ON THE UNIVALENCY OF CERTAIN CLASSES OF ANALYTIC FUNCTIONS

by

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ABSTRACT

In Chapter I we begin by considering a theorem of J. Dieudonné on the minimum radius of starlikeness of a class of analytic functions. We give a simple new proof of this theorem. By this new proof also we find the minimum radius of univalence of this class and we determine all the cases which give the minimum radius of univalence and the minimum radius of starlikeness. We then use a method similar to that in this new proof to obtain the minimum radius of univalence and the minimum radius of starlikeness of some other classes of analytic functions. For each class we determine all the cases giving the minimum radius of univalence and the minimum radius of starlikeness. Then we give some similar results for the minimum radius of convexity.

In Chapter II first we deal with Heawood's Lemma which was established and used by P.J. Heawood to prove the theorem known as the Grace-Heawood Theorem. The same lemma was used by S. Kakeya in the proof of another theorem. We show that Heawood's Lemma is false and we give new proofs of these results. Then for some special cases we improve the value of the radius of univalence

given by Kakeya's Theorem. In this connection we first give L.N. Čakalov's result and then we obtain some improvements of his result.

In Appendix I we give some examples related with Chapter I and Chapter II. In Appendix II we give an example which shows that there is an error in a paper by M. Robertson.

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CONTENTS

CHAPTER I

We begin with the following definitions.

DEFINITION 1.1. The radius of univalence of an analytic function $f(z)$ is the radius r of the greatest circle $|z| < r$ in which $f(z)$ is univalent.

DEFINITION 1.2. The radius of starlikeness of an analytic function $f(z)$ satisfying $f(0) = 0$, $f'(0) \neq 0$ is the radius r of the greatest circle $|z| < r$ in which the following conditions are satisfied

(i) $\frac{zf'(z)}{f(z)}$ is regular,

(ii) $\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$.

DEFINITION 1.3. The radius of convexity of an analytic function $f(z)$ satisfying $f'(0) \neq 0$ is the radius r of the greatest circle $|z| < r$ in which the following conditions are satisfied

(i) $1 + \frac{zf''(z)}{f'(z)}$ is regular,

(ii) $\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0$.

DEFINITION 1.4. The radius of starlikeness of order λ, $0 \leq \lambda \leq 1$, of an analytic function $f(z)$ satisfying $f(0) = 0$, $f'(0) \neq 0$ is the radius r of the greatest circle

$|z| < r$ in which the following conditions are satisfied

(i) $\dfrac{zf'(z)}{f(z)}$ is regular,

(ii) $\text{Re}\left\{\dfrac{zf'(z)}{f(z)}\right\} > \lambda$.

DEFINITION 1.5. The radius of convexity of order λ, $0 \leq \lambda \leq 1$, of an analytic function $f(z)$ satisfying $f'(0) \neq 0$ is the radius r of the greatest circle $|z| < r$ in which the following conditions are satisfied

(i) $1 + \dfrac{zf''(z)}{f'(z)}$ is regular,

(ii) $\text{Re}\left\{1 + \dfrac{zf''(z)}{f'(z)}\right\} > \lambda$.

Let $P(z)$ be a polynomial of degree n all of whose zeros are exterior to or on the circumference of the unit circle. It was proved by J.W. Alexander [1] that the polynomial $f(z) = zP(z)$ is univalent and starlike in $|z| < \frac{1}{n+1}$. A more general result was proved by J. Dieudonné [3] for the functions of the form $f(z) = z\left[P(z)\right]^{\frac{\alpha}{\pi}}$, where $P(z)$ is again a polynomial of degree n and α is any real number, in the following theorem.

THEOREM 1.1. <u>Let</u> $P(z)$ <u>be a polynomial of degree</u> $n(\neq 0)$ <u>all of whose zeros are exterior to or on the circumference of the unit circle. If</u> α <u>is any real number different from zero then the minimum radius of starlikeness of</u>

<u>the function</u> $f(z) = z\,[P(z)]^{\frac{\alpha}{n}}$ <u>has the value</u>:

$$S_0 = 1 \qquad \text{if} \qquad -2 \leq \alpha < 0 \ ;$$

$$S_0 = \frac{1}{|1+\alpha|} \qquad \underline{\text{otherwise}}.$$

First we give Dieudonné's proof then we give a simple new proof of the theorem. By this new proof also we find the minimum radius of univalence of $f(z) = z\,[P(z)]^{\frac{\alpha}{n}}$ and by considering the distribution of the zeros of P(z) outside the unit disc we determine all the cases which give the minimum radius of univalence and the minimum radius of starlikeness. We then use a method similar to that in this new proof to obtain some other results such as the minimum radius of univalence and the minimum radius of starlikeness of a rational function, the minimum radius of univalence and the minimum radius of starlikeness of a polynomial $f(z) = zP(z)$ where the zeros of P(z) lie in an annulus etc. Then we give some similar results for the minimum radius of convexity.

We begin by giving Dieudonné's proof.

<u>Proof</u>. Let D be a simply connected domain in the plane of the complex variable z, and suppose that the function $\varphi(z)$ makes a one-to-one conformal representation onto a convex domain Δ. If $a_1, a_2, \ldots, a_n$ are n points interior to D there exists a single point ζ interior to D such that

$$\varphi(\zeta) = \frac{\varphi(a_1) + \varphi(a_2) + \ldots + \varphi(a_n)}{n} .$$

This property is an immediate consequence of the convexity of Δ. Consider in particular the function $\varphi_1(z) = \frac{1}{z}$ and the domain D to be the interior of a circle which does not contain the origin. Then if $a_1, a_2, \ldots, a_n$ are points of such a domain the point ζ_1 defined by

$$\frac{n}{\zeta_1} = \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} \quad \text{(harmonic mean)} \qquad (1.1)$$

is also interior to the domain.

Now let P(z) be a polynomial of degree n all of whose zeros $x_1, x_2, \ldots, x_n$ are exterior to or on the circumference of the unit circle; by dividing if necessary by a constant it may be supposed that P(0) = 1. Then

$$P(z) = (1 - \frac{z}{x_1})(1 - \frac{z}{x_2}) \ldots (1 - \frac{z}{x_n}) ,$$

and
$$\frac{P'(z)}{P(z)} = \frac{1}{z-x_1} + \frac{1}{z-x_2} + \ldots + \frac{1}{z-x_n} .$$

By the preceding remarks we can show that the function $\Phi(z)$ defined by

$$\frac{P'(z)}{P(z)} = \frac{n}{z - \frac{1}{\Phi(z)}} \qquad (1.2)$$

is regular and of modulus not greater than one in the unit circle. To show this, from (1.2),

$$\phi(z) = \frac{1}{z-n\frac{P(z)}{P'(z)}} = \frac{1}{z-n\frac{1}{\frac{1}{z-x_1}+\dots+\frac{1}{z-x_n}}}$$

$$= \frac{\frac{1}{z-x_1}+\dots+\frac{1}{z-x_n}}{\frac{z}{z-x_1}+\dots+\frac{z}{z-x_n}-n} = \frac{\frac{1}{z-x_1}+\dots+\frac{1}{z-x_n}}{\left(\frac{z}{z-x_1}-1\right)+\dots+\left(\frac{z}{z-x_n}-1\right)}$$

$$= \frac{\frac{1}{x_1-z}+\dots+\frac{1}{x_n-z}}{\frac{1}{1-\frac{z}{x_1}}+\dots+\frac{1}{1-\frac{z}{x_n}}} \tag{1.3}$$

Since $\left|\frac{z}{x_1}\right| < 1, \dots \left|\frac{z}{x_n}\right| < 1$ for $|z| < 1$ then $1-\frac{z}{x_1}, \dots, 1-\frac{z}{x_n}$ lie in the circle $|1-w| < 1$ and by (1.1) the point ζ_1 defined by

$$\frac{n}{\zeta_1} = \frac{1}{1-\frac{z}{x_1}}+\dots+\frac{1}{1-\frac{z}{x_n}}$$

lies in the same circle. Therefore the denominator of (1.3) cannot vanish which implies that $\phi(z)$ is regular in $|z| < 1$. Now

$$z\phi(z) = \frac{\frac{z}{x_1-z}+\dots+\frac{z}{x_n-z}}{\frac{1}{1-\frac{z}{x_1}}\cdots\frac{1}{1-\frac{z}{x_n}}}$$

$$= \frac{\left(\frac{x_1}{x_1 - z} - 1\right) + \ldots + \left(\frac{x_n}{x_n - z} - 1\right)}{\frac{1}{1 - \frac{z}{x_1}} + \ldots + \frac{1}{1 - \frac{z}{x_n}}}$$

$$= 1 - \frac{n}{\frac{1}{1 - \frac{z}{x_1}} + \ldots + \frac{1}{1 - \frac{z}{x_n}}} = 1 - \zeta_1$$

Therefore $|z\,\phi(z)| < 1$ in $|z| < 1$ and by Schwarz's Lemma $|\phi(z)| \leq 1$ in $|z| < 1$.

Now we apply formula (1.2) to seek the minimum radius of starlikeness of the function $f(z) = z[P(z)]^{\frac{\alpha}{n}}$. For this we find the radius r of the greatest circle $|z| \leq r$ such that $\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqq 0$ in the circle. Let

$$u(z) = \frac{zf'(z)}{f(z)} = 1 + \frac{\alpha}{n}\, z\frac{P'(z)}{P(z)} ,$$

or from (1.2),

$$u(z) = 1 + \frac{\alpha z\phi(z)}{z\phi(z) - 1} = \frac{(1+\alpha)z\,\phi(z) - 1}{z\,\phi(z) - 1} .$$

For $|z| \leq r$, we have $|z\,\phi(z)| \leq r$, and by considering lines as limiting cases of circles we can see that the point u(z) remains inside a circle γ whose centre is on the real axis and which cuts this axis in the points $u_1 = \frac{(1+\alpha)r - 1}{r - 1}$, $u_2 = \frac{1 + (1+\alpha)r}{1+r}$.

u(z) actually takes these values when $\phi(z) = e^{i\theta}$ with θ a real constant. The necessary and sufficient condition in order that $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqq 0$ be satisfied by all polynomials P(z) under consideration is that $u_1 \geqq 0$, $u_2 \geqq 0$. If $|1+\alpha| \leqq 1$ these conditions are satisfied whenever $r \leqq 1$, if $|1+\alpha| \geqq 1$ they reduce to $r \leqq \frac{1}{|1+\alpha|}$. This proves the theorem. For $\alpha = n$, this yields Alexander's theorem. If $P(z) = (z - e^{i\theta})^n$ then these minimum values are attained.

Here one can ask the question, considering the distribution of zeros of P(z) outside the unit disc, if P(z) is not of the form $(z-e^{i\theta})^n$, then for each real α is the radius of starlikeness of f(z) greater than the given minimum value? and also, what is the minimum radius of univalence of f(z) and in which cases this is attained? Now we give the new proof(i) and by this simple method answer these questions.

THEOREM 1.2. Let P(z) be a polynomial of degree $n(\neq 0)$ all of whose zeros are exterior to or on the circumference of the unit circle. If α is any real number different from zero then the minimum radius of univalence and the minimum radius of starlikeness of the function $f(z) = z[P(z)]^{\frac{\alpha}{n}}$ have the values:

$$U_0 = S_0 = 1 \qquad \text{if} \qquad -2 \leqq \alpha < 0\ ;$$

$$U_0 = S_0 = \frac{1}{|1+\alpha|} \qquad \text{otherwise.}$$

(i) Theorem 1.2 may also be proved by an argument similar to Dieudonné's proof of Theorem 1.1.

<u>If</u> $-2 \leq \alpha < 0$ <u>then</u> $f(z)$ <u>has this minimum radius of univalence and the minimum radius of starlikeness if and only if</u> $P(z)$ <u>has at least one zero on the circumference of the unit circle; otherwise if and only if all the zeros of</u> $P(z)$ <u>are concentrated at the same point on the circumference of the unit circle</u>.

<u>Proof</u>. Let

$$P(z) = a_0(z-x_1)(z-x_2) \dots (z-x_n) \quad .$$

Then

$$\frac{P'(z)}{P(z)} = \frac{1}{z-x_1} + \frac{1}{z-x_2} + \dots + \frac{1}{z-x_n}$$

so

$$z\frac{f'(z)}{f(z)} = 1 + \frac{\alpha}{n} z\frac{P'(z)}{P(z)} \quad ,$$

and

$$\mathrm{Re}\left\{z\frac{f'(z)}{f(z)}\right\} = \mathrm{Re}\left\{1 + \frac{\alpha}{n}\left(\frac{z}{z-x_1} + \frac{z}{z-x_2} + \dots + \frac{z}{z-x_n}\right)\right\} \tag{1.4}$$

We will consider all possible cases given by real values of α .

<u>Case 1</u>. $\alpha > 0$

If $\mathrm{Re}\left\{\frac{z}{z-x}\right\} \geq -\frac{1}{\alpha}$ for $x = x_1, x_2, \dots, x_n$ then, by (1.4) $\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq 0$. Putting $z = re^{i\theta}$, $x = Re^{i\varphi}$, we have

$$\mathrm{Re}\left\{\frac{z}{z-x}\right\} = \frac{r^2 - rR\cos(\theta-\varphi)}{r^2 - 2rR\cos(\theta-\varphi) + R^2} \geq -\frac{1}{\alpha}$$

$$\Longleftrightarrow \alpha r^2 - rR\alpha\cos(\theta-\varphi) \geq -r^2+2rR\cos(\theta-\varphi)-R^2$$

$$\Longleftrightarrow r^2(\alpha+1)-rR(\alpha+2)\cos(\theta-\varphi)+R^2 \geq 0.$$

But $r^2(\alpha+1)-rR(\alpha+2)\cos(\theta-\varphi)+R^2 \geq r^2(\alpha+1)-rR(\alpha+2)+R^2$,

and the roots of $r^2(\alpha+1)-rR(\alpha+2)+R^2 = 0$

are R and $\frac{R}{\alpha+1}$. Since $R \geq 1$, for $r \leq \frac{1}{\alpha+1}$ we have

$\text{Re}\left\{\frac{z}{z-x}\right\} \geq -\frac{1}{\alpha}$, where equality exists if and only if R=1,

$r = \frac{1}{\alpha+1}$, and $\cos(\theta-\varphi) = 1$. Therefore $S_0 = \frac{1}{\alpha+1}$, and this minimum value is attained if and only if all the zeros of P(z) are concentrated at one point on the circumference of the unit circle. Otherwise, by continuity, $\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$ is satisfied in a larger circle.

Now $\text{Im}\left\{\frac{z}{z-x}\right\} = \dfrac{-rR\sin(\theta-\varphi)}{r^2-2rR\cos(\theta-\varphi)+R^2}$,

which vanishes for $\theta=\varphi$. Therefore if the zeros of P(z) are situated as stated above then $\text{Im}\left\{\frac{zf'(z)}{f(z)}\right\}$ vanishes at the same point on the circumference of $|z| \leq S_0$ where $\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = 0$. This implies that at this point $f'(z) = 0$. Since f(z) is univalent (i) in $|z| < S_0$ then $U_0 = \frac{1}{\alpha+1}$ and this minimum value is attained if and only if all the zeros of P(z) are concentrated at the same point on the circum-

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(i) See, e.g. [10], p.206, Theorem 10.

ference of the unit circle.

Case 2. $0 > \alpha > -1$

Since $\alpha < 0$, if $\mathrm{Re}\left\{\frac{z}{z-x}\right\} < -\frac{1}{\alpha}$ for $x = x_1, x_2, \ldots, x_n$, then by (1.4) $\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$. We have

$$\frac{r^2-rR\cos(\theta-\varphi)}{r^2-2rR\cos(\theta-\varphi)+R^2} < -\frac{1}{\alpha} \qquad (1.5)$$

$$\Longleftrightarrow \alpha r^2-rR\alpha\cos(\theta-\varphi) > -r^2+2rR\cos(\theta-\varphi)-R^2 \qquad (1.6)$$

$$\Longleftrightarrow r^2(\alpha+1)-rR(\alpha+2)\cos(\theta-\varphi)+R^2 > 0 \qquad (1.7)$$

But $\quad r^2(\alpha+1)-rR(\alpha+2)\cos(\theta-\varphi)+R^2 \geq r^2(\alpha+1)-rR(\alpha+2)+R^2 \quad (1.8)$

and the roots of $\quad r^2(\alpha+1)-rR(\alpha+2)+R^2 = 0$

are $\frac{R}{\alpha+1}$, R. Since $R \geq 1$ for $|z| < 1$ we have

$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$. Thus f(z) is univalent and starlike in $|z| < 1$. If the minimum modulus of the zeros of P(z) is 1 then f(z) is not analytic in any larger circle about the origin. Therefore $U_0 = S_0 = 1$ and both of these minimum values are attained if and only if P(z) has at least one zero on the circumference of the unit circle.

Case 3. $\alpha = -1$

We have $\quad \frac{r^2-rR\cos(\theta-\varphi)}{r^2-2rR\cos(\theta-\varphi)+R^2} < 1$

$$\Longleftrightarrow r^2-rR\cos(\theta-\varphi) < r^2-2rR\cos(\theta-\varphi)+R^2$$

$$\Longleftrightarrow rR\cos(\theta-\varphi) < R^2$$

$$\Longleftrightarrow r\cos(\theta-\varphi) < R \quad .$$

Since $f(z)$ is univalent and starlike in $|z|<1$ and if $P(z)$ has a zero on the circumference of the unit circle $f(z)$ is not analytic in any larger circle about the origin, we have $U_0 = S_0 = 1$ and both of these minimum values are attained if and only if $P(z)$ has at least one zero on the circumference of the unit circle.

Case 4. $-1 > \alpha \geq -2$

By (1.5), (1.6), (1.7), (1.8) and the fact that $f(z)$ may not be analytic in any circle about the origin whose radius is greater than 1, we have $U_0 = S_0 = 1$. Both of these minimum values are attained if and only if $P(z)$ has at least one zero on the circumference of the unit circle.

Case 5. $\alpha < -2$

We have $\dfrac{r^2-rR\cos(\theta-\varphi)}{r^2-2rR\cos(\theta-\varphi)+R^2} \leq -\dfrac{1}{\alpha}$

$$\Longleftrightarrow \alpha r^2-\alpha rR\cos(\theta-\varphi) \geq -r^2+2rR\cos(\theta-\varphi)-R^2 \quad .$$

But $\quad r^2(\alpha+1)-rR(\alpha+2)\cos(\theta-\varphi)+R^2 \geq r^2(\alpha+1)+rR(\alpha+2)+R^2$

and the roots of $\quad r^2(\alpha+1)+rR(\alpha+2)+R^2 = 0$

are $-R$ and $-\frac{R}{\alpha+1}$. Since $R \geq 1$, for $r \leq \frac{1}{|\alpha+1|}$ we have $Re\left\{\frac{z}{z-x}\right\} \leq -\frac{1}{\alpha}$, where equality exists if and only if $R=1$,

$r = \frac{1}{|\alpha+1|}$ and $\cos(\theta-\varphi) = -1$. Therefore $S_0 = \frac{1}{|\alpha+1|}$ and this minimum value is attained if and only if all the zeros of P(z) are concentrated at the same point on the circumference of the unit circle. Then by an argument similar to that used in case 1, we have $U_0 = \frac{1}{|\alpha+1|}$ and this minimum value is is attained if and only if all the zeros of P(z) are concentrated at the same point on the circumference of the unit circle. This completes the proof of the theorem.

It is known that the function $w = f(z)$ maps the interior of the unit circle onto a convex domain if and only if the function $zf'(z)$ maps the interior of the unit circle onto a starlike domain. Now we develop a method which enables us to obtain results for the minimum radius of convexity. If f(z) is of the form zQ(z), we shall write g'(z) for Q(z). Thus, having proved a theorem for the minimum radius of starlikeness of f(z) we can deduce a similar result for the minimum radius of convexity of g(z). Therefore we will prove the theorems for the minimum radius of starlikeness and after each proof we will state the corresponding theorem for the minimum radius of convexity.

THEOREM 1.3. _Let_ f'(z) _be a function of the form_ $f'(z) = P(z)^{\frac{\alpha}{n}}$, _where_ P(z) _is a polynomial of degree_ $n(\neq 0)$ _all of whose zeros are exterior to or on the circumference_ of the unit circle. If α is any real number different from zero then the minimum radius of convexity of the function f(z) has the values:

$$C_0 = 1 \quad \text{if} \quad -2 \leq \alpha < 0 \ ;$$

$$C_0 = \frac{1}{|1+\alpha|} \qquad \text{otherwise.}$$

If $-2 \leq \alpha < 0$ then f(z) has this minimum radius of convexity if and only if P(z) has at least one zero on the circumference of the unit circle; otherwise if and only if all the zeros of P(z) are concentrated at the same point on the circumference of the unit circle.

Note. If $-2 \leq \alpha < 0$ then f(z) may not be analytic in any circle about the origin whose radius is greater than 1. Therefore $C_0 = U_0 = 1$ and U_0 is also attained if and only if P(z) has at least one zero on the circumference of the unit circle.(i)

Now we prove other results for the minimum radius of univalence and starlikeness by using arguments similar to that used in Theorem 1.2. First we consider the distribution of the zeros of P(z) relative to an annulus.

THEOREM 1.4. Let f(z) be a polynomial of the form f(z) = zP(z). If m($\neq$0) zeros of the polynomial P(z) lie in

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(i) The first two examples given in Appendix† are related to Theorem 1.2 and Theorem 1.3.

the annulus $0 \angle d \leq |z| \angle D$ and the remaining $n(\neq 0)$ are situated in $|z| \geq D$ then the minimum radius of univalence and the minimum radius of starlikeness of $f(z)$ are $U_o = S_o = \frac{-y_o D}{1-y_o}$ where y_o is the greater root of the equation

$$n(d-D)y^2 + \left[d(1-n) - D(1+m)\right]y - d = 0 \qquad (1.9)$$

$f(z)$ has this minimum radius of univalence and the minimum radius of starlikeness if and only if m zeros of $P(z)$ are concentrated at a point on the circumference of the circle $|z| \leq d$ and n zeros are concentrated at a point on the circumference of the circle $|z| \leq D$, where these two concentration points and the origin are collinear and the two concentration points are on the same side of the origin.

Proof.

We have $P(z) = a_0(z-x_1)(z-x_2)\ldots(z-x_m)(z-x_{m+1})\ldots(z-x_{m+n})$,

and $$\frac{zP'(z)}{P(z)} = \frac{z}{z-x_1} + \frac{z}{z-x_2} + \ldots + \frac{z}{z-x_m} + \frac{z}{z-x_{m+1}} + \ldots + \frac{z}{z-x_{m+n}}$$

so $$\frac{zf'(z)}{f(z)} = \frac{zP(z) + z^2P'(z)}{zP(z)} = 1 + \frac{zP'(z)}{P(z)},$$

and $$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \mathrm{Re}\left\{1 + \frac{z}{z-x_1} + \frac{z}{z-x_2} + \ldots + \frac{z}{z-x_m} + \frac{z}{z-x_{m+1}} + \ldots + \frac{z}{z-x_{m+n}}\right\}$$

First we will show that if y is any negative number and if $|x| \geq d$ then for $r \leq \frac{-yd}{1-y}$ we have $\mathrm{Re}\left\{\frac{z}{z-x}\right\} \geq y$. Putting $z = re^{i\theta}$, $x = Re^{i\varphi}$, we have

$$\mathrm{Re}\left\{\frac{z}{z-x}\right\} = \frac{r^2 - rR\cos(\theta-\varphi)}{r^2 - 2rR\cos(\theta-\varphi) + R^2} \geq y$$

$$\Longleftrightarrow r^2 - rR\cos(\theta-\varphi) \geq yr^2 - 2rRy\cos(\theta-\varphi) + R^2y$$

$$\Longleftrightarrow r^2(1-y) - rR(1-2y)\cos(\theta-\varphi) - yR^2 \geq 0 .$$

Since y is negative, we have

$$r^2(1-y) - rR(1-2y)\cos(\theta-\varphi) - yR^2 \geq r^2(1-y) - rR(1-2y) - yR^2 .$$

The roots of $r^2(1-y)-rR(1-2y)-yR^2 = 0$ are $-\frac{yR}{1-y}$ and R. Since $R \geq d$, for $r \leq \frac{-yd}{1-y}$ we have $\operatorname{Re}\left\{\frac{z}{z-x}\right\} \geq y$, where equality exists if and only if $R = d$, $r = \frac{-yd}{1-y}$, and $\cos(\theta-\varphi) = 1$. Therefore, if y_1, y_2 are any two negative numbers then

for $r \leq \frac{-y_1 d}{1-y_1}$ we have $\operatorname{Re}\left\{\frac{z}{z-x}\right\} \geq y_1$ provided $|x| \geq d$,

and for $r \leq \frac{-y_2 D}{1-y_2}$ we have $\operatorname{Re}\left\{\frac{z}{z-x}\right\} \geq y_2$ provided $|x| \geq D$.

Now we will find negative y_1, y_2 satisfying the system

$$-\frac{y_1 d}{1-y_1} = \frac{-y_2 D}{1-y_2} ,$$

$$my_1 + ny_2 = -1 .$$

To solve this system we write $y_1 = \frac{-1-ny_2}{m}$ from second equation and substitute in the first, obtaining

$$\frac{-d-ny_2 d}{m+1+ny_2} = \frac{y_2 D}{1-y_2}$$

$$\Longleftrightarrow -d - ny_2 d + y_2 d + ny_2^2 d = y_2 Dm + y_2 D + ny_2^2 D$$

$$\Longleftrightarrow ny_2^2(d-D) + y_2\left[d(1-n) - D(1+m)\right] - d = 0 .$$

Both roots of this equation are negative because the product of the roots is $\frac{-d}{n(d-D)} > 0$ and the sum of the roots is $\frac{nd-d+mD+D}{n(d-D)} < 0$. Since $-\frac{1}{n}$ is situated between the roots, if we chose the greater root y_o then $y_o > -\frac{1}{n}$ and $\frac{-1-ny_o}{m} = y_1 < 0$. Hence for $|z| \leq \frac{-y_oD}{1-y_o}$ we have $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq 0$. Equality exists somewhere on the circumference of the circle $|z| \leq \frac{-y_oD}{1-y_o}$ if and only if m zeros of P(z) are concentrated at a point on the circumference of the circle $|z| \leq d$ and n zeros are concentrated at a point on the circumference of $|z| \leq D$ where these two concentration points and the origin are collinear and the two concentration points are on the same side of the origin. Also, if $y = \frac{d}{d-D}$,

then $\quad n(d-D)y^2 + [d(1-n)-D(1+m)]\,y - d > 0$.

Therefore $\quad y_o > \frac{d}{d-D} \Longrightarrow \frac{-y_oD}{1-y_o} < d$.

Consequently $S_o = \frac{-y_oD}{1-y_o}$, and this minimum value is attained if and only if the zeros of P(z) are situated as stated above.

$$\text{Now} \quad \operatorname{Im}\left\{\frac{z}{z-x}\right\} = \frac{-rR\sin(\theta-\varphi)}{r^2-2rR\cos(\theta-\varphi)+R^2},$$

which vanishes for $\theta = \varphi$. Thus if the zeros of P(z) are situated as stated above then $\operatorname{Im}\left\{\frac{zf'(z)}{f(z)}\right\}$ vanishes at the same point on the circumference of $|z| \leq \frac{-y_oD}{1-y_o}$ where $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = 0$. This implies that at the same point

$f'(z) = 0$, which completes the proof of the theorem.

Note. In this proof we assumed that m and n are both different from zero. Now consider the case when just one of them is equal to zero. We assume without loss of generality that m is different from zero. Let y_0 be the root of equation (1.9). The theorem is still valid and is equivalent to Theorem 1.2 in the case $\alpha = n$.

Let us compare this result with the result given by L.N. Čakalov [p. 57, Theorem 2.6] for the radius of univalence of a polynomial f(z). Here we considered the distribution of the zeros of $\frac{f(z)}{z}$ relative to an annulus, whereas he considered the distribution of the zeros of f'(z) relative to an annulus. Čakalov's result is valid for only a special type of annulus about the origin and for a special type of distribution of the zeros relative to this annulus. But the result which we proved is valid for every annulus about the origin and for every type of distribution of the zeros relative to an annulus. Also we find the minimum radius of univalence and the only possible case giving this minimum value. Considering the distribution of the zeros of f'(z) we deduce the following result for the minimum radius of convexity.

THEOREM 1.5. Let f'(z) be a polynomial, $m(\neq 0)$ of whose zeros lie in the annulus $0 < d < |z| < D$ and the remaining $n(\neq 0)$ are situated in $|z| \geq D$. Then the minimum radius of convexity of $f(z)$ is

$$C_0 = \frac{-y_0 D}{1-y_0} ,$$

where y_0 is the greater root of the equation

$$n(d-D)y^2 + [d(1-n)-D(1+m)]\, y - d = 0 .$$

$f(z)$ has this minimum radius of convexity if and only if m zeros of $f'(z)$ are concentrated at a point on the circumference of the circle $|z| \leq d$ and n zeros are concentrated at a point on the circumference of $|z| \leq D$, where these two concentration points and the origin are collinear and the two concentration points are on the same side of the origin.

Note. Consideration of the cases where just one of m, n is zero gives the result corresponding to that explained in the "note" after the proof of Theorem 1.4.

Here $f(z)$ is univalent inside the same circle where it is convex. Therefore we may compare also this result with Čakalov's result. For suitable distributions of the zeros of $f'(z)$ outside $|z| \leq 1$ we obtain better estimates for the minimum radius of univalence.

Now we will determine the minimum radius of starlikeness and convexity of order λ .

THEOREM 1.6. Let $P(z)$ be a polynomial of degree $n(\neq 0)$ all of whose zeros are exterior to or on the circumference of the unit circle. If $0 \leq \lambda \leq 1$ then the minimum radius of starlikeness of order λ of $f(z) = zP(z)$ is

$$S_\lambda = \frac{1-\lambda}{1+n-\lambda} \, .$$

f(z) <u>has this minimum radius of starlikeness of order</u> λ <u>if and only if all the zeros of</u> P(z) <u>are concentrated at the same point on the circumference of the unit circle. If</u> $\lambda = 1$ <u>then the radius of starlikeness of order</u> λ <u>of</u> $f(z) = zP(z)$ <u>is always zero</u>.

<u>Proof</u>. Denoting the zeros of P(z) by $x_1, x_2, \ldots, x_n$, we have
$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \operatorname{Re}\left\{1 + \frac{z}{z-x_1} + \frac{z}{z-x_2} + \ldots + \frac{z}{z-x_n}\right\}.$$

If
$$\operatorname{Re}\left\{\frac{z}{z-x}\right\} \geqq \frac{\lambda-1}{n}$$
for $x = x_1, x_2, \ldots, x_n$ then
$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqq \lambda \, .$$

Putting $z = re^{i\theta}$, $x = Re^{i\varphi}$, we have
$$\operatorname{Re}\left\{\frac{z}{z-x}\right\} = \frac{r^2 - rR\cos(\theta-\varphi)}{r^2-2rR\cos(\theta-\varphi)+R^2} \geqq \frac{\lambda-1}{n}$$

$$\Longleftrightarrow nr^2 - nrR\cos(\theta-\varphi) \geqq (\lambda-1)r^2 - 2rR(\lambda-1)\cos(\theta-\varphi) + (\lambda-1)R^2$$

$$\Longleftrightarrow r^2(n-\lambda+1) - rR\,[n-2(\lambda-1)]\cos(\theta-\varphi) - (\lambda-1)R^2 \geqq 0 \, .$$

But $r^2(n-\lambda+1) - rR\,[n-2(\lambda-1)]\cos(\theta-\varphi) - (\lambda-1)R^2$
$$\geqq r^2(n-\lambda+1) - rR\,[n-2(\lambda-1)] - (\lambda-1)R^2 \, ,$$
and the roots of the equation
$$r^2(n-\lambda+1) - rR\,[(n-2(\lambda-1)] - (\lambda-1)R^2 = 0$$
are $\frac{R(1-\lambda)}{n-\lambda+1}$ and R. Since $R \geqq 1$ for $r < \frac{1-\lambda}{n-\lambda+1}$ we have

$\operatorname{Re}\left\{\frac{z}{z-x}\right\} \geq \frac{\lambda-1}{n}$. Equality exists if and only if $R = 1$, $\cos(\theta-\varphi) = 1$ and $r = \frac{1-\lambda}{n-\lambda+1}$. Therefore, for $r \leq \frac{1-\lambda}{n-\lambda+1}$ we have $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq \lambda$, where equality exists if and only if all the zeros of $P(z)$ are concentrated at the same point on the circumference of the unit circle.

If $\lambda = 1$ then for $z = 0$ we have $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = 1$. By the minimum principle for harmonic functions $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq 1$ cannot be satisfied at all points of a circle about the origin. Therefore for every type of distribution of the zeros of $P(z)$ outside $|z| < 1$ the radius of starlikeness of order λ is zero. Using the fact that $f(z)$ is convex of order λ if and only if $zf'(z)$ is starlike of order λ we have the following theorem.

THEOREM 1.7. *Let* $f'(z)$ *be a polynomial of degree* $n(\neq 0)$ *all of whose zeros are exterior to or on the circumference of the unit circle. If* $0 \leq \lambda < 1$ *then the minimum radius of convexity of order* λ *of* $f(z)$ *is*

$$C_\lambda = \frac{1-\lambda}{1+n-\lambda}.$$

$f(z)$ *has this minimum radius of convexity of order* λ *if and only if all the zeros of* $f'(z)$ *are concentrated at the same point on the circumference of the unit circle. If* $\lambda = 1$ *then the radius of convexity of order* λ *of* $f(z)$ *is always zero.*

Now we seek the minimum radius of univalence and the minimum radius of starlikeness of a rational function. In theorem 1.2 we have shown that if $P(z)$ is a polynomial of degree n all of whose zeros are exterior to or on the circumference of the unit circle then the minimum radius of univalence and the minimum radius of starlikeness of $f(z) = zP(z)$ is $\frac{1}{n+1}$. Now we consider a rational function of the form $f(z) = z\frac{M(z)}{N(z)}$ where $M(z)$ is a polynomial of degree $m(\neq 0)$, $N(z)$ is a polynomial of degree $n(\neq 0)$ and all the zeros of $M(z)$ and $N(z)$ are exterior to or on the circumference of the unit circle. We obtain a similar result, Theorem 1.8, that the minimum radius of univalence and the minimum radius of starlikeness are greater than $\frac{1}{\eta+1}$ where $\eta = m+n$. Then in Theorem 1.10 we determine the exact values of the minimum radius of univalence and the minimum radius of starlikeness.

THEOREM 1.8. Let $f(z)$ be a rational function of the form $f(z) = \frac{zM(z)}{N(z)}$ where $M(z)$ is a polynomial of degree $m(\neq 0)$, $N(z)$ is a polynomial of degree $n(\neq 0)$. If all the zeros of $M(z)$ and $N(z)$ are exterior to or on the circumference of the unit circle then the minimum radius of univalence and the minimum radius of starlikeness of $f(z)$ are greater than $\frac{1}{\eta+1}$, where $\eta = m+n$.

Proof.

Putting $\quad \frac{M(z)}{N(z)} = P(z)$,

$$P(z) = \frac{a_0(z-x_1)(z-x_2)\dots(z-x_m)}{b_0(z-x_{m+1})(z-x_{m+2})\dots(z-x_{m+n})} ,$$

we have $\dfrac{P'(z)}{P(z)} = \dfrac{1}{z-x_1}+\dfrac{1}{z-x_2}+\dots+\dfrac{1}{z-x_m}-\dfrac{1}{z-x_{m+1}}-\dfrac{1}{z-x_{m+2}}-\dots-\dfrac{1}{z-x_{m+n}}$

so $\quad \dfrac{zf'(z)}{f(z)} = 1 + \dfrac{zP'(z)}{P(z)}$,

and $\quad \mathrm{Re}\left\{\dfrac{zf'(z)}{f(z)}\right\} = \mathrm{Re}\left\{1 + \dfrac{z}{z-x_1} + \dfrac{z}{z-x_2} + \dots + \dfrac{z}{z-x_m} - \dfrac{z}{z-x_{m+1}} - \dfrac{z}{z-x_{m+2}} - \dots - \dfrac{z}{z-x_{m+n}}\right\}.$

We will show that if $r \leq \frac{1}{\eta+1}$ then

$$\mathrm{Re}\left\{\frac{z}{z-x}\right\} \geq -\frac{1}{\eta} \quad \text{for} \quad x = x_1, x_2, \dots, x_m$$

and

$$-\mathrm{Re}\left\{\frac{z}{z-x}\right\} \geq -\frac{1}{\eta} \quad \text{for} \quad x = x_{m+1}, x_{m+2}, \dots, x_{m+n},$$

which implies that

$$\mathrm{Re}\left\{1 + \frac{zP'(z)}{P(z)}\right\} \geq 0 \quad \text{for} \quad r \leq \frac{1}{\eta+1} .$$

Putting $z = re^{i\theta}$, $x = Re^{i\varphi}$, we have

$$\mathrm{Re}\left\{\frac{z}{z-x}\right\} = \frac{r^2-rR\cos(\theta-\varphi)}{r^2-2rR\cos(\theta-\varphi)+R^2} \geq -\frac{1}{\eta} \tag{1.10}$$

$$\iff \eta r^2-\eta rR\cos(\theta-\varphi) \geq -r^2+2rR\cos(\theta-\varphi)-R^2$$

$$\iff r^2(\eta+1)-rR(\eta+2)\cos(\theta-\varphi)+R^2 \geq 0 .$$

But $\quad r^2(\eta+1)-rR(\eta+2)\cos(\theta-\varphi)+R^2 \geq r^2(\eta+1)-rR(\eta+2)+R^2$

and the roots of the equation

$$r^2(\eta+1)-rR(\eta+2)+R^2 = 0$$

are $\frac{R}{\eta+1}$ and R. Since $R \geq 1$, for $r \leq \frac{1}{\eta+1}$ we have

$\mathrm{Re}\left\{\frac{z}{z-x}\right\} \geqq 0.$

Also,

$$-\mathrm{Re}\left\{\frac{z}{z-x}\right\} = \frac{-r^2+rR\cos(\theta-\varphi)}{r^2-2rR\cos(\theta-\varphi)+R^2} \geqq -\frac{1}{\eta} \qquad (1.11)$$

$$\iff -r^2\eta + rR\eta\cos(\theta-\varphi) \geqq -r^2+2rR\cos(\theta-\varphi)-R^2$$

$$\iff r^2(1-\eta)+rR(\eta-2)\cos(\theta-\varphi)+R^2 \geqq 0 .$$

But $r^2(1-\eta)+rR(\eta-2)\cos(\theta-\varphi)+R^2 \geqq r^2(1-\eta)-rR(\eta-2)+R^2$ and the roots of the equation

$$r^2(1-\eta)-rR(\eta-2)+R^2 = 0$$

are $\frac{R}{\eta-1}$ and $-R$. Since $R \geqq 1$, for $r \leqq \frac{1}{\eta-1}$ we have $-\mathrm{Re}\left\{\frac{z}{z-x}\right\} \geqq -\frac{1}{\eta}$. Hence $U_0 \geqq \frac{1}{\eta+1}$, $S_0 \geqq \frac{1}{\eta+1}$.

Note. It can be seen that if $m \neq 0$, $n = 0$ then both of these radii are equal to $\frac{1}{\eta+1}$. If $m = 0$, $n \neq 0$ both of them are greater than $\frac{1}{\eta+1}$. In particular if $m = 0$ and $n \geqq 1$ they are both equal to $\frac{1}{\eta-1}$.

For the minimum radius of convexity we have the following theorem.

THEOREM 1.9. _Let_ $f'(z)$ _be a rational function of the form_ $f'(z) = \frac{M(z)}{N(z)}$ _where_ $M(z)$ _is a polynomial of degree_ $m(\neq 0)$, $N(z)$ _is a polynomial of degree_ $n(\neq 0)$. _If all the zeros of_ $M(z)$ _and_ $N(z)$ _are exterior to or on the circumference of the unit circle then the minimum radius of con-_

vexity of $f(z)$ is greater than $\frac{1}{\eta+1}$ where $\eta = m+n$.

Note. Consideration of the cases where just one of m,n is zero gives the results corresponding to those explained in the "note" after the proof of Theorem 1.8.

Now we find the exact values of the minimum radius of univalence and starlikeness.

THEOREM 1.10. Let $f(z)$ be a rational function of the form $f(z) = z\frac{M(z)}{N(z)}$ where $M(z)$ is a polynomial of degree $m(\neq 0)$, $N(z)$ is a polynomial of degree $n(\neq 0)$ and all the zeros of $M(z)$ and $N(z)$ are exterior to or on the circumference of the unit circle. Then the minimum radius of univalence and the minimum radius of starlikeness of $f(z)$ are

$$U_o = S_o = \frac{1}{y_o+1},$$

where y_o is the positive root of the equation

$$y^2 + (2-m-n)y - 2m = 0 .$$

$f(z)$ has this minimum radius of univalence and the minimum radius of starlikeness if and only if all the zeros of $M(z)$ are concentrated at one end of a diameter of the unit circle and all the zeros of $N(z)$ are concentrated at the opposite end of the same diameter.(i)

Proof. Let y_o be the positive root of the equation

$$y^2 + (2-m-n)y - 2m = 0 .$$

(i) The theorem is valid also for the cases $m \neq 0$, $n = 0$ and $m = 0$, $n \geq 2$.

In equation (1.10) substituting η for y_o and in equation (1.11) substituting η for y_o+2, we see that

$$\text{for } r \leqq \frac{1}{y_o+1} \text{ we have } \operatorname{Re}\left\{\frac{z}{z-x}\right\} \geqq -\frac{1}{y_o}.$$

Equality exists if and only if $R = 1$, $r = \frac{1}{y_o+1}$ and $\cos(\theta-\varphi) = 1$.

Also, for $r \leqq \frac{1}{y_o+1}$ we have $-\operatorname{Re}\left\{\frac{z}{z-x}\right\} \geqq -\frac{1}{y_o+2}$.

Equality exists if and only if $R = 1$, $r = \frac{1}{y_o+1}$ and $\cos(\theta-\varphi) = -1$. Since y_o satisfies the equation

$$-\frac{m}{y}-\frac{n}{y+2} = -1 ,$$

it follows that for $r \leqq \frac{1}{y_o+1}$ we have

$$m\operatorname{Re}\left\{\frac{z}{z-x}\right\} - n\operatorname{Re}\left\{\frac{z}{z-x}\right\} \geqq -1 .$$

Therefore, for $r \leqq \frac{1}{y_o+1}$ we have $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqq 0$, where equality exists somewhere on the circumference of the circle $|z| \leqq \frac{1}{y_o+1}$ if and only if all the zeros of $M(z)$ are concentrated at one end of a diameter of the unit circle and all the zeros of $N(z)$ are concentrated at the opposite end of the same diameter. Thus $S_o = \frac{1}{y_o+1}$ and this minimum value is attained if and only if the zeros of $M(z)$ and $N(z)$ are situated as stated above.

$$\text{Now} \qquad \operatorname{Im}\left\{\frac{z}{z-x}\right\} = \frac{-rR\sin(\theta-\varphi)}{r^2-2rR\cos(\theta-\varphi)+R^2} ,$$

so for $\theta = \varphi$ or $\theta - \varphi = \pi$ we have $\operatorname{Im}\left\{\frac{z}{z-x}\right\} = 0$. Therefore if the zeros of M(z) and N(z) are situated as stated above then $\operatorname{Im}\left\{\frac{zf'(z)}{f(z)}\right\} = 0$ at the same point on the circumference of the circle $|z| \leq \frac{1}{y_o+1}$ where $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = 0$. This implies that on the same circumference $\frac{zf'(z)}{f(z)}$ vanishes and so f'(z) vanishes, which completes the proof of the theorem.

For minimum radius of convexity we have

THEOREM 1.11. <u>Let</u> f'(z) <u>be a rational function of the form</u> $f'(z) = \frac{M(z)}{N(z)}$, <u>where</u> M(z) <u>is a polynomial of degree</u> m($\neq$0), N(z) <u>is a polynomial of degree</u> n($\neq$0), <u>and all the zeros of</u> M(z) <u>and</u> N(z) <u>are exterior to or on the circumference of the unit circle. Then the minimum radius of convexity of</u> f(z) <u>is</u>

$$C_o = \frac{1}{y_o+1},$$

<u>where</u> y_o <u>is the positive root of the equation</u>

$$y^2 + (2-m-n)y - 2m = 0 .$$

f(z) <u>has this minimum radius of convexity if and only if all the zeros of</u> M(z) <u>are concentrated at one end of a diameter of the unit circle and all the zeros of</u> N(z) <u>are concentrated at the opposite end of the same diameter.</u>(i)

It is known that if f(z) is regular and Re $f'(z) \geq 0$ in a convex domain then f(z) is univalent in the same domain. By considering the preceding results about the minimum

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(i) The theorem is valid also for the cases $m \neq 0$, $n = 0$ and $m = 0$, $n \geq 2$.

radius of univalence and the minimum radius of starlikeness we deduce the following corollaries.

COROLLARY 1. <u>If the</u> n <u>points</u> $x_1, x_2, \ldots, x_n$ <u>are exterior to or on the circumference of the unit circle then the minimum radius of univalence of the function</u>

$$f(z) = \int \left[1 + \frac{z}{z-x_1} + \frac{z}{z-x_2} + \ldots + \frac{z}{z-x_n} \right] dz$$

$$= \int \left[n+1 + \sum_{i=1}^{n} \frac{x_i}{z-x_i} \right] dz$$

$$= (n+1)z + \sum_{i=1}^{n} x_i \log(z-x_i)$$

$$= (n+1)z + \log \prod_{i=1}^{n} (z-x_i)^{x_i}$$

<u>is</u> $\qquad U_o = \frac{1}{n+1}$.

f(z) <u>has this minimum radius of univalence if and only if</u> $x_1, x_2, \ldots, x_n$ <u>are concentrated at the same point on the circumference of the unit circle</u>.

This follows because in Theorem 1.2 we have shown that in $|z| \leqq \frac{1}{n+1}$, we have

$$\operatorname{Re} \left\{ 1 + \frac{z}{z-x_1} + \frac{z}{z-x_2} + \ldots + \frac{z}{z-x_n} \right\} \geqq 0 ,$$

where equality exists if and only if $x_1, x_2, \ldots, x_n$ are concentrated at one point on the circumference of the unit circle. Also we have shown that if $x_1, x_2, \ldots, x_n$ are situated as stated above then

$$1 + \frac{z}{z-x_1} + \frac{z}{z-x_2} + \ldots + \frac{z}{z-x_n} = 0$$

at a point on the circumference of $|z| \leqq \frac{1}{n+1}$. This gives the required result.

Similarly by Theorem 1.4 we have

COROLLARY 2. <u>If</u> $x_1, x_2, \ldots, x_m$ <u>lie in the annulus</u> $0 < d \leqq |z| < D$ <u>and</u> $x_{m+1}, x_{m+2}, \ldots, x_{m+n}$ <u>are situated in</u> $|z| \geqq D$, <u>then the minimum radius of univalence of the function</u>

$$f(z) = (m+n+1)z + \log \prod_{i=1}^{m+n} (z-x_i)^{x_i}$$

<u>is</u>

$$U_o = \frac{-y_o D}{1-y_o} ,$$

<u>where</u> y_o <u>is the greater root of the equation</u>

$$n(d-D)y^2 + [d(1-n) - D(1+m)]y - d = 0 .$$

$f(z)$ <u>has this minimum radius of univalence if and only if</u> $x_1, x_2, \ldots, x_m$ <u>are concentrated at a point on the circumference of</u> $|z| \leqq d$ <u>and</u> $x_{m+1}, x_{m+2}, \ldots, x_{m+n}$ <u>are concentrated at a point on the circumference of</u> $|z| \leqq D$ <u>where these two concentration points and the origin are collinear and the two concentration points are on the same side of the origin.</u>

By Theorem 1.10 we have

COROLLARY 3. <u>If</u> $x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}$ <u>are outside or on the circumference of the unit circle then the</u>

minimum radius of univalence of the function

$$f(z) = \int \left[1+ \frac{z}{z-x_1} + \frac{z}{z-x_2} + \ldots + \frac{z}{z-x_m} - \frac{z}{z-x_{m+1}} - \ldots - \frac{z}{z-x_{m+n}}\right] dz$$

$$= (m-n+1)z + \log \frac{\prod_{i=1}^{m} (z-x_i)^{x_i}}{\prod_{j=m+1}^{m+n} (z-x_j)^{x_j}}$$

is $U_o = \frac{1}{y_o+1}$,

where y_o is the positive root of the equation

$$y^2 + (2-m-n)y - 2m = 0 .$$

$f(z)$ has this minimum radius of univalence if and only if $x_1, x_2, \ldots, x_m$ are concentrated at one end of a diameter of the unit circle and $x_{m+1}, x_{m+2}, \ldots, x_{m+n}$ are concentrated at the opposite end of this diameter.

Similarly we can state corollaries giving the minimum radii of univalence of the functions

$$\text{(i)} \quad f(z) = \int \left[1 + \frac{\alpha}{n} \left(\frac{z}{z-x_1} + \frac{z}{z-x_2} + \ldots + \frac{z}{z-x_n}\right)\right] dz ,$$

where   is any real number,

and $\quad \text{(ii)} \quad f(z) = \int \left[1-\lambda + \frac{z}{z-x_1} + \frac{z}{z-x_2} + \ldots + \frac{z}{z-x_n}\right] dz \quad 0 < \lambda \leq 1.$

These corollaries follow by Theorems 1.2 and 1.6 respectively.

By using arguments similar to those in the proofs of the preceding theorems we obtain results for other problems such as those stated below.

1. Let $f(z)$ be a function of the form $f(z) = z[P(z)]^{\alpha}$ where $P(z)$ is a polynomial and α is any real number. By considering the distribution of the zeros of $P(z)$ relative to an annulus about the origin to find the minimum radius of univalence and the minimum radius of starlikeness of $f(z)$ and also to determine all possible cases giving these minimum values.

2. Let $f(z)$ be a function of the form $f(z) = z\left[\frac{M(z)}{N(z)}\right]^{\alpha}$. $M(z)$ and $N(z)$ are polynomials all of whose zeros are exterior to or on the circumference of the unit circle and α is any real number. To find the minimum radius of univalence and the minimum radius of starlikeness of $f(z)$ and to determine all possible cases giving these minimum values.

3. Let $f(z)$ be a function of the form $f(z) = z\left[\frac{M(z)}{N(z)}\right]^{\alpha}$ where $M(z)$ and $N(z)$ are polynomials and α is any real number. By considering the distribution of the zeros of $M(z)$ and $N(z)$ relative to an annulus about the origin to find the minimum radius of univalence and the minum radius of starlikeness of $f(z)$ and to determine all possible cases giving these minimum values.

4. By considering the functions given in problems 1,2,3 and for each function $f(z)$ by considering the given distribution of zeros (and poles) of $\frac{f(z)}{z}$ to find the minimum radius of starlikeness of order λ and to determine all

cases which give these minimum values.

5. Alternatively by considering the distribution of the zeros of derivatives outside $|z| < 1$ or relative to an annulus about the origin to find the minimum radii of convexities and also to determine all possible cases giving these minimum values for the functions

(i) $f(z) = \int [P(z)]^{\alpha} dz$,

where P(z) is a polynomial and α is any real number,

and (ii) $f(z) = \int \left[\frac{M(z)}{N(z)}\right]^{\alpha} dz$

where M(z), N(z) are polynomials and α is any real number.

CHAPTER II

THEOREM 2.1. Let $f(z)$ be a polynomial of degree not exceeding n $(n \geq 1)$. If the moduli of all the zeros of $f'(z)$ are greater than cosec $\frac{\pi}{n}$ then $f(z)$ is univalent over $|z| \leq 1$.(i)

This theorem was first conjectured by J.W. Alexander II [1] and then proved by S. Kakeya [7]. Kakeya's proof depended on P.J. Heawood's Lemma(ii) in conjunction with some other theorems. But we will show that Heawood's Lemma is false. This lemma was also used by Heawood himself [6] to prove the theorem known as the "Grace-Heawood Theorem". Therefore we think it is worthwhile first to deal with Heawood's Lemma.

We will state Heawood's proof of his lemma. Then by constructing some counter examples we will show that this lemma is false. In order to prove the Grace-Heawood Theorem, Heawood first established his lemma and by applying his lemma he obtained a preliminary result from which the Grace-Heawood Theorem followed. We will state this preliminary

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(i) The term univalent is restricted usually to functions defined in a domain. We use the term here in an obvious sense for the set $|z| \leq 1$, i.e. we say that $f(z)$ is univalent over $|z| \leq 1$ if $f(z_1) = f(z_2)$, $|z_1| \leq 1$, $|z_2| \leq 1$, implies $z_1 = z_2$.

(ii) [6], so called by Kakeya in [7].

result as Theorem 2.2. We will prove this theorem by another method independent of Heawood's Lemma. Then we will prove Theorem 2.1 directly by applying Theorem 2.2. Also the Grace-Heawood Theorem now follows from Theorem 2.2 as in Heawood's paper. Then we will consider various types of distribution of the zeros of the derivative and for these special cases we will improve the value of the radius of univalence given by Theorem 2.1. In this connection we will first see the result given by L.N. Čakalov [2]. Then by using similar arguments we will obtain some improvements of Čakalov's result.

We begin by stating Heawood's Lemma.

HEAWOOD'S LEMMA. If any two distinct numbers α, β satisfy the relation

$$a\alpha\beta + b(\alpha+\beta) + c = 0$$

then, still preserving this relation, we can make α, β coincide either by increasing the minimum modulus or without altering it. If α, β have different moduli then we can always make them coincide by increasing the minimum modulus.

By applying his lemma Heawood obtained the following result.

THEOREM 2.2. If a polynomial $f(z)$ of degree n takes the same value at the points α, β then there exists a polynomial $F(z)$ of degree n satisfying the following conditions:

i) $F(\alpha) = F(\beta)$;

ii) <u>All the zeros of</u> $F'(z)$ <u>are concentrated at the same point</u>;

iii) <u>The common modulus of the zeros of</u> $F'(z)$ <u>is greater than or equal to the minimum modulus of the zeros of</u> $f'(z)$.

Now we state Heawood's proof of his lemma and Theorem 2.2.

<u>Proof</u>. Let $f(z)$ be a polynomial which has two given zeros. Without loss of generality we may assume that these zeros are +1 and -1. Supposing the derived equation is

$$z^{n-1} - q_1 z^{n-2} + q_2 z^{n-3} - \dots = 0 ,$$

the original equation will be

$$\frac{z^n}{n} - q_1 \frac{z^{n-1}}{n-1} + \dots = C .$$

Since this equation is satisfied by $z = \pm 1$ we have the relation

$$q_{n-1} + \frac{q_{n-3}}{3} + \frac{q_{n-5}}{5} + \dots = 0 , \qquad (2.1)$$

where q_r is the sum of the products, r at a time, of the roots of the derived equation, and q_o is 1.

First we assume that all the zeros of $f'(z)$ except two, viz. α , β , are fixed. Then (2.1) gives between these two zeros a relation of the form

$$\alpha\beta + \lambda(\alpha+\beta) + \mu = 0 ,$$

or say

$$(\alpha - \delta)(\beta - \delta) = k^2 . \qquad (2.2)$$

Geometrically this shows that α, β are "reflected inverses" of each other with respect to centre δ (suppose point C in figure 1.), and radius k in magnitude and direction. Here k is a vector whose magnitude is the radius of inversion and whose line is the line of reflection, CH or CK. α, β may vary in any manner provided this condition is satisfied. In

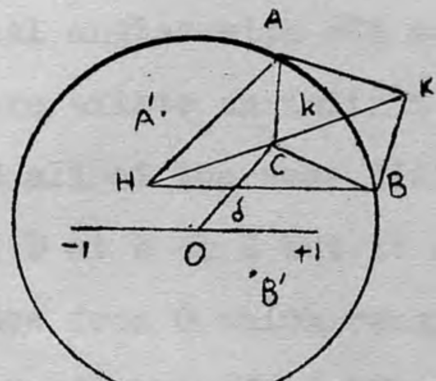

Figure 1.

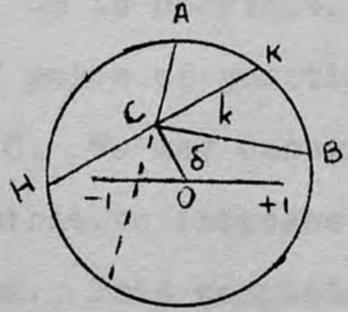

Figure 2.

particular they may both be concentrated at one of the points H and K. In the first place we suppose that they are at unequal distances from the origin, as A', B'. In this case we may obviously increase the distance of the nearer (at the expense of the more remote, if necessary) consistently with the connection between them. Now we suppose that they are at equal distances OA, OB (figure 1). We have the angle BCH = HCA, and the distances $CA.CB = CH^2 = CK^2$. Thus the triangles ACH, HCB; ACK, KCB are similar. Therefore the four angles at H and K are respectively equal to the four at A and B. Hence HAKB are concyclic and then in general one of the points H, K lies inside and one outside the circle centered at O through A, B. By concentrating both zeros at the one lying outside the circle we increase their distances from O. The only exceptional case is when the points H and K are on the circumference of the circle centered at O, in which case CO is perpendicular to HCK (figure 2). In this case for any two lines CA, CB making equal angles with HCK we have CA . CB is constant. Thus there exists an infinite number of pairs of positions of A,B all at the same distance from O. We may concentrate A and B at H or K but it is not possible to increase the distance from O which remains the same. This completes Heawood's proof of his lemma. He then obtained Theorem 2.2 in the following way.

Suppose all the zeros capable of variation, subject to the condition (2.1). By the above considerations, as long as there exist any two zeros at unequal distances from O, then the distance of the nearer may be increased as in the first case. If all the zeros have the same distance from O it may be possible to concentrate a pair at an increased distance as in the second case. Then by the first case the distance of the nearer points may be increased. When the distances of all points from O are the same, let us suppose that each pair is in the case finally considered, where the minimum distance cannot be increased. Now if this state of things persists continually, which seems very unlikely, we can even further concentrate them without alteration of this common distance. But if this state of things does not persist continually we can concentrate them by increasing the distance as before. Hence unless all the zeros are concentrated at one point we can always concentrate them with increased or unaltered minimum distance from 0. It follows that the distribution (or one of the distributions) of the zeros in which the distance of the nearest zero from 0 is greatest must be where all the zeros are concentrated at one point. Thus we obtain a new polynomial such that all the zeros of its derivative are concentrated at the same point. Since the condition (2.1) is satisfied then this new polynomial attains the same value for $z = \pm 1$.

Thus Heawood obtained Theorem 2.2 from which he derived the Grace-Heawood Theorem. Now by proceeding in the same way as in Heawood's proof we construct the following two counter examples:

<u>Example 1</u>. If the derived equation be

$$z^4 - q_1 z^3 + q_2 z^2 - q_3 z + q_4 = 0 ,$$

the original equation will be of the form

$$\frac{z^5}{5} - q_1 \frac{z^4}{4} + q_2 \frac{z^3}{3} - q_3 \frac{z^2}{2} + q_4 z = C .$$

If this is satisfied by $z = \pm 1$, and $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the roots of the derived equation, then

$$\frac{1}{5} + \frac{1}{3} q_2 + q_4 = 0 , \qquad (2.3)$$

$$q_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4,$$

$$q_4 = \alpha_1\alpha_2\alpha_3\alpha_4 .$$

Supposing that all the roots but two, viz. α_1, α_2 are fixed, (2.3) will give between these two a relation of the form

$$\left(\tfrac{1}{3} + \alpha_3\alpha_4\right)\alpha_1\alpha_2 + \tfrac{1}{3}(\alpha_3+\alpha_4)(\alpha_1+\alpha_2) + \left(\tfrac{1}{3}\alpha_3\alpha_4 + \tfrac{1}{5}\right) = 0.$$

If

$$\alpha_3\alpha_4 = -\frac{1}{3},$$

then $\frac{1}{3}(\alpha_3+\alpha_4)(\alpha_1+\alpha_2) + \left(-\frac{1}{9} + \frac{1}{5}\right) = 0$,

or $\alpha_1 + \alpha_2 = -\frac{12}{45} \cdot \frac{1}{\alpha_3+\alpha_4}$.

Since α_3, α_4 are fixed we can write this equation in the form

$$\alpha_1 + \alpha_2 = A .$$

If $\left|\frac{\alpha_1+\alpha_2}{2}\right| < |\alpha_1|$, $\qquad \left|\frac{\alpha_1+\alpha_2}{2}\right| < |\alpha_2|$

then whether $|\alpha_1|$, $|\alpha_2|$ are the same or not, we can only concentrate α_1, α_2 by decreasing the minimum distance (figure 3).

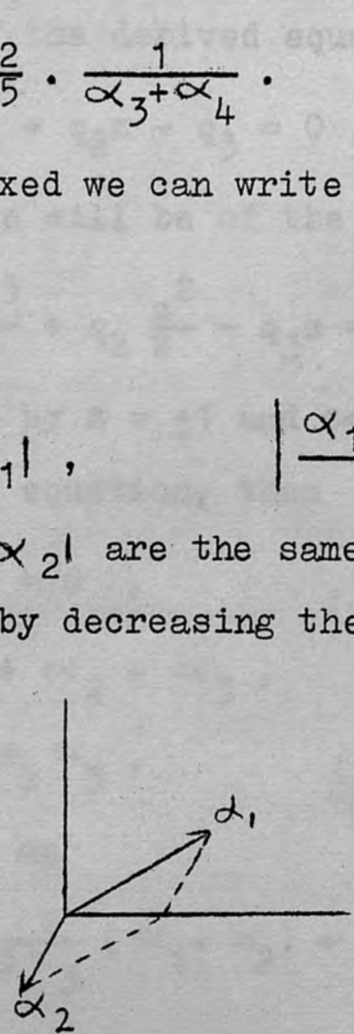

Figure 3.

Suppose that when all roots except α_1, α_2 are fixed the relation (2.1) in the lemma can be written as

$$a\alpha_1\alpha_2 + b(\alpha_1+\alpha_2) + c = 0 ,$$

where $a \neq 0$. Then we can write this equation in the form

$$(\alpha_1 - \delta)(\alpha_2 - \delta) = k^2 .$$

If $|\alpha_2| > |\alpha_1|$ and $\alpha_1 = \delta$ then we can only concentrate α_1, α_2 at δ, so in this case also we see that if two roots have different magnitudes it is not always possible to concentrate them with an increased minimum distance from 0.

Example 2. If the derived equation be

$$z^3 - q_1 z^2 + q_2 z - q_3 = 0 ,$$

the original equation will be of the form

$$\frac{z^4}{4} - q_1 \frac{z^3}{3} + q_2 \frac{z^2}{2} - q_3 z = C .$$

If this is satisfied by $z = \pm 1$ and α_1, α_2, α_3 are the roots of the derived equation, then

$$\frac{1}{3} q_1 + q_3 = 0 , \qquad (2.4)$$

$$q_1 = \alpha_1 + \alpha_2 + \alpha_3 ,$$

$$q_3 = \alpha_1 \alpha_2 \alpha_3 .$$

(2.4) can be written as

$$\alpha_1\alpha_2 + \frac{1}{3\alpha_3}(\alpha_1 + \alpha_2) + \frac{1}{3} = 0 ,$$

so if $\alpha_3 = \frac{1}{\sqrt{3}}$ then

$$(\alpha_1 + \frac{1}{\sqrt{3}})(\alpha_2 + \frac{1}{\sqrt{3}}) = 0 .$$

Thus if $\alpha_1 = -\frac{1}{\sqrt{3}}$ and $|\alpha_2| \geq |\alpha_1|$ then we cannot concentrate α_1, α_2 by increasing the minimum modulus.

These counter examples show that Heawood's Lemma is false. Thus Heawood's proof of Theorem 2.2 is also not valid. We will prove Theorem 2.2 by the principal of apolarity of polynomials.

DEFINITION 2.1. If the coefficients of two polynomials

$$f(z) = a_o + C_n^1 a_1 z + C_n^2 a_2 z^2 + \ldots + a_n z^n , \quad \text{(i)}$$

$$g(z) = b_o + C_n^1 b_1 z + C_n^2 b_2 z^2 + \ldots + b_n z^n ,$$

of degree n satisfy the condition

$$a_o b_n - C_n^1 a_1 b_{n-1} + C_n^2 a_2 b_{n-2} - \ldots + (-1)^n a_n b_o = 0 , \quad (2.5)$$

then f(z) and g(z) are called apolar polynomials.

Let

$$f(z) = a_o + C_n^1 a_1 z + C_n^2 a_2 z^2 + \ldots + a_n z^n ,$$

where the coefficients satisfy the given linear relation

$$a_o \lambda_n + C_n^1 a_1 \lambda_{n-1} + C_n^2 a_2 \lambda_{n-2} + \ldots + a_n \lambda_o = 0 . \quad (2.6)$$

This implies that

$$g(z) = \lambda_o - C_n^1 \lambda_1 z + C_n^2 \lambda_2 z^2 - \ldots + (-1)^n \lambda_n z^n$$

is apolar to f(z). By writing the same relation for the

(i) C_n^P denotes the (p+1)th coefficient of the n'th power of the binomial, i.e. $C_n^P = \frac{n!}{P!(n-P)!}$.

particular polynomial

$$f_o(x) = (x-z)^n = x^n - C_n^1 x^{n-1} z + \dots ,$$

regarding x as a parameter, we find immediately that

$$g(z) = 0 .$$

Thus if the coefficients of a polynomial f(z) satisfy a linear relation then a polynomial g(z) apolar to f(z) can be obtained directly from this relation.(i)

Using this fact, we need only the following theorem of Grace(ii) in order to prove Theorem 2.2.

THEOREM 2.3. <u>If two polynomials are apolar then any circular domain(iii) containing all the zeros of one of these polynomials contains at least one zero of the other.</u>

<u>Proof</u>. Let two polynomials

$$f(z) = a_o + C_n^1 a_1 z + C_n^2 a_2 z^2 + \dots + a_n z^n \tag{2.7}$$

$$g(z) = b_o + C_n^1 b_1 z + C_n^2 b_2 z^2 + \dots + b_n z^n$$

of degree n be apolar. Then their coefficients satisfy the condition of apolarity (2.5). We denote the zeros of f(z) by $\alpha_1, \alpha_2, \dots, \alpha_n$ and the zeros of g(z) by $z_1, z_2, \dots, z_n$. Putting

$$S_k = (-1)^k C_n^k \frac{b_{n-k}}{b_n} ,$$

the relation (2.5) can be written as

(i) See, e.g., [8], p.19.

(ii) [5], [11]; see, e.g., [8], p.16-18.

(iii) By a circular domain we mean the interior or exterior of a circle or half plane.

$$a_0 S_0 + a_1 S_1 + \dots + a_n S_n = 0 \,, \qquad (2.8)$$

where

$$S_0 = 1 \,,$$
$$S_1 = z_1 + z_2 + \dots + z_n \,,$$
$$\text{-------------}$$
$$S_n = z_1 z_2 \dots z_n \,.$$

In this way we associate the relation (2.8) with the equation (2.7). We will show that, if $\alpha_1, \alpha_2, \dots, \alpha_n$ are the zeros of f(z), and **if** $z_1, z_2, \dots, z_n$ is a system of solutions of the equation (2.8), then every circular domain containing all of the points α_k contains at least one point z_k. Clearly this is true for n = 1. By supposing it is true up to the (n-1)th degree we will show that it is also true for the n th degree. The proof will then follow by induction.

Let C be a circular domain containing all the points α_k. We may assume that at least one of the points z_k is exterior to C, otherwise there is nothing to prove. Supposing that this point is $z_n = \zeta$ we will show that one of the points $z_1, z_2, \dots, z_{n-1}$ lies in C. Let us put

$$s_0 = 1 \,,$$
$$s_1 = z_1 + z_2 + \dots + z_{n-1} \,,$$
$$\text{----------}$$
$$s_{n-1} = z_1 z_2 \dots z_{n-1} \,.$$

Then

$$S_0 = s_0 ,$$
$$S_1 = s_1 + \zeta s_0 ,$$

$$S_n = \zeta s_{n-1} .$$

If we substitute these values in (2.8) then we obtain the relation

$$(a_0+a_1\zeta)s_0 + (a_1+a_2\zeta)s_1 +\ldots+ (a_{n-1}+\zeta a_n)s_{n-1} = 0 .$$

But this relation is associated with the equation

$$G(z) = (a_0+a_1\zeta) + C^1_{n-1}(a_1+a_2\zeta)z +\ldots+(a_{n-1}+\zeta a_n)z^{n-1} = 0 .$$

Since we have supposed that the theorem is true up to the (n-1)th degree it is sufficient to show that all the roots of this last equation are in C. Writing G(z) as

$$G(z) = a_0+C^1_{n-1}a_1z+\ldots+a_{n-1}z^{n-1}+\zeta(a_1+C^1_{n-1}a_2z+\ldots+a_nz^{n-1}) ,$$

Since

$$f'(z) = n\left[a_1 + C^1_{n-1}a_2z +\ldots+ a_nz^{n-1}\right] ,$$

we have

$$nG(z) = \zeta f'(z) + n\left[a_0+C^1_{n-1}a_1z +\ldots+ a_{n-1}z^{n-1}\right] ,$$

and subtracting

$$nf(z) = na_0 + n\,C^1_n\,a_1z +\ldots+ na_nz^n$$

we obtain

$$nG(z) = nf(z) + (\zeta-z)f'(z) .$$

Division by f(z) gives

$$h(z) = n + (\zeta-z)\frac{f'(z)}{f(z)} .$$

Thus the zeros of G(z) satisfy the equation h(z) = 0. Now h(z) cannot have a zero exterior to C, for suppose that z_0 is a zero exterior to C. Then

$$\frac{f'(z_0)}{f(z_0)} = \sum_{i=1}^{n} \frac{1}{z_0 - \alpha_i},$$

and

$$h(z_0) = \sum_{i=1}^{n} \left(1 + \frac{\zeta - z_0}{z_0 - \alpha_i}\right) = \sum_{i=1}^{n} \frac{\zeta - \alpha_i}{z_0 - \alpha_i} = 0. \quad (2.9)$$

The image of C under the transformation

$$Z = \frac{\zeta - z}{z_0 - z}$$

is also a circular domain. Let us denote this domain by Γ. Since z_0 is exterior to C, Γ does not contain the point $Z = \infty$. Since ζ is exterior to C, Z = 0 is also exterior to Γ. By (2.9) the sum of the transforms of α_i is zero; so Z = 0 is their centre of gravity. But the transforms of all α_i are in Γ; thus Z = 0 is also in Γ. This contradiction completes the proof.

Using the concept of apolarity of polynomials we will now prove Theorem 2.2.

<u>Proof of Theorem 2.2</u>. The relation

$$f(\alpha) - f(\beta) = 0$$

is a linear relation between the coefficients of f'(z). As we explained in the argument following the definition of apolarity on p. 45, by writing the same linear relation between the coefficients of

$$f_0(x) = (x-z)^{n-1} ,$$

we find the relation

$$g(z) = \int_{\alpha}^{\beta} (x-z)^{n-1} dx = 0 ,$$

where g(z) is apolar to f'(z). Now if Z is the zero of g(z) of maximum modulus then, by Theorem 2.3, in $|z| \leq |Z|$ there exists at least one zero of f'(z). This means that the minimum modulus of the zeros of f'(z) is less than or equal to $|Z|$. Hence the polynomial

$$(z-Z)^n + C$$

satisfies the conditions of F(z).

Thus we have proved Theorem 2.2 by a method independent of Heawood's Lemma. As an immediate consequence we will prove Kakeya's theorem (Theorem 2.1).

Proof of Theorem 2.1. Let f(z) be a polynomial of degree $n \geq 1$ which attains the same value at two distinct points α, β in the unit circle. By theorem 2.2. there exists a polynomial F(z) of the form

$$F(z) = (z-Z)^n + C ,$$

where $|Z|$ is greater than or equal to the minimum modulus of the zeros of f'(z). Also we know that Z is the zero of maximum modulus of the equation

$$(\alpha - z)^n = (\beta - z)^n .$$

Thus $$\left(\frac{\alpha - z}{\beta - z}\right)^n = 1 ,$$

or $$z = \frac{\alpha - \beta\omega}{1 - \omega} ,$$

where ω is an n'th root of unity different from 1. If we allow α, β to vary in the closed unit circle, then we have

$$\max\left|\frac{\alpha-\omega\beta}{1-\omega}\right| = \frac{2}{\min|1-\omega|} = \frac{2}{\left|1-\cos\frac{2\pi}{n}-i\sin\frac{2\pi}{n}\right|} = \frac{2}{\sqrt{(1-\cos\frac{2\pi}{n})^2+\sin^2\frac{2\pi}{n}}}$$

$$= \frac{2}{\sqrt{2(1-\cos\frac{2\pi}{n})}} = \frac{2}{\sqrt{4\sin^2\frac{\pi}{n}}} = \operatorname{cosec}\frac{\pi}{n} .$$

Therefore if the moduli of all the zeros of $f'(z)$ are greater than $\operatorname{cosec}\frac{\pi}{n}$ then $f(z)$ cannot attain the same value at two distinct points inside or on the unit circle. This proves the theorem.

In the above proof of Theorem 2.1 if we allow α, β to vary only in the interior of the unit circle we deduce immediately that:

<u>if the derivative</u> $f'(z)$ <u>of a polynomial of degree not exceeding</u> $n \geq 1$ <u>does not vanish in</u> $|z| < \operatorname{cosec}\frac{\pi}{n}$ <u>then</u> $f(z)$ <u>cannot attain the same value at two distinct points in</u> $|z| < 1$. <u>i.e.</u> $f(z)$ <u>is univalent in</u> $|z| < 1$.

This result is best possible for all $n \geq 1$. This is evident if $n = 2$, as the polynomial $z^2 + 2z$ shows, since we then have $\operatorname{cosec}\frac{\pi}{n} = 1$ and the derivative vanishes at $z = -1$, so that the polynomial is not univalent inside a circle larger than the unit circle. If $n \geq 2$ then we may show this by constructing the following example.

Let $f'(z) = (z - \operatorname{cosec}\frac{\pi}{n})^{n-1}$, where $n \geq 2$. Then $f(z)$ attains the same value at the points $\alpha = e^{\pi\frac{(2-n)i}{2n}}$, $\beta = e^{\pi\frac{(n-2)i}{2n}}$, for if we put $\omega = e^{\frac{2\pi i}{n}}$, then $\alpha = e^{\pi\frac{(2-n)i}{2n}}$,

$\beta = e^{\frac{\pi(n-2)i}{2n}}$ satisfy the equation

$$\operatorname{cosec}\frac{\pi}{n} = \frac{\alpha - \beta\omega}{1-\omega},$$

which implies that

$$\left(\frac{\alpha - \operatorname{cosec}\frac{\pi}{n}}{\beta - \operatorname{cosec}\frac{\pi}{n}}\right)^n = 1 .$$

Thus $$(\alpha - \operatorname{cosec}\frac{\pi}{n})^n = (\beta - \operatorname{cosec}\frac{\pi}{n})^n .$$

We remark that in [9] M. Robertson gives a necessary and sufficient condition (in the form of the vanishing of a certain determinant involving the coefficients of f(z)) for f(z) to be univalent inside $|z| < 1$ but in no larger disc, but the result contains an error.[(i)]

THEOREM 2.4. <u>Let</u> f(z) <u>be a polynomial of degree</u> n <u>which takes the same value at the points</u> -1 <u>and</u> +1. <u>Then the circle with centre</u> 0 <u>and radius</u> $\cot\frac{\pi}{n}$ <u>contains at least one zero of the derivative</u> f'(z).

This theorem is known as the Grace-Heawood Theorem, and it gives a generalization of Rolle's Theorem for the complex plane. It was first proved by Grace [5] and then by Heawood. Grace's proof depended on the principal of apolarity. Heawood's proof is based on his lemma, and so is not valid. As we mentioned before, to prove this theorem Heawood first established his lemma, and by applying it he obtained a preliminary result which we stated as Theorem 2.2. Then the Grace-Heawood Theorem followed from this result. With our

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(i) See Appendix II.

valid proof of Theorem 2.2 the Grace-Heawood Theorem follows from it as in Heawood's paper. To prove Theorem 2.4 it is sufficient to take $\alpha = 1$, $\beta = -1$ in the proof of Theorem 2.1.

For some special types of distribution of the zeros of $f'(z)$ we will now improve the radius of univalence given by Theorem 2.1.[(i)] In this connection we will first state L.N. Čakalov's result [Theorem 2.6] and give his proof. Then by using arguments similar to his we will prove some improvements of his result. By Theorem 2.1, if the derivative of a polynomial $f(z)$, of degree not exceeding $n+1$, $(n \neq 0)$, does not vanish in the disc $|z| < 1$ then $f(z)$ is univalent in $|z| < \sin\frac{\pi}{n+1}$. Čakalov [2] formed a special type of distribution of the zeros of $f'(z)$ outside the unit disc, for which he showed that $f(z)$ is univalent in a larger circle than that given by Theorem 2.1. We will state Čakalov's result as Theorem 2.6. His proof is based on Theorem 2.5 which he obtained by first establishing the following lemma.

LEMMA. <u>Suppose that</u> n <u>is an integer</u>, $n \geq 1$. <u>If</u> $a_1, a_2, \ldots, a_n$ <u>are</u> n <u>positive numbers whose sum</u>

$$S = \sum_{k=1}^{n} a_k \leq \frac{\pi}{2},$$

<u>and</u> $u_1, u_2, \ldots, u_n$ <u>are</u> n <u>non-zero complex numbers whose moduli satisfy the condition</u>

(i) See also the last two examples given in Appendix I.

$$|u_k| \leq \sin a_k \quad , \; k = 1, 2, \ldots, n \; , \tag{2.10}$$

then
$$\operatorname{Re} \prod_{k=1}^{n} (1 + u_k) \geq 0 \; . \tag{2.11}$$

Equality exists in (2.11) if and only if $S = \frac{\pi}{2}$,
$u_k = i\sin a_k e^{ia_k}$, $k = 1, 2, \ldots, n$, or $u_k = -i\sin a_k e^{-ia_k}$
$k = 1, 2, \ldots n$.

Proof. Let $u_k = r_k e^{i\phi_k}$, $1 + u_k = \rho_k e^{i\psi_k}$, where $0 < r_k \leq \sin a_k$ and since $n > 1$, $\rho_k > 0$. Since $\cos\psi_k > 0$ we have, taking principal value of argument, $-\frac{\pi}{2} < \psi_k < \frac{\pi}{2}$. If we eliminate ρ_k from the equations $1 + r_k\cos\phi_k = \rho_k\cos\psi_k$, $r_k\sin\phi_k = \rho_k\sin\psi_k$ we obtain

$$\sin\psi_k = r_k \sin(\phi_k - \psi_k) \; . \tag{2.12}$$

Thus
$$|\sin\psi_k| \leq r_k \leq \sin a_k \; , \quad |\psi_k| \leq a_k \; , \tag{2.13}$$

$$\Big|\sum_{k=1}^{n} \psi_k\Big| \leq \sum_{k=1}^{n} a_k \leq \frac{\pi}{2} \; . \tag{2.14}$$

The first part of the lemma follows from this relation. Then by considering the relations (2.14), (2.13), (2.12) the second part can be seen easily.

THEOREM 2.5. Suppose that $Q(z)$ is a polynomial in z of degree $n > 1$ with zeros $z_1, z_2, \ldots, z_n$, all different from zero. Let $|z_k| = r_k$. Then the polynomial of degree $n+1$

$$P(z) = \int Q(z)dz$$

is univalent in the circle $|z| \leq r_0$ where r_0 is the positive root of the equation

$$\sum_{k=1}^{n} \arcsin \frac{r}{r_k} = \frac{\pi}{2} \; .$$

<u>Proof</u>. Without loss of generality we may assume that $Q(0) = 1$ and so

$$Q(z) = \prod_{k=1}^{n} \left(1 - \frac{z}{z_k}\right) . \qquad (2.15)$$

Let $a_1, a_2, \ldots, a_n$ be n positive numbers subject to the condition $\sum_{k=1}^{n} a_k \leq \frac{\pi}{2}$. By applying the lemma to the product (2.15) we find that $\text{Re } Q(z) \geq 0$ if $\frac{|z|}{r_k} \leq \sin a_k$ or $|z| \leq r_k \sin a_k$, i.e. if the modulus of z does not exceed the least of the products

$$r_k \sin a_k \qquad k = 1, 2, \ldots, n .$$

To be able to choose the a_k in a most advantageous manner we will show the existence of n positive numbers $a_1', a_2', \ldots, a_n'$ subject to the condition $\sum_{k=1}^{n} a_k' \leq \frac{\pi}{2}$ such that the least of the numbers

$$r_1 \sin a_1', \; r_2 \sin a_2', \; \ldots \; r_n \sin a_n' \qquad (2.16)$$

attains a maximum value. We will show that $a_1', a_2', \ldots, a_n'$ are characterized by the following necessary and sufficient conditions

$$\sum a_k' = \frac{\pi}{2}, \quad r_1 \sin a_1' = r_2 \sin a_2' = \ldots = r_n \sin a_n' .$$

In order to show this now suppose that μ is the greatest possible value of the least of the numbers (2.16). If $\sum a_k' < \frac{\pi}{2}$ then by choosing a sufficiently small positive number ϵ and putting $a_k = a_k' + \epsilon$ we obtain $\sum_{k=1}^{n} a_k < \frac{\pi}{2}$. Therefore the least of the numbers $r_k \sin a_k$ will be greater than μ which is a contradiction. On the other hand if the numbers (2.16) are not all equal to each other, then at least one of them for instance $r_m \sin a'_m$ is greater than μ. Let $a_k = a'_k + \epsilon$ for $k \neq m$ and $a_m = a'_m - (n-1)\epsilon$. Then we have $\sum a_k = \sum a'_k$ and for a sufficiently small positive ϵ each of the numbers $r_k \sin a_k$ is greater than μ which is also a contradiction. Let r be the common greatest number in (2.16). If we eliminate $a'_1, a'_2, \ldots, a'_n$ from the equations $\sum_{k=1}^{n} a'_k = \frac{\pi}{2}$, $r_1 \sin a'_1 = r_2 \sin a'_2 = \ldots = r_n \sin a'_n = r$, we find for r the equation

$$\sum_{k=1}^{n} \text{arc sin} \frac{r}{r_k} = \frac{\pi}{2} . \qquad (2.17)$$

This equation has exactly one positive root r_0 because its left side is a monotonic function of r where r ranges from 0 to the least of the numbers r_k. In the circle $|z| < r_0$ we have $\text{Re}\, P'(z) > 0$. Therefore as is known this implies the univalency of P(z) in $|z| < r_0$ which completes the proof.

It can be seen that if Q(z) has at least two zeros of the same modulus but different argument or of the same argument but different modulus then P(z) is univalent in a circle of radius larger than the number r_0 given by Čakalov in this theorem. Because by putting $-\frac{z}{z_k} = u_k$ in

$$Q(z) = \prod_{k=1}^{n} \left(1 - \frac{z}{z_k}\right)$$

and considering the lemma and the proof of the theorem, we deduce that if r_0 is the positive root of equation (2.17) then for $|z| \leq r_o$ we have

$$\operatorname{Re} Q(z) \geq 0$$

where equality exists if and only if

either $$-\frac{z}{z_k} = i\,\frac{r}{r_k}\, e^{i \arcsin \frac{r}{r_k}} \qquad k = 1,2,\ldots, n$$

or $$-\frac{z}{z_k} = -i\,\frac{r}{r_k}\, e^{-i \arcsin \frac{r}{r_k}} \qquad k = 1,2,\ldots, n\ .$$

THEOREM 2.6. Suppose that m is a non-negative integer less than $\frac{n+1}{2}$, and let

$$R = \sin\frac{\pi}{n+1} : \sin\frac{(n+1-2m)\pi}{(n-m)(2n+2)}\ .$$

Let m of the zeros of the polynomial $Q(z) = \prod_{k=1}^{n}\left(1 - \frac{z}{z_k}\right)$ lie in the annulus $1 \leq |z| \leq R$ and the remaining $n-m$ be situated in the region $|z| > R$. Then the polynomial $P(z) = \int Q(z)\,dz$ is univalent in the circle $|z| \leq r_o$ where the radius r_o defined by equation (2.17) is larger than the radius $\sin\frac{\pi}{n+1}$ given by the theorem of Kakeya.

Proof. r_o denotes the positive root of equation (2.17), where $r_k = |z_k|$ and $0 < r_1 \leq r_2 \leq \ldots \leq r_n$. Let us write equation (2.17) as

$$\sum_{1}^{m} \arcsin\frac{r}{r_k} + \sum_{m+1}^{n} \arcsin\frac{r}{r_k} = \frac{\pi}{2}\ .$$

Now by substituting in the first sum 1 and in the second R for r_k we obtain the inequality

$$m \arcsin r + (n-m) \arcsin\frac{r}{R} > \frac{\pi}{2}\ .$$

Therefore the positive root r' of the equation

$$m \text{ arc sin } r + (n-m) \text{ arc sin } \frac{r}{R} = \frac{\pi}{2}$$

is less than r_o. In this equation if we substitute the value R we find that $r' = \sin \frac{\pi}{n+1}$. Therefore $r_o \geq \sin\frac{\pi}{n+1}$. This completes the proof of Theorem 2.6 given by Čakalov. He then states the following Corollary and Remark.

COROLLARY. <u>If</u> $m = 1$ <u>then</u> $R = 2 \cos \frac{\pi}{2n+2}$. <u>Therefore</u> <u>if</u> $|z_1| = 1$ <u>and</u> $|z_k| \geq 2 \cos \frac{\pi}{2n+2}$ <u>for</u> $k = 2, 3, \ldots n$, <u>then</u> $P(z) = \int Q(z)dz$ <u>is univalent in</u> $|z| < r_o$ <u>where</u> $r_o \geq \sin \frac{\pi}{n+1}$.

<u>Remark</u>. It can be easily shown that if $Q_1(z) = \prod_{k=1}^{p} (1 - \frac{z}{z_k})$ and $Q_2(z) = \prod_{k=p+1}^{p+q} (1 - \frac{z}{z_k})$ then the real part of the function $\frac{Q_1(z)}{Q_2(z)}$ is non-negative in the circle $|z| \leq r_o$ where r_o is the positive root of the equation $\sum_{1}^{p+q} \text{arc sin } \frac{r}{|z_k|} = \frac{\pi}{2}$. Therefore $P(z) = \int \frac{Q_1(z)}{Q_2(z)} dz$ is univalent in this circle.

Here we note a few points. It is known that the condition of having a positive real part of the derivative inside a circle is only a sufficient condition for univalency; Theorem 2.6 does not, in fact, give the best possible results for the minimum radius of univalence. In Chapter I, Theorem 1.4 on the other hand, by considering the distribution of the zeros of $\frac{f(z)}{z}$ we proved the best possible results for the minimum radius of univalence and starlikeness. Theorem 1.4 was valid for every annulus about the origin and for every type of distribution of the zeros relative to this annulus. Also we determined the only possible cases giving the best possible results. Then in Theorem 1.5 by considering the distribution of the zeros of $f'(z)$ we obtained the best possible results for the minimum radius of convexity. As we noted there, for suitable distributions of the zeros of $f'(z)$ outside $|z| \leq 1$ we may obtain estimates for the radius of univalence better than that given by Theorem 2.6. Also our best possible results, Theorem 1.10, Theorem 1.11 about the minimum radius of univalence of a rational function are comparable with Čakalov's remark given after the proof of Theorem 2.6. Now by using arguments similar to Čakalov's, we will prove better results than Theorem 2.6. We will give these results as Theorem 2.7, 2.8, 2.9, 2.10.

THEOREM 2.7. Suppose that $n \geq 2$ and

$$R = \frac{\sin \frac{\pi n}{2(2n+1)}}{\sin \frac{(n+1)\pi}{2(n-1)(2n+1)}} .$$

Let one of the zeros of the polynomial $Q(z) = \prod_{k=1}^{n} (1 - \frac{z}{z_k})$ lie in the annulus $1 \leq |z| \leq R$ and the remaining $n-1$ be situated in the region $|z| \geq R$. Then the polynomial $P(z) = \int Q(z)dz$ is univalent in the circle $|z| \leq r_0$ where

$$r_0 \geq \sin \frac{\pi n}{2(2n+1)} .$$

If $n \geq 3$ we have further

$$r_0 > \sin \frac{\pi n}{2(2n+1)} > \sin \frac{\pi}{n+1} .$$

Proof. Since we use arguments similar to those used in the proof of Theorem 2.6, it is sufficient to show that

i) $R > 1$;

ii) $r' = \sin \frac{\pi n}{2(2n+1)}$ is the positive root of the equation $\arcsin r + (n-1) \arcsin \frac{r}{R} = \frac{\pi}{2}$;

iii) if $n > 3$ then $\sin \frac{\pi n}{2(2n+1)} > \sin \frac{\pi}{n+1}$.

i) follows because

$$0 < \frac{\pi n}{2(2n+1)} < \frac{\pi}{2} ,$$

$$0 < \frac{\pi(n+1)}{2(n-1)(2n+1)} < \frac{\pi}{2} \quad \text{for } n > \frac{1+\sqrt{5}}{2} ,$$

and $\frac{\pi n}{2(2n+1)} > \frac{(n+1)\pi}{2(n-1)(2n+1)}$ for $n > 1+\sqrt{2}$.

Thus $\sin \frac{\pi n}{2(2n+1)} > \sin \frac{\pi(n+1)}{2(n-1)(2n+1)}$

ii) follows because on substituting r' in the equation we have

$$\frac{\pi n}{2(2n+1)} + (n-1)\arcsin \frac{\sin\frac{\pi n}{2(2n+1)} \cdot \sin\frac{(n+1)\pi}{2(n-1)(2n+1)}}{\sin\frac{\pi n}{2(2n+1)}} = \frac{\pi}{2} ,$$

i.e., $\frac{\pi n}{2(2n+1)} + (n-1) \frac{\pi(n+1)}{2(n-1)(2n+1)} = \frac{\pi}{2}$,

and so the equation is satisfied.

iii) follows since

$$0 < \frac{\pi n}{2(2n+1)} < \frac{\pi}{2} ,$$

$$0 < \frac{\pi}{n+1} < \frac{\pi}{2} ,$$

and $\frac{\pi n}{2(2n+1)} > \frac{\pi}{n+1}$ for $n > \frac{3+\sqrt{17}}{2}$.

Similarly we will prove the following results, Theorem 2.8, 2.9, 2.10.[(i)]

THEOREM 2.8. *Suppose that* $n > 1$ *and*

$$R = \frac{\sin\frac{\pi n}{2(n+1)}}{\sin\frac{\pi}{2(n-1)(n+1)}} .$$

Let one of the zeros of the polynomial $Q(z) = \prod_{k=1}^{n}(1 - \frac{z}{z_k})$ *lie in the annulus* $1 \leq |z| \leq R$ *and the remaining* $n-1$ *be situated in the region* $|z| > R$. *Then the polynomial* $P(z) = \int Q(z)dz$ *is univalent in the circle* $|z| \leq r_0$, *where*

$$r_0 > \sin\frac{\pi n}{2(n+1)} .$$

If $n > 2$ *we have further*

$$r_0 > \sin\frac{\pi n}{2(n+1)} > \sin\frac{\pi}{n+1} .$$

Proof. i) $R > 1$ because

$$0 < \frac{\pi n}{2(n+1)} < \frac{\pi}{2},$$

$$0 < \frac{\pi}{2(n-1)(n+1)} < \frac{\pi}{2}$$

and
$$\frac{\pi n}{2(n+1)} > \frac{\pi}{2(n-1)(n+1)} .$$

Thus
$$\sin\frac{\pi n}{2(n+1)} > \sin\frac{\pi}{2(n-1)(n+1)} .$$

ii) $r' = \sin\frac{\pi n}{2(n+1)}$ is the positive root of the equation

$$\text{arc}\sin r + (n-1)\text{arc}\sin\frac{r}{R} = \frac{\pi}{2},$$

because on substituting r' in the equation we have

$$\frac{\pi n}{2(n+1)} + (n-1)\text{arc}\sin\left\{\frac{\sin\frac{\pi n}{2(n+1)}}{\sin\frac{\pi n}{2(n+1)}} \cdot \sin\frac{\pi}{2(n-1)(n+1)}\right\} = \frac{\pi}{2},$$

i.e.
$$\frac{\pi n}{2(n+1)} + \frac{\pi}{2(n+1)} = \frac{\pi}{2},$$

and so the equation is satisfied.

iii) If $n > 2$ then $\sin\frac{\pi n}{2(n+1)} > \sin\frac{\pi}{n+1}$, because

$$0 < \frac{\pi n}{2(n+1)} < \frac{\pi}{2},$$

(i) Considering the "remark" given after the proof of Theorem 2.6 we may extend all these results to the rational functions.

$$0 < \frac{\pi}{n+1} < \frac{\pi}{2}$$

and $$\frac{\pi n}{2(n+1)} > \frac{\pi}{n+1} \quad \text{for } n > 2 .$$

THEOREM 2.9 *Suppose that* $n > 1$, x *is a real number such that* $\frac{1}{n^2} < x < \frac{1}{n}$ *and*

$$R = \frac{\sin \frac{x\pi n}{xn+1}}{\sin \frac{-x\pi n+\pi}{2(n-1)(xn+1)}}$$

Let one of the zeros of the polynomial $Q(z) = \prod_{k=1}^{n} \left(1 - \frac{z}{z_k}\right)$ *lie in the annulus* $1 \le |z| \le R$ *and the remaining* $n-1$ *be situated in the region* $|z| > R$. *Then the polynomial* $P(z) = \int Q(z)dz$ *is univalent in the circle* $|z| \le r_0$ *where*

$$r_0 > \sin \frac{x\pi n}{xn+1} > \sin \frac{\pi}{n+1} . \quad \text{(i)}$$

Proof. i) $R > 1$ because

$$0 < \frac{x\pi n}{xn+1} < \frac{\pi}{2} \qquad \text{for } x < \frac{1}{n} ,$$

$$0 < \frac{-x\pi n+\pi}{2(n-1)(xn+1)} < \frac{\pi}{2} \quad \text{for } \frac{1}{n} > x > \frac{2-n}{n^2} ,$$

and $$\frac{x\pi n}{xn+1} > \frac{-x\pi n+\pi}{2(n-1)(xn+1)} \qquad \text{for } x > \frac{1}{2n^2-n} .$$

Thus $$\sin \frac{x\pi n}{xn+1} > \sin \frac{-x\pi n+\pi}{2(n-1)(xn+1)} .$$

ii) $r' = \sin \frac{x\pi n}{xn+1}$ is the positive root of the equation

$$\arcsin r + (n-1) \arcsin \frac{r}{R} = \frac{\pi}{2} ,$$

because on substituting r' in the equation we have

(i) If x is near to $\frac{1}{n}$ then R becomes large and Theorem 2.9 then says that $P(z)$ is univalent in the circle $|z| \le r_0$ where r_0 is less than but near to 1.

$$\frac{x\pi n}{xn+1} + (n-1)\text{arc sin}\left\{\frac{\sin\frac{x\pi n}{xn+1}}{\sin\frac{x\pi n}{xn+1}} \sin \frac{-x\pi n+\pi}{2(n-1)(xn+1)}\right\} = \frac{\pi}{2} \ ,$$

i.e. $\frac{x\pi n}{xn+1} + \frac{-x\pi n+\pi}{2(xn+1)} = \frac{\pi}{2}$,

and so the equation is satisfied.

iii) $\sin \frac{x\pi n}{xn+1} > \sin \frac{\pi}{n+1}$, because

$$0 < \frac{x\pi n}{xn+1} < \frac{\pi}{2} \quad \text{for } x < \frac{1}{n}$$

$$0 < \frac{\pi}{n+1} < \frac{\pi}{2}$$

and $$\frac{x\pi n}{xn+1} > \frac{\pi}{n+1} \quad \text{for} \quad x > \frac{1}{n^2} \ .$$

THEOREM 2.10. Suppose that $n > 1, x$ is a real number such that $x > 1$, k is an integer such that $0 < k < n$ and

$$R = \frac{\sin \frac{x\pi n}{2k(xn+1)}}{\sin \frac{\pi}{2(n-k)(xn+1)}}$$

Let k of the zeros of the polynomial $Q(z) = \prod_{k=1}^{n}(1 - \frac{z}{z_k})$ lie in the annulus $1 \leq |z| \leq R$ and the remaining $n-k$ be situated in the region $|z| > R$. Then the polynomial $P(z) = \int Q(z)dz$ is univalent in the circle $|z| \leq r_o$ where

$$r_o > \sin \frac{x\pi n}{2k(xn+1)} \ . \text{ (i)}$$

If $k < \frac{xn^2+xn}{2(xn+1)}$ we have further

$$r_o > \sin \frac{x\pi n}{2k(xn+1)} > \sin \frac{\pi}{n+1} \ .$$

(i) For k=1 and for large x, R becomes large, and by Theorem 2.10, P(z) is univalent in the circle $|z| \leq r_o$ where r_o is less than but near to 1.

<u>Proof</u>. i) $R > 1$ because

$$0 < \frac{x\pi n}{2k(xn+1)} < \frac{\pi}{2} ,$$

$$0 < \frac{\pi}{2(n-k)(xn+1)} < \frac{\pi}{2}$$

and $$\frac{x\pi n}{2k(xn+1)} > \frac{\pi}{2(n-k)(xn+1)} .$$

Thus $$\sin \frac{x\pi n}{2k(xn+1)} > \sin \frac{\pi}{2(n-k)(xn+1)} .$$

ii) $r' = \sin \frac{x\pi n}{2k(xn+1)}$ is the positive root of the equation

$$k \text{ arc sin } r + (n-k) \text{ arc sin } \frac{r}{R} = \frac{\pi}{2} ,$$

because on substituting r' in the equation we have

$$k \frac{x\pi n}{2k(xn+1)} + (n-k)\text{arc sin}\left\{\frac{\sin\frac{x\pi n}{2k(xn+1)}}{\sin\frac{x\pi n}{2k(xn+1)}} \sin \frac{\pi}{2(n-k)(xn+1)}\right\} = \frac{\pi}{2},$$

i.e. $$k \frac{x\pi n}{2k(xn+1)} + (n-k) \frac{\pi}{2(n-k)(xn+1)} = \frac{\pi}{2}$$

and so the equation is satisfied.

iii) If $k < \frac{xn^2+xn}{2(xn+1)}$ then $\sin \frac{x\pi n}{2k(xn+1)} > \sin \frac{\pi}{n+1}$,

because $$0 < \frac{x\pi n}{2k(xn+1)} < \frac{\pi}{2} ,$$

$$0 < \frac{\pi}{n+1} < \frac{\pi}{2}$$

and $$\frac{x\pi n}{2k(xn+1)} > \frac{\pi}{n+1} \quad \text{for} \quad k < \frac{xn^2+xn}{2(xn+1)} .$$

APPENDIX I

Let f(z) be of the form f(z) = zP(z) where P(z) is a polynomial of degree n. In order to obtain a better idea of how the radius of starlikeness of f(z) varies with the distribution of the zeros of P(z) we now consider some special cases. For these cases we will also find the radius of univalence. Alternatively by considering the distribution of the zeros of f'(z) we may obtain similar results for the radius of convexity. First we consider the case where all the zeros of the polynomial P(z) are situated on the circumference of the unit circle at equal distances from each other. Without loss of generality we may assume that

$$f(z) = z(z^n-1) . \qquad (n \geq 1)$$

Then $$f'(z) = z^n-1+nz^n ,$$

and $$\frac{zf'(z)}{f(z)} = \frac{z\left[(n+1)z^n-1\right]}{z(z^n-1)} = \frac{(n+1)z^n-1}{z^n-1} .$$

Putting $z = re^{i\theta}$, we have

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \text{Re}\left\{\frac{(n+1)r^n e^{ni\theta}-1}{r^n e^{ni\theta}-1}\right\}$$

$$= \frac{\text{Re}\left\{\left[(n+1)r^n e^{ni\theta}-1\right]\left[r^n e^{-ni\theta}-1\right]\right\}}{|r^n e^{ni\theta}-1|^2} .$$

The numerator is

$$\text{Re}\left\{(n+1)r^{2n}-(n+1)r^n e^{ni\theta}-r^n e^{-ni\theta}+1\right\}$$

$$= (n+1)r^{2n}-(n+1)r^n\cos n\theta-r^n\cos n\theta+1 = E, \quad \text{say.}$$

Now, putting $r^n = R$, we have

$$E = (n+1)R^2-(n+2)R\cos n\theta+1 \geq (n+1)R^2-(n+2)R+1 .$$

The roots of $\quad (n+1)R^2-(n+2)R+1 = 0$

are 1 and $\frac{1}{n+1}$, so for $r \leq \frac{1}{\sqrt[n]{n+1}}$, we have $\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq 0$.

Thus <u>the radius of starlikeness of</u> f(z) is $\frac{1}{\sqrt[n]{n+1}}$. Since the derivative vanishes on the circumference of $|z| \leq \frac{1}{\sqrt[n]{n+1}}$ <u>the radius of univalence has also the same value</u>.

Here we see that for $|z| \leq \frac{1}{\sqrt[n]{n+1}}$ we have $\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq 0$ where equality exists if and only if $z^n = \frac{1}{n+1}$, i.e. if and only if z is on the circumference of $|z| \leq \frac{1}{\sqrt[n]{n+1}}$ in such a position that z, one of the zeros of P(z) and the origin are collinear, and z and this zero of P(z) are on the same side of the origin. .. .. (i)

Now let f(z) = zP(z), where P(z) is a polynomial of degree n all of whose zeros are situated on the circumference of the unit circle.

Denoting the zeros of P(z) by $x_1, x_2, \ldots, x_n$, we have

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \text{Re}\left\{1 + \frac{z}{z-x_1} + \frac{z}{z-x_2} + \ldots + \frac{z}{z-x_n}\right\}. \quad ..\text{(ii)}$$

Putting $z = re^{i\theta} \quad x = e^{i\varphi}$

$$\text{Re}\left\{\frac{z}{z-x}\right\} = \frac{r^2 - r\cos(\theta-\varphi)}{r^2 - 2r\cos(\theta-\varphi)+1}.$$

If r is fixed and $r < 1$ then this is a decreasing function of $\cos(\theta-\varphi)$. .. .. (iii)

Let us consider the following cases, represented by figures 1 and 2. Points on the outer circles denote the zeros of P(z).

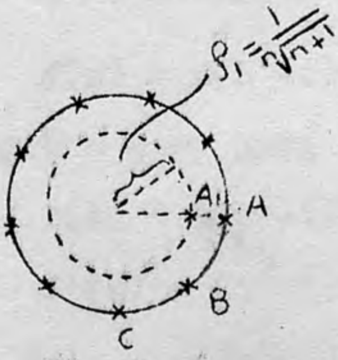

Figure 1.

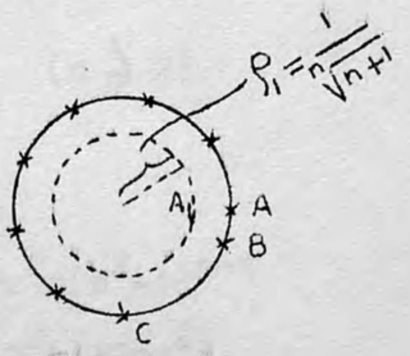

Figure 2.

In figure 1 we consider the case where all the zeros of P(z) are distributed at equal distances from each other on the circumference of the unit circle. Then we keep the positions of all the zeros the same except one, say B. Let B approach one of the other zeros, say A. (Figure 2.)

By (i), (ii) and (iii) we see that

in the first case for $z = A_1$ we have $\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = 0$

in the second case for $z = A_1$ we have $\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < 0$.

If we denote the radius of starlikeness in the first case by ρ_1 and in the second case by ρ_2 then by continuity $\rho_1 > \rho_2$. By moving B nearer to A without altering the positions of the other zeros we obtain the radius of starlikeness smaller than ρ_2. Also by increasing the number of zeros approaching A and keeping the positions of the remaining zeros we decrease the radius of starlikeness.

Now let us consider the case where n is an even number and $\frac{n}{2}$ zeros of P(z) are concentrated at one end of a diameter of the unit circle and $\frac{n}{2}$ of them are concentrated at the opposite end of the same diameter. Again without loss of generality we may assume that

$$f(z) = z(z^2-1)^{\frac{n}{2}} . \quad (n \geq 2)$$

Putting $\frac{n}{2} = m$ we have

$$f(z) = z(z^2-1)^m ,$$

$$f'(z) = (z^2-1)^m + 2z^2 m(z^2-1)^{m-1}$$

$$= (z^2-1)^{m-1}[z^2-1+2mz^2],$$

$$\frac{zf'(z)}{f(z)} = \frac{z(z^2-1)^{m-1}\ z^2(2m+1)-1}{z(z^2-1)^m}$$

$$= \frac{z^2(2m+1)-1}{z^2-1}$$

Putting $z = re^{i\theta}$, we have

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \frac{\mathrm{Re}\{[(2m+1)r^2e^{2i\theta}-1][r^2e^{-2i\theta}-1]\}}{|z^2-1|^2} .$$

The numerator is

$$\mathrm{Re}\left\{(2m+1)r^4-(2m+1)r^2e^{2i\theta}-r^2e^{-2i\theta}+1\right\}$$

$$= (2m+1)r^4+1-r^2(2m+2)\cos 2\theta$$

$$= (n+1)r^4+1-r^2(n+2)\cos 2\theta .$$

since $\quad (n+1)r^4+1-r^2(n+2)\cos 2\theta \geq (n+1)r^4+1-r^2(n+2)$,

and $\quad (n+1)r^4+1-(n+2)r^2 = (r^2-1)[(n+1)r^2-1],$

for $\quad r \leq \frac{1}{\sqrt{n+1}}$ we have $\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq 0$.

Equality exists if and only if $\cos 2\theta = 1$ and $r = \frac{1}{\sqrt{n+1}}$..(iv)

Thus the radius of starlikeness of $f(z)$ is $\frac{1}{\sqrt{n+1}}$. Since the derivative vanishes on the circumference of $|z| \leq \frac{1}{\sqrt{n+1}}$ then the radius of univalence has also the same value.

Let us consider the cases represented by the following figures. Points on the outer circle denote the positions of the zeros of $P(z)$ and the numbers in parenthesis show the number of zeros.

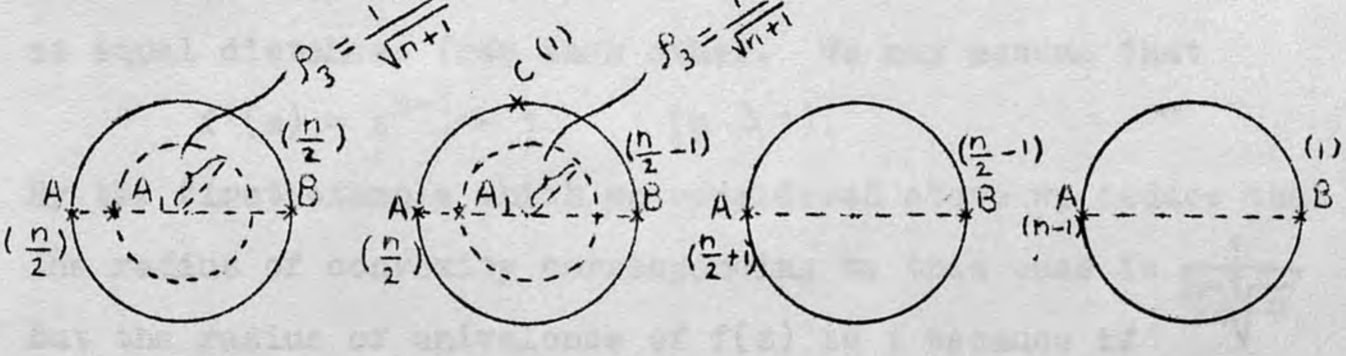

Figure 3. Figure 4. Figure 5. Figure 6.

Figure 3 represents the above case where n is an even number and $\frac{n}{2}$ zeros of P(z) are concentrated at A and $\frac{n}{2}$ zeros are concentrated at B. By moving one of the zeros from B towards A we have figure 4. When this zero reaches A we have Figure 5. By continuing this process we eventually obtain the distribution of zeros represented in Figure 6. By (ii), (iii) and (iv) we see that in the case represented by figure 3

$$\text{for } z = A_1 \quad \text{we have} \quad \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = 0 .$$

But in the case represented by figure 4

$$\text{for } z = A_1 \quad \text{we have} \quad \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < 0 .$$

Let us denote the radii of starlikenesses corresponding to the figures 3, 4, 5, 6 respectively by $\rho_3, \rho_4, \rho_5, \rho_6$. Then $\rho_3 > \rho_4 > \rho_5 > \rho_6$.

Now in the following two examples we will obtain results for the radius of univalence better than that given by the theorem of Kakeya. (Theorem 2.1). First we suppose that f(z) is a polynomial of degree n and all the zeros of

$f'(z)$ are situated on the circumference of the unit circle at equal distances from each other. We may assume that

$$f'(z) = z^{n-1} - 1 \qquad (n \geq 1).$$

By the first example which we considered above we deduce that the radius of convexity corresponding to this case is $\frac{1}{\sqrt[n-1]{n}}$. But the radius of univalence of $f(z)$ is 1 because if $f(z_1) = f(z_2)$ and $z_1 \neq z_2$ where $|z_1| < 1$, $|z_2| < 1$ then

$$\frac{z_1^n}{n} - z_1 = \frac{z_2^n}{n} - z_2 ,$$

so $\qquad \frac{1}{n}(z_1^n - z_2^n) = z_1 - z_2 ,$

or, dividing by $z_1 - z_2$,

$$\frac{1}{n}(z_1^{n-1} + z_1^{n-2}z_2 + \dots + z_1 z_2^{n-2} + z_2^{n-1}) = 1 \quad .$$

But $\qquad \frac{1}{n}|z_1^{n-1} + z_1^{n-2}z_2 + \dots + z_1 z_2^{n-2} + z_2^{n-1}| < \frac{1}{n} \cdot n = 1 ,$

which is a contradiction.

Now we consider the case where $f(z)$ is a polynomial of degree n and one zero of the derivative is located at one end of a diameter of the unit circle and all the others are concentrated at the opposite end of the same diameter. We may assume that

$$f'(z) = (z-1)(z+1)^{n-2} \qquad (n \geq 3).$$

Using the inequality

$$\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} \geq 0$$

we find the radius of convexity of $f(z)$ to be $\frac{3-n+\sqrt{(n-3)^2+4n}}{2n}$. We will now prove that <u>the radius of univalence of</u> $f(z)$ <u>is greater than or equal to</u> $\sin\frac{\pi}{n-1}$. <u>Equality exists only for</u>

n=3. <u>If</u> $n > 3$ <u>then it is always greater than</u> $\sin \frac{\pi}{n-1}$. Since by the previous example we know that for n=3 the radius of univalence is equal to 1 in the following proof we will assume that $n > 3$. A similar proof is valid for n=3 provided that the radius of the circle under consideration is assumed to be less than 1, which implies that f(z) is univalent in every closed circle inside the unit circle; therefore the radius of univalence is equal to 1.

<u>Proof</u>.

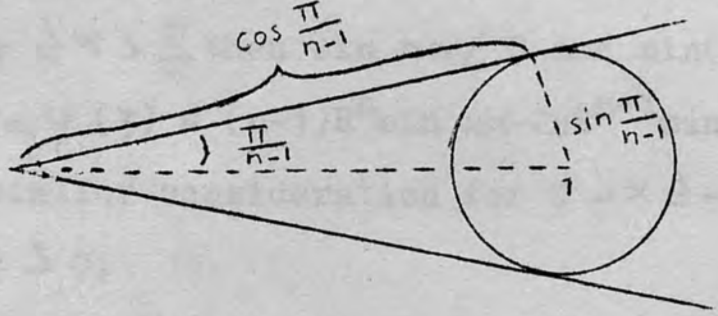

Figure 7.

If $\qquad f'(z) = (z-1)(z+1)^{n-2}$,

then $\qquad n(n-1)f(z) = n(z-1)(z+1)^{n-1}-(z+1)^n$.

Putting $\qquad 1+z = \zeta , \quad \zeta = Re^{i\alpha},$ we have

$$n(n-1)f(z) = \varphi(\zeta) = (n-1)\zeta^n - 2n\zeta^{n-1} ,$$

and taking imaginary parts,

$$\operatorname{Im} \varphi(\zeta) = (n-1)R^n \sin n\alpha - 2nR^{n-1}\sin(n-1)\alpha.$$

First we will show that for $0 < \alpha \leq \frac{\pi}{n-1}$ we have $\operatorname{Im}\varphi(\zeta) < 0$. Therefore the image of the upper half disc $|z| \leq \sin \frac{\pi}{n-1}$, $\operatorname{Im} z > 0$ and the image of the lower half disc $|z| \leq \sin \frac{\pi}{n-1}$, $\operatorname{Im} z < 0$ have not any common point.

For $0 < \alpha \leq \frac{\pi}{n}$, we have

$\cos n\alpha - \cos(n-1)\alpha < 0$.

If $F(\alpha) = (n-1)\sin n\alpha - n\sin(n-1)\alpha$,

then for $0 < \alpha \leq \frac{\pi}{n}$, we have $F'(\alpha) < 0$, and $F(\alpha)$ is a decreasing function of α. Since $F(0) = 0$ it follows that for $0 < \alpha \leq \frac{\pi}{n}$ we have $F(\alpha) < 0$.

But $(n-1)R\sin n\alpha - 2n\sin(n-1)\alpha < 2(n-1)\sin n\alpha - 2n\sin(n-1)\alpha$.

Thus for $0 < \alpha \leq \frac{\pi}{n}$ we have

$$(n-1)R\sin n\alpha - 2n\sin(n-1)\alpha < 0$$

and so $Im\,\varphi(\zeta) < 0$.

If $\frac{\pi}{n-1} \geq \alpha > \frac{\pi}{n}$ then $\sin n\alpha < 0$ and $\sin(n-1)\alpha \geq 0$,

so $Im\,\varphi(\zeta) = (n-1)R^n\sin n\alpha - 2nR^{n-1}\sin(n-1)\alpha < 0$.

By a similar consideration for $0 > \alpha \geq -\frac{\pi}{n-1}$, we have $Im\,\varphi(\zeta) > 0$.

Now we will show that $f(z)$ cannot take the same value at two distinct points z_1, z_2 which are both on the upper half disc $|z| \leq \sin\frac{\pi}{n-1}$, $Im\,z \geq 0$ or on the lower half disc $|z| \leq \sin\frac{\pi}{n-1}$, $Im\,z < 0$.

Writing

$$h(z)=n(n-1)f(z)=n(z-1)(z+1)^{n-1}-(z+1)^n=(z+1)^{n-1}(nz-n-z-1),$$

and $g(z) = \log h(z) = (n-1)\log(z+1) + \log(nz-n-z-1)$,

we have
$$g'(z) = \frac{n-1}{z+1} + \frac{n-1}{nz-n-z-1} .$$

Putting $z = re^{i\gamma}$, we have

$$\mathrm{Re}\, g'(z) = (n-1)\left(\mathrm{Re}\,\frac{1}{re^{i\gamma}+1} + \mathrm{Re}\,\frac{1}{re^{i\gamma}(n-1)-(n+1)}\right).$$

Now

$$\mathrm{Re}\frac{1}{re^{i\gamma}+1} = \frac{r\cos\gamma+1}{(re^{i\gamma}+1)(re^{-i\gamma}+1)} = \frac{r\cos\gamma+1}{r^2+re^{i\gamma}+re^{-i\gamma}+1} = \frac{r\cos\gamma+1}{r^2+2r\cos\gamma+1} ,$$

and

$$\text{Re}\,\frac{1}{re^{i\gamma}(n-1)-(n+1)} = \text{Re}\,\frac{re^{-i\gamma}(n-1)-(n+1)}{[re^{i\gamma}(n-1)-(n+1)][re^{-i\gamma}(n-1)-(n+1)]}$$

$$= \frac{r(n-1)\cos\gamma-(n+1)}{r^2(n-1)^2-r(n^2-1)e^{i\gamma}-r(n^2-1)e^{-i\gamma}+(n+1)^2}$$

$$= \frac{(n-1)r\cos\gamma-(n+1)}{r^2(n-1)^2-2r(n^2-1)\cos\gamma+(n+1)^2},$$

so $\text{Re}\,\frac{1}{re^{i\gamma}+1} + \text{Re}\,\frac{1}{re^{i\gamma}(n-1)-(n+1)}$

$$= \Big[(n-1)^2r^2-2r(n^2-1)\cos\gamma+(n+1)^2+(n-1)^2r^3\cos\gamma-2r^2(n^2-1)$$
$$\cdot\cos^2\gamma+(n+1)^2r\cos\gamma+(n-1)r^3\cos\gamma-(n+1)r^2$$
$$+2r^2(n-1)\cos^2\gamma-2r(n+1)\cos\gamma+(n-1)r\cos\gamma-(n+1)\Big]$$
$$\Big/\Big[[r^2+2r\cos\gamma+1][(n-1)^2r^2-2r(n^2-1)\cos\gamma+(n+1)^2]\Big].$$

The numerator is

$$(n+1)^2+(n-1)^2r^2-(n+1)-(n+1)r^2+\cos\gamma\big[-2r(n^2-1)+(n-1)^2r^3$$
$$+(n+1)^2r+(n-1)r^3-2r(n+1)+(n-1)r\big]$$
$$+\cos^2\gamma\big[-2r^2(n^2-1)+2r^2(n-1)\big]$$
$$= L + M\cos\gamma + N\cos^2\gamma,$$

say. We have

$$M = r\big[-2n^2+2+n^2+1+2n-2n-2+n-1\big] + r^3\big[n^2-2n+1+n-1\big]$$
$$= r(n-n^2)+r^3(n^2-n) = (n^2-n)(r^3-r) < 0,$$
$$N = 2r^2(n-1-n^2+1) < 0,$$

so $L + M\cos\gamma + N\cos^2\gamma \geq L + M + N.$

Since

$$L+M+N = (n+1)^2+(n-1)^2r^2-(n+1)-(n+1)r^2+2r^2(n-n^2)+(r^3-r)(n^2-n)$$
$$= n^2+2n+1-n-1+r^2\big[n^2-2n+1-n-1-2n^2+2n\big] + (n^2-n)r^3-(n^2-n)r$$
$$= (n^2+n)(1-r^2)-r(n^2-n)(1-r^2) = (1-r^2)\big[n^2+n-r(n^2-n)\big] > 0,$$

it follows that if $r < 1$ then $\mathrm{Re}\, g'(z) > 0$. Thus $g(z)$ cannot take the same value at two distinct points inside or on the circle $|z| \leq \sin \frac{\pi}{n-1}$. Now $f(z) = \frac{e^{g(z)}}{n(n-1)}$ and if $f(z_1)=f(z_2)$ then $e^{g(z_1)} = e^{g(z_2)}$ so $g(z_1) = g(z_2) + 2k\pi i$ where k is an integer. But if z_1, z_2 are points on the upper half disc $|z| \leq \sin \frac{\pi}{n-1}$, $\mathrm{Im}\, z \geq 0$, then $\log h(z_1) - \log h(z_2)$ cannot be equal to $2k\pi i$

because $\mathrm{Im} \log h(z) = (n-1)\arg(z+1)+\arg\{(n-1)z-(n+1)\}$,

and so $\mathrm{Im} \{\log h(z_1)-\log h(z_2)\} < 2\pi$.

By a similar consideration z_1, z_2 cannot be points on the lower half disc. Hence $f(z)$ cannot take the same value at two distinct points inside or on the circle $|z| \leq \sin \frac{\pi}{n-1}$. If $n > 3$ then $f'(z)$ cannot vanish on the circumference of this circle, so by Dieudonné's argument(i) the radius of univalence of $f(z)$ is greater than $\sin \frac{\pi}{n-1}$.

(i) If C is the largest circle about the origin inside which a polynomial $f(z)$ is univalent then on the circumference of C either $f(z)$ takes the same value at two distinct points or $f'(z)$ vanishes. Otherwise $f(z)$ is univalent inside a larger circle. [4]; see e.g., [8], p.22.

APPENDIX II

In this appendix we will show that the result given by M. Robertson in [9] contains an error.

Let $f(z) = z + a_2z^2 + \dots + a_nz^n$ and let z_o be the root of smallest modulus of $f'(z) = 0$. Then by the theorem of Kakeya (Theorem 2.1) the radius of univalence of $f(z)$ is greater than or equal to $|z_o| \sin\frac{\pi}{n}$. It is stated by M. Robertson that if

$$D = \begin{vmatrix} b_na_n, & b_{n-1}a_{n-1}, \dots, b_2a_2, & 1, & 0, & 0, \dots\dots, & 0 \\ 0, & b_na_n, \dots\dots, b_3a_3, & b_2a_2, & 1, & 0, \dots\dots, & 0 \\ 0, & 0, \dots\dots, b_4a_4, & b_3a_3, & b_2a_2, & 1, 0, \dots, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & 0, \dots\dots, b_na_n, & b_{n-1}a_{n-1}, & \dots & \dots, b_2a_2, & 1 \\ na_n, & (n-1)a_{n-1}, \dots, 2a_2, & 1, & 0, & 0, \dots\dots, & 0 \\ 0, & na_n, \dots\dots, 3a_3, & 2a_2, & 1, & 0, \dots\dots, & 0 \\ 0, & 0, \dots\dots, 4a_4, & 3a_3, & 2a_2, & 1, 0, \dots, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & 0, \dots\dots, na_n, & (n-1)a_{n-1}, & \dots & \dots, 2a_2, & 1 \end{vmatrix} = 0$$

where $$b_r = \sin^{r-1}\frac{\pi}{n}\,\frac{\sin\frac{r\pi}{2}(1-\frac{2}{n})}{\cos\frac{\pi}{n}},$$

then $f(z)$ takes the same value at two points z_1, z_2 on the circumference of $|z| \leq |z_o| \sin\frac{\pi}{n}$ where $z_1 z_2$ are the points of contact of the tangents drawn from z_o and so the radius of univalence of $f(z)$ is exactly $|z_o| \cdot \sin\frac{\pi}{n}$.

Now we construct the following counter example. Let $f(z) = z + z^2 - \frac{z^4}{4}$. Then

$$D = \begin{vmatrix} b_4a_4 & b_3a_3 & b_2a_2 & 1 & 0 & 0 \\ 0 & b_4a_4 & b_3a_3 & b_2a_2 & 1 & 0 \\ 0 & 0 & b_4a_4 & b_3a_3 & b_2a_2 & 1 \\ 4a_4 & 3a_3 & 2a_2 & 1 & 0 & 0 \\ 0 & 4a_4 & 3a_3 & 2a_2 & 1 & 0 \\ 0 & 0 & 4a_4 & 3a_3 & 2a_2 & 1 \end{vmatrix}$$

where $a_2 = 1$, $a_3 = 0$, $a_4 = -\frac{1}{4}$, $b_2 = 1$, $b_4 = 0$, and so $D = 0$. We will show that $f(z)$ cannot take the same value at two points z_1, z_2 on the circumference of $|z| \leq |z_0| \sin \frac{\pi}{n}$ where z_1, z_2 are the points of contact of the tangents drawn from z_0. Thus according to the argument given in [9] the radius of univalence of $f(z)$ should be greater than $|z_0| . \sin \frac{\pi}{n}$.

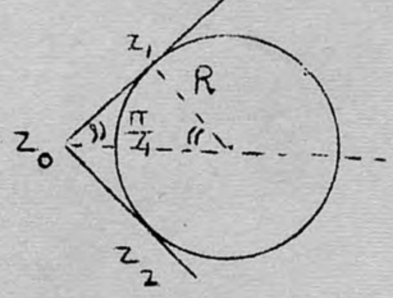

Figure 1.

We have

$$z_0 = \frac{1-\sqrt{5}}{2}, \quad R = |z_0| \cdot \sin\frac{\pi}{n} = \frac{\sqrt{5}-1}{2\sqrt{2}}, \quad z_1 = \frac{\sqrt{5}-1}{2\sqrt{2}} e^{\frac{3\pi i}{4}},$$

$$z_2 = \frac{\sqrt{5}-1}{2\sqrt{2}} e^{-\frac{3\pi i}{4}}.$$

Since $f(z)$ is real for real z and z_1, z_2 are complex conjugates, in order to show that $f(z_1) \neq f(z_2)$ it is sufficient to show that $f(z_1)$ is not real. This follows because

$$f(z_1) = z_1 + z_1^2 - \frac{z_1^4}{4}$$

where $\frac{z_1^4}{4}$ is real, but $z_1^2 + z_1 = \frac{\sqrt{5}-1}{2\sqrt{2}} \left(\frac{\sqrt{5}-1}{2\sqrt{2}} e^{\frac{3\pi i}{2}} + e^{\frac{3\pi i}{4}} \right)$,

which is not real.

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