

DECISION PROBLEMS CONCERNING
SETS OF EQUATIONS

by
Kornilia Kalfa

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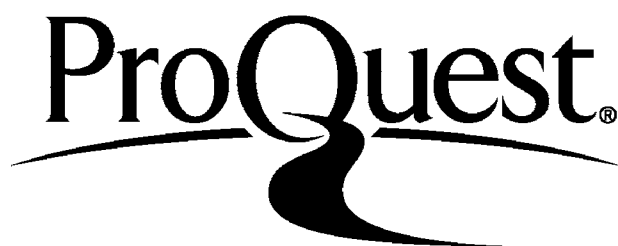
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To my teacher

JOHN ANASTASSIADIS

Professor of Mathematics
at the University of Thessaloniki

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A B S T R A C T

This thesis is about "decision problems concerning properties of sets of equations".

If \mathcal{L} is a first-order language with equality and if P is a property of sets of \mathcal{L} -equations, then "the decision problem of P in \mathcal{L} " is the problem of the existence or not of an algorithm, which enables us to decide whether, given a set Σ of \mathcal{L} -equations, Σ has the property P or not. If such an algorithm exists, P is decidable in \mathcal{L} . Otherwise, it is undecidable in \mathcal{L} .

After surveying the work that has been done in the field, we present a new method for proving the undecidability of a property P , for finite sets of \mathcal{L} -equations. As an application, we establish the undecidability of some basic model-theoretical properties, for finite sets of equations of non-trivial languages. Then, we prove the non-existence of an algorithm for deciding whether a field is finite and, as a corollary, we derive the undecidability of certain properties, for recursive sets of equations of infinite non-trivial languages. Finally, we consider trivial languages, and we prove that a number of properties, undecidable in languages with higher complexity, are decidable in them.

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CHAPTER 0

INTRODUCTION

§0.0. Introduction

An equation is a universal sentence of a first order language with equality of the form $(\forall \bar{v})(\varphi = \psi)$, where φ and ψ are terms. The study of equations, as a separate mathematical discipline, began in 1935 with a paper written by Birkhoff [1]. It has been of great importance to Algebraists, because some of the most interesting theories to them (the theory of groups, the theory of rings, the theory of lattices, to mention just few examples) can be axiomatized by equations.

Global decision problems, concerning properties of sets of equations, were raised, for the first time, by Tarski [28] in 1963. They subsequently received consideration mainly by Perkins [19], McKenzie [12], McNulty [14] and Pigozzi [22]. These are problems of the following kind :

Let \mathcal{L} be a countable language and let P be a property of sets of equations of \mathcal{L} . Is there an algorithm that enables us to decide whether, given a finite (or recursive or singleton) set of equations of \mathcal{L} , it has the property P ?

If such an algorithm exists, P is called decidable for finite (or recursive or singleton) sets of equations in \mathcal{L} . Otherwise, it is called undecidable. Almost all the properties, examined so far, turned out to be undecidable, at least in languages with sufficiently high complexity.

Undecidable properties of sets of equations are the focal point of this thesis. Our approach is Model-theoretical, with a minimal use of formal Recursion theory and an extended use of informal procedures, whenever this is possible.

Chapter 0 is introductory. Preliminary definitions and results from Set theory, Model Theory and Recursion theory are given in §0.1. and a system of notations is introduced. § 0.2. deals with equations.

A classification of the decision problems concerning equations is attempted in §0.3. In §0.4., the existing methods for proving the undecidability of properties of sets of equations are surveyed and the most important results, obtained so far, are given. We conclude the chapter with a discussion of the motivation that led us to pursue this line of research.

In Chapter 1 a new method of proving the undecidability of properties of finite sets of equations Σ is given. As an application, the undecidability of the properties :

- P_0 : The equational theory generated by Σ is equationally complete.
- P_1 : The first order theory of the infinite models of Σ is complete.
- P_2 : The first order theory of the infinite models of Σ is model-complete.
- P_3 : Σ has the joint embedding property.
- P_4 : The first order theory of the non-trivial models of Σ has the joint embedding property.
- P_5 : The first order theory of the infinite models of Σ has the joint embedding property,

for non-trivial languages, is established.

In Chapter 2 the non-existence of an algorithm, for deciding whether a computable field is finite, is proved. As a consequence of this fact, the undecidability of properties :

- P_6 : Σ has finite non-trivial models.
- P_7 : The first order theory of the non-trivial models of Σ is complete.
- P_8 : The first order theory of the non-trivial models of Σ is model-complete,

for recursive sets of equations of infinite strong languages, is established.

The decision problems of properties $P_0 - P_8$, in trivial languages, are examined in Chapter 3 and algorithms for deciding whether a finite set Σ has each one of the properties are constructed.

Indices of symbols and references can be found at the end of the thesis.

A final remark on typography. " \leftrightarrow " is used as an abbreviation of "if and only if" while the symbols " \square " and " $\#$ " indicate "end of the proof" and "contradictory statement", respectively.

§ 0.1. Background material-Notations

A. From Set Theory

We shall assume familiarity with the basic notions of Set theory.

The symbols \cup , \cap , $-$ stand, respectively, for the operations of union, intersection and difference of sets. The symbols \in , \subset , \subsetneq stand, respectively, for the relations of membership, inclusion and proper inclusion between sets.

The Cartesian product of a family of sets $\{X_i : i \in I\}$ is denoted by $\prod_{i \in I} X_i$. We also use the notation $X_0 \times X_1 \times \dots \times X_n$ for the Cartesian product of a finite family of sets. The complement of a set X , is denoted by \bar{X} .

The empty set is denoted by \emptyset , the set of natural numbers by ω and the first ordinal, greater than ω , by ω_1 . Otherwise we use lower case Greek letters $\alpha, \beta, \gamma, \delta, \dots$ for ordinals. The cardinality of an arbitrary set X is denoted by $|X|$. p_i stands for the i^{th} prime number.

$<$ is the usual ordering relation between ordinals. An equivalence relation in a set X is denoted by \sim . The equivalence class of an $x \in X$, with respect to \sim , is denoted by $[x]_{\sim}$.

Let A, B be two sets and $C \subset A$. $f : A \rightarrow B$ is a mapping with domain A and range a subset of B . $f|_C$ is the restriction of f to C . $f[C]$ is the image of C under f . The symbols $f: A \rightarrow B$, $f: A \twoheadrightarrow B$, $f: A \xrightarrow{1} B$ are used to denote that f is, respectively, injective, surjective, bijective.

${}^A B$ stands for the set of all $f: A \rightarrow B$. If f_1 and f_2 belong to ${}^A B$, $f_1 + f_2$ stands for the usual sum, while $f_1 \cdot f_2$ stands for the usual product of the two mappings.

If α is an ordinal and X is a set, an α -termed sequence in X (i.e. a member of ${}^\alpha X$) is denoted by $\langle x_\beta : \beta < \alpha \rangle$, or simply by \bar{x} .

B. From Model Theory

We assume the reader is familiar with the elements of First

order logic and Model theory. Chang's and Keisler's [4] will serve as a constant reference for definitions and basic results in the field.

Throughout the thesis, \mathcal{L} is a countable algebraic language, i.e. a first order language with equality, with countably many operation symbols, at most countably many constant symbols and no relation symbols. In other words, a language of the form

$$\mathcal{L} = \langle \{Q_i\}_{i \in I}, \{c_j\}_{j \in J} \rangle,$$

where I and J are sets of natural numbers, Q_i is an operation symbol and c_j is a constant symbol.

If the number of non-logical symbols in \mathcal{L} is finite, \mathcal{L} is called a finite algebraic language. Otherwise, it is called a denumerable algebraic language.

The rank of Q_i is denoted by $r(i)$. The language is non-trivial if it contains either at least two unary operation symbols or at least one symbol of rank greater than one. Otherwise, it is trivial. The language is strong if it has at least one operation symbol of rank greater than one.

The set of variables is denoted by $Va = \{v_i : i \in \omega\}$. We use " - ", " \vee ", " \wedge ", " \rightarrow ", " \exists ", " \forall ", " = ", respectively, for "not", "or", "and", "implies", "there exists", "for each", "equal".

$Sym_{\mathcal{L}}$ is used for the set of all logical and non-logical symbols of the language. I.e.

$$Sym_{\mathcal{L}} = Va \cup \{ -, \vee, \wedge, \rightarrow, \exists, \forall, = \} \cup \{Q_i\}_{i \in I} \cup \{c_j\}_{j \in J}$$

The set of terms of the language is denoted by $Term_{\mathcal{L}}$. If $t \in Term_{\mathcal{L}}$ and $\bar{s} \in {}^{\omega}Term_{\mathcal{L}}$, then $t[\bar{s}]$ is defined as follows:

- i. $v_i[\bar{s}] = s_i$
- ii. $c_j[\bar{s}] = c_j$
- iii. $Q_i \theta_0 \theta_1 \dots \theta_{r(i)-1}[\bar{s}] = Q_i \theta_0[\bar{s}] \theta_1[\bar{s}] \dots \theta_{r(i)-1}[\bar{s}]$.

$Exp_{\mathcal{L}}$, $Form_{\mathcal{L}}$, $Sent_{\mathcal{L}}$, \exists - $Form_{\mathcal{L}}$, \forall - $Form_{\mathcal{L}}$, $\forall \exists$ - $Form_{\mathcal{L}}$ stand, respectively, for the sets of expressions, formulae, sentences, existential formulae, universal formulae and universal-existential formulae of the language \mathcal{L} . If φ is a formula (or a term) of \mathcal{L} , with exactly the variables v_{α_0} ,

$v_{x_1}, v_{x_2}, \dots, v_{x_{n-1}}$ free in it, we write $\varphi(v_{x_0}, v_{x_1}, \dots, v_{x_{n-1}})$.

An \mathcal{L} -structure is called an \mathcal{L} -algebra. We use Gothic letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \dots$ to denote algebras and the corresponding English letters A, B, C, D, \dots to denote their universes. If Q_i is an operation symbol, c_j is a constant symbol, t is a term and φ is a formula, we use, respectively, the notations $Q_i^{\mathfrak{A}}, c_j^{\mathfrak{A}}, t^{\mathfrak{A}}$ and $\varphi^{\mathfrak{A}}$ for their interpretation in \mathfrak{A} . The cardinality of \mathfrak{A} is denoted by $\|\mathfrak{A}\|$. An algebra is trivial if $\|\mathfrak{A}\| = 1$.

Let \mathfrak{A} be an \mathcal{L} -algebra and $X \subset A$. The expansion of \mathcal{L} by X is the language

$$\mathcal{L}_X = \mathcal{L} \cup \{c_a : a \in X\},$$

obtained from \mathcal{L} by adding a new constant symbol c_a for each $a \in X$. It is understood that, if $a \neq b$ then $c_a \neq c_b$. The expanded algebra \mathfrak{A}_X is the \mathcal{L}_X -structure, obtained by interpreting each new constant symbol c_a by the element a . I.e.

$$\mathfrak{A}_X = \langle A, \{Q_i^{\mathfrak{A}}\}_{i \in I}, \{c_j^{\mathfrak{A}}\}_{j \in J}, \{a\}_{a \in X} \rangle$$

The symbol \models denotes the usual satisfaction predicate. So, if $\varphi(v_{x_0}, v_{x_1}, \dots, v_{x_{n-1}}) \in \text{Form}_{\mathcal{L}}$ and $\langle a_0, a_1, \dots, a_{n-1} \rangle \in {}^n A$, we write

$$\mathfrak{A} \models \varphi[a_0, a_1, \dots, a_{n-1}]$$

for " φ is satisfied by $\langle a_0, \dots, a_{n-1} \rangle$ in \mathfrak{A} ".

If $\Sigma \subset \text{Sent}_{\mathcal{L}}$ and $T \subset \text{Sent}_{\mathcal{L}}$, the symbols $\mathfrak{A} \models \Sigma$ and $T \models \Sigma$ stand, respectively, for " \mathfrak{A} is a model of Σ " and " Σ is a consequence of T ".

A first order theory of \mathcal{L} is conceived of as containing all its consequences. In other words, if $\Phi \subset \text{Sent}_{\mathcal{L}}$, then, Φ is a first order theory of \mathcal{L} , if and only if

$$(\forall \varphi \in \text{Sent}_{\mathcal{L}}) (\varphi \in \Phi \leftrightarrow \Phi \models \varphi)$$

The first order theory generated by a class of \mathcal{L} -algebras \mathfrak{R} , is the set:

$$\text{Th } \mathfrak{K} = \{\varphi \in \text{Sent}_{\mathcal{L}} : (\forall \mathfrak{M} \in \mathfrak{K}) (\mathfrak{M} \models \varphi)\}$$

The first order theory generated by a set $\Sigma \subset \text{Sent}_{\mathcal{L}}$ is the set

$$\theta[\Sigma] = \{\varphi \in \text{Sent}_{\mathcal{L}} : \Sigma \models \varphi\}.$$

$\mathfrak{M} \subset \mathfrak{B}$, $\mathfrak{M} < \mathfrak{B}$, $\mathfrak{M} \cong \mathfrak{B}$ and $\mathfrak{M} \equiv \mathfrak{B}$ stand, respectively, for " \mathfrak{M} is a substructure of \mathfrak{B} ", " \mathfrak{M} is an elementary substructure of \mathfrak{B} ", " \mathfrak{M} is isomorphic to \mathfrak{B} " and " \mathfrak{M} is elementary equivalent to \mathfrak{B} ". If \mathfrak{M} is an \mathcal{L} -structure and $X \subset A$, the substructure of \mathfrak{M} , generated by X , is denoted by $\langle X \rangle$. If $\langle \mathfrak{M}_{\beta} : \beta < \alpha \rangle$ is a chain of \mathcal{L} -structures, its union is denoted by $\bigcup_{\beta < \alpha} \mathfrak{M}_{\beta}$.

The notations $f : \mathfrak{M} \rightarrow \mathfrak{B}$, $f : \mathfrak{M} \twoheadrightarrow \mathfrak{B}$, $f : \mathfrak{M} \hookrightarrow \mathfrak{B}$ and $f : \mathfrak{M} \xrightarrow{\cong} \mathfrak{B}$ are used to denote that f is, respectively, "an homomorphism of \mathfrak{M} to \mathfrak{B} ", "an homomorphism of \mathfrak{M} onto \mathfrak{B} ", "an embedding of \mathfrak{M} to \mathfrak{B} " and "an isomorphism between \mathfrak{M} and \mathfrak{B} ".

Let Φ be a first order theory of \mathcal{L} . Φ is consistent if it has a model. Φ is α -categorical if any two models of it of cardinality α are isomorphic. Φ is complete if it is consistent and it satisfies one of the following equivalent statements:

- i. Any two models of it are elementary equivalent.
- ii. There is no consistent first order theory θ of \mathcal{L} , such that $\Phi \subsetneq \theta$.
- iii. For any $\varphi \in \text{Sent}_{\mathcal{L}}$, either $\varphi \in \Phi$ or $-\varphi \in \Phi$.

Φ is model-complete if for any two models \mathfrak{M} and \mathfrak{B} of it, such that $\mathfrak{M} \subset \mathfrak{B}$, it holds that $\mathfrak{M} < \mathfrak{B}$,

Φ has the joint embedding property if any two models of Φ are embeddable in a third model of it. I.e. if

$$(\forall \mathfrak{M} \models \Phi) (\forall \mathfrak{B} \models \Phi) (\exists \mathfrak{C} \models \Phi) (\exists f : \mathfrak{M} \hookrightarrow \mathfrak{C}) (\exists g : \mathfrak{B} \hookrightarrow \mathfrak{C})$$

Φ is preserved under unions of chains if, for any ordinal α and for any chain $\langle \mathfrak{M}_{\beta} : \beta < \alpha \rangle$ of models of Φ , the union $\bigcup_{\beta < \alpha} \mathfrak{M}_{\beta}$ is a model of Φ .

The following fundamental theorems of Model Theory will be in constant use throughout the thesis:

Theorem 0.1.0 [Los' - Vaught test] Let Φ be a consistent theory

of \mathcal{L} that satisfies the following conditions:

- a. Φ has no finite models
- b. Φ is α -categorical for some infinite cardinal α .

Then Φ is complete.

Theorem 0.1.1. [Lindström's theorem] Let Φ be a consistent theory of \mathcal{L} that satisfies the following conditions:

- a. Φ has no finite models.
- b. Φ is preserved under the union of chains.
- c. Φ is α -categorical for some infinite cardinal α .

Then Φ is model-complete.

Theorem 0.1.2. A first order theory Φ of \mathcal{L} is preserved under the unions of chains if and only if it is axiomatizable by a set of universal-existential sentences. I.e. if there exists $\Sigma \subset \forall \exists - \text{Sent}_{\mathcal{L}}$ such that $\Theta[\Sigma] = \Phi$.

Theorem 0.1.3. [Löwenheim-Skolem theorem] If a set T of \mathcal{L} -sentences has an infinite model, then it has models of any given infinite power α .

Proofs of the above mentioned theorems can be found, respectively, on pages 113, 114, 125 and 67 of Chang's and Keisler's [4].

C. From Recursion Theory and Decision Theory.

We assume the reader understands what is meant by an informal algorithm and a function computable by an informal algorithm. Consequently, he will not find it difficult to understand what a decidable set is: A set A is decidable if there is an informal algorithm that enables us to decide whether an object x belongs to A or not.

Familiarity with the fundamental concepts of Recursion Theory, as developed, for example, in the first chapter of H. Rogers' [24], is desirable. All the same, it is not a sine qua non for the reading of this thesis, because we shall introduce, in a comprehensive way, any notion from Recursion Theory, we are going to use, and informal procedures will be given preference to formal ones, whenever this is

possible.

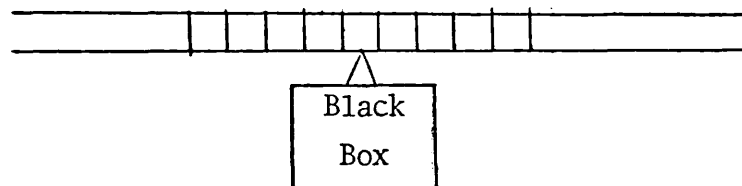
Various formal characterisations of the informal notions of algorithm, function computable by algorithm and decidable set have been obtained since the 1930's. These characterisations have varied widely in form but they have been shown to be equivalent. We give here the Turing Characterization because we find it closer to our intuition and more convenient for our purpose.

We approach the Turing Characterization by the following physical picture : Consider the machine T, that is designed to perform the same task that a human computer performs. It has the following parts:

a. A tape, infinite in both directions, which is divided up into squares. A zero or a one is written in each square and all but finitely many squares have a zero in them at any one time.

b. A black box that takes one of a finite number of internal states, at any one time.

So, the machine looks like this:



The machine is capable of examining only one square at a given time and, according to the number it finds in the square and to the internal state of the box, it performs one of the following operations:

1. It writes a 0 in the square.
2. It writes a 1 in the square.
3. It concentrates on the next square to the left.
4. It concentrates on the next square to the right.

If the machine is given an input (i.e. an initial description on the tape) it performs a uniquely determined succession of operations, which may go on for ever or may terminate after giving an output (i.e. a final description on the tape). For further details about its construction and its operation see Rogers [24] and Davis [5].

What should be understood, for our purpose, is that, for each $n \in \omega - 1$, each Turing Machine computes an n -ary partial function f_T^n from ω to ω , which is defined as follows : Given the machine any input n -tuple $\langle x_0, x_1, \dots, x_{n-1} \rangle \in {}^n\omega$, written in the suitable language, if the machine halts after giving an output, take $f(x_0, x_1, \dots, x_{n-1})$ to be the

total number of ones appearing on the tape.

An n -ary partial function from ω to ω is called recursive if there is a Turing Machine that computes it. A set $A \subset \omega$ is called recursive if its characteristic function χ_A is recursive.

The thesis that Turing Machines, partial recursive functions and recursive sets are an adequate formal explicatum for the intuitive notions of numerical algorithms, number-theoretical functions computable by algorithms and decidable sets of natural numbers, respectively, is known as Church's Thesis. Although such a thesis is not susceptible to a strict mathematical proof, the evidence for its correctness is overwhelming. For a detailed discussion of the evidence we refer the reader to Kleene's [9]. We accept Church's thesis as correct and we are able from now on to treat, in a formal way, questions concerning numerical algorithms, number-theoretical algorithmic functions, and decidable sets of natural numbers.

Since the original informal notions concern much broader classes of non-numerical objects, the following question is naturally raised : Is there any way to apply our formal theory of recursive functions to such broader classes of objects ? There is no difficulty in defining Turing Machines which operate with finite alphabets other than $\{0,1\}$ and the resulting theory is essentially the same. Indeed, we could use any decidable set as an alphabet. But, for our purpose, it will be better to reduce other alphabets to $\{0,1\}$, by Goedel Numberings, as follows :

If A is a countable set of objects, which is decidable in the intuitive sense of the term, a Goedel Numbering of A is an injective mapping $g : A \rightarrow \omega$, with the following properties :

- a. g is an informal algorithm
- b. g^{-1} is an informal algorithm
- c. $g[A]$ is a recursive set.

The number $g(x)$, assigned to x through g , is called the Goedel number of x . A subset B of A is called recursive if its image $g[B]$ is recursive. An n -ary function f from A to A is called recursive if the number-theoretical function f_g , given by the rule:

$$f_g(g(x_0), \dots, g(x_{n-1})) = g(f(x_0, \dots, x_{n-1})),$$

is recursive.

A Turing Machine T can be identified with the finite set of instructions that determines its operation. The finite set of instructions can be expressed, in a purely mathematical way, as a finite set of quadruples of natural numbers. So, a Turing Machine becomes a mathematical object. It is known that the set of all Turing Machines \mathcal{T} (i.e. the set of instructions associated with them) can be given a Goedel Numbering G , such that $G[\mathcal{T}] = \omega$. (See Rogers' [24] , pg 21)

Goedel Numberings have a wide range of application to logic because, given any countable first order language, one can find a Goedel Numbering of the set of its formal expressions. One can consequently ask whether a set or a set of sets of expressions is decidable and, using Recursion Theory, he can have a strict mathematical proof of the correct answer.

Let $\mathcal{L} = \langle \{Q_i\}_{i \in I}, \{c_j\}_{j \in J} \rangle$ be a countable algebraic language. Throughout the thesis, $g_{\mathcal{L}} : \text{Sym}_{\mathcal{L}} \rightarrow \omega$ is taken to be the following mapping:

$$g_{\mathcal{L}}(-) = 0, \quad g_{\mathcal{L}}(\vee) = 1, \quad g_{\mathcal{L}}(\wedge) = 2, \quad g_{\mathcal{L}}(\rightarrow) = 3, \quad g_{\mathcal{L}}(=) = 4, \quad g_{\mathcal{L}}(\forall) = 5, \quad g_{\mathcal{L}}(\exists) = 6$$

$$g_{\mathcal{L}}(v_n) = 7 + 3n, \quad \forall n \in \omega$$

$$g_{\mathcal{L}}(Q_i) = 7 + (3i+1), \quad \forall i \in I$$

$$g_{\mathcal{L}}(c_j) = 7 + (3j+2), \quad \forall j \in J$$

Obviously, $g_{\mathcal{L}}$ is a Goedel numbering of the set of symbols of \mathcal{L} , while $g_{\mathcal{L}}^* : \text{Exp}_{\mathcal{L}} \rightarrow \omega$, which is given by:

$$\forall \sigma = \langle \sigma_0, \sigma_1, \dots, \sigma_{n-1} \rangle \in \text{Exp}_{\mathcal{L}}, \quad g_{\mathcal{L}}^*(\sigma) = \prod_{i < n} p_i^{g_{\mathcal{L}}(\sigma_i)+1},$$

is a Goedel numbering of the set of expressions of the language.

It is intuitively clear to us that the sets $\text{Term}_{\mathcal{L}}$, $\text{Form}_{\mathcal{L}}$ and $\text{Sent}_{\mathcal{L}}$ are recursive. If the reader cannot convince himself with anything but formal proofs, he can find them in Monk's [15]. Then, following the method described there, he can construct his own proofs of facts we will assume to be intuitively clear.

§ 0.2. Equations and equational classes of algebras.

Let \mathcal{L} be a countable algebraic language.

An equation of \mathcal{L} is a universal sentence of the form $(\forall \bar{v}) (\varphi = \psi)$, where φ and ψ are terms. We denote by $\text{Eq}_{\mathcal{L}}$ the set of equations of the language \mathcal{L} . Consequently,

$$\text{Eq}_{\mathcal{L}} = \left\{ (\forall \bar{v}) (\varphi = \psi) : \varphi \in \text{Term}_{\mathcal{L}}, \psi \in \text{Term}_{\mathcal{L}} \right\}$$

If Σ is a set of equations of \mathcal{L} , we denote by $\text{Mod } \Sigma$ the class of all \mathcal{L} -algebras which are models of Σ , i.e.

$$\text{Mod } \Sigma = \{ \mathcal{A} : \mathcal{A} \models \Sigma \}$$

A class \mathfrak{K} of \mathcal{L} -algebras is a variety or an equational class if it consists of all the models of some $\Sigma \subset \text{Eq}_{\mathcal{L}}$. I.e.

$$(\mathfrak{K} \text{ is a variety}) \leftrightarrow (\exists \Sigma \subset \text{Eq}_{\mathcal{L}}) (\mathfrak{K} = \text{Mod } \Sigma)$$

The fragment of first order logic that deals with equations is known as Equational Logic. It has been of great importance to algebraists, because the most interesting classes of algebras (with the remarkable exception of the class of fields) are equational classes. For example, the class of semigroups can be treated as the class of all models of the equation $(\forall v_0 v_1 v_2) (v_0 \cdot (v_1 \cdot v_2) = (v_0 \cdot v_1) \cdot v_2)$ of the language $\mathcal{L} = \langle \cdot \rangle$, while the class of groups can be treated as the class of all models of the set

$$G = \left\{ (\forall v_0 v_1 v_2) ((v_0 \cdot v_1) \cdot v_2 = v_0 \cdot (v_1 \cdot v_2)), (\forall v_0) (v_0 \cdot v_0^{-1} = 1), (\forall v_0) (v_0 \cdot 1 = v_0) \right\}$$

of equations of the language $\mathcal{L} = \langle \cdot, ^{-1}, 1 \rangle$. Abelian groups, Rings, Lattices and Boolean algebras can also be equationally defined; that is to say, in each individual case, the class of algebras can be considered as $\text{Mod } \Sigma$, for a suitable Σ .

We have not given definitions of the above mentioned elementary algebraic notions, but the reader can find them in any standard text of Abstract Algebra.

It is now generally agreed that Equational Logic draws its origin from Birkhoff's [1], which appeared in print in 1935.

In this paper, Equational Logic is treated for the first time as a formal system with the following five axioms of derivation:

1. $\forall \varphi \in \text{Term}_{\mathcal{L}}$, the equation $(\forall \bar{v})(\varphi = \varphi)$ holds.
2. $\forall \langle \varphi, \psi \rangle \in {}^2\text{Term}_{\mathcal{L}}$, if $(\forall \bar{v})(\varphi = \psi)$ holds, then $(\forall \bar{v})(\psi = \varphi)$ holds.
3. $\forall \langle \varphi, \psi, \chi \rangle \in {}^3\text{Term}_{\mathcal{L}}$, if $(\forall \bar{v})(\varphi = \psi)$ and $(\forall \bar{v})(\psi = \chi)$ hold, the equation $(\forall \bar{v})(\varphi = \chi)$ holds.
4. $\forall \langle \varphi, \psi \rangle \in {}^2\text{Term}_{\mathcal{L}}$ and $\forall \bar{n} \in {}^{\omega}\text{Term}_{\mathcal{L}}$, if $(\forall \bar{v})(\varphi = \psi)$ holds, then $(\forall \bar{v})(\varphi[\bar{n}] = \psi[\bar{n}])$ holds.
5. For any operation symbol Q_i of the language and any two sequences $\langle \varphi_0, \varphi_1, \dots, \varphi_{r(i)-1} \rangle$, $\langle \psi_0, \psi_1, \dots, \psi_{r(i)-1} \rangle$ of terms of the language, if the equations $(\forall \bar{v})(\varphi_i = \psi_i)$ hold for all i 's, then $(\forall \bar{v})(Q_i \varphi_0 \varphi_1 \dots \varphi_{r(i)-1} = Q_i \psi_0 \psi_1 \dots \psi_{r(i)-1})$ holds.

As one would expect, axioms 1-5 are part of the axioms of derivation for first order logic. We adopt Birkhoff's axiomatization, and we say that, given $\Sigma \cup \{\varepsilon\} \subset \text{Eq}_{\mathcal{L}}$, ε is derivable from Σ by means of axioms 1-5 (in symbols $\Sigma \vdash_{\text{Eq}} \varepsilon$) if there exists a proof of ε , starting from Σ and using only the rules 1-5.

In the above mentioned remarkable paper, Birkhoff proves a completeness theorem for equational logic, which is entirely analogous to the Goedel Completeness Theorem for first order logic and a characterization theorem, which provides a purely algebraic characterization for equational classes of algebras. These two important theorems, which we are going to use repeatedly, are given here without proofs :

Theorem 0.2.0. (Completeness Theorem for Equational Logic).

For any algebraic language \mathcal{L} , and any $\Sigma \cup \{\varepsilon\} \subset \text{Eq}_{\mathcal{L}}$, it holds:

$$\Sigma \models \varepsilon \text{ if and only if } \Sigma \vdash_{\text{Eq}} \varepsilon$$

Theorem 0.2.1. (Characterization Theorem for Varieties of Algebras)

A class \mathfrak{V} of \mathcal{L} -algebras is a variety iff it is closed under the formation of subalgebras, homomorphic images and direct products.

In other words, \mathfrak{V} is a variety iff it has the following three properties:

1. $(\forall \mathfrak{A} \in \mathfrak{V})(\forall \mathfrak{B} \subset \mathfrak{A})(\mathfrak{B} \in \mathfrak{V})$
2. $(\forall \mathfrak{A} \in \mathfrak{V})(\forall f : \mathfrak{A} \rightarrow \mathfrak{B})(f(\mathfrak{A}) \in \mathfrak{V})$
3. $(\forall \{\mathfrak{A}_i : i \in I\} \subset \mathfrak{V})(\prod_{i \in I} \mathfrak{A}_i \in \mathfrak{V})$

We introduce below some basic notions from Equational Logic, which will be in constant use throughout the thesis:

A set θ of equations of \mathcal{L} is an equational theory if it contains all its equational consequences, i.e. if it holds:

$$(\forall \varepsilon \in \text{Eq}_{\mathcal{L}}) (\theta \vdash \varepsilon \leftrightarrow \varepsilon \in \theta)$$

If \mathcal{R} is a class of \mathcal{L} -algebras, the equational theory generated by \mathcal{R} is the set

$$\text{Th}_{\text{Eq}} \mathcal{R} = \{ \varepsilon \in \text{Eq}_{\mathcal{L}} : (\forall \mathcal{A} \in \mathcal{R}) (\mathcal{A} \models \varepsilon) \}$$

If Σ is a set of equations of \mathcal{L} , the equational theory generated by Σ is the set

$$\theta_{\text{Eq}} [\Sigma] = \{ \varepsilon \in \text{Eq}_{\mathcal{L}} : \Sigma \vdash \varepsilon \}$$

The equational theory θ is called recursively based, or finitely based, or one-based, if there exists $\Sigma \subset \text{Eq}_{\mathcal{L}}$, which is, respectively, recursive or finite or a singleton, with the property $\theta_{\text{Eq}} [\Sigma] = \theta$. The equational theory θ is equationally consistent if it has a non-trivial model or, equivalently, if it is a proper subset of $\text{Eq}_{\mathcal{L}}$. The equational theory θ is equationally complete if it is equationally consistent and satisfies one of the following equivalent statements :

i. There is no equationally consistent equational theory Φ , such that $\theta \subsetneq \Phi$.

ii. Any two non-trivial models of θ satisfy the same equations.

Finally, we give the following definition: Let θ be an equational theory of the language $\mathcal{L} = \langle \{Q_i\}_{i \in I}, \{c_j\}_{j \in J} \rangle$ and let \sim_{θ} be the binary relation, defined as follows :

$$(\forall \langle \varphi, \psi \rangle \in {}^2\text{Term}_{\mathcal{L}}) (\varphi \sim_{\theta} \psi \leftrightarrow (\forall \bar{v}) (\varphi = \psi) \in \theta)$$

It can be easily checked that \sim_{θ} is a congruence relation in the set $\text{Term}_{\mathcal{L}}$, i.e. an equivalence relation in $\text{Term}_{\mathcal{L}}$, with the property "for any operation symbol Q_i and for any $\langle \varphi_0, \dots, \varphi_{r(i)-1} \rangle$ and $\langle \psi_0, \dots, \psi_{r(i)-1} \rangle$ sequences of terms, if the relations $\varphi_n \sim_{\theta} \psi_n$ hold

for all n 's, then the relation $Q_i \varphi_0 \varphi_1 \dots \varphi_{r(i)-1} = Q_i \psi_0 \psi_1 \dots \psi_{r(i)-1}$ holds".

Consider now the quotient set $\text{Term}_{\mathcal{L}} / \sim_{\theta}$. Because of the fact that \sim_{θ} is a congruence relation in $\text{Term}_{\mathcal{L}}$, the algebra

$$\mathfrak{A} = \langle \text{Term}_{\mathcal{L}} / \sim_{\theta}, \{Q_i\}_{i \in I}, \{c_j\}_{j \in J} \rangle,$$

where a) the interpretation of any constant symbol c_j in \mathfrak{A} is its equivalence class under \sim_{θ} (i.e. $c_j^{\mathfrak{A}} = [c_j]$), and

b) the interpretation of any operation symbol Q_i in \mathfrak{A} is the operation defined by the rule:

$$\forall \langle [\varphi_0], [\varphi_1], \dots, [\varphi_{r(i)-1}] \rangle \in {}^{r(i)}\text{Term}_{\mathcal{L}} / \sim_{\theta},$$

$$Q_i^{\mathfrak{A}}([\varphi_0], \dots, [\varphi_{r(i)-1}]) = [Q_i \varphi_0 \dots \varphi_{r(i)-1}]$$

is well-defined. Call \mathfrak{A} the term algebra of θ and denote it by \mathfrak{T}_{θ} .

The reader with an elementary knowledge of universal Algebra will immediately recognise that if θ is equationally consistent, \mathfrak{T}_{θ} is, in fact, the free algebra in ω generators of the class of models of the equational theory θ .

In the course of proving his completeness theorem for equational logic, Birkhoff [1] proves that the equational theory generated by the algebra \mathfrak{T}_{θ} coincides with θ . For any term φ of \mathcal{L} , it is obvious that the following relation holds:

$$[\varphi] = \varphi^{\mathfrak{T}_{\theta}}([v_0], [v_1], \dots, [v_n] \dots).$$

Consequently, the set $\{[v_i] : i \in \omega\}$ is a set of generators for \mathfrak{T}_{θ} . If θ is equationally consistent, then, the above mentioned set is infinite, because the hypothesis that "for some distinct i and j , it holds that $[v_i] = [v_j]$ ", leads to the contradictory statement " $\theta = \text{Eq}_{\mathcal{L}}$ ".

Since these remarks about the term algebras are of great importance to us, we summarize them in the following theorem:

Theorem 0.2.2. For any equational theory θ , the following statements hold:

a. $\text{Th}_{\text{Eq}} \mathfrak{T}_{\theta} = \theta$

b. $\mathfrak{S}_0 = \langle \{[v_n] : n \in \omega\} \rangle$

c. If θ is equationally consistent, then $[v_m] \neq [v_n]$ for any $m \neq n$.

A great number of interesting results concerning sets of equations have been obtained, but a lot of questions still remain open. An exposition of individual results would fall beyond the scope of this thesis. For the interested reader, through, two excellent survey papers of the work done up to 1975 are available. These are Pigozzi's [21] and Taylor's [29].

§ 0.3. Decision problems concerning sets of equations.

Various decision problems about sets of equations naturally arise. They can be classified into two major categories. Local problems, which concern individual sets of equations and global problems, which concern properties of sets of equations in general.

A. Local Decision problems.

If \mathcal{L} is a countable algebraic language and θ is an equational theory of \mathcal{L} , the decision problem of θ is the problem of whether θ is decidable or not. Formally speaking, if $g_{\mathcal{L}}^*$ is the Goedel numbering of the expressions of the language, defined in 0.1, it is the problem of whether the set of natural numbers $g_{\mathcal{L}}^*[\theta]$ is recursive or not.

Obviously, the theory $\text{Eq}_{\mathcal{L}}$ is decidable for any \mathcal{L} . It is also clear that the equational theory $\text{Th}_{\text{Eq}} \mathfrak{A}$, generated by a finite algebra, is decidable. Since any recursively based and equationally complete equational theory is decidable, a number of well-known theories turn out to be decidable. We mention as such the equational theories of Distributive Lattices, of Boolean Algebras, of p-groups for any prime number p (i.e. abelian groups satisfying the equation $(\forall v_0) (p v_0 = 0)$) and the equational theory of p-rings for any prime number p (i.e. commutative rings with unit satisfying the equations $(\forall v_0) (p v_0 = 0)$ and $(\forall v_0) (v_0^{p-1} = 1)$). Equational completeness is not, of course, a necessary condition for the decidability of a recursively based equational theory, as the fact that the equational theories of Lattices and Abelian groups are decidable indicates.

Post [23] was the first to construct a finitely based undecidable equational theory. This was a theory of semigroups. Later, other examples of finitely based undecidable equational theories, in a variety of languages, were given, primarily by Tarski [27], Perkins [20], Malcev [10] and Murskiĭ [16]. Up to now, we don't know whether there exists a finitely based equational theory of groups, which is undecidable.

It should be mentioned here, that the decision problems for

equational theories can be considered as a particular case of the famous word problems, which have been the focal point of intense research by Universal Algebraists (Post [23] and Markov [11] proved that there exists a finite presentation of a semigroup with unsolvable word problem while Boone and Novikov [2,18] proved that there exists a finite presentation of a group with unsolvable word problem, in order to mention only the most famous results). Under the formulation we give below, word problems can be treated as local problems concerning sets of equations :

Let θ be an equational theory, in the countable algebraic language \mathcal{L} . We enlarge the language, by adding a set of new constant symbols $C = \{c_i : i \in I\}$. Let R be a set of equations of the language $\mathcal{L}' = \mathcal{L} \cup C$, with no variables in it. A presentation $\langle C, R \rangle$ of a model of θ is an \mathcal{L}' -algebra \mathcal{U} , which is generated by the set $\{c_i^{\mathcal{U}} : i \in I\}$ and satisfies the equations $\theta \cup R$. The word problem for the presentation $\langle C, R \rangle$ of a model of θ is the problem of the existence or not of an algorithm that enables us to decide whether, given any two c-words (i.e. any two constant terms of \mathcal{L}'), they are equal in this presentation or not. In other words, it is the problem of whether the set

$$\left\{ \langle u, v \rangle \in {}^2(\text{c-words}) : R \cup \theta \models u = v \right\}$$

is decidable or not. We say, accordingly, that the presentation has a solvable word problem or an unsolvable word problem.

If we consider a presentation $\langle C, R \rangle$ of a model of θ , with $|I| = \omega$ and $R = \emptyset$, then this presentation becomes, in fact, the term algebra \mathfrak{F}_θ , expanded in \mathcal{L}' by the set of individual constants $\{[v_n] : n \in \omega\}$: Consequently, the word problem for this presentation is identical with the decision problem for θ , as Theorem 0.2.2. indicates.

B. Global decision problems.

Let \mathcal{L} be a countable algebraic language and let P be a property of sets of equations of \mathcal{L} . The decision problem of P in \mathcal{L} is the problem of the existence or not of an algorithm that enables us to decide whether, given a set $\Sigma \subset \text{Eq}_{\mathcal{L}}$, Σ has the property P or not. If such an algorithm exists, P is called decidable in \mathcal{L} . Otherwise, it is called undecidable in \mathcal{L} .

The above formulation of the decision problem of P in \mathcal{L} , contains the vague statement "given $\Sigma \subset \text{Eq}_{\mathcal{L}}$ ". How is Σ given? Is every form of giving Σ suitable for our purpose? Are there more than one suitable forms?

A first observation is that, in order to be able to formulate our problem in a strict mathematical way and to solve it by the help of Recursion Theory, the relevant sets must be given in a form that implies an indexing of them (i.e. a representation of each one of them by a natural number).

According to the kind of the examined sets (i.e. recursive or finite or singletons), they can be given in various acceptable forms. Recursive sets Σ can be given by an algorithm that calculates their characteristic function χ_{Σ} or by an algorithm that calculates a function with domain Σ and the statement that they are recursive, to mention only two examples. Finite sets and singletons can be given as recursive sets and in, at least, one further way: By writing down their members in the form $\{a_0, a_1, a_2, \dots, a_{n-1}\}$.

Consequently, our original general decision problem of P in \mathcal{L} splits into a number of separate problems dependent upon the kind of the given sets and the form in which the sets are given.

Are these problems related to one another and, if so, what sort of relationship exists?

Suppose that we have chosen a form in which the three kinds of sets can be given (e.g. by a set of instructions for calculating their characteristic functions). Then, obviously, a negative solution for singletons would yield a negative solution for finite sets, and this would yield a negative solution for recursive set. The converse implication holds for positive solutions.

On the other hand, suppose that we have chosen one kind of sets (e.g. finite) and two forms of giving them (e.g. by writing down their members and by giving instruction for their characteristic functions). The corresponding decision problems, say A and B , are not necessarily related to one another. They are related if and only if there is a uniform way of going from the one form to the other; in other words, if and only if there is an algorithm that gives, for each set in the form A , the same set in the form B , or conversely. Indeed, if we can go uniformly from A to B , then a negative solution to A provides a negative solution to B while, a positive solution to B provides a positive solution to A .

After having elucidated any kind of vagueness in the informal conception of the decision problem of a property P in \mathcal{L} , we now advance to give rigorous mathematical formulations of it, for singletons, finite sets and recursive sets of \mathcal{L} -equations :

From our experience we know that the most probable solution to such problems, in equational logic, is the negative one (There are some exceptions : Burris [] gave a positive solution to the decision problem of the property " Σ has a finite non-trivial model" for languages with no operation symbols of rank greater than one. Hence, our effort is to formulate our problems in ways that provide a maximal number of negative solutions to decision problems of P , in one go. Justified by the remarks made above, we make, thus, the following conventions :

From this point onward throughout this thesis

- a. "given a singleton Σ " means "given Σ in the form $\{a\}$ "
- b. "given a finite Σ " means "given Σ in the form $\{a_0, a_1, a_2, \dots, a_{n-1}\}$ and
- c. "given a recursive Σ " means "given an algorithm for calculating its characteristic function \mathcal{X}_Σ ".

Suitable goedel numberings are also required :

Let G be the goedel numbering of the set of Turing Machines (or equivalently, of the set of consistent sets of instructions that are associated with the Turing Machines).

Let g_f be a goedel numbering of the symbols of \mathcal{L} . We have seen how g_f yields a goedel numbering g_f^* of the set of expressions of the language. This also yields a goedel numbering g_f^{**} of the finite sets of expressions of the language, by mapping each $\Sigma = \{a_0, \dots, a_{n-1}\}$ to

$$g_f^{**}(\Sigma) = \prod_{i < n} p_{g_f^*(a_i)},$$

where $p_{g_f^*(a_i)}$ is the $g_f^*(a_i)$ -th prime number.

Using these numberings, we are now in the position to formulate our problems :

- a) Let A be the image of the set $Eq_{\mathcal{L}}$ through g_f^* , and let

$$B = \{x : (x \in A) \wedge (g^{*-1}(x) \text{ has the property } P)\}.$$

The decision problem of P for single equations of \mathcal{L} is the problem of the existence or not of a partial recursive function φ such that

- i. for any $x \in A$, φ is defined and
- ii. $\left\{ \begin{array}{l} \text{for } x \in A \cap B, \varphi(x) = 1 \\ \text{for } x \in A \cap \bar{B}, \varphi(x) = 0. \end{array} \right.$

b) Let A be the image of the set of finite sets of equations through g_f^{**} , and let

$$B = \{x : (x \in A) \wedge (g_f^{**^{-1}}(x) \text{ has the property } P)\}$$

The decision problem of P for finite sets of \mathcal{L} -equations is the problem of the existence or not of a partial recursive function φ with properties i and ii, for these A and B .

c) Let A be the image of the set of Turing Machines under G , which calculate characteristic functions of images of sets of \mathcal{L} -equations under g^* , and let

$$B = \left\{ x : (x \in A) \wedge (G^{-1}(x) \text{ calculates the characteristic function of } g^*[\Sigma] \text{ for a set } \Sigma \text{ with the property } P) \right\}$$

The decision problem of P for recursive sets of \mathcal{L} -equations is as before, for these A and B .

The three formulations are justified by Church's Thesis and by the properties of goedel numberings.

Decision problems of properties of sets of equations appeared for the first time, in 1968, in an expository article by Tarski [28]. There, the decision problems of the properties

- 0. $\theta_{Eq}[\Sigma]$ is equationally consistent
- 1. $\theta_{Eq}[\Sigma]$ is decidable
- 2. $\theta_{Eq}[\Sigma]$ is equationally complete
- 3. there exists a finite algebra \mathcal{A} , so that $Th_{Eq} \mathcal{A} = \theta_{Eq}[\Sigma]$

4. $\theta_{Eq}[\Sigma]$ has a basis of given cardinality κ , which cannot be reduced any further,

for finite sets of equations and for singletons were raised. Perkins [19] gave negative solutions to problem 0 for finite sets of equations in any strong language, and to problems 1-3 for finite sets of equations in any language with at least two binary operation symbols and at least two constant symbols. Later McNulty [14] extended Perkins' results and gave a negative solution to problem 4. A number of other properties were examined by McKenzie [12], McNulty [14] and Pigozzi [22], almost all of which turned out to be undecidable, at least in languages with sufficiently high complexity.

In the above mentioned article by Tarski, another decision problem of P in \mathcal{L} was raised. This is

d) The decision problem of P for finite \mathcal{L} -algebras i.e. the problem of the existence or not of an algorithm that enables us to decide whether the equational theory $Th_{Eq} \mathcal{U}$, generated by a given finite \mathcal{L} -algebra \mathcal{U} , has the property.

Obviously, $Th_{Eq} \mathcal{U}$ is equationally consistent if and only if \mathcal{U} is non trivial. So, the property " Σ is an equationally consistent equational theory" is decidable for finite \mathcal{L} -algebras in any \mathcal{L} . McKenzie proved that the property " Σ is an equationally complete equational theory" is also decidable for finite \mathcal{L} -algebras in any \mathcal{L} . An outstanding open problem of type d, is the decision problem of the property " Σ is a finitely based equational theory" for finite algebras.

We don't give here a strict mathematical formulation of the problem d, because we are not going to deal with it any further. The focal point of this thesis is decision problems of types b and c, and in §0.4, a survey of the existing methods of dealing with them and a summary of the results obtained so far is given.

Before closing this brief exposition of the decision problems concerning equations, we should mention a problem that stands on the borderline between local and global problems :

Let θ be a fixed finitely based equational theory of \mathcal{L} . Is there an algorithm that enables us to decide whether a finite set Σ of equations of \mathcal{L} is a basis of θ ? A finitely based equational theory θ , for which such an algorithm exists, is a base-decidable equational theory and the problem, mentioned above, is the base-decidability problem of θ .

Obviously, any finitely based and undecidable equational theory is base-undecidable. The converse is not true, as Perkins showed that the decidable theory Eq_C , for a language with just one binary operation symbol, is base-undecidable. McNully [13] found a very simple criterion for base-undecidability of equational theories, as an application of which, almost all the well-known equational theories are proved to be base-undecidable.

§0.4. Survey of the existing methods of proving the undecidability of properties of sets of equations.

Undecidable properties of sets of equations are the main concern of this thesis; i.e. properties P of sets of \mathcal{L} -equations, for which there is no algorithm that enables us to decide whether, given a recursive (or finite or singleton) set of equations of \mathcal{L} , it has the property P .

A number of methods for proving the undecidability of properties of sets of equations have been developed since 1963. Although tables of individual results, obtained by these, are available in the literature (McNulty [14] and Taylor [29]), no expository article about the methods themselves is known to me. It seems appropriate here to survey the existing techniques, before moving on to present new ones :

A decision problem is effectively reducible to another if an algorithmic solution to the second yields an algorithmic solution to the first. The common feature of all the existing methods under discussion is that, in order to prove the undecidability of a property P , they effectively reduce a well-known not algorithmically solvable decision problem to the decision problem of P .

None of them uses formal recursion theory but the reader is left in no doubt that the given informal procedures have a formal equivalent, which can be obtained routinely.

A. Perkins' Method (for finite sets of equations) [19]

In Hall [6], it is proved that the word problem of a finite presentation of a semigroup is reducible to the word problem of a presentation of a semigroup on two generators and finitely many relations. Since it is well known that a finite presentation of a semigroup with unsolvable word problem exists (Post [23]), we deduce that there exists a presentation $\langle \{a,b\}, R \rangle$ of a semigroup on two generators and finitely many defining relations with unsolvable word problem. Perkins uses this fact in order to prove the undecidability of the properties 0-3, which we mentioned on pg 24, for finite sets of equations, in languages with at least two binary operation symbols and at least two constant symbols.

The method is the following:

Let P be the property under examination and let \mathcal{L} be the language in which the examination is being performed.

The fact that $\langle \{a,b\}, R \rangle$ has an unsolvable word problem is expressed by the statement that the set :

$$\{ \langle U, V \rangle \in {}^2(\{a,b\}\text{-words}) : R \cup S \models U = V \}, \quad (1)$$

where $S = \{ (\forall v_0 v_1 v_2) ((v_0 \cdot v_1) \cdot v_2 = v_0 \cdot (v_1 \cdot v_2)) \}$, is undecidable.

If we manage to associate, in an algorithmic way, with each pair $\langle U, V \rangle$ of $\{a,b\}$ -words, a finite set $T(U, V) \in \text{Eq}_{\mathcal{L}}$, so that

$$R \cup S \models U = V \leftrightarrow T(U, V) \text{ has the property } P, \quad (2)$$

then, by means of (1) \wedge (2), the undecidability of the property P , for finite sets of equations of \mathcal{L} , will have been established.

B. Perkins' Method (for recursive sets of equations) [19]

A recursive set R of pairs of natural numbers, whose second coordinates form a non-recursive set R' , can be constructed. (For example, consider a recursive set Γ of sentences, which generates an undecidable first order theory. It can be easily proved that the set of Γ -proofs is also recursive, while the set $\text{Th}[\Gamma] = \{y : (\exists x)(x \text{ is a } \Gamma\text{-proof of } y)\}$ is non-recursive. By using suitable goedel numberings, we can take the numerical sets R and R' , in the obvious way).

Perkins reduces the decision problem of such an R' to the decision problem of the properties 0-3 of pg 29, for recursive sets of equations in any strong language. This is done as follows:

Let P be the property under examination and let \mathcal{L} be the language in which the examination is taking place. We associate, in a recursive way, with each natural number n , a recursive set of \mathcal{L} -equations E_n , so that

$$n \in R' \leftrightarrow E_n \text{ has the property } P.$$

So, the needed reduction has been obtained and the undecidability of P , for recursive sets of \mathcal{L} -equations, follows from the undecidability of R' .

C. McNulty's method (for finite sets of equations) [14]

Let \mathcal{L} and \mathcal{L}' be two algebraic languages. A system of definitions for \mathcal{L} in \mathcal{L}' is a mapping δ with domain the set of non-logical symbols of \mathcal{L} and range included in the set $\text{Term}_{\mathcal{L}'}$, such that

- a. $\forall c_j$ constant symbols, $\delta(c_j)$ is a constant term, and
- b. $\forall O_i$ operation symbol, $\delta(O_i)$ contains just the variables $v_0, v_1, \dots, v_{r(i)-1}$

Given δ , a mapping $\text{in}_{\delta} : \text{Term}_{\mathcal{L}} \rightarrow \text{Term}_{\mathcal{L}'}$ is defined by induction:

$$\text{in}_{\delta}(v_n) = v_n$$

$$\text{in}_{\delta}(c_j) = \delta(c_j), \forall j \in J$$

$$\text{in}_{\delta}(O_i \theta_0 \dots \theta_{r(i)-1}) = \delta(O_i)[\text{in}_{\delta} \theta_0, \dots, \text{in}_{\delta} \theta_{r(i)-1}], \forall i \in I.$$

δ is called a universal system of definitions for \mathcal{L} in \mathcal{L}' with respect to $\Phi \subset \text{Eq}_{\mathcal{L}'}$ if, for any $\Sigma \subset \text{Eq}_{\mathcal{L}}$ and for any $(\forall v) (\varphi = \psi) \in \text{Eq}$, the following relation holds :

$$\left\{ \begin{array}{l} \Sigma \models (\forall \bar{v}) (\varphi = \psi) \leftrightarrow \\ (\forall \bar{v}) (\text{in}_{\delta} s = \text{in}_{\delta} t) : \{ (\forall \bar{v}) (s = t) \in \Sigma \} \cup \Phi \models (\forall \bar{v}) (\text{in}_{\delta} \varphi = \text{in}_{\delta} \psi) \end{array} \right\}$$

In Malcev [10], a finite set of equation M , in a language \mathcal{L}_0 with just two unary operation symbols f and g , which contains just one variable v_i , is constructed, such that the set

$$\{ \varepsilon \in \text{Eq}_{\mathcal{L}_0}, \varepsilon \text{ has just the variable } v_i : M \models \varepsilon \} \quad (1)$$

is undecidable. In order to prove the undecidability of a property P , McNulty chooses a suitable universal system of definitions for \mathcal{L} in \mathcal{L}' with respect to $\{ (\forall v_0) (v_0 = v_0) \}$, say δ_0 . Then, by means of δ_0 , he recursively associates with each equation ε , in just the symbols f , g and v_i , a finite set $B(\varepsilon, \delta_0, M)$ with the property

$$M \models \varepsilon \leftrightarrow B(\varepsilon, \delta_0, M) \text{ has the property } P \quad (2)$$

Relation (2) reduces the decision problem of the set (1) to that of the property P , for finite sets of \mathcal{L} -equations, and the undecidability of P follows.

Three general criteria, providing sufficient conditions for the undecidability of properties P (i.e. for the reducibility of the decision problem of the set (1) to the decision problem of P) are established and a large number of individual results are obtained, by a simple application of them. In certain cases, where the criteria are not applicable, individual proofs are given, which, although elaborated and different from one another in the detail, simultaneously use the technique described above.

Here are the most important properties that have been proved to be undecidable by McNulty's method :

0. Σ is ω -categorical, for \mathcal{L} non-trivial and finite
1. Σ is ω -categorical, for \mathcal{L} non-trivial
2. Σ is categorical in all infinite powers, for \mathcal{L} non-trivial and finite
3. $\theta_{Eq}[\Sigma]$ is decidable, for \mathcal{L} non-trivial
4. Σ is irredundant, for \mathcal{L} non-trivial
5. $\theta_{Eq}[\Sigma]$ has an irredundant base of cardinality κ , for \mathcal{L} non-trivial
6. Σ is residually finite, for \mathcal{L} strong
7. Σ is residually small, for \mathcal{L} strong
8. Σ has arbitrarily large simple models, for \mathcal{L} strong
9. Σ has no infinite Jonson models, for \mathcal{L} strong
10. Σ is a base of a primal algebra, for \mathcal{L} strong.

D. Pigozzi's Method (for finite sets of equations) [22]

This method is an elaboration of the previous one. Instead of the conceptually simpler notion of a universal system of definitions for \mathcal{L} in \mathcal{L}' with respect to $\Phi \subset Eq_{\mathcal{L}'}$, the more elaborated notion of a normal universal system of definitions for \mathcal{L} in \mathcal{L}' with respect to $\Phi \subset Eq_{\mathcal{L}'}$, is used. This can be viewed as a universal system of definitions with the extra property : " There is a procedure that gives, for any

$\Sigma \subset \text{Eq}_{\mathcal{L}}$, an algorithm for checking whether, given $\varepsilon \in \text{Eq}_{\mathcal{L}}$,

$$\{(\forall \bar{v})(\text{in}_{\delta} s = \text{in}_{\delta} t) : (\forall \bar{v})(s = t) \in \Sigma\} \cup \Phi \models \varepsilon,$$

assuming that oracles (i.e. external agents that give correct information, when asked) for Σ and Φ have been provided. It is immediately clear from the definitions that normal universal systems provide more information than universal ones.

It is immediately clear from the definition that normal universal systems provide more information than universal ones. Here lies the ability of the method to reach results not accessible by McNulty's method.

Let P be the property under examination and let \mathcal{L} be the language in which the examination is being performed. Consider the language \mathcal{L}_0 with just one binary operation symbol. Choose a suitable $\Phi_0 \subset \text{Eq}_{\mathcal{L}}$ and a suitable normal universal system of definitions for \mathcal{L}_0 in \mathcal{L} with respect to Φ_0 , say δ_0 . Then, in an algorithmic way, associate, with each finite $\Sigma \subset \text{Eq}_{\mathcal{L}}$, a finite set $T(\Sigma, \delta_0) \subset \text{Eq}_{\mathcal{L}}$ so that

$$\theta_{\text{Eq}}[\Sigma] \text{ is equationally inconsistent} \leftrightarrow T(\Sigma, \delta) \text{ has the property } P.$$

So, the decision problem of the property " $\theta_{\text{Eq}}[\Sigma]$ is equationally consistent" for finite sets of \mathcal{L}_0 -equations is reduced to the decision problem of P for finite sets of \mathcal{L} -equation. Since the former has been proved, by Perkins, to be undecidable, the latter is undecidable too.

The two important applications of this method are the undecidability of the Amalgamation property and the undecidability of the Schreier property, both in any strong language.

E. McKenzie's Method (for single equations) [12]

Two disjoint sets of natural numbers A and B are recursively inseparable if there is no $C \subset \omega$ such that $A \subset C$ and $B \subset \omega - C$. Accordingly, two disjoint sets of \mathcal{L} -sentences are recursively inseparable if their images through a goedel numbering are recursively inseparable.

McKenzie proves that, in the language $\mathcal{L} = \langle + \rangle$ with just one binary operation symbol, the sets

$$A = \left\{ (\forall v_0 v_1 \dots v_{n-1}) (\varphi = \psi) \in \text{Eq}_{\mathcal{L}} : (\forall v_0 v_1 \dots v_{n-1}) (\varphi = \psi) \models (\forall v_0 v_1) (v_0 = v_1) \right\}$$

and

$$B = \left\{ (\forall v_0 v_1 \dots v_{n-1}) (\varphi = \psi) \in \text{Eq}_{\mathcal{L}} : (\forall v_0 v_1 \dots v_n) (\varphi = \psi \wedge v_n + v_n \neq v_n) \text{ has a finite model} \right\}$$

are recursively inseparable. Then, he uses this fact in order to prove the undecidability of certain properties P for single equations (or for single universal sentences) of any strong language.

The fact that it is not decidable whether $\varepsilon \in \text{Eq}_{\mathcal{L}}$ has finite non-trivial models, for example, is proved as follows:

If the set

$$C = \{ \varepsilon \in \text{Eq}_{\mathcal{L}} : \varepsilon \text{ has finite non-trivial models} \}$$

was decidable, then, since $B \subset C$ and $A \subset \text{Sent}_{\mathcal{L}} - C$, A and B would not be recursively inseparable #.

This method has apparently a very limited range of applications. The few properties, examined by McKenzie's method, though, are not accessible by any of the previous ones.

§ 0.5. Motivation

We have already mentioned the main decidability results, concerning properties of sets of equations, known in the literature.

What we found astonishing was the fact that, although a large number of properties had been examined, two of the most fundamental model theoretical ones, namely the properties

\underline{P} : the first-order theory, generated by Σ , is complete, and

\overline{P} : the first-order theory, generated by Σ , is model-complete, didn't seem to have received any consideration.

This observation led us to pursue this line of research.

The answer to the decision problem of \underline{P} , for finite sets of equations of any algebraic language, easily emerges:

Since the set $Eq_{\mathcal{L}}$ has only trivial models, they are all elementary equivalent and, consequently, the first-order theory, generated by $Eq_{\mathcal{L}}$, is complete. On the other hand, any first-order theory, axiomatizable by a set of equationally consistent equations, is properly included in $\Theta[Eq_{\mathcal{L}}]$. It is not, thus, complete. Hence we deduce that, given any finite $\Sigma \subset Eq_{\mathcal{L}}$, it holds that

$$\underline{P}(\Sigma) \leftrightarrow \Theta[\Sigma] = \Theta[Eq_{\mathcal{L}}] \leftrightarrow \Theta_{Eq}[\Sigma] = Eq_{\mathcal{L}} \quad (1)$$

Since, as McNulty has proved in [13], in any non-trivial algebraic language, the equational theory $Eq_{\mathcal{L}}$ is base-undecidable, relation (1) implies the following:

Theorem 0.5.0. Let \mathcal{L} be any non-trivial algebraic language. There is no algorithm that enables us to decide whether, given any finite $\Sigma \subset Eq_{\mathcal{L}}$, the first-order theory generated by Σ is complete.

In the last chapter of this thesis, we shall prove that, in any trivial language, $Eq_{\mathcal{L}}$ is a base-decidable equational theory. From this fact and from relation (1) the decidability of \underline{P} , for finite sets of any trivial language, will follow, and the decision problem of \underline{P} for finite sets of equations of all languages will have been completely settled.

In the case of property \bar{P} , the things are not that simple. As we shall see in the last chapter, the only model-complete first-order theory of a trivial language, which is axiomatizable by equations, is $\Theta[\text{Eq}_t]$. This, together with the base-decidability of Eq_t , implies that \bar{P} is decidable for finite sets of equations of trivial languages.

On the other hand, as far as non-trivial languages are concerned, we have been unable either to construct a model-complete first-order theory, different from $\Theta[\text{Eq}_t]$, axiomatizable by equations, or to prove that such a theory doesn't exist.

The decision problem of \bar{P} cannot, thus, be investigated any further, before an answer to the following question is obtained:

Question Is there an equationally consistent set of equations of a non-trivial language, which generates a model-complete first-order theory?

After having examined the decision problems of \underline{P} and \bar{P} , it is natural to raise the corresponding problems for properties

1. the first-order theory of the non-trivial models of Σ is complete,
2. the first-order theory of the infinite models of Σ is complete,
3. the first-order theory of the non-trivial models of Σ is model-complete and
4. the first-order theory of the infinite models of Σ is model-complete.

We have tried to use the already known techniques of §0.4., in order to settle these problems. Since this effort has been unsuccessful, the need to find new techniques, applicable here, has emerged. In what follows, these new methods and their range of application is presented.

CHAPTER 1

UNDECIDABLE PROPERTIES OF FINITE SETS OF EQUATIONS

§1.0 Introduction

Throughout this chapter, \mathcal{L} is a countable algebraic language and P is a property of sets of \mathcal{L} -equations. $P(\Sigma)$ is used to denote the fact that Σ has the property P , while $\neg P(\Sigma)$ is used to denote the fact that Σ doesn't have the property P .

The main concern of this chapter is "undecidable properties P for finite sets of \mathcal{L} -equations". What is meant by this statement, both informally and formally, was explained in the introductory chapter. Here, we present a new method of proving this kind of undecidability and, by applying it, we establish a set of new results.

In §1.1 the general method is presented. In §1.2 the undecidability of properties

- P_0 : the equational theory generated by Σ is equationally complete
- P_1 : the first-order theory generated by the infinite models of Σ is complete
- P_2 : the first-order theory generated by the infinite models of Σ is model-complete
- P_3 : Σ has the joint embedding property
- P_4 : the first-order theory of the non-trivial models of Σ has the joint embedding property and
- P_5 : the first-order theory of the infinite models of Σ has the joint embedding property,

for finite sets of equations of any finite non-trivial algebraic languages (with at least one constant symbol, in cases $P_3 - P_5$) is proved, as an

application of the method. The decision problems of $P_2 - P_5$, in non-trivial infinite languages, are examined in §1.3.

§1.1. A method of proving the undecidability of properties of finite sets of equations.

In this section we present a general criterion that provides a set of sufficient conditions in order for a property P to be undecidable, for finite sets of equations of a non-trivial language \mathcal{L} . The criterion seems complicated, at the first look, but its proof is almost obvious and its application is straightforward. It turns out to be quite powerful, since it gives access to properties which cannot be examined by any of the existing methods. Its wide range of application will be shown in §1.2.

In section 0.3 the meaning of the statement "P is undecidable for finite sets of \mathcal{L} -equations" was given and a formal equivalent of the informal notion, through Goedel numberings and recursive functions, was obtained. We only use informal procedures in this part, but the corresponding formal ones can be easily constructed.

Before advancing to present the criterion (Theorem 1.1.0), certain new notions need to be defined :

Let Γ be a finite set of \mathcal{L} -equations.

In the introductory chapter, what is meant by "the equational theory generated by Γ is decidable" and "the equational theory generated by Γ is base-decidable" was defined. We relativise the two definitions, here, to a new set of \mathcal{L} -equations, Δ , and thus we get two weaker notions :

The equational theory generated by Γ is decidable with respect to Δ , if there is an algorithm that enables us to decide whether, given $\varepsilon \in \Delta$, ε is a consequence of Γ , or not. Otherwise, $\theta_{Eq}[\Gamma]$ is undecidable with respect to Δ . Similarly, the equational theory generated by Γ is base-decidable with respect to Δ , if there is an algorithm that enables us to decide whether, given a finite $\Sigma \subset \Delta$, Σ is a basis of $\theta_{Eq}[\Gamma]$ or not. Otherwise, $\theta_{Eq}[\Gamma]$ is base-undecidable with respect to Δ .

Let Γ be a finite set of \mathcal{L} -equations and let $\text{Sym}(\Gamma)$ be the set of all operation symbols, constant symbols and variables occurring in Γ .

We define $E(\Gamma)$ to be the set of all equations $(\forall \bar{v})(\varphi = \psi)$, in at most $\text{Sym}(\Gamma)$, such that either

- a. φ and ψ contain exactly one and the same variable, or
- b. there exists a term κ , with exactly one variable, and there exists an equation $(\forall \bar{v})(v_1 = v_2)$ in Γ , so that either $\varphi = \kappa[v_1]$ and $\psi = \kappa[v_2]$, or $\varphi = \kappa[v_1]$ and $\psi = v_1$, or $\varphi = \kappa[v_2]$ and $\psi = v_2$.

With the help of the above given definitions, we are now able to formulate our criterion :

Theorem 1.1.0. Let \mathcal{L} be a countable algebraic language, let Γ be a finite set of \mathcal{L} -equations and let P be a property of finite sets of \mathcal{L} -equations. Suppose that the following conditions hold :

- a. $\theta_{\text{Eq}}[\Gamma]$ is decidable with respect to $E(\Gamma)$ and base-undecidable with respect to $E(\Gamma)$.
- b. $\theta_{\text{Eq}}[\Gamma]$ has the property P , and every finite $\Sigma \subset E(\Gamma)$, which generates $\theta_{\text{Eq}}[\Gamma]$, has also the property P .
- c. Every finite $\Sigma \subset E(\Gamma)$, which generates a proper equational subtheory of $\theta_{\text{Eq}}[\Gamma]$, doesn't have the property P .

Then, P is undecidable for finite sets of equations of \mathcal{L} .

Proof

If P was decidable for finite sets of \mathcal{L} -equations, then the set

$$\left\{ (\Sigma \subset \theta_{\text{Eq}}[\Gamma] \cap E(\Gamma)) \wedge (\Sigma \text{ finite}) : P(\Sigma) \right\} \quad (1)$$

would be decidable, too.

Conditions b. and c. of the theorem obviously imply that, given a finite $\Sigma \subset \theta_{\text{Eq}}[\Gamma] \cap E(\Gamma)$, it holds :

$$P(\Sigma) \leftrightarrow \theta_{\text{Eq}}[\Sigma] = \theta_{\text{Eq}}[\Gamma] \quad (2)$$

Relation (2), together with the decidability of set (1), would imply the decidability of the set

$$\left\{ (\Sigma \subset \theta_{\text{Eq}}[\Gamma] \cap E(\Gamma)) \wedge (\Sigma \text{ finite}) : \theta_{\text{Eq}}[\Sigma] = \theta_{\text{Eq}}[\Gamma] \right\} \quad (3)$$

Because of the decidability of $\theta_{\text{Eq}}[\Gamma]$ with respect to $E(\Gamma)$, we get that the set

$$\left\{ (\Sigma \subset E(\Gamma)) \wedge (\Sigma \text{ finite}) : \Sigma \subset \theta_{\text{Eq}}[\Gamma] \right\} \quad (4)$$

is decidable.

Now, the decidability of sets (3) and (4) would yield the following decision procedure, for checking whether a finite $\Sigma \subset E(\Gamma)$ generates $\theta_{\text{Eq}}[\Gamma]$ or not :

<p>Given Σ, check whether Σ is in (4). If no, then Σ is not a basis of $\theta_{\text{Eq}}[\Gamma]$. If yes, then check whether Σ is in (3). Again, if no, Σ is not a base of $\theta_{\text{Eq}}[\Gamma]$. If yes, Σ is a basis of $\theta_{\text{Eq}}[\Gamma]$.</p>

Consequently, $\theta_{\text{Eq}}[\Gamma]$ would be base-decidable with respect to $E(\Gamma)$, which would contradict condition (a) of the theorem. \square

In the course of applying Theorem 1.1.0, in §1.2., we shall use Theorem 1.1.1., given below, whenever we want to prove that the equational theory generated by Γ is base-undecidable with respect to $E(\Gamma)$.

Theorem 1.1.1. is a slightly modified version of the theorem, proved by McNulty [13], that follows :

McNulty's Theorem. Let Γ be a finite set of equations of a countable, non-trivial algebraic language \mathcal{L} , and let θ be a non-trivial term of \mathcal{L} (i.e. a term containing either at least two unary operation symbols or at least an operation symbol of rank greater than one) with at least one variable, such that $\Gamma \models (\forall \bar{v}) (\theta = v_0)$.

Then, the equational theory generated by Γ is base-undecidable.

Outline of the proof

McNulty proves his theorem, using the technique set out on pp 34 and 35 of the introductory chapter. We outline here his proof, only to the extent needed in order to convince the reader that our modification is sound:

Let $\mathcal{L}^* = \{f, g, h, k\}$ be a language with exactly four unary operation symbols.

Consider a universal system δ_0 of definitions for \mathcal{L}^* in \mathcal{L} , with respect to $\{(\forall v_0) (v_0 = v_0)\}$, whose range contains only operation

symbols occurring in θ . Such a system is proved by McNulty to exist always.

Let M be Malcev's set, defined on p.34, whose only symbols are f , g and v_i . Associate with each equation ε of the form $(\forall v_i)(\varphi = \psi)$, in at most f , g and v_i , the finite set of \mathcal{L} -equations

$$B(\varepsilon, \delta_0, M) = \text{in}_{\delta_0} [M] \cup \left\{ \begin{array}{l} (\forall \bar{v}) \left((\text{in}_{\delta_0} h \varphi [kv_0]) [v_1] = (\text{in}_{\delta_0} h \varphi [kv_1]) [v_2] \right) : (\forall \bar{v}) (v_1 = v_2) \in \Gamma \\ (\forall \bar{v}) \left((\text{in}_{\delta_0} h \psi [kv_0]) [v] = \bar{v} \right) : (\forall \bar{v}) (v = v') \in \Gamma \vee (\forall \bar{v}) (v' = v) \in \Gamma \end{array} \right\}$$

McNulty proves that the following holds :

$$M \models \varepsilon \leftrightarrow B(\varepsilon, \delta_0, M) \text{ is a basis of } \theta_{\text{Eq}}[\Gamma].$$

This relation, together with the undecidability of set (1) of p. 34, yield the base-undecidability of $\theta_{\text{Eq}}[\Gamma]$. \square

Theorem 1.1.1. Let Γ be a finite set of equations of a countable, non-trivial algebraic language. Let θ be a non-trivial term of \mathcal{L} , with at least one variable and with all its operation symbols and constant symbols in $\text{Sym}(\Gamma)$. If $\Gamma \models (\forall \bar{v}) (\theta = v_0)$, then the equational theory generated by Γ is base undecidable with respect to $E(\Gamma)$.

Proof

The proof is obtained by a close examination of the sets $B(\varepsilon, \delta_0, M)$ of the previous theorem :

Claim . For every equation ε , in at most the symbols f , g and v_i , the set $B(\varepsilon, \delta_0, M)$ is included in $E(\Gamma)$.

Proof of the claim

Since $\Gamma \models (\forall \bar{v}) (\theta = v_0)$, Γ contains at least one variable . Without loss of generality, we can identify it with the unique variable v_i , which occurs in Malcev's set, M .

Having made this convention, we now observe that the variables , occurring in $B(\varepsilon, \delta_0, M)$, are exactly those occurring in Γ . Also, because of the choice of δ_0 , and because of the construction of $B(\varepsilon, \delta_0, M)$, the operation symbols and the constant symbols in it are among those in Γ . Consequently, $B(\varepsilon, \delta_0, M)$ is in at most $\text{Sym}(\Gamma)$.

On the other hand, since δ_0 interprets, by definition, \mathcal{L} -terms to \mathcal{L}^* -terms with exactly the same variables, $\text{in}_{\delta_0} [M]$ consists of equations $(\forall \bar{v}) (\varphi' = \psi')$ with both φ' and ψ' in exactly the variable v_1 ; so $\text{in}_{\delta_0} [M] \subset E(\Gamma)$.

If we put $k = \text{in}_{\delta_0} h \varphi[kv_0]$ or $k = \text{in}_{\delta_0} h \psi[kv_0]$, we get that the remaining equations in $B((\forall \bar{v}) (\varphi = \psi), \delta_0, M)$ are of the form $(\forall \bar{v}) (k[v_1] = k[v_2])$ or $(\forall \bar{v}) (k[v_1] = v_1)$ or $(\forall \bar{v}) (k[v_2] = v_2)$, for some $(\forall \bar{v}) (v_1 = v_2)$ in Γ ; so they are in $E(\Gamma)$, too. This completes the proof of the claim. \square

Finally, the base-decidability of $\theta_{\text{Eq}}[\Gamma]$, with respect to $E(\Gamma)$, would yield, through the relation

$$M \models \varepsilon \leftrightarrow B(\varepsilon, \delta_0, M) \text{ is a basis of } \theta_{\text{Eq}}[\Gamma]$$

and the claim, a contradiction, exactly as before. So, we are done. \square

§1.2. Applications of the method.

Let Σ be a finite set of equations of the countable algebraic language \mathcal{L} . Let

$$\Sigma^\omega = \Sigma \cup \left\{ (\exists v_0 v_1 \dots v_n) \left(\bigwedge_{0 \leq i < j \leq n} (v_i \neq v_j) \right) : n \in \omega - 1 \right\}$$

and let

$$\Sigma^{>1} = \Sigma \cup \{ (\exists v_0 v_1) (v_0 \neq v_1) \}$$

Obviously, $\theta[\Sigma^\omega]$ is the first order theory generated by the class of infinite models of Σ , while $\theta[\Sigma^{>1}]$ is the first-order theory generated by the class of non-trivial models of Σ .

We make the following notational conventions :

$P_0(\Sigma)$ stands for "the equational theory generated by Σ is equationally complete".

$P_1(\Sigma)$ stands for "the first-order theory generated by Σ^ω is complete".

$P_2(\Sigma)$ stands for "the first-order theory generated by Σ^ω is model-complete".

$P_3(\Sigma)$ stands for " Σ has the joint embedding property".

$P_4(\Sigma)$ stands for " $\Sigma^{>1}$ has the joint embedding property", and

$P_5(\Sigma)$ stands for " Σ^ω has the joint embedding property".

All the notions, used above, were defined in the introductory chapter. In this section, we apply the method, previously described, and we get the undecidability of properties $P_0 - P_5$, for finite sets of equations of any non-trivial finite language (in cases $P_3 - P_5$, the language is also required to have at least one constant symbol). Theorems 1.2.0, 1.2.3 and 1.2.5, establish the above mentioned undecidability results :

Theorem 1.2.0. Let \mathcal{L} be a non-trivial finite algebraic language . There is no algorithm that enables us to decide whether, given any finite $\Sigma \subset \text{Eq}_{\mathcal{L}}$, Σ has the property P_i ($i \in \{0,1\}$).

Lemma 1.2.1. If an equationally consistent equational theory has property P_1 , then it has property P_0 .

Proof.

Suppose that Φ is equationally consistent but not equationally complete. Then, by definition, there is an equational theory θ , such that

$$\Phi \subsetneq \theta \subsetneq \text{Eq}_{\mathcal{L}}.$$

Consider the term algebras \mathfrak{S}_{Φ} and \mathfrak{S}_{θ} . Because of Theorem 0.2.2., they are both infinite models of Φ , but they don't satisfy the same equations. So, we found $\mathfrak{S}_{\Phi} \models \Phi$ and $\mathfrak{S}_{\theta} \models \Phi$ such that $\mathfrak{S}_{\Phi} \neq \mathfrak{S}_{\theta}$. This means that the first order theory, generated by Φ , is not complete. \square

Lemma 1.2.2. If the equational theory, generated by a finite set of equations, Σ , of a countable algebraic language \mathcal{L} , is equationally complete, then it is decidable.

Proof.

Obviously, the following holds :

$$(\forall \varepsilon \in \text{Eq}_{\mathcal{L}})(\Sigma \models \varepsilon \leftrightarrow \Sigma^{>1} \models \varepsilon)$$

Claim. For any $\varepsilon \in \text{Eq}_{\mathcal{L}}$, exactly one of the relations $\Sigma^{>1} \models \varepsilon$ and $\Sigma^{>1} \models \neg \varepsilon$ holds.

Proof of the claim.

Since $\theta_{\text{Eq}}[\Sigma]$ is equationally consistent, Σ has at least one non-trivial model. Consequently, the first order theory generated by $\Sigma^{>1}$ is consistent and it cannot contain the contradictory statement $\varepsilon \wedge \neg \varepsilon$.

Suppose that $\Sigma^{>1} \not\models \varepsilon$. Then, there exists a non-trivial model of Σ , which doesn't satisfy ε . Thus, since any two non-trivial models of Σ satisfy the same equations, it holds

$$(\forall \mathcal{A} \models \Sigma^{>1})(\mathcal{A} \not\models \varepsilon)$$

or, equivalently, it holds

$$(\forall \mathcal{A} \models \Sigma^{>1})(\mathcal{A} \models \neg \varepsilon).$$

This means that $\Sigma^{>1} \models -\varepsilon$. So the proof of the claim is complete. \square

It is well known (see Monk [15], p.173) that, since $\Sigma^{>1}$ is finite, the set of first-order consequences of $\Sigma^{>1}$ can be effectively listed; i.e. the set $\Theta[\Sigma^{>1}]$ can be written as an ω -sequence

$$\varphi_0, \varphi_1, \varphi_2, \dots, \varphi_n, \dots \quad (2)$$

so that, given any $n \in \omega$, the sentence φ_n , that falls at the n^{th} place in the sequence, can be algorithmically found. Thus,

given $\varepsilon \in \text{Eq}_L$, we are assured by the claim that exactly one of ε and $-\varepsilon$ appears in (2), after finitely many steps. If ε appears first, then write $\varepsilon \in \Theta_{\text{Eq}}[\Sigma]$, justified by relation (1). If $-\varepsilon$ appears first, then write $\varepsilon \notin \Theta_{\text{Eq}}[\Sigma]$, justified again by relation (1).

The above described algorithm is a decision procedure for $\Theta_{\text{Eq}}[\Sigma]$. \square

Proof of theorem 1.2.0.

Let

$$\mathcal{L} = \langle \{Q_i\}_{i \in I}, \{c_j\}_{j \in J} \rangle$$

be any algebraic non-trivial language, with both I and J finite. Then the set of \mathcal{L} -equations

$$\Gamma = \left\{ (\forall v_0 v_1 \dots v_{r(i)-1}) (Q_i v_0 \dots v_{r(i)-1} = v_0) : i \in I \cup \{c_j = c_j : \langle j, j \rangle \in {}^2J\} \right\}$$

is also finite. We shall prove that Γ satisfies conditions a, b and c of theorem 1.1.0, for $P = P_0$ and $P = P_1$, respectively :

Condition b. Since $\Theta_{\text{Eq}}[\Gamma]$ is equationally consistent, Γ^∞ has models. Consider any two $\mathcal{A} \models \Gamma^\infty$ and $\mathcal{B} \models \Gamma^\infty$ of the same cardinality α . Because of the restrictions imposed on the constant symbols by Γ , there exist $\langle a, b \rangle \in A \times B$, such that

$$(\forall j \in J) (c_j^{\mathcal{A}} = a \wedge c_j^{\mathcal{B}} = b).$$

Consider now any bijection

$$f : A \xrightarrow{\sim} B;$$

that maps a to b . For any operation symbol Q_i and for any $\langle a_0, a_1, \dots, a_{r(i)-1} \rangle \in {}^{r(i)}A$, it holds:

$$f(Q_i^{\mathcal{U}}(a_0, a_1 \dots a_{r(i)-1})) = f(a_0) = Q_i^{\mathcal{B}}(f(a_0), f(a_{r(i)-1})). \quad (2)$$

Also, because of (1), for any constant symbol c_j , it holds:

$$f(c_j^{\mathcal{U}}) = f(a) = b = c_j^{\mathcal{B}}. \quad (3)$$

Relations (2) and (3) prove that f is an isomorphism between the structures \mathcal{U} and \mathcal{B} .

We have proved that any two models of Γ^∞ of cardinality α are isomorphic; consequently, Γ^∞ is α -categorical. Thus, by Theorem 0.1.0., $\theta[\Gamma^\infty]$ is complete. By lemma 1.2.1., the equational completeness of $\theta_{Eq}[\Gamma]$ follows.

We have proved that

$$P_0(\Gamma) \wedge P_1(\Gamma)$$

holds. The remaining part of condition b holds trivially. \square

Condition a Lemma 1.2.2. proves that the finitely based, equationally complete equational theory $\theta_{Eq}[\Gamma]$ is decidable; hence, $\theta_{Eq}[\Gamma]$ is decidable with respect to $E(\Gamma)$.

The non-trivial language \mathcal{L} contains, by definition, either an operation symbol Q_i of rank greater than one, or two unary operation symbols f and g . In the first case, it holds

$$\Gamma \models (\forall \bar{v})(Q_i v_0 \dots v_{r(i)-1} = v_0), \quad (1)$$

while, in the second case, it holds

$$\Gamma \models (\forall v_0)(fg v_0 = v_0). \quad (2)$$

Relations (1) and (2) show that there exists always an \mathcal{L} -term θ , that satisfies the requirements of Theorem 1.1.1.; so, the base-undecidability of $\theta_{Eq}[\Gamma]$, with respect to $E(\Gamma)$, follows. \square

Condition c. Let Σ be any finite set of \mathcal{L} -equations such that $\theta_{Eq}[\Sigma] \subsetneq \theta_{Eq}[\Gamma]$. Then, by the definition of equational completeness, $\theta_{Eq}[\Sigma]$ is not equationally complete. By Lemma 1.2.1., $\theta[\Sigma^\infty]$ cannot be complete, either. We have proved that it holds :

$$[-P_0(\Sigma)] \wedge [-P_1(\Sigma)] \quad \square$$

Since conditions a, b and c of Theorem 1.1.0. are satisfied by Γ

for both P_0 and P_1 , the two properties are undecidable for finite sets of \mathcal{L} -equations. \square

Theorem 1.2.3. Let \mathcal{L} be a non-trivial finite algebraic language. There is no algorithm that enables us to decide whether, given any finite $\Sigma \subset \text{Eq}_{\mathcal{L}}$, Σ has the property P_2 .

Proof

Let

$$\mathcal{L} = \langle \{Q_i\}_{i \in I}, \{c_j\}_{j \in J} \rangle$$

be any finite non-trivial algebraic language. Then the set of equations

$$\Gamma = \{(\forall \bar{v}) (Q_i v_0 v_1 \dots v_{r(i)-1} = v_0) : i \in I\}$$

is also finite. We shall prove that Γ satisfies conditions a, b and c of Theorem 1.1.0, for the property P_5 :

Condition a. Let $\text{Tsym}(\Gamma)$ be the set of \mathcal{L} -terms in at most $\text{Sym}(\Gamma)$. It can be shown, by two easy inductions on the length of the terms φ and ψ , respectively, that, for any pair $\langle \varphi, \psi \rangle \in {}^2\text{Tsym}(\Gamma)$, it holds :

$$\Gamma \models (\forall \bar{v}) (\varphi = \psi) \leftrightarrow (\text{the first leftmost variable occurring in } \varphi \text{ coincides with the first leftmost variable occurring in } \psi).$$

This, obviously, provides a decision procedure for checking whether, given an equation ε in at most $\text{Sym}(\Gamma)$, it belongs to $\theta_{\text{Eq}}[\Gamma]$ or not. Consequently, $\theta_{\text{Eq}}[\Gamma]$ is decidable with respect to $E(\Gamma)$.

By applying Theorem 1.1.1. exactly as in the previous theorem, we prove that $\theta_{\text{Eq}}[\Gamma]$ is base-undecidable with respect to $E(\Gamma)$. \square

Condition b. Let \mathcal{A} and \mathcal{B} be any two infinite models of Γ^∞ , such that $\mathcal{A} \subset \mathcal{B}$.

Let $a_j \in A$ be the interpretation of the constant symbol c_j in \mathcal{A}_j ; in other words, let

$$\forall j \in J, \quad c_j^{\mathcal{A}} = a_j = c_j^{\mathcal{B}} \quad (1)$$

Consider the expansion of \mathcal{L} by A

$$\mathcal{L}_A = \mathcal{L} \cup \{c_a : a \in A\}$$

and the set of \mathcal{L}_A sentences

$$\Gamma_A = \Gamma \cup \{c_j = c_{a_j} : j \in J\} \cup \{c_a \neq c_b : a \neq b\}.$$

Claim $\Theta[\Gamma_A]$ is complete in \mathcal{L}_A .

Proof of the claim.

Γ_A has the following properties :

a. It has no finite models (since, because of the construction of Γ_A , any model of it has at least as many elements as \mathcal{U}).

b. It is α -categorical, for some infinite α (because if \mathcal{C} and \mathcal{D} are any two \mathcal{L}_A -models of Γ_A of cardinality α , there always exists a bijection

$$f : \mathcal{C} \xrightarrow{\sim} \mathcal{D}$$

that maps, for each $a \in A$, $c_a^{\mathcal{C}}$ to $c_a^{\mathcal{D}}$. We can easily show that f is an isomorphism between the two structures).

Conditions a and b of Theorem 0.1.0. are satisfied by Γ_A ; so Γ_A is complete. \square

Now, by the claim, any two models of Γ_A are elementary equivalent. The \mathcal{L}_A -structures \mathcal{U}_A and \mathcal{B}_A (see §0.1. for their definition) are, because of relations (1), models of Γ_A ; consequently,

$$\mathcal{U}_A \equiv \mathcal{B}_A.$$

It holds, thus, that

$$(\forall \varphi \in \text{Form}_{\mathcal{L}}) (\forall \bar{a} \in {}^\omega A) (\mathcal{U} \models \varphi[\bar{a}] \leftrightarrow \mathcal{B} \models \varphi[\bar{a}]) \quad (2)$$

Relation (2) proves that the substructure \mathcal{U} of \mathcal{B} , is an elementary substructure of \mathcal{B} .

It has been shown that

$$(\forall \mathcal{U} \models \Gamma^\infty) (\forall \mathcal{B} \models \Gamma^\infty) (\mathcal{U} \subset \mathcal{B} \rightarrow \mathcal{U} \prec \mathcal{B})$$

Thus, it has been shown that Γ^∞ has property P_2 .

The remaining part of condition b holds trivially. \square

Condition c. Obviously, it suffices to show that each finite $\Sigma \subset \theta_{\text{Eq}}[\Gamma] \cap E(\Gamma)$, the infinite models of which generate a model-complete first order theory, is a basis of $\theta_{\text{Eq}}[\Gamma]$. In other words it suffices to prove that

$$(\forall \mathcal{M}) (\mathcal{M} \models \Sigma \rightarrow \mathcal{M} \models \Gamma). \quad (1)$$

Towards proving relation (1), let \mathcal{M} be any model of Σ , and let \mathcal{B} be any infinite model of Γ^∞ , such that $A \cap B = \emptyset$. Define the \mathcal{L} -structure

$$\mathcal{M} \cup \mathcal{B} = \langle A \cup B, \{Q_i^{\mathcal{M} \cup \mathcal{B}}\}_{i \in I}, \{c_j^{\mathcal{M} \cup \mathcal{B}}\}_{j \in J} \rangle$$

as follows :

- a. For each operation symbol Q_i and for each

$$\begin{aligned} \bar{a} &= \langle a_0, \dots, a_{r(i)-1} \rangle \in {}^{r(i)}A \cup B, \\ \left[\begin{array}{l} Q_i^{\mathcal{M} \cup \mathcal{B}}(\bar{a}) = Q_i^{\mathcal{M}}(\bar{a}), \text{ if } \bar{a} \in {}^{r(i)}A \\ Q_i^{\mathcal{M} \cup \mathcal{B}}(\bar{a}) = Q_i^{\mathcal{B}}(\bar{a}), \text{ if } \bar{a} \in {}^{r(i)}B \\ Q_i^{\mathcal{M} \cup \mathcal{B}}(a) = Q_i^{\mathcal{M} \cup \mathcal{B}}(a_0, a_0, \dots, a_0), \text{ otherwise.} \end{array} \right. \end{aligned}$$

- b. For each constant symbol c_j ,

$$c_j^{\mathcal{M} \cup \mathcal{B}} = c_j^{\mathcal{B}}.$$

Obviously, $\mathcal{M} \cup \mathcal{B}$ is well-defined.

Claim $\mathcal{M} \cup \mathcal{B}$ is a model of Σ .

Proof of the claim.

Let $k(v_n)$ be any term with no constant symbols and with exactly the variable v_n in it.

It can be shown, by an easy induction on the length of k , that

$$(\forall a \in A) (k^{\mathcal{M} \cup \mathcal{B}}(a) = k^{\mathcal{M}}(a))$$

and

$$(\forall a \in B) (k^{\mathcal{R} \cup \mathcal{B}}(a) = k^{\mathcal{B}}(a)).$$

Thus, because of (2) and (3) and of the definition of $Q_i^{\mathcal{R} \cup \mathcal{B}}$, the following also hold :

$$\begin{aligned} k^{\mathcal{R} \cup \mathcal{B}}(Q_i^{\mathcal{R} \cup \mathcal{B}}(\bar{a})) &= k^{\mathcal{R}}(Q_i^{\mathcal{R}}(\bar{a})), \text{ if } \bar{a} \in r(i)_A \\ k^{\mathcal{R} \cup \mathcal{B}}(Q_i^{\mathcal{R} \cup \mathcal{B}}(\bar{a})) &= k^{\mathcal{R}}(Q_i^{\mathcal{R}}(\bar{a})), \text{ if } \bar{a} \in r(i)_B \\ k^{\mathcal{R} \cup \mathcal{B}}(Q_i^{\mathcal{R} \cup \mathcal{B}}(a)) &= k^{\mathcal{R}}(Q_i^{\mathcal{R} \cup \mathcal{B}}(a_0, \dots, a_0)), \text{ otherwise.} \end{aligned} \quad (4)$$

Since Σ is included in $E(\Gamma)$, each equation $(\forall \bar{v}) (\varphi = \psi)$ in Σ contains no constant symbols and it is of one of the following forms :

i. φ and ψ contain one and the same variable, in which case, the satisfaction of $(\forall \bar{v}) (\varphi = \psi)$ by $\mathcal{R} \cup \mathcal{B}$ is implied by the fact that both \mathcal{R} and \mathcal{B} satisfy $(\forall \bar{v}) (\varphi = \psi)$ and relations (2) and (3) hold.

ii. for some k and some $i \in I$, φ is $k[Q_i v_0 v_1 \dots v_{r(i)-1}]$ and ψ is $Q_i v_0 \dots v_{r(i)-1}$.

Since \mathcal{R} is a model of Σ , it holds :

$$(\forall \bar{a} \in r(i)_A) (k^{\mathcal{R}}(Q_i^{\mathcal{R}}(\bar{a})) = Q_i^{\mathcal{R}}(\bar{a})). \quad (5)$$

Relations (4) and (5) imply that, if $\bar{a} \in r(i)_A$, then it holds that

$$k^{\mathcal{R} \cup \mathcal{B}}(Q_i^{\mathcal{R} \cup \mathcal{B}}(\bar{a})) = k^{\mathcal{R}}(Q_i^{\mathcal{R}}(\bar{a})) = Q_i^{\mathcal{R}}(\bar{a}),$$

and, if $\bar{a} \notin r(i)_A$ but $a_0 \in A$, then it holds that

$$\begin{aligned} k^{\mathcal{R} \cup \mathcal{B}}(Q_i^{\mathcal{R} \cup \mathcal{B}}(\bar{a})) &= k^{\mathcal{R}}(Q_i^{\mathcal{R} \cup \mathcal{B}}(a_0, \dots, a_0)) = Q_i^{\mathcal{R}}(a_0, \dots, a_0) \\ &= Q_i^{\mathcal{R} \cup \mathcal{B}}(\bar{a}). \end{aligned}$$

The remaining cases accept similar treatment.

We've proved, thus, that

$$(\forall \bar{a} \in \omega_{A \cup B}) (\varphi^{\mathcal{R} \cup \mathcal{B}}(\bar{a}) = \psi^{\mathcal{R} \cup \mathcal{B}}(\bar{a})),$$

which means that $\mathcal{R} \cup \mathcal{B}$ satisfies the equation $(\forall \bar{v}) (\varphi = \psi)$.

iii. for some k and some $i \in I$, φ is $k[Q_i v \dots v_{r(i)-1}]$ and ψ is $k[v_0]$

Since \mathfrak{A} is a model of Σ , it holds that

$$(\forall \bar{a} \in r^{(i)}A) (k^{\mathfrak{A}}(Q_i^{\mathfrak{A}}(\bar{a})) = k^{\mathfrak{A}}(a_0)) \quad (6)$$

Consequently, because of relations (2), (4) and (6), we get that, if $\bar{a} \in r^{(i)}A$, then

$$k^{\mathfrak{A} \cup \mathfrak{B}}(Q_i^{\mathfrak{A} \cup \mathfrak{B}}(\bar{a})) = k^{\mathfrak{A}}(Q_i^{\mathfrak{A}}(\bar{a})) = k^{\mathfrak{A}}(a_0)$$

and, if $\bar{a} \notin r^{(i)}A$ but $a_0 \in A$, then

$$\begin{aligned} k^{\mathfrak{A} \cup \mathfrak{B}}(Q_i^{\mathfrak{A} \cup \mathfrak{B}}(\bar{a})) &= k^{\mathfrak{A}}(Q_i^{\mathfrak{A}}(a_0 \dots a_0)) = k^{\mathfrak{A}}(a_0) \\ &= k^{\mathfrak{A} \cup \mathfrak{B}}(a_0). \end{aligned}$$

Similarly, for the remaining cases.

Thus, $\mathfrak{A} \cup \mathfrak{B}$ satisfies the equation $(\forall \bar{v}) (\varphi = \psi)$.

We have proved that every possible equation in Σ is satisfied by $\mathfrak{A} \cup \mathfrak{B}$; so, we have proved the claim. \square

We use the claim in order to prove that \mathfrak{A} itself is a model of Γ :

Since $\mathfrak{A} \cup \mathfrak{B}$ extends \mathfrak{B} , by construction it is infinite. Consequently, we have found two infinite models \mathfrak{B} and $\mathfrak{A} \cup \mathfrak{B}$ of Σ , such that the former is a substructure of the latter. $\mathfrak{A} \cup \mathfrak{B}$ is, thus, elementary equivalent to \mathfrak{B} (because of the fact that Σ has the property P_2 , by hypothesis). Consequently,

$$\mathfrak{A} \cup \mathfrak{B} \models \Gamma.$$

This implies that

$$\mathfrak{A} \models \Gamma.$$

Relation (1) has now been proved and, as an immediate consequence the satisfaction of condition c. of Theorem 1.1.0 by Γ has also been established. \square

Theorem 1.2.3. follows, now, by a simple application of Theorem 1.1.0. \square

Theorem 1.2.4. Let \mathcal{L} be a non-trivial finite algebraic language which contains at least one constant symbol. There is no algorithm that enables us to decide whether, given any finite $\Sigma \subset \text{Eq}_{\mathcal{L}}$, Σ has the property P_i ($i \in \{3,4,5\}$) or not.

Proof

Let

$$\mathcal{L} = \langle \{Q_i\}_{i \in I}, c \rangle$$

be any non-trivial algebraic language, with I finite. Then, the set of \mathcal{L} -equations

$$\Gamma = \{(\forall v_0) (Q_i v_0 v_0 \dots v_0 = v_0) : i \in I\}$$

is also finite. We shall prove that Γ satisfies conditions a, b and c of Theorem 1.1.0. for $P = P_i$ ($i \in \{3,4,5\}$):

Condition a. The base-undecidability of $\theta_{\text{Eq}}[\Gamma]$ with respect to $E(\Gamma)$ follows from the fact that there exists a term θ , satisfying the requirements of Theorem 1.1.1.

Claim. $(\forall \varphi \in \text{Tsym}(\Gamma)) (\Gamma \models (\forall v_0) (\varphi = v_0))$

Proof of the claim. (By induction on the length of φ).

For $\varphi = v_0$, it holds that $\Gamma \models (\forall v_0) (v_0 = v_0)$.

Suppose that for any term θ , of length less than that of φ , it holds that $\Gamma \models (\forall v_0) (\theta = v_0)$. Suppose also that $\varphi = Q_i \theta_0 \theta_1 \dots \theta_{r(i)-1}$, for some $i \in I$ and some $\langle \theta_0, \dots, \theta_{r(i)-1} \rangle \in {}^{r(i)}\text{Tsym}(\Gamma)$. Then, we have that

$$\Gamma \models Q_i \theta_0 \theta_1 \dots \theta_{r(i)-1} = Q_i v_0 v_0 \dots v_0 = v_0.$$

This completes the proof of the claim. \square

By the claim, any equation in at most $\text{Sym}(\Gamma)$ is in $\theta_{\text{Eq}}[\Gamma]$; so $\theta_{\text{Eq}}[\Gamma]$ is decidable with respect to $E(\Gamma)$. \square

Condition b. Let \mathcal{A} and \mathcal{B} be any two models of Γ . By Theorem 0.2.1., $\mathcal{A} \times \mathcal{B}$ is also a model of Γ . Consider the injective mappings

$$f : \mathcal{A} \longrightarrow \mathcal{A} \times \mathcal{B} \quad \text{and} \quad g : \mathcal{B} \longrightarrow \mathcal{A} \times \mathcal{B},$$

given by the rules

$$f(a) = \langle a, c^{\mathfrak{A}} \rangle \quad \text{and} \quad g(b) = \langle c^{\mathfrak{A}}, b \rangle,$$

respectively. Obviously, they are both embeddings. Since any two models of Γ can be embedded in a third, $\Gamma, \Gamma^{>1}$ and Γ^∞ have the joint embedding property. The remaining part of condition b holds trivially. \square

Condition c.

Claim. For any finite set of \mathcal{L} -equations Σ , in at most $\{Q_i\}_{i \in I} \cup \{v_0\}$, and for any $i \in I$, it holds

$$\Sigma \models (\forall v_0) (Q_i v_0 v_0 \dots v_0 = v_0) \leftrightarrow \Sigma \models Q_i c c \dots c = c.$$

Proof of the claim

Direction \rightarrow holds trivially.

I prove direction \leftarrow . Suppose that, for some $i \in I$,

$$\Sigma \models Q_i c c \dots c = c, \quad (1)$$

but

$$\Sigma \not\models (\forall v_0) (Q_i v_0 v_0 \dots v_0 = v_0).$$

Then, there exists a model \mathfrak{A} of Σ , and an element $a \in A$, such that

$$Q_i^{\mathfrak{A}}(a, a, \dots a) \neq a. \quad (2)$$

Consider the structure

$$\mathfrak{A}' = \langle A, \{Q_i^{\mathfrak{A}}\}_{i \in I}, a \rangle.$$

Since Σ doesn't contain c , \mathfrak{A}' is a model of Σ . Also, because of (2),

$$\mathfrak{A}' \not\models Q_i c c \dots c = c.$$

This contradicts relation (1). So, we are done. \square

Let Σ be a finite set of \mathcal{L} -equations, in at most $\{Q_i\}_{i \in I} \cup \{v_0\}$, such that

$$\theta_{\text{Eq}}[\Sigma] \not\subseteq \theta_{\text{Eq}}[\Gamma].$$

Then, there exists an $i \in I$, so that

$$\Sigma \not\models (\forall v_0) (Q_i v_0 \dots v_0 = v_0) \quad (1)$$

Because of the claim, relation (1) implies that

$$\Sigma \not\models Q_i c c \dots c = c.$$

Let \mathfrak{S}_Γ and \mathfrak{S}_Σ be the term algebras of the equational theories $\theta_{\text{Eq}}[\Gamma]$ and $\theta_{\text{Eq}}[\Sigma]$, respectively. Then, because of Theorem 0.2.2., we get

$$(\mathfrak{S}_\Gamma \models \Sigma^\infty \cup \{Q_i c c \dots c = c\}) \wedge (\mathfrak{S}_\Sigma \models \Sigma^\infty \cup \{Q_i c c c \neq c\}) \quad (2)$$

We have found two models of Σ^∞ , that cannot be embedded in a third model of it (because, otherwise, this third model would satisfy the contradictory statement

$$(Q_i c c \dots c = c) \wedge (Q_i c c \dots c \neq c),$$

as relation (2) shows). Thus, Σ^∞ doesn't have the joint embedding property; hence, neither $\Sigma^{>1}$ nor Σ have it. \square

Since conditions a, b and c of Theorem 1.1.0. are satisfied for any P_i ($i \in \{3,4,5\}$), the needed undecidability of the properties, for finite sets of equations of the language $\mathcal{L} = \langle \{Q_i\}_{i \in I}, c \rangle$, is established.

Suppose now that the language contains more than one constant symbols. In other words, suppose that

$$\mathcal{L}' = \langle \{Q_i\}_{i \in I}, \{c\} \cup \{c_j\}_{j \in J} \rangle.$$

For any finite $\Sigma_{\mathcal{L}} \subset \text{Eq}_{\mathcal{L}}$, consider the finite set

$$\Sigma_{\mathcal{L}'} = \Sigma_{\mathcal{L}} \cup \{c_j = c : j \in J\}.$$

Obviously, for each P_i ($i \in \{3,4,5\}$), and for each finite $\Sigma_{\mathcal{L}} \subset \text{Eq}_{\mathcal{L}}$, it holds :

$$P_i(\Sigma_{\mathcal{L}}) \text{ in } \mathcal{L} \leftrightarrow P_i(\Sigma_{\mathcal{L}'}) \text{ in } \mathcal{L}'. \quad (3)$$

Consequently, P_i 's are also undecidable for finite sets of \mathcal{L} -equations. \square

For languages containing constant symbols, properties P_3 , P_4 and P_5 are, thus, undecidable. It is natural to ask what happens if \mathcal{L} contains no constant symbols :

In such languages, the set

$$\Sigma_0 = \{(\forall v_0) (v_0 = v_0)\}$$

has the properties P_i . (because, for any two models \mathcal{A} and \mathcal{B} of it, with $A \cap B = \emptyset$, one can find a common extension of them).

On the other hand, in the language

$$\mathcal{L} = \langle f, g \rangle,$$

where f and g are unary operation symbols, the set

$$\Sigma = \{(\forall v_0 \forall v_1) (f(v_0) = f(v_1))\}$$

has none of properties P_i . (because : if we consider any two infinite models \mathcal{A} and \mathcal{B} of Σ , such that

$$\mathcal{A} \models (\forall v_0) (g(f(v_0)) = f(v_0))$$

and

$$\mathcal{B} \not\models (\forall v_0) (g(f(v_0)) = f(v_0)), \quad (1)$$

and if

$$(\exists \mathcal{C} \models \Sigma) (\exists h : \mathcal{A} \rightarrow \mathcal{C}) (\exists k : \mathcal{B} \rightarrow \mathcal{C}), \quad (2)$$

then, for any $a \in A$ and any $c \in C$, it holds :

$$\begin{aligned} g^{\mathcal{C}}(f^{\mathcal{C}}(c)) &= g^{\mathcal{C}}(f^{\mathcal{C}}(h(a))) = h(g^{\mathcal{A}}(f^{\mathcal{A}}(a))) = \\ &= h(f^{\mathcal{A}}(a)) = f^{\mathcal{C}}(h(a)) = f^{\mathcal{C}}(c). \end{aligned}$$

Consequently,

$$\mathcal{C} \models (\forall v_0) (g(f(v_0)) = f(v_0)). \quad (3)$$

Because of (3) and of the fact that \mathcal{B} is embeddable in \mathcal{C} , it holds that

$$\mathcal{B} \models (\forall v_0) (g(f(v_0)) = f(v_0))$$

which contradicts (1). Σ has, thus, none of properties P_i .

Since we have found sets of \mathcal{L} -equations having the properties P_i ($i \in \{3,4,5\}$) and sets of \mathcal{L} -equations not having them, the answer to the following question is not trivial :

Question. Are the properties P_3 , P_4 and P_5 undecidable for finite sets of equations of a non-trivial finite algebraic language, containing no constant symbols ?

Our method (Theorem 1.1.0) cannot be applied in this case, because, for any $\Gamma \subset \text{Eq}_{\mathcal{L}}$, $\theta_{\text{Eq}}[\Gamma]$ is bound to contain $\Sigma_i = \{(\forall v_i)(v_i = v_i)\} \subset E(\Gamma)$, for some $i \in \omega$.

As we mentioned in the introduction, in Perkins' [19] the undecidability of the property " $\theta_{\text{Eq}}[\Sigma]$ is equationally complete" for finite sets of equations of any finite language with at least two constant symbols and two operation symbols, is proved. We have obtained, hence, an extension of Perkins' result, in this section.

To the best of my knowledge, the results, concerning the remaining properties, are completely new.

§1.3. Denumerable languages.

We have proved, in the previous section, that properties $P_0 - P_5$ are undecidable for finite sets of equations of any finite non-trivial language (with some weak restrictions on it, in certain cases). What happens, though, if \mathcal{L} is infinite? Our general method might be applied here, but before making such an attempt, we should examine whether sets, having the property P_i , exist in these languages :

A. We examine properties P_0, P_1 and P_2 .

If \mathcal{L} is an infinite non-trivial algebraic language, then, either it contains infinitely many operation symbols, or, it contains finitely many operation symbols but infinitely many constant symbols. We examine the two cases, separately :

Case A_1 " \mathcal{L} contains infinitely many operation symbols ".

Let Σ be a finite set of \mathcal{L} -equations and let \mathcal{L}_Σ be the sublanguage of \mathcal{L} , with exactly the non-logical symbols occurring in Σ . Then, since \mathcal{L}_Σ is finite, $\mathcal{L} - \mathcal{L}_\Sigma$ contains operation symbols.

If $\theta_{\text{Eq}_{\mathcal{L}}}[\Sigma]$ is equationally inconsistent, then Σ has none of properties P_i .

If $\theta_{\text{Eq}_{\mathcal{L}}}[\Sigma]$ is equationally consistent, then $\theta_{\text{Eq}_{\mathcal{L}}}[\Sigma]$ is also equationally consistent; so Σ has an infinite \mathcal{L}_Σ -model. Consequently, by Theorem 0.1.3, it has an \mathcal{L}_Σ -model \mathcal{A} , of cardinality ω_1 . Consider any denumerable subset X of A . If \mathcal{A}_0 is the substructure of \mathcal{A} generated by X , it has cardinality ω (as the relations $X \subset A_0$ and $(\forall a \in A_0)(\exists \sigma \in \text{Term}_{\mathcal{L}_\Sigma})(\exists \bar{x} \in {}^\omega X)(a = \sigma^{\mathcal{A}}(\bar{x}))$ imply). So we have

$$(\mathcal{A} \models \Sigma^\infty) \wedge (\mathcal{A}_0 \models \Sigma^\infty) \wedge (\mathcal{A}_0 \subsetneq \mathcal{A}) \quad (1)$$

Consider now any two $a_0 \in A_0$ and $b_0 \in A - A_0$ and any \mathcal{L} -expansion \mathcal{A}' of \mathcal{A} , which satisfies the relations

$$\begin{aligned} (\forall Q_i \in \mathcal{L} - \mathcal{L}_\Sigma) (\forall \bar{a} \in {}^{r(i)}A_0) (Q_i^{\mathcal{A}'}(\bar{a}) = a_0) \\ (\forall Q_i \in \mathcal{L} - \mathcal{L}_\Sigma) (\forall \bar{a} \in {}^{r(i)}A - A_0) (Q_i^{\mathcal{A}'}(\bar{a}) = b_0). \end{aligned} \quad (2)$$

\mathcal{A}_0 is accordingly expanded to a submodel \mathcal{A}'_0 of \mathcal{A}' .
Relation (1) implies that

$$(\mathcal{A}' \models \Sigma^\infty) \wedge (\mathcal{A}'_0 \models \Sigma^\infty) \wedge (\mathcal{A}'_0 \subsetneq \mathcal{A}'),$$

while, relations (2) imply that

$$\mathcal{A}' \models (\forall \bar{v}) (Q_i v_0 \dots v_{r(i)-1} = Q_i v_{r(i)} \dots v_{2r(i)-1}),$$

and

$$\mathcal{A}'_0 \not\models (\forall \bar{v}) (Q_i v_0 \dots v_{r(i)-1} = Q_i v_{r(i)} \dots v_{2r(i)-1}).$$

Consequently, Σ has none of properties P_i .

We have proved that in languages with infinitely many operation symbols, there are no finite sets with any of the properties P_0 , P_1 and P_2 .

Case A_2 " \mathcal{L} contains finitely many operation symbols and infinitely many constant symbols" .

Let, again, Σ be a finite set of \mathcal{L} -equations, and let \mathcal{L}_Σ be the sublanguage of \mathcal{L} with exactly the non-logical symbols occurring in Σ . Then, since Σ is finite, $\mathcal{L} - \mathcal{L}_\Sigma$ contains at least two distinct constant symbols c_{j_1} and c_{j_2} .

If $\theta_{\text{Eq}_{\mathcal{L}}}[\Sigma]$ is equationally inconsistent, then Σ has none of properties P_0 and P_1 .

If $\theta_{\text{Eq}_{\mathcal{L}}}[\Sigma]$ is equationally consistent, then Σ has an infinite \mathcal{L}_Σ -model \mathcal{A} . Obviously, one can always find two \mathcal{L} -expansions \mathcal{A}' and \mathcal{A}'' of \mathcal{A} , such that

$$(\mathcal{A}' \models \Sigma \cup \{c_{j_1} = c_{j_2}\}) \wedge (\mathcal{A}'' \models \Sigma \cup \{c_{j_1} \neq c_{j_2}\})$$

This prove that Σ has none of properties P_0 and P_1 .

We have proved, thus, that in languages with finitely many operation symbols and infinitely many constant symbols, there are no finite sets with any of the properties P_0 and P_1 .

On the other hand, as far as property P_2 is concerned, one can prove exactly as in Theorem 1.1.3 that the finite set

$$\Sigma = \{(\forall \bar{v}) (Q_i v_0 v_1 \dots v_{r(i)-1} = v_0) : i \in I\}$$

has property P_2 and that P_2 is undecidable for finite sets of languages with finitely many operation symbols.

We summarise what we've said above, in the following theorem :

Theorem 1.3.0. Let \mathcal{L} be a denumerable non-trivial algebraic language. Then,

- a. If \mathcal{L} contains infinitely many operation symbols, the properties P_0 , P_1 and P_2 are decidable for finite sets of \mathcal{L} -equations, and
- b. if \mathcal{L} contains finitely many operation symbols, properties P_0 and P_1 are decidable, while property P_2 is undecidable for finite sets of \mathcal{L} -equations.

B. We examine properties P_3 , P_4 and P_5 .

The decision problem of P_i 's, for languages with no constant symbols was discussed at the end of the previous section. Everything said there for finite languages can be repeated here, for the denumerable case. Now we deal with denumerable languages with at least one constant symbol :

Let c be a constant symbol in \mathcal{L} .

Let Σ be any finite set of \mathcal{L} -equations and let \mathcal{L}_Σ be the least language in which Σ can be formulated. $\mathcal{L} - \mathcal{L}_\Sigma$ contains either

- i. an operation symbol Q , or
- ii. a constant symbol c' different from c .

If $\theta_{\text{Eq}_r}[\Sigma]$ is equationally consistent, then Σ has an infinite \mathcal{L}_Σ -model \mathcal{U} . Obviously, there always exist two \mathcal{L} -expansions of \mathcal{U} , say \mathcal{U}' and \mathcal{U}'' , such that, in case (i),

$$(\mathcal{U}' \models \Sigma \cup \{Qcc\dots c = c\}) \wedge (\mathcal{U}'' \models \Sigma \cup \{Qcc\dots c \neq c\})$$

and, in case (ii),

$$(\mathcal{U}' \models \Sigma \cup \{c = c'\}) \wedge (\mathcal{U}'' \models \Sigma \cup \{c \neq c'\}).$$

This proves that Σ has none of properties P_2 .

If $\theta_{Eq_{\mathcal{L}}}[\Sigma]$ is equationally inconsistent, then it has P_3 but not P_4 and P_5 .

We have proved, thus, that there is no finite Σ with the property P_4 or the property P_5 and that it holds

$$P_3(\Sigma) \leftrightarrow \theta_{Eq_{\mathcal{L}}}[\Sigma] = Eq_{\mathcal{L}}$$

Towards examining the decision problem of P_3 , now, let us consider the set

$$\Gamma = \{(\forall v_0 v_1)(Qv_0 \dots v_1 = v_1)\},$$

if \mathcal{L} contains an operation symbol Q of rank greater than one, or the set

$$\Gamma = \{(\forall v_0 v_1)(fg v_0 = v_1)\},$$

if \mathcal{L} contains only unary operation symbols.

We can easily verify, helped by relation (1), that Γ satisfies conditions a, b and c of Theorem 1.1.0. for P_3 ; hence, the undecidability of P_3 , for finite sets of \mathcal{L} -equations follows.

Summarising what we have proved above, we get the following theorem :

Theorem 1.3.1. Let \mathcal{L} be any denumerable non-trivial algebraic language, with at least one constant symbol. Property P_3 is undecidable, while properties P_4 and P_5 are decidable, for finite sets of \mathcal{L} -equations.

In this section and in the previous one, the decision problems of P_i 's, for finite sets of equations of all kinds of non-trivial countable languages, were examined. Negative answers to the majority of the problems were given, which obviously imply negative answers to the corresponding problems, for recursive sets of equations.

All the same in the rare cases, given by Theorems 1.3.0 and 1.3.1, where positive answers were obtained, no answers concerning recursive

sets are implicit. The corresponding decision problem for recursive sets of \mathcal{L} -equations are still to be answered. This is the task of the following theorem :

Theorem 1.3.2. In any denumerable non-trivial algebraic language \mathcal{L} (with at least one constant symbol, in cases P_3, P_4 and P_5), there is no algorithm that enables us to decide whether, given any recursive set $\Sigma \subset \text{Eq}_{\mathcal{L}}$, Σ has the property P_i ($i \in \{0,1,2,4,5\}$).

Proof

Let

$$\mathcal{L} = \langle \{Q_i\}_{i \in I}, \{c_j\}_{j \in J} \rangle.$$

\mathcal{L} contains an operation symbol Q_0 of minimal rank r_0 and possibly a constant symbol c_0 . Consider any non-trivial finite sublanguage \mathcal{L}_0 of \mathcal{L} , such that

$$\{Q_0, c_0\} \subset \mathcal{L}_0,$$

and associate, with each finite $\Sigma_0 \subset \text{Eq}_{\mathcal{L}_0}$, the recursive set of \mathcal{L} -equations

$$\Sigma = \Sigma_0 \cup \{(\forall \bar{v}) (Q_i v_0 v_1 \dots v_{r(i)-1} = Q_0 v_0 \dots v_{r_0-1}) : Q_i \in \mathcal{L} - \mathcal{L}_0\} \cup \{c_j = c_0 : c_j \in \mathcal{L} - \mathcal{L}_0\}$$

If we prove that, for each P_i , it holds that

$$P_i(\Sigma_0) \text{ in } \mathcal{L}_0 \leftrightarrow P_i(\Sigma) \text{ in } \mathcal{L},$$

the reduction of the unsolvable decision problems of P_i 's, for finite sets of \mathcal{L}_0 -equations, to the decision problems, under examination, will have been obtained and the needed undecidability will have been established. It suffices, thus, to prove the following claim:

Claim. For each finite $\Sigma_0 \subset \text{Eq}_{\mathcal{L}_0}$, and for each $i \in \{0,1,2,4,5\}$, it holds :

$$P_i(\Sigma_0) \text{ in } \mathcal{L} \leftrightarrow P_i(\Sigma) \text{ in } \mathcal{L}$$

Proof of the claim.

Direction " \leftarrow " is obvious, for all i 's. We are proving the other direction :

For P_0, P_1 and P_2 .

Suppose that $P_2(\Sigma_0)$ holds; i.e. suppose that

$$(\forall \mathcal{A}_0 \models \Sigma_0^\infty) (\forall \mathcal{B}_0 \models \Sigma_0^\infty) (\mathcal{A}_0 \subset \mathcal{B}_0 \rightarrow \mathcal{A}_0 < \mathcal{B}_0). \quad (1)$$

Consider any two infinite \mathcal{L} -models of Σ , say \mathcal{A} and \mathcal{B} such that \mathcal{A} is included in \mathcal{B} . It is a matter of simple observation that each \mathcal{L} -formula, φ , is equivalent under Σ to an \mathcal{L}_0 -formula ψ_φ . In other words, obviously it holds :

$$(\forall \varphi \in \text{Form}_{\mathcal{L}}) (\exists \psi_\varphi \in \text{Form}_{\mathcal{L}_0}) (\forall \mathcal{C} \models \Sigma) (\forall \bar{c} \in {}^\omega \mathcal{C}) (\mathcal{C} \models \varphi[\bar{c}] \leftrightarrow \mathcal{C} \models \psi_\varphi[\bar{c}]) \quad (2)$$

So, by relations (1) and (2), we get that, for each $\varphi \in \text{Form}_{\mathcal{L}}$, and for each $\bar{a} \in {}^\omega \mathcal{A}$, it holds :

$$\begin{aligned} \mathcal{A} \models \varphi[\bar{a}] &\leftrightarrow \mathcal{A} \models \psi_\varphi[\bar{a}] \leftrightarrow \mathcal{A}/\mathcal{L}_0 \models \psi_\varphi[\bar{a}] \leftrightarrow \\ &\leftrightarrow \mathcal{B}/\mathcal{L}_0 \models \psi_\varphi[\bar{a}] \leftrightarrow \mathcal{B} \models \psi_\varphi[\bar{a}] \leftrightarrow \\ &\leftrightarrow \mathcal{B} \models \varphi[\bar{a}]. \end{aligned}$$

We have proved that

$$(\forall \mathcal{A} \models \Sigma^\infty) (\forall \mathcal{B} \models \Sigma^\infty) (\mathcal{A} \subset \mathcal{B} \rightarrow \mathcal{A}_0 < \mathcal{B}_0).$$

So, we have proved that Σ has the property P_2 .

The two other cases are treated similarly.

For P_4 and P_5

Suppose that Σ_0 has the property P_4 .

Consider any two non-trivial \mathcal{L} -models of Σ , say \mathcal{A} and \mathcal{B} , and their restrictions $\mathcal{A}/\mathcal{L}_0$ and $\mathcal{B}/\mathcal{L}_0$. By hypothesis, these are embeddable in an \mathcal{L}_0 -model \mathcal{C} of Σ_0 , which can be obviously expanded to an \mathcal{L} -model \mathcal{E} of Σ .

We've proved, thus, that

$$(\forall \mathcal{A} \models \Sigma^{>1}) (\forall \mathcal{B} \models \Sigma^{>1}) (\exists \mathcal{C} \models \Sigma^{>1}) (\exists f: \mathcal{A} \rightarrow \mathcal{C}) (\exists g: \mathcal{B} \rightarrow \mathcal{C}).$$

Consequently, Σ has the property P_4 .

Similarly, for P_5 . □

CHAPTER 2

UNDECIDABLE PROPERTIES OF RECURSIVE SETS OF
EQUATIONS§ 2.0. Introduction

The main concern of this chapter is "undecidable properties P of recursive sets of equations".

In §2.1. we reduce a well-known recursively unsolvable problem (namely, the halting problem for Turing Machines) to the problem of the existence of an algorithm for deciding whether a computable field is finite. We establish, thus, the non-existence of such an algorithm.

In §2.2. we further reduce the problem of the existence of the above mentioned decision procedure to the decision problem of each of properties

P_6 : Σ has finite non-trivial models,

P_7 : The first-order theory of the non-trivial models of Σ is complete,

P_8 : The first-order theory of the non-trivial models of Σ is model-complete,

for recursive sets Σ of equations of any strong language with infinitely many operation symbols. The undecidability of the properties, thus, follows.

We conclude § 2.2. with a discussion about the results obtained and the open problems raised in relation to them.

§ 2.1. We cannot decide whether a field is finite

We prove, in this section, that there is no algorithm which enables us to decide whether a field has a finite domain.

We hope it has been made clear in Chapter 0, that, in order for such a problem to accept a mathematical formulation, the objects under consideration (here the fields) must be given in a way that implies an indexing of them (i.e. a representation of each one of them by a natural number). Obviously, the class of all fields is too broad to be indexed; but, for our purpose, an indexing of a subclass of it suffices:

We call a field $\mathcal{F} = \langle F, +_{\mathcal{F}}, \cdot_{\mathcal{F}} \rangle$ computable if F is a recursive set of natural numbers and if $+_{\mathcal{F}}$ and $\cdot_{\mathcal{F}}$ are recursive functions.

Let G be the goedel numbering of the set of Turing Machines, defined on pg. 19. For each natural number x , let φ_x^n be the n -ary recursive function, calculated by $G^{-1}(x)$, and let R_x be the recursive set with characteristic function φ_x^1 . We call x an index of φ_x^n or an index of R_x , respectively.

The function i , that maps each triple of the form

$$\langle R_x, \varphi_y^2, \varphi_z^2 \rangle$$

to the natural number

$$2^x \cdot 3^y \cdot 5^z$$

is, obviously, an injection. It implies, thus, an indexing of the set of all triples. Since every recursive field is a triple of the above kind, i also indexes the class of all computable fields.

We call $i(\mathcal{F}) = 2^x \cdot 3^y \cdot 5^z$ an index of the field $\mathcal{F} = \langle R_x, \varphi_y^2, \varphi_z^2 \rangle$

We now advance to prove the main theorem of this section:

Theorem 2.1.0. There is no algorithm that enables us to decide whether a computable field is finite.

Proof

Let A and B be the following two sets of natural numbers:

$$A = \{w = 2^x \cdot 3^y \cdot 5^z : i^{-1}(w) \text{ is a field}\}$$

$$B = \{w = 2^x \cdot 3^y \cdot 5^z : R_x \text{ is finite}\}$$

It suffices to show that there is no unary partial recursive function ψ with the properties:

- i. For every $w \in A$, $\psi(w)$ is defined
- ii. $\psi(w) = 1$ for every w in $A \cap B$, while $\psi(w) = 0$ for every $w \in A - B$.

Towards establishing the non-existence of such a ψ , we shall prove the following:

Claim An ω -sequence

$$\mathfrak{F}_n = \langle F_n, +^n, \cdot^n \rangle$$

of computable fields can be constructed, such that:

- a. $F_0 = \{0, 1\}$ and any other F_n is infinite
- b. Any two F_n and F_m have exactly the elements 0 and 1 in common
- c. There exists an algorithm that enables us to find, for each $x \in \omega$, the F_n 's to which x belongs.

Proof of the claim

The construction is carried out step by step as follows:

1st step

Let $\langle \mathbb{Z}, +, \cdot \rangle$ be the unique factorisation domain of the integers, and let $f: \mathbb{Z} \rightarrow \omega$ be the bijection, given by the rule

$$\begin{aligned} f(n) &= 2n & , & \text{ if } n \in \omega \\ f(n) &= 2|n|-1 & , & \text{ if } n \in \mathbb{Z} - \omega \end{aligned}$$

The structure $\langle \omega, \oplus, \circ \rangle$, with \oplus and \circ given by the rules

$$\begin{aligned} \forall \langle m, n \rangle \in {}^2\omega & , \quad m \oplus n = f(f^{-1}(m) + f^{-1}(n)) \\ & \quad m \circ n = f(f^{-1}(m) \cdot f^{-1}(n)) \end{aligned}$$

is, obviously, isomorphic with $\langle \mathbb{Z}, +, \cdot \rangle$. It is, thus, a unique factorisation domain with unit 2 and zero 0.

2nd step

Let $y \neq 0$ and let $\text{g.c.d.}(x,y)$ stand for "the greatest common divisor of x and y in $\langle \mathbb{Z}, +, \cdot \rangle$, taken as positive and such that the greatest common divisor of 0 and y is $|y|$ ". Using wellknown facts about integers, since f is algorithmic, we conclude that the following functions and relations are recursive:

- (1) $\text{G.C.D.}(x,y) = f(\text{g.c.d.}(f^{-1}(x), f^{-1}(y)))$
- (2) $\text{G.C.D.}(x,y) = 2$
- (3) $\phi_1(x,y) = f(\text{quotient of the division of } f^{-1}(x) \text{ by the } \text{g.c.d.}(f^{-1}(x), f^{-1}(y)))$
- (4) $\phi_2(x,y) = f(\text{quotient of the division of } f^{-1}(y) \text{ by the } \text{g.c.d.}(f^{-1}(x), f^{-1}(y)))$

The recursiveness of relation (2) immediately implies the recursiveness of the sets

$$\begin{aligned}
 A_0 &= \{0, 2\} \\
 A_1 &= \{2^k \cdot 3^\lambda \cdot p_2 : \text{G.C.D}(k, \lambda) = 2, \lambda \in 2\omega - 1\} \\
 &\dots \\
 &\dots \\
 A_n &= \{2^k \cdot 3^\lambda \cdot p_{n+1} : \text{G.C.D}(k, \lambda) = 2, \lambda \in 2\omega - 1\} \\
 &\dots \\
 &\dots
 \end{aligned}$$

while the recursiveness of functions (3) and (4) imply the recursiveness of the binary partial recursive functions \oplus and \odot , which are defined as follows:

$$\begin{aligned}
 \oplus &\left\{ \begin{aligned} &0 \oplus 0 = 0, 0 \oplus 2 = 2 \oplus 0 = 2, 2 \oplus 2 = 0 \\ &2^x \cdot 3^y \cdot p_i \oplus 2^z \cdot 3^w \cdot p_i = 2^{\phi_1(x^{\circ w} \oplus y^{\circ z}, y^{\circ w})} \cdot 3^{\phi_2(x^{\circ w} \oplus y^{\circ z}, y^{\circ w})} \cdot p_i \end{aligned} \right. \\
 &\qquad \text{for any } \langle y, w \rangle \in {}^2(2\omega - 1) \\
 \odot &\left\{ \begin{aligned} &0 \odot 0 = 0, 0 \odot 2 = 2 \odot 0 = 0, 2 \odot 2 = 2 \\ &2^x \cdot 3^y \cdot p_i \odot 2^z \cdot 3^w \cdot p_i = 2^{\phi_1(x^{\circ z}, y^{\circ w})} \cdot 3^{\phi_2(x^{\circ z}, y^{\circ w})} \cdot p_i \end{aligned} \right. \\
 &\qquad \text{for any } \langle y, w \rangle \in {}^2(2\omega - 1)
 \end{aligned}$$

It can be easily verified that the triple

$$\mathcal{U}_n = \langle A_n, \oplus, \odot \rangle$$

is a computable field which, for $n \neq 0$, has $2^0 \cdot 3^2 \cdot p_{n+1}$ as zero,

$2^2 \cdot 3^2 \cdot p_{n+1}$ as unit, $2^{2^{k-1}} \cdot 3^\lambda \cdot p_{n+1}$ as opposite of $2^{2^k} \cdot 3^\lambda \cdot p_{n+1}$,
 $2^{2^k} \cdot 3^\lambda \cdot p_{n+1}$ as opposite of $2^{2^{k-1}} \cdot 3^\lambda \cdot p_{n+1}$, $2^k \cdot 3^{2^\lambda} \cdot p_{n+1}$ as
 inverse of $2^{2^\lambda} \cdot 3^k \cdot p_{n+1}$ and $2^{k-1} \cdot 3^{2^\lambda} \cdot p_{n+1}$ as inverse of
 $2^{2^{\lambda-1}} \cdot 3^k \cdot p_{n+1}$. . .

3rd step

Let $f_0 : \omega \rightarrow \omega$ be the function that maps 2 to 1 and leaves every other element unchanged. For every $n > 0$, let $f_n : \omega \rightarrow \omega$ be the function that maps $2^0 \cdot 3^2 \cdot p_{n+1}$ to 0, $2^2 \cdot 3^2 \cdot p_{n+1}$ to 1 and leaves every other element unchanged. The structure

$$\mathcal{F}_n = \langle F_n, +^{\mathcal{F}_n}, \cdot^{\mathcal{F}_n} \rangle ,$$

with

$$F_0 = \{0, 1\}$$

$$\forall n > 0, F_n = \{0, 1\} \cup (A_n - \{2^0 \cdot 3^2 \cdot p_{n+1}, 2^2 \cdot 3^2 \cdot p_{n+1}\})$$

and

$$x +^{\mathcal{F}_n} y = f_n(f_n^{-1}(x) \oplus f_n^{-1}(y))$$

$$x \cdot^{\mathcal{F}_n} y = f_n(f_n^{-1}(x) \odot f_n^{-1}(y))$$

is isomorphic with \mathcal{A}_n , and, since f_n is a recursive function, it is a computable field.

It is obvious that the so constructed ω -sequence of \mathcal{F}_n 's, satisfies conditions a, b and c of the claim. \square

For each natural number x , let us consider the recursive function

$$\Phi^x : \omega \rightarrow \omega ,$$

with

$$\Phi^x(0) = \Phi^x(1) = 1$$

and, for $y \notin \{0, 1\}$, $\Phi^x(y)$ calculated by the following algorithm (the

construction of the algorithm is based on the recursiveness of F_n 's and the operation of Turing machines that has been explained in the introductory chapter):

1) Check whether $y \in F_1$

If yes, check whether the Turing machine $G^{-1}(x)$, given input x , halts after 0 steps.

If yes, put $\Phi^x(y) = 1$

If no, put $\Phi^x(y) = 0$

If no,

2) Check whether $y \in F_2$

If yes, check whether the Turing machine $G^{-1}(x)$, given input x , halts after 1 steps.

If yes, put $\Phi^x(y) = 1$

If no, put $\Phi^x(y) = 0$

If no,

.....

n) Check whether $y \in F_n$

If yes, check whether the Turing machine $G^{-1}(x)$, given input x , halts after $n-1$ steps.

If yes, put $\Phi^x(y) = 1$

If no, put $\Phi^x(y) = 0$

If no,

.....

y-1) Check whether $y \in F_{y-1}$

If yes, check whether the Turing machine $G^{-1}(x)$, given input x , halts after $y-2$ steps.

If yes, put $\Phi^x(y) = 1$

If no, put $\Phi^x(y) = 0$

If no, put $\Phi^x(y) = 0$.

Since every $y \notin \{0,1\}$ is in at most one F_n , the function Φ^x is well-defined (If there were a $y \notin \{0,1\}$ both in F_m and F_n , then the Turing machine $G^{-1}(x)$, given input x , might stop working after $m-1$ steps. In this case, we would have $\Phi^x(y) = 1$ and $\Phi^x(y) = 0$. So Φ^x wouldn't be well-defined).

The way, in which Φ^x has been constructed, implies that

- a) Φ^x is the characteristic function of some F_n (not necessarily identical with F_x) and
 b) Φ^x is the characteristic function of F_0 , if $G^{-1}(x)$, given input x , never halts.

The reasoning for this is quite simple: $G^{-1}(x)$, given input x , either stops after, say, κ steps or it never stops. In the first case, Φ^x gets the value 1 for any $y \in F_{\kappa+1}$ and the value 0 for any $y \notin F_{\kappa+1}$. In the second case, Φ^x gets the value 1 exactly for $y = 0$ and $y = 1$.

After having defined Φ^x , we associate with it two binary partial recursive functions $+^x$ and \cdot^x , in the following way:

For $\langle \kappa, \lambda \rangle \in {}^2\omega$, check whether $\Phi^x(\kappa) = \Phi^x(\lambda) = 1$ (this can be done, since Φ^x is recursive).

If no, then $\kappa +^x \lambda$ and $\kappa \cdot^x \lambda$ are not defined.

If yes, κ and λ must belong to the same F_n 's (because of property a. of Φ^x).

Find the F_n 's to which κ and λ belong (this can be done because of property c. of F_n 's) and put

$$\kappa +^x \lambda = \kappa + \delta_{n_0} \lambda \quad \kappa \cdot^x \lambda = \kappa \cdot \delta_{n_0} \lambda,$$

where n_0 is the smallest natural number such that $\langle \kappa, \lambda \rangle \in F_{n_0}$.

Finally, let us consider three recursive functions

$$g : \omega \rightarrow \omega$$

$$y : \omega \rightarrow \omega$$

$$z : \omega \rightarrow \omega$$

such that g maps x to an index of Φ^x , y maps x to an index of $+^x$ and z maps x to an index of \cdot^x . For the three functions it holds, in other words, that

$$\varphi_{g(x)}^1 = \Phi^x, \quad \varphi_{y(x)}^2 = +^x \quad \text{and} \quad \varphi_{z(x)}^2 = \cdot^x$$

Such functions can be easily constructed. (For example, g can be taken by the following procedure: Write down the set of instructions, previously given, for calculating Φ^x . Since it is known that each Turing machine

calculates a unary partial recursive function, find the Turing machine T whose the set of instructions for calculating a unary function coincides with that of Φ^x . Put $g(x) = G(T)$. Functions y and z can be constructed similarly.)

The recursiveness of g , y and z immediately implies the recursiveness of the function

$$w : \omega \rightarrow \omega,$$

given by the following rule:

$$\forall x \in \omega, w(x) = 2^{g(x)} \cdot 3^{y(x)} \cdot 5^{z(x)}.$$

Since, for each $x \in \omega$, if F_n is the set characterised by Φ^x , then $+^x$ and \cdot^x coincide with $+^{\delta_n}$ and \cdot^{δ_n} , respectively, it holds that

$$\langle R_{g(x)}, \phi_{y(x)}^2, \phi_{z(x)}^2 \rangle = \delta_n$$

This fact, together with properties a and b of Φ^x implies that:

- a! $\forall x \in \omega, w(x)$ is the index of a computable field and
- b! $\forall x \in \omega, R_{g(x)}$ is finite if and only if the Turing machine $G^{-1}(x)$, given input x , never halts.

Reconsider now our original problem of pg.71. If A and B are the sets, defined there, properties a' and b' can be rewritten as follows:

$$a'' \cdot \forall x \in \omega, w(x) \in A$$

$$b'' \cdot \forall x \in \omega, w(x) \in B \Leftrightarrow G^{-1}(x), \text{ given input } x, \text{ never halts.}$$

Suppose that a partial recursive function ψ , with properties i. and ii., exists.

Then, properties i and a'' imply that the function

$$\psi \circ w$$

is defined on ω , while properties ii and b'' imply that:

$$\left\{ \begin{array}{l} \psi \circ w(x) = 1 \text{ if } G^{-1}(x), \text{ given input } x, \text{ doesn't halt} \\ \psi \circ w(x) = 0 \text{ if } G^{-1}(x), \text{ given input } x, \text{ halts} \end{array} \right.$$

$\psi \circ w$ is, thus, the characteristic function of the set

$$K = \{x \in \omega : G^{-1}(x), \text{ given input } x, \text{ never halts}\},$$

which must be recursive.

On the other hand, the non-recursiveness of K (i.e. the recursive unsolvability of the halting problem) is a well-known fact (see Rogers' [24] pg 25).

Assuming that ψ exists, we derive a contradiction; so, such a ψ doesn't exist. This implies the non-existence of a procedure for deciding whether a field is finite. \square

§ 2.2. The undecidability of properties P_6 , P_7 and P_8

In this section we establish the undecidability of properties P_6 , P_7 and P_8 , for recursive sets of equations of strong languages with infinitely many operation symbols. We obtain the results by effectively reducing the problem of the existence of an algorithm for deciding whether a computable field is finite to the decision problems of P_i 's and by making use of the fact that the former algorithm doesn't exist:

Let

$$\mathcal{L} = \langle +, - \rangle$$

be the language with a binary operation symbol $+$ and a unary operation symbol $-$. For each field

$$\mathcal{F} = \langle F, +, \cdot, \mathcal{F} \rangle,$$

let us consider the language

$$\mathcal{L}_{\mathcal{F}} = \langle +, -, \{f_{\lambda}\}_{\lambda \in F} \rangle,$$

taken from \mathcal{L} by adding a new unary operation symbol f_{λ} , for each $\lambda \in F$. The theory of vector spaces over \mathcal{F} can be viewed as the first order $\mathcal{L}_{\mathcal{F}}$ -theory, that is generated by the following set of equations:

$$\begin{aligned} V_{\mathcal{F}} = & \{ (\forall v_0 v_1) (v_0 + v_1 = v_1 + v_0), (\forall v_0 v_1 v_2) ((v_0 + v_1) + v_2 = v_0 + (v_1 + v_2)), \\ & (\forall v_0 v_1) (v_0 + (-v_0) = v_1 + (-v_1)), (\forall v_0 v_1) (v_0 + (v_1 + (-v_1)) = v_0) \} \cup \\ & \cup \{ (\forall v_0) (f_{\lambda_1 + \mathcal{F}\lambda_2} v_0 = f_{\lambda_1} v_0 + f_{\lambda_2} v_0) : \langle \lambda_1, \lambda_2 \rangle \in {}^2F \} \cup \\ & \cup \{ (\forall v_0) (f_{\lambda_1 \cdot \mathcal{F}\lambda_2} v_0 = f_{\lambda_1} f_{\lambda_2} v_0) : \langle \lambda_1, \lambda_2 \rangle \in {}^2F \} \cup \\ & \cup \{ (\forall v_0 v_1) (f_{\lambda} (v_0 + v_1)) = f_{\lambda} v_0 + f_{\lambda} v_1 : \lambda \in F \} \cup \\ & \cup \{ (\forall v_0) (f_1 v_0 = v_0) \}. \end{aligned}$$

An $\mathcal{L}_{\mathcal{F}}$ -model of $V_{\mathcal{F}}$ is a vector space over \mathcal{F} . If \mathcal{A} is a vector space over \mathcal{F} , any linearly independent set of generators of it is a basis of \mathcal{A} . All bases of \mathcal{A} are known to have the same cardinality, which is called the dimension of \mathcal{A} and it is denoted by $\dim \mathcal{A}$.

In the course of proving our basic Theorem 2.2.2., we shall make

use of the following two Lemmas, proofs of which can be found in any standard text of Algebra:

Lemma 2.2.0. The following conditions hold:

- a. Any two vector spaces over \mathfrak{F} , with the same dimension, are isomorphic
- b. If \mathfrak{U} is any infinite vector space over \mathfrak{F} , then

$$|\mathfrak{U}| = |\mathfrak{F}| \cdot \dim \mathfrak{U} .$$

Lemma 2.2.1. Consider the $\mathcal{L}_{\mathfrak{F}}$ -structure

$$\bar{\mathfrak{F}} = \langle F, +_{\mathfrak{F}}, -_{\mathfrak{F}}, \{f_{\lambda}^{\mathfrak{F}}\}_{\lambda \in F} \rangle ,$$

with $f_{\lambda}^{\mathfrak{F}}$ defined by the rule

$$\forall \lambda' \in F, f_{\lambda}^{\mathfrak{F}}(\lambda') = \lambda \cdot \bar{\mathfrak{F}} \lambda' .$$

Then, for each natural number n different from zero, the $\mathcal{L}_{\mathfrak{F}}$ -structure

$${}^n \bar{\mathfrak{F}} = \bar{\mathfrak{F}} \times \bar{\mathfrak{F}} \times \dots \times \bar{\mathfrak{F}}$$

is a vector space over \mathfrak{F} with dimension n .

Theorem 2.2.2. The following three conditions hold:

- a. $V_{\mathfrak{F}}$ has non-trivial finite models if and only if \mathfrak{F} is finite
- b. The first-order theory of the non-trivial models of $V_{\mathfrak{F}}$ is complete if and only if \mathfrak{F} is infinite
- c. The first-order theory of the non-trivial models of $V_{\mathfrak{F}}$ is model-complete if and only if \mathfrak{F} is infinite.

Proof

Condition a. If \mathfrak{F} is finite, then $\bar{\mathfrak{F}}$ is a finite non-trivial model of $V_{\mathfrak{F}}$.

Suppose, conversely, that \mathfrak{F} is infinite. Any non-trivial finite model \mathfrak{U} of $V_{\mathfrak{F}}$ would, obviously, have a finite dimension. So, by Lemmas 2.2.1. and 2.2.0.a., it would be isomorphic with ${}^n \bar{\mathfrak{F}}$. \mathfrak{U} should, thus, be infinite, which contradicts the hypothesis that \mathfrak{U} is finite. $V_{\mathfrak{F}}$ has, thus, no finite non-trivial models.

Conditions b. and c. We denote by $|F|^{+}$ the smallest cardinal number which is greater than the cardinality of F . We shall prove the following:

Claim $V_{\mathcal{F}}$ is $\max\{\omega, |F|^+\}$ -categorical.

Proof of the claim

We distinguish two cases:

Case 1 $\max\{\omega, |F|^+\} = \omega$

Let \mathcal{U} be any model of $V_{\mathcal{F}}$, of cardinality ω . If \mathcal{U} had a finite dimension, then it would be isomorphic with the vector spaces

$$\dim_{\mathcal{F}} \mathcal{U},$$

through Lemmas 2.2.1. and 2.2.0.a. But, since $|F|^+$ is at most equal with ω , $|F|$ is a natural number. \mathcal{U} would have, thus, cardinality

$$|F|^{\dim \mathcal{U}} \in \omega,$$

which would contradict the hypothesis that $|A| = \omega$.

The dimension of \mathcal{U} is, thus, an infinite number smaller than its cardinality. We have proved, thus, that

$$(\forall \mathcal{U} \models V_{\mathcal{F}} \text{ with } |A| = \omega) (\dim \mathcal{U} = \omega) \quad (1)$$

Case 2 $\max\{\omega, |F|^+\} = |F|^+ > \omega$

Let, again, \mathcal{U} be any model of $V_{\mathcal{F}}$ of cardinality $|F|^+$. If \mathcal{U} had a finite dimension, then its cardinality would be equal to

$$|F|^{\dim \mathcal{U}}$$

But $|F|$ is at least as large as ω ; so it would hold that

$$|F|^{\dim \mathcal{U}} = |F| \quad (2)$$

From (2) we would deduce that the cardinality of \mathcal{U} would be equal to $|F|$ and we would derive a contradiction. \mathcal{U} has, thus, infinite dimension not greater than its cardinality. I.e. it holds that

$$\omega \leq \dim \mathcal{U} \leq |F|^+ \quad (3)$$

Also, from Lemma 2.2.0.b. we deduce that

$$|A| = \dim \mathcal{U} \cdot |F| = |F|^+. \quad (4)$$

Relations (3) and (4) and cardinal arithmetic (see Suppes [26]) imply that

$$\dim \mathcal{U} = |F|^+$$

We have proved that

$$(\forall \mathfrak{A} \models V_{\mathfrak{F}} \text{ with } |A| = |F|^+)(\dim \mathfrak{A} = |F|^+) \quad (5)$$

Relations (1) and (5) say that any two vector spaces over \mathfrak{F} , of cardinality $\max\{\omega, |F|^+\}$, have the same dimension. They are, thus, by Lemma 2.2.0.a., isomorphic. Hence, $V_{\mathfrak{F}}$ is $\max\{\omega, |F|^+\}$ -categorical and we are done. \square

The first order theory of the non-trivial models of $V_{\mathfrak{F}}$ can, obviously, be considered as the theory

$$\Phi = \emptyset [V_{\mathfrak{F}} \cup \{(\exists v_0 v_1)(v_0 \neq v_1)\}]$$

We have, already, proved that, if \mathfrak{F} is infinite, the following two conditions hold:

- i. Φ has no finite models and
- ii. Φ is $\max\{\omega, |F|^+\}$ -categorical.

This, together with Theorem 0.1.0., implies that Φ is a complete first order theory. Since Φ is axiomatizable just by $\forall \exists$ -sentences, conditions i and ii, together with Theorems 0.1.1. and 0.1.2., imply that Φ is, also, model-complete. We have, thus, proved direction \leftarrow of conditions b and c of the theorem.

We prove the converse direction.

Suppose that \mathfrak{F} is finite. We consider the vector space ${}^3\overline{\mathfrak{F}}$ and its basis

$$\{ \langle 1, 0, 0 \rangle, \langle 0, 1, 0 \rangle, \langle 0, 0, 1 \rangle \},$$

The substructure \mathfrak{A} of ${}^3\overline{\mathfrak{F}}$, generated by the set

$$\{ \langle 1, 0, 0 \rangle, \langle 0, 1, 0 \rangle \},$$

has dimension 2; it is, thus, a proper substructure of ${}^3\overline{\mathfrak{F}}$. Since ${}^3\overline{\mathfrak{F}}$ has $|F|^3$ elements, it holds that

$${}^3\overline{\mathfrak{F}} \models (\exists x_1 x_2 \dots x_{|F|^3}) (x_1 \neq x_2 \neq \dots \neq x_{|F|^3})$$

and

$$\mathfrak{A} \not\models (\exists x_1 x_2 \dots x_{|F|^3}) (x_1 \neq x_2 \neq \dots \neq x_{|F|^3})$$

We have, thus, found two non-trivial models \mathfrak{A} and \mathfrak{B} of $V_{\mathfrak{F}}$ and a sentence σ of $\mathcal{L}_{\mathfrak{F}}$, such that

$$(\mathfrak{A} \subset \mathfrak{B}) \wedge (\mathfrak{B} \models \sigma) \wedge (\mathfrak{A} \not\models \sigma).$$

This means that the theory of the non-trivial models of $V_{\mathfrak{F}}$ is neither complete nor model-complete; so we are done. \square

What we have just proved together with the fact that there is no algorithm, that enables us to decide whether a computable field is finite, helps us establish our undecidability results:

Theorem 2.2.3. Let \mathcal{L}' be any strong algebraic language, with infinitely many operation symbols. There is no algorithm that enables us to decide whether a recursive $\Sigma \subset \text{Eq}_{\mathcal{L}'}$ has each of the following properties:

P_6 : Σ has finite non-trivial models

P_7 : The first-order theory of the non-trivial models of Σ is complete

P_8 : The first-order theory of the non-trivial models of Σ is model-complete.

Proof

Let

$$\mathcal{L}' = \langle Q, \{Q_n\}_{n \in \omega}, \{c_j\}_{j \in J} \rangle$$

be any language, with at least one operation symbol Q of rank greater than one, denumerably many operation symbols Q_n of arbitrary rank and at most denumerably many constant symbols c_j .

For each computable field

$$\mathfrak{F} = \langle F, +_{\mathfrak{F}}, \cdot_{\mathfrak{F}} \rangle,$$

let us consider the system $\delta_{\mathfrak{F}}$ of definitions for $\mathcal{L}_{\mathfrak{F}}$ in \mathcal{L}' , which is given by the following rule:

$$\delta_{\mathcal{F}}(+)=Qv_0v_1v_1\dots v_1$$

$$\delta_{\mathcal{F}}(-)=Q_0v_0v_0\dots v_0$$

$$\delta_{\mathcal{F}}(f_\lambda)=Q_{\lambda+1}v_0v_0\dots v_0, \forall \lambda \in F.$$

As we have seen on pg 34, a function

$$\text{in}_{\delta_{\mathcal{F}}} : \text{Term}_{\mathcal{L}} \rightarrow \text{Term}_{\mathcal{L}},$$

is, then, defined.

If $V_{\mathcal{F}}$ is an defined on pg 78, let us consider the set of \mathcal{L}' -equations

$$\begin{aligned} V'_{\mathcal{F}} = & \{(\forall \bar{v})(\text{in}_{\delta_{\mathcal{F}}} \varphi = \text{in}_{\delta_{\mathcal{F}}} \psi) : (\forall \bar{v})(\varphi = \psi) \in V_{\mathcal{F}}\} \cup \\ & \cup \{(\forall \bar{v})(Q\bar{v} = \text{in}_{\delta_{\mathcal{F}}}(v_0 + v_1)), (\forall \bar{v})(Q_0\bar{v} = \text{in}_{\delta_{\mathcal{F}}}(-v_0))\} \cup \\ & \cup \{(\forall \bar{v})(Q_{\lambda+1}\bar{v} = \text{in}_{\delta_{\mathcal{F}}}(f_\lambda v_0)) : \lambda \in F\} \cup \{(\forall \bar{v})(Q_{\lambda+1}\bar{v} = v_0) : \lambda \notin F\} \cup \\ & \cup \{(\forall v_0)(c_j = \text{in}_{\delta_{\mathcal{F}}}(v_0 + (-v_0))) : j \in J\} \end{aligned}$$

For each

$$\mathcal{A} = \langle A, +^{\mathcal{A}}, -^{\mathcal{A}}, \{f_\lambda^{\mathcal{A}}\}_{\lambda \in F} \rangle$$

\mathcal{L}' -model of $V'_{\mathcal{F}}$, let us define the \mathcal{L}' -structure

$$\mathcal{A}' = \langle A, Q^{\mathcal{A}'}, \{Q_n^{\mathcal{A}'}\}_{n \in \omega}, \{c_j^{\mathcal{A}'}\}_{j \in J} \rangle$$

as follows:

$$\left. \begin{aligned} \forall \bar{a} \in r(Q)A, & \quad Q^{\mathcal{A}'}(\bar{a}) = a_0 +^{\mathcal{A}'} a_1 \\ \forall \bar{a} \in r(Q_0)A, & \quad Q_0^{\mathcal{A}'}(\bar{a}) = -^{\mathcal{A}'} a_0 \\ \forall \bar{a} \in r(Q_{\lambda+1})A, & \quad Q_{\lambda+1}^{\mathcal{A}'}(\bar{a}) = f_\lambda^{\mathcal{A}'}(a_0) \quad \lambda \in F \\ \forall \bar{a} \in r(Q_{\lambda+1})A, & \quad Q_{\lambda+1}^{\mathcal{A}'}(\bar{a}) = a_0 \quad \lambda \notin F \\ & \quad c_j^{\mathcal{A}'} = 0^{\mathcal{A}'} \quad j \in J \end{aligned} \right\} (1)$$

(We have denoted by $0^{\mathcal{A}'}$ the element of A that is equal with $a +^{\mathcal{A}'}(-^{\mathcal{A}'}a)$, for all $a \in A$).

It can be easily checked that \mathcal{A}' is an \mathcal{L}' -model of $V'_{\mathcal{F}}$.

On the other hand, for each

$$\mathcal{A}' = \langle A, Q^{\mathcal{A}'}, \{Q_n^{\mathcal{A}'}\}_{n \in \omega}, \{c_j^{\mathcal{A}'}\}_{j \in J} \rangle$$

\mathcal{L}' -model of $V_{\mathcal{F}}$, let us define the $\mathcal{L}_{\mathcal{F}}$ -structure

$$\mathcal{M} = \langle A, +^{\mathcal{M}}, -^{\mathcal{M}}, \{f_{\lambda}^{\mathcal{M}}\}_{\lambda \in F} \rangle$$

as follows:

$$\left. \begin{aligned} \forall \langle a, b \rangle \in {}^2A, \quad a +^{\mathcal{M}} b &= Q^{\mathcal{M}'}(a, b, b, \dots, b) \\ \forall a \in A, \quad -a^{\mathcal{M}} &= Q_0^{\mathcal{M}'}(a, a, \dots, a) \\ \forall a \in A, \quad f_{\lambda}^{\mathcal{M}}(a) &= Q_{\lambda+1}^{\mathcal{M}'}(a, a, \dots, a) \quad \lambda \in F \end{aligned} \right\} (2)$$

Again, it is easy to verify that \mathcal{M} is an $\mathcal{L}_{\mathcal{F}}$ -model of $V_{\mathcal{F}}$.

Let $\text{Mod } V_{\mathcal{F}}$ and $\text{Mod } V_{\mathcal{F}}'$ be the classes of $\mathcal{L}_{\mathcal{F}}$ -models of $V_{\mathcal{F}}$ and \mathcal{L}' -models of $V_{\mathcal{F}}'$, respectively. We are going to prove the following:

Claim 1. The function

$$f : \text{Mod } V_{\mathcal{F}} \rightarrow \text{Mod } V_{\mathcal{F}}',$$

that maps \mathcal{M} to the structure \mathcal{M}' , given by relations (1), is a bijection.

Proof of the claim

We prove that f is injective; i.e. that it holds:

$$\mathcal{M} \neq \mathcal{N} \rightarrow f(\mathcal{M}) \neq f(\mathcal{N}) \quad (3)$$

If \mathcal{M} is different from \mathcal{N} , either they have different universes or the interpretations of at least one operation symbol in the two structures are different. In the first case, \mathcal{M}' and \mathcal{N}' will have different universes, while, in the second case, the corresponding, through relations (1), \mathcal{L}' -operation symbol will have different interpretations in \mathcal{M}' and \mathcal{N}' . In both cases, $f(\mathcal{M})$ differs from $f(\mathcal{N})$. We have, thus, proved relation (3).

If \mathcal{M}' is an \mathcal{L}' -model of $V_{\mathcal{F}}'$, then it can be easily verified that the $\mathcal{L}_{\mathcal{F}}$ -model of $V_{\mathcal{F}}$, which is given by relations (2), has the property

$$f(\mathcal{M}) = \mathcal{M}'.$$

f is, thus, surjective. This completes the proof of the claim. \square

Claim 2. The following conditions hold:

- a. $V_{\mathcal{F}}'$ has property P_6 if and only if \mathcal{F} is finite

- b. $V_{\mathcal{F}}$ has each of properties P_7 and P_8 if and only if \mathcal{F} is infinite.

Proof of the claim

Condition a. Since \mathcal{A} and $f(\mathcal{A})$ have, by construction, the same universe and because of the fact that f is bijective, it holds that

$$V'_{\mathcal{F}} \text{ has property } P_6 \rightarrow V_{\mathcal{F}} \text{ has property } P_6 \quad (4)$$

Relation (4) and Theorem 2.2.2.a. imply that

$$V'_{\mathcal{F}} \text{ has property } P_6 \Leftrightarrow \mathcal{F} \text{ is finite.}$$

Condition b. and c. Let \mathcal{A} and \mathcal{B} be any two $L_{\mathcal{F}}$ -models of $V_{\mathcal{F}}$ and let

$$i : \mathcal{A} \rightarrow \mathcal{B}$$

be a bijection. It is very easy to show that, if i is an isomorphism between \mathcal{A} and \mathcal{B} , then it is also an isomorphism between the structures \mathcal{A}' and \mathcal{B}' , which correspond to \mathcal{A} and \mathcal{B} , through formation rules (1). It holds, thus, that

$$(\forall \langle \mathcal{A}, \mathcal{B} \rangle \in {}^2\text{Mod } V_{\mathcal{F}}) (\mathcal{A} \cong \mathcal{B} \rightarrow f(\mathcal{A}') \cong f(\mathcal{B}')) \quad (5)$$

In the course of proving Theorem 2.2.2. we have shown that $V_{\mathcal{F}}$ is $\max\{\omega, |F|^+\}$ -categorical. If \mathcal{A}' and \mathcal{B}' are any two models of $V'_{\mathcal{F}}$ of cardinality $\max\{\omega, |F|^+\}$, then $f^{-1}(\mathcal{A}')$ and $f^{-1}(\mathcal{B}')$ are models of $V_{\mathcal{F}}$ with cardinality $\max\{\omega, |F|^+\}$. They are, thus, isomorphic. So are, because of (5), their images through f . Hence, it holds that

$$\mathcal{A}' = f(f^{-1}(\mathcal{A}')) \cong f(f^{-1}(\mathcal{B}')) = \mathcal{B}'$$

We have proved that $V'_{\mathcal{F}}$ is also $\max\{\omega, |F|^+\}$ -categorical.

Suppose, now, that \mathcal{F} is infinite. Then, Theorem 2.2.2. and condition a. of this theorem imply that $V'_{\mathcal{F}}$ has no finite non-trivial models. This fact, combined with the $\max\{\omega, |F|^+\}$ -categoricity of $V'_{\mathcal{F}}$ and Theorem 0.1.0., imply that $V'_{\mathcal{F}}$ has the property P_7 . If we take into account the fact that the first order theory of the non-trivial models of $V'_{\mathcal{F}}$ is axiomatizable by $\forall \exists$ -sentences, we derive from Theorems 0.1.1. and 0.1.2. that $V'_{\mathcal{F}}$ has, also, the property P_8 . Hence, we have proved that:

\mathcal{F} is finite $\rightarrow V_{\mathcal{F}}$ has properties P_7 and P_8

Direction \leftarrow of the claim has, thus, been proved.

We prove the other direction.

Suppose that \mathcal{F} is finite. Then, as we have seen on pg 81, there exist two $\mathcal{L}_{\mathcal{F}}$ -models of $V_{\mathcal{F}}$ \mathcal{A} and \mathcal{B} , such that \mathcal{A} is a substructure of \mathcal{B} , $\|\mathcal{A}\| < |\mathcal{F}|^3$ and $\|\mathcal{B}\| = |\mathcal{F}|^3$

For the images of \mathcal{A} and \mathcal{B} through f , it also holds that

$$(f(\mathcal{A}) \subset f(\mathcal{B})) \wedge (\|f(\mathcal{A})\| < |\mathcal{F}|^3) \wedge (\|f(\mathcal{B})\| = |\mathcal{F}|^3)$$

This, obviously, implies that $V_{\mathcal{F}}$ has neither property P_7 nor property P_8 .

The proof of condition b of the claim is now complete. \square

Up to this point, we have managed to relate with each computable field \mathcal{F} a set $V_{\mathcal{F}}$ of \mathcal{L}' -equations and to find conditions under which it has the required properties.

The computability of \mathcal{F} , obviously, implies the decidability $V_{\mathcal{F}}$; this fact and the computability of $\delta_{\mathcal{F}}$ imply that $V_{\mathcal{F}}$ is decidable. Consequently, if

$$g_{\mathcal{L}'}^* : \text{Exp}_{\mathcal{L}'} \rightarrow \omega$$

is the goedel numbering of the set of expressions of \mathcal{L}' , defined of pg 19, the set

$$g_{\mathcal{L}'}^*[V_{\mathcal{F}}]$$

is a recursive subset of ω .

In the introductory chapter (pg 29), we gave a mathematical formulation of the decision problem of a property p of recursive sets of \mathcal{L} -equations. Under the conventions made in §2.1., the decision problems of P_6 , P_7 and P_8 are the problems of the existence or not of a unary partial recursive function φ , with the properties:

- i. $\forall x \in C$, $\varphi(x)$ is defined
- ii. $\begin{cases} \forall x \in D, \varphi(x) = 1 \\ \forall x \in C-D, \varphi(x) = 0, \end{cases}$

where

$$C = \{x : R_x = g^*[\Sigma] \text{ for some } \Sigma \in Eq_{\mathcal{L}}\}$$

and

$$D = \{x : R_x = g^*[\Sigma] \text{ for some } \Sigma \in Eq_{\mathcal{L}} \text{ with the property: } P_6 \text{ (or } \neg P_7 \text{ or } \neg P_8) \}$$

Let A , B and i are as defined on pg 70. A partial recursive function f , with domain A , that maps each $w \in A$ to the index (see pg 70) of the recursive set

$$g_{\mathcal{L}'}^*[V'_{i^{-1}(w)}]$$

can be constructed, in the obvious way. It holds, thus, for f that

$$\forall w \in A, R_w = g_{\mathcal{L}'}^*[V'_{i^{-1}(w)}]$$

Suppose that a partial recursive function φ , with properties i and ii , given above, exists. Since, for each $w \in A$, $f(w)$ is the index of $g_{\mathcal{L}'}^*[V'_{i^{-1}(w)}]$, for the unary partial recursive function

$$\varphi \circ f$$

it holds that

$$i'. \forall w \in A, \varphi \circ f(w) \text{ is defined}$$

Also claim 2 implies that, if w is the index of a finite field, then $f(w)$ is in D while, if w is the index of an infinite field, then $f(w)$ is in $C-D$. Combining this fact with the properties of φ we derive that

$$ii. \begin{cases} \forall w \in A \cap B, \varphi \circ f(w) = 1 \\ \forall w \in A - B, \varphi \circ f(w) = 0 \end{cases}$$

As we have seen on pg 71, a function with properties i' and ii' provides a procedure for deciding whether a computable field is finite. This contradicts Theorem 2.1.0. We conclude, thus, that φ doesn't exist.

This fact proves that properties P_6 , P_7 and P_8 are undecidable for recursive sets of \mathcal{L}' -equations. \square

In McKenzie's [12] it is proved that, in every strong algebraic language, there is no algorithm which enables us to decide whether a

single equation ε has finite non-trivial models. This results, obviously, implies ours. As we have seen on pg 37, the techniques by which the two results are obtained have no similarity. We have included our weaker result (Theorem 2.2.3.a.) in the thesis, in order to show the range of application of our method. At any rate, we had to follow the same procedure in order to establish the remaining undecidability results, of this section.

The undecidability results concerning properties P_7 and P_8 , which have been obtained in this section, are new in the literature.

The complexity of the language, in which the results are taken, is very high, though. This fact, certainly diminishes the significance of them. Our belief is that they can be extended in any strong language, by the following procedure:

As McNulty has shown in [13], for each computable field \mathcal{F} , a universal system $\delta_{\mathcal{F}}$ of definitions for $\mathcal{L}_{\mathcal{F}}$ in \mathcal{L} , with respect to $\{(\forall v_0)(v_0 = v_0)\}$, exists. If we manage to effectively associate, with each such \mathcal{F} , a suitable $\delta_{\mathcal{F}}$ and a recursive extension $V'_{\mathcal{F}}$ of $V_{\mathcal{F}}$ so that

$$(V'_{\mathcal{F}} \text{ has the property } P_i) \Leftrightarrow (V_{\mathcal{F}} \text{ has the property } P_i)$$

then, we will derive the undecidability of properties P_7 and P_8 , for recursive sets of \mathcal{L} -equations.

Unfortunately, in spite of our intense efforts, we have not been able to find these $\delta_{\mathcal{F}}$'s. We hope someone else will succeed in doing this.

The fact that a property P is undecidable for recursive sets of \mathcal{L} -equations implies no immediate answer to the decision problem of the property, for finite sets of \mathcal{L} -equations.

In the special case of P_7 and P_8 , there is not even hope that one can establish the undecidability of them for finite sets, following the same (or a similar) method that was used for recursive sets. This is because the method is based on the distinction between finite and infinite i.e. because it is based on the fact that

$$V'_{\mathcal{F}} \text{ has } P_i \Leftrightarrow \mathcal{F} \text{ is infinite} \Leftrightarrow V_{\mathcal{F}} \text{ is infinite}$$

Another method should, thus, be found. Maybe this of chapter 1, provided that we have been able to find finite sets Σ with the property P_i (For trivial languages, it will be shown, in chapter 3, that such

sets don't exist). The following problem is, thus, open:

Problem : Investigate the decision problem of P_7 and P_8 , for finite sets of equations of non-trivial languages.

CHAPTER 3

TRIVIAL LANGUAGES

§3.0. Introduction

In the previous chapters, the decision problems of properties

\underline{P} : the first-order theory of Σ is complete,

\overline{P} : the first-order theory of Σ is model-complete,

P_0 : the equational theory of Σ is equationally complete,

P_1 : the first-order theory of the infinite models of Σ is complete,

P_2 : the first-order theory of the infinite models of Σ is model-complete,

P_3 : Σ has the joint embedding property,

P_4 : the first-order theory of the non-trivial models of Σ has the joint embedding property,

P_5 : the first-order theory of the infinite models of Σ has the joint embedding property,

P_6 : Σ has no finite non-trivial models,

P_7 : the first-order theory of the non-trivial models of Σ is complete and

P_8 : the first-order theory of the non-trivial models of Σ is model-complete,

for sets Σ of equations of non-trivial algebraic languages, were examined.

In this chapter, we work in trivial algebraic languages \mathcal{L} (i.e. in algebraic languages, with exactly one unary operation symbol and no operation symbols of rank greater than one) and we construct algorithms that enable us to decide whether a finite $\Sigma \subset \text{Eq}_{\mathcal{L}}$ has each of the above mentioned properties.

The decidability of properties \underline{P} , \overline{P} , P_6 , P_7 and P_8 is proved in §3.1. In §3.2. the general procedure followed, in order for the

decision problems of properties $P_0 - P_5$ to be answered, is explained. The base-decidability of any finitely based equational theory, in any trivial language with at most one constant symbol, is proved in §3.3., and, as a consequence, the decidability of properties $P_0 - P_5$, for finite sets of equations of such languages, is established. Finally, § 3.4. deals with the decision problems of $P_0 - P_5$, in languages with more than one constant symbols.

§ 3.1. The base-decidability of $Eq_{\mathcal{L}}$ and the positive solution to the decision problems of $\underline{P}, \overline{P}, P_6, P_7$ and P_8 .

In McNulty's [13], it is proved that, in any non-trivial language, the equational theory $Eq_{\mathcal{L}}$ is base-undecidable. We prove, in this section, that, in any trivial language, $Eq_{\mathcal{L}}$ is a base-decidable equational theory. We use this result in order to establish the decidability of properties $\underline{P}, \overline{P}$ and P_6 , in such languages. The non-existence of equational \mathcal{L} -theories with properties P_7 or P_8 is also proved.

Let

$$\mathcal{L} = \{f\} \cup \{c_j : j \in J\}$$

be any trivial algebraic language. For any \mathcal{L} -term, φ , let V_{φ} be the set of variables, occurring in φ , and let C_{φ} be the set of constant symbols, occurring in φ . Since f is unary, the two sets are at most singletons.

We define four equational theories, as follows:

$$\Phi_1 = \theta_{Eq}[(\forall v_0)(fv_0 = v_0)]$$

$$\overline{\Phi}_1 = \theta_{Eq}[\{(\forall v_0)(fv_0 = v_0)\} \cup \{c_{j_1} = c_{j_2} : \langle j_1, j_2 \rangle \in {}^2J\}]$$

$$\Phi_2 = \theta_{Eq}[(\forall v_0 v_1)(fv_0 = fv_1)]$$

$$\overline{\Phi}_2 = \theta_{Eq}[\{(\forall v_0)(fv_0 = c_j) : j \in J\}]$$

It is obvious that, if $J = \emptyset$, then $\overline{\Phi}_1$ and $\overline{\Phi}_2$ coincide with Φ_1 and Φ_2 , respectively. It is also obvious that, since non-trivial models of the above theories can be easily constructed, they are equationally consistent. We are going to prove that

Theorem 3.1.0. The following relations hold:

1. $\Phi_1 = \{(\forall v)(\varphi = \psi) : (V_{\varphi} = V_{\psi}) \wedge (C_{\varphi} = C_{\psi})\}$
2. $\overline{\Phi}_1 = \{(\forall v)(\varphi = \psi) : V_{\varphi} = V_{\psi}\}$

3. $\Phi_2 = \{(\forall v_i)(v_i = v_i) : i \in \omega\} \cup \{c_j = c_j : j \in J\} \cup$
 $\{(\forall \bar{v})(\varphi = \psi) : \langle \varphi, \psi \rangle \in {}^2(\text{Term}_f - V_a - \{c_j : j \in J\})\}$
4. $\bar{\Phi}_2 = \{(\forall v_i)(v_i = v_i) : i \in \omega\} \cup \{(\forall \bar{v})(\varphi = \psi) : \langle \varphi, \psi \rangle \in {}^2(\text{Term}_f - V_a)\}$

Proof.

We prove relation (1).

From the axioms of derivation for equational logic (pg 21), it follows that the set

$$\{(\forall \bar{v})(\varphi = \psi) : (V\varphi = V\psi) \wedge (C\varphi = C\psi)\}$$

contains its equational consequences. It is, thus, an equational theory and, since $(\forall v_0)(fv_0 = v_0)$ is in it, it holds that

$$\Phi_1 \subseteq \{(\forall \bar{v})(\varphi = \psi) : (V\varphi = V\psi) \wedge (C\varphi = C\psi)\}. \quad (5)$$

On the other hand, the equation $(\forall v_0)(fv_0 = v_0)$ implies the equations

$$(\forall v_i)(f^n v_i = v_i) \quad \text{and} \quad f^n c_j = c_j,$$

for any n , i and j . It implies, thus, the equation

$$(\forall \bar{v})(\varphi = \psi),$$

for any pair $\langle \varphi, \psi \rangle$ with $V\varphi = V\psi$ and $C\varphi = C\psi$.

We have proved that it holds:

$$\Phi_1 \supseteq \{(\forall \bar{v})(\varphi = \psi) : (V\varphi = V\psi) \wedge (C\varphi = C\psi)\}. \quad (6)$$

From (5) and (6), relation (1) follows

Relation (2) is proved in the same way.

We prove relation (3).

The axioms of derivation for equational logic, obviously, imply that no equation of the forms

$$(\forall v)(v_i = \varphi) \quad \text{and} \quad (\forall \bar{v})(c_j = \psi),$$

where φ is a variable different from v_i or a constant, and ψ is a constant different from c_j or a variable, is implied by the set

$$\{(\forall v_i)(v_i = v_i) : i \in \omega\} \cup \{c_j = c_j : j \in J\} \cup \{(\forall \bar{v})(\varphi = \psi) : \langle \varphi, \psi \rangle \in {}^2(\text{Term}_f - V_a - \{c_j : j \in J\})\} \quad (7)$$

This set is, thus, an equational theory which includes Φ_2 .

On the other hand, for any pair $\langle \varphi, \psi \rangle \in {}^2\text{Term}_L$, it holds that

$$(\forall v_0 v_1) (fv_0 = fv_1) \models (\forall \bar{v}) (f\varphi = f\psi). \quad (8)$$

From this we deduce that each equation in the set (7) is implied by the equation $(\forall v_0 v_1) (fv_0 = fv_1)$. We have proved, thus, that the set (7) is included in Φ_2 .

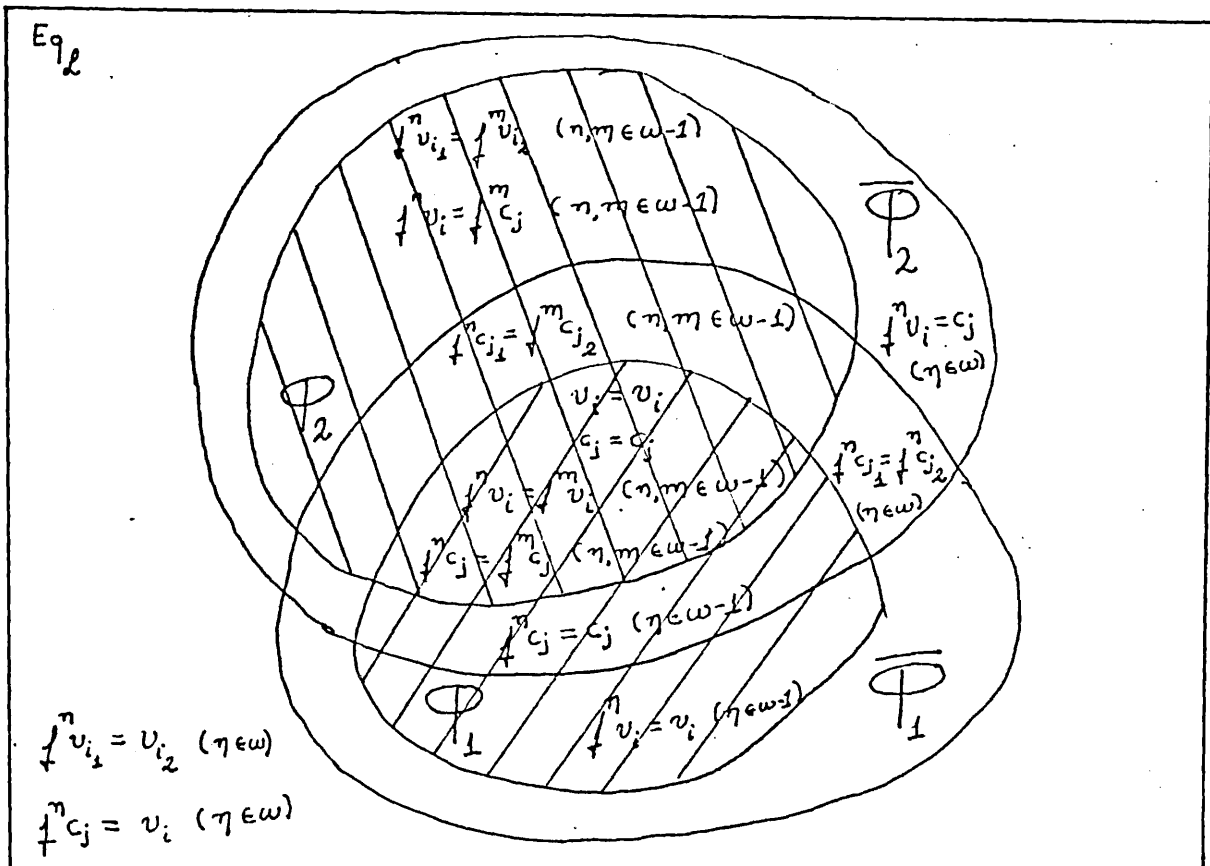
This completes the proof of relation (3).

Relation (4) is proved similarly. □

Procedures for deciding whether an L -equation is in each one of the four theories are tacitly implied by Theorem 3.1.0. and elementary Recursion Theory (see Monk's [15], chapter 10). We have, thus, the following:

Corollary 3.1.1 The equational theories $\Phi_1, \bar{\Phi}_1, \Phi_2$ and $\bar{\Phi}_2$ are decidable.

The diagram below gives the exact position of any L -equation, with respect to $\Phi_1, \Phi_2, \bar{\Phi}_1$ and $\bar{\Phi}_2$. (For simplicity, we omit the universal quantifier and we don't distinguish $\varphi = \psi$ from $\psi = \varphi$):



Making use of this diagram, we are going to prove the following:

Theorem 3.1.2. An equational theory Φ is equationally consistent if and only if it is included in $\bar{\Phi}_1$ or in $\bar{\Phi}_2$.

Proof

Since $\bar{\Phi}_1$ and $\bar{\Phi}_2$ have models of any given cardinality α , so does every subset of each of them. Every subtheory of $\bar{\Phi}_1$ or $\bar{\Phi}_2$ has, thus, non-trivial models; hence, it is equationally consistent

We prove the other direction:

If Φ is not included in any of $\bar{\Phi}_i$'s, then, either it contains an equation not in $\bar{\Phi}_1 \cup \bar{\Phi}_2$, or it is included in $\bar{\Phi}_1 \cup \bar{\Phi}_2$ but it has a non-empty intersection with both $\bar{\Phi}_1 - \bar{\Phi}_2$ and $\bar{\Phi}_2 - \bar{\Phi}_1$.

In the first case, since any equation not in $\bar{\Phi}_1 \cup \bar{\Phi}_2$ is of the form

$$(\forall v_{i_1} \forall v_{i_2}) (f^n v_{i_1} = v_{i_2})$$

or

$$(\forall v_i) (f^n c_j = v_i),$$

for some $n \in \omega$, Φ must contain an equation of the above form. Clearly, such an equation has only trivial models. So does Φ . We have proved, thus, that

$$\Phi = \text{Eq}_f.$$

In the second case, either, for some n, n_1, m_1 in $\omega - 1$,

$$\{(\forall v_0) (f^n v_0 = v_0), (\forall \bar{v}) (f^{n_1} v_0 = f^{m_1} v_1)\} \subset \Phi, \quad (1)$$

or, for some n, n_1 in $\omega - 1$ and for some constant term φ ,

$$\{(\forall v_0) (f^n v_0 = v_0), (\forall v_0) (f^{n_1} v_0 = \varphi)\} \subset \Phi. \quad (2)$$

Relation (1) implies that

$$\{(\forall v_0) (f^{nn_1} v_0 = v_0), (\forall \bar{v}) (f^{nn_1} v_0 = f^{nm_1} v_1)\} \subset \Phi,$$

while relation (2) implies that

$$\{(\forall v_0) (f^{nn_1} v_0 = v_0), (\forall v_0) (f^{nn_1} v_0 = f^n \varphi)\} \subset \Phi.$$

Either

$$(\forall v) (v_0 = f^{nm} v_1)$$

or

$$(\forall v) (v_0 = f^n \phi)$$

is, thus, in Φ . Since these equations have both only trivial models, so does Φ . From this we conclude that

$$\Phi = \text{Eq}_{\mathcal{L}} \quad \square$$

The reader is reminded, at this point, that an equational theory Φ is called base-decidable if there exists an algorithm that enables us to decide whether, given any finite $\Sigma \subset \text{Eq}_{\mathcal{L}}$, Σ is a basis of Φ . We have, thus, as an immediate consequence of Theorem 3.1.2., the following:

Corollary 3.1.3. In any trivial language \mathcal{L} , the equational theory $\text{Eq}_{\mathcal{L}}$ is base-decidable.

We observe now that, since $\bar{\Phi}_1$ and $\bar{\Phi}_2$ have finite non-trivial models extendable to infinite models of them, every equationally consistent equational theory has also, by Theorem 3.1.2., finite non-trivial models that can be extended to infinite models of it. We conclude, thus, that no set of \mathcal{L} -equations has the first-order theory of its non-trivial models complete or model-complete.

We also conclude that the only first-order theory axiomatizable by \mathcal{L} -equations, that either has no finite non-trivial models or is complete or is model-complete, is $\Theta[\text{Eq}_{\mathcal{L}}]$.

Combining the remarks, made above, with Corollary 3.1.3. we finally derive the following:

Corollary 3.1.4. In any trivial language \mathcal{L} , properties \underline{P} , \bar{P} , P_6 , P_7 and P_8 are decidable, for finite sets of \mathcal{L} -equations.

§ 3.2. The decision problems of properties $P_0 - P_5$.

Towards examining the decision problems of properties $P_0 - P_5$ in any trivial language \mathcal{L} , we give below (Theorem 3.2.0.) necessary and sufficient conditions in order for a set of \mathcal{L} -equations to have each one of properties P_i ($i \in \{0,1,2,3,4,5\}$):

Theorem 3.2.0. Let \mathcal{L} be any trivial language and let Σ be any set of \mathcal{L} -equations. Then, the following hold:

- a. Σ has the property $P_0 \Leftrightarrow (\theta_{Eq}[\Sigma] = \bar{\Phi}_1) \vee (\theta_{Eq}[\Sigma] = \bar{\Phi}_2)$.
- b. Σ has the property $P_1 \Leftrightarrow (\theta_{Eq}[\Sigma] = \bar{\Phi}_1) \vee (\theta_{Eq}[\Sigma] = \bar{\Phi}_2)$.
- c. Σ has the property $P_2 \Leftrightarrow (\Phi_1 \subset \theta_{Eq}[\Sigma] \subset \bar{\Phi}_1) \vee (\Phi_2 \subset \theta_{Eq}[\Sigma] \subset \bar{\Phi}_2)$
- d. If \mathcal{L} has no constant symbols, then all Σ 's have the property P_3 . Otherwise,
 Σ has the property $P_3 \Leftrightarrow \Sigma \models \{fc_j = c_j : j \in J\} \cup \{c_{j_1} = c_{j_2} : \langle j_1, j_2 \rangle \in {}^2J\}$.
- e. If \mathcal{L} has no constant symbols, then
 Σ has the property $P_4 \Leftrightarrow \Sigma$ is equationally consistent.
 Otherwise,
 Σ has the property $P_4 \Leftrightarrow (\Sigma \text{ is equationally consistent}) \wedge$
 $\wedge (\Sigma \models \{fc_j = c_j : j \in J\} \cup \{c_{j_1} = c_{j_2} : \langle j_1, j_2 \rangle \in {}^2J\})$.
- f. If \mathcal{L} has no constant symbols, then
 Σ has the property $P_5 \Leftrightarrow \Sigma$ is equationally consistent.
 Otherwise,
 Σ has the property $P_5 \Leftrightarrow (\Sigma \text{ is equationally consistent}) \wedge$
 $\wedge (\Sigma \models \{fc_j = c_j : j \in J\} \cup \{c_{j_1} = c_{j_2} : \langle j_1, j_2 \rangle \in {}^2J\})$.

Proof

We prove relations a. and b.

In Theorem 1.2.0., we proved that $\bar{\Phi}_1$ has properties P_1 and P_2 . Similarly, we can prove that $\bar{\Phi}_2$ has the two properties.

On the other hand, if Σ is neither a basis of $\bar{\Phi}_1$ nor a basis of $\bar{\Phi}_2$, theorem 3.1.2. shows that either

$$\theta_{Eq}[\Sigma] = Eq_{\mathcal{L}} \text{ or } \theta_{Eq}[\Sigma] \subsetneq \bar{\Phi}_1 \text{ or } \theta_{Eq}[\Sigma] \subsetneq \bar{\Phi}_2$$

So, Σ doesn't have the property P_0 and, because of lemma 1.2.1., it doesn't have the property P_1 , either. This completes the proof of relations a. and b. \square

We prove relation c.

In theorem 1.2.3., we proved that any Σ , such that

$$\Phi_1 \subset \theta_{Eq}[\Sigma] \subset \bar{\Phi}_1,$$

has the property P_2 . In a similar way, one can prove that any Σ , with

$$\Phi_2 \subset \theta_{Eq}[\Sigma] \subset \bar{\Phi}_2,$$

has the property P_2 . Direction \leftarrow of relation c has, thus, been shown.

The proof of direction \rightarrow of relation c will be given later, on pg 121. We consider this necessary, because, otherwise, a series of lemmas should be given at this point, which might make the understanding of the general procedure difficult. \square

We prove relations d, e and f.

Suppose, firstly, that $\mathcal{L} = \langle f \rangle$.

If $\theta_{Eq}[\Sigma]$ is equationally inconsistent, then it, obviously, has property P_3 but not properties P_4 and P_5 .

If $\theta_{Eq}[\Sigma]$ is equationally consistent but it contains only variable-uniform equations, then any two models of it \mathfrak{A} and \mathfrak{B} , with $A \cap B = \emptyset$, can be embedded in its model $\mathfrak{A} \cup \mathfrak{B}$, through the inclusion mappings. $\theta_{Eq}[\Sigma]$ has, thus, properties P_i ($i \in \{3,4,5\}$).

If $\theta_{Eq}[\Sigma]$ is equationally consistent but it contains at least one equation of the form

$$(\forall v_i, v_j) (f^m v_i = f^n v_j), \quad j \neq i,$$

then Σ implies the equation

$$(\forall v_i, v_j) (f^m v_i = f^m v_j),$$

which, in its turn, implies

$$(\forall v_i)(f^{m+1}v_i = f^m v_i) .$$

Consequently,

$$\Sigma \models (\exists v_0)(fv_0 = v_0) . \quad (1)$$

Suppose, now, that \mathcal{A} and \mathcal{B} are any two models of Σ . Because of (1), there exists a pair $\langle a, b \rangle \in A \times B$, such that $f^{\mathcal{A}}(a) = a$ and $f^{\mathcal{B}}(b) = b$. Consider the mappings

$$g : \mathcal{A} \rightarrow \mathcal{A} \times \mathcal{B} \quad \text{and} \quad h : \mathcal{B} \rightarrow \mathcal{A} \times \mathcal{B} ,$$

which are given by the rules:

$$\forall x \in A , g(x) = \langle x, b \rangle$$

$$\forall y \in B , h(y) = \langle a, y \rangle .$$

It can be easily shown that g and h are embeddings, and that, since $\mathcal{A} \times \mathcal{B}$ is a model of Σ , Σ has properties P_3 , P_4 and P_5 .

Summarising what we have proved above, we can say that every set of \mathcal{L} -equations has property P_3 and that every equationally consistent set of \mathcal{L} -equations has properties P_4 and P_5 .

Suppose, now, that \mathcal{L} contains arbitrarily many constant symbols.

In this language, if Σ implies the set

$$\{fc_j = c_j : j \in J\} \cup \{c_{j_1} = c_{j_2} : \langle j_1, j_2 \rangle \in {}^2J\} ;$$

any two models \mathcal{A} and \mathcal{B} of it can be embedded in $\mathcal{A} \times \mathcal{B}$.

On the other hand, if Σ doesn't imply the equation $fc_j = c_j$, then it has two models \mathcal{A} and \mathcal{B} such that

$$\mathcal{A} \models fc = c \quad \text{and} \quad \mathcal{B} \not\models fc = c ,$$

and if Σ doesn't imply the equation $c_{j_1} = c_{j_2}$, then it has two models \mathcal{A} and \mathcal{B} such that

$$\mathcal{A} \models c_{j_1} = c_{j_2} \quad \text{and} \quad \mathcal{B} \not\models c_{j_1} = c_{j_2} .$$

In both cases, \mathcal{A} and \mathcal{B} cannot be embedded in a third \mathcal{L} -structure.

We have proved, thus, that the following hold:

$$P_3(\Sigma) \leftrightarrow \Sigma \models \{fc_j = c_j : j \in J\} \cup \{c_{j_1} = c_{j_2} : \langle j_1, j_2 \rangle \in {}^2J\}$$

and

$$P_4(\Sigma) \leftrightarrow (\Sigma \text{ is equationally consistent}) \wedge \\ \wedge (\Sigma \models \{fc_j = c_j : j \in J\} \cup \{c_{j_1} = c_{j_2} : \langle j_1, j_2 \rangle \in {}^2J\}) \leftrightarrow P_5(\Sigma)$$

So, we are done □

It easily follows from theorem 3.2.0. that, if we manage to prove that every finite set of equations of a trivial language generates a base-decidable (hence decidable) equational theory, we can derive a positive solution of the decision problems of P_i 's, as an immediate consequence.

Since, among the numerous base-undecidable equational theories, which are known in the literature, neither can be formulated in a trivial language, and since all our attempts to construct new such theories were unsuccessful, we have good reasons to believe that all equational \mathcal{L} -theories are base-decidable.

In the next sections, a partly successful attempt to prove the required base-decidability result is made, the reasoning of which is exhibited below:

Let G be the set of all variables and constant symbols of \mathcal{L} ; i.e. let

$$G = V_a \cup \{c_j : j \in J\} .$$

We associate, with each finite $\Sigma \subset \text{Eq}_{\mathcal{L}}$, a set of invariants

$$I_{\Sigma} = \{A_{\Sigma}(g) : g \in G\} \cup \{B_{\Sigma}(g_1, g_2) : \langle g_1, g_2 \rangle \in {}^2G, g_1 \neq g_2\} ,$$

which is defined as follows:

$A_{\Sigma}(g) = \langle m, n \rangle$ if m is the smallest natural number, such that, for some $i \neq m$, the equation $f^m g = f^i g$ is implied by Σ and n is the smallest natural number different from zero, such that the equation

$f^m g = f^{m+n} g$ is implied by Σ .

$A_\Sigma(g) = \infty$ if there is no pair of distinct natural numbers $\langle i, j \rangle$, such that the equation $f^i g = f^j g$ is implied by Σ .

$B_\Sigma(g_1, g_2) = \langle m, n \rangle$ if m is the smallest natural number, such that, for some i , the equation $f^m g_1 = f^i g_2$ is implied by Σ and n is the smallest natural number, such that the equation $f^m g_1 = f^n g_2$ is implied by Σ .

$B_\Sigma(g_1, g_2) = \infty$ if there is no pair of natural numbers $\langle i, j \rangle$, such that the equation $f^i g_1 = f^j g_2$ is implied by Σ .

If we prove that the set I_Σ characterises the equational theory, which is generated by Σ (i.e. that $\theta_{Eq}[\Sigma_1] = \theta_{Eq}[\Sigma_2]$, if and only if $I_{\Sigma_1} = I_{\Sigma_2}$) and that the set of invariants I_Σ can be effectively discovered from Σ (i.e. that there exists an algorithmic procedure that gives, for each Σ , the set I_Σ), then we will have a procedure for deciding whether a finite $T \subset Eq_\Sigma$ is a basis of $\theta_{Eq}[\Sigma]$ (since we can find I_Σ and I_T and check whether they are equal or not).

§ 3.3 The base-decidability of all equational theories of a trivial algebraic language with at most one constant symbol and the positive solution to the decision problems of $P_0 - P_5$, in this language.

Let

$$\mathcal{L} = \langle f \rangle \text{ or } \mathcal{L} = \langle f, c \rangle$$

and let Σ be any finite set of \mathcal{L} -equations. We prove, in this section, that the equational theory $\Theta_{Eq}[\Sigma]$ is base-decidable, from which it follows that it is decidable. We make use of this fact and of Theorem 3.2.0. of the previous section in order to give a positive solution to the decision problems of P_i 's, for finite sets of \mathcal{L} -equations.

Let Σ be any set of \mathcal{L} -equations and let $\bar{\Sigma}$ be the result of replacing in Σ each equation of the form

$$f^m c = f^n c$$

by the equation

$$(\forall v_0) (f^m v_0 = f^n v_0),$$

and each equation of the form

$$(\forall v_i) (f^m c = f^n v_i)$$

or

$$(\forall v_i) (f^m v_i = f^n c)$$

by the equation

$$(\forall v_0 v_1) (f^m v_0 = f^n v_1).$$

$\bar{\Sigma}$ contains, thus, no constant symbols. We shall prove the following:

Lemma 3.3.0. For any $\Sigma \subset Eq_{\mathcal{L}}$ and for any pair $\langle m, n \rangle$ of natural numbers, it holds that

$$\Sigma \models f^m c = f^n c \leftrightarrow \bar{\Sigma} \models (\forall v_0) (f^m v_0 = f^n v_0)$$

Proof

Direction \rightarrow is an immediate consequence of the axioms of derivation of *first-order* logic.

We prove the converse direction.

Suppose that Σ doesn't imply the equation $f^m c = f^n c$. Then, there exists a model \mathcal{A} of Σ , such that

$$f^{\mathcal{A}^m}(c^{\mathcal{A}}) \neq f^{\mathcal{A}^n}(c^{\mathcal{A}}).$$

Consider the substructure $\langle c^{\mathcal{A}} \rangle$ of \mathcal{A} , which is generated by $c^{\mathcal{A}}$. Each element of $\langle c^{\mathcal{A}} \rangle$ is of the form

$$f^{\mathcal{A}^{\kappa}}(c^{\mathcal{A}}),$$

for some $\kappa \in \omega$.

For each equation of the form $f^{\mu}c = f^{\nu}c$ in Σ and for each $\kappa \in \omega$, it holds that

$$\Sigma \models f^{\mu+\kappa}c = f^{\nu+\kappa}c.$$

Consequently, since \mathcal{A} is a model of Σ , it follows that

$$f^{\mathcal{A}^{\mu}}(f^{\mathcal{A}^{\kappa}}(c^{\mathcal{A}})) = f^{\mathcal{A}^{\nu}}(f^{\mathcal{A}^{\kappa}}(c^{\mathcal{A}})). \quad (1)$$

For each equation of the form $(\forall v_i)(f^{\mu}v_i = f^{\nu}c)$ in Σ and for each pair $\langle \kappa, \lambda \rangle \in {}^2\omega$, the following series of equalities is deduced from Σ :

$$f^{\mu+\kappa}c = f^{\mu}f^{\kappa}c = f^{\nu}c = f^{\mu}v_i = f^{\mu}f^{\lambda}c = f^{\nu}f^{\lambda}c = f^{\nu+\lambda}c$$

It holds, thus, that

$$f^{\mathcal{A}^{\mu}}(f^{\mathcal{A}^{\kappa}}(c^{\mathcal{A}})) = f^{\mathcal{A}^{\nu}}(f^{\mathcal{A}^{\lambda}}(c^{\mathcal{A}})). \quad (2)$$

From (1) and (2) and from the way in which $\bar{\Sigma}$ has been constructed, we derive that

$$\langle c^{\mathcal{A}} \rangle \models \bar{\Sigma}$$

and that

$$\langle c^{\mathcal{A}} \rangle \not\models (\forall v_0)(f^m v_0 = f^n v_0).$$

This implies that

$$\bar{\Sigma} \neq (\forall v_0)(f^m v_0 = f^n v_0).$$

This completes the proof of the lemma. \square

Lemma 3.3.1. Let Σ be any finite set of \mathcal{L} -equations, which is included in $\bar{\Phi}_1$.

If Σ contains at least one non-tautology with variables, then the equational theory, generated by Σ , equals the equational theory, generated by the set

$$\{(\forall v_0)(f^k v_0 = f^{k+d} v_0), f^l c = f^{l+g} c\},$$

where

- a. k is the smallest natural number such that, for some $i \in \omega$ and some $n \in \omega - 1$, the equation

$$(\forall v_i)(f^k v_i = f^{k+n} v_i) \text{ or } (\forall v_i)(f^{k+n} v_i = f^k v_i)$$

is contained in Σ ,

- b. d is the greatest common divisor of the set

$$\{|m-n| : (\forall v_i)(f^{k+m} v_i = f^{k+n} v_i) \in \Sigma\}$$

- c. l is the smallest natural number such that, for some $i \in \omega$ and some $n \in \omega - 1$, the equation

$$(\forall v_i)(f^l v_i = f^{l+n} v_i) \text{ or } (\forall v_i)(f^{l+n} v_i = f^l v_i)$$

or

$$f^l c = f^{l+n} c \text{ or } f^{l+n} c = f^l c$$

is contained in Σ , and

- d. g is the greatest common divisor of the set

$$\{|m-n| : (\forall v_i)(f^{l+m} v_i = f^{l+n} v_i) \in \Sigma \text{ or } f^{l+m} c = f^{l+n} c \in \Sigma\}$$

If all the non-tautologies, contained in Σ , have no variables, then the equational theory, generated by Σ , equals the equational theory, generated by the set

$$\{f^l c = f^{l+g} c\},$$

where l and g are as before.

Proof

Claim 1. For any pair $\langle m, n \rangle$ of natural numbers and for any $\rho \in \omega$, it holds that

$$(\forall v_i)(f^m v_i = f^n v_i) \models (\forall v_i)(f^m v_i = f^{m+\rho|m-n|} v_i) \quad (1)$$

Proof of the claim (By induction on ρ)

For $\rho = 0$, it holds.

Suppose that (1) holds for $\rho = k$. Since we have that

$$f^{m+(k+1)|m-n|} v_i = f^{m+k|m-n|} f^{|m-n|} v_i,$$

the induction hypothesis leads us to the conclusion that

$$(\forall v_i)(f^m v_i = f^n v_i) \models (\forall v_i)(f^{m+(k+1)|m-n|} v_i = f^{m+|m-n|} v_i). \quad (2)$$

If $m \geq n$, then the relation

$$(\forall v_i)(f^m v_i = f^n v_i) \models (\forall v_i)(f^m f^{|m-n|} v_i = f^n f^{m-n} v_i = f^m v_i),$$

together with (2), imply that

$$(\forall v_i)(f^m v_i = f^n v_i) \models (\forall v_i)(f^m v_i = f^{m+(k+1)|m-n|} v_i).$$

If $m \leq n$, then the relation

$$(\forall v_i)(f^m v_i = f^n v_i) \models (\forall v_i)(f^{m+|m-n|} v_i = f^n v_i = f^m v_i),$$

together with (2), again, imply that

$$(\forall v_i)(f^m v_i = f^n v_i) \models (\forall v_i)(f^m v_i = f^{m+(k+1)|m-n|} v_i).$$

We have, thus, proved that, if relation (1) holds for $\rho = k$, then it holds for $\rho = k+1$.

So, we are done. □

Claim 2. For each quadruple of natural numbers $\langle k, l, m, n \rangle$, with $l \neq 0$, it holds that

$$\{(\forall v_{i_1})(f^k v_{i_1} = f^{k+1} v_{i_1}), (\forall v_{i_2})(f^{k+m} v_{i_2} = f^{k+n} v_{i_2})\} \models (\forall v_0)(f^k v_0 = f^{k+|m-n|} v_0)$$

Proof of the claim

Let us take $m \geq n$ and ρ such that $\rho l \geq n$. By claim 1, the following implications hold:

$$\begin{aligned}
& \left. \begin{array}{l} f^k v_{i_1} = f^{k+1} v_{i_1} \\ f^{k+m} v_{i_2} = f^{k+n} v_{i_2} \end{array} \right\} \rightarrow \left. \begin{array}{l} f^k v_0 = f^{k+\rho 1} v_0 \\ f^{k+m} v_0 = f^{k+n} v_0 \end{array} \right\} \rightarrow \\
& \rightarrow \left. \begin{array}{l} f^k v_0 = f^{k+\rho 1} v_0 \\ f^{k+n+(m-n)} v_0 = f^{k+n} v_0 \end{array} \right\} \rightarrow \left. \begin{array}{l} f^k v_0 = f^{k+\rho 1} v_0 \\ f^{k+\rho 1} f^{m-n} v_0 = f^{k+\rho 1} v_0 \end{array} \right\} \rightarrow \\
& \rightarrow f^{k+(m-n)} v_0 = f^k v_0 .
\end{aligned}$$

So, the equation

$$(\forall v_0) (f^k v_0 = f^{k+|m-n|} v_0)$$

is implied. □

Claim 3 For any triple of natural numbers $\langle k, m, n \rangle$, it holds that

$$\{(\forall v_0) (f^k v_0 = f^{k+m} v_0), (\forall v_0) (f^k v_0 = f^{k+n} v_0)\} \models (\forall v_0) (f^k v_0 = f^{k+\text{g.c.d}(m,n)} v_0)$$

Proof of the claim

Let us take $m \geq n$. If $m = \rho \cdot n$, we are done. If not, let n_1 be the residue of the division of m by n . Then, m is written as $\rho n + n_1$, for some $\rho \in \omega - 1$. The following implications hold:

$$\begin{aligned}
& \left. \begin{array}{l} f^k v_0 = f^{k+m} v_0 \\ f^k v_0 = f^{k+n} v_0 \end{array} \right\} \rightarrow \left. \begin{array}{l} f^k v_0 = f^{k+\rho n + n_1} v_0 \\ f^k v_0 = f^{k+n} v_0 \end{array} \right\} \rightarrow \\
\text{Claim 1} \rightarrow & \left. \begin{array}{l} f^k v_0 = f^{k+\rho n} f^{n_1} v_0 \\ f^k v_0 = f^{k+\rho n} v_0 \end{array} \right\} \rightarrow \left. \begin{array}{l} f^k v_0 = f^{k+\rho n} f^{n_1} v_0 \\ f^k f^{n_1} v_0 = f^{k+\rho n} f^{n_1} v_0 \end{array} \right\} \rightarrow \\
& \rightarrow f^k v_0 = f^{k+n_1} v_0
\end{aligned}$$

So, the equation

$$(\forall v_0)(f^k v_0 = f^{k+n_1} v_0)$$

is implied.

If n_1 is the greatest common divisor of m and n , then we are done. If not, we divide n by n_1 and, if n_2 is the residue of the division, we derive the equation

$$(\forall v_0)(f^k v_0 = f^{k+n_2} v_0)$$

Continuing this procedure, after finitely many steps, we derive the equation

$$(\forall v_0)(f^k v_0 = f^{k+g.c.d(m,n)} v_0) \quad \square$$

We now advance to prove the lemma. For this purpose, let k, d, l and g be as defined on pg 105.

From claim 2 it follows that Σ implies the set

$$\{(\forall v_0)(f^k v_0 = f^{k+|m-n|} v_0) : (\forall v_i)(f^{k+m} v_i = f^{k+n} v_i) \in \Sigma\}$$

and, from claim 3, that the equation

$$(\forall v_0)(f^k v_0 = f^{k+d} v_0)$$

is implied.

Also, if $\bar{\Sigma}$ is as defined on pg 103, we derive, in the same way, that

$$\bar{\Sigma} \models (\forall v_0)(f^l v_0 = f^{l+g} v_0).$$

So, by lemma 3.3.0., we get that

$$\Sigma \models f^l c = f^{l+g} c.$$

We have proved, thus, that the equational theory of Σ includes the equational theory of

$$\{(\forall v_0)(f^k v_0 = f^{k+d} v_0), f^l c = f^{l+g} c\}.$$

Conversely, each equation in Σ is either of the form

$$(\forall v_0)(f^{k+m} v_0 = f^{k+n} v_0)$$

or of the form

$$f^{1+m}c = f^{1+n}c ,$$

where $|m-n|$ is a multiple of d or g , respectively.

In the first case, we have that

$$\begin{aligned} (\forall v_0)(f^k v_0 = f^{k+d} v_0) & \models (\forall v_0)(f^k v_0 = f^{k+pd} v_0) \models \\ & \models (\forall v_0)(f^k v_0 = f^{k+|m-n|} v_0) \models (\forall v_0)(f^{k+m} v_0 = f^{k+n} v_0), \end{aligned}$$

while, in the second case, we have that

$$(\forall v_0)(f^1 v_0 = f^{1+g} v_0) \models (\forall v_0)(f^{1+m} v_0 = f^{1+n} v_0).$$

Through Lemma 3.3.0., we derive that

$$f^1 c = f^{1+g} c \models f^{1+m} c = f^{1+n} c .$$

We have proved that the equational theory, generated by Σ , is included in the equational theory, generated by the set

$$\{(\forall v_0)(f^k v_0 = f^{k+d} v_0) , f^1 c = f^{1+g} c\} .$$

The proof of the lemma has been completed. \square

Lemma 3.3.2. Let Σ be a finite subset of $\overline{\Phi}_2$, which is not included in $\overline{\Phi}_1$. Then, the equational theory, generated by Σ , coincides with the equational theory, generated by the set

$$\{(\forall v_0 v_1)(f^k v_0 = f^k v_1) , (\forall v_0)(f^k v_0 = f^1 c)\},$$

where:

a. k is the smallest natural number such that, for some term φ , the non-tautology

$$(\forall \bar{v})(f^k v_i = \varphi) \quad \text{or} \quad (\forall \bar{v})(\varphi = f^k v_i)$$

is contained in Σ

b. l is the smallest natural number such that, for some term φ , the non-tautology

$$(\forall \bar{v})(f^l v_i = \varphi) \quad \text{or} \quad (\forall \bar{v})(\varphi = f^l v_i)$$

or

$$(\forall \bar{v})(f^1 c = \varphi) \quad \text{or} \quad (\forall \bar{v})(\varphi = f^1 c)$$

is contained in Σ .

Proof

I prove, firstly, that

$$\Sigma \models (\forall v_0 v_1) (f^k_{v_0} = f^k_{v_1}). \quad (1)$$

Because of the construction of $\overline{\Phi}_2$, either for some term φ not containing v_0 , the equation

$$(\forall \overline{v}) (f^k_{v_0} = \varphi)$$

is implied by Σ , or, for some $m \in \omega-1$, the equation

$$(\forall v_0) (f^k_{v_0} = f^{k+m}_{v_0})$$

is implied by Σ .

In the first case, Σ also implies the equation $(\forall \overline{v}) (f^k_{v_1} = \varphi)$. Relation (1) is, thus, deduced.

In the second case, since Σ is not included in $\overline{\Phi}_1$, an equation of the form

$$(\forall v_0 v_1) (f^{k+n}_{v_0} = \psi),$$

where ψ doesn't contain v_0 , is implied by Σ . If we choose ρ , so that $\rho m \geq n$, we deduce from claim 1 of Lemma 3.3.1. that the following implications hold

$$\Sigma \rightarrow \left. \begin{array}{l} f^k_{v_0} = f^{k+\rho m}_{v_0} \\ f^{k+n}_{v_0} = \varphi \end{array} \right\} \rightarrow \left. \begin{array}{l} f^k_{v_0} = f^{k+\rho m}_{v_0} \\ f^{k+\rho m}_{v_0} = f^{\rho m-n}_{\varphi} \end{array} \right\} \rightarrow f^k_{v_0} = f^{\rho m-n}_{\varphi}$$

Hence, exactly as in the previous case, relation (1) is deduced.

We prove that

$$\Sigma \models (\forall v_0) (f^1_c = f^k_{v_0}) \quad (2)$$

If 1 is equal with k , we are done.

Suppose that 1 is smaller than k . Then, either an equation of the form

$$(\forall v_0) (f^1_c = f^{k+m}_{v_0})$$

or an equation of the form

$$f^l c = f^{l+n} c \quad (n \geq 1)$$

is implied by Σ .

In the first case, relation (1) implies that $f^{k+m} v_0$ equals $f^k v_0$ and we are done.

In the second case, we can choose ρ such that $\rho n \geq k$ and, thus, we can have that

$$f^l c = f^{l+n} c = f^{l+\rho n} c = f^k f^{l+\rho n-k} c .$$

Using relation (1), we derive that Σ implies the required equation.

We have proved, thus, relation (2).

Relations (1) and (2), obviously, imply that the equational theory, which is generated by the set

$$\{(\forall v_0 v_1)(f^k v_0 = f^k v_1), (\forall v_0)(f^l c = f^k v_0)\},$$

is included in the equational theory, which is generated by Σ . The converse clearly holds. So we have proved the lemma. \square

We use, now, the lemmas, in order to prove that the set of invariants I_Σ can be effectively discovered from Σ and that it characterises the equational theory generated by Σ . (Theorems 3.3.3. and 3.3.4., respectively) We derive, in the obvious way, that $\theta_{Eq}[\Sigma]$ is base-decidable (Theorem 3.3.5.) and decidable (Theorem 3.3.6.). Finally, we get the decidability of P_i 's, for finite sets of \mathcal{L} -equations (Theorem 3.3.6.):

Theorem 3.3.3. For any finite set Σ of \mathcal{L} -equations, the set of invariants I_Σ can be effectively discovered from Σ ; i.e. there exists an algorithmic procedure that gives, for each finite Σ , the set I_Σ

Proof

Before giving the procedure, we shall prove the following:

Claim If Σ is a finite subset of $\overline{\mathcal{Q}}_1$, which contains at least one equation of the form

$$(\forall v_i)(f^u v_i = f^v v_i) ,$$

for μ and ν distinct, and if the quadruple $\langle k, d, 1, g \rangle$ is as defined on pg 105, then

- a. For each $i \in \omega$, $A_\Sigma(v_i)$ equals $\langle k, d \rangle$ and
- b. $A_\Sigma(c)$ equals $\langle 1, g \rangle$.

Proof of the claim

a. It is obvious that $A_\Sigma(v_i)$ is not ∞ . So, $A_\Sigma(v_i)$ is an ordered pair of natural numbers.

From the axioms of derivation for equational logic (pg. 21), we deduce that the only non-tautologies containing variables, which are implied by Σ , are of the form

$$(\forall v_i)(f^\mu v_i = f^\nu v_i),$$

with μ and ν not smaller than k . Hence, it follows that the first member of $A_\Sigma(v_i)$ is k .

On the other hand, if we consider the equation

$$(\forall v_0)(f^k v_0 = f^{k+d} v_0), \quad (1)$$

any \mathcal{L} -structure \mathcal{A} , with domain $\{a_0, a_1, \dots, a_{d-1}\}$ and $f^{\mathcal{A}}$ defined by the rule

$$\begin{aligned} \forall v < d-1, \quad f^{\mathcal{A}}(a_v) &= a_{v+1} \\ f^{\mathcal{A}}(a_{d-1}) &= a_0, \end{aligned}$$

is a model of it. If ρ is smaller than d , it holds that, for each $v < d$,

$$(f^{\mathcal{A}})^\rho(a_v) \neq a_v.$$

This implies that

$$\forall \rho < d, \mathcal{A} \not\models (\forall v_i)(f^k v_i = f^{k+\rho} v_i).$$

We have, thus, proved that, for any $\rho < d$,

$$(\forall v_0)(f^k v_0 = f^{k+d} v_0) \not\models (\forall v_i)(f^k v_i = f^{k+\rho} v_i) \quad (2)$$

Since, by Lemma 3.3.1., any non-tautology, containing variables, implied by Σ , must be implied by the equation (1), we deduce from relation (2) that

$$\forall \rho < d, \Sigma \not\models (\forall v_i)(f^k v_i = f^{k+\rho} v_i).$$

From this it, obviously, follows that the second member of $A_\Sigma(v_i)$ is d .

We have proved that, for each $i \in \omega$,

$$A_\Sigma(v_i) = \langle k, d \rangle$$

b. Exactly as before, we get that

$$A_{\bar{\Sigma}}(v_i) = \langle 1, g \rangle .$$

Consequently, by Lemma 3.3.0., we have that

$$A_\Sigma(c) = \langle 1, g \rangle$$

This completes the proof of the claim □

We now advance to present the required procedure:

Check whether $\Sigma \subset \bar{\Phi}_1$. This can be done, since Σ is finite and $\bar{\Phi}_1$ is, by corollary 3.1.1., decidable. If yes, then, in order to find I_Σ , do the following:

Check whether there exist two distinct μ and ν such that, for some $i \in \omega$, the equation $(\forall v_i)(f^\mu v_i = f^\nu v_i)$ is in Σ .
If yes, then, by the claim,

$$A_\Sigma(v_i) = \langle k, d \rangle$$

and

$$A_\Sigma(c) = \langle 1, g \rangle ,$$

which, because of lemma 3.3.1., can be recursively found. If no, then check whether there exist two distinct μ and ν such that the equation $f^\mu c = f^\nu c$ is in Σ . If yes, then

$$A_\Sigma(v_i) = \infty$$

and

$$A_\Sigma(c) = \langle 1, g \rangle .$$

If no, then Σ is composed exclusively of tautologies, and

$$\forall g \in G, A_\Sigma(g) = \infty$$

We have, thus, found $A(g)$'s.

It easily follows from Theorem 3.1.0. that, for any two distinct g_1 and g_2 it holds that

$$B_{\Sigma}(g_1, g_2) = \infty$$

If no, then check whether $\Sigma \subset \overline{\Phi}_2$. This can be done, since Σ is finite and $\overline{\Phi}_2$ is, by corollary 3.1.1., decidable. If yes, then, in order to find I_{Σ} , do the following:

Find the pair $\langle k, 1 \rangle$, by the procedure given in Lemma 3.3.2. It easily follows from the rules of derivation of equational logic (pg. 21) that, for any pair of distinct variables $\langle v_i, v_j \rangle$, it holds that

$$B_{\Sigma}(v_i, v_j) = \langle k, k \rangle$$

$$B_{\Sigma}(v_i, c) = \langle k, 1 \rangle$$

$$B_{\Sigma}(c, v_i) = \langle 1, k \rangle .$$

From this we derive that

$$A_{\Sigma}(c) = \langle 1, 1 \rangle$$

and that, for any v_i ,

$$A_{\Sigma}(v_i) = \langle k, 1 \rangle .$$

If no, then, because of Theorem 3.1.2., Σ generates the equational theory Eq_{Σ} . We have, thus, that

$$\forall g \in G, A_{\Sigma}(g) = \langle 0, 1 \rangle$$

and that, for any two distinct $\langle g_1, g_2 \rangle \in {}^2G$,

$$B_{\Sigma}(g_1, g_2) = \langle 0, 0 \rangle$$

This completes the procedure that enables us to effectively find I_{Σ} from Σ . □

Theorem 3.3.4. For any two finite sets of \mathcal{L} -equations Σ and Γ , it holds that

$$I_{\Sigma} = I_{\Gamma} \text{ if and only if } \theta_{Eq}[\Sigma] = \theta_{Eq}[\Gamma]$$

Proof

Direction \leftarrow of the theorem is obvious. We prove the converse direction:

Suppose that $\theta_{Eq}[\Sigma]$ is different from $\theta_{Eq}[\Gamma]$. Since the two theories cannot be, at the same time, equationally inconsistent, and because of Theorem 3.1.2., one of the following four cases holds:

Case 1. The one set, say Σ , generates Eq_c and the other doesn't. Then it holds that

$$B_{\Sigma}(v_0, v_1) = \langle 0, 0 \rangle \neq B_T(v_0, v_1).$$

Case 2. The one set, say Σ , is included in $\bar{\Phi}_1$ and the other is included in $\bar{\Phi}_2$ but not in $\bar{\Phi}_1$. In this case,

$$B_{\Sigma}(v_0, v_1) = \infty \neq B_T(v_0, v_1)$$

Case 3. Both the sets are included in $\bar{\Phi}_1$. Then, either the one set consists exclusively of tautologies and the other doesn't, in which case there exists a $g \in G$ such that

$$A_{\Sigma}(g) \neq A_T(g),$$

or both the sets contain non-tautologies, in which case, by Lemma 3.3.1., we get that

$$\begin{aligned} \theta_{Eq}[\{(\forall v_0)(f^{k_1} v_0 = f^{k_1+d_1} v_0), f^{l_1} c = f^{l_1+g_1} c\}] &= \theta_{Eq}[\Sigma] \neq \\ \theta_{Eq}[\Gamma] &= \theta_{Eq}[\{(\forall v_0)(f^{k_2} v_0 = f^{k_2+d_2} v_0), f^{l_2} c = f^{l_2+g_2} c\}] \end{aligned}$$

From this we deduce that the two quadruples $\langle k_1, d_1, l_1, g_1 \rangle$ and $\langle k_2, d_2, l_2, g_2 \rangle$ cannot be equal. Making use of the procedure, exhibited in the previous theorem, we easily derive that, for some $g \in G$,

$$A_{\Sigma}(g) \neq A_T(g).$$

Case 4. Both the sets are included in $\bar{\Phi}_2$ but not in $\bar{\Phi}_1$. In this last case, if $\langle k_1, l_1 \rangle$ and $\langle k_2, l_2 \rangle$ are as defined in Lemma 3.3.2. for Σ and Γ , respectively, and because of the fact that

$$\theta_{Eq}[\{(\forall v_0 v_1)(f^{k_1} v_0 = f^{k_1} v_1), (\forall v_0)(f^{l_1} v_0 = f^{l_1} c)\}] = \theta_{Eq}[\Sigma] \neq$$

$$\neq \theta_{\text{Eq}}[T] = \theta_{\text{Eq}}[\{(\forall v_0 v_1)(f^{k_2} v_0 = f^{k_2} v_1), (\forall v_0)(f^{k_2} v_0 = f^{l_2} c)\}],$$

we conclude that either $k_1 \neq k_2$ or $l_1 \neq l_2$. Consequently, either

$$B_{\Sigma}(v_0, v_1) \neq B_T(v_0, v_1) \text{ or } B_{\Sigma}(v_0, c) \neq B_T(v_0, c).$$

We have proved that, in all four cases, if the equational theories generated by Σ and T differ, so do the sets of invariants. This completes the proof of the theorem. \square

Theorem 3.3.5. Let Σ be any finite set of \mathcal{L} -equations. The equational theory, generated by Σ , is base-decidable.

Proof

Find the set of invariants I_{Σ} (By theorem 3.3.3., there exists an algorithmic procedure for doing this). Then, for any finite $T \subset \text{Eq}_{\mathcal{L}}$,

Find the set of invariants I_T and check whether $I_{\Sigma} = I_T$ (this can be done, since the two sets are, obviously, recursive). <u>If yes</u> , then T is a basis of $\theta_{\text{Eq}}[\Sigma]$. <u>If no</u> , then T is not a basis of $\theta_{\text{Eq}}[\Sigma]$. \square

Theorem 3.3.6. Let Σ be any finite set of \mathcal{L} -equations. The equational theory, generated by Σ , is decidable.

Proof

The following is, obviously, a procedure for deciding whether an \mathcal{L} -equation ε is implied by Σ :

Check whether $\Sigma \cup \{\varepsilon\}$ is a basis of $\theta_{\text{Eq}}[\Sigma]$ (this can be done, because of the previous theorem). <u>If yes</u> , then $\varepsilon \in \theta_{\text{Eq}}[\Sigma]$. <u>If no</u> , then $\varepsilon \notin \theta_{\text{Eq}}[\Sigma]$. \square

Theorem 3.3.7. There exist algorithms that enable us to decide whether, given a finite $\Sigma \subset \text{Eq}_{\mathcal{L}}$, it has each of the properties P_0, P_1, P_2, P_3, P_4 and P_5 .

Proof

The proof is an immediate consequence of theorems 3.2.0., 3.3.5. and 3.3.6.:

The base-decidability and the decidability of the equational theory, that is generated by an arbitrary finite $\Sigma \subset \text{Eq}_e$, implies that the right-hand parts of the equivalences of Theorem 3.2.0. are decidable predicates. Hence, the left-hand parts also are. \square

§ 3.4. The decision problems of properties $P_0 - P_5$, in trivial languages with more than one constant symbols.

Troughout this section, let

$$\mathcal{L} = \{f\} \cup \{c_j\}_{j \in J}$$

be any trivial language with arbitrarily many constant symbols.

We have tried to use the method, exhibited in §3.2, in order to prove that all finitely based equational \mathcal{L} -theories are base-decidable and, thus, to give a quick positive answer to the decision problems of P_i 's, for finite sets of \mathcal{L} -equations. Although we are almost certain that the method can be applied in the general case, we have not obtained the result, yet.

Since, as Theorem 3.2.0. indicates, the base-decidability of all finitely based equational theories is a much stronger condition than the one required in order for the decidability of P_i ($i \in \{0,1,2,3,4,5\}$) to be taken, in what follows, we shall try to prove the base-decidability of each one P_i , separately:

We prove, firstly, a series of lemmas:

Lemma 3.4.0. Let Σ be any finite set of \mathcal{L} -equations, which is included in Φ_1 . Σ implies Φ_1 if and only if the following two conditions hold:

a. There exists a pair $\langle n, i \rangle \in (\omega-1) \times \omega$, such that either

$$(\forall v_i)(f^n v_i = v_i) \in \Sigma \quad \text{or} \quad (\forall v_i)(v_i = f^n v_i) \in \Sigma$$

b. 1 is the greatest common divisor of the set

$$\{|m-n| : (\forall v_i)(f^m v_i = f^n v_i) \in \Sigma\}$$

Proof.

If Σ satisfies the two conditions, then Lemma 3.3.1. leads us to the conclusion that the subset of Σ , which contains no constant symbols, generates the equational theory

$$\Theta_{Eq} [(\forall v_0)(v_0 = f v_0)] = \Phi_1.$$

It holds, thus, that

$$\Sigma \models \Phi_1 ,$$

and direction \leftarrow of the lemma is proved.

We prove the converse direction.

Suppose that at least one condition is not satisfied by Σ . We shall show that:

Claim Every model \mathcal{M} of $\overline{\Phi}_1$ can be extended to a model \mathcal{M}' of Σ , that is not a model of Φ_1 .

Proof of the claim

Let \mathcal{M} be any model of $\overline{\Phi}_1$.

i. If for Σ condition a. doesn't hold, take as \mathcal{M}' any extension of \mathcal{M} by a new element a' , such that $f^{\mathcal{M}'}(a') \in A$. Since \mathcal{M} is a model of the equation $(\forall v_0)(v_0 = fv_0)$, it is obvious that

$$f^{\mathcal{M}'}(f^{\mathcal{M}'}(a')) = f^{\mathcal{M}'}(a')$$

and, consequently, that

$$\mathcal{M}' \models (\forall v_0)(f^2 v_0 = fv_0) \quad (1)$$

From (1) and the construction of Σ it easily follows that \mathcal{M}' is a model of all the equations of Σ , which contain variables. It is also obvious that \mathcal{M}' satisfies all the equations, without variables, that \mathcal{M} does. We have found, thus, an extension \mathcal{M}' of \mathcal{M} , such that

$$(\mathcal{M}' \models \Sigma) \wedge (\mathcal{M}' \not\models (\forall v_0)(v_0 = fv_0)) ,$$

and, so, we are done.

ii. If for Σ condition a. holds but condition b. doesn't, then lemma 3.3.1. implies that the part of Σ , which contains variables, generates the equational theory

$$\theta_{Eq} [(\forall v_0)(f^d v_0 = v_0)] ,$$

for some $d \in \omega - 2$. Consequently,

$$\Sigma \subset \theta_{Eq} [\{ (\forall v_0)(f^d v_0 = v_0) \} \cup \{ c_{j_1} = c_{j_2} : \langle j_1, j_2 \rangle \in {}^2 J \}] \quad (1)$$

Take now as \mathcal{M}' any extension of \mathcal{M} by d new elements a_0, a_1, \dots, a_{d-1} , such that

$$\forall v \in d - 1, f^{\mathcal{M}'}(a_v) = a_{v+1}$$

and

$$f^{\mathcal{M}'}(a_{d-1}) = a_0$$

It can be easily checked that, because of (1), \mathcal{M}' is a model of Σ , which doesn't satisfy the equation

$$(\forall v_0) (fv_0 = v_0)$$

This completes the proof of the claim □

Since the claim assures us that, if Σ doesn't satisfy at least one of conditions a and b, we can always find a model of it which is not a model of Φ_1 , we derive that, in this case,

$$\Sigma \not\models \Phi_1$$

and the proof of direction \rightarrow of the lemma is finished □

Lemma 3.4.1 Let Σ be any finite set of \mathcal{L} -equations, which is included in $\overline{\Phi}_2$. Σ implies Φ_2 if and only if it is not included in $\overline{\Phi}_1$ and, for some $i \in \omega$ and some term φ , the non-tautology

$$(\forall v) (fv_i = \varphi)$$

or its reverse is contained in Σ .

Proof

Direction \leftarrow of the lemma is proved exactly as in Lemma 3.3.2.

We prove the converse direction:

Suppose that either Σ is also included in $\overline{\Phi}_1$ or there is no term, such that the equation

$$(\forall v) (fv_i = \varphi)$$

or its reverse is contained in Σ . We shall prove the following:

Claim Every model \mathcal{M} of $\overline{\Phi}_2$ can be extended to a model \mathcal{M}' of Σ , which is not a model of Φ_2 .

Proof of the claim

Let \mathfrak{M} be any model of Φ_2 .

i. If Σ is included in Φ_1 , we work as in case i of the previous claim and we construct the required \mathfrak{M}' .

ii. If Σ contains no equations of the form

$$(\forall v) (fv_i = \varphi) \quad \text{or} \quad (\forall v) (\varphi = fv_i),$$

then it is composed exclusively of equations of the form

$$(\forall v_i v_j) (f^m v_i = f^n v_j) \quad \text{or} \quad (\forall v_i) (f^m v_i = \varphi) \quad \text{or} \quad \varphi = \psi,$$

for $\langle m, n \rangle$ in ${}^2(\omega-2)$ and φ, ψ constant terms. In this case, take as \mathfrak{M}' any extension of \mathfrak{M} by two new elements a_1 and a_2 , such that

$$f^{\mathfrak{M}'}(a_1) = f^{\mathfrak{M}}(a_0), \quad \text{for some } a_0 \in A, \quad \text{and} \quad f^{\mathfrak{M}'}(a_2) = a_1$$

It is obvious that, since $f^{\mathfrak{M}'}(a_1)$ is different from $f^{\mathfrak{M}'}(a_2)$, \mathfrak{M}' is not a model of Φ_2 . It is also obvious that, since, for any $n \in \omega-2$, $f^{\mathfrak{M}'n}(a_1) = f^{\mathfrak{M}'n}(a_2) = f^{\mathfrak{M}'n}(a_0)$, \mathfrak{M}' is a model of Σ . This completes the proof of the claim, from which the proof of direction \rightarrow of the lemma follows. □

Now, it is the right time to complete the proof of relation c. of Theorem 3.2.0., which we have left unfinished. It remains to prove direction \rightarrow of the relation:

Suppose that Σ satisfies neither of relations

$$\Phi_1 \subset \theta_{\text{Eq}}[\Sigma] \subset \Phi_1$$

and

$$\Phi_2 \subset \theta_{\text{Eq}}[\Sigma] \subset \Phi_2.$$

Then, Theorem 3.1.2. shows that either $\theta_{\text{Eq}}[\Sigma] = \text{Eq}_c$, in which case Σ doesn't have property P_2 , or

$$(\Sigma \subset \Phi_1) \wedge (\Sigma \not\subset \Phi_1)$$

or

$$(\Sigma \subset \Phi_2) \wedge (\Sigma \not\subset \Phi_2)$$

In the last two cases, the claims, included in the proofs of Lemmas 3.4.0. and

3.4.1., imply that we can find two infinite models \mathfrak{A} and \mathfrak{A}' of Σ , such that

$$(\mathfrak{A} \subset \mathfrak{A}') \wedge (\mathfrak{A} \models (\forall v_0)(fv_0 = v_0)) \wedge (\mathfrak{A}' \not\models (\forall v_0)(fv_0 = v_0))$$

or

$$(\mathfrak{A} \subset \mathfrak{A}') \wedge (\mathfrak{A} \models (\forall v_0 v_1)(fv_0 = fv_1)) \wedge (\mathfrak{A}' \not\models (\forall v_0)(fv_0 = fv_1)) ,$$

respectively. Consequently, Σ doesn't have property P_2 and we are done \square

Lemma 3.4.2. Let Σ be any finite set of \mathcal{L} -equations, such that

$$\Phi_1 \subset \theta_{Eq}[\Sigma] \subset \bar{\Phi}_1 .$$

Σ generates $\bar{\Phi}_1$ if and only if, for any two distinct j_1 and j_2 in J , there exists a finite sequence

$$\varepsilon_0 , \varepsilon_1 , \varepsilon_2 , \dots , \varepsilon_n$$

of equations in $\Sigma \cap (\bar{\Phi}_1 - \Phi_1)$, such that

- i. for every $v \in n$, ε_v and ε_{v+1} have a constant symbol in common, and
- ii. c_{j_1} is in ε_0 and c_{j_2} is in ε_n .

Proof Obvious \square

Lemma 3.4.3. Let Σ be any finite set of \mathcal{L} -equations, such that

$$\Phi_2 \subset \theta_{Eq}[\Sigma] \subset \bar{\Phi}_2 .$$

Σ generates $\bar{\Phi}_2$ if and only if, for every $j \in J$, there exists a term $f\varphi$, such that a finite sequence of equations

$$c_j = c_{j_1} = c_{j_2} = \dots = c_{j_n}$$

and one of the following two equations

$$(\forall v)(c_{j_n} = f\varphi) , (\forall v)(f\varphi = c_{j_n})$$

are in Σ .

Proof Obvious

□

We now advance to construct decision procedures for checking whether, given a finite $\Sigma \subset \text{Eq}_f$, it has each one of properties P_0 , P_1 and P_2 :

For properties P_0 and P_1

Check whether $\theta_{\text{Eq}}[\Sigma] = \overline{\Phi}_1$, using the following procedure:

<p>Check whether $\Sigma \subset \overline{\Phi}_1$ (this can be done, since $\overline{\Phi}_1$ is, by Corollary 3.1.1., decidable). If no, then $\theta_{\text{Eq}}[\Sigma] \neq \overline{\Phi}_1$. If yes, then check, by the procedure given in Lemma 3.4.0., whether $\Sigma \models \Phi_1$. If no, then $\theta_{\text{Eq}}[\Sigma] \neq \overline{\Phi}_1$. If yes, then check, by the procedure given in Lemma 3.4.2., whether $\theta_{\text{Eq}}[\Sigma] = \overline{\Phi}_1$.</p>

If yes, then Theorem 3.2.0. implies that Σ has the properties P_0 and P_1 .

If no, then check whether $\theta_{\text{Eq}}[\Sigma] = \overline{\Phi}_2$, using the following procedure:

<p>Check whether $\Sigma \subset \overline{\Phi}_2$. If no, then $\theta_{\text{Eq}}[\Sigma] \neq \overline{\Phi}_2$. If yes, then check, by the procedure given in Lemma 3.4.1., whether $\Sigma \models \Phi_2$. If no, then $\theta_{\text{Eq}}[\Sigma] \neq \overline{\Phi}_2$. If yes, then check, by the procedure given in Lemma 3.4.3., whether $\theta_{\text{Eq}}[\Sigma] = \overline{\Phi}_2$.</p>

If yes, then Theorem 3.2.0. implies that Σ has the properties P_0 and P_1 .

If no, then Theorem 3.2.0. implies that Σ has neither property P_0 nor property P_1 . □

For property P_2

Check whether $\Phi_1 \subset \Sigma \subset \overline{\Phi}_1$, using the following procedure:

<p>Check whether $\Sigma \subset \overline{\Phi}_1$. If yes; then check whether $\Sigma \models \Phi_1$, using the procedure given in Lemma 3.4.0.</p>

If yes, then Theorem 3.2.0. implies that Σ has the property P_2 .

If no, then check whether $\Phi_2 \subset \Sigma \subset \overline{\Phi}_2$ using the obvious procedure. Theorem 3.2.0. implies that, if yes, then Σ has the property P_2 and if no, then

Σ doesn't have the property P_2 . □

We have, thus, proved that

Theorem 3.4.4. In any trivial language \mathcal{L} , properties P_0 , P_1 and P_2 are decidable for finite sets of \mathcal{L} -equations.

The decision problems of properties P_3 , P_4 and P_5 should now be examined:

In §3.3., we proved that they accept a positive solution, for finite sets of equations of trivial languages with at most one constant symbol

It is also obvious that, if the language \mathcal{L} contains infinitely many constant symbols, the arbitrary finite set of \mathcal{L} -equations, Σ , doesn't contain at least two c_j 's, say c_{j_1} and c_{j_2} . The existence of non-trivial models of Σ , thus, implies the existence of models of Σ not satisfying the equation $c_{j_1} = c_{j_2}$. From this easily follows that, given any finite $\Sigma \subset \text{Eq}_{\mathcal{L}}$, it holds that

$$\Sigma \models \{c_{j_1} = c_{j_2} : \langle j_1, j_2 \rangle \in J\} \leftrightarrow \theta_{\text{Eq}}[\Sigma] = \text{Eq}_{\mathcal{L}} \quad (1)$$

Theorem 3.2.0. and relation (1) obviously imply that there is no finite set of \mathcal{L} -equations with property P_4 or P_5 . They also imply that the only finite sets of \mathcal{L} -equations having property P_3 are the bases of $\text{Eq}_{\mathcal{L}}$. The three properties have again been proved decidable.

In the case of trivial languages, with more than one, but finitely many, constant symbol, we have not yet found an answer. As Theorem 3.2.0 implies, the problem would be positively solved if we were able to construct an algorithm for deciding whether, given a finite $\Sigma \subset \text{Eq}_{\mathcal{L}}$, it holds that

$$\Sigma \models \{fc_j = c_j : j \in J\} \cup \{c_{j_1} = c_{j_2} : \langle j_1, j_2 \rangle \in J^2\},$$

or, equivalently, an algorithm for deciding whether, given a finite $\Sigma \subset \text{Eq}_{\mathcal{L}}$, it holds that

$$\begin{aligned} \forall j \in J, \quad A_{\Sigma}(c_j) &= \langle 0, 1 \rangle \\ \forall \langle j_1, j_2 \rangle \in J^2, \quad B_{\Sigma}(c_{j_1}, c_{j_2}) &= \langle 0, 0 \rangle \end{aligned}$$

Since such algorithms have not been constructed up to now, the following question is still open:

Question Are properties P_3 , P_4 and P_5 decidable for finite sets of equations of trivial languages with more than one but finitely many constant symbols ?

R E F E R E N C E S

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INDEX OF SYMBOLS

\square	11	-	13
\leftrightarrow	11	\vee	13
#	11	\wedge	13
\cup	12	\rightarrow	13
\cap	12	\exists	13
\bar{A}	12	\forall	13
\in	12	=	13
C, \bar{C}	12	$\text{Sym}_{\mathcal{L}}$	13
$\prod_{i \in I} X_i, X_0 \times X_1 \times \dots \times X_n$	12	$\text{Term}_{\mathcal{L}}$	13
\emptyset	12	$t[\bar{s}]$	13
ω, ω_1	12	$\text{Exp}_{\mathcal{L}}$	13
$\alpha, \beta, \gamma, \dots$	12	$\text{Form}_{\mathcal{L}}$	13
$ X $	12	$\text{Sent}_{\mathcal{L}}$	13
P_i	12	$\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$	14
$<$	12	$Q_i^{\mathcal{A}}, c_j^{\mathcal{A}}, t^{\mathcal{A}}, \varphi^{\mathcal{A}}$	14
\sim	12	$\ \mathcal{A}\ $	14
$[x]_{\sim}$	12	\mathcal{L}_x	14
$f: A \rightarrow B$	12	\mathcal{L}_x	14
f/C	12	$\models, \mathcal{A} \models \Sigma, \mathcal{T} \models \Sigma$	14
$f[C]$	12	$\text{Th}\mathcal{K}$	15
$f: A \twoheadrightarrow B$	12	$\theta[\Sigma]$	15
$f: A \rightarrow B$	12	$\mathcal{A} \subset \mathcal{B}$	15
$f: A \twoheadrightarrow B$	12	$\mathcal{A} < \mathcal{B}$	15
A_B	12	$\mathcal{A} = \mathcal{B}$	15
$f_1 + f_2$	12	$\mathcal{A} \equiv \mathcal{B}$	15
$f_1 \cdot f_2$	12	$\langle X \rangle$	15
$\langle X_{\alpha} : \beta < \alpha \rangle, \bar{x}$	12	$\cup_{\beta < \alpha} \mathcal{A}_{\beta}$	15
\mathcal{L}	13	$f: \mathcal{A} \rightarrow \mathcal{B}$	15
Q_i	13	$f: \mathcal{A} \twoheadrightarrow \mathcal{B}$	15
c_j	13	$f: \mathcal{A} \rightarrow \mathcal{B}$	15
$r(i)$	13	$f: \mathcal{A} \twoheadrightarrow \mathcal{B}$	15
v_i	13	f_T^n	17
$\forall a$	13		

χ_A	18	$V_{\mathcal{G}}$	78
g	18	$\dim \mathcal{L}$	78
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$\langle Z, +, \cdot \rangle$	71		
$\oplus \circ$	71		
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$\oplus \odot$	72		