A MARKOV PROPERTY FOR MULTICOMPONENT EUCLIDEAN COVARIANT GAUSSIAN GENERALIZED STOCHASTIC FIELDS

by

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TO MY DEAR MOTHER

sine qua non

TO MY WIFE, ABIEYUWA

toiled and wrought and thought with me

TO MY FAMILY (in the Nigerian sense)

every member of this set is a saint

am gonna be where the lights

are shining on me

PREFACE

The work presented in this Thesis was done between October 1972 and September 1975 in the Department of Mathematics, Bedford College, London, under the supervision of Dr. R.F. Streater, Professor of Applied Mathematics. It is a source of immense pleasure to heartily thank Professor Streater for suggesting the field of investigation, for his constant attention and interest in the work, for his constructive criticisms of earlier versions of this work which led to various improvements, for his sustained encouragement, and for everything he has taught the author. It is no hyperbole to admit that were it not for this amiable and intellectual paragon, who constituted a model of excellence for and driving force behind the author, this work would never have been accomplished. The author also wishes to acknowledge the great benefit he accrued from the highly stimulating seminars organized by Professor Streater for the Quantum Field Theory Group at Bedford College, and to thank him too for offering the author the opportunity of giving lectures on the papers [57][58] of Nelson and [94] of Wong, and also on some aspects of the work presented here.

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Except where acknowledged in the text, the work described here is original and has not been submitted in this or any other university for any other degree.

ABSTRACT

In Chapter 1 to Section 2.3, pertinent probabilistic concepts, as well as notions of Markov property, are briefly discussed. Section 2.4 is an account of the theory of boundary value problems for elliptic systems of linear partial differential operators of arbitrary orders. Particular attention is paid to the solution of the Dirichlet problem for such a system because it intervenes in the analysis of Chapter 4.

In Chapter 3, a spectral representation (THEOREM (3.2.13)) is provided for an arbitrary Euclidean covariant (see (3.2.11)) multicomponent generalized stochastic field. This result, obtained group-theoretically, is then applied to the special case of a three component generalized stochastic field (see (3.2.25)), also needed in Chapter 4.

In Section 4.1, Wong's notion of Markov property is formulated. The rest of Chapter 4 is then concerned with the complete characterization of the class of all three dimensional Euclidean covariant Gaussian generalized stochastic fields which are Markov in the sense of Wong. It is also shown here that some of the latter are not also Markov in the sense of Nelson. Readers familiar with the work of Wong [94] will readily recognize the various results of Chapter 4 as extensions of those of Wong.

In Chapter 5, Wong's notion of Markov property is given abstract formulation (THEOREM (5.2)). Then it is demonstrated that, like Nelson's notion of Markov property, Wong's notion of Markov property is implied by the so-called pre-Markov property.

In Chapter 6, ways of extending the investigations of Chapter 4 to arbitrary multicomponent Euclidean covariant Gaussian generalized stochastic fields are indicated. It is then observed that the required extensions present no new problems or difficulties. Finally, Wong's notion

of Markov property is formulated much more generally and sufficient conditions (THEOREM (6.2.1)) for Markovicity in this extended formulation are furnished. These indicate that quite a large class of mathematically and physically interesting multicomponent Gaussian generalized stochastic fields are Markov in the sense of Wong.

CONTENTS

Page

CHAPTER 0

INTRODUCTION

The important concept of Markov property was first introduced for chains [15] of random variables at the beginning (1906) of this century by the Russian A. Markov, and subsequently extended, in a mathematically rigorous way, to stochastic processes indexed by the real line R by his compatriot A. Kolmogorov [40],[48]. Today, there exists a beautiful and highly developed theory of Markov stochastic processes and their applications [32-34][35][100][9][66]. Naturally, therefore, there is the irresistible urge to attempt to develop a similar theory for the generalizations of stochastic processes indexed by R.

Let $(\Omega,\mathcal{B},\mu)$ be a probability space and let $\langle\cdot,\cdot\rangle_{L^2(\Omega,\mathcal{B},\mu)}$ denote the inner product of $L^2(\Omega,\mathcal{B},\mu)$. In 1948, Lévy [43] defined a Brownian motion indexed by R^d as a mean zero Gaussian stochastic field $\{W(x) : x \in R^d\}$, with underlying probability space $(\Omega,\mathcal{B},\mu)$, such that

$$\langle W(x),W(y)\rangle_{L^2(\Omega,\mathcal{B},\mu)} = \frac{1}{2}(|x| + |y| - |x-y|)$$

$$|z|^2 = \sum_{i=1}^{d} z_i^2 , \qquad z = (z_1,\ldots,z_d) \in R^d .$$

With this Brownian motion in mind, Lévy [44], p.136, introduced the following notion of Markov property for stochastic fields.

MARKOV STOCHASTIC FIELDS

Let D be an open subset of R^d with smooth closed boundary D and complement D'. Then, a stochastic field $\{X(x) : x \in R^d\}$ defined on $(\Omega, \mathcal{B}, \mu)$ is said to be Markov of order $\leq r + 1$, r = nonnegative integer, if each approximation Y to X(x) in a neighbourhood of ∂D, which possesses the property that

$$\lim_{\delta \downarrow 0} \frac{1}{\delta^r} |Y(w) - X(x)(w)| = 0, \quad \delta = \text{distance } (x, \partial D), \text{ all } w \in \Omega,$$

is such that given Y, the random variables X(x) and X(y) are stochastically independent whenever $x \in D$ and $y \in D'$.

If $\{X(x) : x \in R^d\}$ is Markov of order $\leq r + 1$ but not of order $\leq r$, then $\{X(x) : x \in R^d\}$ is said to be Markov of finite order r + 1.

The above definition is the first known attempt to extend the concept of Markov property to stochastic fields.

Lévy [44] pp.167-168, conjectured that the Brownian motion $\{W(x) : x \in R^{2r+1}$, r = nonnegative integer$\}$ is Markov of order r + 1. This conjecture was subsequently proved true by McKean [54] who also established that the Brownian motion $\{W(x) : x \in R^d\}$ has no Markov property at all for even d. Indeed, McKean demonstrated that given

$$\{(\partial^\kappa W)(x) : x \in \partial D, \ \partial = \text{normal derivative on } \partial D, \ \kappa = 0,1,\ldots,r\},$$

then W(x) is stochastically independent of W(y) for $x \in D$ and $y \in D'$. But a Brownian motion is, of course, not even once differentiable, and hence McKean explained the meaning of the phrase "the normal derivatives of a Brownian motion".

Recently, Molchan [52] has furnished an alternative proof of Lévy's conjecture and Pitt [64] has extended McKean's work to arbitrary Gaussian stochastic fields which are Markov of some finite order. Molchan employs aspects of the theory of elliptic partial differential equations and the notion of the reproducing kernel Hilbert space [7] associated with a stochastic field. By a blending of these same ideas with those of McKean in [54] and by applying Peetre's characterization of differential operators [63], Pitt characterized a finite order Markovian Gaussian stochastic field, under some assumptions, by identifying the inner product of its reproducing kernel Hilbert space with the Dirichlet form [46] of an elliptic partial differential operator.

Results of the above type are, of course, certainly of relevance in the initial stages of development of a theory of Markov stochastic fields. But stochastic fields which are Markovian of finite order are clearly only a special class of Markovian fields. Indeed, McKean employs a generalization of Lévy's notion of Markov property in [54], and his extended definition thus accommodates Markov stochastic fields which are not necessarily of finite order. In [41][42], necessary and sufficient conditions for a Euclidean invariant scalar Gaussian stochastic field to be Markov in McKean's sense are presented under some assumptions on the spectral measure of the stochastic field. The last mentioned papers employ the novel theory of hyperfunctions [42].

The results of [41][42] may be viewed as extensions of those of Molchan [52] and Pitt [64]. The next direction of research must now inevitably be that of first removing, if possible, the assumptions made in all the above references; then next one must take on the rather more daunting challenge of extending the analysis to non-Gaussian, as well as generalized, stochastic fields. In this respect, the road to the development of a theory of Markov stochastic fields remains a long and hard one.

Meanwhile, extensions of the concept of Markov property to generalized stochastic fields have been presented independently by Nelson [57] and Wong [94]. Nelson's notion of Markovicity is similar to McKean's mentioned above while Wong's definition is analogous to Lévy's. It is these two concepts - particularly the latter - of Markovicity that are considered by us in this Thesis. In what follows, we first give an account of the motivation for Nelson's notion of Markov property; then we furnish a similar preamble in the case of Wong's notion. Finally, we give our own motivation for considering the problems in this Thesis.

MARKOV GENERALIZED STOCHASTIC FIELDS

It is well-known that Quantum Field Theory [8][71] is a relativistic theory [67]; hence quantum fields [84] possess Minkowski space as their underlying space. Unfortunately, since Minkowski space has indefinite metric, the construction and analysis of Boson quantum fields are, therefore, bedevilled by rather difficult problems arising from the concomitant hyperbolicity of the field equations. It is, consequently, natural to expect that if the time t parameter could be replaced by imaginary time it $(i^2 = -1)$ parameter in all quantum field theoretic equations, so that Minkowski space is transformed into Euclidean space, that the problems would become of elliptic type and hence more tractable. However, the resulting Euclidean theory would be unphysical because it would then correspond to a theory with an unphysical, imaginary energy.

The above heuristic philosophy seems to have been first positively invoked by Dyson [102] in quantum electrodynamical computations. Later Schwinger [72] and Nakano [55] advocated the formulation of a Euclidean operator field theory and, indeed, they constructed Euclidean fields which are operators acting on a Euclidean analogue of the conventional quantum field theoretic Fock space [13]. In his paper [85], Symanzik made the crucially significant discovery - bearing in mind the use it has been put to recently - that the Euclidean fields of Schwinger and Nakano may be considered as generalized stochastic fields, upon whose theory one could then draw. But in the event of a successful completion of the above programme of understanding Euclidean fields, the important question - to which none of the above authors pre-empted or proposed an answer - would then naturally arise of how, if ever possible, to recover the physically interesting quantum fields from the Euclidean fields.

Recently, Nelson addressed himself to the last named problem: he constructed Euclidean fields and then provided a scheme for the recovery of scalar Boson quantum fields from the Euclidean fields [57][58]. Nelson's work immediately lifted the study of Euclidean fields from its previous state of little attention to the limelight, and its results were soon avidly devoured and applied by constructive quantum field theorists [4-6][38][60][81][88]. Furthermore, other authors [29][62] also subsequently produced results on the same theme of constructing Euclidean fields and the recovery of quantum fields from them.

Nelson's work, which represents a rigorous mathematical formulation and extension of Symanzik's ideas, invokes Probability Theory - a subject justly recognized by so many for so long [18][25][37][53][56][73-79] as destined to play a rôle of no small significance in quantum theoretic investigations. Thus, Nelson introduced two important notions [57][58] namely: Markov property and reflection property for scalar generalized stochastic fields. We present these notions in Section 2.3. In [57], it is demonstrated that any scalar Euclidean invariant generalized stochastic field which exhibits the two attributes of Markov property and reflection property of Nelson, and additionally satisfies some other technical assumptions, leads to a scalar Boson quantum field. However, in view of the results of [29] and [62], not all Boson quantum fields are obtained in this way.

An example of a physically interesting scalar Euclidean invariant generalized stochastic field - because it satisfies both the Markov property and the reflection property of Nelson - is afforded by the Gaussian generalized stochastic field whose correlation functional is given by

$$<f^{(2)}, (\alpha_0^2 - \Delta)^{-1} f^{(1)}>_{L^2(R^d,dx)}, \alpha_0 > 0, f^{(i)} \varepsilon \mathcal{S}(R^d), i = 1,2$$

where Δ is the Laplacian in d variables and $\mathcal{S}(R^d)$ is Schwartz space of rapidly decreasing functions. It is this generalized stochastic field which leads via Nelson's Reconstruction Theorem [57] to the free scalar Wightman quantum field.

It is perhaps pertinent to emphasize that Markov property and reflection property are crucial in Nelson's scheme [57] for recovering quantum fields from Euclidean fields. Introduction of the reflection property is necessitated by the need to eliminate the normal derivatives, on the boundary ∂D of an open subset D of R^d, of a given scalar generalized stochastic field. Thus the combination of Markov property + reflection property of Nelson for a scalar generalized stochastic field is analogous to Lévy's notion of "Markov property of order 1".

In order not to proliferate assumptions, it is clear that any definition of Markov property which incorporates Nelson's notion of Markov property + reflection property is mathematically more satisfying because such a definition would suffice in the programme of constructing quantum fields. Thus, one needs a stronger notion of Markov property than Nelson's. Indeed, such a notion is formulated by Nelson himself in [57], but he later abandoned it because of the relative difficulty of employing it in his scheme. In the end, Nelson expressed the fervent hope of the possibility of eventually finding "a better way of imposing a strict notion of locality on a general Markov field" [57].

With the indicated intervention of Markov generalized stochastic fields in Constructive Quantum Field Theory, it is clear that a study of the former will be beneficial to the understanding of the latter. Thus, for example, investigations of sample path continuity problems for the Markov scalar generalized stochastic field described above have been undertaken in [12][65][88]. However, the far more interesting, even exciting, problem of obtaining necessary and sufficient conditions for a generalized stochastic field to be Markov in the sense of Nelson is, to the best of the author's knowledge, still open. We, too, will not be addressing ourselves to this problem here. Instead, we investigate another physically and mathematically interesting notion of Markov property due to Wong [94]; we present this latter concept in Section 4.1.

The motivation for Wong's notion of Markov property is, in contrast to Nelson's, purely mathematical. Wong's notion of Markov property is formulated in [94] only for scalar Euclidean invariant Gaussian generalized stochastic fields, although this can be done much more generally. Wong [94] then obtained that a necessary and sufficient condition for Markovicity, in his sense, of a scalar Euclidean invariant Gaussian generalized stochastic field is that its correlation functional be given by

$$\langle f^{(2)}, (\alpha_0^{\,2} - \Delta)^{-1} f^{(1)} \rangle_{L^2(\mathbb{R}^d, dx)}, \quad \alpha_0 \geq 0.$$

But this is precisely the same scalar Gaussian generalized stochastic field which, as indicated earlier, satisfies Nelson's notion of Markov property + reflection property. Thus, Wong's notion of Markov property is itself also of physical interest because it delivers, in one and only one stroke, the important scalar Gaussian generalized stochastic field which leads by Nelson's Reconstruction Theorem [57] to the free scalar Wightman quantum field. Perhaps, this is precisely what Nelson had in mind - "It is possible that a better way of imposing a strict notion of locality on a general Markov field can be found." [57] - for it follows by what precedes that, at least in the case of a scalar Euclidean invariant Gaussian generalized stochastic field, it is manifestly sufficient to replace Nelson's notion of Markov property + reflection property by Wong's notion of Markov property. This discovery is at the root of our motivation for undertaking a study of Wong's notion of Markov property for Euclidean covariant multicomponent Gaussian generalized stochastic fields (see Section 3.2), and it is both mathematically and physically interesting to provide a complete characterization, if possible, of the class of all such Markov stochastic fields.

The various problems resolved in this Thesis seem best catalogued by presenting the results of each chapter, and this we have already done in the ABSTRACT. We, therefore, refer to the latter for details.

We have endeavoured to make this Thesis accessible to both probabilists and physicists not necessarily familiar with multicomponent Euclidean covariant generalized stochastic fields or the theory of boundary value problems for elliptic systems of linear partial differential operators of arbitrary orders. This has meant the inclusion of some propaedeutic material and adequate references at appropriate points in the text, for completeness.

Finally, equations, theorems and clarifying remarks are numbered consecutively throughout each section of a given chapter; the symbol □ signifies the conclusion of a proof and there is a bibliography at the end of the Thesis.

CHAPTER 1

PROBABILITY THEORY

Probabilistic concepts and theorems, as well as the notational conventions, frequently employed in subsequent chapters are briefly surveyed here. This survey is largely propaedeutic and hence is by no means complete in itself: proofs of well-known theorems whihh are readily found elsewhere are not furnished, but adequate references are supplied for the benefit of the interested reader.

1.1 PROBABILITY SPACE

Fix a nonempty abstract set Ω and let $\mathcal{B}$ denote a nonempty class of subsets of Ω.

(1.1.1) DEFINITION $\mathcal{B}$ is called a Boolean algebra, or simply an algebra, of subsets of Ω provided that

(B1) whenever $A_1 \in \mathcal{B}$ and $A_2 \in \mathcal{B}$, then their set-theoretic union $A_1 \cup A_2 \in \mathcal{B}$;

(B2) whenever $A \in \mathcal{B}$, then its set-theoretic complement A' in Ω also belongs to $\mathcal{B}$.

(1.1.2) DEFINITION An algebra $\mathcal{B}$ is called a σ-Boolean algebra, or simply a σ-algebra, of subsets of Ω if it satisfies the following stronger formulation of condition (B1):

(B1*) whenever $A_i \in \mathcal{B}$, $i = 1,2,\ldots$, then $\bigcup_{i=1}^{\infty} A_i$ belongs to $\mathcal{B}$.

If $\mathcal{B}$ is the σ-algebra of subsets of Ω, then the pair $(\Omega,\mathcal{B})$ is called a measurable space and, furthermore, any $A \in \mathcal{B}$ is called a measurable set.

(1.1.3) DEFINITION Any function whose domain of definition is a class of sets is classed a set function. To us, unless there is a statement to the contrary, a measure μ on $(\Omega,\mathcal{B})$ is a countably additive [27] set function $\mu: \mathcal{B} \to [0,\infty]$.

(1.1.4) REMARK: In what follows, μ will always be assumed complete i.e. μ is such that the conditions $B \varepsilon \mathcal{B}$, $A \subset B$ and $\mu(B) = 0$ together imply that $A \varepsilon \mathcal{B}$ (necessarily with $\mu(A) = 0$).

In case $\mu(\Omega) < \infty$, then μ is said to be finite on $(\Omega,\mathcal{B})$ and the triplet $(\Omega,\mathcal{B},\mu)$ is called a finite measure space.

(1.1.5) DEFINITION A probability space, with probability measure μ, is a finite measure space $(\Omega,\mathcal{B},\mu)$ for which $\mu(\Omega) = 1$.

1.2 RANDOM VARIABLES

In this section, we introduce the notion of a random variable.

For the next definition, suppose that $(\Omega_i,\mathcal{B}_i)$ $i = 1,2$ are two measurable spaces and let f be a function with domain Ω_1 and range in Ω_2.

(1.2.1) DEFINITION f is a measurable function of $(\Omega_1,\mathcal{B}_1)$ into $(\Omega_2,\mathcal{B}_2)$ if for arbitrary $A \varepsilon \mathcal{B}_2$, the set

$$f^{-1}(A) = \{\omega_1 \varepsilon \Omega_1 : f(\omega_1) \varepsilon A\}$$

belongs to $\mathcal{B}_1$.

(1.2.2) REMARK: Let R^n denote the n-fold Cartesian product of the real line R with itself and denote by $\mathcal{R}^n$ the σ-algebra of all subsets of R^n. A measurable function with domain R^n and range in R^m is called a Borel function.

(1.2.3) DEFINITION An n-dimensional R^n-valued random variable on a probability space $(\Omega,\mathcal{B},\mu)$ is a measurable function from $(\Omega,\mathcal{B})$ into

$(R^n, \mathcal{R}^n)$. (A similar definition holds, of course, in the case of an $\mathbb{C}^n$-valued random variable on $(\Omega, \mathcal{B}, \mu)$, where $\mathbb{C}$ is the set of all complex numbers).

(1.2.4) REMARK: Let $X = (X_1, \ldots, X_n)$ be such an R^n-valued random variable on $(\Omega, \mathcal{B}, \mu)$. Then a probability measure μ_X, called the probability distribution of the random vector X, is naturally defined on $(R^n, \mathcal{R}^n)$ in the following way:

$$\mu_X(B) = \mu(X^{-1}(B)) = \mu(\{\omega \in \Omega : X(\omega) \in B,\ B \in \mathcal{R}^n\})$$

Let $x = (x_1, \ldots, x_n) \in R^n$. Then the function $x \to \mu_X(x) = \mu(\{\omega \in \Omega:\ X_i(\omega) \leq x_i,\ i = 1, \ldots, n\})$ is called the joint probability distribution function of the real-valued random variables X_i, $i = 1, \ldots, n$.

μ_X has the following distinguishing properties [95]

(D1) μ_X is nonnegative i.e. $\mu_X(x) \geq 0$ for all $x \in R^n$;

μ_X is nondecreasing i.e. if $a \in R^n$ is a vector with nonnegative components, then $\mu_X(x) \leq \mu_X(x + a)$;

(D2) $$\lim_{\{x_i \to -\infty, i=1,\ldots,n\}} \mu_X(x) = 0$$

$$\lim_{\{x_i \to \infty, i=1,\ldots,n\}} \mu_X(x) = 1$$

Indeed, any measurable function from $R^n \to [0,1]$ satisfying (D1) and (D2) will be called a probability distribution function [95].

(1.2.5) DEFINITION If $F : R^n \to [0,1]$ is a probability distribution function, then a set of random variables $X_1, \ldots, X_n$ on some probability space, possessing F for their joint probability distribution function is called a realization of F. Every probability distribution function admits numerous realizations [95], [81], pages 19-21.

(1.2.6) DEFINITION If $(\Omega,\mathcal{B})$ is a measurable space and μ and ν are σ-finite [27] measures on $(\Omega,\mathcal{B})$, then ν is said to be absolutely continuous with respect to μ, in symbols $\nu << \mu$, provided that for all $A \in \mathcal{B}$, $\mu(A) = 0$ implies $\nu(A) = 0$. If $\nu << \mu$ and $\mu << \nu$ hold simultaneously, ν and μ are said to be equivalent, and we write $\nu \equiv \mu$.

Antithetically, μ and ν are said to be mutually singular, or briefly singular, if there exists a set $A \in \mathcal{B}$ for which $\mu(A) = 0$ and $\nu(B) = \nu(A \cap B)$ for all $B \in \mathcal{B}$. In this case, we write $\nu \perp \mu$.

(1.2.7) RADON-NIKODYM THEOREM If $(\Omega,\mathcal{B},\mu)$ is a σ-finite measure space and if a σ-finite measure ν on $(\Omega,\mathcal{B})$ is absolutely continuous with respect to μ, then there exists a finite-valued μ-measurable nonnegative function ϕ on Ω such that

$$\nu(A) = \int_A \mu(d\omega)\phi(\omega) , \qquad A \in \Omega.$$

The function ϕ is unique in the sense that if also

$$\nu(A) = \int_A \mu(d\omega)\phi^O(\omega), \qquad A \in \Omega$$

then ϕ and ϕ^O differ only on sets $A_O \in \Omega$ satisfying $\mu(A_O) = 0$.

(1.2.8) REMARK: ϕ is called the Radon-Nikodym derivative of ν with respect to μ.

We omit a proof of (1.2.7) which may be found in [27],[39] §6.4, [101] p.93.

1.3 STOCHASTIC INDEPENDENCE

If $(\Omega, \mathcal{B}, \mu)$ is a probability space, then the elements of $\mathcal{B}$ are called events.

(1.3.1) DEFINITION The events $A_i \in \mathcal{B}$, $i = 1,\ldots,n$ are said to be stochastically independent if for any arbitrary subset $\{\alpha(j): j = 1,\ldots,\kappa\}$ of the set $\{1,2,\ldots,n\}$, we have

$$\mu\left(\bigcap_{j=1}^{\kappa} A_{\alpha(j)}\right) = \prod_{j=1}^{\kappa} \mu(A_{\alpha(j)})$$

Let $\mathcal{B}_i$, $i = 1,\ldots,n$ be sub σ-algebras of $\mathcal{B}$. Then in this case, $\mathcal{B}_i$, $i = 1,\ldots,n$ are said to be stochastically independent if for every event $A_i \in \mathcal{B}_i$, the events $\{A_i, i = 1,\ldots,n\}$ are stochastically independent.

(1.3.2) REMARK: The above definitions are applied to random variables in the following way.

Let X_i, $i = 1,\ldots,n$ be random variables on $(\Omega, \mathcal{B}, \mu)$ and suppose that $\mathcal{B}_i$ is the minimal sub σ-algebra of $\mathcal{B}$ with respect to which X_i is measurable. Then the random variables X_i, $i = 1,\ldots,n$ are said to be stochastically independent if the σ-algebras $\mathcal{B}_i$, $i = 1,\ldots,n$ are stochastically independent.

It seems convenient to introduce here the important concept of conditional probability.

If $B \in \mathcal{B}$, $\mu(B) > 0$, then the conditional probability $\mu(A/B)$ of $A \in \mathcal{B}$ given B is defined to be

$$\mu(A/B) = \frac{\mu(A \cap B)}{\mu(B)}$$

By implication, if $A, B \in \mathcal{B}$ are stochastically independent events, then

$$\mu(A/B) = \frac{\mu(A)\mu(B)}{\mu(B)} = \mu(A)$$

(1.3.3) DEFINITION If $X \in L^1(\Omega,\mathcal{B},\mu)$, then

$$EX = \int \mu(d\omega)\ X(\omega)$$

is called the mean or expectation value of X. If $X \in L^2(\Omega,\mathcal{B},\mu)$ then

$$||X - EX||_2^2 = \langle X - EX, X - EX\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

is called the covariance of X. In this latter case, X necessarily belongs to $L^1(\Omega,\mathcal{B},\mu)$. Any random variable possessing a covariance is called a second order random variable.

Next, we bring in the notion of characteristic functions.

(1.3.4) DEFINITION Let $X = (X_1,\ldots,X_n)$ be an R^n-valued random variable on $(\Omega,\mathcal{B},\mu)$ and let $t = (t_1,\ldots,t_n) \in R^n$. Then,

$$t \to C(t) = E\ e^{i\sum_{j=1} t_j X_j} = \int \mu(d\omega)\ e^{i\sum_{j=1}^{n} t_j X_j(\omega)}$$

is called the characteristic function of X.

(1.3.5) REMARK: Every characteristic function C satisfies

(C1) $C(0) = 1$

(C2) C is continuous on R^n

(C3) C is nonnegative definite. This means that for any set $\{\lambda_i : i = 1,\ldots,N\}$ of complex numbers, we have

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_i \overline{\lambda_j}\ C(t^{(i)} - t^{(j)}) \geq 0$$

$(t^{(i)} \in R^n, i = 1,\ldots,N)$.

(1.3.6) REMARK: Suppose we are given a function on R^n to $\mathbb{C}$ which satisfies (C1) and (C2). Then the following theorem is used to check whether or not it satisfies (C3).

(1.3.7) BOCHNER'S THEOREM A bounded continuous function on R^n to $\mathbb{C}$ is nonnegative definite if and only if it is the Fourier transform of a finite positive measure on $(R^n, \mathcal{R}^n)$.

Bochner's Theorem is very important in the probability theory of finite dimensional random variables [95][48]. A proof of (1.3.7) may be found in [95].

1.4 CONDITIONAL EXPECTATION

In the preceding section, we introduced the concept of stochastic independence. Indeed, this concept is only a special form of the more general notion of conditional stochastic independence which we introduce in this section. Much of our work in subsequent chapters actually involves the use of this more general concept.

Fundamental to the rigorous study of conditional stochastic independence is the notion of conditioning with respect to a given σ-algebra of events.

Thus, let $(\Omega, \mathcal{B}, \mu)$ be a probability space and let $\mathcal{B}_0$ be a sub σ-algebra of $\mathcal{B}$. For $f \in L^1(\Omega, \mathcal{B}, \mu)$ set

$$\phi(B) = \int \mu(d\omega)\, \chi_B(\omega) f(\omega), \quad B \in \mathcal{B}_0,$$

where χ_B is the indicator function of the set $B \in \mathcal{B}_0$. Clearly, $\phi : \mathcal{B}_0 \to R$ is a countably additive set function.

Let $\mu_{\mathcal{B}_0}$ denote the restriction of μ to $\mathcal{B}_0$. Since, by definition,

$$\mu_{\mathcal{B}_0}(B) = \mu(B), \qquad \text{for all } B \in \mathcal{B}_0,$$

one sees that ϕ is absolutely continuous with respect to $\mu_{\mathcal{B}_0}$. Hence, by (1.2.7), there exists a unique (up to sets of $\mu_{\mathcal{B}_0}$ measure zero) $\mathcal{B}_0$-measurable function, denoted $E(f \mid \mathcal{B}_0)$ or $E^{\mathcal{B}_0} f$, according to convenience, such that

$$\phi(B) = \int \mu(d\omega)\, \chi_B(\omega) f(\omega) = \int_B \mu_{\mathcal{B}_0}(d\omega) E(f \mid \mathcal{B}_0)(\omega)$$

for all $B \in \mathcal{B}_0$.

(1.4.1) DEFINITION The $\mathcal{B}_0$-measurable function $E(f \mid \mathcal{B}_0)$ is said to be the conditional expectation of the random variable f on $(\Omega, \mathcal{B}, \mu)$ given the sub σ-algebra $\mathcal{B}_0$.

(1.4.2) REMARK: The conditional expectation operator $E(\cdot \mid \mathcal{B}_0)$ or $E^{\mathcal{B}_0}$ is a linear, positivity-preserving operator on $L^1(\Omega, \mathcal{B}, \mu)$, which also possesses the following additional properties.

(1.4.3) THEOREM Let f and g belong to $L^1(\Omega, \mathcal{B}, \mu)$ unless otherwise stated. Then

(i) $E^{\mathcal{B}_0} f = f$, if f is $\mathcal{B}_0$-measurable;

(ii) $E^{\mathcal{B}_0}(E^{\mathcal{B}_0} f) = E^{\mathcal{B}_0} f$

(iii) $E^{\mathcal{B}_{00}}(E^{\mathcal{B}_0} f) = E^{\mathcal{B}_{00}} f$; if $\mathcal{B}_{00}$ is a sub σ-algebra of $\mathcal{B}_0$;

(iv) $\int_\Omega \mu(d\omega) g(\omega) (E^{\mathcal{B}_0} f)(\omega) = \int_\Omega \mu(d\omega) f(\omega) (E^{\mathcal{B}_0} g)(\omega)$

if f,g belong to $L^2(\Omega, \mathcal{B}, \mu)$;

(v) $E^{\mathcal{B}_0} fg = g\, E^{\mathcal{B}_0} f$ if $g \in L^\infty(\Omega, \mathcal{B}_0, \mu)$;

(vi) if $\mathcal{B}_0$ and $\mathcal{B}_{00}$ are stochastically independent sub σ-algebras of $\mathcal{B}$ and if $f \in L^1(\Omega,\mathcal{B}_{00},\mu)$, then $E^{\mathcal{B}_0}f = \int_\Omega \mu(d\omega)f(\omega) = Ef$;

(vii) $||E^{\mathcal{B}_0}f||_p \leq ||f||_p$ if $f \in L^p(\Omega,\mathcal{B},\mu)$, $1 \leq p \leq \infty$, where $||\cdot||_p$ is the norm of $L^p(\Omega,\mathcal{B},\mu)$; hence $E^{\mathcal{B}_0}$ is a contraction on $L^p(\Omega,\mathcal{B},\mu)$.

(1.4.4) REMARK: (iv) and (vii) combine to imply that $E^{\mathcal{B}_0}$ is the orthogonal projection of $L^2(\Omega,\mathcal{B},\mu)$ on to $L^2(\Omega,\mathcal{B}_0,\mu)$. The verification of (i) through (vii) may be found in [82] and [88].

When we have cause to invoke (1.4.3) in our subsequent analysis, we sometimes do so without explicit reference to (1.4.3).

1.5 STOCHASTIC PROCESSES

The theory of stochastic processes, which sometimes feature in our analysis in subsequent chapters, provides a fertile ground for the application of the concepts in abstract probability theory expounded in the preceding sections. Therefore, in this final section of this chapter, we provide a rapid introduction to the theory of stochastic processes.

(1.5.1) DEFINITION Let T be an index set. By a stochastic process, we mean a family

$$H^0(\Omega) = \{X(t) : t \in T\}$$

of real- or complex-valued random variables $X(t)$ indexed by T and possessing $(\Omega,\mathcal{B},\mu)$ for their common probability space.

(1.5.2) REMARK: For simplicity, in what follows we set $T = R$. Let $T_n = \{t_1,\ldots,t_n\}$ be a finite subset of points in R, and if as usual $(R^n, \mathcal{R}^n)$ is the n-dimensional Borel measurable space, let $\mathcal{B}(T_n)$ denote the sub σ-algebra of $\mathcal{B}$ generated by the sets - called cylinder sets - of the form

$$\{\omega : (X(t_1)(\omega),\ldots,X(t_n)(\omega)) \in A, \quad A \in \mathcal{R}^n \}$$

Let μ_{T_n} denote the restriction of μ to $\mathcal{B}(T_n)$. Then μ_{T_n} is a cylinder set measure on $\mathcal{B}$ [23]. Then the probability distribution induced on $(R^n, \mathcal{R}^n)$ is given by

$$\mu_{T_n,X}(A) = \mu_{T_n}(X^{-1}(A)) = \mu_{T_n}(\{\omega: (X(t_1)(\omega),\ldots,X(t_n)(\omega)) \in A, \ A \in \mathcal{R}^n\})$$

Thus $\mu_{T_n,X}$ is the joint probability distribution of the random variables $\{X(t_i) : i = 1,\ldots,n\}$.

(1.5.3) REMARK: The foregoing analysis relates to the rather rare situation where the probability space $(\Omega,\mathcal{B},\mu)$ is known from the beginning. More often than not, the probability space $(\Omega,\mathcal{B},\mu)$ is not known a priori, and usually it is only the family of finite dimensional joint probability distributions $\{\mu_{T_n X} : T_n \in R^n\}$ that are available. The question is then posed as to whether or not there exists a stochastic process $\{X(t) : t \in R\}$ on some probability space realizing these finite dimensional probability distributions. The answer to this query is in the affirmative if the finite dimensional probability distributions $\{\mu_{T_n,X} : T_n \in R^n\}$ satisfy two constraints, known as Kolmogorov Consistency Conditions, which may be formulated as follows:

(K1) If $A \in \mathcal{R}^n$, so that $A \times R \in \mathcal{R}^{n+1}$, then

$$\mu_{T_{n+1},X}(A \times R) = \mu_{T_n,X}(A);$$

(K2) If π denotes a permutation of $(1,2,\ldots,n)$, set

$$T_{\pi(n)} = \{t_{\pi(1)},\ldots,t_{\pi(n)}\} .$$

Let $\gamma : R^n \to R^n$ be the function given by

$$\gamma(x_1,\ldots,x_n) = (x_{\pi(1)},\ldots,x_{\pi(n)})$$

Then, $\mu_{T_{\pi(n)},X} = \mu_{T_n,X} \circ \gamma^{-1}$

Thus permuting elements of the set T_n does not affect the $\mu_{T_n,X}$-measure of any fixed set in $\mathcal{R}^n$.

The theory of stochastic processes owes much to the following result.

(1.5.4) THE KOLMOGOROV EXTENSION THEOREM Let $T_n = (t_1,\ldots,t_n)$ and let $\mu_{T_n,X}$ be as above. If the family of finite dimensional probability distribution $\{\mu_{T_n,X}: T_n \in R^n\}$ satisfy (K1) and (K2) above, then there is a real- or complex-valued stochastic process $H^o(\Omega) = \{X(t) : t \in R\}$ on some probability space $(\Omega,\mathcal{B},\mu)$ realizing the family of finite dimensional probability distributions $\{\mu_{T_n,X} : T_n \in R^n\}$.

(1.5.5) REMARK: The lengthy proof of this important theorem may be found in [39][66]. Whenever we assert that we are given a stochastic process $\{X(t) : t \in R\}$ in our subsequent analysis, it is always tacitly assumed that the finite dimensional probability distributions of the family $\{X(t) : t \in R\}$ of random variables satisfy (K1) and (K2), so that by (1.5.4) there is a common probability space $(\Omega,\mathcal{B},\mu)$ on which $X(t)$ is measurable for all $t \in R$.

(1.5.6) EXAMPLE Let us now furnish an important example of a stochastic process. Let $T_n = \{t_1,\ldots,t_n\} \in R^n$ and for $A \in \mathcal{R}^n$, set

$$\mu_{T_n,X}(A) = \int_A dx \prod_{\nu=1}^{n} \{(2\pi e^{-|t_\nu - t_{\nu-1}|})^{-1/2} \exp- [\frac{(x_\nu - x_{\nu-1})^2}{2e^{-|t_\nu - t_{\nu-1}|}}] \}$$

$(x_0 = t_0 = 0)$, where integration is performed first with respect to x_n, then with respect to x_{n-1}, etc. and finally with respect to x_1. Then $\{\mu_{T_n,X} : T_n \in R^n\}$ is a family of finite dimensional probability measures on $(R^n,\mathcal{R}^n)$. Moreover, $\{\mu_{T_n,X} : T_n \in R^n\}$ satisfies (K1) and (K2) as can readily be verified. Hence, by (1.5.4) there is a real-valued stochastic process $H^0(\Omega) = \{X(t) : t \in R\}$ on some probability space $(\Omega,\mathcal{B},\mu)$ which realize $\{\mu_{T_n,X} : T_n \in R^n\}$. $H^0(\Omega)$ is called a Gaussian stochastic process because its finite dimensional probability distributions $\mu_{T_n,X}$, $T_n \in R^n$, are Gaussian measures [23]. $H^0(\Omega)$ is, however, more commonly called the Ornstein-Uhlenbec process. The Ornstein-Uhlenbec process is the only Gaussian process which is both stationary and Markov [19] (see later for the explanation of these concepts).

(1.5.7) DEFINITION Let $H^0(\Omega) = \{X(t) : t \in R\}$ be a stochastic process on the probability space $(\Omega,\mathcal{B},\mu)$. Then the quantities

$$m(t) = EX(t)$$

$$B(t,s) = \langle X(t),X(s)\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

$$R(t,s) = \langle X(t) - m(t), X(s) - m(s)\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

are called respectively the mean value, correlation function and the covariance function of $H^0(\Omega)$. When $m(t) = 0$, for all $t \in R$, then $B(t,s)$ and $R(t,s)$ are not distinguishable. If $||X(t)||_2 < \infty$ for all $t \in R$, then

$H^O(\Omega)$ is said to be a second order stochastic process.

For the Ornstein-Uhlenbec process considered in (1.5.6), $m(t) = 0$ and $B(t,s) = e^{-|t-s|}$. Furthermore clearly this process is a second order stochastic process.

(1.5.8) DEFINITION Let $H^O(\Omega) = \{X(t) : t \in R\}$ be a Gaussian stochastic process. Then $H^O(\Omega)$ is said to be stationary if

(a) $EX(t) = K$, a constant independent of t.

(b) $\langle X(t) - K, X(s) - K\rangle_{L^2(\Omega,\mathcal{B},\mu)}$ depends only on t - s.

(1.5.9) REMARK: Let $H^O(\Omega) = \{X(t) : t \in R\}$ be a Gaussian stationary stochastic process on $(\Omega,\mathcal{B},\mu)$. We suppose, as we may, that each $X(t) \in H^O(\Omega)$ has mean zero. $H^O(\Omega)$ is a Set of random variables. Let $\mathcal{H}(X)$ denote the completion in $L^2(\Omega,\mathcal{B},\mu)$ of $H^O(\Omega)$. Then $\mathcal{H}(X)$ is a Hilbert space of random variables with inner product given by

$$\langle Y,Z\rangle_{\mathcal{H}(X)} = EY\overline{Z}$$

Next, introduce the shift or translation operator $U(t)$, $t \in (-\infty,\infty)$, defined as follows

$$U(t)X(s) = X(t+s), \qquad s,t \in (-\infty,\infty).$$

Since each Y in $\mathcal{H}(X)$ is a limit in $L^2(\Omega,\mathcal{B},\mu)$ of a sequence of random variables of the form $\sum_{i=1}^{N} \alpha_i X(t_i)$, $U(t)$ may be extended by linearity and continuity to be defined on all of $\mathcal{H}(X)$. Furthermore, $U(t)$ is unitary on $\mathcal{H}(X)$, for we have

$$\begin{aligned}\langle U(t)X(s), U(t)X(s_0)\rangle_{\mathcal{H}(X)} &= \langle X(t+s), X(t+s_0)\rangle_{\mathcal{H}(X)} \\ &= B(s-s_0) \\ &= \langle X(s), X(s_0)\rangle_{\mathcal{H}(X)}\end{aligned}$$

by stationarity. Hence by the penultimate sentence, $U(t)$, $t \in (-\infty,\infty)$, is indeed unitary on all of $\mathcal{H}(X)$.

It is the preceding fact that makes stationary second order stochastic processes interesting to study, and the preceding analysis is at the root of a spectral representation for $X(t) \in \mathcal{H}(X)$. In the next chapter, we extend the concept of a stochastic process by introducing the notion of a generalized stochastic field. In chapter 3 we introduce a generalization of the notion of stationarity. This generalization is called Euclidean covariance. In the same chapter, we obtain a spectral representation for an arbitrary member of a Euclidean covariant generalized stochastic field. It is with such a generalized stochastic field that we are concerned in Chapter 4, and there the spectral representation obtained in Chapter 3 is of paramount relevance.

CHAPTER 2

RANDOM FIELDS AND ELLIPTIC PROBLEMS

Given the theory of the last chapter, we supply generalizations of various concepts and theorems expounded there in the first three sections of this chapter. Notions of Markov property for stochastic processes indexed by R and generalized stochastic fields are then discussed. Because of its relevance in forthcoming chapters, necessary and sufficient conditions for a multicomponent mean zero Gaussian stochastic process indexed by R to have the Markov property are obtained.

The last section of this chapter deals with boundary value problems for elliptic systems of linear partial differential equations. Because it intervenes in Section 4.5, we pay special attention to the exterior Dirichlet problem for such a system.

2.1 GENERALIZED STOCHASTIC FIELDS

In Section 1.5, the important notion of a stochastic process $H^0(\Omega) = \{X(t) : t \in R\}$ indexed by R and realized on some probability space $(\Omega, \mathcal{B}, \mu)$ was introduced and subsequently briefly discussed. It was there presupposed that each $X(t)$ in $H^0(\Omega)$ is a measurable function on $(\Omega, \mathcal{B}, \mu)$ which is defined for each $t \in R$. In many not uninteresting situations, however, the $\mathcal{B}$-measurable function $X(t)$ fails to be well-defined for all $t \in R$ [30][95]. For example, the map $t \to X(t)$ may be a random-variable-valued generalized function [23][88]. Thus in these cases, $X(t)$ is a random variable only in a generalized sense which we make precise in the next definition. Indeed, a large part of our analysis in this and subsequent chapters deals with random variables belonging to this extended class.

(2.1.1) DEFINITION Let $(\Omega,\mathcal{B},\mu)$ be a probability space. Let V be a locally convex topological vector space [68]. Then a continuous linear mapping of V into the set of all random variables on $(\Omega,\mathcal{B},\mu)$, equipped with the topology of convergence in probability, is called a generalized random variable.

The next definition generalizes the notion of a stochastic process introduced in Section 1.5.

(2.1.2) DEFINITION By a generalized stochastic field indexed by V and possessing $(\Omega,\mathcal{B},\mu)$ as common underlying probability space, we mean a family $H^O(\Omega) = \{f \to \xi(f) : f \in V\}$ of generalized random variables on $(\Omega,\mathcal{B},\mu)$.

(2.1.3) REMARK: Let $V_n = \{f_i : i = 1,\ldots,n\}$ be a finite set of elements of V. Then the finite dimensional probability distribution μ_{V_n} of the generalized random variables $\{\xi(f_i) : i = 1,\ldots,n\}$ on $(\Omega,\mathcal{B},\mu)$ is given by $\mu_{V_n}(A) = \mu(\{\omega : (\xi(f_1)(\omega),\ldots,\xi(f_n)(\omega)) \in A;\ A \in \mathcal{R}^n\})$
In comparison with the prevailing situation in the theory of stochastic processes (Section 1.5), given a family $\{\mu_{V_n} : V_n$ = a finite set of elements of V$\}$ of finite dimensional probability distributions, it becomes an open question as to whether or not there is a generalized stochastic field $H^O(\Omega) = \{\xi(f) : f \in V\}$ on some probability space $(\Omega,\mathcal{B},\mu)$ realizing $\{\mu_{V_n}\}$. Asking such a question is tantamount to demanding the conditions under which it is possible to install a countably additivity probability measure on the measurable space $(V^*,\mathcal{V})$ consisting of the topological dual V* of V and the minimal σ-algebra $\mathcal{V}$ containing all open set of V*
Fortunately, the relevant conditions which must be satisfied are well known [14][23][26][73][87]. Again, consistency conditions analogous to Kolmogorov Consistency Conditions intervene, and full details of how to put a probability measure on $(V^*,\mathcal{V})$ are contained in the last named references; hence we shall not pursue the matter further here. In the next

section, we merely describe $(V^*,\mathcal{V})$ and then state the analogue of Bochner's Theorem (1.3.7).

2.2 PROBABILITY MEASURES ON $(V^*,\mathcal{V})$

In this section, V will continue to denote a locally convex topological vector space whose topological dual is V^*. We begin by exhibiting the measurable space $(V^*,\mathcal{V})$.

(2.2.1) DEFINITION Let F be a finite dimensional subspace of V and suppose that $\{f_1,\ldots,f_n\}$ constitutes a basis for F. Denote by $\langle\cdot,\cdot\rangle$ the bilinear pairing of elements of V and V^*. Then the linear span F^o of the set

$$\{\xi \in V^* : \xi(f_i) = \langle \xi, f_i\rangle = 0,\ i = 1,\ldots,n\}$$

is called the annihilator of F.

(2.2.2) REMARK: Notice that since F is finite dimensional, it is isomorphic to its topological dual F^*. On the other hand, the quotient space V^*/F^o is isomorphic to F^* and hence V^*/F^o is itself finite dimensional.

(2.2.3) DEFINITION Let π_F denote the projection of V^* onto the quotient space V^*/F^o. Then, say that a set $A \subset V^*$ is a cylinder set based on F if it admits the following type of representation: $A = \pi_F^{-1}(B)$, where $B \subset V^*/F^o$ is a Borel subset.

(2.2.4) REMARK: The class of all cylinder sets in V^* clearly forms a Boolean algebra $\mathcal{V}^o$. Let $\mathcal{V}$ denote the minimal σ-Boolean algebra generated by $\mathcal{V}^o$. Then the pair $(V^*,\mathcal{V})$ is the sought-for measurable space on which a probability measure μ may live.

Suppose now that μ is a probability measure on $(V^*,\mathcal{V})$. Then the nonlinear functional

$$\exp i\langle\xi,\cdot\rangle = \exp i\xi(\cdot) : V \longrightarrow \mathbb{C}, \qquad \xi \in V^*$$

belongs to $L^1(V^*,\mathcal{V},\mu)$. Hence, it makes sense to investigate the nonlinear functional

$$V \longmapsto \mathbb{C}$$

$$f \longmapsto C(f) = \int_{V^*} d\mu(\xi)\exp i\langle\xi,f\rangle$$

The map $f \to C(f)$ just defined is called the characteristic functional of the probability measure μ. $C(f)$ has the following properties

(C1) $C(0) = 1$

(C2) C is continuous on V

(C3) C is nonnegative definite.

Any nonlinear functional possessing the properties (C1), (C2), (C3) will be called a characteristic functional. Characteristic functionals are important because they afford the speediest method of putting probability measures on measurable spaces, as indicated in the next theorem.

(2.2.5) BOCHNER-MINLOS THEOREM Let $f \to C(f)$ be a complex-valued functional on V which satisfies (C1), (C2) and (C3) of (2.2.4). Then there is a unique probability measure μ on $(V^*,\mathcal{V})$ such that

$$C(f) = \int_{V^*} d\mu(\xi) \quad \exp i\langle\xi,f\rangle.$$

(Of course, it is assumed here that V is nuclear [68]).

Proof of this important theorem may be found in [23][30][51].

(2.2.6) REMARK: In the rest of this Thesis, the generalized stochastic fields we shall encounter will be those indexed by certain spaces of generalized functions [21][22] obtained by completing the Schwartz space $\mathcal{S}(R^d)$ in some specified norms. It seems convenient, therefore, to briefly recall the definition of $\mathcal{S}(R^d)$.

First, we bring in some notations. Let $n = (n_1,\ldots,n_d)$ be a d-tuple of nonnegative integers and set $|n| = n_1 +\ldots+ n_d$

Set $$\frac{\partial^n}{\partial x^n} = \frac{\partial^{n_1}}{\partial x_1^{n_1}} \frac{\partial^{n_2}}{\partial x_2^{n_2}} \cdots \frac{\partial^{n_d}}{\partial x_d^{n_d}}$$

Then $\frac{\partial^n}{\partial x^n}$ is a differential operator monomial of order $|n|$. Next, let $\mathcal{S}^0(R^d)$ be the subspace of the Banach space $C^\infty(R^d)$ such that each $f \in \mathcal{S}^0(R^d)$ satisfies

$$\lim_{|x|\to\infty} |x|^m \left| \left(\frac{\partial^n}{\partial x^n} f\right)(x) \right| = 0$$

for any derivative of order $|n|$ of f and for any nonnegative integer m. We endow $\mathcal{S}^0(R^d)$ with the semi-norm $\rho_{K\ell} : \mathcal{S}^0(R^d) \to R_+ = [0,\infty)$

$$f \to \rho_{K\ell}(f) = \sup_{x\in R^d} \{(1 + |x|^m)\left| \left(\frac{\partial^n}{\partial x^n} f\right)(x)\right| : m \leq K,\ |n| \leq \ell\}$$

Then $\mathcal{S}(R^d)$ is $\mathcal{S}^0(R^d)$ which is already complete in the locally convex topology whose neighbourhood basis of the zero element is generated by sets of the form

$$N(f;\varepsilon_{K_1\ell_1},\ldots,\varepsilon_{K_\nu\ell_\nu}) = \{f : \rho_{K_1\ell_1}(f)<\varepsilon_{K_1\ell_1},\ldots,\rho_{K_\nu\ell_\nu}(f)<\varepsilon_{K_\nu\ell_\nu},$$

where $\varepsilon_{K_i\ell_i} > 0$, $i = 1,\ldots,\}$

$\mathcal{S}(R^d)$ is called the Schwartz space of C^∞ functions on R^d which are rapidly decreasing at infinity.

2.3 NOTIONS OF MARKOV PROPERTY

(2.3.1) Markov property, or probabilistic casuality, is a statement of conditional stochastic independence of random variables, and as mentioned in Chapter 0, the notion of "conditioning" was first introduced by the Russian mathematician A. MARKOV [48]. His compatriot, KOLMOGOROV [40], then gave this important notion very rigorous mathematical basis by invoking Measure Theory. What are now known as Markov stochastic processes intervene in many important physical and mathematical considerations (see [19], Chapter X for various such examples) and a theory of these processes utilizing various mathematical methods [9][35][66] is at an advanced stage of development. However, which mathematical device is most suitable in the investigation of a given Markov stochastic process is largely dependent on what further properties are possessed by the process. Thus, for example, in the case of Markov stochastic processes possessing stationary transition functions, the modern theory of semi-groups of bounded linear operators [31] affords the most powerful and unified tool of study. In this section, we give a definition of Markov property for stochastic processes and then we furnish a necessary and sufficient condition for Markovicity of vector stochastic processes indexed by R, which are of second order and are Gaussian.

The definition of Markov property for stochastic processes indexed by R given below will be seen to explicitly utilize the ordering relation of points of R. In trying to extend the notion of Markov property for stochastic processes so that it may apply to a stochastic field $\{\xi(x) : x \in R^d, d > 1\}$, one is therefore handicapped by the fact that there is no corresponding ordering of the points of R^d, $d > 1$. Below, we give a notion of Markov property due to NELSON [57][58]. In Chapter 4, we present the notion of Markov property due to WONG [94]. It is with these two notions of Markov property, and especially the latter, that we shall be exclusively concerned in subsequent chapters. Markov generalized stochastic fields have intervened in recent investigations in Constructive Quantum Field Theory [81][88]. However, unlike in the case of stochastic processes, the development of a theory of Markov generalized stochastic fields is still very much in its infancy, and our efforts in this work, therefore, constitute only a modest contribution to this fascinating subject.

(2.3.2) DEFINITION Let $H^O(\Omega) = \{X(t) : t \in R\}$ be a real or complex-valued stochastic process on $(\Omega, \mathcal{B}, \mu)$. Let $\mathcal{B}(t)$ denote the minimal σ-algebra with respect to which every $X(s)$, $s \leq t$, is measurable. Let $\mathcal{B}^+(t)$ denote the minimal σ-algebra with respect to which every $X(s)$, $s \geq t$, is measurable. Then $H^O(\Omega) = \{X(t) : t \in R\}$ is said to have the Markov property if for any $\mathcal{B}^+(t)$-measurable random variable Y, we have

$$E(Y|\mathcal{B}(t)) = E(Y \mid X(t)) \tag{2.3.3}$$

up to sets of probability measure zero. In words, this equation states that the future, given the past and the present, is equal to the future given only the present.

(2.3.4) REMARK: An equivalent formulation of Markov property is obtained by requiring $H^O(\Omega)$ to satisfy the following condition if it is indeed Markov: if Y is $\mathcal{B}(t)$-measurable and Z is $\mathcal{B}^+(t)$-measurable, then

$$(2.3.5) \qquad E(ZY \mid X(t)) = E(Z \mid X(t))E(Y \mid X(t))$$

up to sets of probability measure zero. Phenomenologically, (2.3.5) states that the future and the past are conditionally stochastically independent given the present.

The equivalence of (2.3.3) and (2.3.5) is readily established by invoking some of the properties of the conditional operator listed in (1.4.3).

Next, let $\mu(X(r) = \xi(r) \mid X(s) = \xi(s))$ denote the probability that $X(r)(\omega) = \xi(r)$ given that $X(s)(\omega) = \xi(s)$, $r > s$. Set

$$(2.3.6) \qquad \mu(X(r) = \xi(r) \mid X(s) = \xi(s)) = P(\xi(r) \mid \xi(s)).$$

Then $P(\xi(r) \mid \xi(s))$ is called the transition function of the stochastic process $H^O(\Omega) = \{X(r) : r \in R\}$ on the probability space $(\Omega, \mathcal{B}, \mu)$. The transition function $P(\xi(r) \mid \xi(s))$, $r > s$, of every Markov stochastic process $H^O(\Omega)$ satisfies the following equation popularly known as the CHAPMAN-KOLMOGOROV condition

$$(2.3.7) \qquad P(\xi(r) \mid \xi(r_O)) = \int_F P(\xi(r) \mid \xi(s)) dP(\xi(s) \mid \xi(r_O)), \quad r_O < s < r,$$

where $F = R$ or $\mathbb{C}$ is the state space of the stochastic process $H^O(\Omega)$. (2.3.7) is a consequence of (2.3.3). For, if Y is $\mathcal{B}^+(r)$-measurable and $r_O < r$, then

$$(2.3.8) \quad E(Y \mid X(r_O)) = E(E(Y \mid \mathcal{B}(r)) \mid X(r_O)) = E(E(Y \mid X(r)) \mid X(r_O))$$

by (2.3.3). The last equation is actually a disguised form of (2.3.7).

Set $B = \{\omega : X(r) = \xi(r)\}$ and let χ_B denote the indicator function of B. Then (2.3.7) may be written as

$$E(\chi_B \mid X(r_0) = \xi(r_0)) = E(E(\chi_B \mid X(s) = \xi(s)) \mid X(r_0) = \xi(r_0))$$

which is a consequence of (2.3.8).

(2.3.9) REMARK: In the next theorem, we furnish a necessary and sufficient condition for a second order Gaussian vector stochastic process to have the Markov property. The theorem is a generalization of a similar one which already obtains in the case of a scalar Gaussian stochastic process [15, Theorem 8.1] [95]. Presumably, our theorem is well-known, but we have found no proof, explicit or otherwise, of it elsewhere.

(2.3.10) THEOREM Let $H^0(\Omega) = \{X(r) = (X_i(r)) : r \in R\}$ be a mean zero second order Gaussian vector stochastic process indexed by R on the probability space $(\Omega, \mathcal{B}, \mu)$. Set

$$\langle X_j(r), X_k(s)\rangle_{L^2(\Omega,\mathcal{B},\mu)} = B_{jk}(r,s) .$$

Suppose that $B_{jj}(r,r) = ||X_j(r)||^2_{L^2(\Omega,\mathcal{B},\mu)} \neq 0$ for all j and all $r \in R$, and set

$$\frac{B_{jk}(r,s)}{B_{kk}(s,s)} = R_{jk}(r,s)$$

Finally, let $R(r,s)$ denote the matrix whose entries are $R_{jk}(r,s)$.

Then, a necessary and sufficient condition for $H^0(\Omega)$ to have the Markov property is that

$$R(r,s) = R(r,r_0)R(r_0,s) \qquad s < r_0 \leq r$$

Proof: The condition is necessary. Indeed, observe that the random variables $X_i(r_O) - R_{ik}(r_O,s)X_k(s)$ and $X_k(s)$ are mutually orthogonal in $L^2(\Omega,\mathcal{B},\mu)$ and, being Gaussian, are consequently stochastically independent. Hence

$$E(X_i(r_O) \mid X_k(s) = \xi_k(s))$$

$$= E(X_i(r_O) - R_{ik}(r_O,s)X_k(s) + R_{ik}(r_O,s)X_k(s) \mid X_k(s) = \xi_k(s))$$

$$= E(X_i(r_O) - R_{ik}(r_O,s)X_k(s) \mid X_k(s) = \xi_k(s)) + E(R_{ik}(r_O,s)X_k(s) \mid X_k(s) = \xi_k(s))$$

$$(2.3.10) \qquad = R_{ik}(r_O,s)\xi_k(s),$$

because $EX_j(r) = 0$ for all j and all $r \in R$.

Next, set

$$P_{ij}(\xi(r) \mid \xi(s)) = \mu(X_i(r) = \xi_i(r) \mid X_j(s) = \xi_j(s))$$

Then, by the definition of conditional expectation, we have

$$(2.3.11) \qquad E(X_i(r) \mid X_k(s) = \xi_k(s)) = \int_F \xi_i(r)dP_{ik}(\xi(r) \mid \xi(s))$$

At this juncture, we bring in the Markov condition. Thus if $H^O(\Omega) = \{X(r) = (X_i(r)) : r \in R\}$ is indeed Markov, then its transition function necessarily satisfies the Chapman-Kolmogorov condition as it applies to a vector stochastic process, to wit

$$(2.3.12) \quad P_{ik}(\xi(r)(\xi(s)) = \sum_j \int_F P_{ij}(\xi(r) \mid \xi(r_O))dP_{jk}(\xi(r_O) \mid \xi(s))$$

with $s < r_0 \leq r$.

Thus (2.3.11) becomes

$$E(X_i(r) \mid X_k(s) = \xi_k(s))$$

$$= \sum_j \int_F \int_F \xi_i(r) dP_{ij}(\xi(r) \mid \xi(r_0)) dP_{jk}(\xi(r_0) \mid \xi(s))$$

$$= \sum_j \int_F \{ \int_F \xi_i(r) dP_{ij}(\xi(r) \mid \xi(r_0)) \} \; dP_{jk}(\xi(r_0) \mid \xi(s))$$

$$s < r_0 \leq r \; .$$

thanks to Fubini's Theorem [27]. The object enclosed in the chain bracket is $E(X_i(r) \mid X_j(r_0) = \xi_j(r_0))$ and this latter by (2.3.10) is $R_{ij}(r,r_0)\xi_j(r_0)$. Hence

(2.3.13) $$E(X_i(r) \mid X_k(s) = \xi_k(s))$$

$$= \sum_j R_{ij}(r,r_0) \int_F \xi_j(r_0) \; dP_{jk}(\xi(r_0) \mid \xi(s))$$

$$= \sum_j R_{ij}(r,r_0) R_{jk}(r_0,s) \xi_k(s) \qquad s < r_0 \leq r$$

again by (2.3.10) and (2.3.11). On the other hand, by

(2.3.10), $$E(X_i(r) \mid X_k(s) = \xi_k(s)) = R_{ik}(r,s)\xi_k(s) \; .$$

Consequently, (2.3.13) becomes

$$R_{ik}(r,s)\xi_k(s) = \sum_j R_{ij}(r,r_0) R_{jk}(r_0,s)\xi_k(s) \qquad s < r_0 \leq r$$

or equivalently

$$(2.3.14) \qquad R(r,s) = R(r,r_0)R(r_0,s) \qquad s < r_0 \leq r$$

Therefore the condition is indeed necessary.

The condition is also sufficient. Indeed, if $H^0(\Omega) = \{X(r) = (X_i(r)) : r \in R\}$ is a second order mean zero Gaussian vector stochastic process for which (2.3.14) holds then it follows that

$$(2.3.15) \qquad B_{ik}(r,s) = \sum_j \frac{B_{ij}(r,r_0)}{B_{jj}(r_0,r_0)} B_{jk}(r_0,s) \qquad s < r_0 \leq r$$

Equation (2.3.15) says that

$$X_i(r) = \sum_j \frac{B_{ij}(r,r_0)}{B_{jj}(r_0,r_0)} X_j(r_0)$$

is orthogonal to, and hence stochastically independent of, $X_k(s)$ for $s < r_0 \leq r$. By (2.3.15), we have too that

$$E(X_i(r) \mid X(r_0) = (X_j(r_0))) = \sum_j R_{ij}(r,r_0)X_j(r_0)$$

for

$$\langle X_i(r) - \sum_j R_{ij}(r,r_0)X_j(r_0),\ X_k(r_0)\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= B_{ik}(r,r_0) - \sum_j R_{ij}(r,r_0)B_{jk}(r_0,r_0)$$

$$= 0 .$$

Hence

$$\sum_j R_{ij}(r,r_0)X_j(r_0)$$

$$= E(X_i(r) \mid X(r_0) = (X_j(r_0)))$$

$$= E(X_i(r) \mid X(\rho) = (X_j(\rho)) : \rho < r_0 \leq r)$$

by virtue of the penultimate observation. Hence $H^0(\Omega) = \{X(r) = (X_i(r)) ; r \in R\}$ is indeed Markov, and the condition is indeed also sufficient. This completes the proof.

(2.3.16) REMARK: If $R(r,s)$ is invertible for all $(r,s) \in R \times R$, then [1] the most general continuous solution of the functional equation (2.3.14) is of the form

$$\begin{aligned} R(r,s) &= G(r)H(s) \qquad s < r \\ &= G(\max(r,s))H(\min(r,s)) \end{aligned}$$

where $G(\cdot)$ and $H(\cdot)$ are square matrices each of the same dimensionality as $R(\cdot,\cdot)$. However, in our use of (2.3.14) in later chapters, we place no invertibility assumption on $R(\cdot,\cdot)$.

Next, we consider one definition of Markov property for generalized stochastic fields.

(2.3.17) NELSON'S NOTION OF MARKOV PROPERTY [57][58]. We have already made comments concerning this notion of Markov property in Chapter 0. Hence, we now only give the relevant definition.

Let $H^0(\Omega) = \{\xi(f) : f \in \mathcal{S}(R^d)\}$ be a generalized stochastic field on a probability space $(\Omega,\mathcal{B},\mu)$. We always assume linearity for $f \to \xi(f)$, i.e.

$$\xi(\alpha f_1 + \beta f_2) = \alpha\xi(f_1) + \beta\xi(f_2), \quad \alpha, \beta \in \mathbb{C},$$

except possibly on sets of μ-measure zero, and we assume continuity in probability, i.e. if $\{f_n\}$ is a sequence of members of $\mathcal{S}(R^d)$ converging in the topology of $\mathcal{S}(R^d)$ to f in $\mathcal{S}(R^d)$, then $\{\xi(f_n)\}$ converges in probability to $\xi(f)$.

Next, let D be an open subset of R^d with boundary ∂D and complement D'. We denote by $\mathcal{B}(D)$ the minimal σ-algebra generated by the set $\{\xi(f) : f \in \mathcal{S}(R^d)$ with support of $f \subset D\}$, and we set

$$\mathcal{B}(\partial D) = \bigcap \mathcal{B}(O)$$

where the intersection is over all open sets O which contain ∂D. $\mathcal{B}(\partial D)$ is called the boundary data σ-algebra. Then $H^O(\Omega) = \{\xi(f) = f \in \mathcal{S}(R^d)\}$ is said to be Markov in the sense of NELSON if for every positive random variable $u \in \mathcal{B}(D)$, we have

$$\text{(2.3.18)} \qquad E(u \mid \mathcal{B}(D')) = E(u \mid \mathcal{B}(\partial D))$$

except possibly on sets of μ-measure zero.

In [57],[88], Nelson introduced the following concept.

(2.3.19) DEFINITION Let $\rho \to T(\rho)$ be a homomorphism of the group of reflections in the hyperplane R^{d-1} into the group of automorphisms of the measure algebra [27, Section 40] of the probability space $(\Omega, \mathcal{B}, \mu)$. Then $H^O(\Omega) = \{\xi(f) : f \in \mathcal{S}(R^d)\}$ is said to possess the reflection property of Nelson if

$$T(\rho)u = u \qquad \text{for all } u \in \mathcal{B}(R^{d-1})$$

(2.3.20) REMARK: Let $\mathcal{H}(\mathbb{R}^d)$ be the completion of $\mathcal{S}(\mathbb{R}^d)$ in the norm derived from the inner product

$$<\cdot,\cdot>_{\mathcal{H}(\mathbb{R}^d)} : \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}$$

$$(f^{(1)},f^{(2)}) \to <f^{(1)},f^{(2)}>_{\mathcal{H}(\mathbb{R}^d)} = <f^{(1)}, (m^2 - \Delta)^{-1}f^{(2)}>_{L^2(\mathbb{R}^d,dx)}$$

where $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator and $m > 0$. Nelson shows in [57][88] that the Gaussian generalized stochastic field whose correlation functional is given by $<f^{(1)},f^{(2)}>_{\mathcal{H}(\mathbb{R}^d)}$ satisfies his definitions of Markov property and reflection property formulated above. By applying his reconstruction scheme [57][58][88], he then shows that this particular Gaussian generalized stochastic field leads to the free scalar Wightman quantum field [84]. In Chapter 0, we have already indicated that this fact is at the root of our motivation to investigate Wong's definition of Markov property in the case of Gaussian multicomponent generalized stochastic fields.

2.4 BOUNDARY VALUE PROBLEMS

In our study of Wong's notion of Markov property in Chapter 4 for a multicomponent Gaussian generalized stochastic field $H^O(\Omega) = \{\xi(f) = (\xi_i(f)) : f \in \mathcal{S}(\mathbb{R}^d)\}$ we employ aspects of the theory of boundary value problems for elliptic systems of partial differential operators. In this section, we present a rapid introduction to this theory and in the course of doing so we establish our notations. It is perhaps worthwhile to mention that other authors [52][54][64] have also invoked the theory of boundary value problems for elliptic operators in their own study of notions of Markov property different from Wong's.

The expository account presented here follows the treatment in [2] [3] [16] [46] [69-70] [80] [61].

(2.4.1) Let $C^k(R^d)$, $k \in Z_+$ = set of nonnegative integers, denote the Banach space of k times continuously differentiable complex-valued functions on R^d. Then we define the linear partial differential operator $P(\frac{\partial}{\partial x})$ of order ℓ with constant complex coefficients a_n, $n = (n_1,\ldots,n_d) \in Z_+^d$, as follows

$$(2.4.2) \quad P(\frac{\partial}{\partial x})f = \sum_{|n|\leq\ell} a_n \frac{\partial^n}{\partial x^n} f, \quad \text{for all } f \in C^\ell(R^d)$$

To the operator $P(\frac{\partial}{\partial x})$, we make correspond a polynomial $\eta \to P^O(\eta)$, $\eta \in R^d$, in d variables, called the characteristic form of $P(\frac{\partial}{\partial x})$, defined thus

$$(2.4.3) \quad P^O(\eta) = \sum_{|n|=\ell} a_n \eta^n$$

$$\eta^n = \eta_1^{n_1} \eta_2^{n_2} \cdots \eta_d^{n_d}, \quad n = (n_1,\ldots,n_d) \in Z_+^d$$

Next, let $A(\frac{\partial}{\partial x})$ be an N x N matrix whose entries $A_{ij}(\frac{\partial}{\partial x})$, $i,j = 1,\ldots,N$ are linear partial differential operators with constant coefficients. Let $\eta \to A_{ij}^O(\eta)$, $\eta \in R^d$, denote the characteristic form of $A_{ij}(\frac{\partial}{\partial x})$, $i,j = 1,\ldots,N$. Let $A^O(\eta)$ be the matrix whose entries are $A_{ij}^O(\eta)$, $i,j = 1,\ldots,N$. We denote by $|A^O(\eta)|$ the determinant of the matrix $A^O(\eta)$.

(2.4.4) DEFINITION $A(\frac{\partial}{\partial x}) = (A_{ij}(\frac{\partial}{\partial x}) : i,j = 1,\ldots,N)$ is said to be elliptic if

(i) $|A^O(\eta)| \neq 0$ for all $\eta \in R^d$, $\eta \neq 0$;

(ii) there exist integers s_i, t_i, $i = 1,\ldots,N$ such that the order of $A_{ij}(\frac{\partial}{\partial x})$ is $s_i + t_j$, where it is understood that we set $A_{ij}(\frac{\partial}{\partial x}) \equiv 0$ if $s_i + t_j < 0$.

(2.4.5) REMARK: The following definition of ellipticity given by VOLEVICH [90] has been shown by him to be equivalent to (2.4.4). To the matrix $A(\frac{\partial}{\partial x})$ of linear partial differential operators $A_{ij}(\frac{\partial}{\partial x})$, $i,j = 1,\ldots,N$, we make correspond the matrix $\eta \to A(\eta)$, $\eta \in R^d$, whose entries are the polynomials $\eta \to A_{ij}(\eta)$, $i,j = 1,\ldots,N$ in d variables. Then the determinant $|A(\eta)| = L(\eta)$ of $A(\eta)$ is given by

$$(2.4.6) \qquad L(\eta) = \sum_{\pi} \varepsilon_{\pi} A_{\pi(1)1}(\eta) A_{\pi(2)2}(\eta) \ldots A_{\pi(d)d}(\eta)$$

, where π runs through the symmetric group of all permutations of $\{1,\ldots,d\}$ and ε_{π} is the signature of π. Since $\eta \to L(\eta)$ is a polynomial in d variables, we can associate with it a partial differential operator $L(\frac{\partial}{\partial x})$ whose characteristic form we denote by $L^O(\eta)$. Let r denote the degree of $L^O(\eta)$ and let R denote the maximal degree of the summands on the right hand side of (2.4.6). In general $r \leq R$. Then the definition of ellipticity given in (2.4.4) is equivalent to the following formulation [90].

$A(\frac{\partial}{\partial x}) = (A_{ij}(\frac{\partial}{\partial x}) : i,j = 1,\ldots,N)$ is elliptic if

(i) $r = R$

(ii) $L^O(\eta) \neq 0$ for all $\eta \in R^d$, $\eta \neq 0$.

We mention this equivalent definition of ellipticity because it is more readily checked than (2.4.4).

Let us now bring in other important notions.

(2.4.7) DEFINITION $A(\frac{\partial}{\partial x}) = (A_{ij}(\frac{\partial}{\partial x}) : i,j = 1,\ldots,N)$ is said to be properly elliptic if the polynomial $\eta \to |A^o(\eta)|$, $\eta \in R^d$, is of even degree 2m and if for every pair η, $\eta' \in R^d$ of linearly independent vectors, the polynomial $\tau \to |A^o(\eta + \tau\eta')|$ in the complex variable τ has exactly m roots with positive imaginary part.

(2.4.8) REMARK: For $d \geq 3$, every elliptic $A(\frac{\partial}{\partial x}) = (A_{ij}(\frac{\partial}{\partial x}) : i,j = 1,\ldots,N)$ is automatically properly elliptic [2] pp. 631-632. However for d = 2 this is no longer so.

In the next definition, we introduce a special class of properly elliptic operators.

(2.4.9) DEFINITION Let s_i, t_i, $i = 1,\ldots,N$ be as in (2.4.4). Then $A(\frac{\partial}{\partial x}) = (A_{ij}(\frac{\partial}{\partial x}) : i,j = 1,\ldots,N)$ is said to be strongly elliptic if $t_i = s_i > 0$ and if for every complex vector $\alpha = (\alpha_1,\ldots,\alpha_N)$ and every $\eta \in R^d$, $\alpha \neq 0$, $\eta \neq 0$, we have

$$(2.4.10) \qquad \mathrm{Re} \sum_{i,j=1}^{N} (-1)^{s_i} A^o_{ij}(\eta)\alpha_i\bar{\alpha}_j \geq K \sum_{i=1}^{N} |\eta|^{2s_i} |\alpha_i|^2$$

where $K > 0$, is a constant.

(2.4.11) REMARK: In this section, our objective is to discuss boundary value problems for elliptic operators. Suppose, therefore, that $A(\frac{\partial}{\partial x}) = (A_{ij}(\frac{\partial}{\partial x}) : i,j = 1,\ldots,N)$ is a properly elliptic operator of order 2m. Let $B(\frac{\partial}{\partial x})$ be an m x N matrix whose entries are the linear partial differential operators $B_{hj}(\frac{\partial}{\partial x})$, $h = 1,\ldots,m$, $j = 1,\ldots,N$. Let D be an open (not necessarily bounded) subset of R^d whose complement is D', and with boundary ∂D. Let $C_o^{2m}(R^d)$ be the Banach space of elements of $C^{2m}(R^d)$ which vanish at infinity and let $(C^{2m}(D))^N$ denote the N-fold Cartesian product of $C^{2m}(D)$ with itself. For $F = (F_1,\ldots,F_N)$ belonging to $(C(D))^N$, we now consider the problem of finding $f = (f_1,\ldots,f_N)$ belonging to $(C^{2m}(D \cup \partial D))^N$ such that

$$(2.4.12) \qquad \sum_{j=1}^{N} A_{ij}(\frac{\partial}{\partial x}) f_j = F_i \quad \text{in } D,\ i = 1,\ldots,N$$

$$(2.4.13) \qquad \sum_{j=1}^{N} B_{hj}(\frac{\partial}{\partial x}) f_j = \Phi_h \quad \text{on } \partial D,\ h = 1,\ldots,m$$

$$(2.4.14) \qquad f_j = 0 \text{ at infinity, } i = 1,\ldots,N, \text{ if } D \text{ is unbounded.}$$

First we remark that $B(\frac{\partial}{\partial x}) = (B_{hj}(\frac{\partial}{\partial x}) : h = 1,\ldots,m;\ j = 1,\ldots,N)$ is called a matrix of boundary operators associated with $A(\frac{\partial}{\partial x}) = (A_{ij}(\frac{\partial}{\partial x}) : i,j = 1,\ldots,N)$. Next, we mention that (2.4.12), (2.4.13), (2.4.14) constitute what is called a properly elliptic boundary value problem, denoted $\{A(\frac{\partial}{\partial x}), B(\frac{\partial}{\partial x})\}$, for the matrix of operators $A(\frac{\partial}{\partial x})$.

It is well-known [69], even in the classical case of the Laplace operator $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$, that such a boundary value problem may or may not be well-posed. In order to obtain a well-posed boundary value problem $\{A(\frac{\partial}{\partial x}), B(\frac{\partial}{\partial x})\}$, additional conditions must be imposed on $B(\frac{\partial}{\partial x}) = (B_{hj}(\frac{\partial}{\partial x}) : h = 1,\ldots,m;\ j = 1,\ldots,N)$ relative to $A(\frac{\partial}{\partial x}) = (A_{ij}(\frac{\partial}{\partial x}) : i,j = 1,\ldots,N)$; thus $B(\frac{\partial}{\partial x})$ cannot be prescribed arbitrarily. In what follows, we describe the required additional conditions.

Let $\eta \to B^0_{hj}(\eta)$, $\eta \in R^d$, denote the characteristic form of $B_{hj}(\frac{\partial}{\partial x})$ and let $B^0(\eta)$ denote the matrix whose entries are $B^0_{hj}(\eta)$, $h = 1,\ldots,m$; $j = 1,\ldots,N$. Let η, η' be two linearly independent vectors in R^d, and denote by $\tau_k^+(\eta,\eta')$, $k = 1,\ldots,m$, the m roots with positive imaginary part of the characteristic equation $|A^0(\eta + \tau\eta')| = 0$ in the complex variable τ. The existence of these roots is assured by the proper ellipticity of $A(\frac{\partial}{\partial x})$; see (2.4.7). Set

$$\prod_{k=1}^{m} (\tau - \tau_k^+(\eta,\eta')) = M(\eta,\eta',\tau).$$

Let $\mathcal{A}(\eta)$ denote the adjoint of the matrix $A^0(\eta)$; thus $A^0(\eta)\mathcal{A}(\eta) = |A^0(\eta)|I$, I = identity $N \times N$ matrix. Then we introduce the following

(2.4.15) DEFINITION $B(\frac{\partial}{\partial x}) = (B_{hj}(\frac{\partial}{\partial x})$: $h = 1,\ldots,m$; $j = 1,\ldots,N)$ is said to cover $A(\frac{\partial}{\partial x}) = (A_{ij}(\frac{\partial}{\partial x})$: $i,j = 1,\ldots,N)$ if there exist integers σ_h, $h = 1,\ldots,m$, such that $B_{hj}(\frac{\partial}{\partial x})$ is of order $\sigma_h + t_j$ (set $B_{hj}(\frac{\partial}{\partial x}) = 0$ if $\sigma_h + t_j < 0$) and if the rows of the matrix

$$B^0(\eta + \tau\eta')\mathcal{A}(\eta + \tau\eta')$$

(the entries of which are polynomials in τ) are linearly independent modulo the polynomial $M(\eta,\eta',\tau)$, i.e.

$$\sum_{h=1}^{m} C_h \sum_{j=1}^{N} B^0_{hj}(\eta + \tau\eta')\mathcal{A}_{jk}(\eta + \tau\eta') \equiv 0 \pmod{M(\eta,\eta',\tau)} \text{ only if } C_h = 0,$$
$h = 1,\ldots,m$.

(2.4.16) REMARK: We make the assumption in all that follows that $B(\frac{\partial}{\partial x}) = (B_{hj}(\frac{\partial}{\partial x})$: $h = 1,\ldots,m$; $j = 1,\ldots,N)$ covers $A(\frac{\partial}{\partial x}) = (A_{ij}(\frac{\partial}{\partial x})$: $i,j = 1,\ldots,N)$. Then the boundary value problem $\{A(\frac{\partial}{\partial x}), B(\frac{\partial}{\partial x})\}$ given by (2.4.12), (2.4.13), (2.4.14) is well-posed [46][3].

In Chapter 4, we encounter an exterior Dirichlet boundary value problem for a strongly elliptic operator $A(\frac{\partial}{\partial x}) = (A_{ij}(\frac{\partial}{\partial x})$: $i,j = 1,\ldots,N)$. We wish, therefore, to describe the setting and solution of this problem.

Let $A(\frac{\partial}{\partial x}) = (A_{ij}(\frac{\partial}{\partial x})$: $i,j = 1,\ldots,N)$ be strongly elliptic. Then, by definition, the order of $A_{ij}(\frac{\partial}{\partial x})$ is $s_i + t_j$ where s_i, t_i, $i = 1,\ldots,N$ are integers such that $s_i = t_i > 0$. Let D be a bounded open subset of R^d with boundary ∂D and complement D'. Let $\frac{\partial}{\partial \underline{n}}$ denote differentiation in the direction of the outward normal $\underline{n}$ on ∂D. Then the exterior Dirichlet boundary value problem is the following:

given $F = (F_1,\ldots,F_N) \in (C(D'))^N$, find $f = (f_1,\ldots,f_N)$

belonging to $(C^{2m}(D'))^N$ such that

$$(2.4.17) \qquad \sum_{j=1}^{N} A_{ij}(\frac{\partial}{\partial x})f_j = F_i \text{ in } D', \quad i = 1,\ldots,N$$

$$(2.4.18) \qquad (\frac{\partial}{\partial \underline{n}})^{\nu} f_i = \Phi_{\nu i} \text{ on } \partial D, \ \nu = 0,1,\ldots,s_i-1, \ i = 1,\ldots,N$$

$$(2.4.19) \qquad f_i = 0 \text{ at infinity, } i = 1,\ldots,N$$

(2.4.18) and (2.4.19) are called the Dirichlet data for (2.4.17). As shown in [3] p.44, the system of operators $\{(\frac{\partial}{\partial \underline{n}})^{\nu} : \nu = 0,1,\ldots,s_i-1,$ $i = 1,\ldots,N\}$ covers $A(\frac{\partial}{\partial x}) = (A_{ij}(\frac{\partial}{\partial x}) : i,j = 1,\ldots,N)$. Hence the exterior Dirichlet boundary value problem is well-posed.

Below, we give integral representations for the functions f_i, $i = 1,\ldots,N$ which solve (2.4.17), (2.4.18), (2.4.19) for a particular choice of $F = (F_1,\ldots,F_N)$ and $\Phi = (\Phi_{\nu i} : \nu = 0,1,\ldots,s_i-1, \ i = 1,\ldots,N)$.

(2.4.2) GREEN'S MATRIX FOR AN ELLIPTIC SYSTEM Let $A(\frac{\partial}{\partial x}) = (A_{ij}(\frac{\partial}{\partial x}) :$ $i,j = 1,\ldots,N)$, where $A_{ij}(\frac{\partial}{\partial x})$ have as usual constant coefficients, be elliptic. Consider the system of partial differential equations

$$(2.4.21) \qquad \sum_{j=1}^{N} A_{ij}(\frac{\partial}{\partial x}) f_j = F_i , \qquad i = 1,\ldots,N$$

Then a matrix $(x,y) \rightarrow \varepsilon(x,y) = (\varepsilon_{ij}(x,y) : i,j = 1,\ldots,N)$ is called a fundamental matrix for $A(\frac{\partial}{\partial x}) = (A_{ij}(\frac{\partial}{\partial x}) : i,j = 1,\ldots,N)$ if

$$(2.4.22) \qquad \sum_{k=1}^{N} A_{ik}(\frac{\partial}{\partial x}) \varepsilon_{kj}(x,y) = \delta_{ij} \, \delta(x-y)$$

where δ_{ij} is the Kronecker delta function and $\delta(x-y)$ is the Dirac delta "function". The elaborate exercise of constructing a fundamental matrix for an elliptic matrix of partial differential operators whose entries have analytic coefficients has been accomplished by JOHN [36], Chapter 3. Here we give the results of this reference in our particular case where $A_{ij}(\frac{\partial}{\partial x})$, $i,j = 1,\ldots,N$ have constant coefficients.

Thus let

$$s \to g(s) = \begin{cases} \dfrac{|s|}{4!\,(2\pi i)^{d-1}} & \text{for odd } d \\ & s \in R \\ -\dfrac{s^2 \log |s|}{2!\,(2\pi i)^{d}} & \text{for even } d \end{cases}$$

Let $P(\eta)$ be the matrix inverse of $A(\eta)$; and let C denote a path in the complex λ-plane containing all the roots of $|A(\lambda\eta)| = 0$. Then define the functions

$$(2.4.23) \quad v_{ij}(x,\eta,t) = \frac{1}{2\pi i} \oint_C d\lambda \; \frac{1}{\lambda} \; e^{\lambda(x\cdot\eta - t)} P_{ij}(\lambda\eta)$$

$$(2.4.24) \quad V_{ij}(x,\eta,t) = -\int_0^{x\cdot\eta - t} dr \; g(r) \frac{\partial}{\partial t} \; v_{ij}(x,\eta, r+t)$$

$$(2.4.25) \quad W_{ij}(x,y) = \int_{\Omega = \{\eta \in R^d :\, |\eta| = 1\}} d\Omega \; V_{ij}(x,\eta,\eta\cdot y)$$

John [36] shows that a fundamental matrix

$\varepsilon(x,y) = (\varepsilon_{ij}(x,y) : i,j = 1,\ldots,N)$ for $A(\frac{\partial}{\partial x}) = (A_{ij}(\frac{\partial}{\partial x}) : i,j = 1,\ldots,N)$

is given by

$$(2.4.26) \quad \varepsilon_{ij}(x,y) = (\Delta_y)^{(d+k)/2} \; W_{ij}(x,y)$$

where Δ_y is the Laplace operator in the d variables $y = (y_1, \ldots, y_d)$ and $k = 1$ for odd d, $k = 2$ for even d.

Next, let $\mathcal{E}(x,y) = (\mathcal{E}_{ij}(x,y) : i,j = 1,\ldots,N)$ satisfy

$$A(\frac{\partial}{\partial x}) \mathcal{E}(x,y) = O \quad \text{in } D',$$

where O is the null N x N matrix, and assume that $(x,y) \to \mathcal{E}_{ij}(x,y)$ is continuous on $D' \cup \partial D$ with respect to the argument x for all $i,j = 1,\ldots,N$. Suppose further that

$$\mathcal{E}(x,y) = -\varepsilon(x,y) \quad \text{for } x \in \partial D \text{ and } y \in D'$$

Then the matrix $(x,y) \to G(x,y)$ given by

$$G(x,y) = \varepsilon(x,y) + \mathcal{E}(x,y) \tag{2.4.27}$$

is also a fundamental matrix for $A(\frac{\partial}{\partial x}) = (A_{ij}(\frac{\partial}{\partial x}) : i,j = 1,\ldots,N)$. Furthermore, we have

$$G(x,y) = O, \qquad x \in \partial D \tag{2.4.28}$$

The matrix $(x,y) \to G(x,y)$ is called the Green's matrix for $A(\frac{\partial}{\partial x}) = (A_{ij}(\frac{\partial}{\partial x}) : i,j = 1,\ldots,N)$ for the region D' of R^d. The Green's matrix plays a fundamental role in the integral representations of solutions of boundary value problems for elliptic systems. We need such integral representations in Chapter 4; hence we shall next indicate how they are obtained.

(2.4.29) DEFINITION M is said to be a bilinear differential operator of order s if

$$M(f^{(1)}, f^{(2)}) = \sum_{i=1}^{\ell<\infty} a_i P_i(\frac{\partial}{\partial x}) f^{(1)} \overline{Q_i(\frac{\partial}{\partial x}) f^{(2)}}$$

where $\{P_i(\frac{\partial}{\partial x}), Q_i(\frac{\partial}{\partial x}) : i = 1,\dots, \ell < \infty\}$ are linear partial differential operators (not necessarily with constant coefficients) such that

$$\text{order of } P_i(\frac{\partial}{\partial x}) + \text{order of } Q_i(\frac{\partial}{\partial x}) = s, \ i = 1,\dots, \ell$$

and where $\{a_i : i = 1,\dots, \ell < \infty\}$ is a set of complex numbers.

(2.4.30) REMARK: In what follows, always suppose that the elliptic matrix $A(\frac{\partial}{\partial x}) = (A_{ij}(\frac{\partial}{\partial x}) : i,j = 1,\dots,N)$ is also formally self-adjoint. Then, as also indicated in [36], by integration by parts, we readily obtain

$$(2.4.31) \quad \sum_{i,j=1}^{N} \int_{D'} dx f_i^{(1)}(x) \overline{(A_{ij}(\frac{\partial}{\partial x}) f_j^{(2)})(x)} - \sum_{i,j=1}^{N} \int_{D'} dx\, f_j^{(2)}(x) \overline{(A_{ij}(\frac{\partial}{\partial x}) f_i^{(1)})(x)} = \sum_{i,j=1}^{N} \int_{\partial D} d\sigma \; M_{ij}(f_i^{(1)}(x), f_j^{(2)}(x))$$

where M_{ij} is a bilinear differential operator of order $s_i + t_j - 1$. Recall that as in (2.4.4) the order of $A_{ij}(\frac{\partial}{\partial x})$ is $s_i + t_j$. $d\sigma$ is the surface measure on ∂D.

Now, set $f_i^{(1)} = f_i$ and $f_j^{(2)} = G_{jk}(\cdot,y)$ in (2.4.31). Then

$$(2.4.32) \quad f_k(y) = \sum_{i,j=1}^{N} \int_{D'} dx\, G_{jk}(x,y) \overline{(A_{ij}(\frac{\partial}{\partial x}) f_i)(x)} + \sum_{i,j=1}^{N} \int_{\partial D} d\sigma\, M_{ij}(f_i(x), G_{jk}(x,y)) \qquad y \in D'.$$

This is the integral representation we referred to above. Let us now make one, for us, important application of it.

(2.4.33) AN EXTERIOR DIRICHLET BOUNDARY VALUE PROBLEM The following particular boundary value problem arises in our analysis of Markov property in Chapter 4. Let $A(\frac{\partial}{\partial x}) = (A_{ij}(\frac{\partial}{\partial x}) : i,j = 1,\ldots,N)$ be strongly elliptic and formally self-adjoint. Consider the following exterior Dirichlet boundary value problem:

(2.4.34) $$\sum_{j=1}^{N} A_{ij}(\frac{\partial}{\partial x})f_j = 0, \quad \text{in } D', \; i = 1,\ldots,N$$

(2.4.35) $$f_i = 0 \text{ at infinity}, \; i = 1,\ldots,N$$

(2.4.36) $$f_i = f_i^0 \text{ on } \partial D, \quad i = 1,\ldots,N$$

(2.4.37) $$(\frac{\partial}{\partial \underline{n}})^{\nu} f_i = 0, \quad \nu = 1,\ldots,s_i-1, \; i = 1,\ldots,N, \text{ on } \partial D$$

where $\underline{n}$ = outward normal on ∂D. Then by (2.4.34) and (2.4.32) we have

$$f_k(y) = \sum_{i,j=1}^{N} \int_{\partial D} d\sigma \; M_{ij}(f_i(x), G_{jk}(x,y)), \quad y \in D', \quad k = 1,\ldots,N$$

Finally, by application of the Riesz representation theorem for a continuous linear functional on $(C(\partial D))^N$ we see that this last equation may indeed be presented as follows

(2.4.38) $$f_k(x) = \sum_{i=1}^{N} \int_{\partial D} d\sigma \; \mathcal{P}_{ki}(x,y) \; f_i^0(y), \quad x \in D', \quad k = 1,\ldots,N$$

The matrix $\mathcal{P}(x,y) = (\mathcal{P}_{ki}(x,y) : k,i = 1,\ldots,N)$ is such that (2.4.36) holds and it has the following additional properties:

(2.4.39) $$\sum_{k=1}^{N} A_{jk}(\frac{\partial}{\partial x}) \mathcal{P}_{ki}(x,y) = 0 \text{ in } D', \quad i,j = 1,\ldots,N$$

(2.4.40) $x \rightarrow \mathcal{P}_{ki}(x,y)$ vanishes at infinity for all $y \in \partial D$, $k,i = 1,\ldots,N$

(2.4.41) $$\left(\frac{\partial}{\partial \underline{n}_x}\right)^{\nu} \mathcal{P}_{ki}(x,y) = 0 \text{ on } \partial D, \quad \nu = 1,\ldots,s_k-1, \; k = 1,\ldots,N$$

Thus, we have obtained integral representations (2.4.38) for the functions f_i, $i = 1,\ldots,N$ which solve the exterior Dirichlet boundary value problem constituted by (2.4.34)-(2.4.37).

We shall utilize the integral representations (2.4.38) in Chapter 4. Meanwhile, in the next chapter, we provide a spectral representation for an arbitrary Euclidean covariant multicomponent generalized stochastic field.

CHAPTER 3

HARMONIC ANALYSIS OF EUCLIDEAN COVARIANT GENERALIZED STOCHASTIC FIELDS

The analysis in the next chapter centres on a Euclidean covariant multicomponent generalized stochastic field. Here, we provide the spectral representation needed there for an arbitrary such field.

Towards the end of this chapter, our results are put to the test in the case where the field has only three components and we show that our spectral representation coincides with the one furnished by Yaglom in [97].

3.1 GROUP THEORETIC CONCEPTS

Let $|\cdot| \; : \; R^d \rightarrow R_+ = [0,\infty)$

$$x \rightarrow |x| = \left(\sum_{i=1}^{d} x_i^2 \right)^{1/2} = r$$

denote the usual norm on R^d; $|\cdot|$ is derived from the inner product $\langle\cdot,\cdot\rangle \; : \; R^d \times R^d \rightarrow R$

$$(x,y) \rightarrow \langle x,y\rangle = \sum_{i=1}^{d} x_i y_i$$

Then, the finite dimensional Hilbert space $E^d = (R^d, \langle\cdot,\cdot\rangle)$ is called Euclidean space. Sometimes, we simply write R^d for E^d.

The full Euclidean group $M(d)$ is the group, whose group operation is the usual composition of maps, of all nonsingular nonlinear transformations

$$g \; : \; E^d \rightarrow E^d$$

$$x \rightarrow gx$$

such that $\langle gx-gy, gx-gy\rangle = \langle x-y, x-y\rangle$. M(d) is the semi-direct product

$$M(d) = E^d \overset{s}{\otimes} O(d)$$

of E^d, which is an Abelian group under addition, with O(d) — the latter being defined as the group, with matrix multiplication for its group law, of all nonsingular linear transformations

$$h : E^d \to E^d$$
$$x \to hx$$

such that determinant of h $= \pm 1$ and $|hx| = |x|$.

Thus each $g \in M(d)$ is a pair $g = (a,h) \in E^d \overset{s}{\otimes} O(d)$ and $x \to gx = hx + a$, $x \in E^d$. Furthermore if $g_1, g_2 \in M(d)$ with $g_i = (a_i,h_i)$, $i = 1,2$, then

$$g_1 g_2 = (a_1 + h_1 a_2, h_1 h_2) .$$

(3.1.1) DEFINITION Let G be an arbitrary topological group and let X be a topological space. Then G is said to be a group of transformations of X if each $g \in G$ determines a bijective and bi-continuous mapping

$$g : X \to X$$
$$x \to gx$$

satisfying the following

(i) if $e \in G$ is the identity element, then

$e : X \to X$, $x \to ex$, is the identity transformation of X onto itself;

(ii) $g_2(g_1x) = g_2g_1x$ for $(g_1,g_2,x) \in G \times G \times X$

(iii) the map $f: G \times X \to X$

$$(g,x) \to f(g,x) = gx$$

is separately continuous on $G \times X$.

A topological space X possessing a topological group G of transformations is called a group space. A readily available example of a group space is any topological group G. Another example of direct relevance to us is Euclidean space E^d, whose group of transformations is $M(d)$.

(3.1.2) DEFINITION Let X be a group space with G as its group of transformations. Then X is called a homogeneous space if every $x \in X$ can be taken into every point in X by the action of G on X given in (3.1.1) above. Then G is said to act transitively on X. G is said to act effectively on X if for $g \in G$, $g \neq e$ = the identity element, a point $x \in X$ exists such that $gx \neq x$.

(3.1.3) REMARK: It is well-known [24] that if X is a homogeneous space whose group of transformations is G, and if K is the maximal subgroup of G which leaves a certain point $x_0 \in X$ fixed, then X is isomorphic to the quotient space G/K. Indeed, all homogeneous spaces may be described in this way.

(3.1.4) DEFINITION Let $\mathcal{H}$ be an infinite dimensional Hilbert space and let $U(\mathcal{H})$ be the group of all unitary operators on $\mathcal{H}$. Then a unitary representation of the Euclidean group $M(d)$ on $\mathcal{H}$ is a strongly continuous homomorphism of $M(d)$ into $U(\mathcal{H})$.

A unitary representation $g \to T(g)$ of $M(d)$ on $\mathcal{H}$ is said to be irreducible if the only subspaces of $\mathcal{H}$ left invariant under its action on $\mathcal{H}$, for all $g \in M(d)$, consist of the pair $\mathcal{H}$ itself and $\{0\}$.

(3.1.5) REMARK: We require knowledge, in the next section, of the irreducible unitary representations of $M(d)$. We, therefore, wish to describe them in what follows.

Let π denote the integral part of $\frac{1}{2}d$, i.e. π is the largest integer equal to or less than $\frac{1}{2}d$. Then an arbitrary irreducible unitary representation of the rotation group $SO(d)$ is labelled by a π-tuple $s = (s_1,\ldots,s_\pi)$ of integers. We denote such a representation of $SO(d)$ by $V^{(s)}(h)$ or $V^{(s_1,\ldots,s_\pi)}(h)$, $h \in SO(d)$, whichever notation we find more convenient. The integers $s_1,\ldots,s_\pi$ are assumed to satisfy the following conditions: (1) $s_i \geq 0$, $i = 1,\ldots, \pi-1$;
(ii) for d = an even integer, s_π may be any positive or negative integer such that the following inequalities hold

$$s_1 \geq s_2 \geq \ldots s_{\pi-1} \geq |s_\pi| \geq 0;$$

(iii) for d = an odd integer, s_π is always a nonnegative integer such that the following inequalities hold

$$s_1 \geq s_2 \geq \ldots s_{\pi-1} \geq s_\pi \geq 0.$$

Thus, we now know how all the irreducible unitary representations of the rotation group $SO(d)$ may be enumerated. Since $SO(d)$ is a compact topological group, all its irreducible unitary representations are finite dimensional. We denote by $\mathcal{H}^{(s)}$ or $\mathcal{H}^{(s_1,\ldots,s_\pi)}$ the finite dimensional Hilbert space which is the representation space for $V^{(s)}(h)$, $h \in SO(d)$, and let $N(s)$ denote the dimension of $\mathcal{H}^{(s)}$.

Consider next the irreducible unitary representations of the full Euclidean group $M(d)$. Here, the representations are labelled by a pair $\underline{\lambda} = (\lambda,s)$ where $\lambda \in [0,\infty)$ and s is a $(\pi-1)$-tuple $s = (s_1,\ldots,s_{\pi-1})$ of integers whose components satisfy conditions analogous to the three given in the preceding paragraph. We denote an irreducible unitary representation of $M(d)$ by $T^{\lambda}(g)$ or $T^{(\lambda,s)}(g)$ or $T^{(\lambda,s_1,\ldots,s_\pi)}(g)$, $g \in M(d)$, whichever is the most convenient notation in a given context. Since $M(d)$ is locally compact, but not compact, the irreducible unitary representations of $M(d)$ are infinite dimensional. We denote by $\mathcal{H}^{\lambda}$ the infinite dimensional Hilbert space which is the representation space for $T^{\lambda}(g)$, $g \in M(d)$.

Let $g \to T^{\lambda}(g)$ be an irreducible unitary representation of $M(d) = E^d \overset{s}{\otimes} SO(d)$. Then the restriction

$$T^{\lambda}(h) = T^{(\lambda,s)}(h),\ h \in SO(d),\ \text{of } T^{(\lambda,s)}(g) \text{ to } SO(d)$$

is an infinite dimensional reducible unitary representation of $SO(d)$. As shown in [89], $T^{(\lambda,s)}(h)$, $h \in SO(d)$, decomposes as follows into a direct sum of irreducible unitary representations of $SO(d)$;

$$(3.1.6) \qquad T^{(\lambda,s)}(h) = \bigoplus_{n=0}^{\infty} V^{(s^n)}(h)^c$$

where $h \to V^{(s')}(h)^c$ is the complex conjugate representation to $V^{(s')}(h)$ and $s^n = (s_1^n,\ldots,s_\pi^n)$. We employ (3.1.6) in the next section.

Finally, for this section, we remark that a description of how to obtain the matrix elements of the irreducible unitary representations of the rotation group $SO(d)$ and the full Euclidean group $M(d)$ may be found in [89]. For $d = 3$, [86] is also an adequate reference in this connection.

The representations $V^{(s)}(h)$ and $T^{\lambda}(g)$ described above are called single-valued irreducible unitary representations of $SO(d)$ and $M(d)$ respectively, and it is these that we employ in our analysis in the sequel.

3.2 SPECTRAL REPRESENTATIONS FOR EUCLIDEAN COVARIANT GENERALIZED STOCHASTIC FIELDS

We employ the group theoretic concepts enunciated in the last section in the services of this section in which we furnish spectral representations for generalized stochastic fields possessing certain transformation properties described below.

We begin with some definitions.

(3.2.1) DEFINITION Let G be a separable locally compact topological group of type one. Then the map γ_g, $g \in G$,

$$\gamma_g : G \to G$$
$$g_1 \to \gamma_g g_1 = gg_1$$

is called the left action of G on itself.

Next, we consider multicomponent stochastic fields indexed by G. To this end, let $\{\xi(g) = (\xi_j(g)) : g \in G\}$ be a second order multicomponent stochastic field indexed by G on a probability space $(\Omega, \mathcal{B}, \mu)$.

(3.2.2) DEFINITION Let $\mathfrak{M}(g)$ and $B(g_1, g_2)$ denote respectively the mean and correlation matrices of the multicomponent stochastic field $\{\xi(g) = (\xi_j(g) : j = 1,\ldots,N) : g \in G\}$. Let $(L^2(\Omega,\mathcal{B},\mu))^N$ denote the N fold Cartesian product of $L^2(\Omega,\mathcal{B},\mu)$ with itself. Let H denote the Hilbert space obtained by completing the set $\{\xi(g) = (\xi_j(g) : j = 1,\ldots,N) : g \in G\}$ of random vectors in $(L^2(\Omega,\mathcal{B},\mu))^N$. Finally, let $g \to U(g)$ be a strongly continuous representation of G on H. Then $\{\xi(g) = (\xi_j(g)) : g \in G\}$ is said to be a left homogeneous stochastic field whose elements $\xi(g)$, $g \in G$, transform covariantly according to the representation $g \to U(g)$ of G if the following conditions are satisfied:

(i) the induced action V_g on H of the left action γ_g of G on itself is specified as follows

$(V_g\xi)(g_1) = U(g)\xi(\gamma_{g^{-1}}g_1) = U(g)\xi(g^{-1}g_1)$, $g, g_1 \in G$

(ii) The matrices $\mathfrak{M}(g)$ and $B(g_1,g_2)$ are such that

(a) $\mathfrak{M}(g_1) = U(g)\mathfrak{M}(g^{-1}g_1)$

(b) $B(g_1,g_2) = U(g)B(g^{-1}g_1,g^{-1}g_2)U^*(g)$ $g_1,g_2,g \in G$, where $U^*(g)$ is the adjoint of $U(g)$.

(3.2.3) REMARK: Let $g \to T^{\lambda}(g)$ be an arbitrary irreducible unitary representation of G on a Hilbert space $\mathcal{H}^{\lambda}$. We denote by G* the topological dual space to G. Thus G* is the set of all equivalence classes of all irreducible inequivalent unitary representations of G. G* need not be a group [49] but it admits a Borel structure so that integration over G* is a well-defined notion [49].

Next, we remark that (ii)(a) of (3.2.2) is readily satisfied if $\mathfrak{M}(g) = U(g)\mathfrak{M}_0$, $g \in G$, where $\mathfrak{M}_0$ is some constant vector in C^N. In view of our subsequent need for it, we are particularly interested in the most general form of $B(g_1,g_2)$ which ensures that $H^0(\Omega) = \{\xi(g) = (\xi_j(g) : j = 1,\ldots,N) : g \in G\}$ is a left homogeneous stochastic field on $(\Omega,\mathcal{B},\mu)$. Fortunately, the solution of this problem is known and it entails providing a spectral representation for the components $\xi_j(g)$ of each $\xi(g)$ in $H^0(\Omega)$. By taking Mackey's Theorem [50] into consideration and by invoking the most general form in which any positive definite function on G may be represented [20], YAGLOM [98] is finally able to establish the following result. We refer to [98] for the lengthy details of the proof of this theorem.

(3.2.4) THEOREM Let G be a separable locally compact topological group of type one and let $g \to T^{\lambda}(g)$ be a strongly continuous irreducible unitary representation of G on a Hilbert space $\mathcal{H}^{\lambda}$. Let $H^0(\Omega) = \{\xi(g) = (\xi_j(g) : j = 1,\ldots,N) : g \in G\}$ be a multicomponent second order stochastic field indexed by G on the probability space $(\Omega,\mathcal{B},\mu)$. Then $H^0(\Omega)$ is a left homogeneous second order stochastic field whose elements $\xi(g)$ transform covariantly according to the representation $U(g)$ of G on H if

(3.2.5) $$\mathfrak{M}(g) = U(g)\mathfrak{M}_0, \text{ where } \mathfrak{M}_0 \text{ is some constant vector in } \mathbb{C}^N$$

and

(3.2.6) $$B(g_1,g_2) = U(g_1) \int_{G^*} \mathrm{Tr}[T^{\underline{\lambda}}(g_2^{-1}g_1)F(d\underline{\lambda})]U(g_2)^* ,$$

with Tr denoting the trace operator; or, equivalently, in a basis,

(3.2.7) $$B_{mn}(g_1,g_2) = \sum_{s,t=1}^{N} \sum_{i,j=1}^{\infty} U_{ms}(g_1)\overline{U}_{nt}(g_2) \int_{G^*} T^{\underline{\lambda}}_{ij}(g_2^{-1}g_1)F_{ji,st}(d\underline{\lambda})$$

where $F(\Lambda) = (F_{ik,mn}(\Lambda))$, Λ = Borel subset of G^*, is a Hermitian nonnegative operator on the tensor product Hilbert space $\mathcal{H}^{\underline{\lambda}} \otimes H$ of $\mathcal{H}^{\underline{\lambda}}$ and H which satisfies

$$F_{\cdot\cdot,mn}(\underline{\lambda}_0) \geq \mathfrak{M}_{0,m}\,\overline{\mathfrak{M}}_{0,n}\,, \quad \mathfrak{M}_0 = (\mathfrak{M}_{0,i} : i = 1,\ldots,N),$$

with $\{T^{\underline{\lambda}_0}(g) : g \in G\}$ being the identity representation of G, i.e. $T^{\underline{\lambda}_0}(g)$ = identity on $\mathcal{H}^{\underline{\lambda}_0}$. Equation (3.2.7) is subject to the constraint that the matrix elements $\{B_{mn}(e,e) : e$ = identity element of $G\}$ must be convergent.

Moreover, each $\xi(g) \in H^0(\Omega)$ admits the following spectral representation

(3.2.8) $$\xi(g) = U(g) \int_{G^*} \mathrm{Tr}[T^{\underline{\lambda}}(g)Z(d\underline{\lambda})]$$

or, equivalently, in a basis,

$$\xi_m(g) = \sum_{n=1}^{N} \sum_{i,j=1}^{\infty} U_{mn}(g) \int_{G^*} T^{\lambda}_{ij}(g) Z_{ji,n}(d\lambda)$$

where $Z(\Lambda)$, Λ = a Borel subset of G^*, is the vector random operator measure over G^* whose components satisfy

$$(3.2.9) \qquad \langle Z_{ij,m}(\Lambda_1), Z_{k\ell,n}(\Lambda_2)\rangle_{L^2(\Omega,\mathcal{B},\mu)} = \delta_{j\ell} F_{ik,mn}(\Lambda_1 \cap \Lambda_2)$$

Conversely, any matrix of the form (3.2.6) is the correlation matrix of some multicomponent left homogeneous stochastic field indexed by G, on a probability space $(\Omega,\mathcal{B},\mu)$, whose mean is capable of assuming any value of the form (3.2.5). Such a left homogeneous multicomponent stochastic field will be of second order if and only if the entries of the matrix (3.2.6) stay convergent for $g_1 = g_2 = e$ = identity element of G.

(3.2.10) REMARK: If $H^O(\Omega) = \{\xi(g) = (\xi_j(g) : j = 1,\ldots,N) : g \in G\}$ is not of second order, because the convergence constraint may fail to be satisfied, then $H^O(\Omega)$ is of second order only in a generalized sense. Namely, we only require in that case that there exists a suitable class $\mathcal{S}(G)$ of test functions [24] on G such that

$$B_{mn}(f^{(1)}, f^{(2)}) = \int dg_1 dg_2 f^{(1)}(g) B_{mn}(g_1,g_2) \overline{f^{(2)}}(g_2)$$

converges for all $m,n = 1,\ldots,N$, and $f^{(i)} \in \mathcal{S}(G)$, $i = 1,2$. We then say that $H^O(\Omega)$ is a second order left homogeneous generalized stochastic field indexed by $\mathcal{S}(G)$ on $(\Omega,\mathcal{B},\mu)$. Indeed, our interest is primarily in generalized stochastic fields.

(3.2.11) DEFINITION Let $H^O(\Omega) = \{\xi(f) = (\xi_j(f) : j = 1,\ldots,N(s)) : f \in \mathcal{S}(R^d)\}$ be a second order multicomponent generalized stochastic field indexed by $\mathcal{S}(R^d)$ on a probability space $(\Omega,\mathcal{B},\mu)$. Let $H^{(s)}$ denote the

Hilbert space obtained by completing the linear space $H^{O}(\Omega)$ in $(L^{2}(\Omega,\mathcal{B},\mu))^{N(s)}$, and let $h \to V^{(s)}(h)$, $s = (s_1,\ldots,s_\pi)$, be an irreducible unitary representation of SO(d) on $H^{(s)}$. Then, we say that $H^{O}(\Omega)$ is a Euclidean covariant generalized stochastic field transforming according to the representation $h \to V^{(s)}(h)$, $h \in SO(d)$, if it satisfies the following conditions:

(i) the induced action τ_g on $H^{O}(\Omega)$ of the transformation g of R^d onto itself is specified as follows $(\tau_g\xi)(f) = V^{(s)}(h)\xi(V_g f)$

(ii) the mean matrix $\mathfrak{M}(f)$ and the correlation matrix $B(f^{(1)},f^{(2)})$ are such that

(a) $\mathfrak{M}(f) = V^{(s)}(h)\mathfrak{M}(V_g f)$

(b) $B(f^{(1)},f^{(2)}) = V^{(s)}(h)B(V_g f^{(1)},V_g f^{(2)})V^{(s)}(h)^{-1}$

where $(V_g f)(x) = f(g^{-1}x)$ and $g = (a,h) \in M(d) = E^d \overset{s}{\otimes} O(d)$

(3.2.12) REMARK: In what follows, we give a spectral representation for an arbitrary Euclidean covariant multicomponent stochastic field which transforms according to an arbitrary irreducible unitary representation $h \to V^{(s)}(h)$, $h \in SO(d)$. In doing so, we invoke (3.2.4). Our proof and analysis may be compared with that presented by YAGLOM in [97]. The spectral representation which we obtain below plays a significant role in our study of Wong's notion of Markov property in the next Chapter.

(3.2.13) THEOREM Let $H^{O}(\Omega) = \{\xi(f) = (\xi_j(f) : j = 1,\ldots,N(s)) : f \in \mathcal{S}(R^d)\}$ be a second order generalized stochastic field indexed by $\mathcal{S}(R^d)$ on $(\Omega,\mathcal{B},\mu)$. Let $h \to V^{(s)}(h)$ be a strongly continuous irreducible unitary representation of SO(d) on the Hilbert space $H^{(s)}$ obtained by completing the linear space $\{\xi(f) = (\xi_j(f) : j = 1,\ldots,N(s)) : f \in \mathcal{S}(R^d)\}$ of random vectors in $(L^{2}(\Omega,\mathcal{B},\mu))^{N(s)}$.

Then $H^{O}(\Omega)$ is a Euclidean covariant generalized stochastic field whose elements transform according to the representation $h \rightarrow V^{(s)}(h)$ if and only if

(i) $\mathfrak{M}(f) = 0$, all $f \in \mathcal{S}(R^{d})$, where 0 is the null column matrix

(ii) the correlation matrix $B(f^{(1)},f^{(2)})$ of each $\xi(f)$ in $H^{O}(\Omega)$ has entries $B_{kj}(f^{(1)},f^{(2)}) = \int dx\, dy\, f^{(1)}(x)B_{kj}(x-y)\overline{f}^{(2)}(y)$ where the kernels $B_{kj}(x-y)$ are given by

$$B_{kj}(x-y) = \int_{M(d)^*} F(d\underline{\lambda})T^{\underline{\lambda}}_{kj}(x-y) \tag{3.2.14}$$

where $F(\cdot)$ is a measure on the Borel subsets of $M(d)^*$ whose restriction to $R_+ = [0,\infty)$ is tempered and, furthermore, $F(0) = 0$. The matrix elements $T^{\underline{\lambda}}_{kj}(g)$ of $T^{\underline{\lambda}}(g)$ are taken in a basis in which $h \rightarrow V^{(s)}(h)$ is real.

The components $\xi_j(f)$ of each $\xi(f) \in H^{O}(\Omega)$ each has a spectral representation given by

$$\xi_j(f) = \int dx\, \xi_j(x)f(x)\,, \qquad \text{where}$$

$$\xi_j(x) = \sum_{m=1}^{\infty} \int_{M(d)^*} Z_m(d\underline{\lambda})T^{\underline{\lambda}}_{mj}(x) \tag{3.2.15}$$

with

$$\langle Z_m(\Lambda),\, Z_{m'}(\Lambda')\rangle_{L^2(\Omega,\mathfrak{B},\mu)} = \delta_{mm'}F(\Lambda \cap \Lambda').$$

Proof: Recall that R^{d} is a homogeneous space whose group of transformations is the full Euclidean group $M(d)$. Hence $GH^{O}(\Omega) = \{\xi(x) = (\xi_j(x) : j = 1,\ldots,N(s)) : x \in R^{d}\}$ may appropriately be considered as a Euclidean covariant multicomponent stochastic field transforming according to the representation $h \to V^{(s)}(h)$, $h \in SO(d)$, which is indexed by $M(d)$ and which is constant over all left cosets of $M(d)$ modulo $SO(d)$. With this realization, Theorem (3.2.4) is available if we make the identification $G \equiv M(d)$. Then by invoking (3.2.8), it follows that the random variable

$$\eta_i(g) = \sum_{j=1}^{N(s)} \xi_j(g)\overline{V}_{ji}^{(s)}(h) , \quad (h,g) \in SO(d) \times M(d)$$

therefore admits the spectral representation

$$(3.2.16) \quad \eta_i(g) = \sum_{m,n=0}^{\infty} \int_{M(d)^*} Z_{nm,i}(d\underline{\lambda})T^{\underline{\lambda}}_{mn}(g)$$

with $\langle Z_{nm,i}(\Lambda), Z_{qr,j}(\Lambda')\rangle_{L^2(\Omega,\mathcal{B},\mu)} = \delta_{mr}F_{nq,ij}(\Lambda \cap \Lambda')$

where Λ, Λ' are Borel subsets of $M(d)^*$, the topological dual of $M(d)$.

Since we require that $\xi_k(g)$ be constant over all left cosets of $M(d)$ modulo $SO(d)$, therefore, we have

$$\xi_k(g) = \xi_k(gh), \text{ all } h \in SO(d).$$

Hence, for all $h \in SO(d)$,

$$\eta_i(gh) = \sum_{j=1}^{N(s)} \xi_j(gh)\overline{V}_{ji}^{(s)}(h)$$

$$= \sum_{j=1}^{N(s)} \xi_j(g)\, \overline{V}_{ji}^{(s)}(h)$$

$$= \eta_i(g)$$

Thus, by (3.2.16), we now have

$$\sum_{m,n=0}^{\infty} \int_{M(d)^*} Z_{nm,i}(d\underline{\lambda}) T^{\underline{\lambda}}_{mn}(gh) = \sum_{m,n=0}^{\infty} \int_{M(d)^*} Z_{nm}(d\underline{\lambda}) T^{\underline{\lambda}}_{mn}(g)$$

or equivalently,

$$(3.2.17) \qquad \sum_{r,m,n=0}^{\infty} \int_{M(d)^*} Z_{nm,i}(d\underline{\lambda}) T^{\underline{\lambda}}_{mr}(g) T^{\underline{\lambda}}_{rn}(h)$$

$$= \sum_{m,n=0}^{\infty} \int_{M(d)^*} Z_{nm,i}(d\underline{\lambda}) T^{\underline{\lambda}}_{mn}(g)$$

Conversely, it follows from (3.2.17) that $\xi_k(g) = \xi_k(gh)$ for all $h \varepsilon SO(d)$. Hence (3.2.17) is both necessary and sufficient in order that $GH^O(\Omega) = \{\xi(x) = (\xi_j(x)) : x \varepsilon R^d)\}$ be a Euclidean covariant stochastic field indexed by R^d, which is isomorphic to $M(d)/SO(d)$.

Let dg be the Haar measure for the Euclidean group $M(d)$. Since the matrix elements $\{T^{\underline{\lambda}}_{mn}(g) : g \varepsilon M(d)\}$ satisfy

$$\int_{M(d)} dg T^{\underline{\lambda}}_{mn}(g) \overline{T}^{\underline{\lambda}'}_{m'n'}(g) = A\delta_{mm'} \delta_{nn'} \delta(\underline{\lambda} - \underline{\lambda}')$$

where A is some normalization constant and $\underline{\lambda} \to \delta(\underline{\lambda})$ is a point measure on $M(d)^*$ which reduces to the Kronecker delta function if $\underline{\lambda}$ = an integer and which is the Dirac δ-function on R_+, therefore multiplication of both sides of (3.2.17) by $\overline{T}^{\underline{\lambda}}_{m'n'}(g)$ followed by integration with respect to the Haar measure dg for $M(d)$, now furnishes

$$\sum_{r,m,n=0}^{\infty} \int_{M(d)^*} Z_{nm,i}(d\underline{\lambda}) T^{\underline{\lambda}}_{rn}(h) \delta_{mm'} \delta_{rn'} = \sum_{m,n=0}^{\infty} \int_{M(d)^*} Z_{nm,i}(d\underline{\lambda}) \delta_{mm'} \delta_{nn'}$$

Thus

$$\sum_{n=0}^{\infty} \int_{M(d)*} Z_{rm';i}(d\underline{\lambda}) T^{\underline{\lambda}}_{n'n}(h) = \int_{M(d)*} Z_{n'm',i}(d\underline{\lambda})$$

or

$$(3.2.18) \qquad \sum_{r=0}^{\infty} \int_{M(d)*} Z_{rm,i}(d\underline{\lambda}) T^{\underline{\lambda}}_{nr}(h) = \int_{M(d)*} Z_{nm,i}(d\underline{\lambda})$$

Next, by (3.1.6), we have

$$(3.1.6) \qquad T^{\underline{\lambda}}(h) = \sum_{\nu=0}^{\infty} V^{(s^{\nu})}(h)$$

where $h \to V^{(s^{\nu})}(h)$, $\nu = 0,1,\ldots,$ are inequivalent irreducible unitary representations of SO(d), and we assume that basis has been judiciously chosen such that the representations $\{V^{(s^{\nu})}(h)\}$ of SO(d) appear in real form. Thus, we now have

$$(3.2.19) \qquad T^{\underline{\lambda}}_{n_\nu m_\mu}(h) = \delta_{\nu\mu} \; V^{(s^{\nu})}_{n_\nu m_\mu}$$

Employing (3.2.19) in (3.2.18), we have

$$\sum_{\mu=1}^{\infty} \sum_{r_\mu=1}^{N(s^{\mu})} \delta_{\nu\mu} \int_{M(d)*} Z_{r_\mu m,i}(d\underline{\lambda}) V^{(s^{\nu})}_{n_\nu r_\nu}(h)$$

$$= \int_{M(d)*} Z_{n_\nu m,i}(d\underline{\lambda})$$

Thus, if $V^{(s^{\mu_0})}(h)$ denotes the identity representation of SO(d) on $H^{(s)}$, we have from the preceding equation that

$$(3.2.20) \qquad \sum_{r_\nu=1}^{N(s^{\nu})} \int_{M(d)*} Z_{r_\nu m,i_{\mu_0}}(d\underline{\lambda}) V^{(s^{\nu})}_{n_\nu r_\nu}(h) = \sum_{\ell_{\mu_0}=1}^{N(s)} V^{(s^{\mu_0})}_{\ell_{\mu_0} i_{\mu_0}}(h) \int_{M(d)*} Z_{n_\nu m,\ell_{\mu_0}}(d\underline{\lambda})$$

Next, multiply both sides of (3.2.20) by $V^{(s^\mu)}_{p_\mu q_\mu}(h)$ and integrate with respect to the Haar measure dh for SO(d). Then, we obtain

$$\sum_{r_\nu=1}^{N(s^\nu)} \delta_{\nu\mu}\ \delta_{n_\nu p_\nu}\ \delta_{r_\nu q_\mu} \int_{M(d)*} Z_{r_\nu m, i_{\mu_0}}(d\underline{\lambda})$$
$$= \sum_{\ell_{\mu_0}=1}^{N(s)} \delta_{\mu\mu_0}\ \delta_{\ell_{\mu_0} p_\mu}\ \delta_{i_{\mu_0} q_\mu} \int_{M(d)*} Z_{n_\nu m, \ell_{\mu_0}}(d\underline{\lambda})$$

Hence

$$\delta_{n_{\mu_0} p_{\mu_0}} \int_{M(d)*} Z_{q_{\mu_0} m, i_{\mu_0}}(d\underline{\lambda}) = \delta_{i_{\mu_0} q_{\mu_0}} \int_{M(d)*} Z_{n_{\mu_0} m, p_{\mu_0}}(d\underline{\lambda})$$

Consequently, we have

$$Z_{q_\nu m, i_\mu}(\Lambda) = \delta_{\nu\mu}\ \delta_{q_\nu i_\mu}\ Z_m(\Lambda), \quad \Lambda \subset M(d)*$$

where $\Lambda \to Z_m(\Lambda)$ is a new stochastic measure satisfying

$$\langle Z_m(\Lambda), Z_{m'}(\Lambda')\rangle_{L^2(\Omega,\mathcal{B},\mu)} = \delta_{mm'}\ F(\Lambda \cap \Lambda')$$

and $F(\cdot)$ is a countably additive measure on the σ-algebra of subsets of $M(d)*$. $Z_m(\Lambda)$ and $F(\Lambda)$ depend only on the representation $h \to V^{(s)}(h)$ of SO(d).

Therefore, since

$$\eta(g) = V^{(s)}(h^{-1})\xi(g) = \int_{M(d)*} \mathrm{Tr}[T^{\underline{\lambda}}(g)Z(d\underline{\lambda})]$$

it follows that the components of $\xi(x)$ are given by

$$\xi_j(x) = \sum_{\nu=0,m=1}^{\infty} \sum_{n_\nu=1}^{N(s^\nu)} \sum_{i_{\mu_0}=1}^{N(s)} \int_{M(d)^*} Z_{n_\nu m, i_{\mu_0}}(d\underline{\lambda}) T^{\lambda}_{mn_\nu}(g) V^{(s)}_{ji_{\mu_0}}(\underline{h})$$

$$= \sum_{\nu=0,m=1}^{\infty} \sum_{n_\nu=1}^{N(s^\nu)} \sum_{i_{\mu_0}=1}^{N(s)} \delta_{\nu\mu_0} \delta_{n_\nu i_{\mu_0}} \int_{M(d)^*} Z_m(d\underline{\lambda}) T^{\lambda}_{mn_\nu}(g) V^{(s)}_{ji_{\mu_0}}(h)$$

i.e. $\xi_j(x) = \sum_{m=1}^{\infty} \sum_{i_{\mu_0}=1}^{N(s)} \int_{M(d)^*} Z_m(d\underline{\lambda}) T^{\lambda}_{mi_{\mu_0}}(g) V^{(s)}_{ji_{\mu_0}}(h)$

Hence, we may finally write

(3.2.21) $$\xi_j(x) = \sum_{m=1}^{\infty} \sum_{i=1}^{N(s)} \int_{M(d)^*} Z_m(d\underline{\lambda}) T^{\lambda}_{mi}(g) V^{(s)}_{ji}(h)$$

Next, we observe that the functions $(g,h) \to T^{\lambda}_{mi}(g) V^{(s)}_{ji}(h)$ on $M(d) \times SO(d)$ truly depend only on $x \in R^d$ and not on $g \in M(d)$. To see this, notice that

$$\sum_{i=1}^{N(s)} T^{\lambda}_{mi}(g) V^{(s)}_{ji}(h)$$

$$= \sum_{i} T^{\lambda}_{mi}((a,e)(0,h)) V^{(s)}_{ji}(h), \quad \text{since } g = (a,e)(0,h)$$

$$= \sum_{i,\ell} T^{\lambda}_{m\ell}((a,e)) T^{\lambda}_{\ell i}(h) V^{(s)}_{ji}(h)$$

$$= \sum_{i,\ell} T^{\lambda}_{m\ell}((a,e)) V^{(s)}_{\ell i}(h) V^{(s)}_{ji}(h)$$

$$= \sum_{\ell} T^{\lambda}_{m\ell}((a,e)) \delta_{\ell j} \qquad \text{since } h \to V^{(s)}(h) \text{ is real}$$

$$= T^{\lambda}_{mj}((a,e))$$

Let us write $T^{\lambda}_{mj}(a)$ for $T^{\lambda}_{mj}((a,e))$, $a \in R^d$. Then (3.2.21) becomes

$$\xi_j(x) = \sum_{m=1}^{\infty} \int_{M(d)^*} Z_m(d\underline{\lambda}) T^{\lambda}_{mj}(x), \qquad x \in R^d. \tag{3.2.22}$$

From this spectral representation for $\xi_j(x)$, it follows that

$$\begin{aligned}
B_{jk}(x-y) &= \langle \xi_j(x), \xi_k(y) \rangle_{L^2(\Omega,\mathcal{B},\mu)} \\
&= \sum_{m,m'=1}^{\infty} \int_{M(d)^*} \langle Z_m(d\underline{\lambda}), Z_{m'}(d\underline{\lambda}) \rangle_{L^2(\Omega,\mathcal{B},\mu)} T^{\lambda}_{m'j}(x) \overline{T}^{\lambda}_{mk}(y) \\
&= \sum_{m,m'=1}^{\infty} \delta_{mm'} \int_{M(d)^*} F(d\underline{\lambda}) T^{\lambda}_{m'j}(x) \overline{T}^{\lambda}_{mk}(y) \\
&= \sum_{m=1}^{\infty} \int_{M(d)^*} F(d\underline{\lambda}) T^{\lambda}_{mj}(x) \overline{T}^{\lambda}_{mk}(y) \\
&= \sum_{m=1}^{\infty} \int_{M(d)^*} F(d\underline{\lambda}) T^{\lambda}_{mj}(x) T^{\lambda}_{km}(-y)
\end{aligned}$$

Thus

$$B_{jk}(x-y) = \int_{M(d)^*} F(d\underline{\lambda}) T^{\lambda}_{kj}(x-y) \tag{3.2.23}$$

Finally, the temperedness of $F(\cdot)$ on $R_+ = [0,\infty)$ follows from Schwartz Nuclear Theorem [17][23] since $(f^{(1)},f^{(2)}) \to B_{jk}(f^{(1)},f^{(2)})$ is a separately continuous bilinear functional on $\mathcal{S}(R^d) \times \mathcal{S}(R^d)$. This completes proof of the theorem. □

(3.2.24) REMARK: It is perhaps worthy of note that (3.2.23) has been obtained without any assumption concerning the probability distribution of elements of the generalized stochastic field $\{\xi(f) = (\xi_j(f) : j = 1,\ldots,N(s) : f \in \mathcal{S}(R^d)\}$ on the probability space $(\Omega,\mathcal{B},\mu)$. However, in the next chapter where we undertake an exhaustive study of Wong's definition of Markov property for multicomponent Euclidean covariant generalized stochastic fields, we are obliged to assume that $\{\xi(f) = (\xi_j(f) : j = 1,\ldots,N(s)) : f \in \mathcal{S}(R^d)\}$ is Gaussian on $(\Omega,\mathcal{B},\mu)$.

(3.2.25) REMARK: It seems instructive to put (3.2.23) to the test and see what it really gives us in the simple case where d = 3. For d = 3, the inequivalent irreducible unitary representations of M(3) are labelled by the pair $\underline{\lambda} = (\lambda,s) \in R_+ \times Z$, where Z is the set of all integers. Thus

$$M(3)^* = R_+ \times Z$$

The spectral measure $F(\cdot)$ on the measurable subsets of $M(3)^* = R_+ \times Z$ is atomic on Z for all fixed measurable $A \in R_+$, i.e. $s \to F(A,\{s\})$ is atomic for all measurable $A \in R_+$. Furthermore, $F(A,\{s\}) = F(A,\{-s\})$, for all measurable $A \in R_+$.

Next, we must present the matrix elements of $T^{\underline{\lambda}}(x-y)$, $\underline{\lambda} = (\lambda,s) \in R_+ \times Z$, in an orthonormal basis in which the irreducible unitary representations $h \to V^{(n)}(h)$, $n = 0,1,\ldots,\infty$, of SO(3) are real. In what follows, we first present the matrix elements of $T^{\underline{\lambda}}(x-y)$ in a canonical complex orthonormal basis and then we arrive at the matrix elements of $T^{\underline{\lambda}}(x-y)$ in the desired basis mentioned in the foregoing sentence by means of a unitary transformation. To this end, we now introduce certain functions which are very familiar in the representation theory of SO(3).

$$\text{Let} \quad (a)_n = 1 \quad \text{for } n = 0$$
$$= a(a+1)(a+2)\ldots(a+n-1), \text{ for } n > 0$$

with $a \in R$.

Then the hypergeometric function $x \to {}_2F_1(a,b;c;x)$ is defined as follows

$$(3.2.26) \qquad {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}$$

We now define the function $\theta \to d^{\ell}_{mn}(\theta)$, $\theta \in [o,\pi]$, with $-\ell \leq m,n \leq \ell$, $\ell \geq 0$, as follows

$$(3.2.27) \qquad d^{\ell}_{mn}(\theta) = (-1)^{n-m} \frac{1}{(m-n)!} \left[\frac{(\ell+m)!(\ell-n)!}{(\ell-m)!(\ell+n)!} \right]^{1/2} \sin^{m-n} \frac{\theta}{2} \cos^{m+n} \frac{\theta}{2} \ x$$

$$x \ {}_2F_1(\ell+m+1,\ m-\ell;\ m-n+1;\ \sin^2 \frac{\theta}{2})$$

for $m \geq n$. For $n > m$, $d^{\ell}_{mn}(\theta) = (-1)^{n-m} d^{\ell}_{nm}(\theta)$ where $d^{\ell}_{nm}(\theta)$ is given by (3.2.27).

Also there are the associated Legendre functions $P^{\ell}_{m}(z)$ defined as follows

$$(3.2.28) \qquad P^{\ell}_{m}(Z) = \frac{(1-Z^2)^{m/2}}{2^{\ell}\ell!} \frac{d^{\ell+m}}{dZ^{\ell+m}} (Z^2-1)^{\ell}, \qquad 0 \leq m \leq \ell$$

$$= \frac{(1-Z^2)^{-m/2}}{2^{\ell}\ell!} \frac{d^{\ell-m}}{dZ^{\ell-m}} (Z^2-1)^{\ell}, \qquad m < 0.$$

Then, the spherical harmonics, which form an orthonormal basis for $L^2(S^2,d\Omega)$, S^2 = unit sphere in R^3 and $d\Omega$ = surface measure on S^2, are given by

$$(3.2.29) \qquad Y_{\ell m}(\theta,\phi) = (4\pi/(2\ +1))^{1/2} (-1)^m \left[\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} P^{\ell}_{m}(\cos\theta) e^{im\phi}$$

$\ell = 0,1,\ldots,\infty$, $-\ell \leq m \leq \ell$. In Section 4.3 we provide certain recurrence formulae satisfied by the functions $Y_{\ell m}$.

In [86], the matrix elements of $T^{\underline{\lambda}}(x-y)$, $\underline{\lambda} = (\lambda,s) \in R_+ \times Z$, are expressed in the orthonormal basis $\{\Xi^s_{\ell m}(\theta,\phi) : \ell = 0,1,\ldots,\infty, -\ell \leq m \leq \ell\}$ given by

$$(3.2.30) \qquad \Xi^s_{\ell m}(\theta,\phi) = ((2\ell+1)/4\pi)^{1/2} i^{m-s} d^{\ell}_{ms}(\theta) e^{is\phi}$$

In this basis, the matrix elements $t^{\underline{\lambda}}_{mn}(x-y)$ of $T^{\underline{\lambda}}(x-y)$ are given by [86].

$$(3.2.31) \quad t^{\underline{\lambda}}_{mn}(x-y) = 3 \sum_{k=0}^{\infty} (2k+1) i^k (-1)^{m+s} \begin{pmatrix} 1 & 1 & k \\ s & -s & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & k \\ m & -n & n-m \end{pmatrix} \frac{J_{\frac{1}{2}+k}(\lambda|x-y|)}{(\lambda|x-y|)} \cdot Y_{k,m-n}((x-y)')$$

where $(x-y)' = \frac{x-y}{|x-y|}$, $|x-y| \neq 0$, and $\begin{pmatrix} j & k & \ell \\ m & n & p \end{pmatrix}$ are the so-called three-j functions which we define in Section 4.4, equation (4.4.1).

The matrix elements $T^{\underline{\lambda}}_{jk}(x-y)$, $(\lambda,s) \in R_+ \times Z$, are taken in a basis in which the irreducible unitary representations of SO(3) are real; hence they are related to $t^{\underline{\lambda}}_{mn}(x-y)$, $\underline{\lambda} = (\lambda,s)$, given in (3.2.31) by a unitary transformation U as follows

$$(3.2.32) \qquad T^{\underline{\lambda}}_{jk}(x-y) = \sum_{m,n=-1}^{+1} U_{mj} \overline{U}_{nk} t^{\underline{\lambda}}_{mn}(x-y)$$

Employing (3.2.32) in (3.2.23), we see that we must first compute

$$b_{mn}(x-y) = \int_{R_+ \times Z} F(d\lambda, \{s\}) \, t^{(\lambda,s)}_{mn}(x-y)$$

Now, $b_{mn}(x-y)$ is given by

$$b_{mn}(x-y) = 3\sum_{s=-1}^{1}\sum_{k=0}^{\infty}(2k+1)i^k(-1)^{m+s}\begin{pmatrix}1 & 1 & k\\ s & -s & 0\end{pmatrix}\begin{pmatrix}1 & 1 & k\\ m & -n & n-m\end{pmatrix}\int_0^\infty F(d\lambda,\{s\})\ .$$

$$\cdot\ \frac{J_{\frac{1}{2}+k}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}}}\ Y_{k,m-n}((x-y)')$$

Set

$$\sum_{s=-1}^{1}(-1)^s\begin{pmatrix}1 & 1 & k\\ s & -s & 0\end{pmatrix}F(d\lambda,\{s\}) = F^{(k)}(d\lambda)$$

Then, we have

$$F^{(k)}(d\lambda) = \begin{pmatrix}1 & 1 & k\\ 0 & 0 & 0\end{pmatrix}F(d\lambda,\{0\}) - \{(\begin{pmatrix}1 & 1 & k\\ -1 & 1 & 0\end{pmatrix} + \begin{pmatrix}1 & 1 & k\\ 1 & -1 & 0\end{pmatrix}\}F(d\lambda,\{1\})$$

Hence

$$b_{mn}(x-y) = 3\sum_{k=0}^{\infty}(2k+1)i^k(-1)^m\begin{pmatrix}1 & 1 & k\\ m & -n & n-m\end{pmatrix}Y_{k,m-n}((x-y)')\int_0^\infty F^{(k)}(d\lambda)\frac{J_{\frac{1}{2}+k}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}}}$$

$$= 3(-1)^m\begin{pmatrix}1 & 1 & 0\\ m & -n & n-m\end{pmatrix}\ \delta_{mn}Y_{00}((x-y)')\int_0^\infty F^{(0)}(d\lambda)\ \frac{J_{\frac{1}{2}}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}}} +$$

$$+ 15(-1)^{1+m}\begin{pmatrix}1 & 1 & 2\\ m & -n & n-m\end{pmatrix}\ Y_{2,m-n}((x-y)')\int_0^\infty F^{(2)}(d\lambda)\ \frac{J_{\frac{1}{2}+2}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}}}$$

Next, we invoke (4.4.1) to obtain

$$F^{(0)}(d\lambda) = ((-1)/\sqrt{3})(F(d\lambda,\{0\}) + 2F(d\lambda,\{1\}))$$

$$F^{(2)}(d\lambda) = (4/\sqrt{5!})(F(d\lambda,\{0\}) - F(d\lambda,\{1\}))$$

Thus

$$b_{mn}(x-y) = \sqrt{3}(-1)^{1+m}\begin{pmatrix}1 & 1 & 0\\ m-n & & n-m\end{pmatrix}\delta_{mn}Y_{00}((x-y)')\int_0^\infty (F(d\lambda,\{0\}) + 2F(d\lambda,\{1\}))\cdot \cdot \frac{J_{\frac{1}{2}}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}}}$$

$$+ \sqrt{30}(-1)^{1+m}\begin{pmatrix}1 & 1 & 2\\ m & -n & n-m\end{pmatrix}Y_{2,m-n}((x-y)')\int_0^\infty (F(d\lambda,\{0\})-F(d\lambda,\{1\}))\frac{J_{\frac{1}{2}+2}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}}}$$

Set

$$F(d\lambda,\{0\}) + 2F(d\lambda,\{1\}) = d\Phi^{(1)}(\lambda)$$

$$2F(d\lambda,\{0\}) + F(d\lambda,\{1\}) = d\Phi^{(2)}(\lambda)$$

Then, $F(d\lambda,\{0\}) - F(d\lambda,\{1\}) = d(\Phi^{(2)} - \Phi^{(1)})(\lambda)$

Next, we note that

$$\begin{pmatrix}1 & 1 & 2\\ m & -n & n-m\end{pmatrix} = (-1)^{m-n}\frac{1}{\sqrt{30}}\left(\frac{(2+n-m)!\,(2+m-n)!}{(1-n)!\,(1-m)!\,(1+n)!\,(1+m)!}\right)^{1/2}$$

Hence, we have that

$$b_{mn}(x-y) = \sqrt{3}(-1)^{1+m}\begin{pmatrix}1 & 1 & 0\\ m & -n & n-m\end{pmatrix}\delta_{mn}Y_{00}((x-y)')\int_0^\infty d\Phi^{(1)}(\lambda)\,\frac{J_{\frac{1}{2}}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}}} +$$

$$+(-1)^{1-n}\left(\frac{(2+n-m)!\,(2+m-n)!}{(1-n)!\,(1-m)!\,(1+n)!\,(1+m)!}\right)^{1/2}Y_{2,m-n}((x-y)')\int_0^\infty d(\Phi^{(2)}-\Phi^{(1)})(\lambda)\,\cdot$$

$$\cdot\ \frac{J_{\frac{1}{2}+2}(\lambda|x-y|)}{(\lambda|x-y|)}$$

From the preceding, it now follows that

$$(3.2.33)\quad B_{jk}(x-y) = \sum_{m,n,-1}^{+1} U_{mj}\,\overline{U}_{nk} b_{mn}(x-y)$$

$$= \sum_{m,n=-1}^{+1} \sqrt{3}(-1)^{1+m}\begin{pmatrix}1 & 1 & 0\\ m & -n & m-n\end{pmatrix} U_{mj}\overline{U}_{nk}\delta_{mn}\, Y_{00}((x-y)')\int_0^\infty d\Phi^{(1)}(\lambda)\,\frac{J_{\frac{1}{2}}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}}}$$

$$+ \sum_{m,n=-1}^{+1} (-1)^{1-n}\left(\frac{(2+n-m)!\,(2+m-n)!}{(1-n)!\,(1-m)!\,(1+n)!\,(1+m)!}\right)^{1/2} U_{mj}\overline{U}_{nk} Y_{2,m-n}((x-y)')\int_0^\infty d(\Phi^{(2)}-\Phi^{(1)}).$$

$$\cdot\ \frac{J_{\frac{1}{2}+2}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}}}$$

Since $\begin{pmatrix}1 & 1 & 0\\ m & -m & 0\end{pmatrix} = \frac{1}{\sqrt{3}}(-1)^{1-m}$, we have

$$(3.2.34)\quad \sum_{m,n=-1}^{+1} \sqrt{3}\,(-1)^{m+1}\begin{pmatrix}1 & 1 & 0\\ m & -n & m-n\end{pmatrix} U_{mj}\overline{U}_{nk}\delta_{mn}$$

$$= \sum_{m=-1}^{+1} U_{mj}\overline{U}_{mk} = \delta_{jk}$$

Also, we may choose the unitary transformation U such that

$$(3.2.35)\quad \sum_{m,n=-1}^{+1} (-1)^{1-n}\left(\frac{(2+n-m)!\,(2+m-n)!}{(1-n)!\,(1-m)!\,(1+n)!\,(1+m)!}\right)^{1/2} U_{mj}\overline{U}_{nk} Y_{2,m-n}(x')$$

$$= \frac{3}{\sqrt{4\pi}}\,\frac{x_j x_k}{|x|^2} - \frac{1}{\sqrt{4\pi}}\,\delta_{ij}$$

The initarity of U now implies that the above statement is equivalent to the following

$$\sum_{j,k=-1}^{+1} \frac{3}{\sqrt{4\pi}} \bar{U}_{pj} U_{qk} \frac{x_j x_k}{|x|^2} - \frac{1}{\sqrt{4\pi}} \delta_{pq}$$

$$= \sum_{m,n=-1}^{+1} (-1)^{1-n} \left(\frac{(2+n-m)!\,(2+m-n)!}{(1-n)!\,(1-m)!\,(1+n)!\,(1+m)!}\right)^{1/2} \sum_{j,k=-1}^{+1} U_{mj}\bar{U}_{pj}U_{qk}\bar{U}_{nk}Y_{2,m-n}(x')$$

$$= (-1)^{1-q} \left(\frac{(2+q-p)!\,(2+p-q)!}{(1-q)!\,(1-p)!\,(1+q)!\,(1+p)!}\right)^{1/2} Y_{2,p-q}(x')$$

Consequently, by employing (3.2.34) and (3.2.35) in (3.2.33) and recalling that $Y_{00}(x') = \frac{1}{\sqrt{4\pi}}$, we have that $B_{jk}(x-y)$ may be written as

$$(3.2.36)\quad B_{jk}(x-y) = \frac{1}{\sqrt{4\pi}} \delta_{jk} \int_0^\infty d\Phi^{(1)}(\lambda) \frac{J_{\frac{1}{2}}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}}} +$$

$$+ \int_0^\infty d(\Phi^{(2)}-\Phi^{(1)})(\lambda) \frac{J_{\frac{1}{2}+2}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}}} \left(\frac{3}{\sqrt{4\pi}} \frac{(x_j-y_j)(x_k-y_k)}{|x-y|^2} - \frac{1}{\sqrt{4\pi}} \delta_{jk}\right)$$

Now, the second term on the right hand side of (3.2.36) may be rewritten in another form if we invoke the following recurrence relation between Bessel functions:

$$J_{\nu+1}(Z) = 2\nu Z^{-1} J_\nu(Z) - J_{\nu-1}(Z)$$

Put $\nu = 3/2$ in the above relation. Then, we have

$$J_{\frac{1}{2}+2}(Z) = 3Z^{-1}J_{\frac{1}{2}+1}(Z) - J_{\frac{1}{2}}(Z)$$

Thus, the second term on the right hand side of (3.2.36) becomes

$$\int_0^\infty d(\Phi^{(2)}-\Phi^{(1)})(\lambda)\ \frac{J_{\frac{1}{2}+2}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}}}\ \left(\frac{3}{\sqrt{4\pi}}\ \frac{(x_j-y_j)(x_k-y_k)}{|x-y|^2} - \frac{1}{\sqrt{4\pi}}\ \delta_{jk}\right)$$

$$= \frac{3}{\sqrt{4\pi}} \int_0^\infty d(\Phi^{(2)}-\Phi^{(1)})(\lambda)\ \frac{J_{\frac{1}{2}+2}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}}}\ \frac{(x_j-y_j)(x_k-y_k)}{|x-y|^2} -$$

$$- \frac{1}{\sqrt{4\pi}}\ \delta_{jk} \int_0^\infty d(\Phi^{(2)}-\Phi^{(1)})(\lambda)\ \frac{J_{\frac{1}{2}+2}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}}}$$

$$= \frac{3}{\sqrt{4\pi}} \int_0^\infty d(\Phi^{(2)}-\Phi^{(1)})(\lambda)\ \frac{J_{\frac{1}{2}+2}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}}}\ \frac{(x_j-y_j)(x_k-y_k)}{|x-y|^2} -$$

$$- \frac{1}{\sqrt{4\pi}}\ \delta_{jk} \int_0^\infty d(\Phi^{(2)}-\Phi^{(1)})(\lambda)\ \left(3\ \frac{J_{\frac{1}{2}+1}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}+1}} - \frac{J_{\frac{1}{2}}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}}}\right)$$

$$= \frac{3}{\sqrt{4\pi}} \int_0^\infty d(\Phi^{(2)}-\Phi^{(1)})(\lambda)\ \left(\frac{J_{\frac{1}{2}+2}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}}} \cdot \frac{(x_j-y_j)(x_k-y_k)}{|x-y|^2} - \delta_{jk}\ \frac{J_{\frac{1}{2}+1}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}+1}}\right) +$$

$$+ \frac{1}{\sqrt{4\pi}}\ \delta_{jk} \int_0^\infty d(\Phi^{(2)}-\Phi^{(1)})(\lambda)\ \frac{J_{\frac{1}{2}}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}}}$$

Hence (3.2.36) now becomes

$$B_{jk}(x-y) = \frac{1}{\sqrt{4\pi}}\ \delta_{jk} \int_0^\infty d\Phi^{(1)}(\lambda)\ \frac{J_{\frac{1}{2}}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}}} +$$

$$+ \frac{3}{\sqrt{4\pi}} \int_0^\infty d(\Phi^{(2)}-\Phi^{(1)})(\lambda)\ \left(\frac{J_{\frac{1}{2}+2}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}}} \cdot \frac{(x_j-y_j)(x_k-y_k)}{|x-y|^2} - \delta_{jk}\ \frac{J_{\frac{1}{2}+1}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}+1}}\right.$$

$$+ \frac{1}{\sqrt{4\pi}}\ \delta_{jk} \int_0^\infty d(\Phi^{(2)}-\Phi^{(1)})(\lambda)\ \frac{J_{\frac{1}{2}}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}}}$$

and we have finally

$$(3.2.37)\quad B_{jk}(x-y) = \frac{1}{\sqrt{4\pi}} \int_0^\infty d\Phi^{(2)}(\lambda) \frac{J_{\frac{1}{2}}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}}} +$$

$$+ \frac{3}{\sqrt{4\pi}} \int_0^\infty d(\Phi^{(2)}-\Phi^{(1)})(\lambda) \left(\frac{J_{\frac{1}{2}+2}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}}} \cdot \frac{(x_j-y_j)(x_k-y_k)}{|x-y|^2} - \delta_{jk} \frac{J_{\frac{1}{2}+1}(\lambda|x-y|)}{(\lambda|x-y|)^{\frac{1}{2}+1}}\right)$$

The expression (3.2.37) for $B_{jk}(x-y)$ has been obtained also by YAGLOM [97] by a different method. In the indicated reference Yaglom considers a vector generalized stochastic field with an arbitrary number of components.

In the next chapter, we make use of (3.2.37) when we discuss Wong's notion of Markov property for the vector Gaussian generalized stochastic field

$$H^0(\Omega) = \{\xi(f) = (\xi_1(f), \xi_2(f), \xi_3(f)) : f \in \mathcal{S}(R^3)\} .$$

CHAPTER 4

MARKOV PROPERTY

This chapter deals with the complete characterization of the class of all Euclidean covariant multicomponent Gaussian generalized stochastic fields which are Markov in the sense of Wong [94]. The analysis involved in the accomplishment of this task is far more subtle than the one undertaken by Wong [94] in the case of scalar Euclidean invariant Gaussian generalized stochastic fields, and an intimate blend of Functional Analysis, probability theory, group theory and the theory of boundary value problems for elliptic systems of linear partial differential equations is needed in the final solution of the problem.

In order to simplify computations, the random variables and function spaces employed here are all assumed complex-valued. All sesquilinear forms are linear in the first argument and antilinear in the second.

4.1 WONG'S NOTION OF MARKOV PROPERTY

Let $(\Omega,\mathcal{B},\mu)$ be a probability space and let $H^{O}(\Omega) = \{\xi(f) = (\xi_j(f) : j = 1,\ldots,N) : f \in \mathcal{S}(R^d)\}$ be an arbitrary Euclidean covariant generalized stochastic field on $(\Omega,\mathcal{B},\mu)$. In Chapter 3, we obtained a spectral representation for each $\xi(f) \in H^{O}(\Omega)$. In doing so, we made no assumptions concerning the probability distributions of members of $H^{O}(\Omega)$. In this chapter, however, we always assume that $H^{O}(\Omega)$ is indeed Gaussian - i.e. every finite collection of members of $H^{O}(\Omega)$ has Gaussian probability distribution law - as well as being Euclidean covariant. With this assumption, Wong's definition of Markov property for $H^{O}(\Omega)$ may then be formulated and studied. Let us proceed to do this.

By the phrase "boundary data for $H^O(\Omega)$", we mean the set of random variables, constructed from members of $H^O(\Omega)$, which are associated with the boundary ∂D of a given open set D belonging to R^d. In any formulation of Markov property, the mode of prescribing boundary data is of intrinsic importance because it partly determines the class of random variables which satisfy the intended notion of Markov property. In what follows, we give Wong's prescription of boundary data for $H^O(\Omega) = \{\xi(f) = (\xi_j(f) : j = 1,\ldots,N)\}$ and then we introduce his notion of Markov property [94].

(4.1.1) WONG'S ASSUMPTION Let D denote an arbitrary open subset of R^d with boundary ∂D. Let the surface measure for ∂D be $d\sigma$. Then we say that $H^O(\Omega) = \{\xi(f) = (\xi_j(f) : j = 1,\ldots,N) : f \in \mathcal{S}(R^d)\}$ satisfies Wong's assumption [94] if $\xi(\overset{*}{f}) = (\xi_j(\overset{*}{f}) : j = 1,\ldots,N)$ belongs to $(L^2(\Omega,\mathcal{B},\mu))^N$ for all $\overset{*}{f}$ in $L^2(\partial D,d\sigma)$.

In what follows, we always suppose that

$$H^O(\Omega) = \{\xi(f) = (\xi_j(f) : j = 1,\ldots,N) : f \in \mathcal{S}(R^d)\}$$

satisfies Wong's Assumption. Then we can prescribe boundary data for $H^O(\Omega)$ in the spirit of Wong [94].

Let $H(\partial D)$ be the completion of the linear space $\{\xi(\overset{*}{f}) = (\xi_j(\overset{*}{f}) : j = 1,\ldots,N) : \overset{*}{f} \in L^2(\partial D,d\sigma)\}$ of random vectors in $(L^2(\Omega,\mathcal{B},\mu))^N$. Then we take $H(\partial D)$ to be the boundary data for $H^O(\Omega)$ on ∂D.

Next, we introduce Wong's definition of Markov property.

(4.1.2) WONG'S NOTION OF MARKOV PROPERTY Let $H(R^d)$ be the completion of $H^O(\Omega) = \{\xi(f) = (\xi_j(f) : j = 1,\ldots,N) : f \in \mathcal{S}(R^d)\}$ in $(L^2(\Omega,\mathcal{B},\mu))^N$. $H^O(\Omega)$ is said to be Markov in the sense of Wong if given any increasing sequence ∂D_1, ∂D, ∂D_2 of nested boundaries in R^{d-1}, then

(4.13) $\quad (I - P_{H(\partial D)})\, H(\partial D_2) = H(\partial D_2) - P_{H(\partial D)}\, H(\partial D_2)$

is orthogonal to $H(\partial D_1)$,

where $P_{H(\partial D)}$ is the projection of $H(\partial D_2)$ onto $H(\partial D)$ and I is the identity operator on $H(\partial D_2)$.

(4.1.4) REMARK: Of course, in view of the fact that $H^O(\Omega)$ is Gaussian, the orthogonality condition (4.1.3) is equivalent to stochastic independence of $H(\partial D_2) - P_{H(\partial D)}\ H(\partial D_2)$ and $H(\partial D_1)$. Notice too that since $H^O(\Omega)$ satisfies Wong's Assumption, we have that $H(\partial O)$ is a subspace of $H(R^d)$, where O is an arbitrary open subset of R^d.

The next section marks the beginning of our study of Wong's notion of Markov property for a Gaussian Euclidean covariant generalized stochastic field. Our aim is to provide, if possible, necessary and sufficient conditions in order that such a generalized stochastic field be Markov in the sense of Wong.

In the next section, we formulate theorems for a d dimensional Gaussian generalized stochastic field, and thereafter analyse the particular case for which d = 3. In Chapter 6, we show that the analysis which we eventually carry through in this chapter for a three dimensional Gaussian generalized stochastic field may be extended to any Gaussian Euclidean covariant generalized stochastic field. The tools we apply in this chapter are again what are called for, except that the calculations are admittedly more tedious.

4.2. THE d-DIMENSIONAL VECTOR GENERALIZED STOCHASTIC FIELD

Let $(\Omega, \mathcal{B}, \mu)$ be a probability space, and let $H^O(\Omega) = \{\xi(f) = (\xi_j(f) : j = 1,\ldots,d) : f \in \mathcal{S}(R^d)\}$ be a vector generalized stochastic field on $(\Omega, \mathcal{B}, \mu)$. Then, if we supply the relevant matrix elements in equation (3.2.14) and carry through the sort of analysis undertaken by us in (3.2.25), we obtain the following result (see also YAGLOM [97]):

(4.2.1) THEOREM Let $H^0(\Omega)$ be a Euclidean covariant vector generalized stochastic field. Then the mean of each $\xi(f)$ in $H^0(\Omega)$ is always identically the null column vector, and the matrix $B(f^{(2)},f^{(1)})$ of correlation functionals has entries

$$B_{ij}(f^{(2)},f^{(1)}) = \int dxdy\; f^{(2)}(x)B_{ij}(x-y)\bar{f}^{(1)}(y)$$

where

(4.2.2) $$B_{ij}(x-y)$$

$$= b_\nu \int_0^\infty d\Phi_1(\lambda)\left\{ \frac{J_{\nu+1}(\lambda|x-y|)}{(\lambda|x-y|)^{\nu+1}}\delta_{ij} - \lambda^2 \frac{J_{\nu+2}(\lambda|x-y|)}{(\lambda|x-y|)^{\nu+2}}(x_i-y_i)(x_j-y_j)\right\} +$$

$$+ b_\nu \int_0^\infty d\Phi_2(\lambda)\left(\frac{J_{\nu+1}(\lambda|x-y|)}{(\lambda|x-y|)^{\nu}} - \frac{J_{\nu+1}(\lambda|x-y|)}{(\lambda|x-y|)^{\nu+1}}\right)\delta_{ij} +$$

$$+ b_\nu \int_0^\infty d\Phi_2(\lambda)\lambda^2 \frac{J_{\nu+2}(\lambda|x-y|)}{(\lambda|x-y|)^{\nu+2}}(x_i-y_i)(x_j-y_j)$$

$$(b_\nu = 2^\nu\Gamma(1+\nu), \quad \nu = \tfrac{1}{2}(d-2))$$

where $\Phi_i(\cdot)$, $i = 1,2$ are two real nondecreasing functions on $[0,\infty)$, continuous from the left, satisfying

$$\Phi_1(0) = \Phi_2(0) = 0, \quad \Phi_1(+0) = \Phi_2(+0)$$

and temperedness: $\int_0^\infty d\Phi_i(\lambda)(1+\lambda^2)^{-p} < \infty$, $i = 1,2$ for some nonnegative p.

(4.2.3) REMARK: For our subsequent needs, it is convenient to recast (4.2.2) in another equivalent form. To this end, first note that the kernel $(x,y) \to B_{ij}(x-y)$ may be written as follows:

$$B_{ij}(x-y) = b_\nu \int_0^\infty d\Phi_2(\lambda) \frac{J_\nu(\lambda|x-y|)}{(\lambda|x-y|)^\nu} \delta_{ij} + b_\nu \frac{\partial^2}{\partial x_i \partial x_j} \int_0^\infty d\Phi_2(\lambda) \frac{1}{\lambda^2} \frac{J_\nu(\lambda|x-y|)}{(\lambda|x-y|)^\nu} -$$

$$- b_\nu \frac{\partial^2}{\partial x_i \partial x_j} \int_0^\infty d\Phi_1(\lambda) \frac{1}{\lambda^2} \frac{J_\nu(\lambda|x-y|)}{(\lambda|x-y|)^\nu}$$

Thus

$$(4.2.4) \quad B_{ij}(x-y) = b_\nu \int_0^\infty d\Phi_2(\lambda) \frac{J_\nu(\lambda|x-y|)}{(\lambda|x-y|)^\nu} \delta_{ij} +$$

$$+ b_\nu \frac{\partial^2}{\partial x_i \partial x_j} \int_0^\infty d(\Phi_2 - \Phi_1)(\lambda) \frac{1}{\lambda^2} \frac{J_\nu(\lambda|x-y|)}{(\lambda|x-y|)^\nu}$$

Next, let $C_n^\nu(\theta)$, $\theta \in [-1,1]$, be Gegenbauer's polynomial which, we recall, is defined by means of the generating function

$$(4.2.5) \quad (1 - 2\theta Z + Z^2)^{-\nu} = \sum_{n=0}^\infty C_n^\nu(\theta) Z^n, \qquad |z| < 1.$$

Then, there is Gegenbauer's degenerate addition theorem [93] for the Bessel function

$$(4.2.6) \quad e^{ip.x} = 2^\nu \Gamma(\nu) \sum_{\nu=0}^\infty (\nu+n) i^n \frac{J_{\nu+n}(|p||x|)}{(|p||x|)^\nu} C_n^\nu(p'.x')$$

with $p' = \frac{p}{|p|}$, $|p| > 0$ and $\nu = \frac{1}{2}(d-2)$.

From (4.2.6) we now have

$$\int d\Omega\, e^{ip.x} = 2^{\nu}\Gamma(\nu)\nu \frac{J_\nu(|p||x|)}{(|p||x|)^\nu}$$

$$= 2^{\nu}\Gamma(1+\nu) \frac{J_\nu(|p||x|)}{(|p||x|)^\nu}$$

Thus

$$(4.2.7) \quad \int d\Omega e^{ip.x} = b_\nu \frac{J_\nu(|p||x|)}{(|p||x|)^\nu}$$

where $d\Omega$ is the surface measure for S^{d-1}, the unit sphere in R^d.

Employ (4.2.7) in (4.2.4), setting $|p| = \lambda$, $p \in R^d$. Then, we have

$$B_{ij}(x-y) = \int_{R^d} d\Phi_2(|p|)d\Omega e^{ip.(x-y)}\delta_{ij} + \frac{\partial^2}{\partial x_i \partial x_j}\int_{R^d} d(\Phi_2-\Phi_1)(|p|)d\Omega \frac{1}{|p|^2} e^{ip.(x-y)}$$

$$= \int_{R^d} d\Phi_2(|p|)d\Omega e^{ip.(x-y)}\delta_{ij} + \int_{R^d} d(\Phi_2-\Phi_1)(|p|)d\Omega \frac{p_i p_j}{|p|^2} e^{ip.(x-y)}$$

If we now set

$d\Phi_2 = d\Phi^{(1)}$, $d(\Phi_2 - \Phi_1) = d\Phi^{(2)}$ and $d\Phi_2(|p|)d\Omega = \Phi^{(1)}(dp)$,
$d(\Phi_2 - \Phi_1)(|p|)d\Omega = \Phi^{(2)}(dp)$

then we obtain

$$(4.2.8) \quad B_{ij}(x-y) = \int_{R^d} \Phi^{(1)}(dp)e^{ip.(x-y)}\delta_{ij} + \int_{R^d} \Phi^{(2)}(dp) \frac{p_i p_j}{|p|^2} e^{ip.(x-y)}$$

We are now in a position to present the following result.

(4.2.9) THEOREM Let $H^0(\Omega) = \{\xi(f) = (\xi_j(f) : j = 1,\dots,d) : f \in \mathcal{S}(R^d)\}$ be a Euclidean covariant Gaussian vector generalized stochastic field on a probability space $(\Omega,\mathcal{B},\mu)$. Then the mean of each $\xi(f) \in H^O(\Omega)$ is always the null column vector in R^d, and the matrix of correlation functionals of each $\xi(f) \in H^O(\Omega)$ has entries given by

$$(4.2.10) \quad B_{ij}(f^{(2)},f^{(1)}) = \sum_{n=0}^{\infty} \sum_{\ell=1}^{d_n} |A_n|^2 \delta_{ij} \int_0^{\infty} d\Phi^{(1)}(\lambda) f_{n\ell}^{(2)}(\lambda) \overline{f}_{n\ell}^{(1)}(\lambda) +$$

$$+ \sum_{n,n'=0}^{\infty} \sum_{\ell=1}^{d_n} \sum_{\ell'=1}^{d_{n'}} A_n \overline{A}_{n'} b_{ijn\ell n'\ell'} \int_0^{\infty} d\Phi^{(2)}(\lambda) f_{n'\ell'}^{(2)}(\lambda) \overline{f}_{n\ell}^{(1)}(\lambda)$$

where

$$f_{n\ell}(\lambda) = \int dr\, r^{1+2\nu} d\Omega\, \overline{Y}_{n\ell}(x') f(r,x')$$

$$b_{ijn\ell n'\ell'} = \int d\Omega \frac{P_i P_j}{|p|^2} Y_{n\ell}(p')\overline{Y}_{n'\ell'}(p')$$

$$x' = \frac{x}{|x|}, \quad |x| = r > 0, \qquad \nu = \frac{1}{2}(d-2)$$

and the constants $\{A_n : n = 0,1,\ldots,\infty\}$ are given by

$$A_n = 2^{\nu}\Gamma(\nu)(\nu+n)i^n \frac{d_n}{\Omega_{d-1}}$$

with Ω_{d-1} = area of the unit sphere S^{d-1} in R^d.

Proof: We know [59][83] that $L^2(S^{d-1},d\Omega)$ is an orthogonal direct sum $L^2(S^{d-1},d\Omega) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}$ of finite dimensional Hilbert spaces $\mathcal{H}^{(n)}$, $n = 0,1,\ldots,\infty$ where $\mathcal{H}^{(n)}$ is the space of surface spherical harmonics of degree n. We denote the dimension of $\mathcal{H}^{(n)}$ by d_n. The spherical harmonics $\{Y_{n\ell} : n = 0,1,\ldots,\infty, \ \ell = 1,\ldots,d_n\}$ form a basis for $L^2(S^{d-1},d\Omega)$.

Next, we have that [83]

$$(4.2.11) \qquad C_n^{\nu}(p'.x') = \frac{d_n}{\Omega_{d-1}} \sum_{\ell=1}^{d_n} Y_{n\ell}(p')\overline{Y}_{n\ell}(x')$$

where Ω_{d-1} is the area of the unit sphere in R^d. Hence, combining (4.2.11) with (4.2.6), we may now write

$$(4.2.12) \quad e^{ip.x} = \sum_{n=0}^{\infty} \sum_{\ell=1}^{d_n} A_n \frac{J_{\nu+n}(|p||x|)}{(|p||x|)^{\nu}} Y_{n\ell}(p')\overline{Y}_{n\ell}(x')$$

where $A_n = 2^{\nu}\Gamma(\nu) \; (\nu + n)i^n \dfrac{d_n}{\Omega_{d-1}}$.

Finally, if we now employ (4.2.12) in (4.2.8), we obtain (4.2.10). This concludes the proof. □

(4.2.13) REMARK: In the preceding section, we introduced Wong's notion of Markov property and we indicated how boundary data are specified for $H^o(\Omega)$. In what follows we describe the boundary data for $H^o(\Omega)$ associated with the boundary of a ball in R^d.

Define $f^{(1)}_{n\ell}$ and $f^{(2)}_{n\ell}$ as in Theorem (4.2.9). Let $\overset{*}{f}^{(1)}_{n\ell}$ and $\overset{*}{f}^{(2)}_{n'\ell'}$ denote the functions obtained by taking $f^{(1)}$ and $f^{(2)}$ to be respectively

$$f^{(1)}(|x|,x') = \overset{*}{f}^{(1)}(x') \otimes \delta(r_1 - |x|) = \overset{*}{f}^{(1)}(x') \otimes \delta_{r_1}(|x|)$$

$$f^{(2)}(|y|,y') = \overset{*}{f}^{(2)}(y') \otimes \delta(r_2 - |y|) = \overset{*}{f}^{(2)}(y') \otimes \delta_{r_2}(|y|)$$

where $\overset{*}{f}^{(i)} \in L^2(S^{d-1},d\Omega)$, $i = 1,2$ and $\delta_r(\cdot) = \delta(r - \cdot)$, in the definition of $f^{(1)}_{n\ell}$ and $f^{(2)}_{n'\ell'}$.

Then, we have

$$(4.2.14) \quad \overset{*}{f}^{(1)}_{n\ell}(\lambda) = \int d\Omega \; \overline{Y}_{n\ell}(x')\overset{*}{f}^{(1)}(x') \int_0^{\infty} d|x| \, |x|^{1+2\nu} \frac{J_{\nu+n}(\lambda|x|)}{(\lambda|x|)^{\nu}} \delta(r_1-|x|)$$

$$= a_{n\ell}(\overset{*}{f}^{(1)}) r_1^{1+2\nu} \frac{J_{\nu+n}(\lambda r_1)}{(\lambda r_1)^{\nu}}$$

and

$$(4.2.15) \quad \overset{*}{f}^{(2)}_{n'\ell'}(\lambda) = a_{n'\ell'}(\overset{*}{f}^{(2)}) r_2^{1+2\nu} \frac{J_{\nu+n'}(\lambda r_2)}{(\lambda r_2)^{\nu}}$$

where

$$(4.2.16) \quad a_{n\ell}(\overset{*}{f}) = \int d\Omega \overline{Y}_{n\ell}(x')\overset{*}{f}(x'), \quad \overset{*}{f} \in L^2(S^{d-1}, d\Omega)$$

Then, employing (4.2.14) and (4.2.15) in (4.2.10), we now obtain

$$B_{ij}(\overset{*}{f}^{(2)} \otimes \delta_{r_2}, \overset{*}{f}^{(1)} \otimes \delta_{r_1})$$

$$= \sum_{n=0}^{\infty} \sum_{\ell=1}^{d_n} |A_n|^2 a_{n\ell}(\overset{*}{f}^{(2)}) \overline{a}_{n\ell}(\overset{*}{f}^{(1)}) \delta_{ij} (r_1 r_2)^{1+2\nu} \int_0^{\infty} d\Phi^{(1)}(\lambda) \frac{J_{\nu+n}(\lambda r_2)}{(\lambda r_2)^{\nu}} \cdot \frac{J_{\nu+n}(\lambda r_1)}{(\lambda r_1)^{\nu}}$$

$$+ \sum_{n,n'=0}^{\infty} h_{ijn'n}(\overset{*}{f}^{(2)}, \overset{*}{f}^{(1)}) (r_1 r_2)^{1+2\nu} \int_0^{\infty} d\Phi^{(2)}(\lambda) \frac{J_{\nu+n'}(\lambda r_2)}{(\lambda r_2)^{\nu}} \cdot \frac{J_{\nu+n}(\lambda r_1)}{(\lambda r_1)^{\nu}}$$

where

$$(4.2.17) \quad h_{ijn'n}(\overset{*}{f}^{(2)}, \overset{*}{f}^{(1)}) = \sum_{\ell=1}^{d_n} \sum_{\ell'=1}^{d_{n'}} \overline{A}_n A_{n'} a_{n'\ell'}(\overset{*}{f}^{(2)}) \overline{a}_{n\ell}(\overset{*}{f}^{(1)}) b_{ijn\ell n'\ell'}$$

By Wong's Assumption, $B_{ij}(\overset{*}{f}^{(2)} \otimes \delta_{r_2}, \overset{*}{f}^{(1)} \otimes \delta_{r_1})$ is convergent Hence $\xi(\overset{*}{f} \otimes \delta_r)$ belongs to $(L^2(\Omega, \mathcal{B}, \mu))^d$, $\overset{*}{f} \in L^2(S^{d-1}, d\Omega)$.

Let $H(\partial D)$ be the completion in $(L^2(\Omega, \mathcal{B}, \mu))^d$ of the linear space $\{\xi(\overset{*}{f} \otimes \delta_r) = (\xi_j(\overset{*}{f} \otimes \delta_r) : j = 1, \ldots, d) : \overset{*}{f} \in L^2(S^{d-1}, d\Omega)\}$ of random vectors. $H(\partial D)$ then constitutes the boundary data for $H^0(\Omega)$ on $\partial D =$ sphere of radius $r > 0$. $H(\partial D)$ features prominently in subsequent analysis and hence we shall spend some space describing its intrinsic structure.

4.3 THE BOUNDARY DATA HILBERT SPACE $H(\partial D)$

In what follows, we limit ourselves to considering a three component vector Gaussian generalized stochastic field $H^0(\Omega) = \{\xi(f) = (\xi_1(f), \xi_2(f), \xi_3(f)) : f \in \mathcal{S}(R^3)\}$ on the probability space $(\Omega, \mathcal{B}, \mu)$. For ∂D = sphere of radius $r > 0$, about the origin, let $H(\partial D)$ be constructed as in the preceding section. We wish now to describe the boundary data Hilbert space $H(\partial D)$ of three dimensional random vectors in some detail.

As mentioned in the last section, $H(\partial D)$ is the completion in $(L^2(\Omega, \mathcal{B}, \mu))^3$ of the linear space

$$\{\xi(\overset{*}{f} \otimes \delta_r) = (\xi_1(\overset{*}{f} \otimes \delta_r),\ \xi_2(\overset{*}{f} \otimes \delta_r),\ \xi_3(\overset{*}{f} \otimes \delta_r) : \overset{*}{f} \in L^2(S^2, d\Omega)\}$$

of three dimensional random vectors. The norm of $H(\partial D)$ is derived from the following inner product

$$\langle \underline{\xi}(\underline{\overset{*}{f}}^{(2)} \otimes \delta_r),\ \underline{\xi}(\underline{\overset{*}{f}}^{(1)} \otimes \delta_r)\rangle_{H(\partial D)} = \sum_{i,j=1}^{3} \langle \xi_i(\overset{*}{f}{}_i^{(2)} \otimes \delta_r), \xi_j(\overset{*}{f}{}_j^{(1)} \otimes \delta_r)\rangle_{L^2(\Omega, \mathcal{B}, \mu)}$$

$$= \sum_{i,j=1}^{3} B_{ij}(\overset{*}{f}{}_i^{(2)} \otimes \delta_r,\ \overset{*}{f}{}_j^{(1)} \otimes \delta_r),\ r > 0.$$

where $\underline{\overset{*}{f}}^{(i)} = (\overset{*}{f}{}_1^{(i)}, \overset{*}{f}{}_2^{(i)}, \overset{*}{f}{}_3^{(i)}) \in (L^2(S^2, d\Omega))^3$, $i = 1,2$

and $\underline{\xi}(\underline{\overset{*}{f}}^{(i)} \otimes \delta_r) = (\xi_j(\overset{*}{f}{}_j^{(i)} \otimes \delta_r) : \overset{*}{f}{}_j^{(i)} \in L^2(S^2, d\Omega),\ j = 1,2,3),\ i = 1,2.$

(4.3.1) REMARK: In what follows, our aim is to express $H(\partial D)$ as an orthogonal direct sum of Hilbert spaces of random vectors. In the next section, we exhibit the elements of the first two Hilbert spaces in the mentioned orthogonal direct sum expression for $H(\partial D)$, and then we study them. This aim of ours is realized by invoking well-known results in the group theoretic analysis of the classical three dimensional rotation group SO(3) [10][96].

(4.3.2) DISCUSSION Recall that we are dealing with a Euclidean covariant vector generalized stochastic field $H^{O}(\Omega) = \{\xi(f) = (\xi_1(f),\xi_2(f),\xi_3(f)) : f \in \mathcal{S}(R^d)\}$. Hence, if $h \to V^{(1)}(h)$ is the vector irreducible unitary representation of SO(3), then we have the following transformation law:

$$(4.3.3) \quad V^{(1)}(h)B(f^{(2)}\circ g^{-1}, f^{(1)}\circ g^{-1})V^{(1)}(h)^{-1} = B(f^{(2)}, f^{(1)})$$

$$(h,g) \in SO(3) \times M(3)$$

$$f^{(i)} \in \mathcal{S}(R^3), \quad i = 1,2$$

Hence, the operator

$$U(h) : \xi(\overset{*}{f} \otimes \delta_r) \to U(h)\xi(\overset{*}{f} \otimes \delta_r) = V^{(1)}(h)\xi(\overset{*}{f} \circ h^{-1} \otimes \delta_r),$$

$$\overset{*}{f} \in L^2(S^2, d\Omega),$$

which may be readily extended by linearity and continuity so as to be defined on all of $H(\partial D)$ is unitary on $H(\partial D)$. To see this, we have

$$<U(h)\xi(\overset{*}{f}^{(2)} \otimes \delta_r), U(h)\xi(\overset{*}{f}^{(1)} \otimes \delta_r)>_{H(\partial D)}$$

$$= \sum_{i,j=1}^{3} < \sum_k V_{ik}^{(1)}(h)\xi_k(\overset{*}{f}^{(2)}\circ h^{-1} \otimes \delta_r), \sum_m V_{jm}^{(1)}(h)\xi_m(\overset{*}{f}^{(1)}\circ h^{-1} \otimes \delta_r)>_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= \sum_{i,j,k,m=1}^{3} V_{ik}^{(1)}(h)\overline{V}_{jm}^{(1)}(h)B_{km}(\overset{*}{f}^{(2)}\circ h^{-1} \otimes \delta_r, \overset{*}{f}^{(1)}\circ h^{-1} \otimes \delta_r)$$

$$= \sum_{i,j=1}^{3} \sum_{k,m=1}^{3} V_{ik}^{(1)}(h)B_{km}(\overset{*}{f}^{(2)}\circ h^{-1} \otimes \delta_r, \overset{*}{f}^{(1)}\circ h^{-1} \otimes \delta_r)V_{mj}^{(1)}(h^{-1})$$

$$= \sum_{i,j=1}^{3} B_{ij}(\overset{*}{f}^{(2)} \otimes \delta_r, \overset{*}{f}^{(1)} \otimes \delta_r) \quad \text{by} \quad (4.3.3)$$

$$= \langle \xi(\overset{*}{f}^{(2)} \otimes \delta_r), \xi(\overset{*}{f}^{(1)} \otimes \delta_r)\rangle_{H\ (\partial D)} .$$

Hence $h \to U(h)$ is indeed unitary on $H\ (\partial D)$.

We may next undertake a spectral resolution of each $\xi(\overset{*}{f} \otimes \delta_r)$ in $H\ (\partial D)$. This latter objective will be achieved by decomposing the unitary representation $h \to U(h)$ of SO(3) into its irreducible components. Since SO(3) is a Lie group of type one, we know from Mackey's theorem [50] that the mentioned decomposition of $h \to U(h)$ is always possible.

Consider now the random vector $\xi(\overset{*}{f} \otimes \delta_r)$, $r > 0$, $\overset{*}{f} \in L^2(S^2, d\Omega)$, which belongs to $H\ (\partial D)$. Then the components $\xi_i(\overset{*}{f} \otimes \delta_r)$ of $\xi(\overset{*}{f} \otimes \delta_r)$ are given by

$$\xi_i(\overset{*}{f} \otimes \delta_r) = \int dx \xi_i(x) \overset{*}{f}(x') \delta(r - |x|) \qquad (x' = \frac{x}{|x|},\ |x| \neq 0)$$

$$= \int d\Omega d|x| |x|^2 \ \xi_i(|x|, x') \overset{*}{f}(x') \delta(r - |x|)$$

$$= r^2 \int d\Omega \xi_i(r, x') \overset{*}{f}(x')$$

Since $\overset{*}{f}$ belongs to $L^2(S^2, d\Omega)$, we may expand it into an $L^2(S^2, d\Omega)$-convergent series as follows

$$\overset{*}{f} = \sum_{n=0}^{\infty} \sum_{\ell=-n}^{n} a_{n\ell} Y_{n\ell}$$

where $\{Y_{n\ell} : n = 0, 1, \ldots, \infty,\ \ell = -n, -n+1, \ldots, n-1, n\}$ is the set of orthonormal spherical harmonics, which forms a basis for $L^2(S^2, d\Omega)$. Thus

$$(4.3.4) \quad \xi_i(\overset{*}{f} \otimes \delta_r) = r^2 \sum_{n=0}^{\infty} \sum_{\ell=-n}^{n} a_{n\ell} \int d\Omega \xi_i(r, x') Y_{n\ell}(x'), \quad r > 0$$

and the series of random variables on the right hand side of (4.3.4) converges in $L^2(\Omega, \mathcal{B}, \mu)$. It will be convenient in what follows to label the components of the random vector $\xi(\overset{*}{f} \otimes \delta_r)$ in $H(\partial D)$ by the numbers $\{-1,0,1\}$ instead of by the set of numbers $\{1,2,3\}$. We do this because the components of $\xi(\overset{*}{f} \otimes \delta_r)$ form a basis for the vector irreducible unitary representation of SO(3) and the numbers $\{-1,0,1\}$ are conventionally employed to label the basis for this representation.

Next, let $\{h \to V^{(n)}(h) : n = 0,1,\ldots,\infty\}$ be the irreducible unitary representations of SO(3).

Then the random tensor whose components are given by

$$\xi_i(Y_{n\ell} \otimes \delta_r) = r^2 \int d\Omega\, \xi_i(r,x')Y_{n\ell}(x')$$

$i = -1,0,1$, $\ell = -n,-n+1,\ldots,n-1,n$, and n fixed, is transformed by the tensor product representation $h \to (V^{(1)} \otimes V^{(n)})(h)$ of SO(3) as follows

$$(V^{(1)} \otimes V^{(n)})(h)\xi_i(Y_{n\ell} \otimes \delta_r) = r^2 \int d\Omega \sum_{j=-1}^{1} V_{ij}^{(1)}(h)\xi_j(r,x') \sum_{k=-n}^{n} V_{\ell k}^{(n)}(h)Y_{nk}(x')$$

$$= \sum_{j=-1}^{1} \sum_{k=-n}^{n} V_{ij}^{(1)}(h)V_{\ell k}^{(n)}(h)r^2 \int d\Omega \xi_j(r,x')Y_{nk}(x')$$

$$= \sum_{j=-1}^{1} \sum_{k=-n}^{n} V_{ij}^{(1)}(h)V_{\ell k}^{(n)}(h)\xi_j(Y_{nk} \otimes \delta_r)$$

Now, the group SO(3) is simply reducible [28] and we have

$$(V^{(1)} \otimes V^{(n)})(h) = \bigoplus_{N=|1-n|}^{1+n} V^{(N)}(h)^C$$

As a consequence of the indicated decomposition of $(V^{(1)} \otimes V^{(n)})(h)$, the random tensor whose components are $\xi_i(Y_{n\ell} \otimes \delta_r)$, $i = -1,0,1$, $\ell = -n,-n+1,\ldots,n-1,n$ and n fixed, may be resolved into a direct sum of multicomponent random fields which form basis functions for the irreducible unitary representations $h \to V^{(N)}(h)^c$, $N \in [\,1-n\ ,\ 1+n]$, of SO(3) which occur in the reduction of $(V^{(1)} \otimes V^{(n)})(h)$.

Let H(r) be a Hilbert space of three dimensional random vectors and let, as usual, $\mathcal{H}^{(n)}$ denote the finite dimensional Hilbert space of spherical harmonics of degree n. Then the action of the unitary representation $h \to U(h)$ on $H(\partial D)$ is equivalent to the action of the tensor product representation $h \to (V^{(1)} \otimes (\bigoplus_{n=0}^{\infty} V^{(n)}))(h)$

on $H(r) \otimes (\bigoplus_{n=0}^{\infty}\ {}^{(n)})$.

Now

$$(V^{(1)} \otimes (\bigoplus_{n=0}^{\infty} V^{(n)}))(h) = \bigoplus_{n=0}^{\infty} \bigoplus_{N=|1-n|}^{1+n} V^{(N)}(h)^c$$

$$= V^{(0)}(h) \bigoplus_{n=1}^{\infty} \bigoplus_{i=0}^{2} V^{(n,i)}(h)^c$$

where $h \to V^{(0)}(h)$ is the identity representation of SO(3) and $h \to V^{(n,i)}(h)$ is a (2n+1)-dimensional irreducible unitary representation of SO(3) for i = 0,1,2.

Since $H(\partial D)$ is the representation space for the unitary representation $h \to U(h)$, $h \in SO(3)$, which is equivalent to the unitary representation

$$h \to V^{(0)}(h) \bigoplus_{n=1}^{\infty} \bigoplus_{i=0}^{2} V^{(n,i)}(h)^c \ , \qquad h \in SO(3),$$

it is, therefore, seen that the Hilbert space $H(\partial D)$ of random vectors decomposes into an orthogonal direct sum of Hilbert spaces as follows

$$(4.3.5) \qquad H(\partial D) = H^{(0)}(r) \bigoplus_{n=1}^{\infty} H^{(n)}(r) \quad , \quad r > 0$$

in which $H^{(0)}(r)$ is a Hilbert space of scalar random variables and $H^{(n)}(r)$, $n \geq 1$, is a Hilbert space of $3(2n+1)$-dimensional random vectors. Equation (4.3.5) gives the promised spectral resolution of each $\xi(\overset{*}{f} \otimes \delta_r)$ in $H(\partial D)$, $r > 0$. The elements of $H^{(0)}(r)$ are all constant multiples of the only rotationally invariant scalar random variable which may be constructed from $\xi(\overset{*}{f} \otimes \delta_r)$. On the other hand, each element $Y^{(n)}(r)$ in $H^{(n)}(r)$ is of the form $Y^{(n)}(r) = \sum_{i=0}^{2} \alpha_{ni} Y^{(n,i)}(r)$ where $Y^{(n,i)}(r)$ is a $(2n+1)$-dimensional random vector for $i = 0,1,2$. All the preceding imply that each $\xi(\overset{*}{f} \otimes \delta_r)$ in $H(\partial D)$ may be expressed as follows

$$\xi(\overset{*}{f} \otimes \delta_r) = \alpha_0 \Phi_0(r) + \sum_{n=1}^{\infty} \sum_{i=0}^{2} \alpha_{ni} Y^{(n,i)}(r)$$

where the constants $\{\alpha_0, \alpha_{ni} : n = 1,\ldots,\infty,\ i = 0,1,2\}$ depend linearly on $\overset{*}{f} \in L^2(S^2, d\Omega)$. These facts become important in the sequel.

(4.3.6) REMARK: With effect from the next section, we shall be carrying out detailed computations which require knowledge of the recurrence relations between the orthonormal spherical harmonics $\{Y_{n\ell} : n = 0,.,\ldots,\infty;\ \ell \in [-n,n]\}$ which form a basis for $L^2(S^2, d\Omega)$. We, therefore, present here the relevant information.

Let $x = (r,\theta,\phi)$ be the representation of $x \in R^3$ in polar coordinates, and set $x' = \frac{x}{|x|}$, $|x| > 0$. Then we have [86]

(4.3.7) $\cos\theta \; Y_{n\ell}(x') = \alpha_n^{\ell} Y_{n-1,\ell}(x') + \beta_n^{\ell} Y_{n+1,\ell}(x')$

(4.3.8) $\sin\theta \; e^{i\phi} Y_{n\ell}(x') = \gamma_n^{\ell} Y_{n-1,\ell+1}(x') + \theta_n^{\ell} Y_{n+1,\ell+1}(x')$

(4.3.9) $Y_{n,-\ell}(x') = (-1)^{\ell} \overline{Y}_{n\ell}(x')$

where $\theta \in [0,\pi]$, $\phi \in [0,2\pi]$ and

$$(4.3.10) \quad \alpha_n^{\ell} = [(n-\ell)(n+\ell)]^{1/2}[(2n-1)(2n+1)]^{-1/2}$$

$$(4.3.11) \quad \beta_n^{\ell} = [(n+\ell+1)(n-\ell+1)]^{1/2}[(2n+1)(2n+3)]^{-1/2}$$

$$(4.3.12) \quad \gamma_n^{\ell} = [(n-\ell-1)(n-\ell)]^{1/2}[(2n-1)(2n+1)]^{-1/2}$$

$$(4.3.13) \quad \theta_n^{\ell} = (-1)[(n+\ell+1)(n+\ell+2)]^{1/2}[(2n+1)(2n+3)]^{-1/2}$$

In the sequel, we employ the following labelling of the components of $x' = \frac{x}{|x|} = (\frac{x_0}{|x|}, \frac{x_1}{|x|}, \frac{x_{-1}}{|x|})$ in S^2:

$\frac{x_0}{|x|} = \cos\theta$

$\frac{x_1}{|x|} = \sin\theta$ ~~cos φ~~ $e^{i\phi}$

$\frac{x_{-1}}{|x|} = \sin\theta$ ~~sin φ~~ $e^{-i\phi}$ $\qquad \theta \in [0,\pi]$, $\phi \in [0,2\pi]$

Then, we have

$$(4.3.7) \quad \frac{x_0}{|x|} Y_{n\ell}(x') = \alpha_n^{\ell} Y_{n-1,\ell}(x') + \beta_n^{\ell} Y_{n+1,\ell}(x')$$

and by combining (4.3.8) and (4.3.9), we obtain

$$(4.3.14) \quad \frac{x_1}{|x|} Y_{n\ell}(x') = \frac{1}{2}[\gamma_n^{\ell} Y_{n-1,\ell+1}(x') + \theta_n^{\ell} Y_{n+1\ell+1}(x') - \gamma_n^{-\ell} Y_{n-1,\ell-1}(x') - \theta_n^{-\ell} Y_{n+1,\ell-1}(x')]$$

$$(4.3.15)\quad \frac{x_{-1}}{|x|} Y_{n\ell}(x') = \frac{1}{2i}[\gamma_n^{\ell} Y_{n-1,\ell+1}(x') + \theta_n^{\ell} Y_{n+1,\ell+1}(x') + \gamma_n^{-\ell} Y_{n-1,\ell-1}(x') + \theta_n^{-\ell} Y_{n+1,\ell-1}(x')]$$

We make repeated application of (4.3.7), (4.3.14) and (4.3.15) in subsequent computations.

4.4 THE HILBERT SPACES $H^{(0)}(r)$ and $H^{(1)}(r)$, $r > 0$

This section deals with the explicit construction and detailed study of the elements of the Hilbert spaces $H^{(0)}(r)$ and $H^{(1)}(r)$, $r > 0$, which arise in the orthogonal decomposition of $H(\partial D)$, see (4.3.5). Our construction again relies heavily on group theory. Hence we begin by introducing the so-called three-j functions [86].

Let $m \in [-j,j]$, $n \in [-k,k]$ and $p \in [-\ell,\ell]$ where j, k, ℓ are non-negative integers. Then, the three-j function $\begin{pmatrix} j & k & \ell \\ m & n & p \end{pmatrix}$ is defined as follows

$$(4.4.1)\quad \begin{pmatrix} j & k & \ell \\ m & n & p \end{pmatrix} = (-1)^{2j-k+n} \left[\frac{(j+k-\ell)!\,(k+\ell-j)!\,(\ell+j-k)!\,(\ell+p)!\,(\ell-p)!}{(j+k+\ell+1)!\,(j+m)!\,(j-m)!\,(k+n)!\,(k-n)!}\right]^{1/2} \times$$

$$\times \sum (-1)^t \frac{(\ell+j-n-t)!\,(k+n+t)!}{t!\,(\ell+p-t)!\,(t+k-j-p)!\,(\ell-k+j-t)!}$$

Here, the sum on t is over all integer values compatible with the condition that the arguments of all the factorial functions under the summation notation stay nonnegative. The function $\begin{pmatrix} j & k & \ell \\ m & n & p \end{pmatrix}$ vanishes unless $m + n + p = 0$, and is not defined unless $|j-k| \leq \ell \leq j+k$ and $j+k-\ell$ is an integer. Furthermore, the three-j functions satisfy the following identities

$$(4.4.2) \qquad \sum_{m,n} (2\ell+1)^{1/2} \begin{pmatrix} j & k & \ell \\ m & n & p \end{pmatrix} \begin{pmatrix} j & k & \ell' \\ m & n & p' \end{pmatrix} = \delta_{\ell\ell'} \, \delta_{pp'}$$

$$(4.4.3) \qquad \sum_{\ell,p} (2\ell+1)^{1/2} \begin{pmatrix} j & k & \ell \\ m & n & p \end{pmatrix} \begin{pmatrix} j & k & \ell \\ m' & n' & p \end{pmatrix} = \delta_{mm'} \, \delta_{nn'}$$

Next, from the theory of the reduction of the tensor product of two irreducible unitary representations $V^{(j)}(h)$, and $V^{(k)}(h)$ of SO(3) into irreducible components $V^{(N)}(h)^c$, $N \in [\,j-k\,, j+k]$, where we recall $V^{(N)}(h)^c$ = complex conjugate representation to $V^{(N)}(h)$, we have [86]

$$(4.4.4) \qquad \sum_{m',n'} \begin{pmatrix} j & k & N \\ m' & n' & p' \end{pmatrix} V^{(j)}_{m'n}(h)\, V^{(k)}_{n'n}(h) = \sum_{p} \begin{pmatrix} j & k & N \\ m & n & p \end{pmatrix} \overline{V}^{(N)}_{p'p}(h)$$

Of course, the right hand side of (4.4.4) contains just one term, namely that for which $p = -(m+n)$, since as mentioned above, $\begin{pmatrix} j & k & N \\ m & n & p \end{pmatrix}$ vanishes otherwise.

Given basis functions for the representations $V^{(j)}(h)$ and $V^{(k)}(h)$, we readily construct basis functions for the representation $V^{(N)}(h)^c$. Indeed, it is this fact that we invoke in constructing the elements of $H^{(0)}(r)$ and $H^{(1)}(r)$, $r > 0$, in what follows.

(4.4.5) CONSTRUCTION OF THE ELEMENTS OF $H^{(0)}(r)$ and $H^{(1)}$, $r > 0$.

The fact has already been referred to in the last section that the random tensor whose components are given by

$$\xi_i(Y_{n\ell} \otimes \delta_r) = r^2 \int d\Omega \xi_i(r,x')\, Y_{n\ell}(x'),$$

$i = 1,0,1$, $\ell \in [-n,n]$ and n fixed, transforms according to the tensor product (reducible) representation $(V^{(1)} \otimes V^{(n)})(h)$ of SO(3). Therefore, given the components $\xi_i(Y_{n\ell} \otimes \delta_r)$ of the mentioned random tensor, we may construct the random vector $Y(r)$ which transforms according to the representation $V^{(N)}(h))^c$, $N \in [|1-n|, 1+n]$. The random vector $Y(r)$ is (2N+1)-dimensional, and indeed its components are given by

$$(4.4.6) \qquad Y_p(r) = \sum_{i,\ell} (2N+1)^{1/2} \begin{pmatrix} 1 & n & N \\ i & \ell & p \end{pmatrix} \xi_i(Y_{n\ell} \otimes \delta_r)$$

$$r > 0, \quad p \in [-N,N].$$

By invoking (4.4.4), one readily convinces himself that $Y(r)$ does indeed transform according to $V^{(N)}(h)^c$.

We now finally arrive at the primary objective of this section, namely, the specification of the elements of $H^{(0)}(r)$ and $H^{(1)}(r)$, $r > 0$.

Consider first the Hilbert space $H^{(0)}(r)$. The elements of $H^{(0)}(r)$ are scalar random variables and they transform according to the identity representation $V^{(0)}(h)$ of SO(3) which arises in the reduction of the tensor product representation $(V^{(1)} \otimes V^{(1)})(h)$, $h \in SO(3)$. Hence, the random variables in $H^{(0)}(r)$, $r > 0$, are all constant multiples of the scalar random variable $\Phi_0(r)$ given by

$$\Phi_0(r) = \sum_{i,\ell=-1}^{+1} \begin{pmatrix} 1 & 1 & 0 \\ i & \ell & 0 \end{pmatrix} \xi_i(Y_{1\ell} \otimes \delta_r)$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xi_0(Y_{10} \otimes \delta_r) + \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \xi_1(Y_{1,-1} \otimes \delta_r) + \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \xi_{-1}(Y_{11} \otimes \delta_r)$$

Since $\begin{pmatrix} j & j & 0 \\ m & -m & 0 \end{pmatrix} = (-1)^{j-m}(2j+1)^{-1/2}$, it follows that

$$\Phi_0(r) = \frac{1}{\sqrt{3}} \xi_0(Y_{10} \otimes \delta_r) + \frac{1}{\sqrt{3}} \xi_1(Y_{1,-1} \otimes \delta_r) + \frac{1}{\sqrt{3}} \xi_{-1}(Y_{11} \otimes \delta_r)$$

Set $\sqrt{3}\, \Phi_0(r) = \Phi(r)$. Then

(4.4.7) $\Phi(r) = \xi_0(Y_{10} \otimes \delta_r) + \xi_1(Y_{1,-1} \otimes \delta_r) + \xi_{-1}(Y_{11} \otimes \delta_r)$

(4.4.8) REMARK: By invoking (4.3.7), (4.3.14) and (4.3.15), one finds that the random variable $\Phi(r)$ given by (4.4.7) is indeed expressible as follows

$$\sqrt{\frac{4\pi}{3}}\,\Phi(r) = \xi_0\left(\frac{x_0}{|x|}\otimes\delta_r\right) + \frac{1}{\sqrt{2}}(\xi_1-\xi_{-1})\left(\frac{x_1}{|x|}\otimes\delta_r\right) + \frac{(-i)}{\sqrt{2}}(\xi_1+\xi_{-1})\left(\frac{x_{-1}}{|x|}\otimes\delta_r\right)$$

However, for purposes of subsequent calculations, we shall be content with the representation (4.4.7).

(4.4.9) Next, consider the Hilbert space $H^{(1)}(r)$, $r > 0$, The elements of $H^{(1)}(r)$ are nine dimensional random vectors, and an arbitrary such vector $X(r)$ may be written as follows

$$X(r) = \alpha_0 X^{(0)}(r) \underset{a}{\oplus} \alpha_1 X^{(1)}(r) \underset{a}{\oplus} \alpha_2 X^{(2)}(r)$$

($\alpha_i \in \mathbb{C}$, $i = 0,1,2$), where $\underset{a}{\oplus}$ denotes algebraic direct sum and where $X^{(i)}(r)$, $i = 0,1,2$, transforms respectively according to the three-dimensional vector irreducible unitary representation $h \to V^{(1,i)}(h)^c$, $i = 0,1,2$ contained in the reduction of the tensor product representation $h \to (V^{(1)} \otimes V^{(i)})(h)$, $i = 0,1,2$ into irreducibles. The construction of $X(r) \in H^{(1)}(r)$ is consequently reduced to the construction of the vectors $X^{(i)}(r)$, $i = 0,1,2$. This latter task is readily accomplished by invoking (4.4.6).

Thus, consider first the random vector $X^{(0)}(r)$. Then, by (4.4.6), the three components $X_p^{(0)}(r)$, $p = -1,0,1$ of $X^{(0)}(r)$ are given by

$$X_p^{(0)}(r) = \sqrt{3} \sum_{i,\ell} \begin{pmatrix} 1 & 0 & 1 \\ i & \ell & p \end{pmatrix} \xi_i (Y_{0\ell} \otimes \delta_r)$$

$$= \sqrt{3} \sum_{i=-1}^{1} \begin{pmatrix} 1 & 0 & 1 \\ i & 0 & p \end{pmatrix} \xi_i (Y_{00} \otimes \delta_r)$$

Hence

$$X_0^{(0)}(r) = \sqrt{3} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xi_0 (Y_{00} \otimes \delta_r)$$

$$= -\xi_0 (Y_{00} \otimes \delta_r)$$

$$X_1^{(0)}(r) = \sqrt{3} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \end{pmatrix} \xi_{-1} (Y_{00} \otimes \delta_r)$$

$$= \xi_{-1} (Y_{00} \otimes \delta_r)$$

$$X_{-1}^{(0)}(r) = \sqrt{3} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \xi_1 (Y_{00} \otimes \delta_r)$$

$$= \xi_1 (Y_{00} \otimes \delta_r)$$

Hence, we have that

(4.4.10) $$X^{(0)}(r) = (X_{-1}^{(0)}(r),\ X_0^{(0)}(r),\ X_1^{(0)}(r))$$

$$= (\xi_1 (Y_{00} \otimes \delta_r),\ -\xi_0 (Y_{00} \otimes \delta_r),\ \xi_{-1} (Y_{00} \otimes \delta_r))$$

Next, consider the random vector $X^{(1)}(r)$. Then, by (4.4.6), the three components $X_p^{(1)}(r)$, $p = -1,0,1$ of $X^{(1)}(r)$ are given by

$$X_p^{(1)}(r) = \sqrt{3} \sum_{i,\ell=-1}^{1} \begin{pmatrix} 1 & 1 & 1 \\ i & \ell & p \end{pmatrix} \xi_i \ (Y_{1\ell} \otimes \delta_r)$$

Hence

$$X_0^{(1)}(r) = \sqrt{3}\begin{pmatrix}1&1&1\\0&0&0\end{pmatrix}\xi_0(Y_{10}\otimes\delta_r) + \sqrt{3}\begin{pmatrix}1&1&1\\1&-1&0\end{pmatrix}\xi_1(Y_{1,-1}\otimes\delta_r) +$$

$$+ \sqrt{3}\begin{pmatrix}1&1&1\\-1&1&0\end{pmatrix}\xi_{-1}(Y_{11}\otimes\delta_r)$$

$$= \frac{1}{\sqrt{2}}\xi_1(Y_{1,-1}\otimes\delta_r) - \frac{1}{\sqrt{2}}\xi_{-1}(Y_{11}\otimes\delta_r)$$

$$X_1^{(1)}(r) = \sqrt{3}\begin{pmatrix}1&1&1\\-1&0&1\end{pmatrix}\xi_{-1}(Y_{10}\otimes\delta_r) + \sqrt{3}\begin{pmatrix}1&1&1\\0&-1&1\end{pmatrix}\xi_0(Y_{1,-1}\otimes\delta_r)$$

$$= \frac{1}{\sqrt{2}}\xi_{-1}(Y_{10}\otimes\delta_r) - \frac{1}{\sqrt{2}}\xi_0(Y_{1,-1}\otimes\delta_r)$$

$$X_{-1}^{(1)}(r) = \sqrt{3}\begin{pmatrix}1&1&1\\0&1&-1\end{pmatrix}\xi_0(Y_{11}\otimes\delta_r) + \sqrt{3}\begin{pmatrix}1&1&1\\1&0&-1\end{pmatrix}\xi_1(Y_{10}\otimes\delta_r)$$

$$= \frac{1}{\sqrt{2}}\xi_0(Y_{11}\otimes\delta_r) - \frac{1}{\sqrt{2}}\xi_1(Y_{10}\otimes\delta_r)$$

Hence, we have that

$$(4.4.11)\quad X^{(1)}(r) = (X_{-1}^{(1)}(r),\ X_0^{(1)}(r),\ X_1^{(1)}(r))$$

$$= (\frac{1}{\sqrt{2}}\xi_0(Y_{11}\otimes\delta_r) - \frac{1}{\sqrt{2}}\xi_1(Y_{10}\otimes\delta_r),\ \frac{1}{\sqrt{2}}\xi_1(Y_{1,-1}\otimes\delta_r) - \frac{1}{\sqrt{2}}\xi_{-1}(Y_{11}\otimes\delta_r),$$

$$\frac{1}{\sqrt{2}}\xi_{-1}(Y_{10}\otimes\delta_r) - \frac{1}{\sqrt{2}}\xi_0(Y_{1,-1}\otimes\delta_r))$$

(4.4.12) REMARK: Again in a similar way to what was done in (4.4.8), we may express the components of $X^{(1)}(r)$ in a different way by using (4.3.7), (4.3.14) and (4.3.15), but again we prefer (4.4.11) because of our subsequent needs.

Finally, consider the random vector $X^{(2)}(r)$. Then, by (4.4.6), the three components $X_p^{(2)}(r)$, $p = -1,0,1$ of $X^{(2)}(r)$ are given by

$$X_p^{(2)}(r) = \sqrt{3} \sum_{i=-1}^{1} \sum_{\ell=-2}^{2} \begin{pmatrix} 1 & 2 & 1 \\ i & \ell & p \end{pmatrix} \xi_i (Y_{2\ell} \otimes \delta_r)$$

Hence

$$X_0^{(2)}(r) = \sqrt{3} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xi_0 (Y_{20} \otimes \delta_r) + \sqrt{3} \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \end{pmatrix} \xi_1 (Y_{2,-1} \otimes \delta_r) +$$

$$+ \sqrt{3} \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \end{pmatrix} \xi_{-1} (Y_{21} \otimes \delta_r)$$

$$= \frac{2}{\sqrt{10}} \xi_0 (Y_{20} \otimes \delta_r) - \frac{\sqrt{3}}{\sqrt{10}} \xi_1 (Y_{2,-1} \otimes \delta_r) - \frac{\sqrt{3}}{\sqrt{10}} \xi_{-1} (Y_{21} \otimes \delta_r)$$

$$X_1^{(2)}(r) = \sqrt{3} \sum_{i=-1}^{1} \sum_{\ell=-2}^{2} \begin{pmatrix} 1 & 2 & 1 \\ i & \ell & 1 \end{pmatrix} \xi_i (Y_{2\ell} \otimes \delta_r)$$

$$= \sqrt{3} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \end{pmatrix} \xi_0 (Y_{2,-1} \otimes \delta_r) + \sqrt{3} \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix} \xi_{-1} (Y_{20} \otimes \delta_r) +$$

$$+ \sqrt{3} \begin{pmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \end{pmatrix} \xi_1 (Y_{2,-2} \otimes \delta_r)$$

$$= \frac{1}{\sqrt{10}} \xi_{-1} (Y_{20} \otimes \delta_r) + \frac{\sqrt{3}}{\sqrt{5}} \xi_1 (Y_{2,-2} \otimes \delta_r) - \frac{\sqrt{3}}{\sqrt{10}} \xi_0 (Y_{2,-1} \otimes \delta_r)$$

$$X_{-1}^{(2)}(r) = \sqrt{3} \sum_{i=-1}^{1} \sum_{\ell=-2}^{2} \begin{pmatrix} 1 & 2 & 1 \\ i & \ell & -1 \end{pmatrix} \xi_i (Y_{2\ell} \otimes \delta_r)$$

$$= \sqrt{3} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \xi_0 (Y_{21} \otimes \delta_r) + \sqrt{3} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \end{pmatrix} \xi_1 (Y_{20} \otimes \delta_r) +$$

$$+ \sqrt{3} \begin{pmatrix} 1 & 2 & 1 \\ -1 & 2 & -1 \end{pmatrix} \xi_{-1} (Y_{22} \otimes \delta_r)$$

$$= \frac{\sqrt{3}}{\sqrt{5}} \xi_{-1} (Y_{22} \otimes \delta_r) + \frac{1}{\sqrt{10}} \xi_1 (Y_{20} \otimes \delta_r) - \frac{\sqrt{3}}{\sqrt{10}} \xi_0 (Y_{21} \otimes \delta_r)$$

Thus, we have that

$$(4.4.13)\quad X^{(2)}(r) = (X_{-1}^{(2)}(r),\ X_0^{(2)}(r),\ X_1^{(2)}(r))$$

$$= (\frac{\sqrt{3}}{\sqrt{5}}\xi_{-1}(Y_{22}\otimes\delta_r) + \frac{1}{\sqrt{10}}\xi_1(Y_{20}\otimes\delta_r) - \frac{\sqrt{3}}{\sqrt{10}}\xi_0(Y_{21}\otimes\delta_r)\ ,$$

$$\frac{2}{\sqrt{10}}\xi_0(Y_{20}\otimes\delta_r) - \frac{\sqrt{3}}{\sqrt{10}}\xi_1(Y_{2,-1}\otimes\delta_r) - \frac{\sqrt{3}}{\sqrt{10}}\xi_{-1}(Y_{21}\otimes\delta_r)\ ,$$

$$\frac{1}{\sqrt{10}}\xi_{-1}(Y_{20}\otimes\delta_r) + \frac{\sqrt{3}}{\sqrt{5}}\xi_1(Y_{2,-2}\otimes\delta_r) - \frac{\sqrt{3}}{\sqrt{10}}\xi_0(Y_{2,-1}\otimes\delta_r))$$

(4.4.14) REMARK: We have now displayed the vectors $X^{(i)}(r)$, $i = 0,1,2$ each of which transforms according to a vector irreducible unitary representation of SO(3). As mentioned earlier, each $X(r)$ in $H^{(1)}(r)$ is given by the expression

$$X(r) = \alpha_0 X^{(0)}(r) \underset{a}{\oplus} \alpha_1 X^{(1)}(r) \underset{a}{\oplus} \alpha_2 X^{(2)}(r)$$

where $\alpha_i \in \mathbb{C}$, $i = 0,1,2$. Hence, we now know every vector in the Hilbert space $H^{(1)}(r)$, $r > 0$.

By iterating the above mathematical procedure, we readily also display an arbitrary member of $H^{(n)}(r)$, $n \geq 2$. Fortunately, our calculations below require only knowledge of $H^{(0)}(r)$ and $H^{(1)}(r)$.

(4.4.15) REMARK: In what follows, we embark on a further analysis of the scalar random variable $\Phi(r)$ and the three dimensional random vectors $X^{(i)}(r)$, $i = 0,1,2$ given respectively by (4.4.7), (4.4.10), (4.4.11) and (4.4.13). In doing this, we repeatedly use Theorem (4.2.9), which we refer to for notations.

(4.4.16) THEOREM Let $<\Phi(r_2),\ \Phi(r_1)>_{L^2(\Omega,\mathcal{B},\mu)} = B^{(0)}(r_2,r_1)$

Then, we have

$$B^{(0)}(r_2,r_1) = 3|A_1|^2(r_2r_1)^2 \int_0^\infty d\Phi^{(1)}(\lambda)\ \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}}} \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}}} +$$

$$+ \frac{1}{5}(7+\sqrt{2})|A_1|^2(r_2r_1)^2 \int_0^\infty d\Phi^{(2)}(\lambda)\ \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}}} \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}}} .$$

Proof By (4.4.7),

$$\Phi(r) = \xi_0(Y_{10}\otimes\delta_r) + \xi_1(Y_{1,-1}\otimes\delta_r) + \xi_{-1}(Y_{11}\otimes\delta_r)$$

Hence

(4.4.17) $$<\Phi(r_2),\Phi(r_1)>_{L^2(\Omega,\mathcal{B},\mu)} = <\xi_0(Y_{10}\otimes\delta_{r_2}) + \xi_1(Y_{1,-1}\otimes\delta_{r_2}) +$$

$$+ \xi_{-1}(Y_{11}\otimes\delta_{r_2}),\xi_0(Y_{10}\otimes\delta_{r_1}) + \xi_1(Y_{1,-1}\otimes\delta_{r_1}) + \xi_{-1}(Y_{11}\otimes\delta_{r_1})>_{L^2(\Omega)}$$

$$= <\xi_0(Y_{10}\otimes\delta_{r_2}),\xi_0(Y_{10}\otimes\delta_{r_1}) + \xi_1(Y_{1,-1}\otimes\delta_{r_1}) + \xi_{-1}(Y_{11}\otimes\delta_{r_1})>_{L^2(\Omega)} +$$

$$+ <\xi_1(Y_{1,-1}\otimes\delta_{r_2}),\xi_0(Y_{10}\otimes\delta_{r_1}) + \xi_1(Y_{1,-1}\otimes\delta_{r_1}) + \xi_{-1}(Y_{11}\otimes\delta_{r_1})>_{L^2(\Omega)}$$

$$+ <\xi_{-1}Y_{11}\otimes\delta_{r_2}),\xi_0(Y_{10}\otimes\delta_{r_1}) + \xi_1(Y_{1,-1}\otimes\delta_{r_1}) + \xi_{-1}(Y_{11}\otimes\delta_{r_1})>_{L^2(\Omega)}$$

Next, set

$$(r_2 r_1)^2 \int_0^\infty d\Phi^{(i)}(\lambda) \frac{J_{\frac{1}{2}+n}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}}} \cdot \frac{J_{\frac{1}{2}+n'}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}}} = \psi^{(i)}_{nn'}(r_2, r_1) ,$$

$$i = 1,2$$

We may now calculate the various terms on the right hand side of (4.4.17) by invoking Theorem (4.2.9). Thus by (4.2.10) we have

$$\langle \xi_0(Y_{10} \otimes \delta_{r_2}), \xi_0(Y_{10} \otimes \delta_{r_1}) \rangle_{L^2(\Omega, \mathcal{B}, \mu)}$$

$$= \sum_{n=0}^{\infty} \sum_{\ell=-n}^{n} |A_n|^2 \psi^{(1)}_{nn}(r_2, r_1) \int d\Omega Y_{10}(x') \overline{Y}_{n\ell}(x') \int d\Omega \overline{Y}_{10}(y') Y_{n\ell}(y') +$$

$$+ \sum_{n,n'=0}^{\infty} \sum_{\ell=-n}^{n} \sum_{\ell'=-n'}^{n'} A_n \overline{A}_{n'} \psi^{(2)}_{nn'}(r_2, r_1) \int d\Omega \frac{p_0^2}{|p|^2} Y_{n\ell}(p') \overline{Y}_{n'\ell'}(p') \int d\Omega Y_{10}(x') \cdot$$

$$\cdot \overline{Y}_{n\ell}(x') \int d\Omega \overline{Y}_{10}(y') Y_{n'\ell'}(y')$$

$$= |A_1|^2 \psi^{(1)}_{11}(r_2, r_1) + |A_1|^2 \psi^{(2)}_{11}(r_2, r_1) \int d\Omega \frac{p_0^2}{|p|^2} Y_{10}(p') \overline{Y}_{10}(p')$$

By (4.3.7), we have

$$\int d\Omega \frac{p_0^2}{|p|^2} Y_{10}(p') \overline{Y}_{10}(p') = \int d\Omega (\alpha_1^0 Y_{00}(p') + \beta_1^0 Y_{20}(p'))(\alpha_1^0 \overline{Y}_{00}(p') + \beta_1^0 \overline{Y}_{20}(p'))$$

$$= (\alpha_1^0)^2 + (\beta_1^0)^2$$

$$= \frac{1}{3} + \frac{4}{15} = \frac{3}{5}$$

by (4.3.10) and (4.3.11).

Hence

$$\langle \xi_0(Y_{10}\otimes\delta_{r_2}),\xi_0(Y_{10}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)} = |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + \frac{3}{5}|A_1|^2\psi_{11}^{(2)}(r_2,r_1)$$

Similarly, we have

$$\langle \xi_0(Y_{10}\otimes\delta_{r_2}),\ \xi_1(Y_{1,-1}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= |A_1|^2\ \psi_{11}^{(2)}(r_2,r_1)\ \int d\Omega\ \frac{p_0p_1}{|p|^2}\ Y_{10}(p')\overline{Y}_{1,-1}(p')$$

By (4.3.7) and (4.3.14),

$$\int d\Omega\ \frac{p_0p_1}{|p|^2}\ Y_{10}(p')\overline{Y}_{1,-1}(p') = \frac{1}{2}\int d\Omega(\alpha_1^0 Y_{00}(p') + \beta_1^0 Y_{20}(p'))(\gamma_1^{-1}\overline{Y}_{00}(p') + $$

$$+\ \theta_1^{-1}\ \overline{Y}_{20}(p') - \theta_1^1\ \overline{Y}_{2,-2}(p'))$$

$$= \frac{1}{2}\ (\alpha_1^0\gamma_1^{-1} + \beta_1^0\theta_1^{-1})$$

$$= \frac{1}{2}\ (\frac{1}{\sqrt{3}}\cdot\frac{\sqrt{2}}{\sqrt{3}} - \frac{2}{\sqrt{15}}\cdot\frac{\sqrt{2}}{\sqrt{15}})$$

$$= \frac{\sqrt{2}}{10}$$

by (4.3.10), (4.3.11), (4.3.12) and (4.3.13). Hence,

$$\langle \xi_0(Y_{10}\otimes\delta_{r_2}),\xi_1(Y_{1,-1}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)} = \frac{\sqrt{2}}{10}\ |A_1|^2\psi_{11}^{(2)}(r_2,r_1)$$

Also,

$$\langle \xi_0(Y_{10} \otimes \delta_{r_2}),\ \xi_{-1}(Y_{11} \otimes \delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= |A_1|^2\ \psi_{11}^{(2)}(r_2,r_1)\ \int d\Omega\ \frac{p_0 p_{-1}}{|p|^2}\ Y_{10}(p')\overline{Y}_{11}(p')$$

$$= \frac{\sqrt{2}}{10i}\ |A_1|^2\ \psi_{11}^{(2)}(r_2,r_1)$$

since by (4.3.7) and (4.3.15),

$$\int d\Omega\ \frac{p_0 p_{-1}}{|p|^2}\ Y_{10}(p')\ \overline{Y}_{11}(p')$$

$$= \frac{1}{2i} \int d\Omega (\alpha_1^0 Y_{10}(p') + \beta_1^0 Y_{20}(p'))(\gamma_1^{-1}\overline{Y}_{00}(p') + \theta_1^{-1} Y_{20}(p') + \theta_1^1\ Y_{22}(p'))$$

$$= \frac{1}{2i}\ (\alpha_1^0 \gamma_1^{-1} + \beta_1^0 \theta_1^{-1})$$

$$= \frac{\sqrt{2}}{10i}$$

Next

$$\langle \xi_1(Y_{1,-1} \otimes \delta_{r_2}),\ \xi_0(Y_{10} \otimes \delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= |A_1|^2\ \psi_{11}^{(2)}(r_2,r_1) \int d\Omega\ \frac{p_1 p_0}{|p|^2}\ Y_{1,-1}(p')\overline{Y}_{10}(p')$$

$$= \frac{\sqrt{2}}{10}$$

since $\int d\Omega\ \frac{p_1 p_0}{|p|^2}\ Y_{1,-1}(p')\overline{Y}_{10}(p')$ is the complex conjugate of

$\int d\Omega \frac{p_1 p_0}{|p|^2} Y_{10}(p')\overline{Y}_{1,-1}(p')$ which we have computed above.

$$<\xi_1(Y_{1,-1}\otimes\delta_{r_2}), \xi_1(Y_{1,-1}\otimes\delta_{r_1})>_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + |A_1|^2\psi_{11}^{(2)}(r_2,r_1)\int d\Omega \frac{p_1^2}{|p|^2} Y_{1,-1}(p')\overline{Y}_{1,-1}(p')$$

$$= |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + \frac{1}{4}|A_1|^2\psi_{11}^{(2)}(r_2,r_1)\int d\Omega|\gamma_1^{-1}Y_{00}(p') + \theta_1^{-1}Y_{20}(p')-\theta_1^{1}Y_{2,-2}(p')|^2$$

$$= |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + \frac{1}{4}((\gamma_1^{-1})^2 + (\theta_1^{-1})^2 + (\theta_1^{1})^2)|A_1|^2\psi_{11}^{(2)}(r_2,r_1)$$

$$= |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + \frac{2}{5}|A_1|^2\psi_{11}^{(2)}(r_2,r_1)$$

$$<\xi_1(Y_{1,-1}\otimes\delta_{r_2}), \xi_{-1}(Y_{11}\otimes\delta_{r_1})>_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= |A_1|^2\psi_{11}^{(2)}(r_2,r_1)\int d\Omega \frac{p_1 p_{-1}}{|p|^2} Y_{1,-1}(p')\overline{Y}_{11}(p')$$

$$= |A_1|^2\psi_{11}^{(2)}(r_2,r_1)\frac{1}{4i}\int d\Omega\,(\gamma_1^{-1}Y_{00}(p') + \theta_1^{-1}Y_{20}(p')-\theta_1^{1}Y_{2,-2}(p')) \times$$

$$\times (\gamma_1^{-1}\overline{Y}_{00}(p') + \theta_1^{-1}\overline{Y}_{20}(p') + \theta_1^{1}\overline{Y}_{22}(p'))$$

$$= \frac{1}{4i}|A_1|^2((\gamma_1^{-1})^2 + (\theta_1^{-1})^2)\psi_{11}^{(2)}(r_2,r_1)$$

$$= \frac{1}{5i}|A_1|^2\psi_{11}^{(2)}(r_2,r_1)$$

Finally, the remaining three terms in (4.4.17) are given as follows:

$$<\xi_{-1}(Y_{11}\otimes\delta_{r_2}),\ \xi_0(Y_{10}\otimes\delta_{r_1})>_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= |A_1|^2\ \psi_{11}^{(2)}(r_2,r_1)\ \int d\Omega\ \frac{p_{-1}p_0}{|p|^2}\ Y_{11}(p')\overline{Y}_{10}$$

$$= -\frac{\sqrt{2}}{10i}\ |A_1|^2\ \psi_{11}^{(2)}(r_2,r_1)$$

$$<\xi_{-1}(Y_{11}\otimes\delta_{r_2}),\ \xi_1(Y_{1,-1}\otimes\delta_{r_1}\}>_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= |A_1|^2\ \psi_{11}^{(2)}(r_2,r_1)\ \int d\Omega\ \frac{p_{-1}p_1}{|p|^2}\ Y_{11}(p')\ \overline{Y}_{1,-1}(p')$$

$$= -\frac{1}{5i}\ |A_1|^2\ \psi_{11}^{(2)}\ (r_2,r_1)$$

and

$$<\xi_{-1}(Y_{11}\otimes\delta_{r_2}),\ \xi_{-1}(Y_{11}\otimes\delta_{r_1})>_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= |A_1|^2\ \psi_{11}^{(1)}(r_2,r_1) + |A_1|^2\ \psi_{11}^{(2)}(r_2,r_1)\ \int d\Omega\ \frac{(p_{-1})^2}{|p|^2}\ Y_{11}(p')\overline{Y}_{11}(p')$$

$$= |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + \frac{2}{5}\ |A_1|^2\ \psi_{11}^{(2)}(r_2,r_1)$$

since by (4.3.15), (4.3.12) and (4.3.13),

$$\int d\Omega\ \frac{(p_{-1})^2}{|p|^2}\ Y_{11}(p')\overline{Y}_{11}(p')$$

$$= \frac{1}{4}\int d\Omega\ |\gamma_1^{-1}Y_{00}(p') + \theta_1^{-1}Y_{20}(p') + \theta_1^{1}Y_{22}(p')|^2$$

$$= \frac{1}{4}\ ((\gamma_1^{-1})^2 + (\theta_1^{-1})^2 + (\theta_1^{1})^2)$$

$$= \frac{2}{5}$$

As a consequence of all the preceding computations, we have finally that

$$\langle \Phi(r_2), \Phi(r_1) \rangle_{L^2(\Omega, \mathcal{B}, \mu)}$$

$$= B^{(0)}(r_2, r_1)$$

$$= 3|A_1|^2 \psi_{11}^{(1)}(r_2, r_1) + (\tfrac{3}{5} + \tfrac{2\sqrt{2}}{10} + \tfrac{4}{5}) |A_1|^2 \psi_{11}^{(2)}(r_2, r_1)$$

$$= 3|A_1|^2 \psi_{11}^{(1)}(r_2, r_1) + \tfrac{1}{5}(7 + \sqrt{2})|A_1|^2 \psi_{11}^{(2)}(r_2, r_1)$$

$$= 3|A_1|^2 (r_2, r_1)^2 \int_0^\infty d\Phi^{(1)}(\lambda) \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}}} \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}}} +$$

$$+ \tfrac{1}{5}(7 + \sqrt{2})|A_1|^2 (r_2 r_1)^2 \int_0^\infty d\Phi^{(2)}(\lambda) \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}}} \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}}},$$

as we asserted. Hence, we are through. □

(4.4.18): REMARK: We next address ourselves to the analysis of the nine dimensional random vector

$$X(r) = \alpha_0 X^{(0)}(r) \underset{a}{\oplus} \alpha_1 X^{(1)}(r) \underset{a}{\oplus} \alpha_2 X^{(2)}(r), \quad \alpha_i \in C,\ i = 0,1,2$$

which belongs to $H^{(1)}(r)$, $r > 0$. The next result indicates that we may express $H^{(1)}(r)$ as the orthogonal direct sum of two Hilbert spaces. The one is a Hilbert space of three dimensional random vectors and the other is a Hilbert space of six dimensional random vectors.

(4.4.19) THEOREM Let $X^{(0)}(r)$, $X^{(1)}(r)$, and $X^{(2)}(r)$ be as exhibited in (4.4.10), (4.4.11) and (4.4.13) respectively. Then the following assertions are true:

(a) $X^{(0)}(r)$ and $X^{(1)}(r)$ are stochastically independent;

(b) $X^{(1)}(r)$ and $X^{(2)}(r)$ are stochastically independent;

(c) $X^{(0)}(r)$ and $X^{(2)}(r)$ are stochastically dependent.

Proof: (a) $X^{(0)}(r)$ is stochastically independent of $X^{(1)}(r)$. To justify this assertion, we must verify that

$$<X_i^{(0)}(r_2),\ X_j^{(1)}(r_1)>_{L^2(\Omega,\mathcal{B},\mu)} = 0, \quad i,j = -1,0,1\ .$$

Thus, consider $<X_{-1}^{(0)}(r_2),\ X_{-1}^{(1)}(r_1)>_{L^2(\Omega,\mathcal{B},\mu)}$. From (4.4.10) and (4.4.11), we have

$$<X_{-1}^{(0)}(r_2), X_{-1}^{(1)}(r_1)>_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= \frac{1}{\sqrt{2}} <\xi_1(Y_{00}\otimes\delta_{r_2}),\ \xi_0(Y_{11}\otimes\delta_{r_1}) - \xi_1(Y_{10}\otimes\delta_{r_1})>_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= \frac{1}{\sqrt{2}} <\xi_1(Y_{00}\otimes\delta_{r_2}),\xi_0(Y_{11}\otimes\delta_{r_1})>_{L^2(\Omega,\mathcal{B},\mu)} - \frac{1}{\sqrt{2}} <\xi_1(Y_{00}\otimes\delta_{r_2}),\xi_1(Y_{10}\otimes\delta_{r_1})>_{L^2(\Omega,\mathcal{B},\mu)}$$

Now

$$<\xi_1(Y_{00}\otimes\delta_{r_2}),\ \xi_0(Y_{11}\otimes\delta_{r_1})<_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= A_0\bar{A}_1\psi_{01}^{(2)}(r_2,r_1) \int d\Omega\ \frac{p_1 p_0}{|p|^2}\ Y_{00}(p')\bar{Y}_{11}(p')$$

$$= \frac{1}{2} A_0\bar{A}_1\psi_{01}^{(2)}(r_2,r_1) \int d\Omega\ \theta_0^0\beta_1^1(Y_{11}(p') - Y_{1,-1}(p'))\bar{Y}_{21}(p')$$

$$= 0$$

Also, we have

$$<\xi_1(Y_{00}\otimes\delta_{r_2}),\ \xi_1(Y_{10}\otimes\delta_{r_1})>_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= A_0\bar{A}_1\psi_{01}^{(2)}(r_2,r_1)\int d\Omega\ \frac{p_1^2}{|p|^2}\ Y_{00}(p')\bar{Y}_{10}(p')$$

$$= \frac{1}{4}\ A_0\bar{A}_1\psi_{01}^{(2)}(r_2,r_1)\int d\Omega\ \theta_0^0\theta_1^0(Y_{11}(p') - Y_{1,-1}(p'))(\bar{Y}_{21}(p') - \bar{Y}_{2,-1}(p'))$$

$$= 0$$

Hence

$$<X_{-1}^{(0)}(r_2),\ X_{-1}^{(1)}(r_1)>_{L^2(\Omega,\mathcal{B},\mu)} = 0\ .$$

Indeed, an analogous computation to the preceding indicates that, more generally, we have

$$<X_i^{(0)}(r_2), X_i^{(1)}(r_i)>_{L^2(\Omega,\mathcal{B},\mu)} = 0,\quad i = -1,0,1$$

It only remains now to consider the objects

$$<X_i^{(0)}(r_2),\ X_j^{(1)}(r_1)>_{L^2(\Omega,\mathcal{B},\mu)}\ ,\ i,j = -1,0,1,\quad i \neq j$$

Typically, we have

$$<X_{-1}^{(0)}(r_2),\ X_0^{(1)}(r_1)>_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= \frac{1}{\sqrt{2}}<\xi_1(Y_{00}\otimes\delta_{r_2}),\ \xi_1(Y_{1,-1}\otimes\delta_{r_1}) - \xi_{-1}(Y_{11}\otimes\delta_{r_1})>_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= \frac{1}{\sqrt{2}}<\xi_1(Y_{00}\otimes\delta_{r_2}),\ \xi_1(Y_{1,-1}\otimes\delta_{r_1})>_{L^2(\Omega,\mathcal{B},\mu)} - \frac{1}{\sqrt{2}}<\xi_1(Y_{00}\otimes\delta_{r_2}),\xi_{-1}(Y_{11}\otimes\delta_{r_1})>_{L^2(\Omega,\mathcal{B},\mu)}$$

Now

$$\langle \xi_1(Y_{00} \otimes \delta_{r_2}), \xi_1(Y_{1,-1} \otimes \delta_{r_1}) \rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= A_0\bar{A}_1\psi_{11}^{(2)}(r_2,r_1) \int d\Omega \, \frac{p_1^2}{|p|^2} Y_{00}(p')\bar{Y}_{1,-1}(p')$$

$$= \frac{1}{4} A_0\bar{A}_1\psi_{11}^{(2)}(r_2,r_1) \int d\Omega \, \theta_0^0 \, (Y_{11}(p') - Y_{1,-1}(p'))(\gamma_1^{-1}\bar{Y}_{00}(p') + \theta_1^{-1}\bar{Y}_{20}(p') - $$

$$- \theta_1^1\bar{Y}_{2,-2}(p'))$$

$$= 0$$

Also, we have

$$\langle \xi_1(Y_{00} \otimes \delta_{r_2}), \xi_{-1}(Y_{11} \otimes \delta_{r_1}) \rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= A_0\bar{A}_1\psi_{11}^{(2)}(r_2,r_1) \int d\Omega \, \frac{p_1p_{-1}}{|p|^2} Y_{00}(p')\bar{Y}_{11}(p')$$

$$= \frac{-1}{4i} A_0\bar{A}_1\psi_{11}^{(2)}(r_2,r_1) \int d\Omega \, \theta_0^0(Y_{11}(p') - Y_{1,-1}(p'))(\gamma_1^{-1}Y_{00}(p') + \theta_1^{-1}\bar{Y}_{20}(p') + $$

$$+ \theta_1^1\bar{Y}_{22}(p'))$$

$$= 0$$

Hence, again, we have

$$\langle X_{-1}^{(0)}(r_2), X_0^{(1)}(r_1) \rangle_{L^2(\Omega,\mathcal{B},\mu)} = 0$$

Indeed, more generally, we have that

$$\langle X_i^{(0)}(r_2), X_j^{(1)}(r_1) \rangle_{L^2(\Omega,\mathcal{B},\mu)} = 0, \; i,j = -1,0,1, \quad i \neq j \, .$$

By combining this latter information with the one supplied above, we find that we indeed have

$$\langle X_i^{(0)}(r_2),\ X_j^{(1)}(r_1)\rangle_{L^2(\Omega,\mathcal{B},\mu)} = 0,\ i,j = -1,0,1.$$

Hence, the random vectors $X^{(0)}(r)$ and $X^{(1)}(r)$, which we recall are Gaussian, are indeed stochastically independent as we asserted.

(b) We claim that the random vectors $X^{(1)}(r)$ and $X^{(2)}(r)$ are stochastically independent. Again, our claim is justified by carrying out computations similar to those undertaken in (a) above, We make one such computation now.

Consider the correlation function $\langle X_{-1}^{(1)}(r_2),\ X_{-1}^{(2)}(r_1)\rangle_{L^2(\Omega,\mathcal{B},\mu)}$

Then, we have

$$(4.4.20)\quad \langle X_{-1}^{(1)}(r_2),\ X_{-1}^{(2)}(r_1)\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= \frac{1}{\sqrt{20}} \langle \xi_0(Y_{11}\otimes\delta_{r_2}) - \xi_1(Y_{10}\otimes\delta_{r_2}),\ \sqrt{6}\ \xi_{-1}(Y_{22}\otimes\delta_{r_1}) + \xi_1(Y_{20}\otimes\delta_{r_1}) - \sqrt{3}\ \xi_0(Y_{21}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= \frac{1}{\sqrt{20}} \langle \xi_0(Y_{11}\otimes\delta_{r_2}),\ \sqrt{6}\ \xi_{-1}(Y_{22}\otimes\delta_{r_1}) + \xi_1(Y_{20}\otimes\delta_{r_1}) - \sqrt{3}\ \xi_0(Y_{21}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)} -$$

$$- \frac{1}{\sqrt{20}} \langle \xi_1(Y_{10}\otimes\delta_{r_2}),\sqrt{6}\ \xi_{-1}(Y_{22}\otimes\delta_{r_1}) + \xi_1(Y_{20}\otimes\delta_{r_1}) - \sqrt{3}\ \xi_0(Y_{21}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

Consider now, for example, the term $\frac{\sqrt{3}}{\sqrt{10}} \langle \xi_0(Y_{11}\otimes\delta_{r_2}),\ \xi_{-1}(Y_{22}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)}$

which occurs, among others, on the right hand side of (4.4.20). Then, we have

$$<\xi_0(Y_{11}\otimes\delta_{r_2}),\ \xi_{-1}(Y_{22}\otimes\delta_{r_1})>_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= A_1\bar{A}_2\ \psi_{12}^{(2)}(r_2,r_1)\ \int d\Omega\ \frac{p_0 p_{-1}}{|p|^2}\ Y_{11}(p')\bar{Y}_{22}(p')$$

$$= A_1\bar{A}_2\ \psi_{12}^{(2)}(r_2,r_1)\ (\frac{-1}{2i})\ \int d\Omega\ \beta_1^1 Y_{21}(p')(\theta_2^2\bar{Y}_{33}(p') + \gamma_2^{-2}\bar{Y}_{11}(p') + \theta_2^{-2}\bar{Y}_{31}(p')$$

$$= 0 .$$

Similarly, we readily verify that, indeed, every term occurring on the right hand side of (4.4.20) vanishes. Hence, we have

$$<X_{-1}^{(1)}(r_2),\ X_{-1}^{(2)}(r_1)>_{L^2(\Omega,\mathcal{B},\mu)} = 0$$

More generally, one finds, through calculations analogous to the preceding, that

$$<X_i^{(1)}(r_2),\ X_j^{(2)}(r_1)>_{L^2(\Omega,\mathcal{B},\mu)} = 0, \quad i,j = -1,0,1$$

Hence, the random vectors $X^{(1)}(r)$ and $X^{(2)}(r)$ are orthogonal, and since they are Gaussian, they are consequently stochastically independent, as we had claimed.

(c) Finally, we must consider the random vectors $X^{(0)}(r)$ and $X^{(2)}(r)$. For these random vectors, we are claiming stochastic dependence. Hence, we must demonstrate that there is some correlation between $X^{(0)}(r)$ and $X^{(2)}(r)$, i.e. the correlation functions $<X_i^{(0)}(r),\ X_j^{(2)}(r)>_{L^2(\Omega,\mathcal{B},\mu)}$ $i,j = -1,0,1$ are not all zero. That this is, indeed, the case follows by considering, for example, the correlation function $<X_0^{(0)}(r_2),\ X_0^{(2)}(r_1)>_{L^2(\Omega,\mathcal{B},\mu)}$. Then, we have

$$-\langle X_0^{(0)}(r_2),\ X_0^{(2)}(r_1)\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= \frac{1}{\sqrt{10}} \langle \xi_0(Y_{00}\otimes\delta_{r_2}),\ 2\xi_0(Y_{20}\otimes\delta_{r_1}) - \sqrt{3}\,\xi_1(Y_{2,-1}\otimes\delta_{r_1}) - \sqrt{3}\,\xi_{-1}(Y_{21}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= \frac{2}{\sqrt{10}} A_0\bar{A}_2\,\psi_{02}^{(2)}(r_2,r_1) \int d\Omega\,\frac{p_0^2}{|p|^2} Y_{00}(p')\bar{Y}_{20}(p') -$$

$$- \frac{\sqrt{3}}{\sqrt{10}} A_0\bar{A}_2\,\psi_{02}^{(2)}(r_2,r_1) \int d\Omega\,\frac{p_0p_1}{|p|^2} Y_{00}(p')\bar{Y}_{2,-1}(p') -$$

$$- \frac{\sqrt{3}}{\sqrt{10}} A_0\bar{A}_2\,\psi_{02}^{(2)}(r_2,r_1) \int d\Omega\,\frac{p_0p_{-1}}{|p|^2} Y_{00}(p')\bar{Y}_{21}(p')\ .$$

But

$$\int d\Omega\,\frac{p_0p_1}{|p|^2} Y_{00}(p')\bar{Y}_{2,-1}(p')$$

$$= \frac{1}{2}\beta_0^0 \int d\Omega\, Y_{10}(p')(\gamma_2^{-1}\bar{Y}_{10}(p') + \theta_2^{-1}\bar{Y}_{30}(p') - \theta_2^{1}\bar{Y}_{3,-2}(p'))$$

$$= \frac{1}{2}\beta_0^0\gamma_2^{-1} = \frac{1}{\sqrt{30}}$$

$$\int d\Omega\,\frac{p_0^2}{|p|^2} Y_{00}(p')\bar{Y}_{20}(p')$$

$$= \beta_0^0 \int d\Omega\, Y_{10}(p')(\alpha_2^0\bar{Y}_{10}(p') + \beta_2^0\bar{Y}_{30}(p'))$$

$$= \beta_0^0\alpha_2^0 = \frac{2}{\sqrt{45}}$$

and

$$\int d\Omega\,\frac{p_0p_{-1}}{|p|^2} Y_{00}(p')\bar{Y}_{21}(p')$$

$$= \frac{-1}{2i} \int d\Omega\ \beta_0^0 Y_{10}(p')(\theta_2^1 \overline{Y}_{32}(p') + \gamma_2^{-1} \overline{Y}_{10}(p') + \theta_2^{-1} \overline{Y}_{30}(p'))$$

$$= -\frac{1}{2i} \beta_0^0 \gamma_2^{-1} = \frac{i}{\sqrt{30}}$$

Hence

$$-\langle X_0^{(0)}(r_2), X_0^{(2)}(r_1)\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= \left(\frac{4}{\sqrt{450}} - \frac{\sqrt{3}}{\sqrt{300}} - \frac{\sqrt{3}i}{\sqrt{300}}\right) A_0 \overline{A}_2\ \psi_{02}^{(2)}(r_2, r_1)$$

$$= \frac{1}{30}(4\sqrt{2} - 3 - 3i) A_0 \overline{A}_2 \psi_{02}^{(2)}(r_2, r_1)$$

$$\neq 0$$

Thus $\langle X_0^{(0)}(r_2), X_0^{(2)}(r_1)\rangle_{L^2(\Omega,\mathcal{B},\mu)} \neq 0$, and the random variables $X_0^{(0)}(r_2)$ and $X_0^{(2)}(r_1)$ are stochastically correlated.

More evidence of stochastic correlation between components of $X^{(0)}(r)$ and $X^{(2)}(r)$ may now also be obtained by carrying out further computations of the last kind.

Hence, the Gaussian random vectors $X^{(0)}(r)$ and $X^{(2)}(r)$ are indeed stochastically dependent as we claimed. This completes proof of the theorem. □

(4.4.21) REMARK: Theorem (4.4.19) is significant because it reveals that the Hilbert space $H^{(1)}(r)$ of nine dimensional random vectors is, indeed, an orthogonal direct sum

$$H^{(1)}(r) = H_{(1)}(r) \oplus H_{(2)}(r)$$

of two Hilbert spaces $H_{(1)}(r)$ and $H_{(2)}(r)$. $H_{(1)}(r)$ consists of constant multiples of the three dimensional random vector $X^{(1)}(r)$ given by

(4.4.11), while $H_{(2)}(r)$ is a Hilbert space of six dimensional random vectors. Each $Y(r)$ in $H_{(2)}(r)$ is expressible in the form $Y(r) = \alpha_0 X^{(0)}(r) \underset{a}{\oplus} \alpha_1 X^{(2)}(r)$, $\alpha_0, \alpha_1 \in \mathbb{C}$, where $X^{(i)}(r)$, $i = 0,2$ are given by (4.4.10) and (4.4.13) respectively and where, as usual, $\underset{a}{\oplus}$ denotes algebraic direct sum.

In view of the preceding explanation, it now follows that if $P_{N(r)}$ is the orthogonal projection onto the Hilbert space $N(r)$, where $N(r) = H^{(1)}(r)$ or $H_{(1)}(r)$, then

$$(4.4.22)\quad P_{H^{(1)}(r_1)} X^{(1)}(r_2) = P_{H_{(1)}(r_1) \oplus H_{(2)}(r_1)} X^{(1)}(r_2)$$

$$= P_{H_{(1)}(r_1)} X^{(1)}(r_2)$$

We shall invoke (4.4.22) in the sequel.

We wish next to give the structure of the matrix of correlation functions of the random vector $X^{(1)}(r)$.

(4.4.23) THEOREM Let $B(r_2,r_1)$ be the matrix of correlation functions of the random vector $X^{(1)}(r)$ whose components are given by (4.4.11). Then, we have

$$B_{ij}(r_2,r_1) = 0, \qquad i,j = -1,0,1, \quad i \neq j .$$

$$B_{-1-1}(r_2,r_1) = |A_1|^2 (r_2 r_1)^2 \int_0^\infty d\Phi^{(1)}(\lambda) \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}}} \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}}} +$$

$$+ \frac{1}{10}(2+\sqrt{2}) |A_1|^2 (r_2 r_1)^2 \int_0^\infty d\Phi^{(2)}(\lambda) \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}}} \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}}}$$

$$B_{00}(r_2,r_1) = |A_1|^2(r_2r_1)^2 \int_0^\infty d\Phi^{(1)}(\lambda)\, \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}}} \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}}} +$$

$$+ \frac{2}{5}\, |A_1|^2(r_2r_1)^2 \int_0^\infty d\Phi^{(2)}(\lambda)\, \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}}} \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}}}$$

$$B_{11}(r_2,r_1) = |A_1|^2(r_2r_1)^2 \int_0^\infty d\Phi^{(1)}(\lambda)\, \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}}} \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}}} +$$

$$+ \frac{1}{5}\, |A_1|^2(r_2r_1)^2 \int_0^\infty d\Phi^{(2)}(\lambda)\, \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}}} \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}}} .$$

Proof: We begin by demonstrating that the off-diagonal elements of the matrix $B(r_2,r_1)$ vanish. Thus, consider, for example, the entry $B_{-10}(r_2,r_1) = \langle X_{-1}^{(1)}(r_2),\, X_0^{(1)}(r_1)\rangle_{L^2(\Omega,\mathcal{B},\mu)}$. Then by (4.4.11), we have

(4.4.24) $$B_{-10}(r_2,r_1) = \tfrac{1}{2}\langle \xi_0(Y_{11}\otimes\delta_{r_2}) - \xi_1(Y_{10}\otimes\delta_{r_2}),\, \xi_1(Y_{1,-1}\otimes\delta_{r_1}) - \xi_{-1}(Y_{11}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= \tfrac{1}{2}\langle \xi_0(Y_{11}\otimes\delta_{r_2}),\, \xi_1(Y_{1,-1}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)} - \tfrac{1}{2}\langle \xi_0(Y_{11}\otimes\delta_{r_2}),\, \xi_{-1}(Y_{11}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

$$- \tfrac{1}{2}\langle \xi_1(Y_{10}\otimes\delta_{r_2}),\, \xi_1(Y_{1,-1}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)} + \tfrac{1}{2}\langle \xi_1(Y_{10}\otimes\delta_{r_2}),\, \xi_{-1}(Y_{11}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

We may now compute the terms occurring on the right hand side of (4.4.24) one by one. Thus, we have $\langle \xi_0(Y_{11}\otimes\delta_{r_2}),\, \xi_1(Y_{1,-1}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)}$

$$= |A_1|^2 \psi_{11}^{(2)}(r_2,r_1) \int d\Omega \frac{p_0 p_1}{|p|^2} Y_{11}(p')\overline{Y}_{1,-1}(p')$$

$$= \frac{1}{2} |A_1|^2 \psi_{11}^{(2)}(r_2,r_1) \int d\Omega\, \beta_1^1 Y_{21}(p')(\gamma_1^{-1}\overline{Y}_{00} + \theta_1^{-1}\overline{Y}_{20}(p') - \theta_1^1\overline{Y}_{2,-2}(p'))$$

(by (4.3.7) and (4.3.14))

$= 0$

$$<\xi_0(Y_{11}\otimes\delta_{r_2}), \xi_{-1}(Y_{11}\otimes\delta_{r_1})>_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= |A_1|^2 \psi_{11}^{(2)}(r_2,r_1) \int d\Omega \frac{p_0 p_{-1}}{|p|^2} Y_{11}(p')\overline{Y}_{11}(p')$$

$$= \frac{1}{2i} |A_1|^2 \psi_{11}^{(2)}(r_2,r_1) \int d\Omega\, \beta_1^1 Y_{21}(\gamma_1^{-1}\overline{Y}_{00}(p') + \theta_1^{-1}\overline{Y}_{20}(p') + \theta_1^1\overline{Y}_{22}(p'))$$

(by (4.3.7) and (4.3.15)

$= 0$

$$<\xi_1(Y_{10}\otimes\delta_{r_2}), \xi_1(Y_{1,-1}\otimes\delta_{r_1})>_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= |A_1|^2 \psi_{11}^{(2)}(r_2,r_1) \int d\Omega \frac{p_1^2}{|p|^2} Y_{10}(p')\, \overline{Y}_{1,-1}(p')$$

$$= \frac{1}{4}|A_1|^2\psi_{11}^{(2)}(r_2,r_1) \int d\Omega\, (\theta_1^0 Y_{21}(p') - \theta_1^0 Y_{2,-1}(p'))(\gamma_1^{-1}\overline{Y}_{00}(p') + \theta_1^{-1}\overline{Y}_{20}(p') -$$

$$- \theta_1^1\, \overline{Y}_{2,-2}(p'))$$

(by (4.3.14)

$= 0$

Finally, we have

$$<\xi_1(Y_{10}\otimes\delta_{r_2}),\ \xi_{-1}(Y_{11}\otimes\delta_{r_1})<_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= |A_1|^2\ \psi_{11}^{(2)}(r_2,r_1)\ \int d\Omega\ \frac{p_1 p_{-1}}{|p|^2}\ Y_{10}(p')\overline{Y}_{11}(p')$$

$$= \frac{(-1)}{4i}\ |A_1|^2\psi_{11}^{(2)}(r_2,r_1)\ \int d\Omega\ (\theta_1^0 Y_{21}(p') - \theta_1^0 Y_{2,-1}(p'))(\gamma_1^{-1}\overline{Y}_{00}(p')+\theta_1^{-1}\overline{Y}_{20}(p') +$$

$$+ \theta_1^1\ \overline{Y}_{22}(p'))$$

(by (4.3.14) and (4.3.15))

$= 0$

In all the foregoing calculations,

$$(r_2,r_1) \to \psi_{nn'}^{(i)}(r_2,r_1) = (r_2 r_1)^2 \int_0^\infty d\Phi^{(i)}(\lambda)\ \frac{J_{\frac{1}{2}+n}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}}} \cdot \frac{J_{\frac{1}{2}+n'}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}}}$$

$i = 1,2$, as in earlier computations.

Combining all the above results, we now have, indeed, that $B_{-10}(r_2,r_1) = 0$.

In an analogous fashion, we readily verify also that more generally, $B_{ij}(r_2,r_1) = 0$, $i,j = -1,0,1$, $i \neq j$.

To conclude proof of the theorem, it only now remains for us to consider the diagonal entries of the matrix $B(r_2,r_1)$. First, consider

$$B_{-1-1}(r_2,r_1) = <X_{-1}^{(1)}(r_2),\ X_{-1}^{(1)}(r_1)>_{L^2(\Omega,\mathcal{B},\mu)}$$

Then, by (4.4.11),

$$(4.4.25)\quad B_{-1-1}(r_2,r_1) = \tfrac{1}{2}\langle \xi_0(Y_{11}\otimes\delta_{r_2})-\xi_1(Y_{10}\otimes\delta_{r_2}),\xi_0(Y_{11}\otimes\delta_{r_1})-\xi_1(Y_{10}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= \tfrac{1}{2}\langle\xi_0(Y_{11}\otimes\delta_{r_2}),\xi_0(Y_{11}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)} - \tfrac{1}{2}\langle\xi_0(Y_{11}\otimes\delta_{r_2}),\xi_1(Y_{10}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

$$- \tfrac{1}{2}\langle\xi_1(Y_{10}\otimes\delta_{r_2}),\xi_0(Y_{11}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)} + \tfrac{1}{2}\langle\xi_1(Y_{10}\otimes\delta_{r_2}),\xi_1(Y_{10}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

Consider the various terms on the right hand side of (4.4.25) one by one. Then, we have

$$\langle\xi_0(Y_{11}\otimes\delta_{r_2}),\xi_0(Y_{11}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + |A_1|^2\psi_{11}^{(2)}(r_2,r_1)\int d\Omega\,\frac{p_0^2}{|p|^2}\,Y_{11}(p')\bar{Y}_{11}(p')$$

$$= |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + |A_1|^2\psi_{11}^{(2)}(r_2,r_1)\int d\Omega(\beta_1^1)^2\,Y_{21}(p')\bar{Y}_{21}(p'),\quad \text{by (4.3.7)}$$

$$= |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + (\beta_1^1)^2|A_1|^2\psi_{11}^{(2)}(r_2,r_1)\ .$$

Thus,

$$\langle\xi_0(Y_{11}\otimes\delta_{r_2}),\xi_0(Y_{11}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)} = |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + \tfrac{1}{5}|A_1|^2\psi_{11}^{(2)}(r_2,r_1)$$

$$\langle\xi_0(Y_{11}\otimes\delta_{r_2}),\xi_1(Y_{10}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= |A_1|^2\psi_{11}^{(2)}(r_2,r_1)\int d\Omega\frac{p_0p_1}{|p|^2}\,Y_{11}(p')\bar{Y}_{10}(p')$$

$$= \tfrac{1}{2}|A_1|^2\psi_{11}^{(2)}(r_2,r_1)\int d\Omega\beta_1^1 Y_{21}(p')(\theta_1^0\,\bar{Y}_{21}(p') - \theta_1^0\,\bar{Y}_{2,-1}(p')$$

(by (4.3.7) and (4.3.14))

$$= \frac{1}{2}\,\beta_1^1\theta_1^0\,|A_1|^2\psi_{11}^{(2)}(r_2,r_1)$$

$$= -\,\frac{1}{2}\,\frac{\sqrt{2}}{5}\,|A_1|^2\,\psi_{11}^{(2)}(r_2,r_1)$$

$$<\xi_1(Y_{10}\otimes\delta_{r_2}),\ \xi_0(Y_{11}\otimes\delta_{r_1})>_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= |A_1|^2\,\psi_{11}^{(2)}(r_2,r_1)\int d\Omega\,\frac{p_1p_0}{|p|^2}\,Y_{10}(p')\overline{Y}_{11}(p')$$

$$= -\,\frac{1}{2}\,\frac{\sqrt{2}}{5}\,|A_1|^2\psi_{11}^{(2)}(r_2,r_1)$$

Finally,

$$<\xi_1(Y_{10}\otimes\delta_{r_2}),\xi_1(Y_{10}\otimes\delta_{r_1})>_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + |A_1|^2\psi_{11}^{(2)}(r_2,r_1)\int d\Omega\,\frac{p_1^2}{|p|^2}\,Y_{10}(p')\overline{Y}_{10}(p')$$

$$= |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + \frac{1}{4}|A_1|^2\psi_{11}^{(2)}(r_2,r_1)\int d\Omega\,(\theta_1^0Y_{21}(p') - \theta_1^0Y_{2,-1}(p'))\,\cdot$$
$$\cdot\,(\theta_1^0\overline{Y}_{21}(p') - \theta_1^0\overline{Y}_{2,-1}(p'))$$

(by 4.3.14)

$$= |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + \frac{1}{2}|A_1|^2(\theta_1^0)^2\psi_{11}^{(2)}(r_2,r_1)$$

$$= |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + \frac{1}{5}\,|A_1|^2\psi_{11}^{(2)}(r_2,r_1)$$

Hence, by combining all the preceding results, we have

$$B_{-1-1}(r_2,r_1) = |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + \frac{1}{10}(2+\sqrt{2})|A_1|^2\psi_{11}^{(2)}(r_2,r_1)$$

$$= |A_1|^2(r_2r_1)^2 \int_0^\infty d\Phi^{(1)}(\lambda)\, \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}}} \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}}} \; +$$

$$+ \frac{1}{10}(2+\sqrt{2})|A_1|^2(r_2r_1)^2 \int_0^\infty d\Phi^{(2)}(\lambda)\, \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}}} \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}}} \,.$$

Next, consider the entry $B_{00}(r_2,r_1)$. Then by (4.4.11), we have

$$(4.4.26)\quad B_{00}(r_2,r_1) = \frac{1}{2}\langle \xi_1(Y_{1,-1}\otimes\delta_{r_2}) - \xi_{-1}(Y_{11}\otimes\delta_{r_2}), \xi_1(Y_{1,-1}\otimes\delta_{r_1}) - \xi_{-1}(Y_{11}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= \frac{1}{2}\langle \xi_1(Y_{1,-1}\otimes\delta_{r_2}), \xi_1(Y_{1,-1}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)} - \frac{1}{2}\langle \xi_1(Y_{1,-1}\otimes\delta_{r_2}), \xi_{-1}(Y_{11}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

$$- \frac{1}{2}\langle \xi_{-1}(Y_{11}\otimes\delta_{r_2}), \xi_1(Y_{1,-1}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)} + \frac{1}{2}\langle \xi_{-1}(Y_{11}\otimes\delta_{r_2}), \xi_{-1}(Y_{11}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

Again, we embark on computing the various terms on the right hand side of (4.4.26) one by one. Thus, we have

$$\langle \xi_1(Y_{1,-1}\otimes\delta_{r_2}), \xi_1(Y_{1,-1}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + |A_1|^2\psi_{11}^{(2)}(r_2,r_1) \int d\Omega\, \frac{p_1^2}{|p|^2}\, Y_{1,-1}(p')\overline{Y}_{1,-1}(p')$$

$$= |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + \frac{1}{4}|A_1|^2\psi_{11}^{(2)}(r_2,r_1) \int d\Omega\, |\gamma_1^{-1}Y_{00}(p') + \theta_1^{-1}Y_{20}(p') - \theta_1^{1}Y_{2,-2}(p')|^2$$

(by (4.3.14))

$$= |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + \tfrac{1}{4}(\gamma_1^{-1})^2 + (\theta_1^{-1})^2 + (\theta_1^{1})^2)|A_1|^2\psi_{11}^{(2)}(r_2,r_1)$$

$$= |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + \tfrac{2}{5}|A_1|^2\psi_{11}^{(2)}(r_2,r_1)$$

$$<\xi_1(Y_{1,-1}\ \delta_{r_2}),\xi_{-1}(Y_{11}\ \ \delta_{r_1})>_{L^2(\Omega,\ \ ,\mu)}$$

$$= |A_1|^2\psi_{11}^{(2)}(r_2,r_1)\int d\Omega\,\frac{p_1p_{-1}}{|p|^2}\,Y_{1,-1}(p')\,\overline{Y}_{11}(p')$$

$$= \frac{-1}{4i}|A_i|^2\psi_{11}^{(2)}(r_2,r_1)\int d\Omega(\gamma_1^{-1}Y_{00}(p')+\theta_1^{-1}Y_{20}(p') - \theta_1^{1}Y_{2,-2}(p'))(\gamma_1^{-1}\overline{Y}_{00}(p') + $$
$$+ \ {}_1^{-1}\overline{Y}_{20}(p') + {}_1^{1}\overline{Y}_{22}(p')$$

(by (4.3.14) and (4.3.15))

$$= -\frac{1}{4i}|A_1|^2\ ((\gamma_1^{-1})^2 + (\theta_1^{-1})^2 + (\theta_1^{1})^2)\psi_{11}^{(2)}(r_2,r_1)$$

$$= -\frac{2}{5i}|A_1|^2\,\psi_{11}^{(2)}(r_2,r_1)$$

Also,

$$<\xi_{-1}(Y_{11}\otimes\delta_{r_2}),\ \xi_1(Y_{1,-1}\otimes\delta_{r_1})>_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= |A_1|^2\,\psi_{11}^{(2)}(r_2,r_1)\int d\Omega\,\frac{p_{-1}p_1}{|p|^2}\,Y_{11}(p')\,\overline{Y}_{1,-1}(p')$$

$$= \frac{2}{5i}|A_1|^2\,\psi_{11}^{(2)}(r_2,r_1)$$

Finally,

$$<\xi_{-1}(Y_{11}\otimes\delta_{r_2}),\xi_{-1}(Y_{11}\otimes\delta_{r_1})>_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= |A_1|^2 \psi_{11}^{(1)}(r_2,r_1) + |A_1|^2\psi_{11}^{(2)}(r_2,r_1) \int d\Omega \frac{p_{-1}^2}{|p|^2} Y_{11}(p')\overline{Y}_{11}(p')$$

$$= |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + \frac{1}{4} |A_1|^2\psi_{11}^{(2)}(r_2,r_1) \int d\Omega|\gamma_1^{-1}Y_{00}(p')+\theta_1^{-1}Y_{20}(p')+\theta_1^{1}Y_{22}(p')|^2$$

(by (4.3.15))

$$= |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + \frac{1}{4}((\gamma_1^{-1})^2+(\theta_1^{-1}) + (\theta_1^{1})^2)|A_1|^2 \psi_{11}^{(2)}(r_2,r_1)$$

$$= |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + \frac{2}{5} |A_1|^2 \psi_{11}^{(2)}(r_2,r_1)$$

Hence, by taking cognizance of all the preceding results, we have

$$B_{00}(r_2,r_1) = |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + \frac{2}{5} |A_1|^2\psi_{11}^{(2)}(r_2,r_1)$$

$$= |A_1|^2(r_2r_1)^2 \int_0^\infty d\Phi^{(1)}(\lambda) \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}}} \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}}} +$$

$$+ \frac{2}{5} |A_1|^2(r_2r_1)^2 \int_0^\infty d\Phi^{(2)}(\lambda) \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)} \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)}$$

Finally, we consider the entry $B_{11}(r_2,r_1)$. Then,

$$B_{11}(r_2,r_1) = \frac{1}{2}\langle\xi_{-1}(Y_{10}\otimes\delta_{r_2}) - \xi_0(Y_{1,-1}\otimes\delta_{r_2}),\xi_{-1}(Y_{10}\otimes\delta_{r_1}) - \xi_0(Y_{1,-1}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)} \tag{4.4.27}$$

$$= \frac{1}{2}\langle\xi_{-1}(Y_{10}\otimes\delta_{r_2}),\xi_{-1}(Y_{10}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)} - \frac{1}{2}\langle\xi_{-1}(Y_{10}\otimes\delta_{r_2}),\xi_0(Y_{1,-1}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= \frac{1}{2}\langle\xi_0(Y_{1,-1}\otimes\delta_{r_2}),\xi_{-1}(Y_{10}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)} + \langle\xi_0(Y_{1,-1}\otimes\delta_{r_2}),\xi_0(Y_{1,-1}\otimes\delta_{r_1})\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

Next, consider the various terms on the right hand side of (4.4.27) one by one. Then, we have

$$<\xi_{-1}(Y_{10}\otimes\delta_{r_2}),\xi_{-1}(Y_{10}\otimes\delta_{r_1})>_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + |A_1|^2\psi_{11}^{(2)}(r_2,r_1)\int d\Omega\,\frac{(p_{-1})^2}{|p|^2}Y_{10}(p')\overline{Y}_{10}(p')$$

$$= |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + \frac{1}{4}|A_1|^2\psi_{11}^{(2)}(r_2,r_1)\int d\Omega|\theta_1^0Y_{21}(p') + \theta_1^0Y_{2,-1}(p')|^2$$

(by (4.3.15))

$$= |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + \frac{1}{2}|A_1|^2\ (\theta_1^0)^2\psi_{11}^{(2)}(r_2,r_1)$$

$$= |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + \frac{1}{5}|A_1|^2\ \psi_{11}^{(2)}(r_2,r_1)$$

$$<\xi_{-1}(Y_{10}\otimes\delta_{r_2}),\xi_0(Y_{1,-1}\otimes\delta_{r_1})>_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= |A_1|^2\psi_{11}^{(2)}(r_2,r_1)\int d\Omega\,\frac{p_{-1}p_0}{|p|^2}Y_{10}(p')\overline{Y}_{1,-1}(p')$$

$$= \frac{1}{2i}|A_1|^2\psi_{11}^{(2)}(r_2,r_1)\int d\Omega\ \theta_1^0(Y_{21}(p') + Y_{2,-1}(p'))(\beta_1^{-1}\overline{Y}_{2,-1}(p'))$$

(by (4.3.7) and (4.3.15))

$$= \frac{1}{2i}\theta_1^0\beta_1^{-1}|A_1|^2\psi_{11}^{(2)}(r_2,r_1)$$

$$= -\frac{1}{2i}\frac{\sqrt{2}}{5}|A_1|^2\ \psi_{11}^{(2)}(r_2,r_1)$$

Also,

$$<\xi_0(Y_{1,-1}\otimes\delta_{r_2}),\ \xi_{-1}(Y_{10}\otimes\delta_{r_1})>_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= |A_1|^2\psi_{11}^{(2)}(r_2,r_1)\ \int d\Omega\ \frac{p_0 p_{-1}}{|p|^2}\ Y_{1,-1}(p')\overline{Y}_{10}(p')$$

$$= \frac{1}{2i}\ \frac{\sqrt{2}}{5}\ |A_1|^2\ \psi_{11}^{(2)}(r_2,r_1)$$

Finally,

$$<\xi_0(Y_{1,-1}\otimes\delta_{r_2}),\xi_0(Y_{1,-1}\otimes\delta_{r_1})>_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + |A_1|^2\psi_{11}^{(2)}(r_2,r_1)\ \int d\Omega\ \frac{p_0^2}{|p|^2}\ Y_{1,-1}(p')\overline{Y}_{1,-1}(p')$$

$$= |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + |A_1|^2\psi_{11}^{(2)}(r_2,r_1)\ \int d\Omega\ |\beta_1^{-1}\ Y_{2,-1}(p')|^2$$

(by (4.3.7))

$$= |A_1|^2\ \psi_{11}^{(1)}(r_2,r_1) + \frac{1}{5}\ |A_1|^2\psi_{11}^{(2)}(r_2,r_1)$$

Hence, by taking cognizance of all the foregoing results, we have that

$$B_{11}(r_2,r_1) = |A_1|^2\psi_{11}^{(1)}(r_2,r_1) + \frac{1}{5}|A_1|^2\psi_{11}^{(2)}(r_2,r_1)$$

$$= |A_1|^2(r_2r_1)^2\int_0^\infty d\Phi^{(1)}(\lambda)\ \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}}}\cdot\frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}}} +$$

$$+ \frac{1}{5}|A_1|^2(r_2r_1)^2\int_0^\infty d\Phi^{(2)}(\lambda)\ \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}}}\cdot\frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}}}$$

This completes proof of the theorem. □

(4.4.28) REMARK: In the next section, we examine the consequences of demanding Markovicity in the sense of Wong for the Gaussian generalized stochastic field $H^{O}(\Omega) = \{\xi(f) = (\xi_j(f) : j = -1,0,1) : f \in \mathcal{S}(R^d)\}$.

4.5 MARKOV GENERALIZED STOCHASTIC FIELDS

Consider the vector generalized stochastic field

$$H^{O}(\Omega) = \{\xi(f) = (\xi_{-1}(f),\xi_{0}(f),\xi_{1}(f) : f \in \mathcal{S}(R^3)\}$$

defined on the probability space $(\Omega,\mathcal{B},\mu)$. In equation (4.2.8) it is seen that the entries $B_{ij}(f^{(2)},f^{(1)})$ of the matrix $B(f^{(2)},f^{(1)})$ of correlation functionals of $\xi(f) \in H^{O}(\Omega)$ are defined by two spectral measures $d\Phi^{(1)}(|p|)$ and $d\Phi^{(2)}(|p|)$, $p \in R^3$. The following interesting question now naturally arises: "What must the spectral measures $d\Phi^{(1)}(|p|)$ and $d\Phi^{(2)}(|p|)$, $p \in R^3$, be in order that $H^{O}(\Omega) = \{\xi(f) = (\xi_{-1}(f),\xi_{0}(f),\xi_{1}(f)) : f \in \mathcal{S}(R^3)\}$ be Markov in the sense of Wong?" This question is resolved in this section. In doing so, we employ practically every piece of knowledge we have obtained thus far in this and previous chapters.

Questions similar to the ones we answer here seem destined to play a role of great significance in Constructive Quantum Field Theory, on account of the crucial part played thus far by Markov generalized stochastic fields in Nelson's scheme [57][58] for constructing quantum fields. Unfortunately, as already remarked in Chapter 0, to the best of the present author's knowledge, the sort of complete characterization of generalized stochastic fields which are Markov in the sense of Wong which we accomplish in this Thesis has as yet not been considered by any author in the case of generalized stochastic fields which are Markov in the sense of Nelson. However, examples of generalized stochastic fields which are Markov in the sense of Nelson have been furnished in [58][60][45][99][81][92].

(4.5.1) DEFINITION: Let $\{X(r) = (X_j(r) : j = 1,\ldots,N) : r \in [0,\infty)\}$ be a multicomponent Gaussian mean zero stochastic process on the probability space $(\Omega,\mathcal{B},\mu)$. Let $H(r) = \{aX(r) : a \in \mathbb{C}\}$, $r \in [0,\infty)$, $r > 0$, and let $P_{H(r)}$ denote the orthogonal projection onto $H(r)$. Then $\{X(r) = (X_j(r) : j = 1,\ldots,N) : r \in [0,\infty)\}$ is said to be Markov in the sense of Wong if $H(r_1)$ is orthogonal to $H(r_2) - P_{H(r)} H(r_2)$ for $r_1 < r < r_2$.

(4.5.2) THEOREM Let $\{X(r) = (X_j(r) : j = 1,\ldots,N) : r \in [0,\infty)\}$ be as described in (4.5.1). Set

(a) $$P_{H(r)} X_k(r_2) = \sum_{i=1}^{N} Q_{ki}(r_2,r)X_i(r)$$

(b) $$\frac{B_{ij}(r,s)}{B_{jj}(s.s)} = R_{ij}(r,s)$$

(Note that the quantities $\{Q_{ki}(r_2,r) : k,i = 1,\ldots,N\}$ are uniquely determined by the two conditions (i) $P_{H(r)}X_k(r_2)$ belongs to $H(r)$ (ii) $X_k(r_2) - P_{H(r)}X_k(r_2)$ is orthogonal to $X_j(r)$, $j = 1,\ldots,N$). Then $\{X(r) = (X_j(r) : j = 1,\ldots,N) : r \in [0,\infty)\}$ is Markov in the sense of Wong if and only if

$$R(r_2,r_1) = Q(r_2,r)R(r,r_1) \quad \text{for } r_1 < r < r_2 . \tag{4.5.3}$$

Proof: By (4.5.1), $\{X(r) = (X_j(r) : j = 1,\ldots,N) : r \in [0,\infty)\}$ is Markov in the sense of Wong if

$$\langle X_k(r_2) - P_{H(r)}X_k(r_2),\ X_j(r_1)\rangle_{L^2(\Omega,\mathcal{B},\mu)} = 0,\ k,j = 1,\ldots,N$$

for $r_1 < r < r_2$. Thus

$$\langle X_k(r_2),\ X_j(r_1)\rangle_{L^2(\Omega,\mathcal{B},\mu)} = \langle P_{H(r)}X_k(r_2),\ X_j(r_1)\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= \sum_{i=1}^{N} Q_{ki}(r_2,r)\ \langle X_i(r),\ X_j(r_1)\rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

Hence

$$B_{kj}(r_2,r_1) = \sum_{i=1}^{N} Q_{ki}(r_2,r)B_{ij}(r,r_1)$$

Dividing both sides of the last equation by $B_{jj}(r_1,r_1)$, which is non-vanishing for all j and all $r_1 > 0$, we have

$$R_{kj}(r_2,r_1) = \sum_{i=1}^{N} Q_{ki}(r_2,r)R_{ij}(r,r_1)$$

or equivalently,

$$R(r_2,r_1) = Q(r_2,r)R(r,r_1) \quad \text{for } r_1 < r < r_2.$$

Hence the condition (4.5.3) is indeed necessary.

Suppose next that (4.5.3) holds for $r_1 < r < r_2$. Then, we have that

$$B_{kj}(r_2,r_1) = \sum_{i=1}^{N} Q_{ki}(r_2,r)B_{ij}(r,r_1) \qquad r_1 < r < r_2$$

This last equation conveys the information that

$$X_k(r_2) - \sum_{i=1}^{N} Q_{ki}(r_2,r)X_i(r) \text{ is orthogonal to } X_j(r_1),$$

$r_1 < r < r_2$, $j = 1,\ldots,N$. Since

$$\sum_{i=1}^{N} Q_{ki}(r_2,r)X_i(r) = P_{H(r)}X_k(r_2)$$

by definition, then we have that

$X_k(r_2) - P_{H(r)}X_k(r_2)$ is orthogonal to $X_j(r_1)$, $r_1 < r < r_2$, $j = 1,\ldots,N$. But by (4.5.1) this implies that $\{X(r) = (X_j(r)) : r \in [0,\infty)\}$ is Markov in the sense of Wong. Hence, the condition (4.5.3) is indeed also sufficient. □

(4.5.4) REMARK: Our next result gives necessary conditions in order that the vector generalized stochastic field

$$H^O(\Omega) = \{\xi(f) = (\xi_{-1}(f), \xi_0(f), \xi_1(f)) : f \in \mathcal{S}(R^3)\}$$

be Markov in the sense of Wong. In Theorem (4.5.36), we demonstrate that the conditions we obtain are indeed also sufficient for Markovicity.

(4.5.5) THEOREM Let $H^O(\Omega) = \{\xi(f) = (\xi_{-1}(f), \xi_0(f)\xi_1(f)) : f \in \mathcal{S}(R^3)\}$ be a Euclidean covariant Gaussian generalized stochastic field which satisfies Wong's Assumption (4.1.1). Then, in order that $H^O(\Omega)$ be Markov in the sense of Wong, it is necessary that the functions

$$r \to B_i(r) = \int_0^\infty d\Phi^{(i)}(\lambda)\lambda^2 \frac{J_{\frac{1}{2}+1}(\lambda r)}{(\lambda r)^{\frac{1}{2}+1}}, \quad r > 0,$$

are twice continuously differentiable and satisfy the differential equations

$$(\triangle B_i)(r) = \alpha B_i(r), \; r > 0, \; i = 1,2 \text{ and } \alpha = \text{constant, where}$$

$$\triangle = \frac{1}{r^4}\frac{d}{dr}(r^4 \frac{d}{dr}) .$$

Proof: The conditions of the theorem are necessary. To see this, let D_1, D and D_2 be the open subsets of R^3 whose boundaries ∂D_1, ∂D and ∂D_2 are spheres of radius r_1, r and r_2 respectively, with $r_1 < r < r_2$. Then ∂D_1, ∂D and ∂D_2 is an increasing family of nested boundaries. If the generalized stochastic field $H^O(\Omega) = \{\xi(f) = (\xi_{-1}(f), \xi_0(f), \xi_1(f)) : f \in \mathcal{S}(R^3)\}$ is indeed Markov in the sense of Wong, then it is necessarily so when the special boundaries ∂D_1, ∂D and ∂D_2 just described are those employed in constructing the boundary data Hilbert spaces $H(\partial D_1)$, $H(\partial D)$ and $H(\partial D_2)$ respectively.

Thus, by definition, we must have that $H(\partial D_2) - P_{H(\partial D)} H(\partial D_2)$ is stochastically independent of $H(\partial D_1)$, $r_1 < r < r_2$.

Now, by the analysis of Section 4.3, we have

$$(4.3.5) \qquad H(\partial D) = H^{(0)}(r) \bigoplus_{n=1}^{\infty} H^{(n)}(r) \quad , \qquad r > 0 .$$

The elements of $H^{(0)}(r)$ are complex multiples of the scalar random variable $\Phi(r)$ given by (4.4.7), while the elements of $H^{(n)}(r)$, $n \geq 1$, are $3(2n+1)$-dimensional random vectors. A detailed description of $H^{(0)}(r)$ and $H^{(1)}(r)$ is given in Section 4.4.

Let $Y(r_i) = (Y_j(r_i) : j = 1,\ldots,3(2n+1))$ belong to $H^{(n)}(r_i)$, $i = 1,2$. Then a necessary condition for Markovicity in the sense of Wong of $H^0(\Omega) = \{\xi(f) = (\xi_{-1}(f), \xi_0(f), \xi_1(f)) : f \in \mathcal{S}(R^3)\}$ is that $Y(r_2) - P_{H(\partial D)} Y(r_2)$ be orthogonal to $Y(r_1)$, $r_1 < r < r_2$. But because of (4.3.5),

$$P_{H(\partial D)} Y(r_2) = P_{H^{(n)}(r)} Y(r_2)$$

where $P_{H^{(n)}(r)}$ is the orthogonal projection onto $H^{(n)}(r)$, $r > 0$.

By Theorem (4.5.2), $Y(r_2) - P_{H^{(n)}(r)} Y(r_2)$ is orthogonal to $Y(r_1)$, $r_1 < r < r_2$, if and only if the vector stochastic process $\{Y(r) = (Y_j(r) : j = 1,\ldots,3(2n+1)) : r \in [0,\infty)\}$ is Markov in the sense of Wong. By the same theorem, it now follows that

$$(4.5.6) \qquad \langle Y_j(r_2), Y_k(r_1) \rangle_{L^2(\Omega,\mathcal{B},\mu)} = B_{jk}^{(n)}(r_2,r_1)$$

$$= \sum_{i=1}^{3(2n+1)} Q_{ji}^{(n)}(r_2,r) B_{ik}^{(n)}(r,r_1), \quad r_1 < r < r_2$$

Next, we move on to use the above considerations to provide necessary conditions on the spectral measures $d\Phi^{(1)}(|p|)$ and $d\Phi^{(2)}(|p|)$, $p \in R^3$, associated with the generalized stochastic field $H^0(\Omega) = \{\xi(f) = (\xi_{-1}(f), \xi_0(f), \xi_1(f)) : f \in \mathcal{S}(R^3)\}$.

(A) Suppose then that $Y(r)$ belongs to $H^{(0)}(r)$, $r > 0$. Then $Y(r)$ is a complex multiple of the random variable $\Phi(r)$ given by (4.4.7), and it, therefore, suffices to set $Y(r) = \Phi(r)$. By Theorem (4.4.16),

$$(4.5.7) \quad \langle\Phi(r_2),\Phi(r_1)\rangle_{L^2(\Omega,\mathcal{B},\mu)} = B^{(0)}(r_2,r_1)$$

$$= 3|A_1|^2(r_2r_1)^2 \int_0^\infty d\Phi^{(1)}(\lambda) \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}}} \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}}} +$$

$$+ \frac{1}{5}(7 + \sqrt{2}|A_1|^2(r_2r_1)^2 \int_0^\infty d\Phi^{(2)}(\lambda) \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}}} \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}}}$$

But, by (4.5.6), we must have that

$$(4.5.8) \quad B^{(0)}(r_2,r_1) = Q^{(0)}(r_2,r)B^{(0)}(r,r_1), \quad r_1 < r < r_2$$

By Aczel [1], the solution of this last functional equation is of the form

$$B^{(0)}(r_2,r_1) = H(r_2)G(r_1), \quad r_1 < r_2$$

Now, set

$$(4.5.9) \quad \sqrt{5}(|A_1|r_1^{\,3})^{-1}G(r_1) = G_0(r_1)$$

$$(4.5.10) \quad \sqrt{5}(|A_1|r_2^{\,3})^{-1} H(r_2) = H_0(r_2)$$

If we combine (4.5.7), (4.5.8), (4.5.9) and (4.5.10), we obtain:

$$H_0(r_2)G_0(r_1) = 15 \int_0^\infty d\Phi^{(1)}(\lambda)\lambda^2 \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}+1}} \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}+1}} +$$

$$+ (7+\sqrt{2}) \int_0^\infty d\Phi^{(2)}(\lambda)\lambda^2 \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}+1}} \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}+1}}$$

Let Δ denote the linear differential operator

$$\frac{d^2}{dr^2} + \frac{4}{r}\frac{d}{dr} = \frac{1}{r^4}\frac{d}{dr} r^4 \frac{d}{dr}$$

Then, we have

$$(\Delta H_0)(r_2)G_0(r_1) = 15 \int_0^\infty d\Phi^{(1)}(\lambda)\lambda^2(-\lambda^2) \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}+1}} \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}+1}} +$$

$$+ (7+\sqrt{2}) \int_0^\infty d\Phi^{(2)}(\lambda)\lambda^2(-\lambda^2) \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}+1}} \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}+1}}$$

$$= 15 \int_0^\infty d\Phi^{(1)}(\lambda)\lambda^2 \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}+1}}(-\lambda^2) \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}+1}} +$$

$$+ (7+\sqrt{2}) \int_0^\infty d\Phi^{(2)}(\lambda)\lambda^2 \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}+1}} \cdot (-\lambda^2) \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}+1}}$$

$$= H_0(r_2)(\Delta G_0)(r_1)$$

Thus

$$(\Delta H_0)(r_2)G_0(r_1) = H_0(r_2)(\Delta G_0)(r_1) , \qquad r_1 < r_2 ,$$

and we have

$$(H_0(r_2))^{-1}(\Delta H_0)(r_2) = (G_0(r_1))^{-1}(\Delta G_0)(r_1) = \alpha \text{ , a constant}$$

Hence,

$$(\Delta H_0)(r) = \alpha H_0(r)$$
$$r > 0 .$$
$$(\Delta G_0)(r) = \alpha G_0(r)$$

Thus

$$(\Delta H_0)(r_2)G_0(r_1) = H_0(r_2)(\Delta G_0)(r_1) = \alpha H_0(r_2)G_0(r_1)$$

Hence, the function

$$\text{(4.5.11)} \quad r \to R(r) = H_0(r)G_0(0)$$

$$= 15 \int_0^\infty d\Phi^{(1)}(\lambda)\lambda^2 \frac{J_{\frac{1}{2}+1}(\lambda r)}{(\lambda r)^{\frac{1}{2}+1}} + (7 + \sqrt{2}) \int_0^\infty d\Phi^{(2)}(\lambda)\lambda^2 \frac{J_{\frac{1}{2}+1}(\lambda r)}{(\lambda r)^{\frac{1}{2}+1}}$$

satisfies the following differential equation

$$\text{(4.5.12)} \qquad (\Delta R)(r) = \alpha R(r), \quad r > 0.$$

In (4.5.11), $G_0(0)$ is defined by a limiting procedure.

(B) The equation (4.5.12) puts a condition on the spectral measures $d\Phi^{(1)}(\lambda)$ and $d\Phi^{(2)}(\lambda)$. One obtains other nontrivial conditions on the spectral measures $d\Phi^{(1)}(\lambda)$ and $d\Phi^{(2)}(\lambda)$ by considering the Hilbert space $H^{(1)}(r)$, $r > 0$, consisting of nine dimensional random vectors.

Thus, let now $Y(r)$ belong to $H^{(1)}(r)$, $r > 0$. We have already remarked in Section 4.4 that $H^{(1)}(r)$ is an orthogonal direct sum

$$\text{(4.4.22)} \qquad H^{(1)}(r) = H_{(1)}(r) \oplus H_{(2)}(r)$$

in which $H_{(1)}(r)$ consists of constant multiples of the three dimensional random vector $X^{(1)}(r)$ given by (4.4.11) and $H_{(2)}(r)$ is the Hilbert space of six dimensional random vectors of the form $\alpha_0 X^{(0)}(r) \underset{a}{\oplus} \alpha_1 X^{(2)}(r)$, $\alpha_0, \alpha_1 \varepsilon \mathbb{C}$, where $X^{(i)}(r)$, $i = 0,2$, are given by (4.4.10) and (4.4.13) respectively, and we recall that $\underset{a}{\oplus}$ denotes algebraic direct sum. We may now let, in particular, $Y(r) = X^{(1)}(r)$, $r > 0$. in (4.5.6).

Next, we have that a necessary condition for Markovicity in the sense of Wong of the generalized stochastic field $H^0(\Omega) = \{\xi(f) = (\xi_{-1}(f), \xi_0(f), \xi_1(f)) : f \varepsilon \mathcal{S}(R^3)\}$ is that $X^{(1)}(r_2) - P_{H(\partial D)} X^{(1)}(r_2)$ be orthogonal to $X^{(1)}(r_1)$ for $r_1 < r < r_2$. By (4.4.22), we have

$$P_{H(\partial D)} X^{(1)}(r_2) = P_{H^{(1)}(r)} X^{(1)}(r_2) = P_{H_{(1)}(r)} X^{(1)}(r_2), \; r_1 < r < r_2$$

Let $B(r_2, r_1)$ denote the matrix of correlation functions of the random vector $X^{(1)}(r)$. Then, by (4.5.6), we must now have

$$(4.5.13) \quad B(r_2, r_1) = Q^{(1)}(r_2, r) B(r, r_1) \qquad r_1 < r < r_2$$

Let us proceed to exploit (4.5.13). By Theorem (4.4.23) the matrix $B(r_2, r_1)$ is diagonal. Hence (4.5.13) may be re-written as the following three equations

$$(4.5.14) \quad B_{ii}(r_2, r_1) = Q_{ii}^{(i)}(r_2, r) B_{ii}(r, r_1), \; r_1 < r < r_2, \quad i = -1,0,1$$

The functional equations (4.5.14) have solutions of the forms

$$(4.5.15) \quad B_{ii}(r_2, r_1) = H_{ii}^{(0)}(r_2) G_{ii}^{(0)}(r_1), \; r_1 < r_2, \quad i = -1,0,1.$$

We next consider the three equations in (4.5.15) in turn. For $i = -1$, we have from Theorem (4.4.23)

$$(4.5.16)\quad B_{-1-1}(r_2,r_1) = |A_1|^2(r_2r_1)^2 \int_0^\infty d\Phi^{(1)}(\lambda)\, \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}}} \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}}} +$$

$$+ \frac{1}{10}(2+\sqrt{2})\,|A_1|^2(r_2r_1)^2 \int_0^\infty d\Phi^{(2)}(\lambda)\, \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}}} \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}}}$$

Set

$$(4.5.17)\quad \sqrt{10}\,(|A_1|r_1^{\,3})^{-1} G^{(0)}_{-1-1}(r_1) = G_{-1-1}(r_1)$$

$$(4.5.18)\quad \sqrt{10}\,(|A_1|r_2^{\,3})^{-1} H^{(0)}_{-1-1}(r_2) = H_{-1-1}(r_2)$$

Then

$$H_{-1-1}(r_2)G_{-1-1}(r_1) = 10\int_0^\infty d\Phi^{(1)}(\lambda)\lambda^2\, \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}+1}} \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}+1}} +$$

$$+ (2+\sqrt{2})\int_0^\infty d\Phi^{(2)}(\lambda)\lambda^2\, \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}+1}} \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}+1}} .$$

Hence, as in the considerations under (A) above, we have that the function

$$(4.5.19)\quad r \to S_1(r) = H_{-1-1}(r)G_{-1-1}(0) = 10\int_0^\infty d\Phi^{(1)}(\lambda)\lambda^2\, \frac{J_{\frac{1}{2}+1}(\lambda r)}{(\lambda r)^{\frac{1}{2}+1}} +$$

$$+ (2+\sqrt{2})\int_0^\infty d\Phi^{(2)}(\lambda)\lambda^2\, \frac{J_{\frac{1}{2}+1}(\lambda r)}{(\lambda r)^{\frac{1}{2}+1}}$$

satisfies the differential equation

$$(4.5.20)\qquad (\Delta S_1)(r) = \alpha S_1(r), \qquad r > 0$$

where α is an arbitrary constant.

Similarly, from the relations

$$B_{00}(r_2,r_1) = H_{00}^{(0)}(r_2)G_{00}^{(0)}(r_1)$$

$$= |A_1|^2(r_2r_1)^2 \int_0^\infty d\Phi^{(1)}(\lambda) \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}}} \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}}} +$$

$$+ \frac{2}{5} |A_1|^2(r_2r_1)^2 \int_0^\infty d\Phi^{(2)}(\lambda) \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}}} \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}}}$$

$$(r_1 < r_2)$$

$$B_{11}(r_2,r_1) = H_{11}^{(0)}(r_2)G_{11}^{(0)}(r_1)$$

$$= |A_1|^2(r_2r_1)^2 \int_0^\infty d\Phi^{(1)}(\lambda) \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}}} \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}}} +$$

$$+ \frac{1}{5} |A_1|^2(r_2r_1)^2 \int_0^\infty d\Phi^{(2)}(\lambda) \frac{J_{\frac{1}{2}+1}(\lambda r_2)}{(\lambda r_2)^{\frac{1}{2}}} \cdot \frac{J_{\frac{1}{2}+1}(\lambda r_1)}{(\lambda r_1)^{\frac{1}{2}}} \qquad (r_1 < r_2)$$

given by Theorem (4.4.23), we obtain that the functions

(4.5.21) $$r \to S_2(r) = H_{00}(r)G_{00}(0)$$

$$= 5 \int_0^\infty d\Phi^{(1)}(\lambda)\lambda^2 \frac{J_{\frac{1}{2}+1}(\lambda r)}{(\lambda r)^{\frac{1}{2}+1}} +$$

$$+ 2 \int_0^\infty d\Phi^{(2)}(\lambda)\lambda^2 \frac{J_{\frac{1}{2}+1}(\lambda r)}{(\lambda r)^{\frac{1}{2}+1}}$$

and

(4.5.22) $$r \to S_3(r) = H_{11}(r)G_{11}(0)$$

$$= 5 \int_0 d\Phi^{(1)}(\lambda)\lambda^2 \frac{J_{\frac{1}{2}+1}(\lambda r)}{(\lambda r)^{\frac{1}{2}+1}} +$$

$$+ \int_0^\infty d\Phi^{(2)}(\lambda)\lambda^2 \frac{J_{\frac{1}{2}+1}(\lambda r)}{(\lambda r)^{\frac{1}{2}+1}}$$

satisfy respectively the differential equations

(4.5.23) $(\Delta S_2)(r) = \alpha S_2(r)$, $r > 0$ and α = arbitrary constant

(4.5.24) $(\Delta S_3)(4) = \alpha S_3(r)$, $r > 0$ and α = arbitrary constant.

Finally, by solving for the functions

(4.5.25) $$r \to B_i(r) = \int_0^\infty d\Phi^{(i)}(\lambda)\lambda^2 \frac{J_{\frac{1}{2}+1}(\lambda r)}{(\lambda r)^{\frac{1}{2}+1}}, \quad r > 0, \quad i = 1,2$$

from any two of the equations (4.5.19), (4.5.21), (4.5.22) and (4.5.11) and by employing the differential equations (4.5.20), (4.5.23), (4.5.24) and (4.5.12), we conclude that the functions $r \to B_i(r)$, $i = 1,2$, of (4.5.25) satisfy the differential equations

(4.5.26) $(\Delta B_i)(r) = \alpha B_i(r)$, $r > 0$, $i = 1,2$ and α = constant.

Hence, the conditions of the theorem are indeed necessary. □

(4.5.27) DISCUSSION Let $H^O(\Omega) = \{\xi(f) = (\xi_{-1}(f),\xi_0(f),\xi_1(f)) : f \in \mathcal{S}(R^3)\}$ be Markov in the sense of Wong. We shall now use the last theorem to determine the spectral measures $d\Phi^{(1)}(|p|)$ and $d\Phi^{(2)}(|p|)$, $p \in R^3$, which describe the generalized stochastic field $H^O(\Omega)$. To this end, we must solve the two differential equations given by (4.5.26). First, one recognizes two distinct cases, namely,

(i) Case $\alpha = 0$ and (ii) Case $\alpha \neq 0$.

We consider these cases in turn.

(i) CASE $\alpha = 0$

Then, we have the differential equations

(4.5.28) $(\Delta B_i)(r) = (\frac{1}{r^4}\frac{d}{dr}\, r^4 \frac{dB_i}{dr})(r) = 0, \; r > 0, \; i = 1,2.$

We need now to solve (4.5.28) subject to the condition $B_i(r) = 0$ at $r = \infty$, $i = 1,2$. Then, we obtain

(4.5.29) $B_i(r) = B_i r^{-3}, \; i = 1,2$

where B_i = constant, $i = 1,2$.

Next, by invoking the definitions (4.5.25) of the functions $r \to B_i(r)$, $i = 1,2$, we have

(4.5.30) $$B_i r^{-3} = \int_0^\infty d\Phi^{(i)}(\lambda)\lambda^2 \frac{J_{\frac{1}{2}+1}(\lambda r)}{(\lambda r)^{\frac{1}{2}+1}}$$

$$= (2^{-3/2}B_i) \int_0^\infty d\lambda \; \lambda^2 \frac{J_{\frac{1}{2}+1}(\lambda r)}{(\lambda r)^{\frac{1}{2}+1}}$$

Hence, by the uniqueness of the Hankel transform, we have, after setting $2^{-3/2}B_i = A_i$, that

(4.5.31) $d\Phi^{(i)}(\lambda) = A_i \, d\lambda, \; i = 1,2.$

CASE $\alpha \neq 0$.

Here, the equations (4.5.26) have the following solutions

(i) $$B_i(r) = B_i \frac{J_{\frac{1}{2}+1}(\alpha_i r)}{(\alpha_i r)^{\frac{1}{2}+1}}, \quad \alpha_i > 0, \quad i = 1,2$$

and B_i = constant, $i = 1,2$, $r > 0$.

(ii) $$B_i(r) = B_i \frac{K_{\frac{1}{2}+1}(\alpha_i r)}{(\alpha_i r)^{\frac{1}{2}+1}}, \quad \alpha_i > 0, \quad i = 1,2$$

and B_i = constant, $i = 1,2$, $r > 0$.

In (ii), $z \to K_\nu(z)$, $\nu \in \mathbb{C}$, is the modified Bessel function of the second kind and it is defined in terms of the Bessel function

$$z \to J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{\nu+2n}}{n!\Gamma(\nu+n+1)}$$

as follows

$$K_\nu(z) = \frac{1}{2}\pi[\sin(\pi\nu)]^{-1}[e^{\frac{i\pi\nu}{2}} J_{-\nu}(ze^{\frac{i\pi}{2}}) - e^{-\frac{i\pi\nu}{2}} J_\nu(ze^{\frac{i\pi}{2}})]$$

Now, consider solution (i) above. Then, employing (4.5.25), we obtain

$$B_i \frac{J_{\frac{1}{2}+1}(\alpha_i r)}{(\alpha_i r)^{\frac{1}{2}+1}} = \int_0^\infty d\Phi^{(i)}(\lambda)\lambda^2 \frac{J_{\frac{1}{2}+1}(\lambda r)}{(\lambda r)^{\frac{1}{2}+1}}$$

$$= B_i \int_0^\infty d\lambda\delta(\lambda - \alpha_i) \frac{J_{\frac{1}{2}+1}(\lambda r)}{(\lambda r)^{\frac{1}{2}+1}}$$

Hence, we find that

$$d\Phi^{(i)}(\lambda) = B_i\lambda^{-2} \delta(\lambda - \alpha_i)d\lambda, \quad \alpha_i > 0, \; i = 1,2.$$

But in order that (4.5.8) and other similar equations encountered in the proof of Theorem (4.5.5) are satisfied we must have $\alpha_1 = \alpha_2 = \alpha_0$, say. Hence

$$(4.5.32) \quad d\Phi^{(i)}(\lambda) = B_i\lambda^{-2}\delta(\lambda - \alpha_0)d\lambda, \quad \alpha_0 > 0, \; i = 1,2.$$

Employing (4.5.32) in Theorem (4.4.16), we find that the random variable $\Phi(r)$ given by (4.4.7) has correlation function given by

$$<\Phi(r_2),\Phi(r_1)>_{L^2(\Omega,\mathcal{B},\mu)} = B^{(0)}(r_2,r_1)$$

$$= B(r_2 r_1)^2 \frac{J_{\frac{1}{2}+1}(\alpha_0 r_2)}{(\alpha_0 r_2)^{\frac{1}{2}}} \cdot \frac{J_{\frac{1}{2}+1}(\alpha_0 r_1)}{(\alpha_0 r_1)^{\frac{1}{2}}}$$

where B is some constant. From the preceding one concludes that the random variable $\Phi(r)$, $r \in R_+$, has the very simple representation

$$(4.5.33) \quad \Phi(r) = \sqrt{B}\, r^2 \frac{J_{\frac{1}{2}+1}(\alpha_0 r)}{(\alpha_0 r)^{\frac{1}{2}}} X \quad , \ \alpha_0 > 0,\ r \in R_+ ,$$

where X is a random variable with

$$||X||^2_{L^2(\Omega,\mathcal{B},\mu)} = 1 .$$

But as indicated in the proof of Theorem (4.5.5) a necessary condition for $H^0(\Omega) = \{\xi(f) = (\xi_{-1}(f),\xi_0(f),\xi_1(f))\}$ to be Markov in the sense of Wong is that $\{\Phi(r) : r \in R_+ = [0,\infty)\}$ is Markov in the sense of Definition (4.5.1). This is the case if and only if $\{\Phi(r) : r \in R_+ = [0,\infty)\}$ is Markov in the usual sense. From the fact that we must then have

$$B^{(0)}(r_2,r_1) = \frac{B^{(0)}(r_2,r)B^{(0)}(r,r_1)}{B^{(0)}(r,r)} , \ r_1 < r < r_2$$

it follows that $\Phi(r)$ cannot vanish for any nonzero $r \in R_+ = [0,\infty)$. But $\Phi(r)$ given by (4.5.33) vanishes at infinitely many nonzero points in R_+. Hence, solution (i) is inadmissible.

Consider solution (ii) above, for which

$$B_i(r) = B_i \frac{K_{\frac{1}{2}+1}(\alpha_i r)}{(\alpha_i r)^{\frac{1}{2}+1}} , \quad r > 0, \qquad i = 1,2.$$

Then, by (4.5.25), it follows that

$$B_i \frac{K_{\frac{1}{2}+1}(\alpha_i r)}{(\alpha_i r)^{\frac{1}{2}+1}} = \int_0^\infty d\Phi^{(i)}(\lambda)\lambda^2 \frac{J_{\frac{1}{2}+1}(\lambda r)}{(\lambda r)^{\frac{1}{2}+1}}, \quad i = 1,2, \quad r > 0$$

$$= \alpha_i^{-3} B_i \int_0^\infty d\Phi \frac{\lambda^4}{\alpha_i^2+\lambda^2} \frac{J_{\frac{1}{2}+1}(\lambda r)}{(\lambda r)^{\frac{1}{2}+1}}$$

Hence, by the uniqueness of the Hankel transform, we have

$$(4.5.34) \qquad d\Phi^{(i)}(\lambda) = \alpha_i^{-3} B_i \frac{\lambda^2}{\alpha_i^2+\lambda^2} d\lambda, \quad i = 1,2$$

Since

$$K_\nu(mr_2)I_\nu(mr_1) = \int_0^\infty d\lambda \frac{\lambda}{m^2+\lambda^2} J_\nu(\lambda r_2)J_\nu(\lambda r_1)$$

$r_2 \geq r_1$, real part of $\nu > -1$, $m > 0$, where $z \to I_\nu(z)$ is the modified Bessel function of the first kind [93], we readily see that equation (4.5.8) and (4.5.14) are satisfied if and only if $\alpha_1 = \alpha_2$ in (4.5.34). Thus, setting $\alpha_1 = \alpha_2 = \alpha_0$ and $\alpha_0^{-3}B_i = A_i$, $i = 1,2$, we may re-write (4.5.34) as follows

$$(4.5.35) \qquad d\Phi^{(i)}(\lambda) = A_i \frac{\lambda^2}{\alpha_0^2+\lambda^2} d\lambda, \quad i = 1,2, \quad \alpha_0 > 0$$

Thus, we have obtained explicit expressions for the spectral measures $d\Phi^{(i)}(\lambda)$, $i = 1,2$, in the two distinct cases for which $\alpha = 0$ and $\alpha \neq 0$ mentioned above. We may combine the results (4.5.31) and (4.5.35) if we understand α_0 in (4.5.35) to be nonnegative i.e. $\alpha_0 \geq 0$ rather than $\alpha_0 > 0$.

The following question now presents itself: "Are the conditions of Theorem (4.5.5) also sufficient for Markovicity in the sense of Wong of the generalized stochastic field

$H^0(\Omega) = \{\xi(f) = (\xi_{-1}(f), \xi_0(f), \xi_1(f)): f \in \mathcal{S}(R^3)\}$? "

The next result answers this question in the affirmative.

(4.5.36) THEOREM Let $H^{O}(\Omega) = \{\xi(f) = (\xi_{-1}(f),\xi_{0}(f),\xi_{1}(f)) : f \in \mathcal{S}(R^{3})\}$ be a Euclidean covariant Gaussian generalized stochastic field which satisfies Wong's Assumption (4.1.1). Then the conditions of Theorem (4.5.5) are also sufficient in order that $H^{O}(\Omega)$ be Markov in the sense of Wong.

Proof: Let $\langle\xi_i(x),\xi_j(y)\rangle_{L^2(\Omega,\mathcal{B},\mu)} = B_{ij}(x-y)$, and let $B(x-y)$ denote the matrix whose entries are $B_{ij}(x-y)$. Let D be an arbitrary open subset of R^3 whose boundary is ∂D and let $H(\partial D)$ be the associated boundary data Hilbert space. Then

$$B(f^{(2)}_{\partial D},f^{(1)}_{\partial D}) = \int d\sigma(x)d\sigma(y)\ f^{(2)}_{\partial D}(x)B(x-y)\overline{f^{(1)}_{\partial D}}(y) ,$$

$f^{(i)}_{\partial D} \in L^2(\partial D,d\sigma)$, $i = 1,2$, is the matrix of correlation functionals of $\xi(f_{\partial D})$ in $H(\partial D)$.

Suppose now that the conditions of Theorem (4.5.5) hold. Then, as demonstrated above (see equation (4.5.35)),

$$d\Phi^{(i)}(\lambda) = A_i \frac{\lambda^2}{\alpha_0^{\,2} + \lambda^2}\, d\lambda,\ \alpha_0 \geq 0,\ A_i = \text{constant},\ i = 1,2.$$

By (4.2.8), we, therefore, have

$$B_{ij}(x-y) = A_1\int d|p|d\Omega|p|^2(\alpha_0^{\,2} + |p|^2)^{-1}\, e^{ip.(x-y)}\delta_{ij} +$$

$$+ A_2 \int d|p|d\Omega|p|^2(\alpha_0^{\,2} + |p|^2)^{-1}\frac{p_ip_j}{|p|^2}\, e^{ip.(x-y)}$$

$$= A_1 \int dp\ (\alpha_0^{\,2} + |p|^2)^{-1}\, e^{ip.(x-y)}\delta_{ij} +$$

$$+ A_2 \int dp(\alpha_0^2 + |p|^2)^{-1} \frac{p_i p_j}{|p|^2} e^{ip.(x-y)}$$

Hence, it follows that

(4.5.37) $((-\Delta)(\alpha_0^2 - \Delta)B_{ij})(x-x_0) = 0, \ x \neq x_0, \ \alpha_0 \geq 0$, where $\Delta = \sum_{i=-1}^{+1} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator in three variables. (4.5.37) is a system of partial differential equations as readily transpires if we set

$$\delta_{ij}(-\Delta)(\alpha_0^2 - \Delta) = A_{ij}(\tfrac{\partial}{\partial x}), \quad i,j = -1,0,1 ,$$

for then (4.5.37) may be re-written as follows

$$(4.5.38) \quad \sum_{k=-1}^{1} (A_{ik}(\tfrac{\partial}{\partial x})B_{kj})(x-x_0) = 0, \ x \neq x_0, \ \alpha_0 \geq 0.$$

The operator $A(\frac{\partial}{\partial x}) = (A_{ij}(\frac{\partial}{\partial x}) : i,j = -1,0,1)$ is strongly elliptic (see (2.4.9)), as is trivially checked.

Let ∂D be a smooth two dimensional surface separating x and x_0, $x \neq x_0$. ∂D separates R^3 into a bounded part D containing x_0 and an unbounded part D' containing x. Then (4.5.38) may be solved as an exterior Dirichlet problem as in (2.4.33) with the boundary conditions (2.4.35), (2.4.36), (2.4.18) on ∂D. Then, by (2.4.32), the solution of this Dirichlet problem may be presented as follows as an integral equation*

$$(4.5.39) \quad B_{ij}(x-x_0) = \sum_{k=-1}^{1} \int_{\partial D} d\sigma(z) \mathcal{P}_{ik}(x,z) B_{kj}(z-x_0), \quad x \in D'$$

Let D_2, D, D_1 be any three bounded open subsets of R^3 such that $D_2 \supset D \supset D_1$. Then the boundaries ∂D_1, ∂D, ∂D_2 form an increasing family of nested surfaces in R^3.

* see Appendix

Next, suppose that $f_{\partial D_2}$ belongs to $L^2(\partial D_2, d\sigma)$. Then the map $L^2(\partial D_2, d\sigma) \to L^2(\partial D, d\sigma)$, $f_{\partial D_2} \to h_{\partial D,ij}$, specified as follows

$$(4.5.40) \qquad h_{\partial D,ij}(x) = \int_{\partial D_2} d\sigma_2(z) f_{\partial D_2}(z)\, \mathcal{P}_{ij}(z,x) \ ,$$

$x \in \partial D$, is well-defined.

Let $X(h_{\partial D})$ be the random vector whose components are $X_j(h_{\partial D}) = \sum_{i=-1}^{1} \xi_i(h_{\partial D,ji})$, $j = -1,0,1$. Then, we claim that $X(h_{\partial D})$ is the orthogonal projection of $\xi(f_{\partial D_2})$ onto $H(\partial D)$. To demonstrate this, we first note that $X(h_{\partial D})$ belongs to $H(\partial D)$. Next, we must demonstrate that $\xi(f_{\partial D_2}) - X(h_{\partial D})$ is orthogonal to every member of $H(\partial D)$. Indeed, we have that if $\xi(g_{\partial D}) = (\xi_i(g_{\partial D}) : i = -1,0,1)$ belongs to $H(\partial D)$, then

$$<\xi_j(f_{\partial D_2}) - \sum_i \xi_i(h_{\partial D,ji}),\ \xi_k(g_{\partial D})>_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= <\xi_j(f_{\partial D_2}),\ \xi_k(g_{\partial D})>_{L^2(\Omega,\mathcal{B},\mu)} - \sum_i <\xi_i(h_{\partial D,ji}),\ \xi_k(g_{\partial D})>_{L^2(\Omega,\mathcal{B},\mu)}$$

$$= \int d\sigma_2(x) d\sigma(y) f_{\partial D_2}(x) B_{jk}(x-y) \overline{g}_{\partial D}(y) -$$

$$- \sum_i \int d\sigma(x) d\sigma(y) h_{\partial ji}(x) B_{ik}(x-y) \overline{g}_{\partial D}(y)$$

$$= \int d\sigma_2(x) d\sigma(y) f_{\partial D_2}(x) B_{jk}(x-y) \overline{g}_{\partial D} -$$

$$- \sum_i \int d\sigma(x) d\sigma(y) \int d\sigma_2(z) f_{\partial D_2}(z)\, \mathcal{P}_{ji}(z-x) B_{ik}(x-y) \overline{g}_{\partial D}(y)$$

(by (4.5.40))

$$= \int d\sigma_2(z)d\sigma(y)f_{\partial D_2}(z)[B_{jk}(z-y) - \sum_i \int d\sigma(x)\mathcal{P}_{ji}(z-x)B_{ik}(x-y)]\bar{g}_{\partial D}(y)$$

$$= 0$$

by (4.5.39).

Hence, $\xi(f_{\partial D_2}) - X(h_{\partial D})$, where $X_i(h_{\partial D}) = \sum_j \xi_j(h_{\partial D,ij})$, is orthogonal to every member of $H(\partial D)$ of the form $\xi(g_{\partial D})$. By linearity and continuity, it now follows that $\xi(f_{\partial D_2}) - X(h_{\partial D})$ is indeed orthogonal to all of $H(\partial D)$. Thus $X(h_{\partial D})$ is the projection of $\xi(f_{\partial D_2})$ onto $H(\partial D)$ i.e. $X(h_{\partial D}) = P_{H(\partial D)}\,\xi(f_{\partial D_2})$. Hence, $H(\partial D_2) - P_{H(\partial D)}H(\partial D_2)$ is stochastically independent of $H(\partial D)$.

In an analogous fashion to all of the preceding, we arrive also at the assertion that $H(\partial D_2) - P_{H(\partial D)}H(\partial D_2)$ is stochastically independent of $H(\partial D_1)$, where we recall that ∂D_1, ∂D, ∂D_2 is an increasing family of nested surfaces in R^3. But stochastic independence of $H(\partial D_1)$ and $H(\partial D_2) - P_{H(\partial D)}H(\partial D_2)$ is Wong's definition of Markov property. Hence, the conditions of the theorem are indeed also sufficient, and this completes proof of the theorem. □

(4.5.41) REMARK: We may now finally combine Theorem (4.5.5) and Theorem (4.5.36) to obtain the following single result.

(4.5.42) THEOREM Let $H^0(\Omega) = \{\xi(f) = (\xi_{-1}(f), \xi_0(f), \xi_1(f)) : f \in \mathcal{S}(R^3)\}$ be a Euclidean covariant Gaussian generalized stochastic field which satisfies Wong's Assumption (4.1.1). Then, the necessary and sufficient conditions for $H^0(\Omega)$ to be Markov in the sense of Wong are that the functions

$$r \to B_i(r) = \int_0^\infty d\Phi^{(i)}(\lambda)\lambda^2 \frac{J_{\frac{1}{2}+1}(\lambda r)}{(\lambda r)^{\frac{1}{2}+1}}, \quad r > 0,\ i = 1,2$$

are twice continuously differentiable and satisfy the differential equations

$$(\Delta B_i)(r) = \left(\frac{1}{r^4}\frac{d}{dr} r^4 \frac{dB_i}{dr}\right)(r) = \alpha_0{}^2 B_i(r)$$

$\alpha_0 \geq 0$, $r > 0$, $i = 1,2$.

(4.5.43) CONCLUSION Theorem (4.5.42) is, of course, a major result because it represents a unique characterization of the class of Euclidean covariant Gaussian vector generalized stochastic fields in R^3 which are Markov in the sense of Wong. As we remarked in the introduction to this section, a complete characterization, even under a constraint such as Wong's Assumption (4.1.1), of generalized stochastic fields which are Markov in the sense of Nelson has as yet not been undertaken by any author. We leave this latter problem open to other investigators.

We have seen in the preceding discussions of this section that Theorem (4.5.42) gives rise to two classes of generalized stochastic fields which are Markov in the sense of Wong. They are the generalized stochastic fields defined by means of the following unsmeared matrix elements

(a) $\tilde{B}_{ij}(p) = (A_1\delta_{ij} + A_2 \frac{p_i p_j}{|p|^2})(|p|^2 + \alpha_0{}^2)^{-1}, \quad \alpha_0 > 0$

(b) $\tilde{B}_{ij}(p) = (A_1\delta_{ij} + A_2 \frac{p_i p_j}{|p|^2}) \; |p|^{-2}$

$i,j = 0,1,2$, where A_k = constant, $k = 1,2$.

Let $A \in (0,1)$ and set $A_1 = 1$, $A_2 = A$ in (b) above. Then, the vector Gaussian generalized stochastic field defined by (b) is also Markov in the sense of Nelson. Indeed, the inverse of the matrix whose entries are (b) has entries given by $\delta_{ij} \; |p|^2 + \frac{A}{1 - A}\, p_i p_j$, and a proof very similar to that given by Nelson in [58] now justifies our claim.

On the other hand, the vector generalized stochastic field defined by (a) is Markov in the sense of Nelson only when $A_2 = 0$. Hence, Wong's notion of Markov property gives rise to some generalized stochastic fields which do not satisfy Nelson's definition of Markovicity; thus the two notions of Markovicity are manifestly not equivalent.

In Chapter 5, we provide an abstract formulation of Wong's notion of Markovicity, and in Chapter 6, we discuss the problem of generalizing this notion of Markovicity.

CHAPTER 5

WONG'S NOTION OF MARKOV PROPERTY REFORMULATED

In this relatively brief chapter, we present an abstract formulation of Wong's notion of Markov property and then examine its relationship, if any, with Nelson's notion of Markovicity. We saw in the last chapter that there are vector generalized stochastic fields which are Markov in the sense of Wong but not in the sense of Nelson. This indicates that these two notions of Markovicity are not equivalent.

In what follows, we again assume that the generalized stochastic fields under consideration are Gaussian and satisfy Wong's Assumption (4.1.1).

(5.1) NOTATION Let $H^{O}(\Omega) = \{\xi(f) = (\xi_j(f) : j = 1,\ldots,N) : f \in \mathcal{S}(R^d)\}$

be an N-component generalized stochastic field on the probability space $(\Omega,\mathcal{B},\mu)$, and let $H(R^d)$ denote the completion of the linear space $H^{O}(\Omega)$ in $(L^2(\Omega,\mathcal{B},\mu))^N$. For any bounded open subset $D \subset R^d$ with boundary ∂D and complement D', let $H(D)$ and $H(\partial D)$ denote respectively the completion of

$$\{\xi(f) = (\xi_j(f) : j = 1,\ldots,N) : f \in \mathcal{S}(R^d), \text{ support of } f \subset D\}$$

and

$$\{\xi(f) = (\xi_j(f) : j = 1,\ldots,N) : f \in L^2(\partial D, d\sigma)\}$$

in $(L^2(\Omega,\mathcal{B},\mu))^N$.

Finally, let $Q_{\partial D}$ denote the orthogonal projection of $H(R^d)$ onto $H(\partial D)$.

The following result is an abstract formulation of Wong's concept of Markovicity.

(5.2) THEOREM Let ∂D_1, ∂D and ∂D_2 be an increasing triplet of nested boundaries and let $H(\partial D_1)$, $H(\partial D)$ and $H(\partial D_2)$ be respectively the associated boundary data Hilbert spaces. Let $P_{\partial D}$ denote the orthogonal projection of $H(\partial D_2)$ onto $H(\partial D)$. Then Wong's notion of Markovicity is equivalent to the following condition:

$$Q_{\partial D_1} Q_{\partial D_2} = Q_{\partial D_1} P_{\partial D} Q_{\partial D_2}$$

as an operator equation on $H(R^d)$.

Proof Wong's definition of Markovicity is the following statement: $H(\partial D_2) - P_{\partial D} H(\partial D_2)$ is always orthogonal to $H(\partial D_1)$. This is equivalent to the following

$$\langle Q_{\partial D_2} u, Q_{\partial D_1} v\rangle_{H(R^d)} = \langle P_{\partial D} Q_{\partial D_2} u, Q_{\partial D_1} v\rangle_{H(R^d)}$$

for every u, v belonging to $H(R^d)$.

Thus, it follows that

$$\langle Q_{\partial D_1} Q_{\partial D_2} u, Q_{\partial D_1} v\rangle_{H(R^d)} = \langle Q_{\partial D_1} P_{\partial D} Q_{\partial D_2} u, Q_{\partial D_1} v\rangle_{H(R^d)}$$

Hence

$$\langle (Q_{\partial D_1} Q_{\partial D_2} - Q_{\partial D_1} P_{\partial D} Q_{\partial D_2}) u, v\rangle_{H(R^d)} = 0$$

for every $u, v \in H(R^d)$. In particular, this is true for every $v \in H(R^d)$ and for arbitrary but fixed $u \in H(R^d)$.

Next, set

$$(Q_{\partial D_1} Q_{\partial D_2} - Q_{\partial D_1} P_{\partial D} Q_{\partial D_2})u = \Phi(u)$$

Then, since u is fixed in $H(R^d)$, so is $\Phi(u)$. From the preceding, we have then that the bounded linear functional

$$F_u : H(R^d) \to R$$

$$v \to F_u(v) = \langle \Phi(u), v \rangle_{H(R^d)}$$

is identically zero on $H(R^d)$. Hence by the uniqueness of any bounded linear functional on a Hilbert space, it follows that

$$\Phi(u) = (Q_{\partial D_1} Q_{\partial D_2} - Q_{\partial D_1} P_{\partial D} Q_{\partial D_2})u = 0$$

for all u in $H(R^d)$.

Hence,

$$(5.2^*) \qquad Q_{\partial D_1} Q_{\partial D_2} = Q_{\partial D_1} P_{\partial D} Q_{\partial D_2}$$

as we claimed. □

(5.3) REMARK: Notice that every element of each of the Hilbert spaces $H(R^d)$, $H(\partial D)$, $H(D)$ and $H(D')$ is an N-component random vector.

(5.4) DEFINITION We say that $H^0(\Omega) = \{\xi(f) = (\xi_j(f) : j = 1,\dots,N) : f \in \mathcal{S}(R^d)\}$ has the restricted Markov property of Wong if

$$E(u_i v_j | H(\partial D)) = E(u_i | H(\partial D)) E(v_j | H(\partial D))$$

for every $u = (u_i, i = 1,\dots,N) \in H(D)$ and

$$v = (v_j, j = 1,\dots,N) \in H(D') .$$

(5.5) THEOREM Let $H^0(\Omega)$ have the restricted Markov property of Wong. Then $H^0(\Omega)$ is Markov in the sense of Wong.

Proof: Let D_1, D, D_2 be bounded open subsets of R^d such that $D_2 \supset D \supset D_1$.

Then, the boundaries ∂D_1, ∂D, ∂D_2 form an increasing family of nested surfaces in R^{d-1}.

Now, Wong's Assumption (4.1.1) implies that $H(\partial D_1)$, $H(\partial D)$, and $H(\partial D_2)$ are subspaces of $H(R^d)$. We have too that $H(\partial D_1)$ is a subspace of $H(D)$ and $H(\partial D_2)$ is a subspace of $H(D')$.

Hence, since $H^0(\Omega)$ has the restricted Markov property of Wong, we have

$$E(u_i v_j | H(\partial D)) = E(u_i | H(\partial D)) E(v_j | H(\partial D))$$

for every $u = (u_i : i = 1,\ldots,N) \in H(\partial D_1) \subset H(D)$ and

$$v = (v_j : j = 1,\ldots,N) \in H(\partial D_2) \subset H(D')$$

This last equation implies

$$\langle u_i, v_j \rangle_{L^2(\Omega,\mathcal{B},\mu)} = \langle u_i, E(v_j | H(\partial D)) \rangle_{L^2(\Omega,\mathcal{B},\mu)}$$

or, equivalently,

$$\langle u_i, v_j - E(v_j | H(\partial D)) \rangle_{L^2(\Omega,\mathcal{B},\mu)} = 0$$

Hence, we have that for every $u \in H(\partial D_1)$ and every $v \in H(\partial D_2)$, $v - E(v| H(\partial D))$ is stochastically independent of u. This is Wong's definition of Markovicity, and hence the claim is justified. □

(5.6) REMARK: The restricted Markov property of Wong (5.4) which implies Wong's notion of Markovicity is in turn implied by another notion of Markov property which we now introduce.

(5.7) DEFINITION Say that $H^0(\Omega) = \{\xi(f) = (\xi_j(f) : j = 1,\ldots,N) : f \in \mathcal{S}(R^d)\}$ has the pre-Markov property if for any $u = (u_i : i = 1,\ldots,N)$ belonging to $H(D)$, we have

$$E(u_i | H(D')) = E(u_i | H(\partial D)), \quad i = 1,\ldots,N$$

(5.8) REMARK: Let us now show that the pre-Markov property implies the restricted Markov property of Wong. From (5.7), it follows that for $u = (u_i : i = 1,\ldots,N) \in H(D)$ and $v = (v_j : j = 1,\ldots,N) \in H(D')$, then

$$E(u_i v_j | H(D')) = v_j E(u_i | H(D')) = v_j E(u_i | H(\partial D))$$

by (5.7). But $H(\partial D)$ is a subspace of $H(D')$. Hence

$$E(E(u_i v_j | H(D')) | H(\partial D))$$

$$= E(u_i | H(\partial D)) E(v_j | H(\partial D))$$

$$= E(u_i v_j | H(\partial D))$$

Thus, we have established that

$$E(u_i v_j | H(\partial D)) = E(u_i | H(\partial D)) E(v_j | H(\partial D))$$

for all $u = (u_i : i = 1,\ldots,N) \in H(D)$ and all

$v = (v_j : i = 1,\ldots,N) \in H(D')$

But this is the restricted Markov property of Wong as we have defined it in (5.4). Hence our claim is justified.

(5.9) REMARK: Let Q_D, $Q_{D'}$ denote respectively the orthogonal projection of $H(R^d)$ onto $H(D)$, $H(D')$. Then (5.7) may be expressed in the following abstract form

$$Q_{D'} Q_D = Q_{\partial D} Q_D \qquad (5.10)$$

Let us now make contact with Nelson's notion of Markov property. To this end, it is well-known [81] that the pre-Markov property implies the Markov property of Nelson. We, therefore, have the following sequence of logical implications

PRE-MARKOV PROPERTY $\Longrightarrow$ RESTRICTED MARKOV PROPERTY OF WONG $\Longrightarrow$ WONG'S NOTION OF MARKOV PROPERTY

$\Downarrow$

NELSON's NOTION OF MARKOV PROPERTY

The results of Chapter 4 now lead us to conclude that Wong's notion of Markov property is genuinely weaker than the restricted Markov property of Wong. Hence, the notions of Markovicity due to Nelson and Wong are different extensions of the pre-Markov property.

CHAPTER 6

CONCLUSION AND OUTLOOK

Throughout this chapter, unless there is an explicit statement to the contrary, we again assume that all generalized stochastic fields under consideration are Gaussian.

Let $(\Omega, \mathcal{B}, \mu)$ be a probability space and let

$$H^{O}(\Omega) = \{\xi(f) = (\xi_{-1}(f), \xi_{0}(f), \xi_{1}(f)) \ : \ f \in \mathcal{S}(R^3)\}$$

be a Euclidean covariant generalized stochastic field on $(\Omega, \mathcal{B}, \mu)$. In Chapter 4, $H^{O}(\Omega)$ was studied in considerable detail, and necessary and sufficient conditions for it to be Markov in the sense of Wong were presented. By means of these conditions, we were then able to obtain explicit expressions for the two spectral measures $d\Phi^{(i)}(|p|)$, $p \in R^3$, $i = 1,2$, associated with $H^{O}(\Omega)$. Because of the mathematical and physical importance of the results of Chapter 4, it is, therefore, interesting to know whether or not the methods employed there are again available in the more general case of arbitrary Euclidean covariant generalized stochastic fields. This particular question has already been answered in the affirmative in (4.1.4). In Section 6.1, we indicate how to extend our results to arbitrary Euclidean covariant generalized stochastic fields.

In Section 6.2, we discuss how Wong's notion of Markov property may be modified so as to accommodate a much wider class of interesting - whether from a mathematical or physical point of view - generalized stochastic fields. Then we provide examples of generalized stochastic fields which satisfy this modified concept of Markovicity in the sense of Wong. These examples include many generalized stochastic fields which are Markov in the sense of Nelson.

6.1 ARBITRARY EUCLIDEAN COVARIANT GENERALIZED STOCHASTIC FIELDS

We refer to (3.2.11) for the definition of Euclidean covariance for generalized stochastic fields.

Let $H^{O}(\Omega) = \{\xi(f) = (\xi_j(f) : j = 1,\dots,N) : f \in \mathcal{S}(R^d)$ be a Euclidean covariant generalized stochastic field on $(\Omega, \mathcal{B}, \mu)$ which transforms according to an irreducible unitary representation $h \rightarrow V^{(s)}(h)$, $s = (s_1,\dots,s_\pi)$, π = integral part of $\frac{1}{2}d$, of SO(d). Then by (3.2.11), if $B(f^{(2)},f^{(1)})$ is the matrix of correlation functionals of $\xi(f)$ in $H^{O}(\Omega)$, we have

$$B(f^{(2)},f^{(1)}) = V^{(s)}(h)B(V_g f^{(2)},V_g f^{(1)})V^{(s)}(h)^{-1}$$

where $(V_g f)(x) = g^{-1}x$ and $g = (a,h) \in E^d \overset{s}{\otimes} SO(d) = M(d)$.

By way of anticipating the need to extend the results of Chapter 4 to arbitrary Euclidean covariant generalized stochastic fields, we have ensured that all the results in Chapters 1 to 3 are formulated in their most general forms.

The analysis of Chapter 4 requires generalization only in two respects. Since we now have to consider an N-dimensional generalized stochastic field which transforms covariantly according to an irreducible unitary representation $h \rightarrow V^{(s)}(h)$ of SO(d), d and N arbitrary, we must have available

(A) The spectral representation for an arbitrary Euclidean covariant generalized stochastic field.

(B) The reduction formula for tensor products of irreducible unitary representations of SO(d). Let us consider (A) and (B) separately.

Now, (A) is already adequately catered for by Theorem (3.2.13) which furnishes a spectral representation for an arbitrary Euclidean covariant generalized stochastic field that transforms according to a given irreducible unitary representation $h \rightarrow V^{(s)}(h)$, $s = (s_1,\dots,s_\pi)$, π = integral part of

$\frac{1}{2}$ d, of SO(d). By invoking Theorem (3.2.13), like we did in (3.2.25), one readily writes down explicitly the matrix $B(f^{(2)},f^{(1)})$ of correlation functionals of any Euclidean covariant generalized stochastic field. It only now remains for us to indicate where matrix elements of irreducible unitary representations of the full Euclidean group may be found, because they intervene in Theorem (3.2.13). To this end, we refer to [89] where a procedure for constructing the mentioned matrix elements is discussed.

Next, consider (B). By comparison with what obtains in the case of SO(3), the reduction of tensor products of irreducible unitary representations of SO(d), d > 3, is a far more difficult undertaking, and the subject gives rise to interesting problems in Combinatorial Mathematics. However, by employing the S-function methods of Littlewood[47], Butler and Wybourne [11], have succeeded in providing a procedure for solving the reduction problem for SO(d). We refer to the latter work for full details. Thus, the reduction formula for the tensor product of two irreducible unitary representations of SO(d) can again be written down explicitly.

It is perhaps instructive to write down the reduction formula for the tensor product of two irreducible unitary representations of SO(4). Here, if

$$h \to V^{(s_1,s_2)}(h) \text{ and } h \to V^{(s_1^0,s_2^0)}(h) \text{ are two such representations}$$

of SO(4), then

$$V^{(s_1,s_2)}(h) \otimes V^{(s_1^0,s_2^0)}(h)$$

$$= \bigoplus_{n=0}^{\alpha} \bigoplus_{m=0}^{\beta} V^{(s_1+s_1^0-n-m,\, s_2+s_2^0-n+m)}(h)^c$$

where

α = minimum of $s_1 + s_2$ and $s_1^0 + s_2^0$

β = minimum of $s_1 - s_2$ and $s_1^0 - s_2^0$

and $h \rightarrow V^{(a,b)}(h)^c$ is a representation which is complex conjugate to $V^{(a,b)}(h)$.

CONCLUSION With the implementation of the generalizations (A) and (B), and again under Wong's Assumption (4.1.1), results analogous to those announced in Chapter 4 are now readily obtained for an arbitrary Euclidean covariant generalized stochastic field. The only significant difference is an increase in the tediousness of the very similar analysis that must be done in the latter case.

We may thus conclude as follows.

As in the analysis of Chapter 4, we can provide necessary and sufficient conditions for an arbitrary Euclidean covariant generalized stochastic field to be Markov in the sense of Wong. These conditions then enable us to write down the explicit forms of the spectral measures associated with such a Markovian generalized stochastic field.

Every massive (i.e. $\alpha_0 > 0$) Euclidean covariant generalized stochastic field which satisfies Wong's notion of Markov property reduces to the Ornstein-Uhlenbec process on R [19][91].

6.2 OUTLOOK

In any definition of Markov property, the mode of specifying boundary data plays a significant part in determining which generalized stochastic fields actually satisfy the given notion of Markovicity. Thus, for example, on account of Wong's Assumption (4.1.1), Wong's notion of Markov property studied in Chapter 4 admits only generalized stochastic fields which are defined for and only for sharp-time. It is, therefore, of interest to try to weaken Wong's Assumption. To do this, one must give alternative, more general methods of specifying boundary data.

BOUNDARY DATA AND MARKOV PROPERTY Let

$H^{O}(\Omega) = \{\xi(f) = (\xi_j(f) : j = 1,\ldots,N) : f \in \mathcal{S}(R^d)\}$

be a not necessarily Gaussian, but Euclidean covariant, generalized stochastic field on the probability space $(\Omega,\mathcal{B},\mu)$. Then, to relax Wong's Assumption (4.1.1), we now merely require as follows:

"There exists a nonempty set of boundary data for $H^{O}(\Omega)$."

For the Markov condition, we again retain Wong's original formulation. Thus, let ∂D_1, ∂D, ∂D_2 be an increasing family of nested boundaries and let $H(\partial D_1)$, $H(\partial D)$ and $H(\partial D_2)$ be respectively the associated boundary data Hilbert spaces, defined in some acceptable way. Then, we require that $H(\partial D_2) - P_{H(\partial D)} H(\partial D_2)$ be stochastically independent of $H(\partial D_1)$.

Let us now conclude this chapter by furnishing examples of generalized stochastic fields which are Markov in the sense of Wong, when we employ the above recipe for obtaining boundary data.

(6.2.1) THEOREM Let $H^{O}(\Omega) = \{\xi(f) = (\xi_j(f) : j = 1,\ldots,N) : f \in \mathcal{S}(R^d)\}$ be a Euclidean covariant Gaussian generalized stochastic field on $(\Omega,\mathcal{B},\mu)$. Set $\langle \xi_i(f^{(2)}), \xi_j(f^{(1)}) \rangle_{L^2(\Omega,\mathcal{B},\mu)} = B_{ij}(f^{(2)},f^{(1)})$. Let $\mathcal{H}(R^d)$ denote the completion of $(\mathcal{S}(R^d))^N$ in the norm derived from the following inner product:

$$\langle \cdot,\cdot \rangle_{\mathcal{H}(R^d)} : (\mathcal{S}(R^d))^N \times (\mathcal{S}(R^d))^N \to R$$

$$(\underline{f}^{(2)},\underline{f}^{(1)}) \to \langle \underline{f}^{(2)},\underline{f}^{(1)} \rangle_{\mathcal{H}(R^d)} = \sum_{i,j=1}^{N} B_{ij}(f_i^{(2)},f_j^{(1)})$$

$$\underline{f}^{(k)} = (f_1^{(k)},\ldots,f_N^{(k)}), \quad k = 1,2.$$

Then, let $H(\partial D)$ be the completion in $(L^2(\Omega,\mathcal{B},\mu))^N$ of the linear space

$$\{\xi(f) = (\xi_1(f),\ldots,\xi_N(f)) : \underline{f} = (f,\ldots,f) \in \mathcal{H}(R^d),\ \text{supp } f \subset \partial D\}$$

of random vectors. We take $H(\partial D)$ to represent the Hilbert space of boundary data for $H^0(\Omega)$ on ∂D. $H(\partial D)$ is indexed by the Hilbert space $\mathcal{H}(\partial D)$ which is the completion of $\{\underline{f} \in \mathcal{H}(R^d) : \text{supp } \underline{f} \subset \partial D\}$ in $\mathcal{H}(R^d)$.

Let $B_{ij}(x-y)$ be the kernel of $B_{ij}(f^{(2)}, f^{(1)})$.

Finally, suppose that $H^0(\Omega)$ is such that

$$\sum_{k=1}^{N} (A_{ik}(\frac{\partial}{\partial x})B_{kj})(x-y) = 0, \quad x \neq y, \; i,j = 1,\ldots,N$$

where

$$A(\frac{\partial}{\partial x}) = (A_{ik}(\frac{\partial}{\partial x}) : i,k = 1,\ldots,N)$$

is a strongly elliptic matrix of linear partial differential operators with constant coefficients.

Then, $H^0(\Omega)$ is Markov in the sense of Wong.

REMARK: The proof of this theorem is completely analogous to the proof of Theorem (4.5.36), provided that we replace $(L^2(\partial D, d\sigma))^N$ by $\mathcal{H}(\partial D)$ in the latter proof. Hence, we will not reproduce the proof here.

Quite a wide class of Euclidean covariant Gaussian generalized stochastic fields satisfy the hypothesis of the last theorem, and hence are Markov in the sense of Wong. Furthermore, if a Euclidean covariant Gaussian generalized stochastic field satisfies the hypothesis of the last theorem and is additionally Markov in the sense of Nelson, then it is also Markov in the sense of Wong. This indicates that the theorem delivers many physically interesting Euclidean covariant Gaussian generalized stochastic fields; the latter include those recently studied, under physical motivation, in [45] and [99].

Needless to mention, Theorem (6.2.1) may be formulated more generally by requiring only that $A(\frac{\partial}{\partial x}) = (A_{ij}\frac{\partial}{\partial x}) : i,j = 1,\ldots,N)$ be a properly elliptic matrix of linear partial differential operators with constant coefficients, for which the Dirichlet problem is well-posed [70]. However, the formulation of Theorem (6.2.1) given above seems adequate for several physical applications.

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APPENDIX

By (2.4.32) and (4.5.38), it follows that

$$B_{ij}(x-x_0) = \sum_{k,l=-1}^{1} \int_{\partial D} d\sigma(z)\, M_{kl}\left(B_{kj}(z-x_0), G_{li}(z,x)\right)$$

Since the principal part of $A_{ij}\left(\frac{\partial}{\partial x}\right)$ is of order 4, the highest order of the normal derivative $\frac{\partial}{\partial n_z}$ of $B_{kj}(z-x_0)$ which occurs on the right hand side of the last equation is one, by (2.4.18). Hence, $B_{ij}(x-x_0)$ has the following integral representation

$$B_{ij}(x-x_0) = \sum_{k=-1}^{1} \int_{\partial D} d\sigma(z) \left[P^{(1)}_{ik}(x,z)\, B_{kj}(z-x_0) + P^{(2)}_{ik}(x,z) \frac{\partial}{\partial n_z} B_{kj}(z-x_0) \right]$$

in which $P^{(i)}(x,z)$ is a uniquely determined matrix of kernels, for $i=1,2$, in view of the uniqueness of the solution of the Dirichlet problem.

Now, let $\left(\frac{\partial}{\partial n_z}\right)^*$ denote the operator adjoint of $\frac{\partial}{\partial n_z}$ relative to the usual inner product of $L^2(\partial D, d\sigma)$. Then, we have finally

$$B_{ij}(x-x_0) = \sum_{k=-1}^{1} \int_{\partial D} d\sigma(z) \left[\left(P^{(1)}_{ik}(x,z) + \left(\frac{\partial}{\partial n_z}\right)^* P^{(2)}_{ik}(x,z) \right) B_{kj}(z-x_0) \right]$$

$$= \sum_{k=-1}^{1} \int_{\partial D} d\sigma(z)\, P_{ik}(x,z)\, B_{kj}(z-x_0)$$

which is (4.5.39), if we set

$$P^{(1)}_{ik}(x,z) + \left(\frac{\partial}{\partial n_z}\right)^* P^{(2)}_{ik}(x,z) = P_{ik}(x,z).$$