^2\alpha\beta^2\beta(\vec{s})})$ and $x(\overrightarrow{\alpha^2\alpha\beta^2\beta(\vec{s})}) = x(\overrightarrow{\alpha^2\alpha\beta^2\beta(\vec{s_0})})$ which means that $y(\overrightarrow{\alpha^2\alpha\gamma^2\gamma(\vec{s_0})}) \leqslant x(\overrightarrow{\alpha^2\alpha\beta^2\beta(\vec{s_0})})$, that is, $y(\overrightarrow{a^2c^2}) \leqslant x(\overrightarrow{a^2b^2})$, a contradiction. The second case is done similarly, getting a contradiction to (A) this time. This proves (D).

Now define a new relation $\trianglelefteq$ on X as follows:

$z_1 \trianglelefteq z_2$ iff $z_1 \leqslant z_2$ or there exist $x(\overrightarrow{a^0b^0})$ in X and $y(\overrightarrow{a^0c^0})$ in Y such that $z_1 \leqslant x(\overrightarrow{a^0b^0})$ and $y(\overrightarrow{a^0c^0}) \leqslant z_2$.

Note that we have used the same $\vec{a^0}$ in both x and y in this definition. Clearly $\trianglelefteq$ is an element of ∇O and properly extends $\leqslant$. The proof will be complete if we show that $\trianglelefteq$ is a partial order since then the maximality of $\leqslant$ is contradicted.

Clearly $u \trianglelefteq u$ for all $u \in X$, and we only have to check:

Transitivity: Suppose $u \trianglelefteq v$, $v \trianglelefteq w$. The only non-immediate case is when $u \leqslant x(\overrightarrow{a^0b^0})$, $y(\overrightarrow{a^0c^0}) < v$, $v \leqslant x(\overrightarrow{a^1b^1})$, $y(\overrightarrow{a^1c^1}) \leqslant w$. Since $y(\overrightarrow{a^0c^0}) \leqslant x(\overrightarrow{a^1b^1})$, $\vec{a}^0 \neq \vec{a}^1$ by (A). Hence, by (C), either $x(\overrightarrow{a^0b^0}) \leqslant y(\overrightarrow{a^1c^1})$ in which case $u \leqslant w$, or $y(\overrightarrow{a^1c^1}) \leqslant x(\overrightarrow{a^0b^0})$, which contradicts (D).

Antisymmetry: Suppose $u \trianglelefteq v$, $v \trianglelefteq u$. The only non-immediate cases are

(a) $u \leqslant v$ and $v \leqslant x(\overrightarrow{a^1b^1})$, $y(\overrightarrow{a^1c^1}) \leqslant u$;

(b) $u \leqslant x(\overrightarrow{a^0b^0})$, $y(\overrightarrow{a^0c^0}) \leqslant v$ and $v \leqslant x(\overrightarrow{a^1b^1})$, $y(\overrightarrow{a^1c^1}) \leqslant u$.

Now, (a) implies $y(\overrightarrow{a^1c^1}) \leqslant x(\overrightarrow{a^1b^1})$, which contradicts (A), while (b) implies $y(\overrightarrow{a^1c^1}) \leqslant x(\overrightarrow{a^0b^0})$, so $\vec{a}^1 \neq \vec{a}^0$ (by (A)) and from (B) it follows that $x(\overrightarrow{a^1b^1})$ and $y(\overrightarrow{a^0c^0})$ are comparable. If $x(\overrightarrow{a^1b^1}) < y(\overrightarrow{a^0c^0})$, then $y(\overrightarrow{a^0c^0}) \leqslant x(\overrightarrow{a^1b^1})$ and so $x(\overrightarrow{a^1b^1}) = y(\overrightarrow{a^0c^0})$, but by (C) $y(\overrightarrow{a^0c^0})$ and $y(\overrightarrow{a^1c^1})$ are comparable and by (A), $y(\overrightarrow{a^0c^0}) = x(\overrightarrow{a^1b^1}) < y(\overrightarrow{a^1c^1})$, a contradiction. So $y(\overrightarrow{a^0c^0}) < x(\overrightarrow{a^1b^1})$ which is a contradiction, since this means we are in the situation described in (D). So (a) and (b) do not occur.

Thus $\trianglelefteq$ is a partial order and the proof is complete. //

We have the following strengthening:

<u>Corollary 7</u>: Global OEP holds in N.

Proof: Since VO has a definable well-order, we can find a canonical maximal member of Ω in lemma 4, which is total by the theorem. //

§5 Divisible hulls in N

In this section we show that $(g_A;g_+)$ has no divisible hull in N.

Lemma 1: Suppose S is a finite subgroup of g_A and that $r_1 \in g_A$ has support contained in S. Then $r_1 \in S$.

Proof: Suppose not. Let $s_1,\dots,s_n$ list the non-zero members of S and let $r_2,\dots,r_m$ list the remaining non-zero members of $gp(S,r_1)$. Let $\Gamma(s_1,\dots,s_n,r_1,\dots,r_m)$ be a type for this group. Let $\alpha \in$ Ord be such that $r_1 = F'(S,\alpha)$, where F' is as in theorem 2.8. Applying theorem 3.6 gives us intervals $H_1,\dots,H_m$ such that $r_i \in H_i$ for $i = 1,\dots,m$, and for any $t_1,\dots,t_m$ such that $t_i \in H_i$ for $i = 1,\dots,m$, if $\Gamma(s_1,\dots,s_n,t_1,\dots,t_m)$ holds, then so does $t_1 = F'(S,\alpha)$. For such a t_1 of course we must have $r_1 = t_1$. But theorem 3.7 guarantees that there is such a t_1, different from r_1, a contradiction. So $r_1 \in S$. //

Theorem 2: In N, $(g_A;g_+)$ has no divisible hull.

Proof: Suppose $D \in N$ is a divisible hull. Let Δ be a finite subgroup of g_A and α an ordinal such that $D = F'(\Delta,\alpha)$. Let r be an element of g_A such that r has order 4 and $\langle r\rangle \cap \Delta = (0)$. Let $d \in D$ be such that $2d = r$ and let δ be a finite subgroup of g_A and β an ordinal such that $d = F'(\delta,\beta)$.

Now r has order 4, so $\langle r\rangle$ is a direct summand of any finite subgroup of g_A that contains r. So let U be a subgroup of g_A such that $U \geq \Delta$ and

$$\Delta + \delta + \langle r\rangle = U \oplus \langle r\rangle.$$

Let $u_1,\dots,u_n$ list the non-zero members of $U \oplus \langle 2r \rangle$ and let $r = r_1, r_2, \dots, r_m$ list the remaining non-zero members of $U \oplus \langle r \rangle$. Let $\Gamma(u_1,\dots,u_n,r_1,\dots,r_m)$ be a type for $U \oplus \langle r \rangle$. Now, the statement

"D is a divisible hull of g_A, $r = 2d$ and $D = F'(\Delta,\alpha)$,
$d = F'(\delta,\beta)$" ...(1)

is of the form of ϕ in theorem 3.6, so there are intervals $H_1,\dots,H_m$ such that $r_i \in H_i$ for $i = 1,\dots,m$, and for all $t_1,\dots,t_m$ such that $t_i \in H_i$ for $i = 1,\dots,m$,

if $\Gamma(u_1,\dots,u_n,t_1,\dots,t_m)$ holds, then D is a divisible hull of g_A, and $2d' = t_1$ ...(2),

where $d' = F'(\delta',\beta)$ for δ' the image of δ under the isomorphism $r_i \mapsto t_i$ for $i = 1,\dots,m$.

Now there is $s \in g_A$ of order 4 such that $\langle s \rangle \cap (U \oplus \langle r \rangle) = (0)$, and if we put $r' = r + 2s$ we see that $U \oplus \langle r \rangle \simeq U \oplus \langle r' \rangle$, by an isomorphism fixing U pointwise. Thus there are $t_2,\dots,t_m$ such that $\Gamma(u_1,\dots,u_n,r',t_2,\dots,t_m)$ holds and $r' = r + 2s$. By theorem 3.7, we can assume that $r' \in H_1$ and $t_i \in H_i$ for $i = 2,\dots,m$, and hence that $2d' = r'$ where d' is as in (2).

In summary, we have

$$2d = r,\ 2d' = r',\ r' = r + 2s,$$

and

$$\langle s \rangle \cap (U \oplus \langle r \rangle) = (0).$$

Hence

$$2(d - d') = 2s$$

and

$$d - d' = s + e \qquad \dots(3),$$

where $e \in \operatorname{soc}(D) = \operatorname{soc}(g_A)$, and hence $d - d' \in g_A$.

Now, $d - d'$ clearly has support inside $U + \langle r \rangle + \langle r' \rangle$, and so by lemma 1,

$$d - d' \in U + \langle r \rangle + \langle r' \rangle \qquad \ldots(4).$$

From (3), then,

$$e \in U + \langle r \rangle + \langle r' \rangle + \langle s \rangle,$$

that is,

$$e \in U + \langle r \rangle + \langle s \rangle,$$

and since $2e = 0$, it follows that

$$e \in U + \langle r \rangle + \langle r' \rangle \qquad \ldots(5).$$

From (3),(4) and (5), we see that

$$s \in U + \langle r \rangle + \langle r' \rangle,$$

which it is not. This contradiction proves the theorem. //

PART II

NEARLY FREE ALGEBRAS

§0 Introduction

This section is concerned with free and nearly free algebras. The axiom of choice is used in this Part, in line with usual mathematical practice.

In §1, the concepts of freeness, κ-freeness and $L_{\infty\kappa}$-freeness are analysed in a general setting. This section contains nothing startling, but is necessary because there does not seem to be any similar analysis available in the literature.

In §2, the construction due to Eklof of a κ^+- not κ^{++}-free abelian group from a κ- not κ^+-free abelian group (for regular κ) is presented, for motivation of what follows.

In §3, a construction is presented which retains some of the features of Eklof's construction, in particular the iterability, but which is applicable to many varieties.

The results of the preceding section are applied in §4 to torsion-free varieties of groups, and in §5 they are applied in varieties of rings.

In §6, the implications of the analysis in §1 are worked out for nilpotent varieties of groups.

The concept of parafreeness in a variety of groups is introduced in §7 and some large cardinal axioms are used to show that some κ-free groups must be parafree.

§1 Free, λ-free and $L_{\infty\lambda}$-free algebras

In the sequel we shall be concerned principally with groups. However, in this section we shall deal with the topics of the title in varieties of algebras. There are two reasons for this generality: the first is that by suppressing the algebraic particularities we may be more able to see the generalities underlying our conclusions, and the second, that there are applications in areas other than groups (as in §5). In any case, the more general framework is no harder to handle than the purely group-theoretic.

Throughout this section $\underline{V}$ denotes a variety of algebras in a countable language.

We shall be concerned with the construction of certain algebras in $\underline{V}$ by unions of chains. The important chains will be well-ordered and continuous. Thus we will have little control over what happens at limit steps in the construction, but we shall use the successor steps to control the results of the construction. We show that we can attach an invariant to constructions, and then that from this invariant can be obtained an invariant of the algebra constructed. This in turn will lead to a criterion for the algebra constructed to be free.

Let α be an ordinal. An <u>α-tower</u> is a family

$$A = \{ A_i : i \leq \alpha \}$$

of algebras in satisfying

(a) $A_i \leq A_j$, whenever $i \leq j \leq \alpha$;

(b) if $\delta \leq \alpha$ is a limit, then $A_\delta = \bigcup_{i<\delta} A_i$;

and (c) for each successor $i < \alpha$, A_{i+1} is free.

Some towers are of special interest:

Let λ be a cardinal. We say that an algebra A in $\underline{V}$ is <u>λ-generated</u> iff A is generated by a subset of cardinality less than λ. Clearly if λ is uncountable, this is equivalent to $|A| < \lambda$. We say A is exactly <u>λ^+-generated</u> iff A is λ^+-generated but not λ-generated. If λ is infinite, this implies $|A| = \lambda$, and if λ is uncountable, A is exactly λ^+-generated iff $|A| = \lambda$. These equivalences of course rely on the fact that the language of $\underline{V}$ is countable.

Suppose λ is an infinite cardinal of cofinality μ. Let A in $\underline{V}$ be exactly λ^+-generated. A <u>near-filtration</u> of A is a μ-tower $\mathcal{A} = \{ A_i : i \leqslant \mu\}$ with $A = A_\mu$, such that for all $i < \mu$ A_i is λ-generated. A <u>filtration</u> of A is a near-filtration $\mathcal{A}$, as above, such that for each limit ordinal $\delta < \mu$ A_δ is a free algebra.

<u>Theorem 1:</u> Suppose δ is a limit ordinal and $\mathcal{A} = \{ A_i : i \leqslant \delta\}$ is a δ-tower with A_i free for all $i < \delta$. If, for all $i < j < \delta$, A_i is a free factor of A_j, then A_δ is free, and each A_i is a free factor of A_δ.

Proof: The conditions imply that we can find X_i a basis of A_i for each $i < \delta$ such that if $i < j < \delta$ and $\alpha < \delta$ is a limit ordinal:

$$X_i = X_j \cap A_i \quad \text{and} \quad X_\alpha = \bigcup_{i<\alpha} X_i.$$

It follows that $\bigcup_{i<\delta} X_i$ is a basis of A_δ and each A_i is a free factor of A_δ. //

Theorem 2: Suppose λ is an infinite cardinal of cofinality μ, and let A in $\underline{V}$ be exactly λ^+-generated. Then

A is free iff A has a filtration $A = \{ A_i : i \leq \mu \}$ such that A_i is a free factor of A_{i+1} for all $i < \mu$.

If A is free, then the filtration also has the property that each A_i is a free factor of A.

Proof: The sufficiency of the condition follows from theorem 1 and an easy induction.

For the necessity, suppose A is free on X, where $|X| = \lambda$. We can write $X = \bigcup_{i<\mu} X_i$ where $X_i \subseteq X_j$ for $i<j<\mu$, $X_\delta = \bigcup_{i<\delta} X_i$ if $\delta<\mu$ is a limit, and for all $i<\mu$, $|X_i| < \lambda$. Clearly, then, taking A_i to be the subalgebra of A generated (in fact, freely) by X_i gives us the filtration we need. //

We shall call a filtration of the sort described in the theorem as a <u>filtration by free factors</u>. Thus an infinitely- but not finitely-generated algebra is free iff it has a filtration by free factors.

Of course not every algebra in $\underline{V}$ can be expected to have a filtration. We turn now to those with near-filtrations.

Suppose B is a free subalgebra of the algebra C in $\underline{V}$. If A is a subalgebra of B, then we say that B is a <u>block for A in</u> C iff there is no free $D \leq C$ such that $B \leq D$ and A is a free factor of D. If A is not free then of course any free B containing A is a block for A. Note too that if B is a block for A in C, then so too is any free B', with $B \leq B' \leq C$.

Suppose κ is a regular uncountable cardinal and A in $\underline{V}$ is exactly κ^+-generated and has a near-filtration $\mathcal{A} = \{ A_i : i \leq \kappa \}$. Put

$$\Delta(\mathcal{A}) = \{ i < \kappa : \text{there is } j \text{ with } i<j<\kappa \text{ such that } A_j \text{ is a block for } A_i \text{ in } A \}.$$

Clearly $\Delta(\mathcal{A})$ is an invariant of the construction $\mathcal{A}$.

<u>Lemma 3:</u> Let κ be a regular uncountable cardinal, and suppose A in $\underline{V}$ is exactly κ^+-generated, and that

$$\mathcal{A} = \{ A_i : i \leq \kappa \} \qquad \text{and } \mathcal{B} = \{ B_i : i \leq \kappa \}$$

are near-filtrations of A (so that $A = A_\kappa = B_\kappa$). Then

$$\Delta(\mathcal{A}) \equiv \Delta(\mathcal{B}) \qquad \text{mod } NS_\kappa.$$

Proof: Let $C = \{ i<\kappa : A_i = B_i \}$. It is easy to see that C is closed in κ. We show C is unbounded too. Suppose $i_0<\kappa$. Since κ is regular and $|A_{i_0}| < \kappa$, there is j_0, $i_0 < j_0 < \kappa$, such that $A_{i_0} \leq B_{j_0}$. In the same way we can find i_1 with $j_0<i_1<\kappa$ such that $B_{j_0} \leq A_{i_1}$. In this fashion we construct two sequences, $(i_n : n<\omega)$ and $(j_n : n<\omega)$, such that for all $n<\omega$

$$A_{i_n} \leq B_{j_n} \leq A_{i_{n+1}}.$$

If δ is the (common) supremum of these two sequences, then $A_\delta = B_\delta$, since towers are continuous. Thus $\delta \in C$, and C is unbounded.

To prove the lemma it is sufficient to show that $C \cap \Delta(\mathcal{A}) = C \cap \Delta(\mathcal{B})$. Suppose $i \in C \cap \Delta(\mathcal{A})$. Then there is j such that A_j is a block for A_i in A. Let k in C be such that $A_j \leq A_k$. Then A_k is a block for A_i in A, too. Since i and k are in C, B_k is a block for B_i in A, and hence $i \in C \cap \Delta(\mathcal{B})$. Thus $C \cap \Delta(\mathcal{A}) \subseteq C \cap \Delta(\mathcal{B})$, and by symmetry, the reverse inclusion holds too. This proves the lemma. //

Thus we are justified in writing $\Delta(A)$ for the equivalence class of $\Delta(\mathcal{A})$ mod NS_κ, whenever $\mathcal{A}$ is a near-filtration of the exactly κ^+-generated algebra A, since the lemma tells us that $\Delta(A)$ is an invariant of A.

If $X \subseteq \kappa$, we write X for the equivalence class of X mod NS_κ.

<u>Theorem 4:</u> Suppose A in $\underline{V}$ is exactly κ^+-generated, where κ is a regular uncountable cardinal, and that A has a near-filtration. Then A is free if $\Delta(A) = \tilde{0}$.

Proof: If $\mathcal{A}$ is a filtration of A by free factors, then $\Delta(\mathcal{A}) = 0$. //

This theorem, then, gives us a criterion for an algebra of uncountable regular cardinality with a near-filtration to be free. We turn now to the concept of λ-freeness.

If λ is a cardinal and A an algebra in $\underline{V}$, then A is <u>λ-free</u> iff every λ-generated subalgebra of A is contained in a free subalgebra of A.

<u>Theorem 5:</u> Let λ be an infinite cardinal and suppose A in $\underline{V}$ is exactly λ^+-generated. Then

(i) if A has a near-filtration, then A is λ-free;

(ii) if λ is uncountable, then A is λ-free iff A has a near-filtration.

Proof: The claim in (i) is obvious, and is the sufficiency in (ii). For the necessity in (ii), suppose A is λ-free, and let $\mu = cf(\lambda)$. There is a family $\{a_i : i<\mu\}$ of subsets of A such that $|a_i| < \lambda$ for all $i<\mu$, and $A = \bigcup_{i<\mu} a_i$. (This family exists because $|A| = \lambda$.)

Let A_0 be λ-generated and free, containing a_0. If A_i has been defined, let A_{i+1} be λ-generated and free, containing A_i and $\bigcup_{j<i+1} a_j$. If $\delta<\mu$ is a limit ordinal, and A_i has been defined for all $i<\delta$, then put $A_\delta = \bigcup_{i<\delta} A_i$. Then $\{A_i : i<\mu\} \cup \{A\}$ is a near-filtration of A. //

It is not of course claimed in the above theorem that the A_δ are free for limit δ. However, sometimes they will be. A variety is a <u>Schreier variety</u> iff every subalgebra of a free algebra is free. In a Schreier variety the A_δs of the above proof will also be free. Thus we have:

<u>Corollary 6:</u> If A in $\underline{V}$ is an exactly λ^+-generated algebra, where λ is an uncountable cardinal, and $\underline{V}$ is a Schreier variety, then A is λ-free iff A has a filtration. //

We shall say that the variety $\underline{V}$ has the <u>filtration property</u> (FP) iff the conclusion of the above corollary holds in the variety whenever λ is an uncountable cardinal.

We turn now to $L_{\infty\kappa}$-freeness. Let λ be an infinite cardinal, and L the language for $\underline{V}$. We say an algebra A in $\underline{V}$ is <u>$L_{\infty\kappa}$-free</u> iff A is $L_{\infty\kappa}$-elementary-equivalent to a free algebra in $\underline{V}$.

Let κ be a regular cardinal, and suppose A in $\underline{V}$ is exactly κ^+-generated. If C is a well-ordered chain (not necessarily continuous) of κ-generated free subalgebras of A with $\bigcup C = A$, then we say $\mathcal{C}$ is a <u>κ-pure chain for</u> A iff whenever C_1, C_2 are members of $\mathcal{C}$ with $C_1 < C_2$, C_1 is a free factor of C_2. Notice that if A has a κ-pure chain, then A is κ-free.

<u>Theorem 7:</u> Suppose κ is a regular cardinal and that A in $\underline{V}$ is exactly κ^+-generated. If A has a κ-pure chain, then A is $L_{\infty\kappa}$-free.

Proof: Suppose that $\mathcal{C}$ is a κ-pure chain for A. Let F be free on κ generators. Let J be the set of all isomorphisms $f:S\to T$ where $S \in \mathcal{C}$ and T is a free factor of F. We show that J has the $<\kappa$-back-and-forth property. Let * denote the free product in $\underline{V}$.

First the forth direction:

Let $f \in J$, $f:S\to T$. Let X A, $|X| < \kappa$. Since $\cup\mathcal{C} = A$, there is $S_1 \in \mathcal{C}$ such that S is a free factor of S_1 and $S \cup X \subseteq S_1$. Suppose $S_1 = S*S_0$. Choose T_0 a free factor of F such that there is an isomorphism f_0 of S_0 onto T_0, and the subalgebra of F generated by T and T_0 is the free product $T*T_0$.

Then $f*f_0 : S*S_0 \to T*T_0$ is in J and extends f, as required for the forth direction.

For the back direction, let f be as above and suppose $X \subseteq F$, $|X|<\kappa$. There is S_1 in C such that $S_1 = S*S_0$ and S_0 is exactly μ^+-generated for some $\mu<\kappa$ with $\mu>|X|$. There is then a free factor T_0 of F which is exactly μ^+-generated and such that the subalgebra generated by T and T_0 is $T*T_0$ and contains X. There is then an isomorphism f_0 of S_0 onto T_0, and $f*f_0$ is in J and extends f, while range$(f*f_0) \supseteq X$. This does the back direction.

Hence A is $L_{\infty\kappa}$-free. //

In an unpublished preprint, /Kueker2/, it is shown that the converse of this theorem is true when $\kappa = \omega_1$. It is suggested in that preprint that some form of converse is true for other regular values of κ. It seems (to this author) that it is most likely that the direct converse is true when κ is a successor cardinal greater than ω_1. However, we shall not discuss this here any further since we shall be mainly concerned with proving that the algebras we construct are $L_{\infty\kappa}$-free, rather than decomposing $L_{\infty\kappa}$-free ones. The exception to this is in §6, where we shall want to analyse nilpotent varieties of groups. For these varieties, there are easier methods available. We shall turn to these methods after the next theorem.

We used before the notion of a block. We introduce now a related concept. If A and B are free algebras of $\underline{V}$ and $A \leqslant B$, then B is an <u>ff-block</u> for A iff there is no free C in $\underline{V}$ such that $B \leqslant C$ and both A and B are free factors of C.

<u>Theorem 8:</u> Let κ be a regular cardinal, and suppose $\{A_i : i{\leqslant}\kappa\}$ is a filtration of the exactly κ^+-generated algebra A in $\underline{V}$ such that for all successors $i{\leqslant}\kappa$, A_i is a free factor of A_j whenever $i<j<\kappa$. Put $E = \{i<\kappa : A_{i+1}$ is an ff-block for $A_i\}$.

Then A is $L_{\infty\kappa}$-free, but if E is stationary, A is not free.

Proof: The set of A_i such that i is a successor is clearly a κ-pure chain for A, so A is $L_{\infty\kappa}$-free, by theorem 7. If A is free, then there is a filtration of A by free factors, and, as in the proof of lemma 3, this means there is a club of i such that A_i

belongs to the filtration by free factors. If E is stationary, there is i in E such that A_i belongs to the filtration by free factors. This is a contradiction, since there is then also a $j > i$ such that A_j belongs to the filtration by free factors, which means that A_i and A_{i+1} are both free factors of A_j. //

For some varieties we can analyse the $L_{\infty\kappa}$-free algebras in more detail.

Suppose A and B are in $\underline{V}$, $A \leqslant B$. If λ is a cardinal, we say B is <u>λ-generated</u> <u>over</u> A iff there is $X \subseteq B$ with $|X| < \lambda$ such that $A \cup X$ generates B. We say A is <u>λ-pure</u> in B iff whenever $A \leqslant B \leqslant C$ and C is free and λ-generated over A, A is a free factor of C. Note that if B is λ-free and A is λ-generated, and λ-pure in B, then A is free and pure in B.

In order to use these definitions, we need to have some restrictions on the variety $\underline{V}$. We say that $\underline{V}$ <u>has</u> <u>(FF)</u> iff, whenever A, B and C are free algebras, $A \leqslant B \leqslant C$ and A is a free factor of C, A is also a free factor of B.

Now we can give a necessary and sufficient condition for $L_{\infty\kappa}$-freeness in some varieties.

<u>Theorem 9:</u> Let λ be an infinite cardinal and suppose $\underline{V}$ has (FF). Let A be an algebra in $\underline{V}$ which is not λ-generated. Then A is $L_{\infty\kappa}$-free iff A is λ-free and every λ-generated subalgebra of A is contained in a λ-generated λ-pure subalgebra of A. λ

Proof: The proof of the sufficiency is very similar to the proof of theorem 7, so we only indicate the differences. We take J to be the set of isomorphisms $f:S \to T$, where S is λ-generated and λ-pure in **A**, and T is a free factor of the free algebra F on λ generators. In theorem 7, we had to use the regularity of κ to obtain S_1; here the hypotheses give us S_1 immediately, with S_1 free, λ-generated and λ-pure. Since S is λ-pure, S is a free factor of S_1 and the rest is easy.

For the necessity, suppose J is a family of partial isomorphisms from **A** to the free algebra F, with the $<\lambda$-back-and-forth property. Since **A** is not λ-generated, neither is F. Suppose $B < \mathbf{A}$ is λ-generated. There is $f \in J$ such that $\text{range}(f) \supseteq G$, $f(B) \subseteq G$ and G is a λ-generated free factor of F. Since $f^{-1}(G)$ is free and contains B, **A** is λ-free. We show that $f^{-1}(G)$ is λ-pure in **A**. Suppose that $C \geqslant f^{-1}(G)$, $C \leqslant \mathbf{A}$, and that C is free and λ-generated over $f^{-1}(G)$. There is $g \in J$ such that $g \supseteq f$ and $\text{range}(g) \supseteq H$, where H is a λ-generated free factor of F such that $H \supseteq g(C)$. By (FF), G is a free factor of H. Hence $f^{-1}(G) = g^{-1}(G)$ is a free factor of $g^{-1}(H)$, and hence, by (FF) again, $f^{-1}(G)$ is a free factor of C. //

This proof is essentially that of theorem 1.1 of /Eklof1/, where it was given for the variety of abelian groups. The next theorem introduces no new information but is in the style of /Eklof2/ and /Mekler/ and is given here to indicate the connection with those two papers.

Theorem 10: Let κ be a regular uncountable cardinal. Suppose $\underline{V}$ has (FF) and (FP) and that **A** in $\underline{V}$ is exactly κ^+-generated and is κ-free. Then

(i) A is $L_{\infty\kappa}$-free iff A has a filtration $\{A_i : i \leq \kappa\}$ such that for all $i<\kappa$ A_{i+1} is κ-pure in A;

(ii) A is free iff for every filtration $\{A_i : i \leq \kappa\}$ of A, the set $\{i<\kappa : A_i$ is not κ-pure$\}$ is not stationary in κ;

(iii) A is free | for some filtration $\{A_i : i \leq \kappa\}$ of A, the set $\{i<\kappa : A_i$ is not κ-pure$\}$ is not stationary in κ.

Proof: (i): The sufficiency follows from theorem 7 or 9. For the necessity, suppose A is $L_{\infty\kappa}$-free. By (FP), there is a filtration $\{B_i : i \leq \kappa\}$ of A. We produce a filtration satisfying the condition, as follows: if $i<\kappa$ is a successor or 0, then A_i is κ-generated and κ-pure (hence also free) containing A_{i-1} (or 0, if i = 0) and also B_j where j is least such that B_j is not contained in A_{i-1}; if δ is a limit $A_\delta = \bigcup_{i<\delta} A_i$. The A_i clearly form a near-filtration satisfying the condition. Since for limits δ A_δ is clearly $B_{\delta'}$, for some δ', A_δ is free for all limits δ, too.

(ii): It follows easily from the property (FF) that, if $\{A_i : i \leq \kappa\}$ is a filtration of A, A_i is not κ-pure in A iff there is $j>i$ such that A_j is a block for A_i in A. Now apply theorem 4.

(iii): This follows from the first sentence of (ii) and lemma 3.

//

We have not talked much about λ-freeness when λ is singular. The following theorem appeared in /Shelah2/. There is a more comprehensible proof in /Hodges2/.

<u>Theorem 11:</u> Let λ be a singular cardinal and $\underline{V}$ a variety with the Schreier property. Then every λ-free algebra in $\underline{V}$ is λ^+-free.

//

§2 Abelian groups and Eklof's construction

If κ is an infinite cardinal and $\underline{V}$ is a variety of algebras, let us say that an algebra A in $\underline{V}$ is <u>κ-almost free</u> iff A is κ-free, is not free and has cardinality κ.

In the paper /Eklof2/ it is shown how to construct in the variety <u>Abgps</u> for any regular κ a κ^+-almost free group from a κ-almost free group. A substantial part of what follows is based on Eklof's construction, so we analyse the construction now.

There are two ingredients: the first is of general application (that is, works in any variety of algebras), while the second is apparently peculiar to <u>Abgps</u>. In the theorem below we see the first ingredient and in the lemma and discussion which follow the theorem we see the features special to <u>Abgps</u>. It is a consequence of the special features that the whole construction can be iterated in <u>Abgps</u> any finite number of times to obtain a κ^{++}-almost free group, a κ^{+++}-almost free group, etc. It is this iteration which we seek to recover in varieties other than <u>Abgps</u>, and the next section is devoted to this.

First, let us make a definition. If κ is a regular cardinal and $\mathcal{A} = \{A_i : i \leq \kappa+1\}$ is a $(\kappa+1)$-tower in the variety $\underline{V}$, then $\mathcal{A}$ is a <u>blocking $(\kappa+1)$-tower</u> iff:

(a) each A_i, $i \leq \kappa+1$, is free on κ generators;

(b) A_i is a free factor, with exactly κ^+-generated complement, of both A_j and $A_{\kappa+1}$, whenever $i < j \leq \kappa$;

and (c) $A_{\kappa+1}$ is an ff-block for A_κ.

Theorem 1: Let κ be a regular cardinal and $\underline{V}$ be a variety of algebras in a countable language. If there is a blocking $(\kappa+1)$-tower in $\underline{V}$, then there is in $\underline{V}$ a κ^+-almost free algebra which is $L_{\infty\kappa^+}$-free.

Proof: Let $E = \{\delta<\kappa^+ : cf(\delta) = \kappa\}$. Then E is stationary in κ^+, and for any limit $\delta<\kappa^+$, $E\cap\delta$ is not stationary in δ. Let $A = \{A_i : i\leq\kappa+1\}$ be a blocking $(\kappa+1)$-tower in $\underline{V}$.

We construct a κ^+-tower $B = \{B_i : i\leq\kappa^+\}$ such that B is a filtration of $B = B_{\kappa^+}$ and

(i) if $i \notin E$, then B_i is a free factor of B_j with an exactly κ^+-generated complement, for all j, $i<j<\kappa^+$;

(ii) if $i \in E$, then B_{i+1} is an ff-block for B_i.

This will be enough, since by theorem 1.8 and the fact that $\kappa^+ - E$ is unbounded, B is κ^+-free and $L_{\infty\kappa^+}$-free, but by theorem 1.8, again, and the fact that E is stationary, B is not free. Let * denote the free product in $\underline{V}$.

Let B_0 be free on κ generators. Suppose that for all $i<j$, B_i has been constructed so that the following holds:

(1) $B|j$ is a j-tower, where $B|j = \{B_i : i<j\}$;

(2) B_i is free on κ generators, for all $i<j$;

(3) if $i \in j - E$, then B_i is a free factor, with exactly κ^+-generated complement, of each B_k with $i<k<j$;

(4) if $i \in j\cap E$, then B_{i+1} is an ff-block for B_i.

Then put $B_j = \bigcup_{i<j} B_i$ if j is a limit. Clearly in this case (1), (3) and (4) continue to hold. To see that B_j is free, observe that since $E\cap j$ is not stationary in j, there is a

$\mathrm{cf}(j)$-tower $C \subseteq B|j$ with $\cup C = B_j$, such that if $C_1 < C_2$ are in C, C_1 is a free factor of C_2, and by theorem 1.1 B_j is free. So (2) holds for $j+1$. There are two cases left:

Case 1: If $j = i+1$ with $i \notin E$, then let $B_{i+1} = B_i * F$, where F is free on κ generators. Clearly in this case (1) - (4) continue to hold.

Case 2: If $j = i+1$ with $i \in E$, then i is a limit ordinal of cofinality κ. Since $i \cap E$ is not stationary in i, there is a κ-tower $C \subseteq B|i$ with $\cup C = B_i$, such that if $C = \{C_i : i \leqslant \kappa\}$, C_{i_1} is a free factor with exactly κ^+-generated complement of C_{i_2} whenever $i_1 < i_2 \leqslant \kappa$. Then there is a sequence h_i, $i \leqslant \kappa$, of isomorphisms such that

$$\begin{tikzcd} C_{i_1} \arrow[r, hook] \arrow[d, "h_{i_1}"', "\wr"] & C_{i_2} \arrow[d, "h_{i_2}", "\wr"'] \\ A_{i_1} \arrow[r, hook] & A_{i_2} \end{tikzcd} \qquad \ldots(5)$$

commutes for all $i_1 < i_2 \leqslant \kappa$. Then h_κ is an isomorphism of B_i onto A_κ, and there is an isomorphism h' and an algebra B_{i+1} making the following diagram commute:

$$\begin{tikzcd} B_i \arrow[r, hook] \arrow[d, "h_\kappa"'] & B_{i+1} \arrow[d, "h'"] \\ A_\kappa \arrow[r, hook] & A_{\kappa+1} \end{tikzcd} \qquad \ldots(6)$$

and we choose this B_{i+1} for our filtration. Clearly (1), (2) and (4) still hold, so it remains to check (3). Suppose that

$k < i+1$ and $k \notin E$. There is B_t in C such that $k < t$ and hence B_k is a free factor with exactly κ^+-generated complement of B_t. Since A is a blocking tower and diagram (5) and (6) are commutative, B_t is a free factor of B_{i+1}, and hence so is B_k, as required.

So the construction can be carried out preserving (1) - (4); hence (i) and (ii) hold, and the theorem is proved. //

So, in order to construct a κ^+-almost free algebra in $\underline{V}$, it is enough to have a blocking $(\kappa+1)$-tower. This brings us to the second ingredient - the construction of a suitable blocking tower. For abelian groups, /Eklof2/ takes advantage of the following fact:

<u>Lemma 2:</u> Suppose F is a free abelian group of cardinality κ, a regular uncountable cardinal, and that $K \leqslant F$ is such that F/K is κ-almost free. If H is a κ-generated direct summand of K, then H is a direct summand of F.

Proof: Choose a κ-generated direct summand G of F such that $G \geqslant H$. Since $G/G \cap K$ is κ-generated and $G+K/K \simeq G/G \cap K$, $G+K/K$ is free and K is a direct summand of $G+K$. Hence H is a direct summand of $G+K$, and since $H \leqslant G$, H is a direct summand of G and hence of F. //

This proof uses the facts that G is a direct summand of the free group F iff F/G is free, and that if G is a direct summand of the free group F, then G is also a direct summand of any group H with $G \leqslant H \leqslant F$ (that is (FF) holds in <u>Abgps</u>).

Given F and K as in the lemma, it is easy to obtain a filtration $\mathcal{K} = \{K_i : i \leq \kappa\}$ of K such that K_i is a free factor of both K and F, and since F/K is not free, K is not a free factor of F. Of course all this is relative to the variety Abgps. This gives us a $(\kappa+1)$-tower which fails to be blocking only because the cardinalities of its members are wrong. In /Eklof2/, this problem is solved by forming $K_i \oplus F$ and inducing $K_i \oplus F \hookrightarrow K_j \oplus F$. Then the $K_i \oplus F$ form a blocking $(\kappa+1)$-tower. In the next section another "fattening" process is given, in lemma 3.4.

Since any κ-almost free abelian group can be written in the form F/K, we see that the existence of a blocking $(\kappa+1)$-tower is necessary for the existence of a κ-almost free abelian group and sufficient for the existence of a κ^+-almost free abelian group. It follows then from theorem 1 that in Abgps the construction can be iterated any finite number of times.

Now lemma 2 is certainly not true in all varieties, even those admitting quotients by certain subalgebras, and nor is it obvious that an algebra in such a variety has a presentation F/K with K free. We shall see in the next section that there is for many varieties a way round this difficulty.

§3 Freely filtered presentations

It is shown in this section that the idea behind theorem 2.1 can be employed in varieties other than Abgps in a way which allows the construction to be iterated. The main concept is that of a freely filtered presentation of a κ-almost free algebra. Our main theorem (theorem 5) will show, among other things, that if κ is regular and there is a κ-almost free algebra with a freely filtered presentation, then there is a κ^+-almost free algebra with a freely filtered presentation. Thus the construction will iterate. Of course we shall have to place some restrictions on the variety in which we are working.

Let us fix a variety $\underline{V}$ of algebras in a countable language and write $\underline{V}$ too for the category of all $\underline{V}$-algebras and $\underline{V}$-homomorphisms. In our earlier sections on freeness, we considered algebras which could be written as unions of continuous chains (of inclusions) but in this section it will be more convenient to work in a more category-theoretic language, so we extend the definitions of §1 in the obvious way to allow embeddings as well as inclusions. Hence an α-tower (α an ordinal) is a functor T from the ordered set $\alpha+1$ (regarded as a category in the natural way) to the category $\underline{V}$ which is cocontinuous and such that all the maps in its image are monic; further, it is required that if $i<\alpha+1$ is a successor, then $\mathsf{T}(i)$ is a free algebra. The definitions of the near-filtration and filtration are modified in the analogous way. It will be convenient to make the notational convention that sanserif letters denote towers whose algebras are denoted by the corresponding capital letter in this typeface, and whose maps are denoted by the corresponding lower case letter. Thus for the α-tower T above the maps would be $t_{ij}: T_i \longrightarrow T_j$, for $i<j<\alpha+1$. If T is also a filtration, then T_α will often be written just T.

An α-tower T is <u>strictly increasing</u> if no t_{ij} is an isomorphism (onto T_j) for $i<j<\alpha+1$.

We make the following <u>assumption on $\underline{V}$</u>: $\underline{V}$ has a zero object, $\underline{0}$.

Here, by a zero object we mean one that is both initial and terminal in $\underline{V}$. It is unique up to isomorphism. We denote by 0 the unique map between any algebras which factors through $\underline{0}$. Note that $\underline{0}$ is the one element algebra and is free on the empty set, because the first is terminal and the second is initial. Thus $\underline{V}$ has exactly one constant operation.

Now the coproduct in $\underline{V}$ is just the free composition as defined in /Cohn1/, p113, and it follows from the fact that $\underline{V}$ has $\underline{0}$ that this is also the free product since $\underline{0}$ is minimal and trivial (/Cohn1/, p186). Hence the free product of any set of algebras in $\underline{V}$ exists and is just the coproduct. This means that the coproduct injections are actually 1-1, as may be verified directly.

If λ is a cardinal, we write $F(\lambda)$ for the free algebra on λ generators. We shall have to deal with coproducts and canonical injections, in situations where it is not obvious which injection is being used; in these cases we number the "slots" from the left and write, for example:

$$X \xrightarrow{(1)} X \amalg X \qquad \text{and} \qquad X \amalg X \xrightarrow{(1,3)} X \amalg X \amalg X \amalg X,$$

which means in the first case that the injection is into the left cofactor, and in the second case that the injection takes the left cofactor into the extreme left cofactor and the right cofactor into the cofactor which is third from the left. In cases where it is obvious which cofactor we mean, we say nothing. Thus in $A \rightarrowtail A \amalg B \leftarrowtail B$, both maps are assumed to be coproduct injections unless some indication to the contrary is given. If $f:A \to B$ and $g:C \to D$, then $f \amalg g$ is the map induced from $A \amalg C$ to $B \amalg D$.

Let us extend the notion of free factor as follows. If $e:A \rightarrowtail B$ is an embedding of the free algebra A into the free algebra B, then e is an <u>ff-map</u> iff there is an isomorphism h and a free algebra C such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, tail, "e"] \arrow[dr, tail] & B \arrow[d, "h", "\wr"'] \\ & A \amalg C \end{tikzcd} .$$

<u>Lemma 1:</u> In the above definition C is unique up to isomorphism.

Proof: Consider the diagram

$$A \underset{0}{\overset{}{\rightrightarrows}} A \amalg C \xrightarrow{0 \amalg 1_C} C,$$

and observe that $0 \amalg 1_C$ is the coequaliser of the coproduct injection and 0. Coequalisers are unique up to isomorphism and $A \amalg C$ is isomorphic to B. //

This lemma justifies the following definition. If $e:A \rightarrowtail B$ is an ff-map and C as in the definition above is free on λ generators, then we say e is $\underline{\lambda\text{-complemented}}$.

The notion of ff-block is extended as follows. If A and B are free and e is an embedding of A into B, then e is an ff-block iff there is no ff-map $f:B \rightarrowtail C$ such that fe is an ff-map. Thus, in our earlier language, B is an ff-block for A iff the inclusion of A into B is an ff-block in our new language.

It is now clear that all the results of §1 carry over, mutatis mutandis, to our new definitions, and we shall apply them without further comment.

Now, each algebra A in $\underline{V}$ has a presentation by generators and relations. This means that there is a free algebra F and a congruence Q on F such that A is the quotient of F by Q. For some algebras A there is a subalgebra R of A such that Q is the congruence generated by $R \times \underline{0}$. Put another way, for such algebras A there is a diagram:

$$R \overset{m}{\rightarrowtail} F \overset{p}{\twoheadrightarrow} A \qquad \ldots(1)$$

such that F is free and p is the coequaliser of m and 0. We shall refer to diagrams such as (1) as presentations. Note that in the variety, Gps, of all groups, for example, R is not necessarily the kernel of p, but is merely a subgroup of F whose normal closure is the kernel of p. We shall be interested in a special type of presentation, namely, one in which R is free. We call such a presentation a free presentation.

Lemma 2: Suppose $R \overset{m}{\rightarrowtail} F \overset{p}{\twoheadrightarrow} A$ is a free presentation. Let H be an arbitrary algebra in $\underline{V}$ and let Q be the coequaliser of 0 and fm, where $f:F \rightarrowtail F \amalg H$ is the canonical injection. Then Q and $A \amalg H$ are isomorphic.

Proof: Consider the following commuting diagram, with a and r canonical:

$$\begin{tikzcd} R \arrow[r, tail, "m"] \arrow[d, tail, "r"] & F \arrow[r, two heads, dashed, "p"] \arrow[d, tail, "f"] & A \arrow[d, tail, "a"] \\ R \amalg H \arrow[r, tail, dashed, "m \amalg 1"] & F \amalg H \arrow[r, two heads, "p \amalg 1"] & A \amalg H \end{tikzcd}$$

Let q be the canonical projection onto Q, and observe that since $(p \amalg 1)fm = 0$, there is a unique $g:Q \rightarrow A \amalg H$ such that $gq = p \amalg 1$. Since $qfm = 0$, there is a unique $g':A \rightarrow Q$ such that $g'p = qf$.

Now let h,h' be the canonical injections of H into $F \amalg H$ and $A \amalg H$, respectively, and let t be the unique fill-in making the following diagram commute:

$$\begin{tikzcd} A \arrow[r, tail, "a"] \arrow[dr, "g'"'] & A \amalg H \arrow[d, dashed, "t"] & H \arrow[l, tail, "h'"'] \arrow[dl, dashed, "qh"] \\ & Q & \end{tikzcd}$$

We have constructed now two maps between Q and $A \amalg H$. An easy calculation shows $gg' = a$, and it follows $gt = 1$. It is also easy to see that $tg = 1$. The lemma is proved. //

The next concept is easier to describe in pictures than in words. Let κ be a regular cardinal, and consider the directed graph below, which consists of a κ-by-3 array of points and arrows as shown:

$$\begin{tikzcd}
* \arrow[r] & * \arrow[r] & * \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] \\
* \arrow[u] \arrow[r] & * \arrow[u] \arrow[r] & * \arrow[u] \\
* \arrow[u] \arrow[r] & * \arrow[u] \arrow[r] & * \arrow[u] \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] \\
* \arrow[u] \arrow[r] & * \arrow[u] \arrow[r] & * \arrow[u]
\end{tikzcd} \qquad \ldots(2)$$

The intention of the picture is that the vertical columns are well-ordered in type κ and the horizontal arrows join points on the same level, with the 0 level at the bottom and the κ level at the top. What we are going to do is to consider a diagram like (2) with algebras at the points and morphisms for the arrows which will be such that the horizontal arrows will form free presentations, while the vertical arrows will be such that each column is a filtration of the algebra at the top; the whole thing will be called a filtered presentation of the algebra in the top right-hand corner of the diagram.

We make this formal as follows. Suppose **A** is an exactly κ^+-generated non-free algebra. Then a <u>filtered</u> <u>presentation</u> of **A** is an ordered quintuple $\underline{P} = (R, F, A, \rho, \phi)$ where A is a filtration of **A** and R and F are filtrations such that ρ and ϕ are natural transformations, $\rho: R \to F$, $\phi: F \to A$, with the property that for all $i \leqslant \kappa$:

$$R_i \overset{\rho_i}{\rightarrowtail} F_i \overset{\phi_i}{\twoheadrightarrow} A_i$$

is a free presentation of A_i. We say $\underline{P}$ is a <u>freely filtered presentation</u> of A iff ρ_κ is an ff-block and in addition ρ_i, r_{ij} and f_{ij} are ff-maps for all $i<j\leq\kappa$ (Recall our notational convention.)

Now we can begin to see where our blocking $(\kappa+1)$-tower is going to come from. If we have a freely filtered presentation of the κ-almost free algebra A, then in the notation of the last paragraph, the filtration R with $R \xrightarrow{\rho_\kappa} F$ added at the top is very like the $(\kappa+1)$-tower used in the last section in <u>Abgps</u> to obtain the blocking tower. In fact the only thing that prevents it from being a blocking tower is the fact that the cardinalities are wrong. ($R \xrightarrow{\rho_\kappa} F$ is an ff-block by definition.) So to get a blocking tower we have to increase the cardinalities. We do this in the next lemmas.

Let λ be a cardinal. Let λ-cop be the functor from $\underline{V}$ to $\underline{V}$ that sends each algebra in $\underline{V}$ to its λ-fold coproduct, and does the same thing on maps in $\underline{V}$.

If λ and μ are cardinals with $\lambda\leq\mu$, then there is an embedding $\eta^A(\lambda,\mu)$ of λ-cop A into μ-cop A which is natural in λ, μ and A. Of course $\eta^A(\lambda,\mu)$ is an ff-map and is μ-complemented if $\lambda<\mu$, and $\mu>\omega$.

<u>Lemma 3:</u> (a) If λ is a cardinal, then λ-cop preserves monics, unions of chains of monics, ff-maps, ff-blocks and freeness.

(b) Suppose δ is a limit ordinal, $R = \{R_i \xrightarrow{r_{ij}} R_j : i<j<\delta\}$ is a δ-directed system of embeddings in $\underline{V}$ and that $\{\lambda_i : i<\delta\}$ is a set of cardinals with $\lambda_i<\lambda_j$ for $i<j<\delta$. Let R be the colimit of R, and put $\lambda = \bigcup_{i<\delta}\lambda_i$. Write η^k_{ij} for $\eta^{R_k}(\lambda_i,\lambda_j)$ and let f_{ij} be the composition of:

$$\lambda_i\text{-cop } R_i \xrightarrow{\lambda_i\text{-cop } r_{ij}} \lambda_i\text{-cop } R_j \xrightarrow{\eta^j_{ij}} \lambda_j\text{-cop } R_j.$$

Then $\{f_{ij} : i<j<\delta\}$ is a δ-directed system with colimit λ-cop R.

Proof: (a) It is easy to see that λ-cop preserves monics. We note that λ-cop is a left adjoint (to the composition: diagonal functor from $\underline{V}$ to the product category $\underline{V}^\lambda$ followed by the functor from $\underline{V}^\lambda$ to $\underline{V}$ which sends each λ-tuple to its product) and hence it follows that λ-cop preserves unions. It is clear that λ-cop preserves ff-maps, ff-blocks, and freeness.

(b) To see that the f_{ij} really are a δ-directed system, consider the diagram below:

$$\begin{tikzcd}
\lambda_i\text{-cop } R_i \arrow[rr, tail, "f_{ij}"] \arrow[dr, tail, "\lambda_i\text{-cop } r_{ij}"] & & \lambda_j\text{-cop } R_j \arrow[rr, tail, "f_{jk}"] \arrow[dr, tail, "\lambda_j\text{-cop } r_{jk}"] & & \lambda_k\text{-cop } R_k \\
& \lambda_i\text{-cop } R_j \arrow[ur, tail, "\eta^j_{ij}"'] \arrow[dr, tail, "\lambda_i\text{-cop } r_{jk}"] \arrow[rr, phantom, "(N)"] & & \lambda_j\text{-cop } R_k \arrow[ur, tail, "\eta^k_{jk}"'] & \\
& & \lambda_i\text{-cop } R_k \arrow[ur, tail, "\eta^k_{ij}"'] & &
\end{tikzcd}$$

The square (N) commutes by naturality, while the two inner triangles commute by definition. The whole outer triangle is the definition of f_{ik}, as required. For the second claim, notice that λ-cop R is a double colimit:

$$\lambda\text{-cop } R = \mathrm{colim}_{j<\delta}\ \mathrm{colim}_{i<\delta}\ \lambda_j\text{-cop } R_i \ .$$

Since the system of f_{ij} is final in this double system, λ-cop R is the colimit of the f_{ij}s too. //

Our next construction is a generalisation of the construction in Case 2 of theorem 2.1.

<u>Lemma 4:</u> Let δ be a limit ordinal of cardinality λ and cofinality κ, and for each $i\leqslant\delta$, put $\lambda_i = \max(\omega, |i|)$. Let R be a strictly increasing filtration of the algebra R, which is free on κ generators, and suppose $f:R\rightarrowtail F$ is an ff-block, with F free on κ generators. Assume too that r_{ij} and fr_i are ff-maps for all $i<j\leqslant\kappa$.

Suppose that T is a δ-tower such that for all $i\leqslant\delta$, T_i is free on λ_i generators, and that there is $C\subseteq\delta$ such that if $\kappa = \omega$, C is unbounded in δ, while if $\kappa>\omega$, C is a club in δ, and, in either case, for all i,j in C, if $i<j$, t_{ij} is a λ_j-complemented ff-map.

Then there is an embedding $e:T_\delta\rightarrowtail\lambda$-cop F which is an ff-block and such that for all i in C, $et_{i\delta}$ is a λ-complemented ff-map.

Proof: We do first as an illustration the case which is simplest to describe. The argument is in effect in this case the combination of the argument in Case 2 of theorem 2.1 and the construction of a "fattened" blocking tower that we alluded to in §2.

Case 1: $\delta>\lambda$ and $\kappa>\omega$. In this case we can find a club $C'\subseteq C$ such that all the members of C' have cardinality λ. Then for i,j in C', if $i<j$, t_{ij} is a λ-complemented ff-map. Let $(c(i): i<\kappa)$ list the members of C' in strictly increasing order. It is clear that for each $i<\kappa$ there is an isomorphism h_i of $T_{c(i)}$ onto λ-cop R_i such that, putting s_{ij} = λ-cop r_{ij}, the following diagram commutes for all i,j with $i<j<\kappa$:

$$\begin{tikzcd}
T_{c(i)} \arrow[r, tail, "t_{c(i)c(j)}"] \arrow[d, "h_i"'] & T_{c(j)} \arrow[d, "h_j"] \\
\lambda\text{-cop } R_i \arrow[r, tail, "s_{ij}"'] & \lambda\text{-cop } R_j
\end{tikzcd}$$

The commutativity of these diagrams and the fact that R and T are towers mean that there is an isomorphism $h_\delta:T_\delta\longrightarrow R$ such that the above diagram commutes with κ in place of j and δ in place of $c(j)$. Define e to be the composition of h with λ-cop f. In view of lemma 3, the conclusion now holds with C' in place of C. However, it is easy to see that if i is in C, there is j in C', $i<j$, and since $t_{i\delta} = t_{j\delta}t_{ij}$, $t_{i\delta}$ is λ-complemented and an ff-map.

Case 2: $\delta>\lambda$ and $\kappa = \omega$. In this case we proceed exactly as in case 1 except that C and C' are merely unbounded in δ. The remaining details present no difficulty.

Case 3: $\delta = \lambda$ and $\kappa>\omega$. List the members of C as $(c(i) : i<\kappa)$

in strictly increasing order. Let η_{ij} be $\eta^{R_j}(\lambda_{c(i)},\lambda_{c(j)})$ from $\lambda_{c(i)}$-cop R_j into $\lambda_{c(j)}$-cop R_j, for $i<j<\kappa$. Thus η_{ij} is an ff-map and is $\lambda_{c(j)}$-complemented if $\lambda_{c(i)} < \lambda_{c(j)}$. Then there are isomorphisms h_i, $i<\kappa$, such that the following diagram commutes:

$$\begin{tikzcd} T_{c(i)} \arrow[r, tail, "t_{c(i)c(j)}"] \arrow[d, "h_i"'] & T_{c(j)} \arrow[d, "h_j"] \\ \lambda_{c(i)}\text{-cop } R_i \arrow[r, tail, "s_{ij}"'] & \lambda_{c(j)}\text{-cop } R_j \end{tikzcd},$$

where s_{ij} is $\lambda_{c(i)}$-cop r_{ij} followed by η_{ij}. In the same way as in case 1, these h_i induce h_δ an isomorphism of T_δ onto λ-cop R, by lemma 3, such that for all $i<\kappa$ the above diagram commutes with κ in place of j and δ in place of c(j). Define e to be the composition of h_δ with λ-cop f. This is enough.

Case 4: $\delta = \omega$. This is done exactly as in case 3 except that C is unbounded now and $\lambda_{c(i)}$ is just ω for all i. Since R is strictly increasing, the construction still works. //

Before producing our main theorem, we have one more definition to make. We say a variety satisfies (EP) iff the following holds in it:

Suppose A and B are free algebras and e: $A \rightarrowtail B$ is an embedding, and let p be the unique fill-in such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[dr, tail, "e"'] & A \amalg B \arrow[d, "p"] & B \arrow[l, tail] \arrow[dl, "1_B"] \\
& B &
\end{tikzcd}$$

Then there is an isomorphism $h: C \longrightarrow A$, such that the following diagram commutes:

$$\begin{tikzcd}
C \amalg B \arrow[r, "h \amalg 1"] \arrow[dr, "0 \amalg 1_B"'] & A \amalg B \arrow[d, "p"] \\
& B
\end{tikzcd}$$

The point of this condition is that C is free; we shall see in the next section that there are varieties for which the condition holds. The term (EP) is explained by the fact that this is an exchange principle for bases.

Now we have our main theorem:

<u>Theorem 5:</u> Let $\underline{V}$ be a variety of algebras in a countable language, and suppose there is a zero algebra in $\underline{V}$, and that $\underline{V}$ satisfies (EP). Let μ be a regular uncountable cardinal and suppose there is $E \subseteq \mu$, such that E is stationary in μ, contains only limit ordinals and satisfies:

(i) $E \cap \delta$ is not stationary in δ for any limit ordinal $\delta<\mu$.

(ii) for each $\delta \in E$, if $cf(\delta) = \kappa$, there is a κ-almost free algebra in $\underline{V}$ which has a freely filtered presentation.

Then there is in $\underline{V}$ a μ-almost free algebra, which is $L_{\infty\mu}$-free and has a freely filtered presentation.

Proof: We build a freely filtered presentation (S,G,B,σ,γ) of a μ-almost free algebra. We do this by building its restrictions $S|j$ etc by induction on $j<\mu$. So we anticipate our success, and write $S|j, \sigma|j$ where $j<\mu$ for $\{S_i \xrightarrow{s_{ik}} S_k : i<k<j\}$ and $\{\sigma_i : i<j\}$, respectively, assuming that all these things have so far been defined. We use similar notation $G|j$, etc, without further comment.

The construction is by induction on $j<\mu$. Put $\lambda_i = \max(\omega, |i|)$ for each $i<\mu$. The following are to hold by induction:

(a) $S|j$, $G|j$, $B|j$ are cocontinuous functors from j to $\underline{V}$, and $\sigma|j, \gamma|j$ are natural transformations from $S|j$ to $G|j$ and $G|j$ to $B|j$, respectively;

(b) if $i<j$, then S_i, G_i and B_i are free on λ_i generators, G_i is the free product of the other two, σ_i is the coproduct injection, and γ_i is the factoring-out map, that is, the coequaliser of 0 and σ_i (inverse on the left to the coproduct injection α_i);

(c) for all $i<k<j$, s_{ik} and g_{ik} are λ_k-complemented ff-maps;

(d) if $i \in j - E$ and $i<k<j$, then b_{ik} is a λ_k-complemented ff-map and the following diagram commutes (with α_i as in (b)):

$$\begin{tikzcd} G_k & B_k \arrow[l, tail, "\alpha_k"'] \\ G_i \arrow[u, tail, "g_{ik}"] & B_i \arrow[l, tail, "\alpha_i"] \arrow[u, tail, "b_{ik}"'] \end{tikzcd};$$

(e) if $\delta \in j \cap E$, then $b_{\delta\delta+1}$ is an ff-block.

For each κ such that there is $\delta \in E$ with $\mathrm{cf}(\delta) = \kappa$, let $(R^{\kappa}, F^{\kappa}, A^{\kappa}, f^{\kappa}, \gamma^{\kappa})$ be a freely filtered presentation of the κ-almost free algebra A^{κ}. By adding extra free generators into F^{κ} and the same generators into R^{κ}, we can arrange that R^{κ} is strictly increasing for all κ. This notation is used in case 3 below.

Recall that F(X) is the free algebra on X.

There are four cases.

Case 0: Let $S_0 \xrightarrow{\sigma_0} G_0 \xrightarrow{\gamma_0} B_0$ be $F(\omega) \xrightarrow{(1)} F(\omega) \amalg F(\omega) \xrightarrow{0 \amalg 1} F(\omega)$.

Suppose that the construction has been carried out for all $i<j$ such that (a) - (e) hold for j. Define as follows.

Case 1: j is a limit ordinal. Put

$$S_j = \operatorname{colim} S|j$$
$$G_j = \operatorname{colim} G|j$$
$$B_j = \operatorname{colim} B|j,$$

and let s_{ij}, g_{ij} and b_{ij} be the colimit injections for $i<j$. Let σ_j and γ_j be the induced maps.

Case 2: $j = i+1$, where $i \notin E$. Let $S_j \xrightarrow{\sigma_j} G_j \xrightarrow{\gamma_j} B_j$ be the top row of the following diagram, and let s_{ii+1}, g_{ii+1} and b_{ii+1} be the vertical maps (in order from the left):

$$\begin{tikzcd}
S_i \amalg F(\lambda_j) \arrow[r, "\sigma_i \amalg (1)"] & G_i \amalg (F(\lambda_j) \amalg F(\lambda_j)) \arrow[r, two heads, "\gamma_i \amalg 0 \amalg 1"] & B_i \amalg F(\lambda_j) \\
S_i \arrow[u, "(1)"] \arrow[r, "\sigma_i"] & G_i \arrow[u, "(1)"] \arrow[r, two heads, "\gamma_i"] & B_i \arrow[u, "(1)"]
\end{tikzcd}$$

NB Assumes lemma 2?

Case 3: $j = \delta+1$, where $\delta \in E$. Thus δ is a limit ordinal. Let $cf(\delta) = \kappa$, and suppose for the moment that $\kappa > \omega$. By hypothesis (i), there is a club C in δ which avoids E. This case is the crux of the matter, so we examine it in more detail. For each i in C, by (d), if $i < k$, then b_{ik} is a λ_k-complemented ff-map. To satisfy (e), we have to put in an ff-block $b_{\delta\delta+1}: B_\delta \rightarrowtail B_{\delta+1}$ as our next step in the construction of the tower B, and to satisfy (d) we have to make sure that for all i in $\delta - E$, $b_{i\delta+1}$ is an ff-map and in particular that this is true for all i in C. Of course, C is well-ordered in type κ so we are looking for a way of constructing a blocking $(\kappa+1)$-tower as in the last section. As hinted after the definition of a freely filtered presentation, we could do this if we could identify R^κ with $B|C$ and then add ρ^κ at the top. The difficulty here is that the cardinalities are wrong in general. This is where lemma 4 enters the picture, because what it does is to make the cardinalities agree, thus "matching" R^κ to $B|C$. Now, the hypotheses of lemma 4 are satisfied with R^κ, F^κ and ρ^κ in place of R, F and f, and with $B|j$ in place of T. Let $B_{\delta+1}$ be λ_δ-cop F and let $b_{\delta\delta+1}$ be the embedding e of the lemma.

Now let p be the unique fill-in making the following diagram commute:

$$\begin{tikzcd}
B_\delta \arrow[r, tail] \arrow[dr, "b_{\delta\delta+1}"'] & B_\delta \amalg B_{\delta+1} \arrow[d, "p"] & B_{\delta+1} \arrow[l, tail] \arrow[dl, "1_{B_{\delta+1}}"] \\
 & B_{\delta+1} &
\end{tikzcd}$$

By (EP), the following diagram commutes for some isomorphism $d: B_\delta \longrightarrow D$:

$$
\begin{tikzcd}
D \amalg B_{\delta+1} \arrow[r, two heads, "0 \amalg 1"] & B_{\delta+1} \\
B_\delta \amalg B_{\delta+1} \arrow[u, "d \amalg 1"] \arrow[r, two heads, "p"'] & B_{\delta+1} \arrow[u, "1_{B_{\delta+1}}"']
\end{tikzcd}
.
$$

Now, it is easy to see that, for any free A, $0 \amalg A \cong A$, by a natural isomorphism. It follows from this that the following diagram commutes too:

$$
\begin{tikzcd}
S_\delta \amalg D \amalg B_{\delta+1} \arrow[r, two heads, "0 \amalg 0 \amalg 1"] & 0 \amalg B_{\delta+1} \cong B_{\delta+1} \\
S_\delta \amalg B_\delta \amalg B_{\delta+1} \arrow[u, "1 \amalg d \amalg 1"] \arrow[r, two heads, "0 \amalg p"'] & 0 \amalg B_{\delta+1} \cong B_{\delta+1} \arrow[u, "1"']
\end{tikzcd}
$$

Now let $S_j \overset{\sigma_j}{\rightarrowtail} G_j \overset{\gamma_j}{\twoheadrightarrow} B_j$ be the top row of the next diagram, and $s_{\delta\delta+1}$ and $g_{\delta\delta+1}$ the indicated maps:

$$
\begin{tikzcd}
S_\delta \amalg D \arrow[r, tail] & S_\delta \amalg (D \amalg B_{\delta+1}) \arrow[r, two heads, "0 \amalg 0 \amalg 1"] & B_{\delta+1} \\
 & S_\delta \amalg (B_\delta \amalg B_{\delta+1}) \arrow[u, "1 \amalg d \amalg 1"] \arrow[r, two heads, "0 \amalg p"] & B_{\delta+1} \arrow[u, "1_{B_{\delta+1}}"'] \\
 & S_\delta \amalg B_\delta \arrow[u, tail] \arrow[d, equal] & \\
S_\delta \arrow[uuu, bend left, "s_{\delta\delta+1}"] \arrow[r, tail, "\sigma_\delta"] & G_\delta \arrow[uuu, bend left=60, "g_{\delta\delta+1}"] \arrow[r, two heads, "\gamma_\delta"] & B_\delta \arrow[uu, "b_{\delta\delta+1}"']
\end{tikzcd}
.
$$

This does the construction in case 3 for $\kappa > \omega$. If $\kappa = \omega$, proceed as above, using the fact that there is an unbounded set C in δ, avoiding E. The rest is then exactly as above.

This completes the description of the construction. We now check that the construction preserves the inductive assumptions (a) - (e). Cases 0 and 2 are trivial, so we turn to

Case 1: Clearly (a) holds. Using (i) and (d), there is C either a club (if $cf(j)>\omega$) or unbounded in j (if $cf(j) = \omega$) such that for all i in C, if $i<k<j$, the following diagram commutes, with b_{ik} a λ_k-complemented ff-map:

$$\begin{tikzcd}
S_k \arrow[r, tail, "\sigma_k"] & G_k & B_k \arrow[l, tail, "\alpha_k"'] \\
S_i \arrow[u, "s_{ik}"] \arrow[r, tail, "\sigma_i"] & G_i \arrow[u, "g_{ik}"] & B_i \arrow[l, tail, "\alpha_i"'] \arrow[u, "b_{ik}"']
\end{tikzcd}$$

From the commutativity of these diagrams for i in C and the fact that colimits and coproducts commute, it follows that G_j is the coproduct of S_j and B_j, with coproduct injections σ_j and α_j. Since coequalisers and colimits commute, γ_j is the coequaliser of 0 and σ_j, and is thus a left inverse for α_j. Finally, because for i, k in C the above diagram has s_{ik}, g_{ik}, b_{ik} all ff-maps, it follows from theorem 1.1 that S_j, G_j, B_j are all free on λ_j generators. This proves (b), (c) and (d). Clearly (e) continues to hold.

Case 3: The only non-obvious part is (d). We applied lemma 4 in the construction, in order to guarantee (e); however, the second part of the conclusion of that lemma gives us (d) for a club or unbounded set of i in δ. This is clearly enough.

So the construction preserves the condition and therefore can be carried out. We are left to define the top row of the presentation. Let

$$S = \operatorname{colim} S$$

$$G = \operatorname{colim} G$$

$$B = \operatorname{colim} B$$

and let $\sigma_\mu, \gamma_\mu, s_{i\mu}, g_{i\mu}, b_{i\mu}$ be the obvious induced maps.

It remains to check that $(S, G, B, \sigma, \gamma)$ is actually a freely filtered presentation of B, and that B is μ-almost free and $L_{\infty\mu}$-free. For the first of these, observe that by (c), (b) S and G are free on μ generators. By (a), (b) and (c), S, G, B are filtrations of the required type.* Clearly B is exactly μ^+-generated. Because E is stationary in μ and (e) holds, it follows from theorem 1.8 that B is not free. Because $\mu - E$ is unbounded in μ, and (d) holds, it follows also from theorem 1.8 that B is $L_{\infty\mu}$-free. //

Thus the construction of theorem 2.1 can be refined in such a way as to make it iterable, as we see in the next result.

<u>Corollary 6:</u> Suppose <u>V</u> satisfies (EP) and has <u>0</u>. Let $\aleph_\alpha$ be a regular cardinal. If there is an $\aleph_\alpha$-almost free algebra in with a freely filtered presentation, then for each n, $0<n<\omega$, there is an $\aleph_{\alpha+n}$-almost free algebra in <u>V</u>, which is moreover $L_{\infty\aleph_{\alpha+n}}$-free.

Proof: Apply the theorem with $E = \{\delta < \aleph_{\alpha+1} : cf(\delta) = \aleph_\alpha\}$, to obtain such an algebra for $\aleph_{\alpha+1}$ with a freely filtered presentation. It is easy to see that this construction works inductively to give the result. //

* Insert here correction on p111,

It may seem to the reader that the proof of theorem 5 is unnecessarily involved. Why was it necessary to introduce freely filtered presentations? Perhaps the comments of the next paragraph will convince the sceptical reader that there were good reasons for proceeding as we did.

Let us remark immediately that it is the demand that the theorem give rise to an itereative procedure supplying almost free algebras (as in corollary 6) that leads us to freely filtered presentations. This means that the constructed object must itself satisfy the hypotheses of the theorem (so that the theorem can be used to provide the induction step in, for example, corollary 6). Although we could start with blocking towers and proceed as in theorem 2.1 to obtain an almost free algebra, the process would terminate at this point since in our general setting there seems no way of going from almost free algebras to blocking towers (or even <u>free</u> presentations). So what we have done is to strengthen our hypotheses in order to be able to prove more - including the statement that the constructed objects satisfy the hypotheses of the theorem. Thus the circle is completed and the inductive argument of corollary 6 will work. In the next sections we shall see that these stronger hypotheses can be satisfied in particular cases.

Notice that we have not included in the hypotheses of the corollary that the given algebra be $L_{\infty\aleph_\alpha}$-free, but the algebras we obtain from the use of theorem 5 all are equivalent to free ones. The construction in the corollary does not go through limit cardinals - we can only obtain λ-almost free algebras for λ less than the next limit ordinal. This is not surprising in view of the fact that for many varieties (e.g. Abgps) there are no λ-almost free algebras for singular λ. (See theorem 1.11 (Shelah).) So at successors λ^+ of singular cardinals, we have to start again to obtain λ^+-almost free algebras, and will probably have to appeal to principles of a different kind from that employed in theorem 5. Such principles will presumably involve assumptions about the sort of set theory we use. We shall not discuss this further here. We note however that we

---

Correction for p109 (insert at *):

To see that C_μ is an ff-block, proceed as follows. Suppose not. Then there are free algebras H and K such that $G_\mu \amalg H = S_\mu \amalg K$. By lemma 2, $K \not\cong B_\mu \amalg H$. Now we can produce a filtration of $B_\mu \amalg H$ in the following way: start with a filtration $\{H_i : i \leq \mu\}$ of H by free factors, and take $B_0 \amalg H_0$, then $B_1 \amalg H_0$, then $B_1 \amalg H_1$, etc, with all the obvious inclusions, and take colimits as required on the way up. Since K is free, there is a club of i such that $B_i \amalg H_i$ is a free factor of $B_{i+1} \amalg H_i$. Since E is stationary, some such i must also be in E, which contradicts the construction of $b_{i\,i+1}$ for i in E.

would not expect the existence of an ω-almost free algebra with freely filtered presentation to depend on our choice of set theory. In the next section we shall consider varieties of groups and the existence of ω-almost free groups with freely filtered presentations in such varieties.

Now, theorem 5 gives us apparently just one μ-almost free algebra. We shall see now that there are many more - in fact 2^{μ}.

<u>Corollary 7</u>: Under the hypotheses of theorem 5, there are 2^{μ} pairwise non-isomorphic algebras in $\underline{V}$ which are μ-almost free and $L_{\infty\mu}$-free.

Proof: If E is as in the theorem then there are μ subsets E_i, $i<\mu$, of E which are pairwise disjoint, stationary in μ, and such that $E = \bigcup_{i<\mu} E_i$. All this is true by a theorem of Solovay given as theorem 85, p433, of /Jech2/. Let $\{J_i : i<2^{\mu}\}$ be a family of 2^{μ} subsets of μ such that if $i \neq j$ $J_i \nsubseteq J_j$. Let $E^i = \bigcup\{E_j : j \in J_i\}$. Then each E^i is a stationary (in μ) subset of E such that (i) and (ii) of theorem 5 hold; and if $i \neq j$, there is no club C in μ such that $C \cap E^i \subseteq E^j$ (since C intersects each $E_k \subseteq E^i$, and there is no E_k not included in E^j).

Apply theorem 5 with E^i to obtain a μ-almost free algebra B^i with a filtration $\mathcal{B}^i$ such that, for each $k \in \mu - E^i$, B^i_k is a free factor of each B^i_t with $k<t<\mu$, while, for each $\delta \in E^i$, $b^i_{\delta\delta+1}$ is an ff-block.

Now suppose $i \neq j$ and f is an isomorphism of B^i onto B^j. As in the proof of lemma 1.3, there is a club C in μ such that, for all k in C, $f(B_k^i) = B_k^j$. There is $k \in C \cap E^i$ such that $k \notin E^j$. Let $t \in C$ be such that $t>k$. Now B_{k+1}^i is a free factor of B_t^i, so B_k^i is not. On the other hand, since $k \notin E^j$, B_k^j is a free factor of B_t^j. Since k, $t \in C$, this contradicts the assumption that f is an isomorphism. //

The device used in this corollary is due to Shelah and can be found in Section 1 of /Shelah1/.

<u>Corollary 8:</u> Suppose $\underline{V}$ satisfies (EP) and has $\underline{O}$. Let $\aleph_\alpha$ be a regular cardinal. If there is an $\aleph_\alpha$-almost free algebra in $\underline{V}$, then there are, for each n, with $0<n<\omega$, $2^{\aleph_{\alpha+n}}$ pairwise non-isomorphic $\aleph_{\alpha+n}$-almost free algebras in $\underline{V}$ which are $L_{\infty\aleph_{\alpha+n}}$-free.

Proof: Obvious from corollaries 6 and 7. //

Let us note that the restriction that the language of the variety be countable is not necessary: all the above theorems remain true provided we replace in the definition of κ-almost free the requirement that the algebra have cardinality κ, by the requirement that the algebra be exactly κ^+-generated. The only place we used the fact that the language was countable was to check at various points that a "small" subalgebra was contained in some member of a chain before the end; clearly the concept of exactly κ^+-generated works in the same way. (Compare the remarks on p77.) These comments will enable us to apply these results in §5 below.

§4 Varieties of groups

In this section the results of §3 are appled to some varieties of groups.

First we check that any variety of groups satisfies (EP). In fact we prove something more general, to avoid repeating the proof in the next section. In the paper /Higgins/, the concept of an Ω-group is introduced. Let Ω be a similarity type containing a binary operator +, a unary operator -, and a nullary operator 0. Then an algebra $(G;\Omega)$ is an <u>Ω-group</u> iff

(G1) $(G;+,-,0)$ is a group;

(G2) for any $\alpha \in \Omega$, $\alpha(0,0,\ldots,0) = 0$.

The group in (G1) is not assumed to be commutative, despite the notation.

Thus, an Ω-group is just an Ω-algebra which is a group with respect to some operators +,-,0 in Ω and which has the additional property that $\{0\}$ is a subalgebra. Clearly any group is an Ω-group. More interestingly, so is each ring, R-module, K-algebra, for a suitable choice of Ω. For example, a K-algebra is a group with respect to addition and has a binary operator (multiplication) and a set of unary operators (corresponding to multiplication by members of K - one for each $k \in K$).

The conditions in (G1) and (G2) are expressible as equations, so we can talk about the variety of all Ω-groups for a fixed Ω.

<u>Lemma 1</u>: If $\underline{V}$ is a variety of Ω-groups, then $\underline{V}$ satisfies (EP).

Proof: Suppose A and B are free and that $e: A \rightarrowtail B$ is an embedding. Let p be the unique fill-in making the following diagram commute (with the natural embeddings):

$$
\begin{tikzcd}
A \arrow[r, tail] \arrow[dr, tail, "e"'] & A \amalg B \arrow[d, "p"] & B \arrow[l, tail] \arrow[dl, "1_B"] \\
& B &
\end{tikzcd}
$$

Let X and Y be bases for A and B respectively, and let T be the following subset of $A \amalg B$:

$$T = \{x - e(x) : x \in X\} \ .$$

Clearly $T \cup Y$ generates $A \amalg B$, and putting C equal to the subalgebra generated by T will do once we have shown that $T \cup Y$ is a basis of $A \amalg B$.

Suppose that $w(u_1,\ldots,u_n,v_1,\ldots,v_m)$ is a word in the variables shown and that $w(x_1-e(x_1),\ldots,x_n-e(x_n),y_1,\ldots,y_m) = 0$ in $A \amalg B$, where $x_i \in X$, $y_i \in Y$. We must show that

$$w(u_1,\ldots,u_n,v_1,\ldots,v_m) = 0$$

is a law in $\underline{V}$.

Since each $e(x_i)$ is a word $w_i(y_1,\ldots,y_k)$ in the members of Y, there is a word $w'(u_1,\ldots,u_n,v_1,\ldots,v_k)$ with variables as shown (and $k \geq m$) such that

$$w'(u_1,\ldots,u_n,v_1,\ldots,v_k) = w(u_1-w_1,\ldots,u_n-w_n,v_1,\ldots,v_m)$$

is a law (where w_i abbreviates $w_i(v_1,\ldots,v_k)$).

Since $w'(x_1,\ldots,x_n,y_1,\ldots,y_k) = 0$, and $X \cup Y$ is a basis for $A \amalg B$, it follows that $w'(u_1,\ldots,u_n,v_1,\ldots,v_k) = 0$ is a law, too.

Hence

$$w'(x_1+e(x_1),\dots,x_n+e(x_n),y_1,\dots,y_k) = 0,$$

from which it follows that

$$w(x_1,\dots,x_n,y_1,\dots,y_n) = 0.$$

Since $X \cup Y$ is a basis for $A \amalg B$, this means that

$$w(u_1,\dots,u_n,v_1,\dots,v_n) = 0$$

is a law, as required.

Thus $T \cup Y$ is a basis for $A \amalg B$, and (EP) holds with C as above, and h as the inclusion. //

Of course, this means that (EP) is satisfied in any variety of groups. To apply the methods of §3, we also need to know that one can take qoutients by kernels. This is one of the main points of /Higgins/. We go over to multiplicative notation now, and so we write the trivial group as (1). However, the trivial homomorphism will still be written 0, since 1 is reserved for the identity homomorphism. If A and B are groups in the variety $\underline{V}$ of groups, we shall write A*B for the free product in $\underline{V}$ (=coproduct) of A and B. So * changes with $\underline{V}$, but we shall leave it to the context to make clear which free product we are working with.

So in order to apply corollary 3.8 in varieties of groups, all we have to do is produce some freely filtered presentations of almost free groups. We start in the variety Gps of all groups.

In the definitions of the following paragraph, terms such as "free" are to be understood relative to Gps: so "free" is to mean what is often referred to as "absolutely free". We produce an ω-almost free group with a freely filtered presentation. The group will be isomorphic to the subgroup of the rationals under addition generated by $\{1/2^n : n<\omega\}$ (although it will be written multiplicatively). We have to define filtrations $\mathcal{R}$, $\mathcal{F}$ and $\mathcal{A}$ and two natural transformations ρ and ϕ. We adopt the same <u>notational</u> <u>convention</u> as on the first page of §3.

Let F be the free group on $\{x_n : n<\omega\}$ and let $\mathcal{F}$ be the filtration $\{F_i \hookrightarrow F_j : i<j\leq\omega\}$ where $F = F_\omega$, $F_i = \langle x_n : n\leq i\rangle \leq F$ and f_{ij} is the inclusion. Let $\mathcal{R}$ be the filtration $\{R_i \hookrightarrow R_j : i<j\leq\omega\}$ where $R = R_\omega$, $R_i = \langle x_n x_{n+1}{}^{-2} : n<i\leq\omega\rangle \leq F$ and r_{ij} is the inclusion. Let

$$\mathcal{A} = \{A_i \xrightarrow{a_{ij}} A_j : i<j\leq\omega\}$$

be such that $A_i = F_i/N_i$, where N_i is the normal closure in F_i of R_i, for all $i\leq\omega$, and $a_{ij}: A_i \longrightarrow A_j$ is given by $a_{ij}: xN_i \longmapsto xN_j$, for all $i<j\leq\omega$. Finally, let $\rho_i : R_i \longrightarrow F_i$ be the inclusion, and let $\phi_i : F_i \twoheadrightarrow A_i$ be the canonical projection, for each $i\leq\omega$.

<u>Lemma 2:</u> In <u>Gps</u>, $(\mathcal{R},\mathcal{F},\mathcal{A},\rho,\phi)$ is a freely filtered presentation of the group $A = A_\omega$, which is ω-almost free.

Proof: Clearly A is countable and not free. Any finitely generated ($=\omega$-generated) subgroup of A is infinite cyclic and hence free. So A is ω-free.

Since <u>Gps</u> is a Schreier variety, it is clear that R_i and F_i are free for all i. It is also clear that F_i is a free factor of F_j for all $i<j\leq\omega$. It remains to check several things:

(1) R_i is a free factor of R_j whenever $i<j\leq\omega$.

Let K_i be the normal subgroup of F generated by $\{x_n : i \leqslant n < \omega\}$. Then $R_i \cap K_i = F_{i-1} \cap K_i = (1)$, and hence $F = R_i K_i = F_{i-1} K_i$. Thus R_i is free on i generators, and since $\{x_0 x_1^{-2}, \ldots, x_{i-1} x_i^{-2}\}$ is a generating set for R_i with i members, this is actually a free generating set, by 41.52 and 41.33 of /Neumann/. Thus (1) holds.

(2) R_i is a free factor of F_i for all $i<\omega$. From the proof of (1), it is clear that $F_i = R_i * \langle x_i \rangle$.

(3) A_i is free for all $i<\omega$. This follows immediately from (2), and we observe that A_i is free on $x_i N_i$.

(4) each a_{ij} is well-defined, $i<j\leqslant\omega$. We have to check that if $xN_i = yN_i$, then $xN_j = yN_j$. This follows immediately from the fact that $N_i \subseteq F_i \cap N_j$, and this is true because $F_i \cap N_j$ is normal in F_i and includes R_i, while N_i is least with these properties.

(5) each a_{ij} is an embedding. By the remark at the end of the proof of (3), we need only consider what happens to $x_i N_i$. Since $x_i x_{i+1}^{-2} \in R_{i+1}$, $x_i N_{i+1} = x_{i+1}^{2} N_{i+1}$, and since A_{i+1} is free on $x_{i+1} N_{i+1}$, we see that $a_{i\,i+1}$ is an embedding for all $i<\omega$. Hence $a_{i\omega}$ is also an embedding, since it is clearly the colimit-induced map, it being clear that $a_{ik} = a_{jk} a_{ij}$ for all $i<j<k\leqslant\omega$.

From (1) to (6)* it follows that (R,F,A,ρ,ϕ) is a freely filtered presentation of A. //

Now, the above construction and proof can also be carried out

* For (6), see footnote correction on p 120

in the variety Abgps, of all abelian groups, to produce the same A (although the filtrations will obviously be different). Thus we have already

Theorem 3: (/Eklof2/, /Mekler/) If $\underline{V}$ is either Gps or Abgps, then there are $2^{\aleph_n}$ $L_{\infty\aleph_n}$-free groups in $\underline{V}$, for each n, $0<n<\omega$.

Proof: By corollary 3.8 and the above. //

However, we can offer more. Recall that a variety of groups is torsion-free iff $x^n = 1$ is not a law in the variety for any n. This is equivalent to the variety's having all of Abgps contained in it, and hence also to the variety's having $\mathbb{Z}$ as its free group on one generator. If $\underline{V}$ is a variety of groups and G is a group, then we write $V(G)$ for the verbal subgroup of G corresponding to $\underline{V}$ (as in 14.31 of /Neumann/). The assignment $G \longmapsto G/V(G)$ induces a functor $\underline{\text{Gps}} \longrightarrow \underline{V}$, and we denote this functor too by $\underline{V}$. This functor is left adjoint to the inclusion of $\underline{V}$ into Gps.

Lemma 4: If $\underline{V}$ is a torsion-free variety of groups and (R,F,A,ρ,ϕ) is as defined above for Gps, then $(\underline{V}R,\underline{V}F,\underline{V}A,\underline{V}\rho,\underline{V}\phi)$ is a freely filtered presentation of the group $\underline{V}A$, which is ω-almost free in $\underline{V}$.

Proof: Since $\underline{V}$ is left adjoint, it preserves freeness, free factors and colimits. The only things to check are that a_{ij} is monic, and that A is ω-free but not free in $\underline{V}$. First we note that since A is abelian, and $\underline{V}$ is torsion-free, $\underline{V}A = A$; if A were free in $\underline{V}$, then A would have to be isomorphic to $\mathbb{Z}$, which it is not. So A is not free. On the other hand, since each A_i is abelian and free in Gps, $\underline{V}A_i$ is free in $\underline{V}$ and abelian and hence just A_i. Hence a_{ij} is still the map induced by $\phi_i(x_i) \longmapsto \phi_{i+1}(x_{i+1})^2$ and embeds. It follows that $\underline{V}A$ is ω-free.** //

Now we obtain our main theorem of this section:

Theorem 5: If $\underline{V}$ is a torsion-free variety of groups and $0<n<\omega$, then there are $2^{\aleph_n}$ pairwise non-isomorphic groups in $\underline{V}$ of cardinality $\aleph_n$ which are $L_{\infty\aleph_n}$-free in $\underline{V}$.

Proof: By lemma 1, lemma 4 and corollary 3.8. //

The obvious question raised by this theorem is: what happens in varieties with torsion? It is clear that the theorem is not true in general without some further restriction on $\underline{V}$, since in the variety of abelian groups in which $x^p = 1$ is a law for one prime p, every group is free. The problem is in the proof of lemma 4: if $x^2 = 1$ is a law in $\underline{V}$, then $\underline{V}A = 1$; if $x^3 = 1$ is a law in $\underline{V}$, then $\underline{V}A$ is cyclic of order 3 and hence free in $\underline{V}$. So this question must remain open.

Correction for p 118 (between lines −4 and −3):

(6) $\mathcal{F}_\omega$ is an ff-block. For, if not, there is a free group B with presentation $R \rightarrowtail H \twoheadrightarrow B$, in which A is embeddable, which contradicts the non-freeness of A.

** Insert here:

To see that $\underline{V}\mathcal{F}_\omega$ is an ff-block, suppose not and consider the abelianisation of the counterexample. It is now clear that we are in the situation of (6) above (because Abgps has the Schreier property).

§5 Varieties of K-rings

In this section, the methods of §3 are applied in varieties of K-rings and theorems similar to those of §4 are deduced for varieties of rings, K-algebras over a field K, and Lie algebras.

We have to begin with discussion of a triviality. Suppose we want to apply the methods of §3 to the variety of rings. It is usual to require that rings have a 1, although not that this is distinct from the additive identity - making it an axiom that $0 \neq 1$ prevents the class of rings from being a variety. The zero ring has a 1, then, which of course coincides with the 0; but it is not a subring of any other ring since the obvious inclusion is not a homomorphism (it does not send 1 to 1). The upshot of this is that factoring out kernels makes no sense in the category of all rings with a 1 and their homomorphisms, and on the face of it the methods of §3 do not apply.

There are several ways out of this difficulty. The naive approach is to go back to §3 and try to isolate all the points at which we used the fact that the variety in question has a zero algebra, and then to try to get away with something less. This would work, at least for the special case of rings, but is more trouble than it is worth. Instead, we drop the requirement that rings have 1s, and then show after we have dealt with this case that we can reintroduce the requirement without changing our results. This means that we work first in the variety of rings which need not have a 1 (in which variety the zero ring <u>is</u> a zero object) and then look at the subvariety of rings with a 1 and show that if we add a 1 in the right way we turn free rings (without 1s) into free rings with 1s, while at the same time we do not turn any non-free rings without 1s into free rings with 1s.

The remarks above related to rings, but in fact they apply to the more general concept of K-rings, to which we now turn.

We use the word ring for a ring with a 1 and the subword rng for a ring which may not have a 1; to be more precise, a rng does not have a distinguished 1. We use Rings and Rngs for the corresponding categories (and varieties). Let K be a ring. A K-rng is a K-bimodule R which is also a rng; that is, if we denote the multiplication and both K-actions by juxtaposition, R is a rng and $(xy)z = x(yz)$ for all x, y and z from either K or R. If the K-rng R is also a ring, we call R a K-ring. Thus a K-ring is a ring R with a homomorphism $k \mapsto k1_R$ of K into R. If K is commutative, then a K-ring is called a K-algebra. We note that a $\mathbb{Z}$-rng (where $\mathbb{Z}$ is the ring of integers) is just a rng, and that a $\mathbb{Z}$-algebra is just a ring. (Compare /Anderson & Fuller/, ex 1.11,p24.) The corresponding categories (and varieties) will be denoted by K-rngs, K-rings, etc, and their commutative versions by writing C in front of these names. (The underline is this time part of the definition.)

The connection between the categories K-rngs and K-rings which is important for us in this section is the following result.

Lemma 1: Let K be a ring. There is a functor add1 from K-rngs to K-rings which preserves and reflects freeness and takes embeddings to embeddings.

Proof: As its name suggests, add1 adjoins a 1 to a rng. The construction is analogous to the standard one for embedding a rng in a ring (compare /Anderson & Fuller/, ex 1.1, p22). Form $R \times K$, and give this set a bimodule structure by defining addition and the actions co-ordinatewise (regarding K as a K-bimodule in the natural way). Define multiplication by

$$(r,k)(r',k') = (rr' + kr' + rk', kk').$$

It is easy to check that this makes $R \times K$ a K-ring, which we shall denote by add1(R). It is clear how to define add1 on homomorphisms. It is easy to verify directly that add1 preserves freeness (it amounts to showing that add1 is left adjoint to the functor "forget the 1" from <u>K-rings</u> to <u>K-rngs</u>), while if add1(R) were free (in <u>K-rings</u>) on X, then add1(F) and add1(R) would be isomorphic, where F is a free <u>K-rng</u> on X, from which it is clear that R and F are isomorphic. The last requirement is clear from the definition of add1. //

The point of the lamma is that it enables us to to work in <u>K-rngs</u> to construct filtrations, and then to apply add1 to obtain filtrations in <u>K-rings</u>. (It is easy to see that add1 preserves direct limits of embeddings - or use the fact that it is a left adjoint.)

The next lemma is the point of our excursion into <u>K-rngs</u>:

<u>Lemma 2:</u> Let K be a ring. The category <u>K-rngs</u> has a zero object.

Proof: Clearly the zero K-bimodule is a zero. //

The final condition to be checked before we can use §3 is that (EP) holds.

<u>Lemma 3:</u> Let K be a ring. Then <u>K-rngs</u> satisfy (EP).

Proof: K-rngs can be regarded as Ω-algebras. Apply lemma 4.1.//

We are now almost ready to produce a freely filtered presentation. For our applications, it is sufficient to assume that K is an integral domain (that is, a not-necessarily-commutative ring without zero divisors). Making this assumption has the consequence that we have available the concept of a degree function as in /Cohn2/, p32, p34. This makes it easy to see whether the algebras we construct in lemma 4 are free. The definition of a degree in /Cohn2/ works for K-rings and we extend it to K-rngs by defining the degree of an element of the K-rng R to be the degree of its image in add1(R), if add1(R) has a degree function.

In the following definition, K is assumed to be an integral domain. The definition is modelled on that preceeding lemma 4.2.

Let F be the free K-rng on $\{x_n : n<\omega\}$, and let $\mathcal{F}$ be the filtration $\{F_i \hookrightarrow F_j : i<j\leqslant\omega\}$, where $F_\omega = F$, $F_i = \langle x_n : n\leqslant i\rangle \leqslant F$ and f_{ij} is the inclusion of F_i into F_j. Let $R_i = \langle x_n - x_{n+1}^2 : n<i\rangle \leqslant F$ for $i\leqslant\omega$, put $R = R_\omega$ and let r_{ij} be the inclusion for $i<j\leqslant\omega$. Let $\mathcal{R}$ be the filtration $\{R_i \hookrightarrow R_j : i<j\leqslant\omega\}$. (We shall show below that it <u>is</u> a filtration.) For each $i\leqslant\omega$ let A_i be the quotient of F_i by the ideal generated by R_i and let a_{ij} be the map induced from A_i to A_j for $i<j\leqslant\omega$. Put $A = A_\omega$, and let $\mathcal{A}$ denote the directed system $\{A_i \xrightarrow{a_{ij}} A_j : i<j\leqslant\omega\}$. Finally let

ρ_i be the inclusion of R_i into F_i and let ϕ_i be the natural map of F_i onto A_i, for $i \leqslant \omega$.

<u>Lemma 4:</u> In the above notation, (R, F, A, σ, ϕ) is a freely filtered presentation of the ω-almost free K-rng A.

Proof: First we show that $\{x_0 - x_1^2, x_1 - x_2^2, \ldots, x_{i-1} - x_i^2, x_i\}$ is a basis for F_i. Suppose p is a word in i+1 variables (a polynomial) such that:

$$p(x_0 - x_1^2, \ldots, x_{i-1} - x_i^2, x_i) = 0.$$

Since $\{x_0, \ldots, x_i\}$ is a basis for F_i, the following is a law (in the variables $u_0, \ldots, u_i$):

$$p(u_0 - u_1^2, \ldots, u_{i-1} - u_i^2, u_i) = 0.$$

If we substitute x_i for u_i and

$$(\ldots((x_i^2 + x_{i-1})^2 + x_{i-2})^2 + \ldots)^2 + x_{i-k} \qquad \text{for } u_{i-k}$$

$(k = 1, \ldots, i)$, we see that

$$p(x_0, \ldots, x_{i-1}, x_i) = 0,$$

and hence that this is a law and the set in question is a basis.

It is now easy to check the facts that r_{ij} and ρ_i are ff-maps, that A_i is free on $\phi_i(x_i)$, and that a_{ij} is an embedding, for all $i < j \leqslant \omega$.

Since K is an integral domain, if A were free it would have a degree function, since then add1(A) would be free and would have a degree function, by /Cohn2/, p34. This would lead to a contradiction since the degree of $\phi_\omega(x_0)$ would then be undefined because $\phi_\omega(x_0) = \phi_\omega(x_n)^{2^n}$ for all n. This argument shows A is not embeddable in a free, and hence by the method of (6), p. 120, that ρ_ω is an ff-block.

Any finite set of generators of A would lie in some A_n with

$n<\omega$, and A is clearly ω_1-generated, so A is exactly ω_1-generated, as required. //

Whether it can be shown that A in this lemma is not free when K is not an integral domain is not clear (to this author). It seems that some condition on K may be necessary in view of what happens in torsion varieties of groups (comapre the remarks at the end of §4).

We now have:

Theorem 5: Let K be an integral domain. If $0<n<\omega$, then there are $2^{\aleph_n}$ pairwise non-isomorphic K-rngs which are exactly $\aleph_{n+1}$-generated and $L_{\infty\aleph_n}$-free.

Proof: By lemmas 2, 3 and 4 and corollary 3.8. //

And we deduce

Theorem 6: Let K be an integral domain, and let n be a positive integer. If $\underline{V}$ is one of the following categories:

K-rings, K-algebras (with K commutative), Rings,

CK-algebras (K commutative) or CRings,

then there are $2^{\aleph_n}$ pairwise non-isomorphic $\underline{V}$-algebras which are exactly $\aleph_{n+1}$-generated and $L_{\infty\aleph_n}$-free in $\underline{V}$.

Proof: For K-rings this follows using the fact that add1 takes filtrations of non-free K-rngs to filtrations of non-free K-rings.

Again [illegible] add1 [illegible]

The special cases of K-algebras and Rings now follow. The commutative versions follow by repeating word for word all that has been said so far in this section, since we have not assumed non-commutativity anywhere as an axiom. //

It would be possible to prove analogues of the above for some subvarieties of K-rngs and their images under add1 in K-rings. The reader can be spared this.

The following corollary may be of interest however.

Theorem 7: Let K be a commutative field and n a positive integer. There are $2^{\aleph_n}$ pairwise non-isomorphic Lie algebras over K which are exactly $\aleph_{n+1}$-generated as Lie algebras and $L_{\infty\aleph_n}$-free as Lie algebras.

Proof: Let U be the functor which sends each Lie algebra over K to its universal enveloping algebra. Then U is left adjoint to the functor from K-algebras to the category of all Lie algebras over K which sends the algebra A to its Lie algebra $[A]$ of commutators (that is, A with $[x,y] = xy - yx$ as the new product). This is proved in §§V.1,2 of /Jacobson/, for example. Moreover, if A is free, so is $[A]$. It follows that a filtration of a non-free K-algebra induces a filtration of a non-free Lie algebra in the following way. Let $\{A_i : i \leqslant \kappa\}$ be a filtration of the non-free K-algebra A. Each A_i is the universal enveloping algebra of $[A_i]$, so $[A_i]$ is free. Let L be the union of the $[A_i]$. Since U is a left adjoint it preserves colimits, and $U(L) = A$. If L were free, so would A be. It is not claimed (and not denied) that $\{[A_i] : i \leqslant \kappa\}$ is a filtration of L, but by taking unions in L and using the fact that Lie algebras have the Schreier property, it can be completed to one. The theorem follows. //

§6 Nilpotent varieties of groups

In this section we show how the concepts of §1 apply in these varieties. For the variety of all abelian groups a lot is known, and the extension to the other abelian varieties presents little difficulty. The point of this section is that the nilpotent varieties can be handled by referrring to their abelian parts. This is made precise in theorem 13.

Unfortunately, it does not seem possible to deal with nilpotent varieties in a uniform way, and so we have to distinguish three cases. First we consider abelian varieties, and we further distinguish cases based on the exponent of the variety. Recall that a variety is torsion-free iff it has exponent 0; we shall say a variety is <u>primary</u> iff it has prime-power exponent, and <u>composite</u> iff its exponent is not zero and is divisible by two different primes. These three possibilities are of course mutually exclusive.

This section is organised as follows: first we deal with torsion-free and primary abelian varieties and work out what free subgroups and free factors of free groups look like. Second, we do the same thing for torsion-free and primary nilpotent varieties. Third, we work out what λ-free groups look like in composite nilpotent varieties, with λ uncountable. Then we consider λ-purity for torsion-free and primary nilpotent varieties and work out what the $L_{\infty\lambda}$-free groups are in composite nilpotent varieties. Finally, we combine everything in theorem 13.

In this section, we abbreviate Abgps to Ab.

Theorem 1: Let p be a prime, n a positive integer.

(a) In Ab, every subgroup of a free group is free and its free factors are precisely its direct summands;

(b) In Ab(p^n), if F is free, then for any $G \leqslant F$ the following are equivalent:

(i) G is free

(ii) G is pure in F

(iii) G is a direct summand of F

(iv) G is a free factor of F.

Proof: Of course, (a) is standard. (See, for example, /Fuchs/, 14.5.) For (b), note that G is pure in F iff the p-heights of each $x \in G$ are the same in G and in F iff G is free. So (i) and (ii) are equivalent. The equivalence of (ii) and (iii) is an easy consequence of 27.5 of /Fuchs/. If $F = G \oplus H$, then by what we have just shown H is free and the equivalence of (iii) and (iv) is established. //

Corollary 2: The varieties in theorem 1 have (FP) and (FF).

Proof: (FF) follows from (b) on p38 of /Fuchs/ in all cases, and (FP) from the fact that Ab is Schreier in case (a). In case (b) we note that in any near-filtration the groups appearing at successor steps are free, hence pure, which means that the groups appearing at limit steps are pure, and hence free. So every near-filtration is a filtration. //

Corollary 3: In $\underline{Ab}(p^n)$, if A is 2-free, then A is free.

Proof: By 17.2 of /Fuchs/, A is a direct sum of cyclic groups. If $\langle a\rangle$ is a direct summand of A, then $\langle a\rangle$ is a direct summand of some free subgroup of A. By the theorem, $\langle a\rangle$ is free. Thus A is the free product of free groups and is free. //

Now we turn to the nilpotent torsion-free and primary varieties.

Lemma 4: Let H be a pure subgroup of the nilpotent group G. If S is a subset of H such that S/G' $(= \{sG' : s \in S\})$ generates HG'/G', then S generates H.

Proof: Since H is pure in G, $HG'/G' \simeq H/H'$ (by lemma 0.11.1) by an isomorphism sending $sG' \longmapsto sH'$ sor all $s \in S$. Thus S/H' generates H/H' and by lemma 0.13.1 S generates H. //

Theorem 5: Let $\underline{V}$ be a nilpotent torsion-free or primary variety of groups. Let F be free in $\underline{V}$ and suppose $G \leqslant F$ is a pure subgroup which is not contained in F'. Then G is free.

Proof: F^{ab} is free in $\underline{V} \cap \underline{Ab}$ and $\underline{V} \cap \underline{Ab}$ is one of the varieties of theorem 1. Since G^{ab} is pure in F^{ab}, by lemma 0.11.1, it follows in all cases that G^{ab} is free in $\underline{V} \cap \underline{Ab}$. Let $S \subseteq G$ be such that S/F' is a $\underline{V} \cap \underline{Ab}$-basis for G^{ab}. Since $G \nleqslant F'$, S is not empty. By lemma 4, S generates G. If the exponent of $\underline{V}$ is 0, then S is a basis of G by theorem 0.13.2, and in the other case, S is a basis of G by theorems 1 and 0.13.3. So G is free. //

Corollary 6: If $\underline{V}$ is as in theorem 5, then $\underline{V}$ has (FP). Further, in a filtration, all the groups may be taken to be pure subgroups.

Proof: Suppose we wish to construct a filtration of the λ-free group A in $\underline{V}$, where $|A| = \lambda$. We take $a_0 \subseteq A$ such that $a_0 \subsetneq A'$ and $|a_0| < \lambda$. Then proceed as in the proof of theorem 1.4, except that instead of taking A_{i+1} free we take A_{i+1} pure. This constructs a near-filtration of A, all of whose members are pure and not contained in A'. By the theorem, this is a filtration. //

Having worked out what at least some of the free subgroups look like, let us consider the free factors.

Lemma 7: Let $\underline{V}$ be as in theorem 5. Suppose B in $\underline{V}$ is free and A is a pure subgroup of B. Then A is a free factor of B iff A is free in $\underline{V}$ and A^{ab} is a free factor of B^{ab} in $\underline{V} \cap \underline{Ab}$.

Proof: The necessity follows from the fact that the functor $-^{ab}$ is left adjoint to the inclusion of $\underline{V} \cap \underline{Ab}$ into $\underline{V}$.

For the sufficiency, note first that A^{ab} is $\underline{V} \cap \underline{Ab}$-free. Let X be a basis of A. Then, X/B' is a basis for A^{ab} in $\underline{V} \cap \underline{Ab}$ and so can be extended to $(X \cup Y)/B'$, a basis of B^{ab}. Using theorem 0.13.2 or theorem 0.13.3 as appropriate, we see $X \cup Y$ generates freely a free subgroup of B, which by lemma 4 is B. So A is a free factor of B. //

Corollary 8: Suppose $\underline{V}$ is nilpotent and primary. Let A in $\underline{V}$ be uncountable and $|A|$-free. Then A is free.

Proof: $\underline{V}$ has (FP) by corollary 6 and a filtration (with $\lambda = |A|$)

$\{A_i : i \leqslant cf(\lambda)\}$ may be chosen so that each A_i is pure in A. Hence each A_i^{ab} is a free factor of A_{i+1}^{ab}, by theorem 1(b) and lemma 0.11.1 and hence, by lemma 7, A_i is a free factor of A_{i+1} for each i. It follows then from theorem 1.1 that A is free.

//

<u>Theorem 9</u>: If $\underline{V}$ is as in theorem 5, then $\underline{V}$ has (FF).

Proof: Suppose A,B,C are free in $\underline{V}$ with $A \leqslant B \leqslant C$, and that A is a free factor of C. We have to show that A is a free factor of B.

Since A is pure in C, A is pure in B and by lemma 7 it is enough to show that A^{ab} is a free factor in $\underline{V} \cap \underline{Ab}$ of B^{ab}. By theorem 1 this is equivalent to showing that A^{ab} is a direct summand of B^{ab}. Since A^{ab} is a direct summand of C^{ab}, this now follows, and we are done. //

So far we have said nothing about composite varieties. We turn to these now. The existence and uniqueness of the S_i in the next theorem are guaranteed by theorem 0.13.4.

<u>Theorem 10</u>: Suppose $\underline{V}$ is a nilpotent variety of groups of exponent $p_1^{n_1} \ldots p_r^{n_r}$ where the p_i are distinct primes and the n_i are positive integers. Let λ be an uncountable cardinal, and suppose A in $\underline{V}$ has cardinality λ and is λ-free. For $i = 1,\ldots,r$ let S_i be the p_i-Sylow subgroup of A. Then

(i) S_i is $\underline{V} \cap \underline{B}(p_i^{n_i})$-free for each $i = 1,\ldots,r$;

(ii) if λ is a limit cardinal, then each S_i is of cardality λ, and A is free;

(iii) if $\lambda = \rho^+$, then at least one S_i has cardinality λ, and A is not free iff at least one has cardinality ρ.

Proof: It is easy to see from the fact that $A = S_1 \times \ldots \times S_r$ that each S_i is λ-free in $\underline{V} \cap \underline{B}(p_i^{n_i})$. If $|S_i| < \lambda$, this means that S_i is free, while if $|S_i| = \lambda$, then since λ is uncountable, we have (i) by corollary 8. The remainder now follows, using the fact that $A = S_1 \times \ldots \times S_r$ and A is free iff all the S_i are of cardinality λ. //

Now we get some control over λ-purity.

<u>Lemma 11</u>: Let $\underline{V}$ be nilpotent and torsion-free or primary, and let λ be an uncountable cardinal. Suppose B in $\underline{V}$ is λ-free and A is a λ-generated subgroup of B, not contained in B'. Then A is λ-pure in B iff A^{ab} is λ-pure in B^{ab}.

Proof: Suppose A is λ-pure in B. Let C be such that $A^{ab} \leqslant C \leqslant B^{ab}$ and C is λ-generated over A^{ab}. There is $D \leqslant B$ such that A is a free factor of D, D is free and $C \leqslant D^{ab}$. (This follows from theorem 9 and the λ-freeness of B.) Hence A^{ab} is a $\underline{V} \cap \underline{Ab}$-free factor of D^{ab} and thus of C, by corollary 2. So A^{ab} is λ-pure in B^{ab}.

For the converse, let $X \subseteq B$ have cardinality $< \lambda$. Let D be pure in B such that $A \cup X \subseteq D$ and $|D| = |A \cup X| + \omega$. Then D is free since B is λ-free (using theorem 5). Now A^{ab} is a free factor (in $\underline{V} \cap \underline{Ab}$) of D^{ab} and by lemma 7, A is a free factor of D. Thus A is λ-pure in B. //

Now we show that if $\underline{V}$ is a composite nilpotent variety, then the only $L_{\infty\lambda}$-free groups in $\underline{V}$ of cardinality λ, uncountable, are the free ones.

Lemma 12: Suppose $\underline{V}$ is a composite nilpotent variety and λ is an uncountable cardinal. If A in $\underline{V}$ is λ-free and has cardinality λ but is not free, then A is not $L_{\infty\lambda}$-free.

Proof: By theorem 10, we may assume $\lambda = \rho^+$, and

$$A = S_1 \times \ldots \times S_r,$$

where S_i is $\underline{V} \cap \underline{B}(p_i^{n_i})$-free of cardinality ρ or ρ^+. Without loss of generality, assume $|S_1| = \rho$. Let $F_1 \times \ldots \times F_r$ be free of cardinality ρ^+, where each F_i is the p_i-Sylow subgroup of F. Suppose J is a family of isomorphisms between subgroups of A and subgroups of F with the $<\rho^+$-back-and-forth property. Let $f \in J$ be such that $S_1 \subseteq \mathrm{dom}(f)$. Then if $X \subseteq F_1$ has cardinality ρ and is disjoint from $f(S_1)$, it is easy to see that there is no $g \in J$ such that $g \supseteq f$ and $\mathrm{range}(g) \supseteq X$. This is a contradiction. //

And now the main theorem of this section:

Theorem 13: Let $\underline{V}$ be a nilpotent variety of groups and let λ be an uncountable cardinal. Suppose $A \in \underline{V}$ is λ-free of cardinality λ. Then

(i) A is free in $\underline{V}$ iff A^{ab} is free in $\underline{V} \cap \underline{Ab}$;

(ii) A is $L_{\infty\lambda}$-free in $\underline{V}$ iff A^{ab} is $L_{\infty\lambda}$-free in $\underline{V} \cap \underline{Ab}$.

Proof: Note first that if λ is singular, then A is free by theorem 3.1 of /Hodges2/ (which appears below as theorem 7.17, with a different proof). Hence A^{ab} is $\underline{V} \cap \underline{Ab}$-free and (i) and (ii) follow immediately. If $\underline{V}$ is composite, then either both A and A^{ab} are free or both are not, since in the notation of theorem 10, S_i and S_i^{ab} have the same cardinality for each i. Thus (i) holds for composite $\underline{V}$, and lemma 12 supplies us with (ii) for these varieties.

We are left with the case of regular λ and $\underline{V}$ torsion-free or primary. In (i) the necessity follows from the fact that the functor $-^{ab}$ preserves freeness. So to do the sufficiency, suppose A^{ab} is free and let $\{A_i : i \leqslant \lambda\}$ be a filtration of A by pure subgroups A_i such that $A_0 \nleqslant A'$. Then $\{A_i^{ab} : i \leqslant \lambda\}$ is a filtration of A^{ab}, and since A^{ab} is free,

$$E = \{i < \lambda : A_i^{ab} \text{ is not } \lambda\text{-pure in } A^{ab}\}$$

is not stationary in λ. By lemma 11,

$$E = \{i < \lambda : A_i \text{ is not } \lambda\text{-pure in } A\},$$

so A is free, as required.

For (ii), both $\underline{V}$ and $\underline{V} \cap \underline{Ab}$ have (FF), by theorem 9 and corollary 2, so by lemma 11 and theorem 1.7, the result follows easily.

//

§7 Parafree almost free groups

In this section we see that in some cases groups which are λ-almost free in a variety $\underline{V}$ of groups must also be $\underline{V}$-parafree, and we obtain as a corollary the extension of theorem 1.11 (Shelah) to nilpotent varieties of groups (Theorem 3.3 of /Hodges2/).

In this section the word "variety" will always mean variety of groups.

In /Baumslag1,2/ the concept of a $\underline{V}$-parafree group in a variety $\underline{V}$ was introduced. These groups share many of the properties of free groups in $\underline{V}$, at least if the free groups in $\underline{V}$ are residually nilpotent. We shall employ the methods of /Stammbach/ (especially §IV.5), and the reader can find there and in the two papers of Baumslag a summary of the properties of $\underline{V}$-parafree groups. The definition of $\underline{V}$-parafree given below comes from /Stammbach/.

Up till now, we have used the notation $G^{(i)}$ to represent the ith member of the lower central series of the group G, but in this section it will be convenient to drop the brackets and write G^i for this.

Two groups K and G ahve the <u>same</u> <u>lower</u> <u>central</u> <u>sequence</u> iff there exists for each positive integer i an isomorphism f_i of K/K^i onto G/G^i such that for each $i \geqslant 2$ the following square commutes:

$$\begin{array}{ccc} K/K^i & \twoheadrightarrow & K/K^{i-1} \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_{i-1}} \\ G/G^i & \twoheadrightarrow & G/G^{i-1} \end{array} \qquad \ldots(1)$$

Notice that we do not require the existence of a homomorphism from K to G; but of course if there is one then it will induce homomorphisms such that the square (1) commutes (since projection onto G/G^i is natural in G) and K and G will have the same lower central sequence if these homomorphisms are isomorphisms onto.

If $\underline{V}$ is a variety, then the group G is $\underline{V}$-parafree iff

(i) G is in $\underline{V}$;

(ii) G is residually nilpotent;

and (iii) G has the same lower central sequence as a group which is free in $\underline{V}$.

The rank of a free group of $\underline{V}$ which witnesses (iii) must be rank(G^{ab}), so there is at most one. In /Baumslag1,2/ examples are given of $\underline{V}$-parafree groups which are not free in $\underline{V}$. The techniques of /Stammbach/ express condition (iii) of this definition inhomological terms, and since these techniques are probably not generally known to non-specialists, there follows a sketch of the approach adopted in this book.

The idea of the book is to produce a homological algebra which is relative to a variety of groups. So instead of working out the homology of abelian groups or the homology of groups, one works out the homology of groups in $\underline{V}$. This is done as follows. If G is a group and M is a right G-module (that is, a right $\mathbb{Z}G$-module, where $\mathbb{Z}G$ is the integral group ring), then a homomorphism $f:K \to G$ induces a right K-module structure on M and a homomorphism (of groups)

$$f_*: H_2(K,M) \longrightarrow H_2(G,M),$$

where H_2 denotes the usual second homology group. (Compare /Stammbach/, p12.)

Stammbach defines a functor $V(-,M)$ from the category of all groups in the variety $\underline{V}$ to the category of abelian groups which behaves with respect to groups in $\underline{V}$ very much as $H_2(-,M)$ does. We shall not give the full definition here; it is sufficient for our purposes to define it on groups, and to know that it is a functor. Let $f:F \twoheadrightarrow G$ be an epimorphism in $\underline{V}$, with F free in $\underline{V}$. Then for any right G-module M we put

$$V(G,M) = \operatorname{coker}\,(f_*: H_2(F,M) \longrightarrow H_2(G,M)),$$

where f_* is the map induced by f.

Finally, regarding $\mathbb{Z}$ (the integers) as a trivial right G-module, we abbreviate

$$VG = V(G,\mathbb{Z}) \quad \text{and} \quad H_2(G) = H_2(G,\mathbb{Z}).$$

The main results of /Stammbach/ that we shall use are the following four theorems, which appear there as 5.4, 5.5, 5.6 and 5.8 of chapter IV.

<u>Theorem 1:</u> Let $\underline{V}$ be a variety of groups and let G be a group in $\underline{V}$. Let $g^i: G \twoheadrightarrow G/G^i$ denote the canonical map for $i \geqslant 2$. Then the following are equivalent:

(i) G has the same lower central sequence as a group which is free in $\underline{V}$;

(ii) G^{ab} is free in $\underline{V} \cap \underline{\text{Abgps}}$ and the induced map

$g_*^i: \mathcal{V}G \longrightarrow (G/G^i)$ is zero for all $i \geqslant 2$;

(iii) there exists a group F, free in $\underline{V}$, and a homomorphism $f: F \rightarrow G$ such that the induced maps $f^i: F/F^i \longrightarrow G/G^i$ are isomorphisms onto for all $i \geqslant 1$. //

<u>Theorem 2:</u> let $\underline{V}$ be a variety of groups and let G be a residually nilpotent group in $\underline{V}$ such that G^{ab} is free in $\underline{V} \cap \underline{\text{Abgps}}$ and $\mathcal{V}G = 0$. Then G is $\underline{V}$-parafree. //

Given a group G and a non-negative integer q, we may define the <u>lower central q-series</u> of G as follows:

$$G^{1,q} = G, \quad G^{i+1,q} = G \circ_q G^{i,q} \qquad i = 1,2\ldots$$

where for any $K \leqslant G$, $G \circ_q K$ is the subgroup generated by all the elements

$$x^{-1}y^{-1}xyz^q \qquad \text{with } x,y \in G \text{ and } z \in K.$$

(If $q = 0$, this is of course just the lower central series.)

The next theorems show how like a free group in $\underline{V}$ a $\underline{V}$-parafree group can be.

<u>Theorem 3:</u> Let $\underline{V}$ be a torsion-free variety of groups, and suppose the free groups of $\underline{V}$ are residually nilpotent. If G contain a group F which is free in $\underline{V}$ and such that the inclusion $f: F \hookrightarrow G$ induces isomorphisms $f^{i,q}: F/F^i \longrightarrow G/G^i$ for all $i = 1,2,\ldots$ and $q = 0,1,2,\ldots$. //

(If the conclusion holds then of course all the squares analogous to (1) commute.)

<u>Theorem 4:</u> Let $\underline{V}$ be a variety of groups and suppose $f: K \longrightarrow G$

is a homomorphism in $\underline{V}$. If G is $\underline{V}$-parafree and K is residually nilpotent and the induced map $f^1: K^{ab} \to G^{ab}$ is an isomorphism onto, then K is $\underline{V}$-parafree and f is a monomorphism. //

We note the following easy consequence of the definitions.

Lemma 5: If $\underline{V}$ is a nilpotent variety and K and G are groups in $\underline{V}$ with the same lower central sequence, then K and G are isomorphic.

Proof: For some integer n, $K^n = G^n = (1)$ and
$K \simeq K/K^n \simeq G/G^n \simeq G$. //

We shall apply theorems 1 - 4 to groups which are λ-free in and of cardinality λ, with λ uncountable. In view of the next lemma, and theorem 1, for such a group G to be $\underline{V}$-parafree, it will be necessary and sufficient that G be residually nilpotent and G^{ab} free in $\underline{V} \cap \underline{Abgps}$.

Lemma 6: Let $\underline{V}$ be a variety and G a group in $\underline{V}$. If G is ω-free in $\underline{V}$, then $VG = 0$.

Proof: It is an easy consequence of the definitions that if G is free in $\underline{V}$, $VG = 0$. Now, any group can be regarded as the colimit of the directed family consisting of all its finitely (that is, ω-) generated subgroups together with the inclusions, and if G is ω-free, the subgroups which are free in $\underline{V}$ and finitely generated are cofinal in this directed system. It will then follow that $VG = 0$ if we show that V commutes with colimits over directed systems.

Suppose G is the colimit of the directed system $\{G_i : i \in I\}$ of its subgroups. Then there is a free $\underline{V}$-group F which is the colimit of the directed system of free groups $\{F_i : i \in I\}$ and such that there is for each $i \in I$ an epimorphism $f^i: F_i \twoheadrightarrow G_i$ and the system induces an epimorphism $f: F \twoheadrightarrow G$.

Since $H_2(-)$ commutes with colimits over directed systems (Prop. VI.1.3, p107 of /Cartan & Eilenberg/), it follows that

$$H_2(F) \xrightarrow[f_*]{} H_2(G) = \operatorname{colim}_I \left(H_2(F_i) \xrightarrow[f_*^i]{} H_2(G_i)\right).$$

Since cokernels are colimit constructions and colimits commute with colimits, it follows that

$$\begin{aligned} G &= \operatorname{coker} f_* \\ &= \operatorname{coker} \operatorname{colim}_I f^i \\ &= \operatorname{colim}_I \operatorname{coker} f^i \\ &= \operatorname{colim}_I G_i, \end{aligned}$$

and we are done. //

Next we examine residual nilpotence.

<u>Lemma 7:</u> Let $\underline{V}$ be a variety whose free groups are residually nilpotent. If G in $\underline{V}$ is ω_1-free in $\underline{V}$, then G is residually nilpotent.

Proof: Let G be ω_1-free and suppose G is not residually nilpotent. Then there is $g \in G$ such that $g \neq 1$ but $g \in G^i$ for all i. Let X be the set of generators of G. For each $i<\omega$ there is a commutator word w_i of weight i in the generators X such that $g = w_i \in G^i$. Choose one such w_i for each i. Let Y be the subset of X consisting of all those elements which occur in some w_i.

Then Y is countable and there is a free subgroup F of G containing Y. Hence $g \in \bigcap_{i<\omega} F^i = (1)$, a contradiction. //

The last requirement is that G^{ab} be free. We turn to this now. For the next lemma, recall that if K is a pure subgroup of G, then K^{ab} is naturally embeddable as a pure subgroup of G^{ab}. (This is lemma 0.11.1.)

<u>Lemma 8:</u> Let $\underline{V}$ be a variety and λ an uncountable cardinal of cofinality μ. If G is a λ-free groups in $\underline{V}$ of cardinality λ, then G may be written as the union of a continuous chain $\{A_i : i<\mu\}$ such that:

(i) each A_i is a λ-generated pure subgroup of G;

and (ii) if $a_{ij}: A_i^{ab} \rightarrowtail A_j^{ab}$ denotes the natural embedding for $i<j<\mu$, then the directed system $\{a_{ij} : i<j<\mu\}$ is continuous and G^{ab} is its direct limit.

Proof: By theorem 1.5, G has a near-filtration $\{G_i : i\leqslant\mu\}$, and we can assume that $|G_0| \geqslant \omega$. Let A_0 be a pure subgroup of G of cardinality $|G_0|$, containing G_0. If $\delta<\mu$ is a limit ordinal, let $A = \bigcup_{i<\delta} A_i$. Let A_{i+1} be a pure subgroup of G containing, and of the same cardinality as, $A_i \cup \bigcup_{j\leqslant i+1} G_j$. Since the union of a chain of pure subgroups is pure, this proves (i).

Since abelianisation is a left adjoint to the forgetful functor from $\underline{V} \cap$ <u>Abgps</u> into $\underline{V}$, it preserves colimits, and so (ii) holds. //

We now deduce easily

Lemma 9: Suppose $\underline{V}$ is a variety of exponent 0 or a prime power and that G is a λ-free group in $\underline{V}$ of uncountable cardinality, λ. Then

(i) if $|G^{ab}| = \lambda$, then G^{ab} is λ-free in $\underline{V} \cap \underline{Abgps}$;

and (ii) if $|G^{ab}| < cf(\lambda)$, then G^{ab} is free in $\underline{V} \cap \underline{Abgps}$.

Proof: Let $\{A_i : i<cf(\lambda)\}$ be as in lemma 8. For each $i<cf(\lambda)$ there is a subgroup F of G, free in $\underline{V}$ and containing A_i. Since A_i is pure in G, it is also pure in F. It follows that A_i^{ab} is embeddable as a pure subgroup of F^{ab}, which is free in $\underline{V} \cap \underline{Abgps}$. The conditions on $\underline{V}$ now show that A_i^{ab} is free in $\underline{V} \cap \underline{Abgps}$ (by corollary 6.2). It now follows easily that if case (i) holds, $\{A_i^{ab} : i<cf(\lambda)\}$ is a filtration of G^{ab} (in $\underline{V} \cap \underline{Abgps}$). In case (ii), it is easily seen that the sequence of A_i^{ab} is eventually constant. This constant value must be G^{ab}, which is therefore free in $\underline{V} \cap \underline{Abgps}$, since the A_i^{ab} are. //

We can put all this together now to obtain our first positive result.

Theorem 10: Let $\underline{V}$ be a variety of groups whose free groups are residually nilpotent. Let κ be a regular uncountable cardinal, and suppose G is a κ-free group in $\underline{V}$ of cardinality κ. If

either (a) the exponent of $\underline{V}$ is a prime power;

or (b) the exponent of $\underline{V}$ is 0 and κ has the property that every κ-free abelian group is κ^+-free;

then G is $\underline{V}$-parafree.

Proof: If $|G^{ab}| < \kappa$, then since κ is regular, it follows from lemma 9 that G^{ab} is free in $\underline{V} \cap \underline{Abgps}$ in both cases. If $|G^{ab}| = \kappa$

then G^{ab} is κ-free by lemma 9, and if (a) holds, then, by corollary 6.3, G^{ab} is free in $\underline{V} \cap \underline{Abgps}$, whilst if (b) holds, this is so by hypothesis. By lemmas 6 and 7 and theorem 2, it now follows that in all cases G is $\underline{V}$-parafree. //

We shall give some examples of varieties $\underline{V}$ and cardinals κ satisfying the hypotheses of the theorem at a later stage. Meanwhile, in a slightly different style, we have:

Theorem 11: Let $\underline{V}$ be a variety of groups whose free groups are residually nilpotent, and suppose the exponent of $\underline{V}$ is either 0 or a prime power. Let κ be a λ-compact cardinal. If G is a κ-free group in $\underline{V}$ of cardinality λ, then G is $\underline{V}$-parafree.

Proof: Let U denote the universe of sets and let j be an elementary embedding of U into M, witnessing λ-compactness of κ. Now, in M, j(G) is $j(\kappa)$-free, while in U, j"G is a subgroup of j(G), of cardinality λ, and since κ is λ-compact, there is a subset B of j(G) such that $B \in M$, $j"G \subseteq B$ and the cardinality in M of B is less than $j(\kappa)$. In M, B may be extended to a free subgroup F of j(G), since j(G) is $j(\kappa)$-free in M. However, F really (that is, in U) is a free group in $\underline{V}$ and j"G really is a pure subgroup of F (since j"G is even an elementary submodel of F), and of course G and j"G are isomorphic. It follows that G is isomorphic to a pure subgroup of F, which is free in $\underline{V}$, and hence G^{ab} is isomorphic to a pure subgroup of F^{ab}, which means that G^{ab} is free in $\underline{V} \cap \underline{Abgps}$, by corollary 6.2 (since F^{ab} is).

By lemma 6, G = 0, and by lemma 7, G is residually nilpotnet. It follows from theorem 2 that G is $\underline{V}$-parafree. //

And another variation on the same theme:

Theorem 12: Let $\underline{V}$ be a variety of groups whose free groups are residually nilpotent, and suppose the exponent of $\underline{V}$ is either 0 or a prime power. Let λ be a singular cardinal and suppose G is a λ-free group in $\underline{V}$ of cardinality λ such that $|G^{ab}| = \lambda$. Thne G is $\underline{V}$-parafree.

Proof: By lemma 9, G^{ab} is λ-free in $\underline{V} \cap \underline{Abgps}$ and hence free, by theorem 1.11 (Shelah) if $\underline{V}$ has exponent 0, and by corollary 6.2 otherwise. By lemmas 6 and 7 and theorem 2, G is $\underline{V}$-parafree. //

We give now some applications of the last three theorems. In order to apply these theorems, one has to check two types of hypothesis: the condition on the variety, and that on the cardinalities. We begin with the varieties.

Recall that if $\underline{U}$ and $\underline{V}$ are varieties of groups, a group G belongs to the product variety $\underline{UV}$ iff G contains a normal subgroup N with $N \in \underline{U}$ and $G/N \in \underline{V}$. (Compare /Neumann/, 21.11, p38.) We have already defined (§0.12) the variety $\underline{N}_c$ of all groups nilpotent of class at most c (c a positive integer). If $c_1, c_2, \ldots, c_k$ are positive integers, the variety $\underline{P}(c_1, c_2, \ldots, c_k)$ is the product variety $\underline{N}_{c_k}\underline{N}_{c_{k-1}} \ldots \underline{N}_{c_1}$, called the variety of all groups polynilpotent of class $(c_1, c_2, \ldots, c_k)$. If $c_1 = c_2 = \ldots = c_k = 1$, then since $\underline{N}_1 = \underline{Abgps}$, we obtain the variety of all groups which are soluble of length at most k as a special case of this definition.

The first condition we consider is residual nilpotence.

<u>Theorem 13</u>: The following varieties have the property that their free groups are residually nilpotent:

(a) $\underline{Gps}$; (b) any nilpotent variety; (c) $\underline{P}(c_1,c_2,\ldots,c_k)$.

The varieties in (a) and (c) have exponent 0.

Proof: The claim is obvious for (b). The other two cases are covered by 41.51 and 26.33 on p112 and p76 of /Neumann/. //

The following appears as 42.56 on p123 of /Neumann/.

<u>Theorem 14</u>: (Šmel'kin) If F is free in $\underline{V} = \underline{P}(c_1,c_2,\ldots,c_k)$, then a subset X of F consisting of at least two elements generates freely a $\underline{V}$-free subgroup of F iff X is independent modulo F'. //

From this we obtain

<u>Corollary 15</u>: Let $\underline{V}$ be as in theorem 14 and suppose $\lambda > \omega$ is limit cardinal. If G is λ-free in $\underline{V}$ and has cardinality λ, then $|G^{ab}| = \lambda$.

Proof: Let $\{\lambda_i : i < cf(\lambda)\}$ be a strictly increasing sequence of uncountable cardinals with limit λ. Then there is a (not necessarily continuous) chain $\{G_i : i < cf(\lambda)\}$ of subgroups of G whose union is G and which is such that each G_i is free in $\underline{V}$ and of cardinality λ_i. Let X_i be a basis of G_i for each $i < cf(\lambda)$. Then $|X_i| = \lambda_i$ and since $G_i < G_j$ whenever $i < j$, it follows from theorem 14 that X_i is independent modulo G_j' whenever $i < j$. It will then be sufficient to show that X_i is independent modulo G' for all $i < cf(\lambda)$, for then a maximal independent subset of G^{ab} has cardinality at least λ_i for each $i < cf(\lambda)$, from which it follows that $|G^{ab}| = \lambda$.

So suppose $x_1, x_2, \ldots, x_n \in X_i$ and

$$x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n} = g \in G'.$$

Since $G' = \bigcup G'_j$, there is $j > i$ such that $g \in G'_j$, which means $x_1, x_2, \ldots, x_n$ are dependent modulo G'_j, unless $r_1 = \ldots = r_n = 0$. So X_i is independent modulo G'. //

Lemma 16: If $\underline{V}$ is a nilpotent variety and G in $\underline{V}$ is uncountable, then $|G^{ab}| = |G|$.

Proof: If T is a transversal for G/G', then $\langle T\rangle G' = G$, so by lemma 0.13.1, $\langle T\rangle = G$, and since G is uncountable $|G| = |T| = |G^{ab}|$. //

Now we turn to the conditions on the cardinalities involved. The following was noticed by several people. For example, /Mekler/ lists Mekler, Shelah, Gregory and Kueker. There is a proof in /Eklof3/.

Theorem 17: If κ is a weakly compact cardinal, then every κ-free abelian group is κ^+-free. //

It is consistent that for regular κ the converse is true. (See, for example, /Mekler/.)

For singular λ we have the following extension of theorem 1.11.

Theorem 18: (Hodges) Suppose $\underline{V}$ is a nilpotent variety of groups and λ is a singular cardinal. If G is λ-free in $\underline{V}$ and of cardinality λ, then G is free in $\underline{V}$.

Proof: If the exponent of $\underline{V}$ is not zero and not a prime power, then this follows from theorem 6.10. Otherwise, by lemma 16, $|G^{ab}| = \lambda$, and by theorem 12, G is $\underline{V}$-parafree. By lemma 5, G is free in $\underline{V}$. //

This theorem was given a different proof in /Hodges2/.

The next theorem is a consequence of theorem 6.13, but the methods of this section allow a different proof, which we now give.

Theorem 19: Let $\underline{V}$ be a nilpotent variety of groups, of exponent 0 or a prime power. Suppose G is κ-free in $\underline{V}$. If

either (a) κ is weakly compact and $|G| = \kappa$,

or (b) κ is strongly compact (and $|G|$ is arbitrary),

then G is free in $\underline{V}$.

Proof: If (a) holds, then by theorem 10 G is $\underline{V}$-parafree and hence by lemma 5 free in $\underline{V}$. If (b) holds, then G is $\underline{V}$-parafree by theorem 11 and hence free in $\underline{V}$ by lemma 5. //

And finally we have the following application (which covers too the variety of all groups soluble of length at most k):

Theorem 20: Let $\underline{V} = \underline{P}(c_1, c_2, \ldots, c_k)$. Suppose G is κ-free in and

either (a) κ is singular and $|G| = \kappa$,

or (b) κ is weakly compact and $|G| = \kappa$,

or (c) κ is strongly compact (and $|G|$ is arbitrary)

Then G is $\underline{V}$-parafree.

Proof: This is similar to the proofs of theorems 18 and 19 from theorems 10-15. //

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