

P R E F A C E

The problems discussed in this thesis arise in the geometry of numbers, a part of the theory of numbers which originated under the inspiration of Minkowski at the turn of the last century.

The first chapter is introductory and provides the background to the main part of the thesis. The well known theorems which are needed later are collected in an appendix.

In each of the remaining three chapters a different problem is considered and is preceded by a short summary. The theorems are numbered according to the chapter in which they occur.

The subject matter of this thesis was first suggested by my supervisor, Dr. J.H.H. Chalk. I am very deeply indebted to him for all his invaluable help and encouragement throughout its preparation.

I am very grateful to D. M. E. F O S T E R for his help in the solution of the problem discussed in Chapter 3 and for making a number of suggestions leading to the improvement of Chapter 4. My thanks are also

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to thank Professor W.H. Bailey for suggesting the theory of numbers as a subject for research, and the University of London for their award of a postgraduate studentship for fifteen months.

M A Y 1 9 6 0

London, May 1960.

D.M.E.F.

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D.M.E.F.

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CHAPTER 1.

The geometry of numbers is that part of the theory of numbers which is concerned with the solubility of various types of inequalities in integers. For instance, given a function $f(u_1, \dots, u_n)$ of n real variables u_1, \dots, u_n it is sometimes possible to find constants λ_n, μ_n depending upon n and possibly upon some invariant of $f(u_1, \dots, u_n)$ such that the inequalities

$$\lambda_n \leq f(u_1, \dots, u_n) \leq \mu_n \quad (1.1)$$

are always soluble in integers u_1, \dots, u_n . There are certain functions for which the best possible constants λ_n, μ_n are known, for example quadratic forms with real coefficients in a small number of variables. In particular, the cases when either one of λ_n or μ_n is zero or $\lambda_n = -\mu_n$ have been extensively studied. We shall consider, in this thesis, slightly more general functions of the form

$$f(u_1, \dots, u_n) = q(u_1, \dots, u_n) + l(u_1, \dots, u_n)$$

where

$$q(u_1, \dots, u_n) = \sum_{\lambda=1}^n \sum_{s=1}^n a_{\lambda s} u_\lambda u_s \quad (a_{\lambda s} = a_{s\lambda})$$

is a quadratic form in the variables u_1, \dots, u_n with real coefficients $a_{\lambda s}$ ($\lambda, s = 1, \dots, n$), not all zero, having determinant

$$D_n = \|\| a_{\lambda s} \|\|_{\lambda, s=1, \dots, n} \quad \text{and}$$

$$l(u_1, \dots, u_n) = 2a_{n+1,1}u_1 + \dots + 2a_{n+1,n}u_n + a_{n+1,n+1}$$

is a real linear polynomial in u_1, \dots, u_n . Such a function

$f(u_1, \dots, u_n)$ we will call a quadratic polynomial.

We now introduce the idea of an integral unimodular substitution of the variables u_1, \dots, u_n .

Definition 1.1. * A substitution of the form

$$u_\mu = \sum_{s=1}^n p_{\mu s} u_s' + p_{\mu, n+1} \quad (\mu=1, \dots, n)$$

where the $p_{\mu s}$ ($\mu=1, \dots, n; s=1, \dots, n+1$) are integers with determinant $|\| p_{\mu s} \|_{\mu, s=1, \dots, n} = \pm 1$, is said to be an integral unimodular substitution of the variables u_1, \dots, u_n .

Definition 1.2. Two functions $f(u_1, \dots, u_n)$ and $g(u_1, \dots, u_n)$ are said to be equivalent, or $f(u_1, \dots, u_n) \sim g(u_1, \dots, u_n)$, if one can be transformed into the other by an integral unimodular substitution of the variables.

To illustrate our methods, we indicate briefly the type of inductive argument which leads to inequalities of the form (1.1.) for the polynomial $f(u_1, \dots, u_n)$. Some knowledge about small values of the quadratic form $q(u_1, \dots, u_n)$ may allow us to suppose, after applying a suitable unimodular substitution to the variables u_1, \dots, u_n , that

$$0 < q(1, 0, \dots, 0) = a_{11} < \delta,$$

where δ depends only upon the determinant of $q(u_1, \dots, u_n)$. Thus, on completing the square for u_1 , we may rewrite

$$f(u_1, \dots, u_n) = a_{11} (u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n + \alpha_{n+1})^2 + \bar{f}(u_2, \dots, u_n),$$

* When considering values of quadratic forms the integral unimodular substitutions which are used are always homogeneous, that is

$$p_{1, n+1} = \dots = p_{n, n+1} = 0.$$

where $\alpha_2, \dots, \alpha_{n+1}$ are appropriate real numbers and $\bar{f}(u_2, \dots, u_n)$ is a quadratic polynomial in the variables u_2, \dots, u_n . If we can now use an inductive hypothesis for $\bar{f}(u_2, \dots, u_n)$ to establish the existence of integers u_2, \dots, u_n satisfying

$$\lambda_{n-1} \leq \bar{f}(u_2, \dots, u_n) \leq \mu_{n-1}$$

for appropriate constants λ_{n-1}, μ_{n-1} we still have one variable, namely u_1 , at our disposal. We will show that a suitable choice of the integer u_1 will lead to inequalities of the type required in each of the cases considered. Hence, provided that we can find λ_{n_0}, μ_{n_0} for some integer n_0 , an inductive argument will give a result for general $n \geq n_0$.

To give a geometrical interpretation of the inequalities (1.1) we are led naturally to the introduction of an n -dimensional lattice. Referred to a rectangular Cartesian system of axes in some n -dimensional Euclidean space E_n , every point is uniquely determined by a set of n inhomogeneous coordinates (x_1, \dots, x_n) , and the aggregate of all points with integral coordinates is said to form the fundamental lattice $\mathcal{L}_n^{(0)}$. Thus if we denote by K the set of points (x_1, \dots, x_n) in E_n , defined by

$$\lambda_n \leq f(x_1, \dots, x_n) \leq \mu_n$$

it is clear that the solubility of (1.1) in integers implies the existence of a point of $\mathcal{L}_n^{(0)}$ in K and conversely.

More generally, an n -dimensional lattice \mathcal{L}_n in E_n consists

4.

of a set of all points whose coordinates (x_1, \dots, x_n) satisfy a relation of the form

$$x_\lambda = \sum_{s=1}^n \alpha_{\lambda s} u_s \quad (\lambda = 1, \dots, n)$$

where the $\alpha_{\lambda s}$ ($\lambda, s = 1, \dots, n$) are fixed real numbers whose determinant $\|\alpha_{\lambda s}\|_{\lambda, s=1, \dots, n} \neq 0$ and u_1, \dots, u_n are allowed to take the values $0, \pm 1, \pm 2, \dots$. The determinant Δ_n of the lattice is given by the absolute value of $\|\alpha_{\lambda s}\|_{\lambda, s=1, \dots, n}$. Clearly a homogeneous, integral, unimodular substitution applied to the variables u_1, \dots, u_n leaves the lattice \mathcal{L}_n invariant.

With the usual vector notation, if A_1, \dots, A_n are the n points of \mathcal{L}_n having coordinates $(\alpha_{11}, \alpha_{21}, \dots, \alpha_{n1}), \dots, (\alpha_{1n}, \alpha_{2n}, \dots, \alpha_{nn})$ respectively, then every point of \mathcal{L}_n may be expressed in the form

$$u_1 A_1 + \dots + u_n A_n$$

and A_1, \dots, A_n , together with the origin O , are said to generate \mathcal{L}_n . This particular set of generating points of \mathcal{L}_n is not unique.

It may be proved that a necessary and sufficient condition that n points of \mathcal{L}_n should generate the lattice is that the $n \times n$ determinant formed by their x -coordinates should be $\pm \Delta_n$, or equivalently, that the $n \times n$ determinant formed by their corresponding u -coordinates should be ± 1 . We observe finally that \mathcal{L}_n is affinely equivalent to $\mathcal{L}_n^{(0)}$ since the non-singular linear transformation of coordinates given by

$$x_\lambda = \sum_{s=1}^n \alpha_{\lambda s} x'_s \quad (\lambda = 1, \dots, n)$$

transforms \mathcal{L}_n into $\mathcal{L}_n^{(0)}$.

The foundations of the geometry of numbers were set by Minkowski [21], who introduced ideas of considerable generality. His theorem on convex bodies in E_n , for example, occupies a central position in the subject and can often be used to give a crude estimate for the types of inequalities required. A convex body may be defined as follows;

Definition 1.3. A convex body K in E_n is a closed and bounded set of points, having at least one inner point, and such that if P, Q are any two points of K then the mid-point, $\frac{1}{2}(P+Q)$, of the segment PQ also belongs to K . If a convex body K contains some point, say the origin O , which is such that if $P \neq O$ belongs to K , the point $O - P$ also belongs to K , then K is said to be symmetrical about O . It is known that every convex body has a volume in the Jordan sense. We can now state the fundamental theorem of which there are several proofs by Minkowski and others, [16]. Although we shall not give any of these proofs, it is perhaps interesting to observe that some of them depend only on ideas of elementary geometry and convexity.

THEOREM 1.1. If K is a convex body which is symmetrical about the origin O and has volume $V(K) \geq 2^n \Delta_n$, then K contains a point of Λ_n , other than O .

We shall only be directly concerned with two of the many applications of Minkowski's theorem. These are stated explicitly in Corollaries 1.1 and 1.2.

COROLLARY 1.1. If $\lambda_1, \dots, \lambda_n$ are n positive real numbers satisfying

$$\lambda_1 \cdots \lambda_n = \Delta_n,$$

there is a point of \mathcal{L}_n , other than 0, in the n -dimensional rectangular parallelepiped consisting of all points (x_1, \dots, x_n) whose coordinates satisfy

$$|x_1| \leq \lambda_1, \dots, |x_n| \leq \lambda_n.$$

Proof. The proof is immediate, since the volume of the rectangular parallelepiped under consideration is $2^n \lambda_1 \cdots \lambda_n = 2^n \Delta_n$.

Now suppose that $q(u_1, \dots, u_n)$ is a positive definite quadratic form, so that in particular $D_n > 0$. Denote by γ_n the least upper bound of the minima of all positive definite quadratic forms in n variables of determinant 1 for integral values of the variables, not all zero.

COROLLARY 1.2.

$$\gamma_n \leq \frac{4}{\pi} \left\{ \Gamma\left(1 + \frac{1}{2}n\right) \right\}^{\frac{2}{n}}.$$

Proof. Again the proof follows easily when we observe that

$q(u_1, \dots, u_n)$ may be written in the form

$$q(u_1, \dots, u_n) = x_1^2 + \dots + x_n^2$$

where

$$x_\nu = \sum_{s=1}^n \alpha_{\nu s} u_s \quad (\nu = 1, \dots, n)$$

for appropriate real $\alpha_{\nu s}$ ($\nu, s = 1, \dots, n$). On comparison of determinants, we see that $\pm D_n^{\frac{1}{2}} = \|\alpha_{\nu s}\|_{\nu, s=1, \dots, n}$. An application of Minkowski's fundamental theorem to the n -dimensional sphere

$$x_1^2 + \dots + x_n^2 \leq \frac{4}{\pi} \left\{ \Gamma\left(1 + \frac{1}{2}n\right) \right\}^{\frac{2}{n}} D_n^{\frac{1}{n}}$$

which has volume $2^n D_n^{\frac{1}{2}}$ now gives the result.

Although Corollary 1.2. is only best possible when $n=1$, in which case $\delta_1=1$, the following values of δ_n for $2 \leq n \leq 8$ are known, by theorems of Lagrange, Gauss, Korkine and Zolotareff [33] Blichfeldt [5] (see Van der Waerden [27] for further references).

$$\delta_2 = 2/3^{1/2}, \quad \delta_3 = 2^{1/3}, \quad \delta_4 = 2^{1/2}, \quad \delta_5 = 2^{3/5}, \quad \delta_6 = 2/3^{1/6}, \\ \delta_7 = 2^{6/7}, \quad \delta_8 = 2.$$

For values of $n > 9$ the problem becomes more difficult and remains unsolved, although Corollary 1.2. has since been superseded by theorems of Blichfeldt, Rankin and Rogers [26] who have shown that, for large values of n , $\overline{\lim}_{n \rightarrow \infty} (\delta_n/n) \leq 1/(\pi e)$. [It may be verified that this is an improvement of Minkowski's asymptotic estimate by a factor $1/2$].

At present the best known result in the opposite direction,

$$\underline{\lim}_{n \rightarrow \infty} (\delta_n/n) \geq 1/(2\pi e) \text{ is a consequence of a theorem of Hlawka [17].}$$

We now turn our attention to indefinite quadratic forms, and suppose henceforth that $q(u_1, \dots, u_n)$ is indefinite. A classical theorem ([15], Theorem 59) due to Meyer states that if $q(u_1, \dots, u_n)$ has integral coefficients, then it assumes the value zero for integral values of the variables, not all zero, provided that $n \geq 5$. This suggested the following conjecture.

CONJECTURE. If the coefficients of $q(u_1, \dots, u_n)$ are not all in a rational ratio, then $q(u_1, \dots, u_n)$ can be made arbitrarily small for integral values of the variables, not all zero, provided that $n \geq 5$.

Until recently no substantial progress on this problem had been made except in two or three special cases considered by Davenport and Heilbronn [12], Watson [30] and Oppenheim [23]. In particular we

mention the following two theorems of Oppenheim as they will be required for Chapters 3 and 4 .

THEOREM 1. 3. Let $\epsilon > 0$. If $q(u_1, \dots, u_n)$ is an indefinite quadratic form in $n \geq 4$ variables whose coefficients are not all in a rational ratio and which represents zero non-trivially, the inequalities

$$0 < |q(u_1, \dots, u_n)| < \epsilon$$

are always soluble in integers u_1, \dots, u_n .

THEOREM 1. 4. If $q(u_1, \dots, u_n)$ is an indefinite quadratic form such that, for every $\epsilon > 0$, the inequalities

$$0 < q(u_1, \dots, u_n) < \epsilon$$

are soluble in integers u_1, \dots, u_n , then if $n \geq 3$, the inequalities

$$0 < -q(u_1, \dots, u_n) < \epsilon$$

are also soluble in integers u_1, \dots, u_n for every positive ϵ .

Although the proof of Theorem 1. 4 is fairly straightforward, the proof of Theorem 1. 3 is not elementary. For it depends upon properties of quadratic forms with integral coefficients, and the following simple case of a general theorem [31] of Weyl on the uniformity of distribution of the values $(\text{mod } 1)$ of a polynomial in a single integral variable.

THEOREM 1. 5. If at least one of θ_1, θ_2 is irrational and ϵ is any assigned positive number, the inequalities

$$|\theta_1 x^2 + \theta_2 x + y + \theta_3| < \epsilon$$

have an infinity of solutions in integers x, y .

Returning to the conjecture a very considerable advance has been made during the last four years by Davenport, Birch and Ridout, using analytical methods ([27], [32])

We now consider the minimum value of $|q(u_1, \dots, u_n)|$ for all integers $u_1, \dots, u_n \neq 0, \dots, 0$ in the cases when $n = 2, 3$ and state two classical results due to Markoff [35] and Korkine and Zolotareff [34].

THEOREM 1. 6. The inequalities

$$|q(u_1, u_2)| \leq \left(\frac{4}{5} |D_2|\right)^{\frac{1}{2}}$$

are always soluble in integers $u_1, u_2 \neq 0, 0$ with strict inequality unless $q(u_1, u_2) \sim \lambda(u_1^2 + u_1 u_2 - u_2^2)$.

THEOREM 1. 7. The inequalities

$$|q(u_1, u_2, u_3)| \leq \left(\frac{2}{3} |D_3|\right)^{\frac{1}{3}}$$

are always soluble in integers $u_1, u_2, u_3 \neq 0, 0, 0$ with strict inequality, unless $q(u_1, u_2, u_3) \sim \lambda(u_1^2 + u_1 u_2 + u_2^2 - 2u_3^2)$.

Proofs of these are given in Dickson's books; ([14], Theorem 119) and ([15], Theorem 83), respectively. A recent theorem of

Watson, which gives an inequality for the values of a non-zero binary quadratic form and depends partly upon Theorem 1. 6, is also needed in Chapter 4.

THEOREM 1. 8. If P, N denote the lower bounds of the positive values of $q(u_1, u_2), -q(u_1, u_2)$ respectively for all integers $u_1, u_2 \neq 0, 0$, then

$$PN \leq \frac{4}{5} |D_2|,$$

provided that $q(u_1, u_2)$ does not represent zero non-trivially.

The equality sign is required when $q(u_1, u_2) \sim \lambda(u_1^2 + u_1 u_2 - u_2^2)$.

A more general result, obtained by Blaney ([3], Lemma 1) and stated in the next theorem, provides an estimate for the upper bound of the least positive value of $q(u_1, \dots, u_n)$ and will be useful in Chapter 2.

THEOREM 1.9. There exists a constant β_n depending only on n , such that for any indefinite quadratic form $q(u_1, \dots, u_n)$ of determinant D_n , there are relatively prime integers u_1, \dots, u_n satisfying

$$0 < q(u_1, \dots, u_n) \leq \beta_n |D_n|^{1/n}.$$

This holds in particular with $\beta_n = 2^{n-1}$.

The best possible value of $\beta_2 (= 2)$ is well known [28].

Davenport [10] and Oppenheim [24] later found the best possible values of $\beta_3 = (\frac{27}{4})^{\frac{1}{3}}$ and $\beta_4 = 2$. For values of $n \geq 5$, the inequalities are probably soluble with any $\beta_n > 0$, however small, if the coefficients of $q(u_1, \dots, u_n)$ are not all in a rational ratio; otherwise the problem of finding the best possible value of β_n has been solved by Watson, as we shall show presently.

It is well known that an indefinite quadratic form in the variables u_1, \dots, u_n may be expressed in an infinity of ways as a sum of signed squares of real linear forms in u_1, \dots, u_n . By Sylvester's law of inertia, the numbers μ and ϵ say, of positive and negative signs, respectively, are invariant; hence so also are the signature

and rank of the form, namely $s = r - t$ and $\rho = r + t$, respectively.

Consider the special case where $a_{\lambda\lambda}$ ($\lambda = 1, \dots, n$) and $2a_{rs}$ ($r < s$; $r, s = 1, \dots, n$) are integers. Then the indefinite form $q(u_1, \dots, u_n)$ has integral coefficients, determinant D_n and signature $s(q)$, say. We write

$$d_n(q) = \begin{cases} 2^n |D_n|, & \text{if } n \text{ is even} \\ 2^{n-1} |D_n|, & \text{if } n \text{ is odd.} \end{cases}$$

A weaker form of the following theorem [29] of Watson, whose proof depends upon the classical theory of integral quadratic forms, is essential in the solution of the problem considered in Chapter 3, (see Lemma 3.4).

THEOREM 1. 10. There exists integers u_1, \dots, u_n

satisfying

$$0 < \frac{|q|}{\{d_n(q)\}^{\frac{1}{n}}} \leq \begin{cases} 1 \\ \left(\frac{1}{2}\right)^n \\ \left(\frac{1}{3}\right)^n \\ \left(\frac{1}{4}\right)^n \end{cases} \quad \text{if} \quad s(q) \equiv \begin{cases} 0 \text{ or } \pm 1 \\ \pm 3 \\ \pm 2 \\ \pm 4 \end{cases} \pmod{8}$$

If we introduce the quadratic form in $n+1$ variables

$$Q(u_1, \dots, u_{n+1}) = \sum_{r=1}^{n+1} \sum_{s=1}^{n+1} a_{rs} u_r u_s \quad (a_{rs} = a_{sr})$$

with determinant D_{n+1} , say, clearly $f(u_1, \dots, u_n)$ may now be written

$$f(u_1, \dots, u_n) = Q(u_1, \dots, u_n, 1).$$

In Chapter 2, we consider the case when $q(u_1, \dots, u_n)$ is an indefinite quadratic form, whence $D_n \neq 0$. Thus we may define $\Omega = |D_{n+1} D_n^{-1}|$.

The following theorem, significant only for large values of Ω , was first obtained by Blaney ([3], Theorem 2), in a slightly different form.

THEOREM 1. 11. There exist constants C_n, C'_n depending only on n , such that the inequalities

$$0 < |D_n|^{-\frac{1}{n}} Q(u_1, \dots, u_n, 1) < C_n \Omega^{\nu_n} + C'_n$$

are always soluble in integers u_1, \dots, u_n with $\nu_n = 1/2$.

The special case $n=2$ was later studied by Blaney, [4], who obtained an improved value of $\nu_2 = 1/3$, a result which can also be deduced from a theorem [20] of Macbeath. We shall show that the theorem is true, possibly for different constants C_n, C'_n , with $\nu_n = 1/2^n$. The proof, which is by induction on n , is similar to that of Blaney, and in fact makes use of his Theorem 1. 9.

The remaining case $D_n = 0$ is considered separately in Chapter 3. In fact we suppose that the rank of $q(u_1, \dots, u_n)$ is $n-1$ so that

$$\wp(u_1, \dots, u_n) = \pm L_1^2 \pm \dots \pm L_{n-1}^2 + L_n + c \quad (1.2)$$

where L_1, \dots, L_n are n homogeneous real linear forms in the n variables u_1, \dots, u_n having a non-zero determinant of coefficients Δ , say, and c is an arbitrary constant.

The polynomial $\wp(u_1, \dots, u_n)$ in (1.2) has already been studied in some detail by Macbeath. In 1950, he proved the following two theorems relating to the case $n=2$ ([18], Theorems 1 and 2).

THEOREM 1. 12.There exists integers u_1, u_2 satisfying

$$|\wp(u_1, u_2)| < \left(\frac{\sqrt{13}}{8} |\Delta|\right)^{2/3},$$

unless $\dagger \wp(u_1, u_2)$ is equivalent to one of the following:

$$\wp_1 = \left(\frac{1}{2} |\Delta|\right)^{2/3} (u_1^2 + u_1 + 2u_2 + c'),$$

$$\wp_2 = \left(\frac{1}{6} |\Delta|\right)^{2/3} (u_1^2 + u_1 + 6u_2 + c'),$$

$$\wp_3 = \left(\frac{1}{4} |\Delta|\right)^{2/3} (u_1^2 + 4u_2 + c')$$

for some real number c' . If $c' = 1$, $|\wp_1| \geq \left(\frac{1}{2} |\Delta|\right)^{2/3}$; if $c' = 4$, $|\wp_2| \geq \left(\frac{\sqrt{2}}{3} |\Delta|\right)^{2/3}$; and if $c' = 5/2$, $|\wp_3| \geq \left(\frac{3}{8} \sqrt{\frac{3}{2}} |\Delta|\right)^{2/3}$.THEOREM 1. 13.Given $\epsilon > 0$, there is a finite set ofpolynomials \wp_1, \dots, \wp_N such that if \wp is not equivalent to amultiple of a polynomial of the form $\wp_c + c$ ($c = 1, \dots, N$)then integers u_1, u_2 exist such that

$$|\wp(u_1, u_2)| \leq \epsilon.$$

The proof of Theorem 1. 12 is arithmetical while that of Theorem 1. 13 depends upon two important lemmas, one of which has already been mentioned as Theorem 1. 5. The other lemma, which is concerned with the approximation to an irrational number by a fraction with a square denominator was first proved by Hardy and Littlewood and subsequently improved by Vinogradov and Heilbronn (see [18] and references given there).

A later theorem[†] of Macbeath [19], considerably more general in scope, and yielding information about the position of those points

[†] For a simple proof of Macbeath's theorem, see Rogers [36].

of a non-homogeneous lattice in E_n (that is one which does not necessarily contain the origin), which are near the boundary of a given convex region leads, in particular, to an inequality for a general polynomial, provided that the signature of its quadratic section is $n - 1$:

THEOREM 1. 14. The inequalities

$$|L_1^2 + \dots + L_{n-1}^2 + L_n + c| \leq (C_n |\Delta|)^{2/(n+1)}$$

are always soluble in integers u_1, \dots, u_n with

$$C_n^{\frac{1}{2}(n+1)} = 2^{\frac{1}{2}(n-5)} \pi^{-\frac{1}{2}(n-1)} (n+1) \Gamma\left\{\frac{1}{2}(n+1)\right\}.$$

It is our object in Chapter 3 to establish two further theorems, analogous to Theorems 1. 12, 1. 13 for the general polynomial

$$f(u_1, \dots, u_n).$$

THEOREM 1. 15. If the coefficients of $f-c$ are not all in a rational ratio, f assumes arbitrarily small values for integers u_1, \dots, u_n .

THEOREM 1. 16. If the coefficients of $f-c$ are in a rational ratio, then there are integers u_1, \dots, u_n satisfying

$$|f(u_1, \dots, u_n)| \leq \left(\frac{1}{2} |\Delta|\right)^{2/(n+1)}$$

except possibly when $|s| = n-1$ and $n \geq 10$: the equality sign being required when $f \sim \lambda f_0$ where

$$f_0 = \pm u_1(u_1 + 1) \pm \dots \pm u_{n-1}(u_{n-1} + 1) + 2u_n + 1.$$

The proof of Theorem 1. 15, although based upon Weyl's result

as stated in Theorem 1. 5, is not immediate in the cases where the quadratic section of $f(u_1, \dots, u_n)$ represents zero non-trivially, and has signature $\pm(n-1)$. These arise only when $n \geq 3$ and our treatment (ad hoc for $n=3,4$) requires theorems of Oppenheim [23, II and III] on the representation of arbitrarily small numbers by indefinite zero quadratic forms in four or more variables. Theorem 1. 16 is proved by induction on n ; the case $n=2$ being contained in Theorem 1. 12. As we need this special case in Chapter 4, we include a version of Macbeath's proof.

The remaining problem, considered in Chapter 4, is concerned with the values of an indefinite quadratic form $q(u_1, u_2, u_3)$ of determinant D_3 . It is shown that $q(u_1, u_2, u_3)$ is equivalent to a form, each of whose diagonal coefficients is bounded above in terms of $|D_3|$. A result of this type clearly gives some information about generating points of \mathcal{L}_3 in the three-dimensional region $|x_1^2 + x_2^2 - x_3^2| \leq 1$, and may indeed be stated in geometrical terms. Problems of a similar type have already been considered by Minkowski and others. In particular, Minkowski [22] proved by simple geometrical arguments that the region

$$|x_1, x_2| \leq \frac{1}{2} \Delta_2$$

always contains two generating points of \mathcal{L}_2 . This result was extended by Chalk [7], who in a later paper [8] gave the following conjecture, proving it for $n=3$ and 4.

CONJECTURE. There exist n lattice points, generating Λ_n , in the region

$$|x_1 \dots x_n| \leq \frac{1}{2^{n-1}} \Delta_n.$$

In terms of quadratic forms the two theorems of Chapter 4 are as follows.

THEOREM 1. 17. If $q(u_1, u_2, u_3)$ represents zero non-trivially, it is equivalent to a form for which

$$|a_{ii}| \leq |D_3|^{1/3} \quad (i = 1, 2, 3)$$

with strict inequality unless $q \sim \lambda q_0$ or λq_1 where

$$q_0(u_1, u_2, u_3) = 2u_1u_2 + u_3^2$$

and

$$q_1(u_1, u_2, u_3) = 2u_1u_2 + u_2^2 + u_2u_3 + u_3^2.$$

THEOREM 1. 18. If $q(u_1, u_2, u_3)$ does not represent zero non-trivially, it is equivalent to a form for which

$$|a_{ii}| \leq \left(\frac{27}{25} |D_3|\right)^{1/3} \quad (i = 1, 2, 3)$$

with strict inequality unless $q \sim \lambda q_2$ where

$$q_2(u_1, u_2, u_3) = u_1^2 + u_1u_2 - u_2^2 + \frac{5}{2}u_3^2.$$

The proof of Theorem 1. 17 is not difficult, and depends on an elementary result in the theory of continued fractions and on the case $n = 2$ of Theorem 1. 16. In order to prove Theorem 1. 18 we use Theorems 1. 6, 1. 7 and 1. 8 together with an extension of Minkowski's theorem [22] on generating points of Λ_2 .

CHAPTER 2.

2.1 Let $q(u_1, \dots, u_n) = \sum_{\lambda=1}^n \sum_{s=1}^n a_{\lambda s} u_\lambda u_s \quad (a_{\lambda s} = a_{s\lambda})$

denote an indefinite quadratic form in n variables with real coefficients and with determinant $D_n \neq 0$. Blaney ([3], Theorem 2) proved that for any $\delta \geq 0$ there is a number $\Gamma = \Gamma(\delta, n)$ such that the inequalities

$$\delta |D_n|^{1/n} < q(u_1 + \alpha_1, \dots, u_n + \alpha_n) < \Gamma |D_n|^{1/n} \quad (2.1)$$

are soluble in integers u_1, \dots, u_n for any real $\alpha_1, \dots, \alpha_n$.

The object of this chapter is to establish an estimate for Γ as a function of δ . The result obtained, which is naturally only significant if δ is large, is as follows.

THEOREM 2.1. The inequalities (2.1) are always soluble in integers u_1, \dots, u_n if

$$\Gamma(\delta, n) = \delta + C_n \delta^{1/2^n} + C'_n, \quad (2.2)$$

where C_n, C'_n are suitable positive numbers depending only on n .

The value of Γ obtained by Blaney depends on an arbitrary parameter $c > 0$ and is not given explicitly, but examination of the proof shows that any value so found would exceed δ by an amount of order at least $\delta^{1/2}$ when δ is large. The more precise estimate (2.2) is obtained by a refinement of Blaney's argument. His proof is by induction on n and falls into two cases. In one case, after a reduction of the homogeneous form $q(u_1, \dots, u_n)$ not depending on δ , the choice of only one of u_1, \dots, u_n in (2.1) is made to depend on δ . At the corresponding stage in our proof, we allow at least one more

variable to depend on γ . The other case is treated on different lines from Blaney's and avoids the use of the arbitrary parameter c .

The case $n=2$ was considered by Blaney in a later paper ([4], Theorem 3), and the value

$$\Gamma(\gamma, 2) = \gamma + 2\gamma^{1/3} + O(1)$$

was obtained by methods of the geometry of numbers. The same estimate can be deduced from a result of Macbeath [20].

It is convenient to express Theorem 2.1 in a slightly more general form in terms of quadratic polynomials. Let $Q(u_1, \dots, u_{n+1})$ be a real quadratic form in $n+1$ variables, of determinant D_{n+1} , and suppose that the section $Q(u_1, \dots, u_n, 0)$ is an indefinite form in u_1, \dots, u_n of determinant $D_n \neq 0$.

Let

$$\Omega = |D_{n+1} D_n^{-1-1/n}|. \quad (2.3)$$

THEOREM 2.2. There exist integers u_1, \dots, u_n satisfying

$$0 < |D_n|^{-1/n} Q(u_1, \dots, u_n, 1) < C_n \Omega^{1/2^n} + C'_n. \quad (2.4)$$

Theorem 2.1 is an easy consequence of Theorem 2.2, as will be shown in §2.2 below.

2.2. The deduction of Theorem 2.1 from Theorem 2.2 is elementary.

Given the indefinite form $q(u_1, \dots, u_n)$ of determinant D_n , and the numbers $\alpha_1, \dots, \alpha_n, \gamma$, we apply Theorem 2.2 to the form

$$Q(u_1, \dots, u_{n+1}) = q(u_1 + \alpha_1 u_{n+1}, \dots, u_n + \alpha_n u_{n+1}) - \gamma (D_n)^{1/n} u_{n+1}^2.$$

This form has determinant

$$D_{n+1} = -D_n \gamma (D_n)^{1/n},$$

so that $\Omega = \gamma$ from (2.3). Theorem 2.2 asserts the existence of

integers u_1, \dots, u_n such that

$$0 < |D_n|^{-1/n} \{ q(u_1 + \alpha_1, \dots, u_n + \alpha_n) - \gamma |D_n|^{1/n} \} < C_n \gamma^{1/2^n} + C_n',$$

and this is (2.1) with Γ as in (2.2).

2.3. For the proof of Theorem 2.2 we need two lemmas.

The first is Lemma 1 of Blaney [3].

LEMMA 2.1. There exists p_n such that, for any indefinite quadratic form $q(u_1, \dots, u_n)$ of determinant D_n , there are integers u_1, \dots, u_n with highest common factor 1 satisfying

$$0 < q(u_1, \dots, u_n) \leq p_n |D_n|^{1/n}. \quad (2.5)$$

This holds in particular when $p_n = 2^{n-1}$.

COROLLARY. If $Q(u_1, \dots, u_n)$ is such that $Q(u_1, \dots, u_n, 0)$ is indefinite and has determinant D_n , and if $\epsilon = \pm 1$ is given, then there exists an integral unimodular substitution on u_1, \dots, u_n such that the coefficient of u_1^2 , say a_{11} , in the transformed form satisfies

$$0 < \epsilon a_{11} \leq p_n |D_n|^{1/n}. \quad (2.6)$$

LEMMA 2.2. For any real α , $a > 0$, $\beta > 0$, there exists an integer u such that

$$0 < a(u + \alpha)^2 - \beta \leq 2(a\beta)^{1/2} + a, \quad (2.7)$$

and provided $\beta \geq a$ there also exists an integer u such that

$$0 < -a(u + \alpha)^2 + \beta \leq 2(a\beta)^{1/2}. \quad (2.8)$$

Proof. (2.7) holds if

$$(\beta/a)^{1/2} < u + \alpha \leq (\beta/a)^{1/2} + 1,$$

and (2.8) holds if

$$(\beta/a)^{1/2} - 1 \leq u + \alpha < (\beta/a)^{1/2},$$

assuming $\beta \geq \alpha$.

2.4. Proof of Theorem 2.2 when $n=2$. If $D_3 = 0$ then

$Q(u, u_2, 1)$ is the product of two inhomogeneous linear forms of determinant $2|D_2|^{1/2}$. In this case, (2.4) is known to be soluble for any $C_2' > 2$ (see [13], Theorem 1). Hence we may suppose $D_3 \neq 0$. Since $Q(u, u_2, 0)$ is indefinite we have $D_2 < 0$.

By the Corollary to Lemma 2.1 we may suppose, without loss of generality, that

$$0 < |a_{11}| \leq 2|D_2|^{1/2}, \quad a_{11} D_3 < 0. \quad (2.9)$$

We have, identically,

$$Q(u, u_2, u_3) = a_{11}(u_1 + b_2 u_2 + b_3 u_3)^2 + \frac{D_2}{a_{11}}(u_2 + c_3 u_3)^2 + \frac{D_3}{D_2} u_3^2$$

for appropriate b_2, b_3, c_3 . Hence

$$Q(u, u_2, 1) = a_{11}(u_1 + b_2 u_2 + b_3)^2 - \frac{|D_2|}{a_{11}} \left\{ (u_2 + c_3)^2 - \frac{|a_{11} D_3|}{D_2^2} \right\}.$$

If $a_{11} > 0$, we choose the integer u_2 so that

$$u_2 + c_3 = \frac{|a_{11} D_3|^{1/2}}{|D_2|} + \theta, \quad (2.10)$$

where $0 < \theta \leq 1$. Then

$$Q(u, u_2, 1) = a_{11}(u_1 + \alpha)^2 - \beta,$$

where $\beta > 0$ and

$$\begin{aligned} a_{11} \beta &\leq |D_2| + 2|a_{11} D_3|^{1/2} \\ &\leq |D_2| + 2^{3/2} |D_2|^{1/4} |D_3|^{1/2} \\ &= |D_2| (1 + 2^{3/2} \Omega^{1/2}) \end{aligned}$$

by (2.9) and (2.3). By Lemma 2.2 there exists u_1 such that

$$\begin{aligned} 0 < Q(u, u_2, 1) &\leq a_{11} + 2(a_{11} \beta)^{1/2} \\ &< 2|D_2|^{1/2} + 2|D_2|^{1/2} (1 + 2^{3/4} \Omega^{1/4}), \end{aligned}$$

which is of the form (2.4).

If $a_{11} < 0$ we choose u_2 so that $2 < \theta \leq 3$ in (2.10). This gives

$$Q(u_1, u_2, 1) = -|a_{11}|(u_1 + 2)^2 + \beta,$$

where $\beta > 0$ and

$$|a_{11}|\beta = \theta (2 |a_{11}| D_3^{1/2} + |D_2| \theta).$$

Since $|a_{11}|\beta > 4|D_2| > a_{11}^2$

by (2.9), we have $\beta > |a_{11}|$ and the second part of Lemma 2.2 is applicable, giving

$$0 < Q(u_1, u_2, 1) < 2(|a_{11}|\beta)^{1/2}.$$

As before, this is of the form (2.4), but with different constants.

Since we do not know whether the exponent of Ω in (2.4) is best possible or not, there is little point in working out good values for the constants C_2, C_2' .

2.5. Proof of Theorem 2.2 when $n \geq 3$.

We proceed by

induction and assume the truth of the result when n is replaced by $n - 1$. Let the form $Q(u_1, \dots, u_n, 0)$ be expressed as a sum of n signed squares of real linear forms, and consider two cases according as (i) there are at least two positive signs, or (ii) there is just one positive sign.

Case 1. By the Corollary to Lemma 2.1 we can suppose that

$$0 < a_{11} \leq |p_n| |D_n|^{1/n}. \quad (2.11)$$

We have, identically,

$$Q(u_1, \dots, u_{n+1}) = a_{11} (u_1 + b_2 u_2 + \dots + b_{n+1} u_{n+1})^2 + Q'(u_2, \dots, u_{n+1})$$

for appropriate b_2, \dots, b_{n+1} , where Q' is a quadratic form in n variables of determinant $D_{n+1} a_{11}^{-1}$. Also $Q'(u_2, \dots, u_n, 0)$ is indefinite, by the hypothesis of the present case, and has determinant $D_n a_{11}^{-1}$. By the

inductive hypothesis, applied to $-Q'(u_2, \dots, u_n, 1)$, there exist integers u_2, \dots, u_n such that $-Q'(u_2, \dots, u_n, 1) = \beta$ where

$$0 < \beta < (|D_n| a_n^{-1})^{1/(n-1)} (C_{n-1} \Omega_1^{1/2^{n-1}} + C_{n-1}'),$$

where

$$\begin{aligned} \Omega_1 &= |D_{n+1} D_n^{-n/(n-1)} a_n^{1/(n-1)}| \\ &\leq p_n^{1/(n-1)} |D_{n+1} D_n^{-1-1/n}| = p_n^{1/(n-1)} \Omega, \end{aligned}$$

by (2.11) and (2.3). Hence, again by (2.11),

$$\begin{aligned} a_n \beta &< |D_n|^{1/(n-1)} (p_n |D_n|^{1/n})^{1-1/(n-1)} (E_n \Omega^{1/2^{n-1}} + E_n') \\ &= |D_n|^{2/n} (F_n \Omega^{1/2^{n-1}} + F_n'), \end{aligned}$$

for suitable constants E_n, E_n', F_n, F_n' . By Lemma 2.2 we can choose u_1 so that

$$0 < Q(u_1, \dots, u_n, 1) = a_n (u_1 + \alpha)^2 - \beta \leq 2(a_n \beta)^{1/2} + a_n;$$

and by (2.11) and the above estimate for $a_n \beta$, this gives a result of the form (2.4).

Case 2. By the Corollary to Lemma 2.1 we can suppose that

$$0 < -a_n \leq p_n |D_n|^{1/n} \quad (2.12)$$

As in Case 1,

$$Q(u_1, \dots, u_{n+1}) = -|a_n| (u_1 + b_2 u_2 + \dots + b_{n+1} u_{n+1})^2 + Q'(u_2, \dots, u_{n+1}),$$

and the determinant of $Q'(u_2, \dots, u_{n+1})$ is again $D_{n+1} a_n^{-1}$. The form

$Q'(u_2, \dots, u_n, 0)$ is indefinite, by the hypothesis of the present case,

and its determinant is $D_n a_n^{-1}$. Let

$$Q''(u_2, \dots, u_{n+1}) = Q'(u_2, \dots, u_{n+1}) - p_n |D_n|^{1/n} u_{n+1}^2.$$

Plainly $Q''(u_2, \dots, u_n, 0) = Q'(u_2, \dots, u_n, 0)$ and is therefore indefinite

with determinant $D_n a_n^{-1}$. We require the determinant of the form

$Q''(u_2, \dots, u_{n+1})$; since this differs only from that of $Q'(u_2, \dots, u_{n+1})$ by

having an amount $-p_n |D_n|^{1/n}$ added to the element in the bottom right hand corner, its value is

$$D_{n+1} a_n^{-1} - p_n |D_n|^{1/n} D_n a_n^{-1}.$$

By the inductive hypothesis, applied to Q'' , we can satisfy

$$0 < Q''(u_2, \dots, u_n, 1) < |D_n a_n^{-1}|^{1/(n-1)} (C_{n-1} \Omega_2^{1/2^{n-1}} + C_{n-1}'),$$

where

$$\Omega_2 = |D_{n+1} a_n^{-1} - p_n |D_n|^{1/n} D_n a_n^{-1}| \cdot |D_n a_n^{-1}|^{-n/(n-1)}.$$

Now

$$\begin{aligned} \Omega_2 &\leq |a_n|^{1/(n-1)} |D_n|^{-n/(n-1)} (|D_{n+1}| + p_n |D_n|^{1+1/n}) \\ &\leq p_n^{1/(n-1)} |D_n|^{-1-1/n} (|D_{n+1}| + p_n |D_n|^{1+1/n}) \\ &= p_n^{n/(n-1)} + p_n^{1/(n-1)} \Omega, \end{aligned}$$

by (2.12) and (2.3).

Thus $Q'(u_2, \dots, u_n, 1) = \beta$ where

$$p_n |D_n|^{1/n} < \beta < p_n |D_n|^{1/n} + |D_n a_n^{-1}|^{1/(n-1)} (G_n \Omega^{1/2^{n-1}} + G_n')$$

for suitable constants G_n, G_n' . Now

$$Q(u_1, \dots, u_n, 1) = -|a_{11}|(u_1 + \alpha)^2 + \beta,$$

and since the condition $\beta > |a_{11}|$ is satisfied by (2.12), it follows from the second part of Lemma 2.2 that there is an integer v , such that

$$0 < Q(u_1, \dots, u_n, 1) < 2(|a_{11}| \beta)^{1/2}.$$

We have

$$\begin{aligned} |a_{11}| \beta &< p_n |D_n|^{1/n} |a_{11}| + |D_n|^{1/(n-1)} |a_{11}|^{1-1/(n-1)} (G_n \Omega^{1/2^{n-1}} + G_n') \\ &< |D_n|^{2/n} (H_n \Omega^{1/2^{n-1}} + H_n'), \end{aligned}$$

and we again obtain a result of the form (2.4). This completes the proof.

CHAPTER 3.

3.1. Let L_1, \dots, L_n denote n homogeneous linear forms in n variables u_1, \dots, u_n with real coefficients and determinant $\Delta \neq 0$, and let c be any real number. We write

$$\mathcal{F} = \mathcal{F}(u_1, \dots, u_n) = \pm L_1^2 \pm \dots \pm L_{n-1}^2 + L_n + c, \quad (3.1)$$

$$q = q(u_1, \dots, u_n) = \pm L_1^2 \pm \dots \pm L_{n-1}^2, \quad (3.2)$$

where q is a singular quadratic form with rank $n-1$ and signature $s = 2\mu + 1 - n$, μ being the number of positive signs in (3.2). If \mathcal{F} and \mathcal{F}' are any two such polynomials, we say that they are equivalent, writing $\mathcal{F} \sim \mathcal{F}'$, if they can be transformed into one another by an integral, inhomogeneous, unimodular substitution of the variables u_1, \dots, u_n . From his general theorem [19] on infinite convex regions Macbeath has deduced, with a slightly different normalization, that, if $|s| = n-1$, the inequality

$$|\mathcal{F}| < (C_n |\Delta|)^{2/(n+1)} \quad (3.3)$$

is soluble in integers u_1, \dots, u_n with

$$C_n^{\frac{1}{2(n+1)}} = 2^{\frac{1}{2}(n-5)} \pi^{-\frac{1}{2}(n-1)} (n+1) \Gamma\left\{\frac{1}{2}(n+1)\right\}. \quad (3.4)$$

This constant is not best-possible and indeed, for large values of n , a substantial improvement can be obtained by a modification of his argument.† In this chapter we prove the following two theorems.

THEOREM 3.1. If the coefficients of $\mathcal{F} - c$ are not all in a rational ratio, \mathcal{F} assumes arbitrarily small values for integers u_1, \dots, u_n

† See next page.

THEOREM 3.2. If the coefficients of β -c are in a rational ratio, then there are integers u_1, \dots, u_n satisfying

$$|\beta| \leq \left(\frac{1}{2} |\Delta|\right)^{2/(n+1)} \quad (3.5)$$

except possibly when $|s| = n-1$ and $n \geq 10$.

We observe that when

$$\beta = \beta_0 = \pm u_1(u_1+1) \pm \dots \pm u_{n-1}(u_{n-1}+1) + 2u_n + 1,$$

the equality sign in (3.5) is necessary. For $\beta_0 \equiv 1 \pmod{2}$ when u_1, \dots, u_n are integers, and so

$$|\beta_0| \geq 1 = \left(\frac{1}{2} |\Delta|\right)^{2/(n+1)}$$

The proof of each theorem is to a large extent by induction on n , starting with the case $n=2$. This was studied by Macbeath [18], who obtained, in addition, some isolation results for the minimum of a quadratic polynomial in two variables. For completeness, we prove both theorems for $n=2$ in § 3.2. As in the general case these proofs depend on Lemmas 3.1 and 3.2. Lemma 3.1 is stated without proof and is a consequence of a general theorem [31] due to Weyl on the uniformity of distribution of the quadratic polynomial $\theta_1 x^2 + \theta_2 x + \theta_3 \pmod{1}$. Lemma 3.2, which gives an inequality for a quadratic in a single integral variable, is a corollary of Lemma 5 given by Davenport [11], but

† An application of Macbeath's theorem [19], combined with Blichfeldt's classical estimate for the critical determinant of an n -dimensional sphere, leads to a constant $C'_n \sim \frac{1}{2} C_n \sim \frac{n}{2\pi e}$ as $n \rightarrow \infty$.
 as $n \rightarrow \infty$. such that $C'_n \sim \frac{1}{2} C_n \sim \frac{n}{2\pi e}$ as $n \rightarrow \infty$.

for convenience a proof is given in § 3.2. The proof of Theorem 3.2, for general n [see § 3.3], is based upon the fact that a suitable integral unimodular substitution of the variables u_1, \dots, u_n reduces $q(u_1, \dots, u_n)$ to a non-singular quadratic form in $n-1$ variables of determinant D say, with the property that its first coefficient (say a) satisfies

$$0 < |a| \leq 2 |D|^{1/(n-1)}.$$

In the final section § 3.4 we prove Theorem 3.1, the proof depending mainly on Lemmas 3.1 and 3.5, and at one point on Theorem 3.2.

3.2. LEMMA 3.1. If at least one of θ_1, θ_2 is irrational, and ϵ is any assigned positive number, the inequality

$$|\theta_1 x^2 + \theta_2 x + \gamma + \theta_3| < \epsilon$$

has an infinity of solutions in integers x, y .

LEMMA 3.2.[†] If $n \geq 2$ and a, α, ϵ are any real numbers satisfying

$$0 < a \leq 2 \tag{3.6}$$

$$0 \leq \epsilon^2 \leq \frac{1}{4} a^2 - a + 2a^{1-1/n}, \tag{3.7}$$

then there is an integer u for which

$$|a(u + \alpha)^2 - a^{-1}\epsilon^2| \leq 1. \tag{3.8}$$

Proof. Put $f(u) = a(u + \alpha)^2 - a^{-1}\epsilon^2$

for convenience. Since

$$2a^{-1}(\epsilon^2 + a)^{1/2} \geq 2a^{-1/2} \gg \sqrt{2} > 1,$$

† See next page.

by (3.6) and (3.7), we can choose an integer u which satisfies

$$|u + \alpha| < a^{-1} (t^2 + a)^{1/2},$$

and then $f(u)$ satisfies

$$-a^{-1}t^2 \leq f(u) < 1.$$

Hence, if $t^2 < a$, (3.8) is valid with strict inequality. Suppose now that $t^2 \geq a$, and let u denote the integer for which

$$a^{-1} (t^2 + a)^{1/2} - 1 \leq u + \alpha < a^{-1} (t^2 + a)^{1/2}.$$

Clearly, if it also satisfies

$$u + \alpha > a^{-1} (t^2 - a)^{1/2},$$

(3.8) is again satisfied with strict inequality. Hence we may suppose that

$$a^{-1} (t^2 + a)^{1/2} - 1 \leq u + \alpha \leq a^{-1} (t^2 - a)^{1/2},$$

where $a^{-1} (t^2 + a)^{1/2} - 1 \leq a^{-1} (t^2 - a)^{1/2}$. (3.9)

The condition (3.9) combined with (3.7) gives us a lower bound for a : for on rearranging (3.9) and squaring we have

$$2(t^4 - a^2)^{1/2} \geq 2(t^2 - a^2),$$

where $2(t^2 - a^2) \geq 2a - a^2 = a(2 - a) \geq 0$.

Squaring once again, we then have

$$t^2 \geq 1 + \frac{1}{4}a^2$$

and so, on using (3.7), $2a^{1-1/n} - a \geq 1$,

which gives $a \geq 1$. (3.10)

The statement of the lemma remains true if we replace ~~(3.6)~~, (3.7) by

$$0 \leq t^2 \leq \frac{1}{4}a^2 - a + 2a^{1/2} \quad \text{for } 0 < a \leq 1$$

$$\text{and } 0 \leq t^2 \leq \frac{1}{4}a^2 + a \quad \text{for } 1 \leq a \leq 2.$$

since $n \geq 2$. Consider now $f(u-1)$. From our present choice of u , we have

$$a^{-1} (t^2 + a)^{1/2} - 2 \leq u - 1 + \alpha \leq a^{-1} (t^2 - a)^{1/2} - 1,$$

where

$$a^{-1} (t^2 - a)^{1/2} - 1 \leq \left(\frac{1}{4} - \frac{2}{a} + \frac{2}{a^{1+1/n}} \right)^{1/2} - 1 \leq \frac{1}{2} - 1 < 0,$$

by (3.10). Hence

$$a \{ a^{-1} (t^2 - a)^{1/2} - 1 \}^2 - a^{-1} t^2 \leq f(u-1) \leq a \{ a^{-1} (t^2 + a)^{1/2} - 2 \}^2 - a^{-1} t^2,$$

where

$$a \{ a^{-1} (t^2 + a)^{1/2} - 2 \}^2 - a^{-1} t^2 = 1 + 4 \{ a - (t^2 + a)^{1/2} \} \leq 1,$$

since

$$t^2 \geq a \geq a^2 - a$$

follows from $a \leq 2$, and where

$$a \{ a^{-1} (t^2 - a)^{1/2} - 1 \}^2 - a^{-1} t^2 = a - 1 - 2 (t^2 - a)^{1/2} \geq -1$$

since

$$a + \frac{1}{4} a^2 \geq \frac{1}{4} a^2 + 2a^{1-1/n} - a \geq t^2$$

follows from (3.7) and (3.10).

Note that equality in (3.8) occurs only when

either $t^2 = 5/4, a = 1, \alpha \equiv \frac{1}{2} \pmod{1}$

or $t^2 = 2, a = 2, \alpha \equiv 0 \pmod{1}, n \geq 3.$

Proof of Theorems 3.1 and 3.2 ($n = 2$). By considering a

suitable multiple of f in place of f if necessary we may assume

that $|\Delta| = 2$ (3.11)

Then it is sufficient to show that $f(u_1, u_2)$ assumes arbitrarily

small values for integers u_1, u_2 unless the coefficients of $\mathcal{F} - c$ are in a rational ratio; in which case, we show that $|\mathcal{F}| \leq 1$ is soluble in integers u_1, u_2 . Let $\epsilon > 0$. Then, by Minkowski's theorem on linear forms, there are coprime integers u_1^*, u_2^* satisfying

$$|L_1| \leq \epsilon^{1/2}, \quad |L_2| \leq 2 \epsilon^{-1/2}. \quad (3.12)$$

After applying a suitable integral unimodular substitution to the variables u_1, u_2 we can ensure that $(u_1^*, u_2^*) = (1, 0)$, and then $\pm \mathcal{F}$ can be expressed as

$$\pm \mathcal{F} = a(u_1 + b_2 u_2 + b_3)^2 + c_1 u_1 + c_2 u_2 + c_3 \quad (3.13)$$

for suitable real $a, b_2, b_3, c_1, c_2, c_3$, where

$$\text{either (i) } 0 < a \leq \epsilon, \quad c_1 = 0$$

$$\text{or (ii) } a > 0, \quad b_2 = b_3 = 0.$$

Note that (i) is an obvious consequence of (3.12) provided that $L_1 \neq 0$, while (ii) arises naturally when $L_1 = 0$.

Case (i). By (3.11), we have $a^{1/2} |c_2| = 2$. Choosing u_2, u_1 successively, and writing $c_2 u_2 + c_3 = -\lambda$, we can ensure that

$$0 \leq \lambda < |c_2| = 2a^{-1/2}, \quad (3.14)$$

$$|u_1 + b_2 u_2 + b_3 - \lambda^{1/2} a^{-1/2}| \leq \frac{1}{2}.$$

Then

$$|\mathcal{F}| \leq (a \lambda)^{1/2} + \frac{1}{4} a$$

$$< (2 a^{1/2})^{1/2} + \frac{1}{4} a,$$

by (3.14), and, since $a \leq \epsilon$, where ϵ is arbitrary, it follows that \mathcal{F} assumes arbitrarily small values for integral u_1, u_2 .

Case (ii) We may apply Lemma 3.1 directly to

$$\pm \wp = au_1^2 + c_1 u_1 + c_2 u_2 + c_3,$$

where $a^{\frac{1}{2}}|c_2| = 2$, by taking

$$\theta_1 = ac_2^{-1}, \quad \theta_2 = c_1 c_2^{-1}.$$

Thus, unless the coefficients of $\wp - c_3$ are in a rational ratio, \wp assumes arbitrarily small values. Otherwise, on writing $\pm \wp$ in the alternative form

$$\pm \wp = a(u_1 + \alpha)^2 + c_2 u_2 + \beta$$

with appropriate α, β we can establish the required inequality $|\wp| \leq 1$.

For, if $|c_2| < 2$, we may select any value of u_1 and then choose u_2 to satisfy $|\wp| \leq \frac{1}{2}|c_2| < 1$. If $|c_2| \geq 2$, we apply Lemma 3.2; note that since $a^{\frac{1}{2}}|c_2| = 2$, we have

$$a \leq 1. \quad (3.15)$$

Writing $-\lambda = c_2 u_2 + \beta$ for convenience, we choose u_2 first to satisfy

$$\frac{1}{4}a - 1 < \lambda \leq \frac{1}{4}a - 1 + 2a^{-1/2}. \quad (3.16)$$

Then, if $\lambda < 0$, the integer u_1 may be selected so that

$$|u_1 + \alpha| \leq \frac{1}{2}$$

and then $|\wp| < \frac{1}{4}a + (1 - \frac{1}{4}a) = 1$.

But, if $\lambda \geq 0$, we may introduce $t^2 = a\lambda$ and then the choice of u_1 in Lemma 3.2 ensures that $|\wp| \leq 1$, by (3.15) and (3.16); and in fact we again have strict inequality unless

$$\wp \sim C(u_1^2 + u_1 + 2u_2 - 1).$$

3.3. For the proof of Theorem 3.2 when $n > 2$ we require two

further lemmas.

LEMMA 3.3 If $q = q(u_1, \dots, u_n)$ is a singular quadratic form in the n variables u_1, \dots, u_n with rational coefficients and rank $n-1$, it is equivalent to a non-singular quadratic form in $n-1$ variables with rational coefficients.

Proof. By Lagrange's reduction of a quadratic form we can express q in the form

$$q = a_1 L_1^2 + \dots + a_{n-1} L_{n-1}^2$$

where a_1, \dots, a_{n-1} are rational non-zero numbers and L_1, \dots, L_{n-1} are $n-1$ independent linear forms in u_1, \dots, u_n with rational coefficients. If we write

$$L_\lambda = \sum_{s=1}^n \alpha_{\lambda s} u_s \quad (\lambda = 1, \dots, n-1),$$

where the $\alpha_{\lambda s}$ are rational, we may suppose, by permuting the variables, that

$$\|\alpha_{\lambda s}\|_{\lambda, s=1, \dots, n-1} \neq 0.$$

Then there exist rational numbers u_1, \dots, u_{n-1} satisfying the equations

$$L_1 = \dots = L_{n-1} = 0 \quad (3.17)$$

for any prescribed rational value of u_n . Since the equations (3.17) are linear, it follows that they have a solution in coprime integers

u_1^*, \dots, u_n^* , and, if we now apply a suitable unimodular substitution to the variables u_1, \dots, u_n , we may suppose that $(u_1^*, \dots, u_{n-1}^*, u_n^*) = (0, \dots, 0, 1)$. Hence L_1, \dots, L_{n-1} reduce to linear forms in $n-1$ variables with rational coefficients, and q is of the required form.

LEMMA 3.4. If $q = q(u_1, \dots, u_n)$ is an indefinite quadratic form in n variables u_1, \dots, u_n with rational coefficients and determinant D_n , then there are integers u_1, \dots, u_n satisfying

$$0 < q \leq 2 |D_n|^{1/n}.$$

Remark. We give the proof in two parts. The first is elementary and gives the required inequality for forms with small signature. This partial result interested Dr. G.L. Watson and he has recently found a proof of a rather better inequality valid for any signature. A brief outline of his argument is given in the second part of the proof. [See G.L. Watson, Quart. J. of Math. (Oxford) (2) 9 (1958) 99 - 108 for details.] This proof is not elementary and it would be of interest to find a direct proof † of Lemma 3.4.

Proof. Let the signature be s .

(i) Suppose $|s| \leq 8$. The lemma has been established for $n = 2, 3$ and 4 by Segre [28], Davenport [10] and Oppenheim [24], without the restriction that the forms be rational. Thus it is sufficient to consider values of $n \geq 5$, and the proof is inductive, depending on the cases $n = 3$ and 4 . By considering a suitable multiple of q in place of q , if necessary, we may suppose that $|D_n| = 1$,

† It seems difficult to modify our argument for $|s| \geq 9$.

and it suffices to show that the inequalities

$$0 < q \leq 2 \quad (3.18)$$

are soluble in integers u_1, \dots, u_n . By a classical theorem of Meyer's, q represents zero for relatively prime integers u_1^*, \dots, u_n^* and after applying a suitable integral unimodular substitution to the variables u_1, \dots, u_n we can take $(u_1^*, u_2^*, \dots, u_n^*) = (1, 0, \dots, 0)$. Then

$$q \sim a u_1 (\alpha_2 u_2 + \dots + \alpha_n u_n) + q_1(u_2, \dots, u_n)$$

for appropriate rational $a \neq 0$, relatively prime integers $\alpha_2, \dots, \alpha_n$, and q_1 . By a further integral unimodular substitution of the type

$$\begin{aligned} u_2' &= \alpha_2 u_2 + \dots + \alpha_n u_n, \\ u_n' &= \alpha_2^{(n)} u_2 + \dots + \alpha_n^{(n)} u_n \quad (n = 3, \dots, n), \end{aligned}$$

we see that $q \sim a u_1 u_2 + q_2(u_2, \dots, u_n)$,

or $q \sim a u_2 \{u_1 + b_2 u_2 + b_3 (\beta_3 u_3 + \dots + \beta_n u_n)\} + q_3(u_3, \dots, u_n)$,

for appropriate b_2, b_3 , relatively prime integers β_3, \dots, β_n and q_3 .

With a third integral unimodular substitution, on the variables u_3, \dots, u_n we have

$$q \sim a u_2 (u_1 + b_2 u_2 + b_3 u_3) + q_4(u_3, \dots, u_n), \quad (3.19)$$

where $q_4 = q_4(u_3, \dots, u_n)$ is a quadratic form in the $n-2$ variables u_3, \dots, u_n with rational coefficients and determinant \bar{D} , say.

Comparing determinants we see that

$$|\bar{D}| = 4 a^{-2}. \quad (3.20)$$

By changing the sign of u_2 , if necessary, we can ensure that

$$a > 0.$$

If $a \leq 2$, let $(u_2, u_3, \dots, u_n) = (1, 0, \dots, 0)$

and choose u_1 to be an integer satisfying

$$0 < a(u_1 + b_2) \leq a,$$

which gives (3.18). If $a > 2$, then

$$|\bar{D}| < 1, \tag{3.21}$$

by (3.20). Since the signature of q is an algebraic invariant, it follows, by (3.19), that q_4 has the same signature as q . Since the lemma holds for $n = 2, 3, 4$, we suppose that it is true for indefinite quadratic forms in at most $n-1 \geq 4$ variables, and then prove it true for indefinite quadratic forms in n variables. Thus the result will follow by induction on n . Since $|S| \leq 8$, $q_4(u_3, \dots, u_n)$ is a quadratic form in $n-2$ variables which is indefinite for $n \geq 11$ but may be either indefinite or definite when $5 \leq n \leq 10$. If $q_4(u_3, \dots, u_n)$ is indefinite, there are integers u_3, \dots, u_n satisfying

$$0 < q_4(u_3, \dots, u_n) \leq 2 |\bar{D}|^{1/(n-2)} \leq 2 \tag{3.22}$$

by the inductive hypothesis, which gives (3.18) with $u_1 = u_2 = 0$.

If, however, $q_4(u_3, \dots, u_n)$ is definite, in which case $5 \leq n \leq 10$, it is well known [5] by classical estimates that there are integers u_3, \dots, u_n satisfying (3.22) with q_4 replaced by $|q_4|$. Then (3.18) is again true with $u_1 = u_2 = 0$ unless q_4 is negative definite.

In this case there are relatively prime integers

$$(u_1, u_2, u_3, \dots, u_n) = (0, 0, u_3^*, \dots, u_n^*)$$

for which $q = -b$ say, where

$$0 < b \leq 2. \tag{3.23}$$

If we now apply a suitable unimodular substitution to u_1, \dots, u_n we may write

$$q = -b(u_1 + \alpha_{12}u_2 + \dots + \alpha_{1n}u_n)^2 + q'(u_2, \dots, u_n),$$

where $q' = q'(u_2, \dots, u_n)$ is an indefinite quadratic form in u_2, \dots, u_n of determinant

$$D' = \pm b^{-1}. \quad (3.24)$$

Then there are integers u_2^*, \dots, u_n^* for which $q' = \lambda$, where

$$0 < \lambda \leq 2b^{-1/(n-1)}, \quad (3.25)$$

by the inductive hypothesis. Hence

$$q(u_1, u_2^*u_2, \dots, u_n^*u_n) = -b(u_1 + \alpha_2' u_2)^2 + \lambda u_2^2,$$

for some α_2' . By the case $n=2$ of the lemma we can satisfy

$$0 < -b(u_1 + \alpha_2' u_2)^2 + \lambda u_2^2 \leq 2(b\lambda)^{1/2}$$

in integers u_1, u_2 ; this is sufficient for our purpose unless

$$\lambda b > 1 \quad (3.26)$$

Thus we suppose (3.26) true. If $\lambda \leq 2$, we choose an integer u_1 to satisfy

$$|u_1 + \alpha_2'| \leq \frac{1}{2}$$

and, since $\lambda - 2 \leq 0$, $\frac{1}{4} \leq b^{-2} \leq b^{-1}\lambda$,

by (3.23) and (3.26), we have

$$b^{-1}(\lambda - 2) \leq (u_1 + \alpha_2')^2 < b^{-1}\lambda, \quad (3.27)$$

which gives (3.18) with $u_2 = 1$. If $\lambda > 2$, (3.25) implies that

$$0 < b < 1.$$

We therefore have the inequality

$$(b+2)^2 > 8b^{1-1/(n-1)}, \quad \text{if } n \geq 5,$$

and so, by (3.25), $\lambda < (b+2)^2 / 4b$.

Thus

$$\lambda^{1/2} + (\lambda-2)^{1/2} < \frac{b+2}{2b^{1/2}} + \frac{2-b}{2b^{1/2}} = 2b^{-1/2},$$

and so $\lambda^{1/2} - (\lambda-2)^{1/2} = 2 / \{ \lambda^{1/2} + (\lambda-2)^{1/2} \} > b^{1/2}$.

Hence there is an integer u_1 satisfying (3.27). The lemma is now established when $|s| \leq 8$.

(ii) General case. Let q be expressed in the form

$$q = \xi_1^2 + \dots + \xi_n^2 - \xi_{n+1}^2 - \dots - \xi_{n+t}^2 \quad (n = r+t, s = r-t),$$

where ξ_1, \dots, ξ_n are real linear forms in the variables u_1, \dots, u_n .

Then we say that q is of type (r, t) . Let (r', t') be any positive integers satisfying

$$r' \equiv r \pmod{4}, \quad t' \equiv t \pmod{4}, \quad r' + t' = n.$$

Then the following principle due to Watson is sufficient for our purpose.

For $n \geq 4$, there exists a form q' in the n variables u_1, \dots, u_n with the same determinant as q and of type (r', t') with the property that it represents, for integral values of the variables, the same set of values as q .

Thus, in particular, there is such a form q' with signature s' satisfying $|s'| \leq 4$, and clearly it must be indefinite.

Proof of Theorem 3.2 when $n > 2$. We suppose that the theorem is valid for polynomials in $n-1$ variables, and then prove it valid for n variables. Since the theorem has been established for two variables, the result follows by induction.

There is again no loss of generality in taking $|\Delta| = 2$, and it now suffices to show that there are integers u_1, \dots, u_n for which

$$|\mathcal{F}| \leq 1 \quad (3.28)$$

provided that $n \leq 9$ when $|s| = n-1$. By Lemma 3.3, we may suppose that l_1, \dots, l_{n-1} are linear forms in the $n-1$ variables u_1, \dots, u_{n-1} with determinant $\Delta' \neq 0$. If $|s| = n-1 \leq 8$, it is well known [5] that we can apply a further unimodular substitution to the variables u_1, \dots, u_{n-1} to ensure that the coefficient of u_1^2 , say a , satisfies

$$0 < \kappa a \leq 2 |\Delta'|^{2/(n-1)}, \quad (3.29)$$

where
$$\kappa = \begin{cases} 1 & \text{if } s = n-1, \\ -1 & \text{if } s = -(n-1). \end{cases} \quad (3.30)$$

Otherwise, if $|s| = n-1$, we can ensure that a satisfies (3.29) with

$$\kappa = \begin{cases} 1 & \text{if } -(n-3) < s \leq n-3, \\ -1 & \text{if } s = -(n-3), \end{cases} \quad \text{by Lemma 3.4.} \quad (3.31)$$

Thus in either case we may suppose, by considering $-\mathcal{F}$ in place of \mathcal{F} if necessary, that

$$0 < a \leq 2 |\Delta'|^{2/(n-1)}. \quad (3.32)$$

We write
$$\mathcal{F} = a \bar{l}_1^2 + \bar{l}_2^2 + \dots + \bar{l}_{n-1}^2 + \bar{l}_n + c,$$

where

$$\bar{l}_\lambda = \sum_{s=1}^{n-1} \alpha_{\lambda s} u_s \quad (\lambda = 1, \dots, n-1), \quad \bar{l}_n = \sum_{s=1}^n \alpha_{ns} u_s,$$

$$\alpha_{11} = 1, \quad \alpha_{\lambda 1} = 0 \quad (\lambda = 2, \dots, n-1).$$

To evaluate the absolute value of the determinant $\bar{\Delta}$, say, of the linear forms $a^{\frac{1}{2}} \bar{L}_1, \bar{L}_2, \dots, \bar{L}_n$ we consider the quadratic form

$$\pm L_1^2 \pm \dots \pm L_{n-1}^2 + L_n u_{n+1} + c u_{n+1}^2$$

in the original n variables u_1, \dots, u_n and a new variable u_{n+1} .

Its determinant ± 1 is clearly left invariant under unimodular substitutions applied to u_1, \dots, u_n . Thus the quadratic form

$$a \bar{L}_1^2 \pm \bar{L}_2^2 \pm \dots \pm \bar{L}_{n-1}^2 + \bar{L}_n u_{n+1} + c u_{n+1}^2$$

also has determinant $\pm \frac{1}{a} \bar{\Delta}^2 = \pm 1$, and so $|\bar{\Delta}| = 2$. Hence,

comparing determinants,

$$|\Delta' \alpha_{nn}| = 2. \quad (3.33)$$

It is convenient to express \mathcal{F} in the form

$$\mathcal{F} = a (\bar{L}_1 + b)^2 \pm \bar{L}_2^2 \pm \dots \pm \bar{L}_{n-1}^2 + L_n' + c',$$

where the absolute value of the determinant of $\bar{L}_2, \dots, \bar{L}_{n-1}, L_n'$ in the $n-1$ variables u_2, \dots, u_n is

$$a^{-\frac{1}{2}} |\Delta' \alpha_{nn}| = 2 a^{-\frac{1}{2}},$$

by (3.33). The theorem is trivial if $|\alpha_{nn}| \leq 2$ since for any values of u_1, \dots, u_{n-1} we can always choose an integer u_n so that

$$|\mathcal{F}| \leq \frac{1}{2} |\alpha_{nn}| \leq 1.$$

We therefore suppose that $|\alpha_{nn}| > 2$, which implies that $|\Delta'| < 1$, by (3.33), and hence

$$0 < a < 2. \quad (3.34)$$

Consider the polynomial

$$\bar{\mathcal{F}} = \bar{\mathcal{F}}(u_2, \dots, u_n) = \pm \bar{L}_2^2 \pm \dots \pm \bar{L}_{n-1}^2 + L_n' + c'.$$

If $|s| = n-1 \leq 8$, the quadratic section of \bar{f} is a definite quadratic form in u_2, \dots, u_n , while, if $|s| \neq n-1$, our choice of a in (3.29) and (3.31) ensures that the quadratic section of \bar{f} is an indefinite quadratic form in u_2, \dots, u_{n-1} . In either case, therefore, we can apply the inductive hypothesis to \bar{f} to establish the existence of integers u_2, \dots, u_n for which

$$\bar{f} = -\lambda,$$

where $\frac{1}{4}a - 1 < \lambda \leq \frac{1}{4}a - 1 + 2 \left(\frac{1}{2} \cdot 2a^{-\frac{1}{2}} \right)^{2/n}$.

With these values of u_2, \dots, u_n we have

$$f = a(u_1 + \alpha)^2 - \lambda, \quad (3.35)$$

for an appropriate value of α . If

$$\frac{1}{4}a - 1 < \lambda < 0,$$

we can always choose an integer u_1 so that

$$|u_1 + \alpha| \leq \frac{1}{2},$$

and then $|f| < \frac{1}{4}a + (1 - \frac{1}{4}a) = 1$.

Thus we can suppose that

$$0 \leq \lambda \leq \frac{1}{4}a - 1 + 2a^{-1/n},$$

or, if $t^2 = a\lambda$, that

$$0 \leq t^2 \leq \frac{1}{4}a^2 - a + 2a^{1-1/n}. \quad (3.36)$$

Hence, by Lemma 3.2, there is an integer u_1 satisfying

$$|a(u_1 + \alpha)^2 - a^{-1}t^2| \leq 1,$$

by (3.34), (3.35), (3.36). This completes the proof of the theorem.

Remark. If $|s| = n-1 \geq 9$, the argument above gives a constant

$$\left\{ \frac{1}{2} \left(\frac{\gamma_{n-1}}{2} \right)^{\frac{1}{2}(n-1)} \right\}^{2/(n+1)}$$

in place of $2^{-2/(n+1)}$ on the right-hand side of (3.5). This is asymptotically equivalent to Macbeath's constant $c_n^{2/(n+1)}$ in (3.3), for large values of n , when we introduce Blichfeldt's estimate for γ_{n-1} .

3.4. Before proving the general case of Theorem 3.1 we require a further lemma.

LEMMA 3.5. For $n \geq 3$ suppose that the equations

$L_1 = \dots = L_{n-1} = 0$ have no solution in integers u_1, \dots, u_n other than $0, \dots, 0$. Then the singular quadratic form

$$\pm L_1^2 \pm \dots \pm L_{n-1}^2$$

in the variables u_1, \dots, u_n assumes an arbitrarily small non-zero value for integers u_1, \dots, u_n .†

Proof. If we write

$$L_n = \sum_{s=1}^n \alpha_{ns} u_s \quad (n=1, \dots, n-1), \quad (3.37)$$

the matrix $(\alpha_{ns})_{n-1, n}$ has rank $n-1$, by the hypothesis, and we may suppose, without loss of generality, that

$$\Delta_1 = \|\alpha_{ns}\|_{n, s=1, \dots, n-1} \neq 0.$$

For any $\epsilon > 0$, there are an infinity of integers u_1, \dots, u_n satisfying

$$|L_c| \leq \{\epsilon / (n-1)\}^{1/2} \quad (c=1, \dots, n-1), \quad |u_n| \leq \{\epsilon / (n-1)\}^{-(n-1)/2} |\Delta_1|, \quad (3.38)$$

by Minkowski's theorem on linear forms and our hypothesis concerning

L_1, \dots, L_{n-1} . Hence

$$|\pm L_1^2 \pm \dots \pm L_{n-1}^2| \leq \epsilon$$

† It is assumed that L_1, \dots, L_{n-1} are linearly independent.

for integers u_1, \dots, u_n and, unless

$$\pm L_1^2 \pm \dots \pm L_{n-1}^2 = 0,$$

when $\pm L_1^2 \pm \dots \pm L_{n-1}^2 \neq \pm (L_1^2 + \dots + L_{n-1}^2),$

there is nothing further to prove. Suppose that there are integers

$$(u_1, \dots, u_n) \neq (0, \dots, 0)$$

for which $\pm L_1^2 \pm \dots \pm L_{n-1}^2 = 0.$

Consider first the cases $n = 3$ and $4.$

If $n = 3,$ it is well known ([6], II, Theorem 3) that there are integers u_1, u_2, u_3 satisfying

$$0 < (L_1 - L_2)(L_1 + L_2) < \epsilon.$$

If $n = 4,$ each of the integer-vectors u_1, \dots, u_4 satisfying (3.38) gives rise to a point $(x_1, x_2, x_3) = (L_1, L_2, L_3)$ in 2-dimensional projective space $S_2.$ Two at least of these points, say $P(\xi_1, \xi_2, \xi_3)$ and $Q(\eta_1, \eta_2, \eta_3),$ must be distinct, and, unless they both lie on the conic

$$x_1^2 \pm (x_2^2 - x_3^2) = 0,$$

there is nothing further to prove. If, however, they both lie on this conic, then $P+Q$ does not, for we cannot have three collinear points on a conic, and hence

$$0 < |(\xi_1 + \eta_1)^2 \pm \{(\xi_2 + \eta_2)^2 - (\xi_3 + \eta_3)^2\}| \leq 4\epsilon.$$

In other words, there are integers u_1, \dots, u_4 satisfying

$$0 < |L_1^2 \pm (L_2^2 - L_3^2)| \leq 4\epsilon.$$

which is again the required result.

We now suppose that $n \geq 5$ and, by interchanging the l_i ($i=1, \dots, n-1$) if necessary, that

$$\pm l_1^2 \pm \dots \pm l_{n-1}^2 = \pm (l_1^2 \pm \dots - l_{n-1}^2) = 0$$

for relatively prime integers (u_1^*, \dots, u_n^*) . By applying a suitable integral unimodular substitution to u_1, \dots, u_n we may assume further that

$$(u_1^*, u_2^*, \dots, u_n^*) = (1, 0, \dots, 0).$$

If we again write

$$L_\lambda = \sum_{s=1}^n \alpha_{\lambda s} u_s \quad (\lambda = 1, \dots, n-1),$$

where the $\alpha_{\lambda s}$ are not necessarily the same as before, the rank of the matrix $(\alpha_{\lambda s})_{n-1, n}$ is $n-1$, by hypothesis. Let \underline{a}_i ($i=1, \dots, n$) denote the i 'th column vector in $(\alpha_{\lambda s})$; and let A_i ($i=1, \dots, n$) denote the $(n-1) \times (n-1)$ matrices formed from $(\alpha_{\lambda s})$ by omitting the i 'th column. Since the rank of $(\alpha_{\lambda s})$ is $n-1$, at least one of $|A_i|$ ($i=1, \dots, n$) does not vanish. Consider the effect on $(\alpha_{\lambda s})$ of a substitution of the type

$$u_\lambda = u_\lambda' + \delta_{\lambda i} u_j' \quad (\lambda = 1, \dots, n), \quad (3.39)$$

where $1 \leq i, j \leq n$ ($i \neq j$), $\delta_{\lambda i} = \begin{cases} 1 & \text{if } \lambda = i, \\ 0 & \text{otherwise,} \end{cases} \quad (3.40)$

applied to the forms L_1, \dots, L_{n-1} . For convenience of notation, denote the new matrix of coefficients by $(\alpha'_{\lambda s})$ and use \underline{a}'_i, A'_i with their natural meanings. Then clearly $\underline{a}'_k = \underline{a}_k$ for $k \neq i$, and $\underline{a}'_j = \underline{a}_j + \underline{a}_i$; and so

$$|A'_k| = |A_k| \quad (k \neq i), \quad |A'_i| = |A_i| \pm |A_j|.$$

If, for any $i \geq 2$, $|A_i| = 0$, we can find j so that $|A_j| \neq 0$, and then the substitution (3.39) makes $|A'_i| \neq 0$.

We are allowed to perform any substitution of type (3.39) with $i \geq 2$, and so it follows that the set of forms given by (3.37) may be replaced by an equivalent set for which

$$\alpha_{11}^2 \pm \dots - \alpha_{n-1,1}^2 = 0, \quad |A_k| \neq 0 \quad (k=2, \dots, n).$$

If we put $u_n = 0$, $L_1^2 \pm \dots - L_{n-1}^2$ is a quadratic form in $n-1 \geq 4$ variables u_1, \dots, u_{n-1} with determinant $|A_n| \neq 0$. Moreover it represents zero for integers $(u_1, u_2, \dots, u_{n-1}) = (1, 0, \dots, 0) \neq (0, \dots, 0)$ and so it is indefinite. Hence, by theorems given by Oppenheim ([23], II and III), it represents arbitrarily small non-zero values unless it is a multiple of a form with rational coefficients. In the latter case, all the terms in the original form $\pm (L_1^2 \pm \dots - L_{n-1}^2)$ which do not involve u_n are in a rational ratio. If we repeat the argument with $u_{n-1} = 0$ and with $u_{n-2} = 0$, since there is no term in $\pm (L_1^2 \pm \dots - L_{n-1}^2)$ involving three variables, we deduce that either $\pm (L_1^2 \pm \dots - L_{n-1}^2)$ assumes an arbitrarily small non-zero value for integers u_1, \dots, u_n or it is equivalent to a multiple of a rational form in n variables. In the latter case, by Lemma 3.3, $\pm (L_1^2 \pm \dots - L_{n-1}^2)$ is equivalent to a multiple of a non-singular rational quadratic form in $n-1$ variables, and we now show that this is impossible by our hypothesis concerning L_1, \dots, L_{n-1} . For, after applying a suitable unimodular substitution to the variables u_1, \dots, u_n we may assume that $\pm (L_1^2 \pm \dots - L_{n-1}^2)$ is a non-singular quadratic form in the $n-1$ variables u_1, \dots, u_{n-1} . We may write

$$L_n = \sum_{s=1}^n \alpha_{ns} u_s \quad (n=1, \dots, n),$$

where $\Delta = \|\alpha_{rs}\|_{r,s=1,\dots,n} \neq 0$. Then, by differentiating $\pm (L_1^2 \pm \dots \pm L_{n-1}^2)$ partially with respect to u_n , we have

$$\alpha_{1n} L_1 \pm \alpha_{2n} L_2 \pm \dots \pm \alpha_{n-1,n} L_{n-1} = 0$$

since the quadratic form, when expressed in terms of the u_i ($i=1,\dots,n$), is independent of u_n . Hence

either (i) L_1, \dots, L_{n-1} are linearly dependent
or (ii) $\alpha_{1n} = \alpha_{2n} = \dots = \alpha_{n-1,n} = 0$.

Clearly (i) is impossible since $\Delta \neq 0$. In (ii), none of the forms L_1, \dots, L_{n-1} involve u_n , and so the equations $L_1 = \dots = L_{n-1} = 0$ are soluble with

$$(u_1, \dots, u_{n-1}, u_n) = (0, \dots, 0, 1),$$

which is contrary to the hypothesis of the lemma. This completes the proof.

Proof of Theorem 3.1 when $n > 2$. As in the proof of

Theorem 3.2 we suppose that the theorem is true for $n-1$ variables and then prove it is true for n variables, the case $n=2$ having been established independently. The proof divides into two parts according as the $n-1$ equations

$$L_1 = \dots = L_{n-1} = 0 \tag{3.41}$$

have, or do not have, a non-trivial solution in integers u_1, \dots, u_n .

If the equations (3.41) have a non-trivial solution in integers u_1^*, \dots, u_n^* , which we may take to be relatively prime, we may suppose, by applying a suitable unimodular substitution to the variables

u_1, \dots, u_n that

$$(u_1^k, \dots, u_{n-1}^k, u_n^k) = (0, \dots, 0, 1).$$

Then L_1, \dots, L_{n-1} are linear forms in the $n-1$ variables u_1, \dots, u_{n-1} only. We may write

$$f = q(u_1, \dots, u_{n-1}) + L_n + c, \quad (3.42)$$

where $q(u_1, \dots, u_{n-1}) = \sum_{\lambda=1}^{n-1} \sum_{s=1}^{n-1} \alpha_{\lambda s} u_\lambda u_s$ ($\alpha_{\lambda s} = \alpha_{s\lambda}$),

$$L_n = \sum_{s=1}^n \alpha_{ns} u_s,$$

and, if $D = \|\alpha_{\lambda s}\|_{\lambda, s=1, \dots, n-1}$, comparing determinants we see that

$$\pm \Delta = |D|^{\frac{1}{2}} \alpha_{nn}.$$

Since $\Delta \neq 0$, by hypothesis, it follows that $\alpha_{nn} \neq 0$. If we take all the variables to be zero except u_λ, u_s, u_n where $1 \leq \lambda, s \leq n-1$, we have

$$f = \alpha_{\lambda\lambda} u_\lambda^2 + 2\alpha_{\lambda s} u_\lambda u_s + \alpha_{ss} u_s^2 + \alpha_{n\lambda} u_\lambda + \alpha_{ns} u_s + \alpha_{nn} u_n + c.$$

Now put $u_s = 0$. Then, by Lemma 3.1 with

$$\theta_1 = \alpha_{nn}^{-1} \alpha_{\lambda\lambda}, \quad \theta_2 = \alpha_{nn}^{-1} \alpha_{n\lambda},$$

it follows that f assumes an arbitrarily small value for integers u_λ, u_n unless the ratios

$$\alpha_{nn}^{-1} \alpha_{\lambda\lambda}, \quad \alpha_{nn}^{-1} \alpha_{n\lambda} \quad (\lambda = 1, \dots, n-1)$$

are all rational. We therefore suppose that this is the case and now take $u_\lambda = u_s$. By a further application of Lemma 3.1 with

$$\theta_1 = \alpha_{nn}^{-1} (\alpha_{\lambda\lambda} + 2\alpha_{\lambda s} + \alpha_{ss}), \quad \theta_2 = \alpha_{nn}^{-1} (\alpha_{n\lambda} + \alpha_{ns})$$

we deduce that f again assumes an arbitrarily small value for

integers u_s, u_n unless the ratios

$$\alpha_{nn}^{-1} u_{ns}, \quad \alpha_{nn}^{-1} \alpha_{nn} \quad (s, n = 1, \dots, n-1)$$

are all rational. However, this is impossible since the coefficients of $\mathcal{F} - c$ are not all in a rational ratio, and we therefore have the required result.

If the equations (3.41) have no solution in integers (u_1, \dots, u_n) other than $(0, \dots, 0)$, then, by Lemma 3.5, $\pm L_1^2 \pm \dots \pm L_{n-1}^2$ assumes an arbitrarily small non-zero value for integers (u_1^*, \dots, u_n^*) , say. We may assume that these integers are relatively prime, and by applying a suitable unimodular substitution to the variables u_1, \dots, u_n we may suppose further that $(u_1^*, u_2^*, \dots, u_n^*) = (1, 0, \dots, 0)$. Thus, for any $\epsilon > 0$, we may write

$$\pm \mathcal{F} = a (\bar{L}_1 + \alpha)^2 + \bar{\mathcal{F}}(u_2, \dots, u_n),$$

where $0 < a < \epsilon$.

Here \bar{L}_1 is a linear form in u_1, \dots, u_n with the coefficient of u_1 equal to unity and $\bar{\mathcal{F}} = \bar{\mathcal{F}}(u_2, \dots, u_n)$ is a polynomial of the form

$$\bar{\mathcal{F}} = \pm \bar{L}_2^2 \pm \dots \pm \bar{L}_{n-1}^2 + \bar{L}_n + \bar{c},$$

$\bar{L}_2, \dots, \bar{L}_n$ being linear forms in u_2, \dots, u_n of determinant $\pm a^{-\frac{1}{2}} \Delta$.

If the coefficients of $\bar{\mathcal{F}} - \bar{c}$ are not all in a rational ratio, by our inductive hypothesis, there exist integers u_2, \dots, u_n satisfying

$$|\bar{\mathcal{F}}| < \frac{1}{2} \epsilon.$$

If we now choose an integer u_1 so that

$$|\bar{L}_1 + \alpha| \leq \frac{1}{2},$$

we have, for these values of u_2, \dots, u_n

$$|\mathcal{F}| < \frac{1}{2} \epsilon + \frac{1}{2} \epsilon = \epsilon.$$

Alternatively, if the coefficients of $\bar{p} - \bar{c}$ are all in a rational ratio, we apply Theorem 3.2 or Macbeath's theorem [19] for polynomials in $n-1$ variables. Thus there are integers u_1, \dots, u_n for which $\bar{p} = -\lambda$, where

$$0 \leq \lambda \leq 2 (C_n |\Delta| a^{-\frac{1}{2}})^{2/n},$$

$$\text{or } 0 \leq a\lambda \leq 2 a^{1-1/n} (C_n |\Delta|)^{2/n} \quad (3.43)$$

for some constant C_n depending only on n . For these values of u_1, \dots, u_n we have

$$\pm \bar{p} = a(u_1 + \alpha)^2 - \lambda.$$

We can always choose an integer u_1 such that

$$u_1 + \alpha = (a^{-1}\lambda)^{1/2} + \theta,$$

where θ is a suitable real number satisfying

$$|\theta| \leq \frac{1}{2}. \quad (3.44)$$

With this value of u_1 we have

$$\pm \bar{p} = 2(a\lambda)^{1/2} \theta + a\theta^2,$$

and hence $|\bar{p}| \leq (a\lambda)^{1/2} + \frac{1}{4}a = O(a)$

by (3.43) and (3.44). Since $0 < a < \epsilon$ and ϵ is arbitrary, it follows that $|\bar{p}|$ can be made arbitrarily small for integers u_1, \dots, u_n .

This completes the proof of the theorem.

CHAPTER 4.

4.1. Let \mathcal{L}_n be a lattice of determinant $\Delta_n > 0$ in some n -dimensional Euclidean space E_n . The problem of finding infinite regions in E_n which contain the origin and n further generating points of \mathcal{L}_n has already been considered by Minkowski. In particular, Minkowski [22] proved by simple geometrical arguments that the region

$$|x_1, x_2| \leq \frac{1}{2} \Delta_2$$

always contains two generating points of \mathcal{L}_2 . Chalk [7] obtained a generalisation of this result, and later suggested the following conjecture [8], which he proved for $n=3$ and 4.

CONJECTURE. There exist n lattice points, generating \mathcal{L}_n in the region

$$|x_1, \dots, x_n| \leq \frac{1}{2^{n-1}} \Delta_n.$$

Clearly the conjectured inequality, if true for general n , would be best possible when the lattice \mathcal{L}_n is of the form

$$x_i = u_i + \frac{1}{2} u_n \quad (i=1, \dots, n-1), \quad x_n = u_n.$$

Further results of a slightly different nature concerning generating points in E_2 and E_3 have also been obtained by Chalk and Rogers [9], Barnes [1] and Oppenheim [25].

Our object is to prove the following two theorems, which

yield information about sets of generating points of \mathcal{L}_3 in the three dimensional region $|x_1^2 + x_2^2 - x_3^2| \leq 1$.

THEOREM 4.1. If \mathcal{L}_3 has a point, other than the origin, on the surface $x_1^2 + x_2^2 - x_3^2 = 0$, the region

$$|x_1^2 + x_2^2 - x_3^2| \leq \Delta_3^{2/3} \quad (4.1)$$

contains a set of three generating points of \mathcal{L}_3 .

THEOREM 4.2. If \mathcal{L}_3 has no point, other than the origin, on the surface, $x_1^2 + x_2^2 - x_3^2 = 0$, the region

$$|x_1^2 + x_2^2 - x_3^2| \leq \left(\frac{27}{25} \Delta_3^2\right)^{1/3} \quad (4.2)$$

contains a set of three generating points of \mathcal{L}_3 .

We shall show that the inequalities (4.1) and (4.2) are best possible. Before doing so, however, it is convenient to restate Theorems 4.1 and 4.2 in terms of indefinite quadratic forms in three variables. For, if \mathcal{L}_3 is given by equations of the form

$$x_\lambda = \sum_{s=1}^3 \alpha_{\lambda s} u_s \quad (\lambda = 1, 2, 3),$$

where $\Delta_3 = \|\alpha_{\lambda s}\|_{\lambda, s=1, 2, 3}$, clearly $x_1^2 + x_2^2 - x_3^2$ may be expressed as an indefinite quadratic form

$$q(u_1, u_2, u_3) = \sum_{\lambda=1}^3 \sum_{s=1}^3 a_{\lambda s} u_\lambda u_s \quad (a_{\lambda s} = a_{s\lambda})$$

for appropriate $a_{\lambda s}$ ($\lambda, s = 1, 2, 3$), with determinant

$D = \|\alpha_{\lambda s}\|_{\lambda, s=1, 2, 3}$. On comparison of determinants, we see that

$$D_3 = -\Delta_3^2 < 0.$$

The following Theorems 4.1^{*}, 4.2^{*}, which are expressed in terms of

quadratic forms, contain the assertions of Theorems 4.1, 4.2 respectively, and we prove them in this form.

THEOREM 4.1 ^{*}. If $q(u_1, u_2, u_3)$ represents zero non-trivially, it is equivalent to a form for which

$$|a_{ii}| \leq |D_3|^{1/3} \quad (i=1, 2, 3) \quad (4.1)^*$$

with strict inequality unless $q \sim \lambda q_0$ or λq_1 where

$$q_0(u_1, u_2, u_3) = 2u_1u_2 + u_3^2$$

and

$$q_1(u_1, u_2, u_3) = 2u_1u_2 + u_2^2 + u_1u_3 + u_3^2.$$

THEOREM 4.2 ^{*}. If $q(u_1, u_2, u_3)$ does not represent zero non-trivially, it is equivalent to a form for which

$$|a_{ii}| \leq \left(\frac{27}{25} |D_3|\right)^{1/3} \quad (i=1, 2, 3) \quad (4.2)^*$$

with strict inequality unless $q \sim \lambda q_2$ where

$$q_2(u_1, u_2, u_3) = u_1^2 + u_1u_2 - u_2^2 + \frac{5}{2}u_3^2.$$

The proof of Theorem 4.1 ^{*} in § 4.2 divides into two cases, in one of which we use an elementary result in the theory of continued fractions (Lemma 4.1) to replace the inequalities (4.1) ^{*} by

$$|a_{ii}| < \epsilon \quad (i=1, 2, 3)$$

[†] We observe that the unimodular substitution

$$u_1 = U_1, \quad u_2 = U_2 + U_3, \quad u_3 = U_3$$

transforms q_2 into Q_2 where

$$Q_2(U_1, U_2, U_3) = U_1^2 - U_2^2 + \frac{3}{2}U_3^2 + U_1U_2 + U_1U_3 - 2U_2U_3.$$

for any $\epsilon > 0$. The other case is less trivial and the proof depends upon a theorem ([18], Theorem 1) of Macbeath (Lemma 4.2) on values of a quadratic polynomial in two variables.

The proof of Theorem 4.2* is rather different and is based upon four further lemmas. Lemma 4.3, which is needed as a starting point for the proof of the theorem, is classical and gives the first 'minimum' for an indefinite quadratic form in three variables. Lemma 4.4 is a straightforward extension, to a two-dimensional asymmetric hyperbolic region, of Minkowski's original theorem on generating points of \mathcal{L}_2 . The result stated in Lemma 4.5 is a special case of a recent theorem of Watson on values of a non-zero binary quadratic form. Finally, Lemma 4.6 gives a useful inequality for a quadratic in a single integral variable, and its proof is contained in that of Lemma 3.2.

4.2. For the proof of Theorem 4.1* we require the following two lemmas.

LEMMA 4.1. If α is a given positive irrational number and $\epsilon > 0$, the inequalities

$$0 < |q_n \alpha - p_n| < \epsilon \quad \text{and} \quad 0 < |q_{n+1} \alpha - p_{n+1}| < \epsilon$$

are always soluble in integer pairs (p_n, q_n) and (p_{n+1}, q_{n+1}) with

$$\underline{p_n q_{n+1} - p_{n+1} q_n = 1.}$$

Proof. Take $p_n/q_n, p_{n+1}/q_{n+1}$ to be successive convergents to the continued fraction for α , with n odd and sufficiently large.

LEMMA 4.2. If L_1, L_2 are two real linear forms in the variables u_1, u_2 with determinant $\Delta \neq 0$, and c is any real number, the inequalities

$$|L_1^2 + L_2 + c| < \left(\frac{1}{2} |\Delta|\right)^{2/3}$$

are always soluble in integers u_1, u_2 unless

$$L_1^2 + L_2 + c \sim \lambda (u_1^2 + u_1 + 2u_2 + 1).$$

The proof is contained in Theorems 3.1 and 3.2.

Proof of Theorem 4.1 ^{*}. By considering a positive multiple

of $q = q(u_1, u_2, u_3)$ in place of q , if necessary, we may assume that $|D_3| = 1$. Then it suffices to prove that, unless $q \sim \lambda q_0$ or λq_1 , the inequalities

$$|q(u_{1s}, u_{2s}, u_{3s})| < 1 \quad (s=1,2,3) \quad (4.3)$$

are soluble in integers (u_{1s}, u_{2s}, u_{3s}) with $\|u_{\lambda s}\|_{\lambda, s=1,2,3} = 1$, since the integral unimodular substitution

$$u_{\lambda} = \sum_{s=1}^3 u_{\lambda s} u_s \quad (\lambda=1,2,3)$$

will transform q into a form each of whose diagonal coefficients is less than 1 in absolute value.

As q represents zero non-trivially, we may suppose, after applying an integral unimodular substitution to the variables, that

$a_{11} = 0$ and hence q now takes the form

$$q(u_1, u_2, u_3) = 2(a_{12}u_2 + a_{13}u_3)u_1 + a_{22}u_2^2 + 2a_{23}u_2u_3 + a_{33}u_3^2.$$

Since $D_3 \neq 0$, the coefficients a_{12}, a_{13} cannot both be zero.

By interchanging u_2, u_3 if necessary we may suppose that $a_{12} \neq 0$.

Two cases now arise according as the ratio a_{13}/a_{12} is irrational or rational.

Suppose first that a_{13}/a_{12} is irrational and let $\epsilon > 0$. By changing the signs of u_2, u_3 if necessary we may assume that $a_{12} > 0, a_{13} < 0$.

Choose $(u_{11}, u_{21}, u_{31}) = (1, 0, 0)$. By Lemma 4.1, since $(\epsilon/a_{12}) > 0$ there exist integer pairs (u_{22}, u_{32}) and (u_{23}, u_{33}) with $u_{22}u_{33} - u_{32}u_{23} = 1$ satisfying

$$0 < \left| u_{2s} + \frac{a_{13}}{a_{12}} u_{3s} \right| < \frac{\epsilon}{a_{12}}, \quad (s=2,3).$$

For each pair (u_{2s}, u_{3s}) ($s=2,3$) we can always choose a corresponding integer $u_1 = u_{1s}$ ($s=2,3$) satisfying

$$|q(u_{1s}, u_{2s}, u_{3s})| \leq \epsilon$$

and (4.3) follows, with the triads $(1, 0, 0), (u_{12}, u_{22}, u_{32})$ and (u_{13}, u_{23}, u_{33}) , since ϵ may be arbitrarily small.

Now suppose that $a_{13}/a_{12} = p/q$ where p, q are integers with $(p, q) = 1$. It is known that there exist integers p', q' with $(p', q') = 1$ satisfying $pq' - p'q = 1$. Then the integral unimodular substitution given by

$$u_1' = u_1, \quad u_2' = pu_2 + qu_3, \quad u_3' = p'u_2 + q'u_3$$

will reduce q to the form

$$q(u_1, u_2, u_3) = 2b_{12}u_1u_2 + b_{22}u_2^2 + 2b_{23}u_2u_3 + b_{33}u_3^2$$

for appropriate b_{12}, \dots, b_{33} . Comparing determinants we see that

$$b_{12}^2 |b_{33}| = |D_3| = 1. \quad (4.4)$$

If $|b_{12}| < 1$ the result is easily proved. For we choose the triads $(1, 0, 0)$, $(u_{12}, 1, 0)$ and $(u_{13}, 1, 1)$ where (u_{12}, u_{13}) are the integers satisfying

$$|2b_{12}u_{12} + b_{22}| \leq |b_{12}| < 1$$

and $|2b_{12}u_{13} + b_{22} + 2b_{23} + b_{33}| \leq |b_{12}| < 1$.

Now suppose that $|b_{12}| > 1$ and hence $|b_{33}| < 1$, by (4.4). We first choose the triads $(1, 0, 0)$ and $(0, 0, -1)$. Then taking $u_2 = u_{23} = 1$, we have on rearranging,

$$q(u_1, 1, u_3) = b_{33} \left(u_3 + \frac{b_{23}}{b_{33}} \right)^2 + 2b_{12}u_1 + b_{22} - \frac{b_{23}^2}{b_{33}}.$$

By Lemma 4.2, there are integers u_{13}, u_{33} satisfying

$$|q(u_{13}, 1, u_{33})| < \left(\frac{1}{2} \sqrt{|b_{33}|} \cdot 2|b_{12}| \right)^{2/3},$$

$$= 1,$$

by (4.4). Thus (4.3) now follows with the triads $(1, 0, 0)$, $(0, 0, -1)$ and $(u_{13}, 1, u_{33})$.

It remains to consider the case when $|b_{12}| = 1, |b_{33}| = 1$.

By changing the signs of q, u_1 if necessary we may suppose that

$$q(u_1, u_2, u_3) = 2u_1u_2 + b_{22}u_2^2 + 2b_{23}u_2u_3 + u_3^2.$$

Further by absorbing integral multiples of u_2, u_3 into u_1 and changing the sign of u_3 if necessary we may suppose that

$$|b_{22}| \leq 1 \quad \text{and} \quad 0 \leq 2b_{23} \leq 1.$$

If $|b_{22}| < 1$, the congruences

$$b_{22} + 2b_{23} \equiv 0 \pmod{2}$$

together imply that $b_{22} = b_{23} = 0$. Thus if u_{13}, u_{13}' are

integers satisfying

$$|2u_{13} + b_{22} + 2b_{23} + 1| \leq 1$$

and $|2u'_{13} - b_{22} + 2b_{23} - 1| \leq 1$

respectively, it follows that $|q| < 1$ for the triads $(1, 0, 0)$, $(0, 1, 0)$ and $(u_{13}, 1, 1)$ or $(u'_{13}, -1, 1)$ unless

$$q = q_0 = 2u_1 u_2 + u_3^2.$$

If $|b_{22}| = 1$, then q is equivalent to the form

$$q(u_1, u_2, u_3) = 2u_1 u_2 + u_2^2 + 2b_{23} u_2 u_3 + u_3^2.$$

Let u_{13} be an integer satisfying

$$|u_{13} + b_{23} - 1| \leq \frac{1}{2}.$$

Thus $|q| < 1$ for the triads $(1, 0, 0)$, $(-1, 2, -1)$ and $(u_{13}, -1, 1)$ unless $2b_{23} = 1$, in which case

$$q = q_1 = 2u_1 u_2 + u_2^2 + u_2 u_3 + u_3^2.$$

4.3. In this section we prove Theorem 4.2^{*}. The proof is independent of Theorem 4.1^{*} and uses the following four lemmas.

LEMMA 4.3. The inequalities

$$|q(u_1, u_2, u_3)| \leq \left(\frac{2}{3} |D_3|\right)^{1/3}$$

are always soluble in integers $(u_1, u_2, u_3) \neq (0, 0, 0)$.

For a proof of this classical result, which is the first of a sequence of minima of an indefinite quadratic form in three variables, see [15], Theorem 83. We observe, in passing, that the particular form relating to the fourth minimum arises as the critical form $q_2(u_1, u_2, u_3)$ in Theorem 4.2^{*}.

LEMMA 4.4. For any $\Gamma > 0$, the region

$$-\Gamma \Delta_2 \leq x_1, x_2 \leq \frac{1}{4\Gamma} \Delta_2$$

always contains two generating points of Λ_2 .

Proof. Consider the tangent parallelogram Π_ϵ defined by

$$|\epsilon^{-1}x_1 + \epsilon x_2| \leq \sqrt{\frac{\Delta_2}{\Gamma}}, \quad |\epsilon^{-1}x_1 - \epsilon x_2| \leq 2\sqrt{\Gamma\Delta_2}.$$

Clearly Π_ϵ is symmetrical about the origin, and since it may be transformed by a linear substitution of determinant 2 into a rectangle having area $8\Delta_2$ its area is $4\Delta_2$. By Minkowski's fundamental theorem, Π_ϵ contains a point of Λ_2 , other than the origin 0. Further, by varying ϵ continuously, we can obtain a parallelogram $\Pi_{\epsilon'}$ which contains two independent points P, Q say of Λ_2 other than 0. Let P', Q' be the reflections of P, Q respectively in 0. If the parallelogram $PQP'Q'$ contains points of Λ_2 , other than 0, we simply replace it by a smaller parallelogram. Thus we assume that $PQP'Q'$ does not contain any point of Λ_2 , other than 0.

Since P, Q are lattice points, it follows that the area of the parallelogram with sides OP, OQ is an integral multiple of Δ_2 , say $m\Delta_2$. The area of the parallelogram $PQP'Q'$ is $2m\Delta_2 \leq 4\Delta_2$ and so two possibilities arise according as $m=1$ or 2 . If $m=1$, the parallelogram with sides OP, OQ has area Δ_2 and hence P, Q , together with 0, generate Λ_2 . If $m=2$, the parallelogram $PQP'Q'$ coincides with the original tangent parallelogram Π_ϵ and $Q, \frac{1}{2}(P+Q)$ together with 0, generate Λ_2 .

We observe that the two generating points obtained lie entirely inside the region considered if there is no point of λ_2 on either bounding hyperbola. However if there is a point of λ_2 on one of these hyperbolae the tangent parallelogram Π_ϵ , for suitable ϵ , through that point will have on its boundary two basis points lying inside the region unless there is a primitive point of λ_2 on the other hyperbola. In this case λ_2 is of the form

$$\begin{aligned}x_1 &= \frac{\epsilon}{2} \sqrt{\frac{\Delta_2}{\Gamma}} u_1 - \epsilon \sqrt{\Gamma \Delta_2} u_2 \\x_2 &= \frac{1}{2\epsilon} \sqrt{\frac{\Delta_2}{\Gamma}} u_1 + \epsilon^{-1} \sqrt{\Gamma \Delta_2} u_2.\end{aligned}$$

Restating the result, with $\mu = \frac{1}{2\Gamma}$, we obtain the following corollary,

COROLLARY. If $\mu > 0$ and if $q(u_1, u_2) = (\alpha u_1 + \beta u_2)(\gamma u_1 + \delta u_2)$ is an indefinite quadratic form in u_1, u_2 of determinant $d = -\frac{1}{4}(\alpha\delta - \beta\gamma)^2$, the inequalities

$$-\frac{1}{\mu} |d|^{1/2} < q(u_{1s}, u_{2s}) < \mu |d|^{1/2} \quad (s=1,2)$$

are always soluble in integers (u_{1s}, u_{2s}) ($s=1,2$) with

$$\|u_{\lambda s}\|_{\lambda, s=1,2} = 1 \quad \text{unless}$$

$$q(u_1, u_2) \sim \sqrt{|d|} (\mu u_1^2 - \mu^{-1} u_2^2).$$

A proof of the next lemma, due to Watson, is given for convenience as his has not been published. Let

$$q = q(u_1, u_2) = a u_1^2 + 2b u_1 u_2 + c u_2^2$$

denote an indefinite quadratic form in u_1, u_2 which does not represent zero non-trivially and has determinant

$$d = ac - b^2 < 0.$$

Denote by P, N the lower bounds of the positive values of $q, -q$ respectively for all integers $(u_1, u_2) \neq (0, 0)$.

LEMMA 4.5. $PN \leq (4|d|)/5,$

with equality when $q(u_1, u_2) = \lambda(u_1^2 + u_1 u_2 - u_2^2)$.

Proof. † We suppose $PN \neq 0$, for otherwise the result is obvious. Also if $P=N$ the result is well known ([14], Theorem 119) since

$$P = N \leq \sqrt{\frac{4}{5}|d|}.$$

By changing the sign of q , if necessary, we may suppose that

$$N < P$$

Hence $N \leq \sqrt{\frac{4}{5}|d|}$. (4.5)

If we consider a suitable multiple of q , instead of q , we may take $P=1$, and it now suffices to prove that

$$N \leq \frac{4}{5}|d|. \quad (4.6)$$

Let $\epsilon > 0$. After applying an appropriate unimodular substitution to the variables u_1, u_2 we may assume that

$$1 \leq a < 1 + \epsilon, \quad \frac{1}{2}a \leq b \leq a \quad (4.7)$$

† The proof given here is an adaptation of that of Dr. Watson, who has very kindly let me include it in my thesis.

By our hypothesis concerning P and N , it follows that either $q > 1$ or $q \leq -N$ for all integers $(u_1, u_2) \neq (0, 0)$.

The inequality (4.6) follows easily if $|d| \geq \frac{5}{4}$. For in this case we have

$$N \leq \sqrt{\frac{4}{5}|d|} \leq \frac{4}{5}|d|,$$

by (4.5). Thus suppose now that

$$|d| < \frac{5}{4}. \quad (4.8)$$

Since $ac - b^2 = d < 0$

we have $ac < b^2 \leq a^2$

by (4.7), and hence

$$c < a. \quad (4.9)$$

Thus either (i) $1 \leq c < 1 + \epsilon$

or (ii) $c < 0$.

In the first case

$$q(-1, 1) = a - 2b + c$$

and by (4.7), (4.9) and the choice of ϵ , we have

$$1 - 2(1 + \epsilon) + 1 < q(-1, 1) < 1 + \epsilon - 2 + 1 + \epsilon,$$

or $-2\epsilon < q(-1, 1) < 2\epsilon$, since, using (4.7), $b > \frac{1}{2}a > 0$ and $(1 + \epsilon)^2 > b^2 > ac > 1$.

which is impossible if ϵ is sufficiently small. Thus only the

second case can arise, and we have therefore

$$c \leq -N \quad (4.10)$$

Now

$$|d| = a|c| + b^2 \geq aN + \frac{1}{4}a^2, \text{ by (4.10),}$$

$$\geq N + \frac{1}{4}, \text{ by (4.7),}$$

so that

$$N \leq |d| - \frac{1}{4}$$

$$< \frac{4}{5}|d|,$$

by (4.8).

LEMMA 4.6. If a, α, ϵ are any constants satisfying

$$0 < a < 1$$

$$0 \leq \epsilon^2 < 1 + \frac{1}{4}a^2,$$

the inequalities

$$|a(u + \alpha)^2 - a^{-1}\epsilon^2| < 1$$

are always soluble for an integer u .

The proof follows immediately from Lemma 3.2.

Proof of Theorem 4.2^{*}. By considering a suitable

positive multiple of $q = q(u_1, u_2, u_3)$ in place of q if necessary, we can take $D_3 = -25/27$. Then, as in the proof of Theorem 4.1^{*}, it suffices to prove that, unless $q \sim \lambda q_2$, the inequalities

$$|q(u_{1s}, u_{2s}, u_{3s})| < 1 \quad (s=1, 2, 3) \quad (4.11)$$

are always soluble in integers (u_{1s}, u_{2s}, u_{3s}) , $(s=1, 2, 3)$,

with $\|u_{\lambda s}\|_{\lambda, s=1, 2, 3} = 1$.

If M denotes the lower bound of $|q(u_1, u_2, u_3)|$ over all

integer triads $(u_1, u_2, u_3) \neq (0, 0, 0)$, then by a weaker form of Lemma 4.3 we have

$$0 \leq M < 9/10.$$

Suppose first that $M = 0$. Then, for any $\epsilon > 0$, the inequalities

$$0 < |q(u_1, u_2, u_3)| < \epsilon$$

are always soluble in integers u_1, u_2, u_3 , and it follows that the inequalities

$$0 < q(u_1, u_2, u_3) < \epsilon$$

are also soluble for any $\epsilon > 0$, by Theorem 1.4 of Oppenheim ([23], I).

Now suppose that $M \neq 0$, and choose ϵ so that

$$0 \leq \epsilon < 1/81.$$

By the definition of M , there are coprime integers u_1, u_2, u_3 satisfying

$$0 < M \leq |q| < M/(1-\epsilon) < 1.$$

Thus, if the inequalities $0 < q < 1$ are insoluble in integers u_1, u_2, u_3 , then the inequalities $0 < -q < 1$ are soluble in integers u_1, u_2, u_3 .

In either case, therefore, after applying a suitable unimodular substitution to the variables u_1, u_2, u_3 we may ensure that

$$\text{either (i) } 0 < a_{11} < 1$$

$$\text{or (ii) } 0 < -a_{11} < M/(1-\epsilon) < 1 \quad \text{and the}$$

inequalities $0 < q < 1$ are insoluble in integers u_1, u_2, u_3 .

Case (i). We may write

$$q(u_1, u_2, u_3) = a_{11}(u_1 + c_2 u_2 + c_3 u_3)^2 + q_1(u_2, u_3),$$

for suitable constants c_2, c_3 and $q_1(u_2, u_3)$ which is an indefinite quadratic form in u_2, u_3 of determinant $-25/(27a_{11})$.

By the corollary to Lemma 4.4, with $\mu = \{(4 - a_{11})(27a_{11})^{1/2}\}/20$, there are integer pairs (u_{2s}, u_{3s}) and (u_{13}, u_{33}) with $u_{22}u_{33} - u_{32}u_{23} = 1$ satisfying

$$-\frac{100}{(4 - a_{11})27a_{11}} < q_1(u_{2s}, u_{3s}) < \frac{4 - a_{11}}{4} \quad (s = 2, 3) \quad (4.12)$$

unless

$$q_1(u_2, u_3) \sim (\mu u_2^2 - \mu^{-1} u_3^2) \{25/(27a_{11})\}^{1/2}.$$

If

$$0 \leq q_1(u_{2s}, u_{3s}) < \frac{4 - a_{11}}{4}$$

for some $s = 2, 3$ we choose an integer u_{1s} satisfying

$$|u_{1s} + c_2 u_{2s} + c_3 u_{3s}| \leq \frac{1}{2},$$

and then

$$|q(u_{1s}, u_{2s}, u_{3s})| < \frac{1}{4} a_{11} + \frac{1}{4} (4 - a_{11}) = 1.$$

Now suppose that $q_1(u_{2s}, u_{3s}) = -\lambda$ for some $s = 2, 3$ where

$$0 < \lambda < 100 / \{27a_{11}(4 - a_{11})\}, \quad \text{by (4.12),}$$

or

$$0 < a_{11} \lambda < 100 / \{27(4 - a_{11})\}.$$

In this case we have

$$q(u_1, u_{2s}, u_{3s}) = a_{11}(u_1 + c_2 u_{2s} + c_3 u_{3s})^2 - a_{11}^{-1}(a_{11} \lambda).$$

Since $0 < a_{11} < 1$, we have successively,

$$(3a_{11} - 2)^2 (3a_{11} - 8) \leq 0,$$

$$27a_{11}^3 - 108a_{11}^2 + 108a_{11} - 32 \leq 0,$$

$$400 - 27(4 - a_{11})(4 + a_{11}^2) \leq 0,$$

$$100 / \{27(4 - a_{11})\} \leq (4 + a_{11}^2) / 4.$$

By Lemma 4.6, with $q = a_{11}$, $c^2 = a_{11} \lambda$, there is an integer u_{15} satisfying

$$|q(u_{15}, u_{25}, u_{35})| < 1.$$

Thus the inequalities (4.11) follow with the triads $(1, 0, 0)$, (u_{12}, u_{22}, u_{32}) and (u_{13}, u_{23}, u_{33}) .

It remains to consider the case when

$$q(u_1, u_2) = (\mu u_2^2 - \mu^{-1} u_3^2) \{25 / (27a_{11})\}^{1/2},$$

where $\mu = \{(4 - a_{11})(27a_{11})^{1/2}\} / 20$. If we choose $(u_{22}, u_{32}) = (0, -1)$ and $(u_{23}, u_{33}) = (1, 1)$, then

$$-\frac{100}{(4 - a_{11})27a_{11}} \leq q_1(u_{25}, u_{35}) < \frac{4 - a_{11}}{4}, \quad (s = 2, 3),$$

and (4.11) again follows with the triads $(1, 0, 0)$, $(u_{12}, 0, -1)$ and

$(u_{13}, 1, 1)$ unless $a_{11} = 2/3$. In this case $\mu = 1/\sqrt{2}$ and q

is equivalent to

$$q(u_1, u_2, u_3) = \frac{2}{3} (u_1 + c'_2 u_2 + c'_3 u_3)^2 + \frac{5}{6} (u_2^2 - 2u_3^2)$$

for some constants c'_2, c'_3 . By absorbing integral multiples of

u_2, u_3 into u_1 and changing the sign of u_2 , if necessary, we

may assume that

$$0 \leq c'_2 \leq \frac{1}{2} \quad \text{and} \quad 0 \leq c'_3 < 1.$$

We shall show that there are three triads of determinant 1 for which $|q| < 1$ unless $c_2' = \frac{1}{2}$ and $c_3' = 0$.

If $c_2' \neq \frac{1}{2}$ and $c_3' \neq 0$, we choose the triads $(1, 0, 0)$, $(0, 1, 0)$ and $(1, 0, 1)$; if $c_2' = \frac{1}{2}$ and $c_3' \neq 0$, we choose the triads $(1, 0, 0)$, $(1, 1, -1)$ and $(1, 0, 1)$; finally if $c_2' \neq \frac{1}{2}$, $c_3' = 0$ we choose the triads $(1, 0, 0)$, $(1, 1, -1)$ and $(0, -1, 0)$.

In the remaining case when $(c_2', c_3') = (\frac{1}{2}, 0)$, the unimodular substitution

$$u_1 = U_1 + U_3, \quad u_2 = U_2 - 2U_3, \quad u_3 = U_2 - U_3$$

will transform q into the equivalent form $Q = Q(U_1, U_2, U_3)$, where

$$\frac{3}{2} Q(U_1, U_2, U_3) = U_1^2 + U_1 U_2 - U_2^2 + \frac{5}{2} U_3^2.$$

It may be verified that $\frac{3}{2} Q$ does not represent zero and has absolute minimum 1, attained only when $U_2 \equiv 0 \pmod{2}$.

Case (ii).

In this case we write

$$q(u_1, u_2, u_3) = -|a_{11}|(u_1 + d_2 u_2 + d_3 u_3)^2 + q_2(u_2, u_3),$$

for suitable constants d_2, d_3 and $q_2(u_2, u_3)$, which is a positive definite quadratic form in u_2, u_3 of determinant $25/(27|a_{11}|)$, and

$$0 < |a_{11}| < 1.$$

After applying an integral unimodular substitution to the variables u_2, u_3 it is known ([14], Theorem 51) that we can ensure that

$$q_2(u_2, u_3) = A u_2^2 + 2B u_2 u_3 + C u_3^2$$

where

$$AC - B^2 = \frac{25}{27|a_{11}|}, \quad |2B| \leq A \quad \text{and} \quad 0 < A \leq \min \left\{ C, \sqrt{\frac{4}{3} \cdot \frac{25}{27|a_{11}|}} = \sqrt{\frac{100}{81|a_{11}|}} \right\}. \quad (4.13)$$

We again choose $(u_{11}, u_{21}, u_{31}) = (1, 0, 0)$. We next choose

$(u_{22}, u_{32}) = (1, 0)$, so that

$$-q(u, 1, 0) = |a_{11}|(u_1 + d'_2)^2 - |a_{11}|^{-1}(|a_{11}|A),$$

for appropriate d'_2 , where

$$\begin{aligned} 0 < |a_{11}|A &< \sqrt{\frac{5}{4}}|a_{11}|, \quad \text{by (4.13),} \\ &< \frac{5}{8} + \frac{1}{2}|a_{11}|, \end{aligned}$$

by the inequality of the arithmetic and geometric mean. Since

$0 < |a_{11}| < 1$, we have

$$|a_{11}|(2 - |a_{11}|) < \frac{3}{2}$$

or $\frac{5}{8} + \frac{1}{2}|a_{11}| < 1 + \frac{1}{4}|a_{11}|^2$,

and hence $0 < |a_{11}|A < 1 + \frac{1}{4}|a_{11}|^2$.

By Lemma 4.6, with $a = |a_{11}|$, $\epsilon^2 = |a_{11}|A$, we can always choose an integer u_{12} satisfying

$$|q(u_{12}, 1, 0)| < 1.$$

Finally we take $(u_{23}, u_{33}) = (0, 1)$, so that

$$-q(u, 0, 1) = |a_{11}|(u_1 + d'_3)^2 - |a_{11}|^{-1}(|a_{11}|C),$$

for some constant d'_3 . We now show, with the help of Lemma 4.5, that A cannot be too small, and then deduce that $|a_{11}|C$ is bounded above in terms of $|a_{11}|$.

Consider the quadratic section

$$q(u_1, u_2, 0) = -|a_{11}| (u_1 + d_2' u_2)^2 + A u_2^2$$

of $q(u_1, u_2, u_3)$. This is an indefinite quadratic form in u_1, u_2 of determinant $-|a_{11}|A$, which does not represent zero non-trivially. Thus if P, N denote the lower bounds of the positive values of $q(u_1, u_2, 0)$, $-q(u_1, u_2, 0)$ respectively, it follows, by Lemma 4.5, that

$$PN \leq \frac{4}{5} |a_{11}| A. \quad (4.14)$$

By hypothesis

$$P \geq 1 \quad \text{and} \quad N \geq M > |a_{11}|(1-\epsilon). \quad (4.15)$$

Thus by (4.14) and (4.15) we have

$$|a_{11}|(1-\epsilon) < M \leq N \leq PN \leq \frac{4}{5} |a_{11}| A,$$

and hence

$$A > \frac{5}{4} (1-\epsilon).$$

But since $|2B| \leq A \leq C$, by (4.13), we have

$$\frac{3}{4} \cdot \frac{5}{4} (1-\epsilon) C < \frac{3}{4} AC \leq AC - B^2 = \frac{25}{24 |a_{11}|},$$

which leads to

$$|a_{11}| C < \frac{80}{81(1-\epsilon)} < \left(1 + \frac{1}{4} a_{11}^2\right)$$

since $\epsilon < 1/81$. A final application of Lemma 4.6, with $Q = |a_{11}|$, $t^2 = |a_{11}| C$, shows that

$$|q(u_1, 0, 1)| < 1$$

for some integer u_{13} . The inequalities (4.11) now follow with the triads $(1, 0, 0)$, $(u_{12}, 1, 0)$ and $(u_{13}, 0, 1)$.

Note. If M' denotes the lower bound of the positive values of $q(u_1, u_2, u_3)$ taken over all integer triads $(u_1, u_2, u_3) \neq (0, 0, 0)$, by a theorem [2] of Barnes, we have

$$M' \leq \left(\frac{4}{3} \cdot \frac{25}{24} \right)^{1/3} = \left(\frac{100}{81} \right)^{1/3}.$$

It may be remarked that this is inadequate to ensure and thereby exclude case (ii) of Theorem 4.2^{*}.

$$0 < a_{ii} < 1,$$

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