# ON THE NOTION OF NEGATION IN CERTAIN NON-CLASSICAL PROPOSITIONAL LOGICS

by

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## ABSTRACT

E.A. Nemesszeghy, ON THE NOTION OF NEGATION IN CERTAIN NON-CLASSICAL PROPOSITIONAL LOGICS.

The purpose of this study is to investigate some aspects of how negation functions in certain non-classical propositional logics. These include the intuitionistic system developed by Heyting(1), the minimal calculus proposed by Johansson(2), and various intermediate logics between the minimal and the classical systems. Part I contains the new results which can be grouped into two classes: extension-criteria results and infinite chain results. In the first group criteria are given for answering the question: when do formulae added to the axioms of the minimal calculus as extra axioms extend the minimal calculus to various known intermediate logics? One of the results in this group (THEOREM 1 in Chapter II, Section 1) is a generalization of a result of Jankov(3). In the second group certain intermediate logics are defined which form infinite chains between well-known logical systems. One of the results here (THEOREM 1 in Chapter II, Section 2) is a generalization of a result of McKay(4). In Part II the new results are discussed from the viewpoint of negation. It is rather difficult, however, to draw definite conclusions which are acceptable to all. For these depend on, and are closely bound up with, certain basic philosophical presuppositions which are neither provable, nor disprovable in a strict sense. Taking an essentially classical position, it is argued that the logics appearing in the defined infinite chains are such that they diverge only in the vicinity of negation, and the notions of negation in them are simply ordered in a sense which is specified during the discussion. In Appendix I a number of conjectures are formulated in connection with the new results.

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My thanks are due to my brother Dr. E.Z. Nemesszeghy who first raised my interest in the problems of negation. I am also grateful to my older brother, G. Nemesszeghy for allowing me to enclose our joint paper 'On a Technical Interpretation of Three-Valued Logic' in Appendix III. To him is due the basic idea of this new interpretation. My contribution includes the background information about Lukasiewicz's three-valued logic and certain simplifications concerning the functional completeness.

Finally, I thank Miss Freda W. Taylor and Mrs. Moira Hughes for willingly typing a difficult manuscript.

# INTRODUCTION

> "Negation, one might say, is
> a gesture of exclusion,
> rejection. But such a
> gesture is used in a great
> variety of ways."
>
> (L.Wittgenstein)

From ancient times philosophers found something puzzling about negation. Plato speaks of these puzzles in his _Theaetetus_(1) and _Sophist_(2). In the latter he expresses the problem through the words of the Stranger:

> "My dear friends, we are engaged in a very difficult speculation - there can be no doubt of that; for how a thing can appear and seem and not to be, or how a man can say a thing which is not true, has always been and still remains a very perplexing question."

The problem of "non-existence" and that of "non-truth"(falsity), to which Plato refers in the quoted text, exercised the minds of philosopers up to the present age. Although J.L. Austin warns us that one should not confuse falsity with negation(3), nonetheless, the fact that many philosophers tried to define negation in terms of truth and falsity raises the question of the interrelation between these concepts. That negation is a basic logical notion, is a point which hardly needs elaboration. Still to illustrate the fundamental character of this notion I single out four items:

i. Logical consistency is commonly defined in terms of negation.

ii. The principle of excluded middle essentially depends on the notion of negation.

iii. The question of the relation of negation to falsity raises some fundamental epistemological and ontological issues.

iv. The problems of logical and semantical paradoxes, at least indirectly relate to the question of the notion of negation.

In the present century, however, a new problem arose. A great number of logical systems have been proposed in which certain classical laws concerning negation do not hold, showing thereby that the notion of negation in these systems is different from that of the classical logic. I here refer to the intuitionistic system developed by Heyting(4), the minimal logic proposed by Johansson(5), and the various intermediate logics between the minimal logic and the classical logic. The question arises whether the different notions of negation implicit in these systems relate to each other, and if so, how? One way to answer this question is to look at the use of the negation-symbol in the different logics and try to make precise "the difference in notion" through "the difference in use". The purpose of the present study is precisely to investigate the variety of ways in which negation functions in certain non-classical propositional logics. This is done with the help of a concept which is called <u>specific theorem of negation</u>. A formula F, by definition, is a specific theorem of negation of a logic $L_1$ with respect to a logic $L_2$ if and only if F contains at least one negation-symbol and is a theorem of $L_1$ but not a theorem of $L_2$. For instance, $(p \vee \neg p)$ is a specific theorem of negation of the classical logic with respect to the intuitionistic logic because it contains a negation-sign and is a theorem of the classical logic but not a theorem of the intuitionistic logic. If we adopt this terminology then the present study may be described as an investigation of, and a comment on, the specific theorems of negation in certain non-classical logics relative to each other. Since these logics can be ordered by certain relations, and under certain interpretations, it is hoped that one can outline a similar ordering among the various notions of negation.

In the third chapter the extension-criteria results are applied to the nine formulae which are given by Johansson in his paper and which in fact are all specific theorems of negation of the intuitionistic logic with respect to the minimal logic. The second part of the thesis begins with Chapter IV in which general philosophical questions are discussed. The fifth chapter examines the intuitionistic account of the connectives. The sixth and final chapter is a comment on the new results of Part I. In Appendix I several conjectures are formulated in connection with the new results.

In the presentation my chief aim was clarity. This is why I used a slight variation of the Peano-Russell symbolism which is more transparent than the Polish notation, although in discussing certain observations of Lukasiewicz in the third chapter, I kept to the Polish notation. Since both vagueness and pedantic accuracy can hinder the clarity of presentation, I tried to be exact to the degree which avoids ambiguities as far as possible but does not over-burden the text with unnecessary qualifications. This was the reason why I did not use different symbols for the logical connectives in the different logics. The signs ' $\neg$ ', 'v', '&', ' $\rightarrow$ ' should always be understood within the defined systems. The signs ' $\neg$ ' and ' $\rightarrow$ ' are also used to denote "complementation" and "relative pseudo-complement" in implicative lattices with standard negation, in order to bring out their similarities with the notions of negation and of implication. They should not cause confusion for the context will always indicate clearly how they should be understood. If any ambiguities still remain the reader is referred to the Index and the List of Symbols. Reference to the items listed in the Bibliography is given by the name of the author (in capitals) followed by the year of publication both in one pair of brackets, as for instance (JOHANSSON, 1936). Reference within the text is made by the help of sections and sub-sections of the chapters. Chapters are indicated by Roman numerals, sections with Arabic numerals without brackets, sub-sections with Arabic numerals within brackets. Thus, for instance, II.3. (9) refers to Chapter II, section 3, sub-section (9). When reference is made within the same chapter and within the same section, then only the sub-section is indicated. For example, (2) refers to sub-section (2) within the same section and chapter. Reference to items listed in the Notes is given by Arabic numerals in brackets as superscripts. Thus, e.g. Heyting(4) refers to the fourth item in the Notes. For the sake of convenience Part I and Part II each has a separate Bibliography.

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(2)..... $(p \,\&\, \neg p) \longrightarrow q$

(3)..... $((p \,\&\, \neg p) \vee q) \longrightarrow q$

(4)..... $((p \vee q) \,\&\, \neg p)$ --- $q$

(5)..... $(q \vee \neg q) \longrightarrow (\neg \neg q \longrightarrow q)$

(6)..... $(\neg p \vee q) \longrightarrow (p \longrightarrow q)$

(7)..... $(p \vee q) \longrightarrow (- p$ --- $q)$

(8)..... $(p \longrightarrow (q \vee \neg r)) \longrightarrow ((p \,\&\, r) \longrightarrow q)$

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It is rather difficult to draw definite conclusions which are acceptable to all for the outcome of several questions depends on certain basic philosophical positions which are outside the realm of logic, and are neither provable or disprovable in a strict sense.
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# PART I.

## CHAPTER I BASIC CONCEPTS AND THEIR NOTATIONS

> "It is an old idea that the more pointedly and logically we formulate a thesis, the more irresistibly it cries out for its antithesis."
> (H.Hesse)

The object of this chapter is to define certain concepts which will be used later, and to establish their notation. Many of these concepts are familiar; others, less familiar, can be easily understood from those which are. In this chapter I shall also list some well-known results which will be referred to in subsequent chapters.

### 1. Formulae and negation-schemata

The alphabet of our formal languages contains (as letters) countably many propositional variables, p, q, r, ... (with or without subscripts), and some or all of the following logical connectives, $\neg$ (not), & (and), v (or), $\longrightarrow$ (implies). Formulae are built up from the letters of our alphabet by the well-known formation rules.(1) I shall use metavariables P, Q, R, ...; F, for any well-formed formulae; $F_1$, $F_2$, ..., $F_n$, ...; $F^*$, $F^{**}$, $F'$, $F''$ will designate special formulae. I shall use brackets as auxiliary symbols to indicate the scope of the connectives but adopt the convention that any connective in the list above binds more strongly than any subsequent one. This convention enables us to omit brackets when no confusion will result. By the degree of a formula F is meant the number of occurrences of the logical connectives in F. I denote a formula of degree 'k' by $F^{(k)}$.

(1).....By a negation-schema I understand a formula which contains at least one negation-sign ' $\neg$ ' and possibly some other logical connectives. Thus, for instance, the following are negation-schemata:

$$\neg p$$
$$p \vee \neg p$$
$$p \,\&\, \neg q$$
$$\neg \neg p \rightarrow p$$
$$\neg p \rightarrow (p \rightarrow q)$$
$$\neg p \rightarrow (p \rightarrow p)$$
$$(\neg p \rightarrow p) \rightarrow p$$

## 2. Certain non-classical propositional logics

By classical propositional logic I mean any propositional calculus that has the same set of theorems(2) as the propositional calculus of Principia Mathematica (WHITEHEAD - RUSSELL 1910). Thus I call classical logic(3), for instance, the system presented by Frege (FREGE 1879), the logic given by Hilbert and Ackermann (HILBERT - ACKERMANN 1928), the system of Łukasiewicz (ŁUKASIEWICZ 1929). I abbreviate classical logic by CL.

By non-classical propositional logic I understand any calculus that does not have the same set of theorems as CL. Perhaps the simplest example for such a logic is the system in which every well-formed formula is a theorem. This system is called the absolute inconsistent logic by some logicians. Another, much more important, logic has been presented by Heyting (HEYTING, 1930) and was represented by him with slight and inessential alterations in his book Intuitionism.(HEYTING, 1966). He used $\neg$, &, v, $\rightarrow$, as primitive symbols and the following formulae as axioms:

(1).....I. $\vdash p \rightarrow (p \,\&\, p)$

II. $\vdash (p \,\&\, q) \rightarrow (q \,\&\, p)$

III. $\vdash (p \rightarrow q) \rightarrow ((p \,\&\, r) \rightarrow (q \,\&\, r))$

IV. $\vdash ((p \rightarrow q) \,\&\, (q \rightarrow r)) \rightarrow (p \rightarrow r)$

V. $\vdash q \rightarrow (p \rightarrow q)$

VI. $\vdash (p \,\&\, (p \rightarrow q)) \rightarrow q$

VII. $\vdash p \rightarrow (p \vee q)$

VIII. $\vdash (p \vee q) \rightarrow (q \vee p)$

IX. $\vdash ((p \rightarrow r) \,\&\, (q \rightarrow r)) \rightarrow ((p \vee q) \rightarrow r)$

X. $\vdash \neg p \rightarrow (p \rightarrow q)$

XI. $\vdash ((p \rightarrow q) \,\&\, (p \rightarrow \neg q)) \rightarrow \neg p$

The rules of deduction are substitution and modus ponens.(4)

We may observe that there are only two negation-schemata among these axioms, namely X. and XI. It is well-known that neither $(p \vee \neg p)$ nor $\neg \neg p \rightarrow p$, nor $(p \rightarrow q) \vee (q \rightarrow p)$ is derivable in this system.

I abbreviate any calculus which has the same set of theorems as Heyting's logic by HL. For instance, Kleene and Łukasiewicz have given different presentations of HL.[5] (KLEENE, 1952 and ŁUKASIEWICZ, 1952).

In 1936 Johansson published a paper (JOHANSSON, 1936) in which he developed a calculus called by him minimal calculus. This we get from Heyting's system by dropping axiom X. above and leaving the remaining axioms and rules unchanged. Johansson showed in the same paper that a number of negation-schemata which are derivable in HL are underivable in the minimal calculus. For instance $((p \vee q) \ \& \ \neg p) \longrightarrow q$; $(\neg p \vee q) \longrightarrow (p \longrightarrow q)$, $\neg\neg(\neg\neg p \longrightarrow p)$ are all unprovable.[6]

I abbreviate any propositional logic that has the same set of theorems as Johansson's minimal logic by ML.

In my study I am particularly interested in propositional logics which we get by adding extra-axioms to the axioms of ML and leaving the rules of ML unchanged. From the way Johansson defined his minimal logic it is obvious that we can get HL by adding $\neg p \longrightarrow (p \longrightarrow q)$ as an extra axiom to the axioms of ML. In symbols:

(2).....(ML, $\neg p \longrightarrow (p \longrightarrow q)$) = HL

It is also well-known[7] that we can get the classical calculus CL by adding the following two extra-axioms to the axiom of ML: $\neg p \longrightarrow (p \longrightarrow q)$ ; $(p \vee \neg p)$, and retaining the rules of ML.

(3).....(ML, $\neg p \longrightarrow (p \longrightarrow q)$, $(p \vee \neg p)$) = CL

Let me immediately define a calculus which we get by adding to ML the extra axiom $(p \vee \neg p)$ only[8] and which I designate $Cal^7$:

(4).....$(ML, p \vee \neg p) = Cal^7$ by def.

If we leave out axioms X and XI from the calculus of Heyting[9] and restrict the vocabulary of logical constants to the binary connectives, we get a calculus which is called positive logic. I shall denote any calculus which has the same set of theorems as positive logic[10] by PL.

PL, ML and HL are closely related. This can perhaps be best seen, if instead of the unary-connective "$\neg$", we introduce a Zero-order connective, i.e. a fixed proposition " f ", usually interpreted as something false, or absurd, and define $\neg p$ in terms of " $\longrightarrow$ " and " f " as follows:

(5)..... $\neg p = p \longrightarrow f$

In this case the axioms of PL with definition (5) give us a system in which all and only the theorems of ML are derivable[11]. And if we add to this system of ML the following formula as an extra axiom

(6)..... $f \longrightarrow q$

then we get HL[12].

This result gives us a rough characterization of ML with respect to HL: ML is such a subsystem of HL, in which a false proposition does not imply every proposition.

The relationships between PL, ML, HL, CL are simple in that we can get ML, HL, CL by certain simple extensions of PL. To give an exact formulation of what I mean by an extension of a calculus, I introduce the following definition:

(7)..... Let $Cal_1$ be a propositional calculus given as an axiomatic system in a formal language $L_1$ and let $F_1, F_2 \ldots, F_n$ be formulae in a formal language $L_2 \supseteq L_1$ then by definition

$Cal_2 = (Cal_1, F_1, F_2, \ldots\ldots, F_n)$ is an extension[13] of $Cal_1$

In particular, if $F_1, F_2, \ldots\ldots, F_n$ are all derivable in $Cal_1$ then $Cal_2$ collapses into $Cal_1$. In this case I speak about a non-proper extension of $Cal_1$. A little more generally,

(8).....If $Cal_2$ is an extension of $Cal_1$, and $Cal_2 = Cal_1$, then by definition $Cal_2$ is a non-proper extension of $Cal_1$, On the other hand, if $Cal_2 \supset Cal_1$, then again by definition $Cal_2$ is a proper extension of $Cal_1$.

Let me introduce now some other technical expressions which are evidently related to the definitions of extension and proper-extension of a calculus.

(9)..... If $Cal_1$ has the same set of theorems as $Cal_2$ ( $Cal_1 = Cal_2$) then $Cal_1$ and $Cal_2$ are said to be equivalent.

(10)..... If $Cal_1 \subseteq Cal_2$ (i.e. if the set of the theorems of $Cal_1$ is a subset of the theorems of $Cal_2$) then $Cal_1$ is said to be a fragment of $Cal_2$.

(11).....If $Cal_1 \subset Cal_2$, then $Cal_1$ is said to be a proper fragment of $Cal_2$. Alternatively, I say that $Cal_2$ strictly succeeds $Cal_1$.

(12).....If $Cal_1 \nsubseteq Cal_2$ and $Cal_2 \nsubseteq Cal_1$, then I say that $Cal_1$ and $Cal_2$ are incomparable.

(13),.... A calculus Cal* is an intermediate logic(14) between HL and CL iff HL $\subseteq$ Cal* $\subseteq$ CL. Plainly any such intermediate logic Cal* which is a proper fragment of CL is a non-classical logic.

More generally

(14).....A calculus Cal* is an intermediate logic between two calculi $Cal_1$ and $Cal_2$ iff $Cal_1 \subseteq Cal^* \subseteq Cal_2$

Finally in this paragraph I introduce a definition(15) which will be useful later.

(15)..... A calculus Cal* is a predecessor of $Cal_2$ over $Cal_1$ iff $Cal_1 \subseteq Cal^* \subset Cal_2$.

Thus, for instance, both ML, HL are predecessors of CL over ML. It should be remembered that predecessors of a logic over another can be incomparable. For example, HL and $Cal^7$ (defined by (2) and (4)) are incomparable predecessors of CL over ML.

Another class of non-classical logics is usually referred to as many-valued logics.(16) The first three-valued logic has been invented by Łukasiewicz (ŁUKASIWICZ 1920). His ideas were motivated by certain considerations of modality, namely that statements expressing future-contingent events (that are possible but not necessary) are neither, strictly speaking, "true" or "false"; so they must possess a third value (say "neutral", "indifferent") which he designated by "$\frac{1}{2}$". He used two primitive functors C and N corresponding to implication and negation.(17) An axiomatic presentation of this system was given by Wajsberg (WAJSBERG 1931). Other many-valued propositional logics were given by Post (POST, 1921), by Słupecki (SŁUPECKI, 1936), by D.A. Bochwar (BOCHWAR, 1939) by Kleene (KLEENE, 1938) and by Reichenbach (REICHENBACH, 1944).

## 3. Specific theorems of negation

As has already been mentioned in the Introduction, one way of registering the different notions of negation implicit in different calculi is to look at and compare the different sets of negation-schemata derivable in the calculi under investigation. If, for instance, two calculi $Cal_1$ and $Cal_2$ have the same formal language, the same rules, but different sets of derivable negation-schemata then the notion of negation implicit in $Cal_1$ and $Cal_2$ must be different.(18) Taking a concrete example, each of the following negation-schemata is derivable in CL but underivable in HL (and in ML).

(1)..... $(p \vee \neg p)$

(2)..... $(\neg \neg p \longrightarrow p)$

(3)..... $(\neg p \vee \neg \neg p)$

(4)..... $(\neg p \longrightarrow p) \longrightarrow p$

These negation-schemata are sometimes called in the literature "excluded middle", "law of double negation", "weakened form of excluded middle" and the "law of Clavius" respectively. I call (1) (2) (3) (4) instances of specific theorems of negation of CL with respect to HL (and to ML). Still these specific theorems are not all equivalent in derivative strength with respect to HL or ML. This can be seen if we extend HL or ML by some of the negation-schemata of (1), (2), (3), (4) and compare the calculi we get in such a way. For example.

(5)..... (HL ,(1)) = CL but (HL ,(3)) $\neq$ CL

On the other hand, although

(6)..... (HL ,(1)) = CL and (HL ,(2)) = CL

nevertheless

(7)..... (ML, (1)) $\neq$ CL but (ML ,(2)) = CL

These suggest the careful definitions of the derivative strength of formulae in a calculus relative to another. Here are the needed definitions:

(8)..... F is a <u>specific theorem of negation</u> of $Cal_1$ with respect to $Cal_2$, iff F is a negation-schema that is derivable in $Cal_1$ but underivable in $Cal_2$. In symbols: $\vdash_{Cal_1} F$ and $\nvdash_{Cal_2} F$

(9).....Formula $F_1$ is equivalent to $F_2$ in $Cal_1$ with respect to $Cal_2$ iff $(Cal_2F_1) = (Cal_2F_2) = Cal_1$.

(10).....Formulae $F_1, F_2, \ldots, F_k$ are jointly equivalent to $F_o$ in $Cal_1$ with respect to $Cal_2$ iff

$$(Cal_2, F_1, F_2, \ldots, F_k) = (Cal_2, F_o) = Cal_1$$

By using this terminology we may say, for instance, that the "excluded middle" is jointly equivalent to all the specific theorems of negation in CL with respect to HL.

4. Semantics

Since I shall be using matrices and lattices as models for the propositional calculi under investigation, it will be convenient to define certain concepts in connection with them.

Following the terminology of Łukasiewicz and Tarski (ŁUKASIEWICZ-TARSKI, 1930,p.39), slightly adapted by Jaskowski (JASKOWSKI, 1936 p. 259) I give the definition of a (logical) matrix as follows:

(1).....A (logical) matrix is an ordered sextuple $M = \langle A, B, \longrightarrow_M, \&_M, \vee_M, \neg_M \rangle$ which consists of two disjoint sets (with elements of any kind whatever) A and B (usually called the set of non-designated and designated[19] elements respectively), four functions $\longrightarrow_M$, $\&_M$, $\vee_M$, $\neg_M$ defined for all elements A + B and taking values elements of A + B exclusively.

(2).....The matrix M is called normal if when $x \in B$ and $y \in A$ then

$x \rightarrow_M y \in A$

Let F be a formula. Replace in F the propositional connectives $\rightarrow$, &, V, ¬, by the functional symbols $\rightarrow_M$, $\&_M$, $V_M$ $\neg_M$, respectively, and put in place of the propositional variables p, q,.... occurring in F the elements of A + B. We obtain a function from the elements of A + B into the elements of A + B. We call this function the value-function of F and designate it by V (F). A value of V(F) is called valuation of F.

(3).....By definition F is <u>valid</u> in matrix M iff the range of V(F) is within B; i.e. iff $V(F) \in B$. In other words, F is valid iff every valuation of F is a designated element. In symbols $\models_M F$.

(4)..... For example, let M be such that A = $\{0\}$, B = $\{1\}$, and let the four functions be defined by the following value-tables:

| $\rightarrow$ | 1 | 0 |
|---|---|---|
| 1 | 1 | 0 |
| 0 | 1 | 1 |

| & | 1 | 0 |
|---|---|---|
| 1 | 1 | 0 |
| 0 | 0 | 0 |

| V | 1 | 0 |
|---|---|---|
| 1 | 1 | 1 |
| 0 | 1 | 0 |

| ¬ | |
|---|---|
| 1 | 0 |
| 0 | 1 |

I designate this particular matrix by $M_2$. Let F* be ((p & p $\rightarrow$q)) $\rightarrow$ q. Since the value-function of this formula is identically 1, F* is valid in $M_2$.

It is well-known that for any F if $\vdash_{CL}$ F then $\models_{M_2}$ F. I call $M_2$ a matrix model for CL. More generally, (5).....By definition M is a (matrix) model for a calculus Cal iff the following holds

$$\text{If } \vdash_{Cal} F \text{ then } \models_{M} F$$

And again by definition

(6)..... M is a characteristic matrix (model) for a calculus Cal iff the following holds

$$\vdash_{Cal} F \text{ iff } \models_{M} F$$

Instead of saying that M is a charecteristic matrix (model) of Cal, I sometimes say that M characterizes Cal.

It is well-known, for instance, that $M_2$ is a characteristic matrix for CL.

(7)..... $\{M_i\}$ , i = 1, 2,... is a characteristic matrix-set for a calculus Cal iff the following holds

$$\vdash_{Cal} F \text{ iff } \models_{\{M_i\}} F \qquad i = 1, 2, \ldots$$

Instead of saying that $\{M_i\}$ is a characteristic matrix-set for Cal, I sometimes say that the set $\{M_i\}$, i = 1, 2,... characterizes Cal.

We need to define further two operations on matrices: the $\Gamma^1$ operation on matrices and matrix-multiplication. These were first introduced by Jaskowski (JASKOWSKI, 1966. pp. 260-261).

(8).....Let M and N be two matrices having the same element b for their sole designated element: $B_M = B_N = \{b\}$. The set $A_N$ is composed of the elements of $A_M$ and one additional element $\{a\}$: $A_N = A_M + \{a\}$. If $\alpha$ is a function defined by the following two conditions:

(i) $\alpha(b) = a$

(ii) if $x \in A$ $\alpha(x) = x$

the functions of the matrix N are defined in terms of those of M and that of $\alpha$ by the following tables:

| $\rightarrow$ | b | $\alpha(y)$ |
|---|---|---|
| b | $b \rightarrow b$ | $\alpha(b \rightarrow y)$ |
| $\alpha(x)$ | $x \rightarrow b$ | $x \rightarrow y$ |

| & | b | $\alpha(y)$ |
|---|---|---|
| b | b & b | $\alpha$(b & y) |
| $\alpha(x)$ | $\alpha$(x & b) | $\alpha$(x & y) |

| v | b | $\alpha(y)$ |
|---|---|---|
| b | b v b | b v y |
| (x) | x v b | $\alpha$(x v y) |

| $\neg$ | |
|---|---|
| b | $\alpha(\neg b)$ |
| (x) | $\neg x$ |

Under these conditions, by definition, N is the result of the $\ulcorner\,\urcorner$ operation performed on M . In symbols: $N = \ulcorner M \urcorner$

(9).....As a simple example let us now form $\ulcorner M_2 \urcorner$. Let the additional element be 'a' and the sole designated element 1. The function $\alpha$ will be

(i) $\alpha(1) = a$

(ii) $\alpha(0) = 0$

Thus the function-tables of $\ulcorner M_2 \urcorner$ will be according to (8):

| $\rightarrow$ | 1 | a | 0 |
|---|---|---|---|
| *1 | 1 | a | 0 |
| a | 1 | 1 | 0 |
| 0 | 1 | 1 | 1 |

| & | 1 | a | 0 |
|---|---|---|---|
| 1 | 1 | a | 0 |
| a | a | a | 0 |
| 0 | 0 | 0 | 0 |

| v | 1 | a | 0 |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| a | 1 | a | a |
| 0 | 1 | a | 0 |

| $\neg$ | |
|---|---|
| 1 | 0 |
| a | 0 |
| 0 | 1 |

I call this matrix $M_3$. Thus $M_3 = \Gamma M_2$

In the same way we can form from any n-element matrix an n + 1 element matrix by using the $\Gamma$ operation.

(10).....Let M and N be two matrices. By definition M x N is a product of M and N iff the elements of M x N are the ordered pairs $\langle m, n \rangle$, $m \in M$ and $n \in N$, and the four functions defined are as follows:

M x N M N

$$\langle m_1, n_1 \rangle \rightarrow \langle m_2, n_2 \rangle = \langle m_1 \rightarrow m_2, n_1 \rightarrow n_2 \rangle$$

$$\langle m_1, n_1 \rangle \;\&\; \langle m_2, n_2 \rangle = \langle m_1 \;\&\; m_2, n_1 \;\&\; n_2 \rangle$$

$$\langle m_1, n_1 \rangle \vee \langle m_2, n_2 \rangle = \langle m_1 \vee m_2, n_1 \vee n_2 \rangle$$

$$\neg \langle m_1, n_1 \rangle = \langle \neg m_1, \neg n_1 \rangle$$

(11).....For example, let us form $M_2 \times M_2$. Since $M_2$ contains two distinct elements 1, 0, $M_2 \times M_2$ will contain the following four elements $\langle 1, 1\rangle, \langle 1, 0\rangle, \langle 0, 1\rangle, \langle 0, 0\rangle$. If we rename these elements by 1, a, b, 0, respectively then the function tables can be given as:

| $\rightarrow$ | 1 | a | b | 0 |
|---|---|---|---|---|
| 1 | 1 | a | b | 0 |
| a | 1 | 1 | b | b |
| b | 1 | a | 1 | a |
| 0 | 1 | 1 | 1 | 1 |

| & | 1 | a | b | 0 |
|---|---|---|---|---|
| 1 | 1 | a | b | 0 |
| a | a | a | 0 | 0 |
| b | b | 0 | b | 0 |
| 0 | 0 | 0 | 0 | 0 |

| V | 1 | a | b | 0 |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| a | 1 | a | 1 | a |
| b | 1 | 1 | b | b |
| 0 | 1 | a | b | 0 |

| $\neg$ | |
|---|---|
| 1 | 0 |
| a | b |
| b | a |
| 0 | 1 |

I shall designate this matrix by $M_2^2$.

(12).....More generally, a matrix M multiplied by itself k times will be denoted by $(M)^k$ or simply $M^k$.

(13).....Jaskowski has proved (JASKOWSKI, 1936) that the intuitionist logic HL is characterized by an infinite set of matrices which can be given recursively in terms of $M_2$, the $\Gamma$ operation and matrix-multiplication as follows:

(14).....

$$I_0 = M_2$$
$$I_{k+1} = \Gamma I_k^{k+1}$$

For $k = 0, I_{k+1} = I_1 = \Gamma I_0 = \Gamma M_2 = M_3$

Thus we get the matrix given by (9). This is a three-element matrix.

For k = 1 we have $I_2 = \Gamma I_1^2 = \Gamma(M_3 \times M_3)$. This is a ten-element matrix. And the following matrix in the sequence has 1001 elements. In any case, Jankowski's result gives us a completeness result: the set of matrices $\{I_i\}$ given by (14) characterizes HL.

It should be observed that all the matrices I spoke about in this paragraph form lattices, and hence can be given in lattice-representation by using Hasse-diagrams. Since these lattice-representations are visually far more suggestive than matrices given in value-tables, I shall use them whenever I can.

First we need some basic definitions about lattices:

(15).....By definition a lattice is an ordered couple $\langle A, \leq \rangle$ where A is a set, '$\leq$' is a reflexive, antisymmetric and transitive relation defined for arbitrary elements a, b, c $\in$ A, and for each a, b $\in$ A the greatest lower bound (denoted by $a \cap b$) and the least upper bound (denoted by $a \cup b$) exist. A lattice is called degenerate iff it has only one element.

(16).....A subset A' of A is a sublattice of A iff it is closed under operations $\cap$, $\cup$ i.e. iff

$$a \cap b \in A' \text{ and } a \cup b \in A' \quad \text{for any } a, b \in A'.$$

(17).....A lattice homomorphism h from lattice A into lattice B is a mapping h of A into B such that for any a, b $\in$ A

$$h(a \cap b) = h(a) \cap h(b)$$
$$h(a \cup b) = h(a) \cup h(b)$$

(18).....If a lattice homomorphism h is such that the homomorphic-image h(A) is not a proper subset of B, then we say that h is a lattice-homomorphism of A onto B.

(19).....The top and the bottom element of a lattice A will be denoted by $1_A$ and $0_A$ respectively whenever they exist. A lattice may or may not have a top or bottom element.

(20).....A non-empty set $\nabla$ of elements of a lattice A is said to be a filter in A provided for any a, b $\in$ A

$$a \cap b \in \nabla \quad \text{iff } a \in \nabla \text{ and } b \in \nabla$$

(21).....A non-empty set $\Delta$ of elements of a lattice A is said to be an ideal in A provided for any a, b $\in$ A

$$a \cup b \in \Delta \quad \text{iff } a \in \Delta \text{ and } b \in \Delta$$

(22).....For every fixed element $a_0 \in A$, the set of all elements $a \geq a_0$ ($a \leq a_0$) is a filter (an ideal) called the principal filter (ideal) generated by $a_0$.

(23).....A lattice A is said to be relatively pseudo-complemented if for all a, b $\in$ A there is a greatest element: $x \in A$ such that $a \cap x \leq b$. This element x is called the pseudo complement of "a" relative to "b", and is denoted by $a \rightarrow b$.

Obviously, every relatively pseudo-complemented lattice can be conceived as an algebra

$\langle A, \cap, \cup, \rightarrow \rangle$ with three binary operations $\cap$, $\cup$, $\rightarrow$

It is well-known that[20]

(24).....Every relatively pseudo-complemented lattice has the unit element, is distributive, and in it the following relations hold:

(a) $x \leq a \rightarrow b$ iff $a \cap x \leq b$ for any $x, a, b \in A$

(b) $a \rightarrow b = 1_A$ iff $a \leq b$ for any $a, b \in A$

(c) $1_A \rightarrow b = b$ for any $b \in A$

(d) $a \rightarrow (b \rightarrow c) = (a \cap b) \rightarrow c = b \rightarrow (a \rightarrow c)$

(e) $(a \cup b) \rightarrow c = (a \rightarrow c) \cap (b \rightarrow c)$

(f) $a \rightarrow (b \rightarrow c) \leq (a \rightarrow b) \rightarrow (a \rightarrow c)$

(g) $a \leq b \rightarrow (a \cap b)$

(h) $a \cap (a \rightarrow b) = a \cap b$

(i) $b \leq (a \rightarrow b)$

(25).....It is also known[21] that positive logic PL is characterized by the set of relatively pseudo-complemented lattices.

(26).....A lattice homomorphism h from a relatively pseudo-complemented lattice A into a relatively pseudo-complemented lattice B is a mapping h of A into B such that

$$h(a \cap b) = h(a) \cap h(b)$$

$$h(a \cup b) = h(a) \cup h(b)$$

$$h(a \rightarrow b) = h(a) \rightarrow h(b)$$

for any $a, b \in A$.

(27).....If a relatively pseudo-complemented lattice A is a lattice with a unary function ' $\neg$ ' and a distinguished element $c_0$ such that for any $a \in A$, $\neg a = a \rightarrow c_0$, then ' $\neg$ ' is called a standard negation.

(28).....Another name for a relatively pseudo-complemented lattice is implicative lattice.

(29) .....For any implicative lattice A with standard negation the following relations hold[22]:

(a) $c_o = \neg 1_A$

(b) $a \rightarrow \neg b = b \rightarrow \neg a$

(c) $a \leq \neg\neg a$

(d) $\neg\neg a \cap \neg a = c_o$

(30).....Minimal logic ML is characterized by the set of implicative lattices with standard negation. A formula F is derivable in ML iff F is valid in every implicative lattice with standard negation with at most $2^{2^{r+1}}$ elements, where 'r' is the number of subformulae in F. (RASIOWA-SIKORSKI, 1953).[23]

I shall also use a completeness theorem concerning the propositional theories of ML. (RASIOWA, 1974 p. 256):

(31).....For every consistent propositional theory of ML, there is a non-degenerate implicative lattice A with standard negation such that for any formula F, F is derivable in the theory iff F is valid in A.

(32).....It is well-known[24] that the set of implicative lattices with standard negation such that the distinguished element $c_o = 0$, is characteristic of HL.

(33).....Another name for such lattices is pseudo-complemented lattices.

Evidently, any such pseudo-complemented lattice can be conceived as an algebra $\langle A, \cap, \cup, \rightarrow, 0_A \rangle$ with three binary operations $\cap, \cup, \rightarrow$ and a distinguished element $0_A$. These algebras are also called pseudo-Boolean algebras or simply Heyting algebras.

(34).....Lattices will be represented by Hasse-diagrams, arrows indicating complementation (negation). For instance, the classical matrix model $M_2$ (see (4)) can be represented as a two-element implicative lattice in which '1' is the unit element and '0' is the zero element. (Fig. 1)

(35).....I shall call the lattice represented in Fig.1 '$I_o$'

Let me represent now all the four different unary functions on a two-element implicative lattice. (see Fig.2)

(36).....I wish to name the implicative lattices represented in Fig.2 as $I_o$, $M_o$, $S_o$ and $P_o$ respectively. Observe that apart from $I_o$, only $M_o$ is a lattice with standard negation.

(37).....Fig.3 represents the 27 different unary functions on a three-element implicative lattice. Observe that there are only three lattices among the 27 which have standard negations:

(a) The lattice in the first row and first column.

(b) The lattice in the second row and first column.

(c) The lattice in the third row and third column.

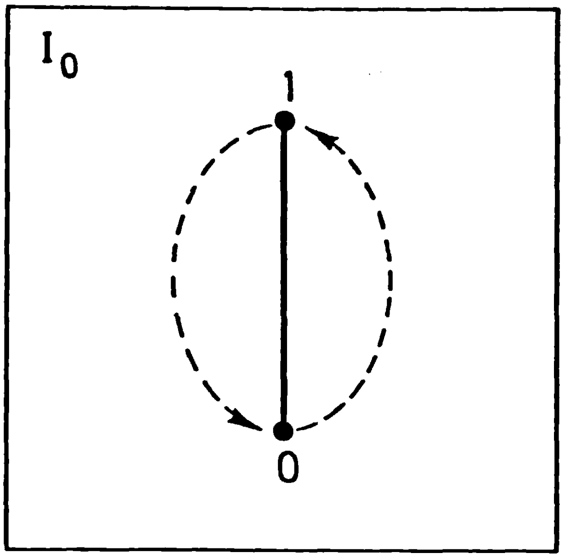


Fig. 1

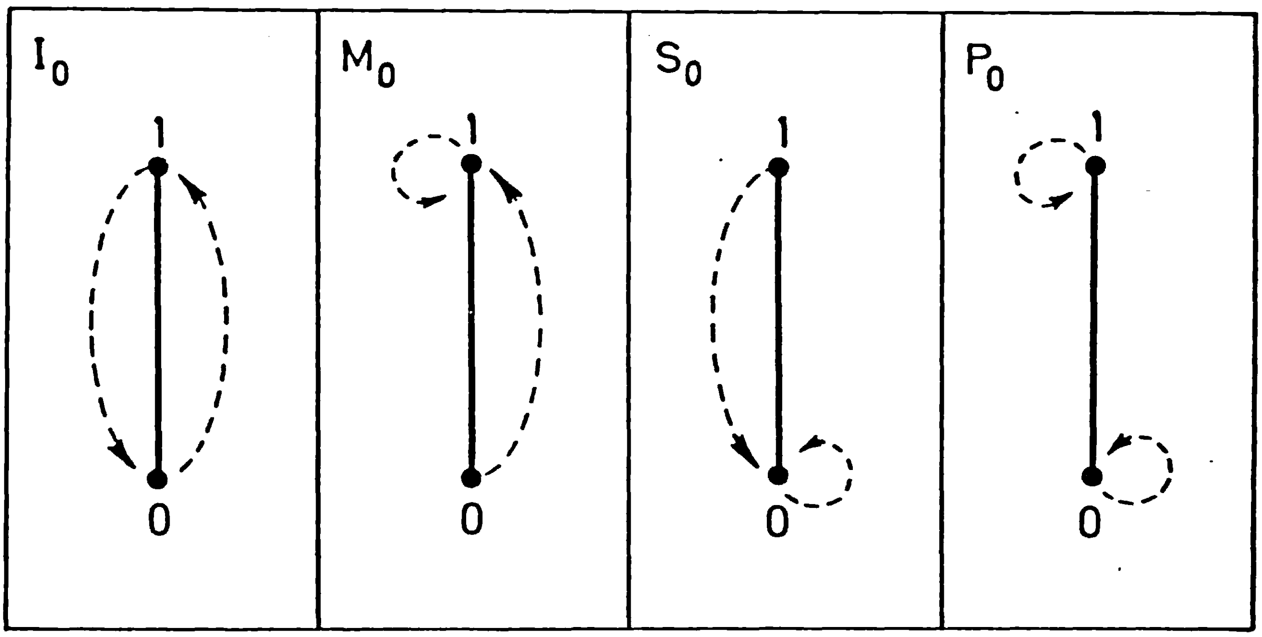


Fig. 2

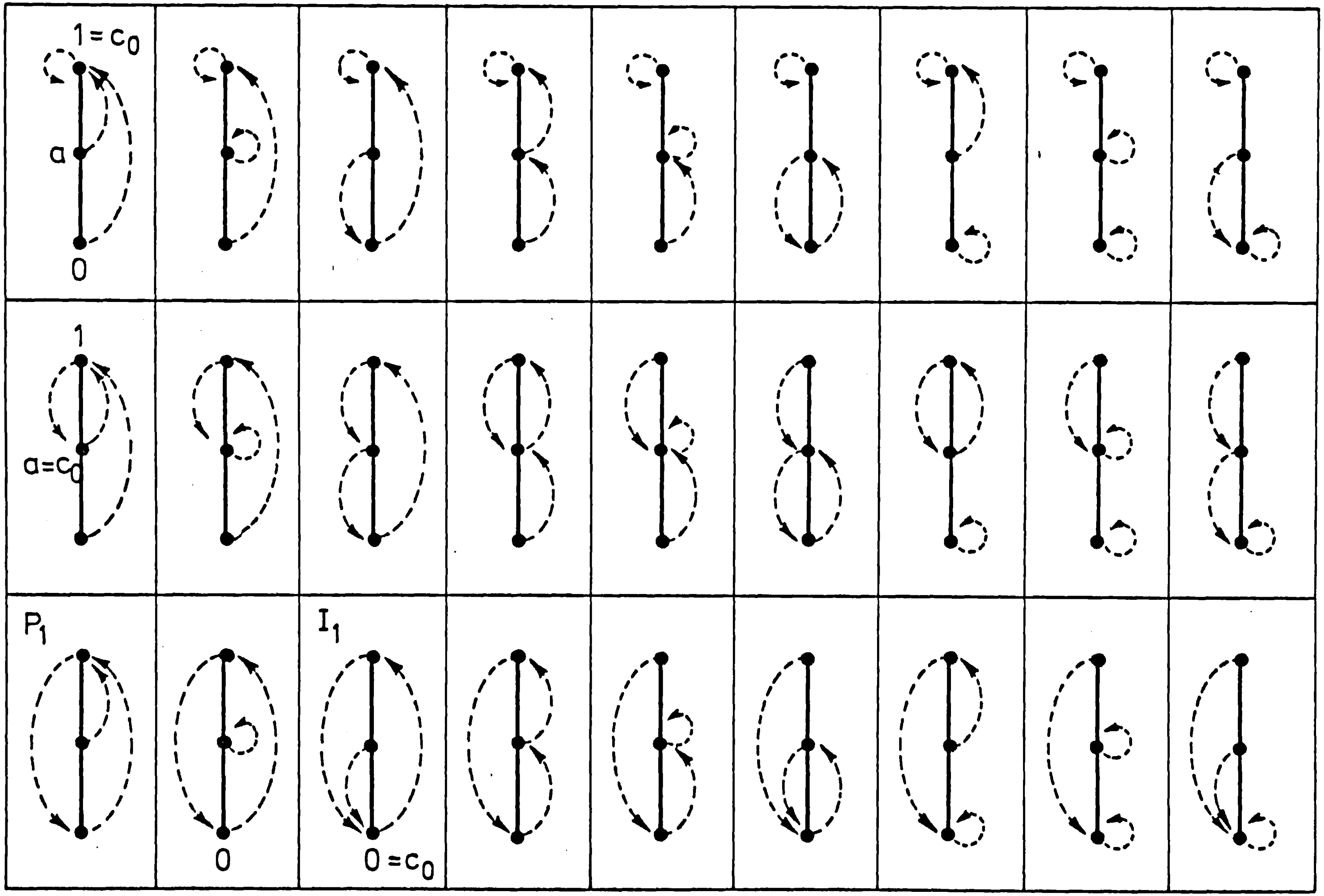


Fig. 3

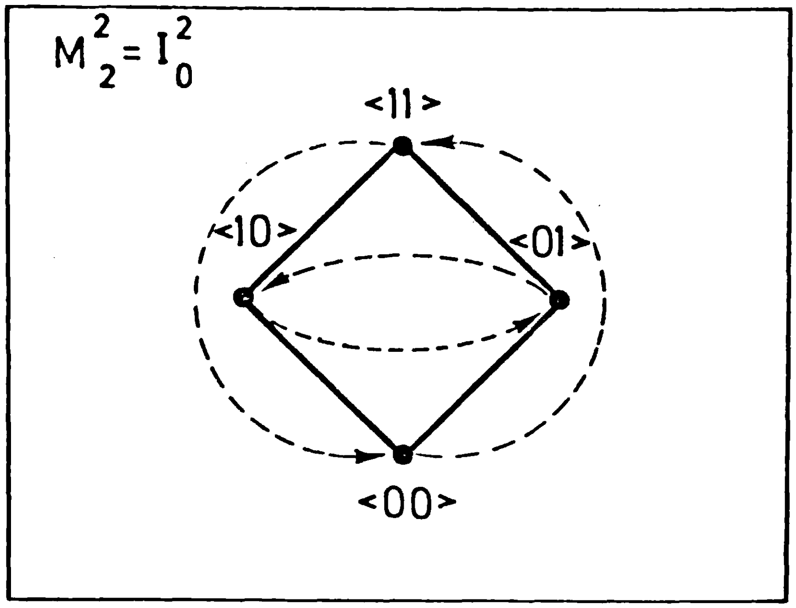
$M_2^2 = I_0^2$
<11>
<10>
<01>
<00>

Fig. 4

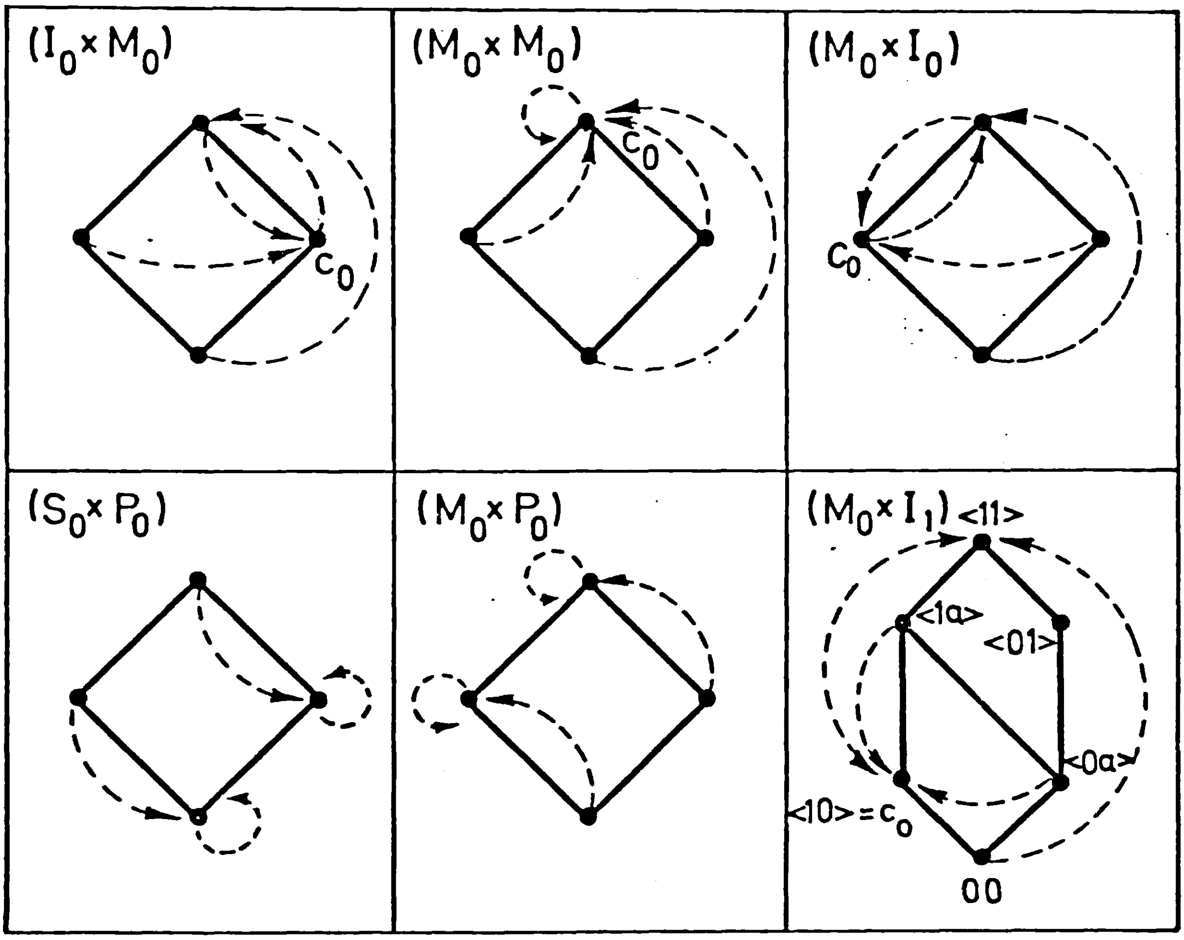
$(I_0 \times M_0)$
$c_0$
$(M_0 \times M_0)$
$c_0$
$(M_0 \times I_0)$
$c_0$
$(S_0 \times P_0)$
$(M_0 \times P_0)$
$(M_0 \times I_1)$
<11>
<1a>
<01>
<0a>
<10> = $c_0$
00

Fig. 5

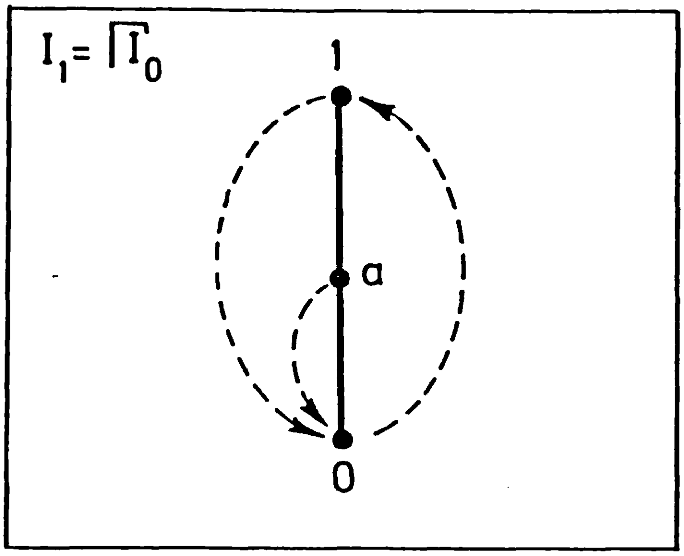


Fig. 6

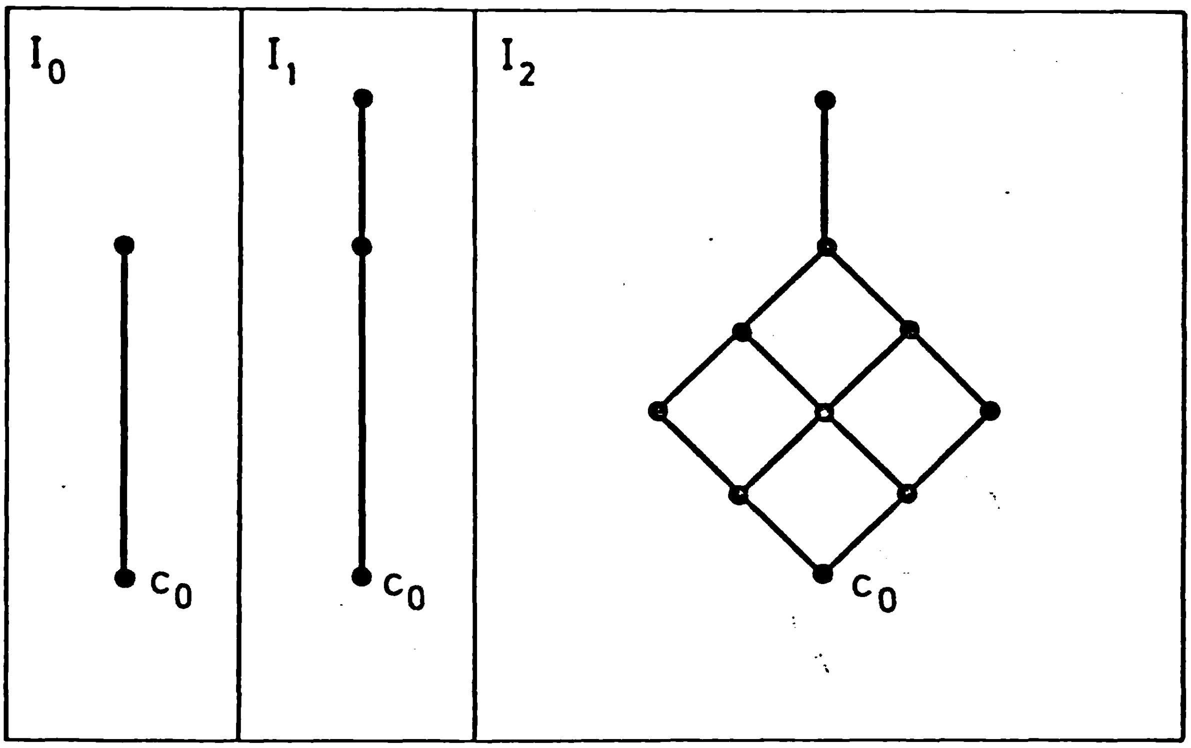


Fig. 7

I name this last lattice as $I_1$. In view of what has been said in (33) this is the only three-element implicative lattice which represents an intuitionistic type of negation. I denote a lattice of Fig.3 appearing in the ith row and jth column by $^{(3)}A_{ij}$.

(38).....The lattice in the first column and third row in Fig.3 will be denoted as $P_1$.

(39).....The four-element matrix $M_2 \times M_2 = M_2^2$ is represented in Fig. 4.

(40).....In Fig.5 I give the lattice representations of $(I_o \times M_o)$, $(M_o \times M_o)$, $(M_o \times S_o)$, $(S_o \times P_o)$ and $(M_o \times I_1)$.[25]

(41).....If A is an implicative lattice represented in a Hasse-diagram, we can easily represent $\Gamma A$ where $\Gamma$ is Jaskowski's matrix-operator. For instance, in Fig.6 we find the Hasse diagram of $\Gamma I_o$. Notice that $\Gamma I_o$ is the three-element implicative lattice which I have named $I_1$. (see above (37)(c)).

(42).....In Fig.7 I give the lattice-representations of the three first matrices in the sequence of Jaskowski's matrices that characterize HL. (see (14) above). These lattices are, of course, implicative lattices with standard negations and distinguished elements $c_o = 0$.

Finally, since in one of the proofs I use Zorn's lemma, I formulate it here in a convenient form.

(43).....If every chain of elements of an ordered set S has an upper bound in S, then S contains a maximal element. (ZORN, 1935)

# CHAPTER II NEW RESULTS

"There is always a possibility
of error; though in the case
of some logical and mathematical
proofs, this possibility may be
considered slight." (K.R. Popper)

The new results can be grouped in two distinct types: extension-criteria results and infinite chain-results.

## 1. EXTENSION-CRITERIA RESULTS

In (JANKOV, 1963 p.1103) Jankov gave the definition of a super-constructive propositional calculus as follows: "By a super-constructive propositional calculus (sc.-calculus) we will mean any calculus that is constructed from a language of logical propositions in the following manner: as axioms of the sc.-calculus one takes all the basic axioms of Heyting's intuitionistic calculus and a certain finite set of additional axioms; as deduction rules of the sc.-calculus one takes modus ponens and substitution." In the same paper Jankov announced[(1)] the following result: "In order that the sc.-calculus P be equivalent to the classical calculus it is necessary and sufficient that all the additional axioms of P be valid over $I_o$ and at least one of them be disproved over $I_1$." This result can be formulated according to our terminology as

$(HL, F_1, F_2, \ldots, F_n) = CL$ iff each additional axiom $F_i$ $(1 \leq i \leq n)$ is valid in $I_o$ and there is at least one $F_i$ such that it is invalid in $I_1$.

Harrop commented on this result of Jankov in (HARROP, 1965 p.289). J.G. Anderson gave a proof of this theorem of Jankov in (ANDERSON, 1969) but he used Kripke-type semantics, developing a theory of structures which he called <u>contracted Kripke models</u> (2) as a preliminary to the proof. Consequently his proof is rather long. I shall prove Jankov's result in a simple manner. It will come out as an easy consequence of the following more general theorem:

(1).....<u>THEOREM 1</u>: <u>$(ML, F_1, F_2, \ldots, F_n) \supseteq Cal^7$ iff there is an additional axiom $F_i$ $(1 \leq i \leq n)$ such that $F_i$ is invalid in $I_1$.</u>

I recall(3) that $Cal^7$ is defined by $(ML, p \vee \neg p)$ and that $I_1$ is the three-element implicative lattice with standard negation and distinguished element $c_o = 0$.

<u>Proof:</u> First, observe that without loss of generality it is sufficient to give the proof for the case in which there is one additional axiom, say F, because we can replace $F_1, F_2, \ldots, F_n$ additional axioms by the single axiom $F = F_1 \& F_2 \& \ldots \& F_n$ since this single axiom F will be invalid in $I_1$ iff there is at least one $F_i$ $(1 \leq i \leq n)$ which is invalid in $I_1$. Thus I shall prove <u>THEOREM 1</u> in the form <u>$(ML, F) \supseteq Cal^7$ iff F is invalid in $I_1$.</u>

Necessity: If $(ML,F) \supseteq Cal^7$ then F is invalid in $I_1$. This easy part of the theorem follows immediately from the definition of $Cal^7$ and the fact that $p \vee \neg p$ is invalid in $I_1$ because if the value of 'p' is $a_{I_1}$ then $V(p \vee \neg p) = a_{I_1} \cup \neg a_{I_1} \neq 1_{I_1}$ .

Sufficiency: If F is invalid in $I_1$ then $(ML,F) \supseteq Cal^7$. To prove this, suppose indirectly that F is invalid in $I_1$ but $(ML,F) \not\supseteq Cal^7$. Then by the completeness theorem concerning the propositional theories of ML (see I.4.(31)), there is a non-degenerate implicative lattice A with standard negation and distinguished element $c_o$, such that F is valid in A but $p \vee \neg p$ is invalid in A. Let one of the values of 'p' be a , $a \in A$, for which $a \cup \neg a \neq 1_A$. I shall show that there is a sublattice A' of A, which is closed under ' $\rightarrow$ ' and is isomorphic to $I_1$. The existence of such a sublattice entails that F is valid in $I_1$, and this contradicts our supposition.

It remains to be shown that there is such a sublattice A'. Take the following three elements of A: $1_A$, d, $c_o$ , where $d = a \cup \neg a$. These three elements of A are distinct, and on them the operations '$\cap$', '$\cup$', ' $\rightarrow$ ' are closed as the following considerations demonstrate:

(I)..... $d \neq 1_A$, for if $d = 1_A$ then $a \cup \neg a = 1_A$ and this contradicts $a \cup \neg a \neq 1_A$

(II)..... $c_o \neq 1_A$, for if $c_o = 1_A$ then $d = a \cup \neg a = a \cup (a \rightarrow c_o)$ $= a \cup (a \rightarrow 1_A) = a \cup 1_A = 1_A$ and this contradicts (I).

(III).....$c_o \neq d$, for if $c_o = d$, then by I.4 (24) $1_A = \neg c_o = \neg d =$

$$= (a \vee \neg a) \rightarrow c_o = (a \rightarrow c_o) \wedge (\neg a \rightarrow c_o) =$$

$(\neg a \wedge \neg \neg a) = c_o$ and this contradicts (II)

Thus the elements of $1_A$, $d$, $c_o$ are indeed distinct. In order to show that they are closed for the operations '$\wedge$', '$\vee$', '$\rightarrow$' it will be useful to observe that

(IV).....$c_o \leq d$, for

$$c_o = \neg a \wedge \neg \neg a \leq \neg a \leq \neg a \vee a = d$$

Hence the lattice-ordering among our three elements is $c_o \leq d \leq 1_A$, so for any two elements $a'$, $b'$ of these three elements $a' \vee b' = \max (a', b')$ and $a' \wedge b' = \min (a', b')$. Thus these three elements form a sublattice $A'$.

Finally, again by using I.4. (24) we see that the operation '$\rightarrow$' is closed on the elements of $A'$:

$$c_o \rightarrow c_o = 1_A$$
$$c_o \rightarrow d = 1_A$$
$$c_o \rightarrow 1_A = 1_A$$
$$d \rightarrow 1_A = 1_A$$
$$1_A \rightarrow 1_A = 1_A$$
$$1_A \rightarrow d = d$$
$$1_A \rightarrow c_o = c_o$$
$$d \rightarrow c_o = c_o$$

Since F is valid in A, it remains valid in the sublattice $A'$. But $A'$ is evidently isomorphic to $I_1$, thus F is valid in $I_1$ and this contradicts our original supposition.

(2).....COROLLARY: $(ML, F_1, F_2, \ldots, F_n) = Cal^7$ iff each of the additional axioms $F_i$ is derivable in $Cal^7$ and there is at least one $F_i$ among them which is invalid in $I_1$.

This follows directly from THEOREM 1 and the fact that $(ML, F_1, F_2, \ldots, F_n) \subseteq Cal^7$ iff each additional axiom is derivable in $Cal^7$.

(3).....REMARK: Both THEOREM 1 and its COROLLARY can be given in a more general form: THEOREM 1 and its COROLLARY remain true if there are infinitely many $F_1, F_2 \ldots$ additional axioms in the formulations. The sufficiency part is immediate from what has been proved. The necessity part follows again from the fact that $Cal^7$ is finitely axiomatizable over ML by a single additional axiom which is invalid in $I_1$.

Let us immediately see how we can use the extension-criteria expressed in THEOREM 1 and its COROLLARY.

(4).....EXAMPLE 1: Since formula $F^* = \neg p \vee \neg\neg p$ is valid in $I_1$, $Cal^* = (ML, F^*) \not\supseteq Cal^7$ by THEOREM 1. From this it follows that $p \vee \neg p$ is underivable in $Cal^*$. On the other hand, it is clear that $F^*$ is derivable in $Cal^7$, for substitution $\neg p$ in place of 'p' in formula $p \vee \neg p$ gives us $F^*$. Hence by COROLLARY $ML \subseteq Cal^* \subset Cal^7$. Thus $Cal^*$ is an intermediate logic between ML and $Cal^7$, such that it is a proper fragment of $Cal^7$.

(5).....EXAMPLE 2: Formula F** = $(p \rightarrow q) \rightarrow (\neg p \vee q)$ is invalid in $I_1$ because if the value of both 'p' and 'q' in F** is '$a_{I_1}$', then $(a_{I_1} \rightarrow a_{I_1}) \rightarrow (\neg a_{I_1} \cup a_{I_1}) = a_{I_1} \neq 1_A$. By THEOREM 1 Cal** = (ML,F**)$\supseteq$ $Cal^7$. On the other hand, since F** is a classical tautology, we know that Cal** is an intermediate logic between $Cal^7$ and CL.

I shall now prove Jankov's result as an easy consequence of THEOREM 1. The proof is given here again, without loss of generality, for the case in which there is one additional axiom:

(6).....THEOREM 2: (HL,F)$\supseteq$ CL iff F is invalid in $I_1$.

Proof: If F is invalid in $I_1$ then (HL,F) = (ML, $\neg p \rightarrow (p \rightarrow q)$,F)$\supseteq$ ($Cal^7$, $\neg p \rightarrow (p \rightarrow q)$) = CL by THEOREM 1. This proves the sufficiency part.

The necessity part is immediate from (HL, $p \vee \neg p$) = CL and the fact that $p \vee \neg p$ is invalid in $I_1$, for if the value of 'p' is '$a_{I_1}$' then $a_{I_1} \cup \neg a_{I_1} = a_{I_1} \neq 1_{I_1}$.

(7).....COROLLARY: (HL,F) = CL iff F is a classical tautology and invalid in $I_1$.

This follows directly from THEOREM 2 and the fact that (HL,F)$\subseteq$ CL iff F is a classical tautology.

(8).....REMARK: THEOREM 2 and its COROLLARY remain true if there are infinitely many additional axioms to HL in the formulations. (See previous remark under (3)).

(9).....<u>EXAMPLE 1:</u> Since formula $F' = (p \longrightarrow q) \vee (q \longrightarrow p)$ is valid in $I_1$, by THEOREM 2, Cal' = $(HL,F') \not\supseteq CL$. Thus, Cal' is an <u>intermediate logic</u> between HL and CL, such that it is a proper fragment of CL. M. Dummett investigated this <u>intermediate logic</u> in (DUMMETT, 1959).

(10).....<u>EXAMPLE 2:</u> The formula $F'' = (\neg p \longrightarrow p) \longrightarrow p$ is invalid in $I_1$ because if the value of 'p' is '$a_{I_1}$' then $V(F'') = a_{I_1} \neq 1_A$. By <u>THEOREM 2</u> $(HL,F'') \supseteq CL$. Since F'' is a classical tautology, by (7) $(ML,F) = CL$. Łukasiewicz calls F'' the law of Clavius.[4]

(11).....<u>THEOREM 3:</u> <u>$(ML,F_1,F_2,\ldots,F_n) \supseteq HL$ iff there is an $F_i$ $(1 \leq i \leq n)$ such that $F_i$ is invalid in $M_o$.</u>

<u>Proof:</u> I recall[5] that $M_o$ is the two-element implicative lattice with standard negation and distinguished element $c_o = 1_{M_o}$. Since it is again sufficient to give the proof[6] for the case in which there is one additional axiom, say F, I prove the theorem in the simpler form:

<u>$(ML,F) \supseteq HL$ iff F is invalid in $M_o$</u>

<u>Necessity:</u> If $(ML,F) \supseteq HL$ then F is invalid in $M_o$. This easy part of the theorem follows immediately from the definition of HL: $(ML,F^*) = HL$ where $F^* = \neg p \longrightarrow (p \longrightarrow q)$ and the fact that $F^*$ is invalid in $M_o$.

<u>Sufficiency:</u>[7] If F is invalid in $M_o$, then $(ML,F) \supseteq HL$.

Lemma: If A is an implicative lattice with a distinguished element $c_o$ and a zero element $O_A$, such that $c_o \neq O_A$ then there is a lattice homomorphism h from A onto $M_o$, such that $h(c_o) = 1_{M_o}$.

Proof of Lemma: Let S be the set of proper filters $\{\nabla_i\}$ in A such that for each $\nabla_i$, $c_o \in \nabla_i$. S is not empty for the principal filter generated by $c_o$ is a member of S. This filter is proper because $c_o \neq O_A$. Since each $\nabla_i$ is a subset of A, S can be partially ordered by set-inclusion. Each chain in S has an upper-bound in S for the union of the proper filters occurring in each chain is such an upper bound. By Zorn's lemma there is a maximal element in S, say $\nabla_o$

It is well-known that each filter $\nabla$ in an implicative lattice A determines an equivalence relation $\approx_{\nabla}$(for all a, b $\in$ A, a is equivalent to b iff $(a \rightarrow b) \in \nabla$ and $(b \rightarrow a) \in \nabla$, and there is a natural homomorphism h from A onto the subalgebra of A determined by $\approx_{\nabla}$, which I shall denote by $A/_{\nabla}$. It can be shown that whenever $\nabla$ is maximal $A/_{\nabla}$ contains exactly two elements (RASIOWA-SIKORSKI, 1963) p.66.

The proof of the lemma is completed by the application of this fact from the existence of the maximal filter $\nabla_o$ : $A \overset{h}{\dashrightarrow} A/_{\nabla_o} = M_o$, $h(c_o) = 1_{M_o}$ and $h(O_A) = O_{M_o}$.

We continue now the proof of the sufficiency. Suppose indirectly that F is invalid in $M_o$ but $(ML,F) \not\supset HL$. If $(ML,F) \not\supset HL$, then by the completeness theorem concerning the propositional theories of ML, there is an implicative lattice A with standard negation (distinguished element: $c_o$) such that F is valid in A, and F* is invalid in A. Notice that if there is a zero element $O_A$ of A then $c_o \neq O_A$ because if

$c_o = 0_A$, F* is valid in A. The validity of F in A means that the value-function of F in A is identically $1_A$. If we were able to find a lattice A' in which this remains true and A' has a zero element, then by the lemma we would arrive at a contradiction. Take a principal filter A' in A generated by some element below $c_o$, say c'. A' fulfils the requirements we are looking for: it has a zero element, namely c'. The value function of F in A' is identically $1_{A'} = 1_A$ for A', being a filter, if a, b $\in$ A', $a \cap b$, $a \cup b$, $a \longrightarrow b$ define the same elements in A' as in A. Thus by the lemma, there is a lattice homomorphism h from A' onto $M_o$ such that $h(c_o) = 1_{M_o}$, which means that F is valid in $M_o$ and this contradicts our supposition that F is invalid in $M_o$.

(12).....COROLLARY: $(ML, F_1, F_2, \ldots, F_n) = HL$ iff each of the additional axioms $F_i$ $(i = 1,2,\ldots,n)$ is derivable in HL and there is at least one $F_i$ such that $F_i$ is invalid in $M_o$.

This follows directly from THEOREM 3 and the fact that $(ML, F_1, \ldots, F_n) \subseteq HL$ iff each additional axiom $F_i$ is derivable in HL.

(13).....REMARK: Both THEOREM 3 and its COROLLARY remain true if there are infinitely many additional axioms.(8)

Let us now see how we can use the extension-criteria expressed in THEOREM 3 and its COROLLARY.

(14).....EXAMPLE 1: Since formula $F^* = \neg\neg(\neg\neg p \longrightarrow p)$ is valid in $M_o$, $Cal^* = (ML, F^*) \not\supseteq HL$ by THEOREM 3, and since F* is derivable in HL and underivable in ML, Cal* is an intermediate logic between ML and HL such that it is a proper extension of ML and a proper fragment of HL.

(15).....EXAMPLE 2: Formula $F^{**} = (\neg p \vee q) \longrightarrow (p \longrightarrow q)$ is invalid in $M_o$ because if the value of 'p' is $1_{M_o}$ and that of 'q' is $0_{M_o}$, then $V(F^{**}) = 0_{M_o} \neq 1_{M_o}$. Hence $Cal^{**} = (ML, F^{**}) \supseteq HL$. On the other hand, since F** is a classical tautology, Cal** is an intermediate logic between HL and CL.

(16).....THEOREM 4: $(ML, F_1, \ldots, F_n) \supseteq CL$ iff there is an $F_i$ $(1 \leq i \leq n)$ which is invalid in $M_o$, and there is an $F_i$ $(1 \leq i \leq n)$ which is invalid in $I_1$.

Proof: This is a direct consequence of THEOREM 3 and THEOREM 1.

(17).....COROLLARY: $(ML, F_1, \ldots, F_n) = CL$ iff each $F_i$ $(1 \leq i \leq n)$ is a classical tautology, and there is an $F_i$ $(1 \leq i \leq n)$ such that $F_i$ is invalid in $M_o$, and there is an $F_i$ $(1 \leq i \leq n)$ such that $F_i$ is invalid in $I_1$.

This is a direct consequence of THEOREM 4 and the fact that $(ML, F_1, \ldots, F_n) \subseteq CL$ iff each $F_i$ $(1 \leq i \leq n)$ is derivable in CL.

(18).....REMARK: THEOREM 4 and its COROLLARY remain true if there are infinitely many additional axioms $F_i$.

(19).....EXAMPLE 1: Since $F^* = \neg p$ is valid in $M_o$, $(ML, F^*) \not\supseteq CL$ by THEOREM 4.

(20).....<u>EXAMPLE 2:</u> Since formula $F^{**} = \neg p \rightarrow p$ is invalid both in $M_o$ and in $I_1$ because if the value of 'p' is $0_{M_o}$ and $0_{I_1}$ respectively, $V(F^{**}) = 0_{M_o} \neq 1_{M_o}$ and $V(F^{**}) = 0_{I_1} \neq 1_{I_1}$, hence $(ML,F^{**}) \supseteq CL$ by <u>THEOREM 4.</u>

(21).....<u>EXAMPLE 3</u>: Formula $F^{***} = (\neg\neg p \rightarrow p)$ is a classical tautology and is invalid both in $M_o$ and in $I_1$, for if the value of 'p' is $0_{M_o}$ and $0_{I_1}$ respectively then $V(F^{***}) \neq 1_{M_o}$ and $V(F^{***}) \neq 1_{I_1}$. Thus by <u>COROLLARY (17)</u> $(ML,F^{***}) = CL$. Similar simple tests show that each of the following negation-schemata extend the minimal calculus exactly to the classical calculus, some of which have also been proved by K. Segerberg (SEGERBERG, 1968).

(i).....$(\neg p \rightarrow \neg q) \rightarrow (q \rightarrow p)$

(ii).....$\neg(\neg p \,\&\, \neg q) \rightarrow p \vee q$

(iii).....$\neg(\neg p \vee \neg q) \rightarrow p \,\&\, q$

(iv).....$\neg(p \,\&\, \neg q) \rightarrow (p \rightarrow q)$

(22).....<u>THEOREM 5</u>: <u>$(Cal^7,F_1,F_2,\ldots,F_n) \supseteq CL$ iff there is an $F_i$ ($1 \leq i \leq n$) such that it is invalid in $M_o$.</u>

The proof is again given here in the simpler form:
<u>$(Cal^7,F) \supseteq CL$ iff F is invalid in $M_o$.</u>

The <u>necessity part</u> follows from $(Cal^7,F^*) = CL$ where $F^* = \neg p \rightarrow (p \rightarrow q)$ and the fact that $F^*$ is invalid in $M_o$.

The <u>sufficiency part</u> is an easy consequence of <u>THEOREM 3</u>:

$(Cal^7,F) = (ML, p \vee \neg p, F) \supseteq (ML, p \vee \neg p) = CL$ if F is invalid in $M_o$.

(23).....<u>COROLLARY:</u> $(Cal^7, F_1, F_2, \ldots, F_n) = CL$ iff each $F_i$ $(1 \leq i \leq n)$ is a classical tautology and there is an $F_i$ such that it is invalid in $M_o$.

(24).....<u>REMARK:</u> <u>THEOREM 5</u> and its <u>COROLLARY</u> remain true if there are infinitely many additional axioms.(9)

## 2. Infinite chain results

In 1968 C.G.McKay proved (McKAY, 1968) that there exist denumerably many distinct logics between HL and CL, which strictly succeed(10) HL and which have the same ICD fragment(11) as HL. The proof presented by McKay was constructive: he has given countably many formulae $F_1$, $F_2$,... by a recursive definition and showed that

$$CL \supseteq (HL, F_1) \supseteq (HL, F_2) \supseteq \ldots, \supseteq HL$$

In a similar way I could prove that there exist denumerably many distinct logics between ML and HL, ML and $Cal^7$, $Cal^7$ and CL. But instead of proving these separately I shall prove two more general theorems from which any of the above results come out as an easy consequence(12).

(1).....<u>THEOREM 1: If F* is a theorem of HL, then there are countably many distinct logics between (ML,F*) and(Cal$^7$,F*).</u>

<u>Proof:</u> Let F** be $\neg p_1 \vee \neg\neg p_1$, and consider the following formulae $F_1, F_2, \ldots$

$$F_1 = F^{**}$$

$$F_2 = ((p_2 \rightarrow F_1) \rightarrow p_2) \rightarrow p_2$$

$$\vdots$$

$$F_{n+1} = ((p_{n+1} \rightarrow F_n) \rightarrow p_{n+1}) \rightarrow p_{n+1}$$

$$\vdots$$

<u>First,</u> observe that

$$(ML, F^*, F_1) \supseteq (ML, F^*, F_2) \supseteq \ldots$$

To prove this, suppose indirectly that for some value of 'n'

$$(ML, F^*, F_n) \not\supseteq (ML, F^*, F_{n+1})$$

Then by the completeness theorem concerning the propositional theories of ML, there is an implicative lattice A with standard negation such that $F_n$ is valid in A but $F_{n+1}$ is invalid in A. The validity of $F_n$ in A entails that the value-function

$$V((p_{n+1} \rightarrow F_n)) = 1_A \qquad \text{by } a \rightarrow 1_A = 1_A$$

$$V((p_{n+1} \rightarrow F_n) \rightarrow p_{n+1}) = V(p_{n+1}) \text{ by } 1_A \rightarrow a = a$$

$$V(F_{n+1}) = V(p_{n+1}) \rightarrow V(p_{n+1}) = 1_A \text{ by } a \rightarrow a = 1_A$$

and this contradicts the supposition that $F_{n+1}$ is invalid in A.

Secondly, I shall prove that each of these logics is distinct, i.e. for each n = 1, 2, ...

$$(ML, F^*, F_n) \neq (ML, F^*, F_{n+1})$$

To prove this it is sufficient to show that there is an implicative lattice with standard negation for each n = 1, 2,... such that the axioms of ML, $F^*$ and $F_{n+1}$ are valid in it, but $F_n$ is invalid.

Consider the sequence of implicative lattices $A_1$, $A_2$, ... shown on Fig.8. The axioms of ML, and $F^*$ are valid in each of these lattices because the distinguished element $c_o$ is the zero element. On the other hand, $F_n$ is invalid in $A_n$ but $F_{n+1}$ is valid in $A_n$. This statement can be proved by induction:

Basic step: For n = 1, $F_1 = \neg p_1 \vee \neg\neg p_1$ is invalid in $A_1$ because if the value of '$p_1$' is either 'a' or 'b' then $V(F_1) = d \neq 1_{A_1}$. Note that for any other valuations $V(F_1) = 1_{A_1}$. On the other hand, $F_2$ is valid in $A_1$ because

$$F_2 = ((p_2 \longrightarrow F_1) \longrightarrow p_2) \longrightarrow p_2$$

and if the value of $p_2$: $V(p_2) \leq d$, then

$$V(F_2) = (1_{A_1} \longrightarrow V(p_2)) \longrightarrow V(p_2) = 1_{A_1} ;$$

and if the value of $p_2$: $V(p_2) \not\leq d$, then $V(p_2) = 1_{A_1}$ and then clearly $V(F_2) = 1_{A_1}$.

Induction hypothesis: Suppose the statement is true for n = k, i.e. $F_k$ is invalid in $A_k$ but $F_{k+1}$ is valid in $A_k$. I would like to show that the statement remains true for n = k + 1, i.e. $F_{k+1}$ is invalid

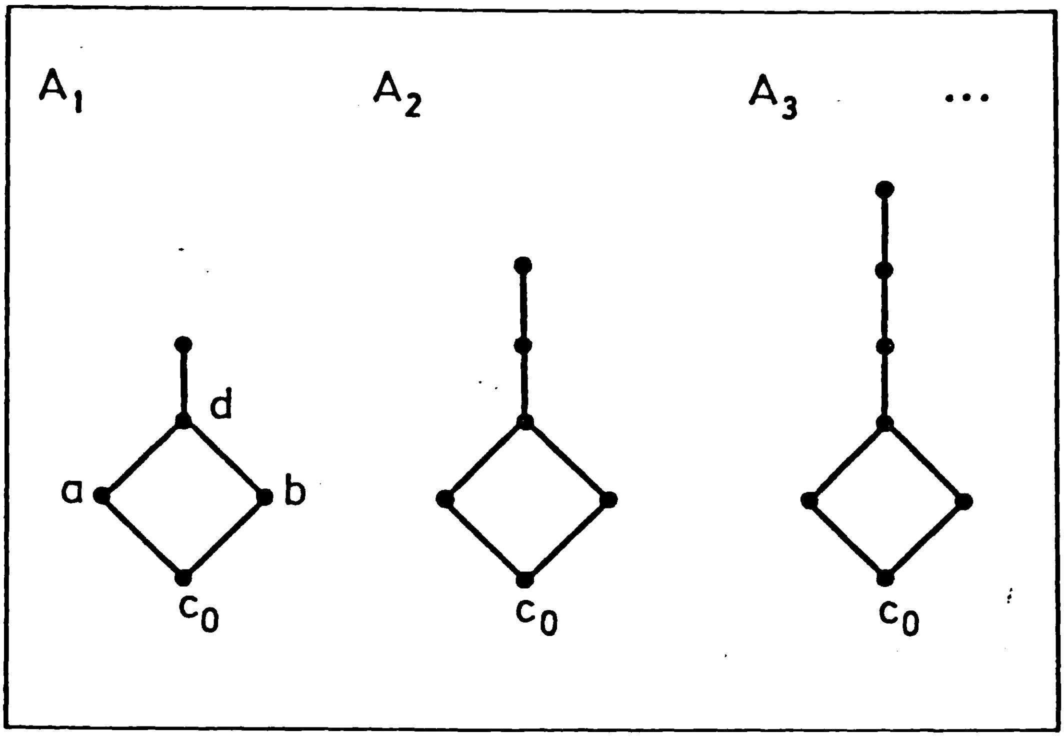


Fig. 8

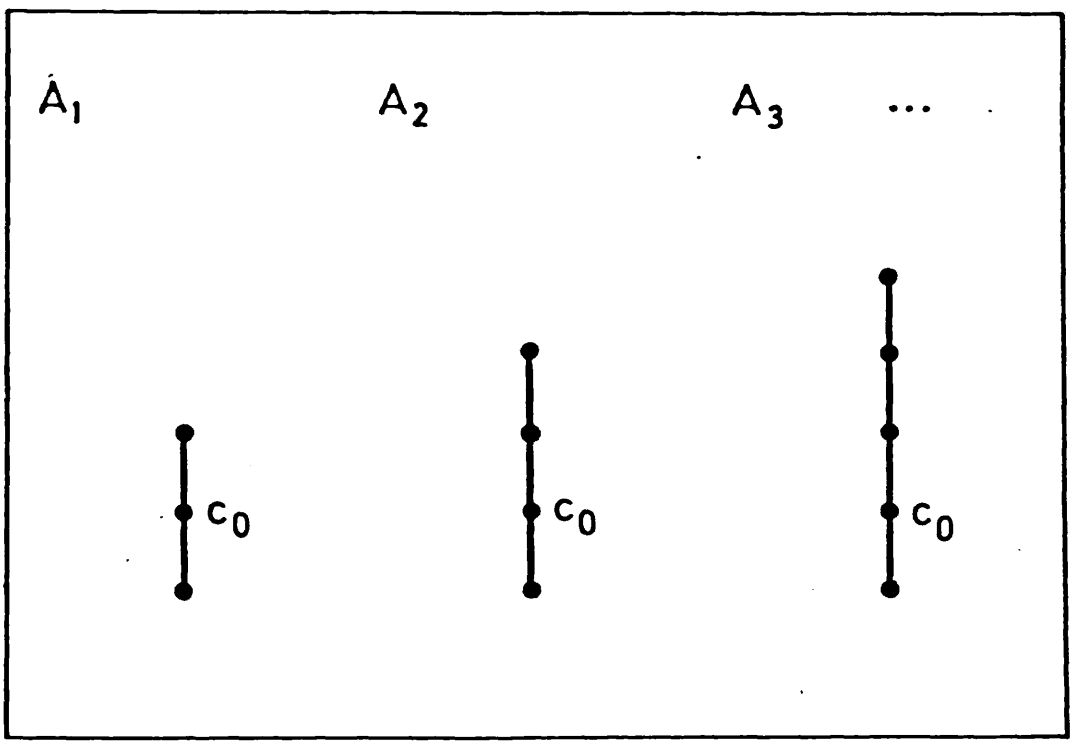


Fig. 9

in $A_{k+1}$ but $F_{k+2}$ is valid in $A_{k+1}$.

First observe that by definition $A_{k+1}$ contains $5 + k$ elements, say 0, a, b, $d_i$ $(i = 1,2,\dots,k)$, 1 ; one element more than $A_k$ whose elements I denote by $0^*$, $a^*$, $b^*$, $d^*_i$ $(i = 1, 2,\dots,k-1)$, $1^*$.(See Fig.8) Notice that $d_i \angle d_j$ and $d^*_i \angle d^*_j$ iff $i \angle j$.

<u>Lemma</u>: There is a valuation of $F_k$ in $A_{k+1}$ such that $V(F_k) \angle d_k$ ; and for all valuations of $F_{k+1}$ in $A_{k+1}$ : $V(F_{k+1}) \geq d_k$.

To prove the first part of the lemma, suppose indirectly that for all valuations of $F_k$ in $A_{k+1}$: $V(F_k) \geq d_k$. Consider the following map h from $A_{k+1}$ onto $A_k$:

$$h(1) = 1^*$$
$$h(d_k) = 1^*$$
$$h(d_i) = d^*_i \quad (i = 1, 2, \dots, k-1)$$
$$h(a) = a^*$$
$$h(b) = b^*$$
$$h(0) = 0^*$$

Since h is a lattice-homomorphism, it ensures that $F_k$ is valid in $A_k$ and this contradicts the induction hypothesis.[13]

To prove the second part of the lemma, suppose indirectly that there is a valuation of $F_{k+1}$ in $A_{k+1}$, say $V^*(F_{k+1})$ such that $V^*(F_{k+1}) \not\geq d_k$. In other words $V^*(F_{k+1}) \leq d_{k-1}$. Consider the valuation of $F_{k+1}$ in $A_k$ determined by $V^*(F_{k+1})$ and the defined lattice-homomorphism h. Clearly for this valuation, which I shall denote by $V^{**}(F_{k+1})$, is true that $V^{**}(F_{k+1}) \leq d^*_{k-1}$. But this means that $F_{k+1}$ is invalid in $A_k$, and this contradicts the 2nd part of the induction hypothesis.

To prove that $F_{k+1}$ is invalid in $A_{k+1}$ we exhibit a valuation of $F_{k+1}$ in $A_{k+1}$ such that it does not equal the top element 1. By definition

$$F_{k+1} = ((p_{k+1} \longrightarrow F_k) \longrightarrow p_{k+1}) \longrightarrow p_{k+1}$$

where $p_{k+1}$ is a new propositional variable, not occurring in $F_k$. By the lemma there is a valuation of $F_k$ in $A_{k+1}$ such that $V(F_k) \angle d_k$. Taking this valuation for $F_k$ and the value $d_k$ for the propositional variable $p_{k+1}$, we have

$$\begin{aligned} ((d_k \longrightarrow V(F_k)) \longrightarrow d_k) \longrightarrow d_k &= \\ (V(F_k) \longrightarrow d_k) \longrightarrow d_k &= \\ 1_{A_{k+1}} \longrightarrow d_k = d_k \neq 1_{A_{k+1}} \end{aligned}$$

This proves that $F_{k+1}$ is indeed invalid in $A_{k+1}$.

Finally I demonstrate that $F_{k+2}$ is valid in $A_{k+1}$. By definition

$$F_{k+2} = ((p_{k+2} \longrightarrow F_{k+1}) \longrightarrow p_{k+2}) \longrightarrow p_{k+2}$$

where $p_{k+2}$ is a new propositional variable, not occurring in $F_{k+1}$. By the lemma, for all valuations of $F_{k+1}$ in $A_{k+1}$: $V(F_{k+1}) \geq d_k$. Hence if $V(p_{k+2}) \nleq V(F_{k+1})$, then $V(p_{k+2}) = 1$, and then clearly the value of $F_{k+2}$ is 1. On the other hand, if $V(p_{k+2}) \leq V(F_{k+1})$, then again the value of $F_{k+2}$ is the top element of $A_{k+1}$.(14)

(2).....<u>REMARK:</u> It is also true that (ML,F*,$F_n$) have the same ICD fragment as (ML,F*). This is clear from the way McKay proved the same for his intermediate logics between HL and CL[15].

(3).....<u>COROLLARY 1:</u> <u>There are denumerably many distinct logics between HL and CL.</u>

Take $\neg p \longrightarrow (p \longrightarrow q)$ for F*, then (ML,F*) = HL and ($Cal^7$,F*) = CL.

(4).....<u>COROLLARY 2:</u> <u>There are denumerably many distinct logics between and $Cal^7$.</u>

Take for F* a formula which is derivable in ML, then (ML,F*) = ML and ($Cal^7$,F*) = $Cal^7$.

(5).....<u>THEOREM 2:</u> <u>If F* is a theorem of $Cal^7$ then there are countably many distinct logics between (ML,F*) and (HL,F*).</u>

<u>Proof:</u> It goes exactly the same way as the proof of THEOREM 1 except that for F** we now take $\neg\neg(\neg\neg p_1 \longrightarrow p_1)$ and for the implicative lattices $A_1, A_2, \ldots, A_j, \ldots$ we use the lattices in Fig. 9.

(6).....<u>REMARK:</u> <u>Mutatis mutandis</u> REMARK (2) applies also to the result of <u>THEOREM 2</u>[16].

(7).....<u>COROLLARY 1:</u> <u>There are denumerably many distinct logics between $Cal^7$ and CL.</u>

If for $F^*$ we take $(p \vee \neg p)$, then $(ML,F^*) = Cal^7$ and $(HL,F^*) = CL$.

(8).....COROLLARY 2: There are denumerably many distinct logics between ML and HL.

If for $F^*$ we take a formula which is derivable in ML, then $(ML,F^*) = ML$ and $(HL,F^*) = HL$.

## 3. More extension-criteria results.

In the previous section it was proved that the following sequence of logics form infinite chains between ML and $Cal^7$, HL and CL, ML and HL, $Cal^7$ and CL respectively.

$$Cal^7 \supset (ML,F_1) \supset (ML,F_2) \supset \dots \supset (ML,F_{n+1}) \supset \dots \supset ML$$

$$CL \supset (HL,F_1) \supset (HL,F_2) \supset \dots \supset (HL,F_{n+1}) \supset \dots \supset HL$$

$$HL \supset (M,F'_1) \supset (ML,F'_2) \supset \dots \supset (ML,F'_{n+1}) \supset \dots \supset ML$$

$$CL \supset (Cal^7,F'_1) \supset (Cal^7,F'_2) \supset \dots \supset (Cal^7,F'_{n+1}) \supset \dots \supset Cal^7 .$$

where formulae $F_i$ and $F'_i$ are defined recursively as

$$F_1 = \neg p_1 \vee \neg \neg p_1$$

$$F_{n+1} = ((p_{n+1} \rightarrow F_n) \rightarrow p_{n+1}) \rightarrow p_{n+1}$$

$$F'_1 = \neg \neg ( \neg \neg p_1 \rightarrow p_1)$$

$$F'_{n+1} = ((p_{n+1} \rightarrow F'_n) \rightarrow p_{n+1}) \rightarrow p_{n+1}$$

The theorems proved in the first part of this chapter gave criteria for extending ML to $Cal^7$, HL to CL, ML to HL and $Cal^7$ to CL. The question may be asked whether it is possible to give similar criteria with respect to certain other logics contained in these four infinite chains? The answer to this question is affirmative.

(1).....<u>THEOREM 1</u>: For any i = 1, 2,...,n + 1,...
$(ML,F_i,F) \supseteq Cal^7$ iff F is invalid in $I_1$.
This is a direct consequence of II.1.(1) and the fact that $F_i$ is valid in $I_1$ .

(2).....<u>COROLLARY</u>: For any i = 1, 2,...,n+1,...
$(ML,F_i,F) = Cal^7$ iff F is invalid in $I_1$ and F is derivable in $Cal^7$.
This is a direct consequence of II.1.(2) and the fact that $F_i$ is derivable in $Cal^7$.

(3).....<u>THEOREM 2</u>: For any i = 1, 2,...,n+1,...
$(HL,F_i,F) \supseteq$ CL iff F is invalid in $I_1$. This is a direct consequence of II,1,(6) and the fact that $F_i$ is valid in $I_1$.

(4).....<u>COROLLARY:</u> For any i = 1, 2,...,n+1,...
$(HL,F_i,F)$ = CL iff F is invalid in $I_i$ and F is a classical tautology.
This is a direct consequence of II,1,(7) and the fact that $F_i$ is a classical tautology.

(5).....<u>THEOREM 3:</u> For any i = 1, 2,...,n+1,...
$(ML,F_i',F) \supseteq$ HL iff F is invalid in $M_o$. This directly follows from II,1,(11) and the fact that $F_i'$ is valid in $M_o$.

(6).....<u>COROLLARY</u>: For any $i = 2,\dots, n+1,\dots$
$(ML,F_i',F) = HL$ iff F is invalid in $M_o$ and is derivable in HL. This directly follows from I,1,(12) and the fact that $F_i'$ is derivable in HL.

(7).....<u>THEOREM 4:</u> For any $i = 1, 2,\dots, n+1\dots$
$(Cal^7,F_i'F) \supseteq CL$ iff F is invalid in $M_o$. This again follows from II.1.(22) and the fact that $F_i'$ is valid in $M_o$.

(8).....<u>COROLLARY</u>: For any $i = 1, 2,\dots, n+1,\dots$
$(Cal^7,F_i',F) = CL$ iff F is invalid in $M_o$ and F is a classical tautology. This again follows from II.1.(23) and the fact that $F_i'$ is a classical tautology.

(9).....<u>THEOREM 5</u>: $(ML,F) \supseteq (ML,F_1)$ iff F is invalid in the five-element implicative lattice $A_1$ defined in II,2.(1).

<u>Necessity</u>: If $(ML,F) \supseteq (ML,F_1)$ then F is invalid in $A_1$. This easy part of the theorem follows immediately from the fact that $F_1$ is invalid in $A_1$.

<u>Sufficiency:</u> If F is invalid in $A_1$ then $(ML,F) \supseteq (ML,F_1)$. Suppose indirectly that F is invalid in $A_1$ but $(ML,F) \not\supseteq (ML,F_1)$. Then by the completeness theorem concerning zero-order theories of ML, there is a non-degenerate implicative lattice A with standard negation, such that F is valid in $A_1$ but $F_1$ is invalid in $A_1$. Let 'a' be a value of '$p_1$' such that

(10).....$V(F_1) = \neg a \cup \neg\neg a \neq 1_A$. The following five elements of A:

$1_A$, $d = \neg a \vee \neg\neg a$, $\neg a$, $\neg\neg a$, $c_o$ are distinct as the subsequent considerations demonstrate:

I. $1_A \neq d$, for if $1_A = d$, this contradicts (10).

II. $1_A \neq \neg a$ and $1_A \neq \neg\neg a$, for otherwise $d = 1_A$, and this contradicts (I).

III. $1_A \neq c_o$ for otherwise $d = 1_A$ and this again contradicts (I).

IV. $d \neq \neg a$, for suppose $d = \neg a$ then $1_A = \neg\neg d = \neg\neg\neg a = \neg a$ and this contradicts II.

V. $d \neq \neg\neg a$, for if $d = \neg\neg a$, then $1_A = \neg\neg d = \neg\neg\neg\neg a = \neg\neg a$ and this contradicts II.

VI. $d \neq c_o$, for if $d = c_o$ then $1_A = \neg\neg d = \neg\neg c_o = c_o$ and this contradicts III.

VII. $\neg a \neq \neg\neg a$, for suppose $\neg a = \neg\neg a$, then $d = \neg a$ and this contradicts IV.

VIII. $\neg a \neq c_o$, for if $\neg a = c_o$ then $\neg\neg a = \neg c_o = 1_A$ and this contradicts II.

IX. $\neg\neg a \neq c_o$, for if $\neg\neg a = c_o$ then $\neg a = \neg\neg\neg a = \neg c_o = 1_A$ and this contradicts VIII.

Thus the five elements are indeed distinct. I assert that they also form a sublattice A' of A, i.e. they are closed under operations '$\vee$', '$\wedge$'. By definition $\neg a \vee \neg\neg a = d$ and by I.4.(29)(d) $\neg a \wedge \neg\neg a = c_o$. For the remaining elements $a', b' \in A'$, $a' \vee b' = \max(a', b')$ and $a' \wedge b' = \min(a', b')$ because the lattice ordering among the elements are as follows: $1 \geq d \geq \neg a \geq c_o$ and $1 \geq d \geq \neg\neg a \geq c_o$.

Finally I show that the operation '$\longrightarrow$' is closed on the elements of

A':

$d \to \neg a = (\neg a \cup \neg\neg a) \to \neg a = (\neg a \to \neg a) \cap (\neg\neg a \to \neg a)$

$= \neg\neg a \to \neg a$ by I.4.(25) (e)

$\neg\neg a \to \neg a = \neg\neg a \to (a \to c_o) = (\neg\neg a \cap a) \to c_o = a \to c_o =$

$= \neg a$ by I.4 (25) (d) & I.4.(29) (c)

$d \to \neg\neg a = (\neg a \cup \neg\neg a) \to \neg\neg a = (\neg a \to \neg\neg a) \cap (\neg\neg a \to$

$\neg\neg a) = \neg a \to \neg\neg a$

$\neg a \to \neg\neg a = \neg a \to (\neg a \to c_o) = (\neg a \cap \neg a) \to c_o = \neg\neg a$

by I.4.(25) (d)

$d \to c_o = (\neg a \cup \neg\neg a) \to c_o = (\neg a \to c_o) \cap (\neg\neg a \to c_o) = c_o$

by I.4.(24) (d)

$\neg a \to c_o = \neg\neg a$

$\neg\neg a \to c_o = \neg a$

It is easy to see by I.4.(24) (b) & (c) that the operation $\to$ is closed under the remaining elements of A'. Since F is valid in A, it is valid also in the sublattice A'. But A' is evidently isomorphic with $A_1$. So F is valid in $A_1$ and this contradicts our original supposition that F is invalid in $A_1$.

(11).....<u>COROLLARY</u>: $(ML,F) = (ML,F_1)$ iff F is invalid in $A_1$ and is derivable in $(ML,F_1)$.

(12).....<u>THEOREM 6</u>: $(ML,F) \supseteq (ML,F_1)$ iff F is invalid in $A_1$.

<u>Necessity</u> is evident from the fact that $F_1$ is invalid in $A_1$.

<u>Sufficiency</u>: $(ML,F) = (ML, \neg p \to (p \to q),F) \supseteq (ML, \neg p \to (p \to q), F_1) = (ML,F_1)$ by (9) if F is invalid in $A_1$.

(13)..... <u>COROLLARY</u>: (HL,F) = (HL,$F_1$) iff F is invalid in $A_1$ and F is derivable in (HL,$F_1$).

# CHAPTER III: APPLICATION OF THE NEW RESULTS

> "The aggregate of all
> application of logic will
> not compare with the
> treasure of the pure theory
> itself."
>
> (C.S. Peirce)

In this chapter I apply the extension-criteria to certain formulae. In the first section I consider those formulae which appear in Johansson's paper[(1)] and are said to be unprovable in ML but provable in HL. In the second section I undertake the investigation of classical tautologies not greater than degree three.

## 1. Some formulae which are provable in HL but not provable in ML.

When Johansson compares his paper with Heyting's[(2)], he remarks that the following 9 formulae fail to be theorems of ML;

(1)..... $\neg p \rightarrow (p \rightarrow q)$

(2)..... $(p \,\&\, \neg p) \rightarrow q$

(3)..... $((p \,\&\, \neg p) \vee q) \rightarrow q$

(4)..... $((p \vee q) \,\&\, \neg p) \rightarrow q$

(5)..... $(q \vee \neg q) \rightarrow (\neg \neg q \rightarrow q)$

(6)..... $(\neg p \vee q) \rightarrow (p \rightarrow q)$

(7)..... $(p \vee q) \rightarrow (\neg p \rightarrow q)$

(8)..... $(p \rightarrow (q \vee \neg r)) \rightarrow ((p \,\&\, v) \rightarrow q)$

(9)..... $\neg \neg (\neg \neg p \rightarrow p)$

With the exception of (9), each of these formulae is invalid on $M_o$ as the following valuations demonstrate:

$$\neg 1 \rightarrow (1 \rightarrow 0) = 1 \rightarrow 0 = 0 \neq 1$$

$$(1 \cap \neg 1) \rightarrow 0 = 1 \rightarrow 0 = 0 \neq 1$$

$$((1 \cap \neg 1) \cup 0) \rightarrow 0 = 1 \rightarrow 0 = 0 \neq 1$$

$$((1 \cup 0) \cap \neg 1) \rightarrow 0 = 1 \rightarrow 0 = 0 \neq 1$$

$$(0 \cup \neg 0) \rightarrow (\neg \neg 0 \rightarrow 0) = 1 \rightarrow 0 = 0 \neq 1$$

$$(\neg 1 \cup 0) \rightarrow (1 \rightarrow 0) = 1 \rightarrow 0 = 0 \neq 1$$

$$(1 \cup 0) \rightarrow (\neg 1 \rightarrow 0) = 1 \rightarrow 0 = 0 \neq 1$$

$$(1 \rightarrow (0 \cup \neg 1)) \rightarrow ((1 \cap 1) \rightarrow 0) = 1 \rightarrow 0 = 0 \neq 1$$

Applying the Corollary of Theorem 3 of Chapter II i.e. II.1.(12) we can state that each of these 8 formulae extend the minimal calculi to Heyting's calculus. In other words for i = 1,...,8

(10)..... $(ML, (i)) = HL$

This means that in Heyting's calculus the axiom (1) can be replaced by any of (2), (3), (4), (6), (6), (7) and (8). The fact that (2) can replace (1) in HL is not surprising for already in ML

(11)..... $p \rightarrow (q \rightarrow r) \leftrightarrow (p \,\&\, q) \rightarrow r$ holds,[3] and, thus in particular

(12)..... $\neg p \rightarrow (p \rightarrow q) \leftrightarrow (p \,\&\, \neg p) \rightarrow q$

But it is somewhat interesting that (4) can replace (2). This is because intuitively (2) gives rise to problems which have been named "paradoxes of material implication", but (4) which is sometimes called "scheme of disjunctive syllogism" seems to be intuitively more innocuous. The fact that (4) can replace either (2) or (1) in HL without altering the strength of the axiom system shows that in spite of our "intuitions" (4) is equivalent to (2), or, alternatively to (1). Formula (5) is interesting in that it is the only formula among the eight which is a formula of one variable. S. Kanger has given a formula of one variable[4] which is simpler than (5) and still extends ML to HL:

(13)..... $\neg\neg p \rightarrow (\neg p \rightarrow p)$

which can be more conveniently written as $(\neg p \;\&\; \neg\neg p) \rightarrow p$. It is easy to see that this formula is less than degree 6, and is invalid on $M_o$ as the following valuation shows

(14)..... $(\neg 0 \cap \neg\neg 0) \rightarrow 0 = (1 \cap 1) \rightarrow 0 = 0 \neq 1$

Formula (6) is provable in HL but its converse

(15)..... $(p \rightarrow q) \rightarrow (\neg p \vee q)$

is not provable in HL. However (15) is a classical tautology and is invalid in $I_1$:

$$(a \rightarrow a) \rightarrow (\neg a \cup a) = 1 \rightarrow (0 \cup a) = 1 \rightarrow a = a \neq 1$$

Hence by applying the Corollary of Theorem 4 in Chapter II, i.e. II.1 (17) we get the result that (6) and (15) jointly extends the minimal calculus to the classical calculus.

(16)..... (ML, (6), (15)) = CL

One may ask the question whether there is any other formula among the 8 formulae (i), i = 1,...,8 which has the same property, i.e. which together with its converse extends ML exactly to CL. The answer to this question is affirmative. The only other formula which has that property is (7).

(17)..... $(\neg p \rightarrow q) \rightarrow (p \vee q)$ is a classical tautology, and is invalid in $I_1$ as the following valuation demonstrates.

$$(\neg a \rightarrow a) \rightarrow (a \cup a) = (0 \rightarrow a) \rightarrow a = 1 \rightarrow a = a \neq 1$$

Thus by II.1.(17)

(18)..... (ML, (7), (17) ) = CL

Finally we remark that formula (9) is valid in $I_1$ as the following value-table demonstrates

| $p$ | $\neg p$ | $\neg\neg p$ | $\neg\neg p \rightarrow p$ | $\neg(\neg\neg p \rightarrow p)$ | $\neg\neg(\neg\neg p \rightarrow p)$ |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | 0 | 1 |
| a | 0 | 1 | a | 0 | 1 |
| 0 | 1 | 0 | 1 | 0 | 1 |

Thus by II.1. (1)

(ML, (9)) $\not\supseteq$ Cal[7]. Yet since (9) is provable in HL and is valid in $M_o$, (19)..... (ML, (9)) $\subset$ HL by II.1.(11). Thus the set of theorems of (ML, (9)) is a proper part of the intuitionistic theorems.

With this remark we end the comments on the formulae (i) i = 1,...,9.

# PART II

The object of the second part of the dissertation is to comment on the results of the previous chapters and to discuss certain philosophical problems about negation. First, I would like to raise the question 'what is a negation-sign?'. In answering this question in Chapter IV, I adopt a basically classical standpoint even when I consider Heyting's intuitionistic logic and the various intermediate logics between the minimal logic and the classical logic. Then as an alternative view I discuss the intuitionistic account of logical connectives and the philosophical basis of intuitionistic logic. Finally I make some comments on the results of the previous chapters and explore the feasibility of ordering the different negation-signs occurring in the various investigated intermediate logics.

A word must be said, however, why this apparently reverse order has been chosen to attain the object of Part II. In the first draft of Part II, I began with the comments on the results of Part I, and the attempt to order the negation-signs. But I found that the whole undertaking was dependent on, and bound up with, definite standpoints on basic philosophical issues such as what is a negation-sign, what is a connective, what are the philosophical motivations and intentions in constructing different logical systems, etc. So I came to realize that at least some brief discussion of some of these issues were necessary before the rest. At the same time the reverse order enables me to delimit the questions in which I am mainly interested.

# CHAPTER IV: WHAT IS A NEGATION-SIGN?

"It is just as important to neglect distinctions that do not touch the heart of the matter as to make distinctions which concern what is essential. But what is essential depends on one's purpose."

(G. Frege)

In order to answer the question 'what is a negation-sign?', I intend to deal briefly with the following questions: 1. What is a definition for negation? 2. What is a connective? 3. Can inference rules or axioms define a connective? 4. Do rules which give the truth-values of sentences containing the connective determine the meaning of the connective? 5. How does negation fare in the light of the discussion above?

## 1. What is a definition for negation?

In discussing the usual definition of negation found in many logic book, G.E.M. Anscombe expresses a certain uneasiness: 'I can define something as the so-and-so only if I am justified in being sure, first that there is a so-and-so, and second that there is only one. If I have no such assurance, it is not certain that I am succeeding in defining anything.'(1) Miss Anscombe's uneasiness is certainly well-founded if we intend to define a thing. Her requirement about the definition would then ensure that the definition must pin you down to exactly one object. But if we are giving the definition of an abstract term such as 'liberty' or 'culture' would it be sound to seek an assurance that there is at least one thing and at most one thing which is 'liberty' or 'culture'? Or, if we are defining an English adjective such as 'large' or 'black', does the question 'there is one and only one so-and-so' really arise? Obviously not. It is also evident that in defining logical words like 'and', 'or', 'if... then', and in particular 'not', it would be incorrect to seek the same referential definiteness that we rightly seek in defining a thing.(2)

It should be also noted that a definition is to a certain extent relative to the degree of accuracy we seek. Thus it counts as a definition (at least in certain standard dictionaries) if the definition pins down the meaning to within reasonable limits; e.g. one could give the same correct definition for 'fright' and 'terror', though a more precise definition would distinguish between them.(3) The same applies when we are considering a family of related notions, and our concern is less with one parricular notion than with what is common to them all.

When I introduced the different logics in Chapter I, I simply said that the formal languages in which our logics were formulated, contained in their vocabularies the connective $\neg$, and suggested the reading of it as 'not'. According to this if P is a metavariable for a proposition, then $\neg P$ should be read as not-P. We may say that by definition

(1)..... $\neg P$ is the <u>negation</u> of P.

The question immediately arises whether the suggestion that ¬ should be read as 'not' is philosophically significant. The answer to this question is affirmative. The sign ¬ is introduced with the philosophical motivation that ¬ may serve as a rational reconstruction of negation of (parts of) English <u>for certain purposes</u>. The qualification 'for certain purposes' indicates the admission that ¬ cannot capture the flexibility and functional multiplicity of negation in English.(4) But within the limits of the qualification the conviction is expressed that in simple languages such as in giving information in certain automatic systems, the negation-sign ¬ may adequately represent the word 'not'. This in turn may throw light how negation functions in a natural language. The idea that an artificial, formal language can help to understand a natural language is in a strong philosophical tradition which goes back, even in modern times, to Frege or Boole.(5)

In spite of the above remark about the connection between ¬ and 'not', (1) remains notably unenlightening for at least three reasons. First, the negation-sign ¬ must be understood relative to the formal language L in which the propositional logic is formulated. This raises the questions, what is a connective of L?, is its occurrence arbitrary?, what the symbols and formulae of a formal language are used for? (These questions will be discussed in Section 2) Secondly, the negation-sign must be understood relative to the <u>logical system</u>(6) in which it occurs. This is why certain authors use different signs for negation in CL, in HL and in ML though the formal language L to which these different logical systems pertain is the same.(7) This makes us ask what special requirements are imposed on the negation-sign by the different logical systems and which way; through special axioms and inference-rules? are these axioms or rules sufficient to fix the 'meaning' of a connective like negation? (I deal with these questions in Section 3) Thirdly, (1) does not make clear, supposing a truth-condition theory of meaning how $\neg P$ makes sense if P is replaced by a concrete proposition, or more precisely what is the role of $\neg$ in making sense of $\neg P$, and whether this role could determine the meaning of the connective. (This problem is treated in Section 4)

But before we deal with these questions in detail, one may ask why an explicit definition of negation would not do. In some books of logic the negation-sign is introduced by the following explicit definition

(2)..... $P = P \longrightarrow f$

Here the meaning of the *definiendum* (the combination of the symbols to the left of the definitional equation) is given by the *definiens* (the combination of the symbols to the right of the definitional equation). But this definition is informative only if the meaning of the combination of the signs in the *definiens* is already known. In (2) the *definiens* is $P \longrightarrow f$ where 'f' is usually interpreted as a fixed false (or absurd) proposition, and thus negation is reduced to a special case of material implication.

Johansson has shown[8] that in his minimal calculus if we introduce negation by (2), then we could interpret 'f' as $\neg p \,\&\, \neg\neg p$ for in that logic these two expressions imply each other. That $(\neg p \,\&\, \neg\neg p) \longrightarrow f$ is derivable in ML is shown by substituting $\neg p$ for p, and f for q in $(p \,\&\, (p \longrightarrow q)) \longrightarrow q$; and the reverse implication $f \longrightarrow (\neg p \,\&\, \neg\neg p)$ is proved by substituting f for q in $q \longrightarrow (p \longrightarrow q)$, and f for q and $\neg p$ for p again in $q \longrightarrow (p \longrightarrow q)$ and combining the two results.

Yet it would be a mistake to think that in (2) we may replace f by $\neg p \& \neg\neg p$ for if we did that, (2) would become a viciously circular definition. (2) avoids being circular by having f as a primitive sign in it.

It is also worthy of notice that f is in a different category from a propositional variable, hence the addition of f to the vocabulary of a language L which contains propositional variables and the connectives &, v, $\longrightarrow$ only, is a definite extension of that language.(9) It was already mentioned in Chapter I that if we add f to the vocabulary of the positive logic then with definition (2) we get a system in which all and only the theorems of the minimal logic are derivable.(10) This can be shown by substituting f for R in the following thesis of the positive logic

(3)..... $(P \longrightarrow (Q \longrightarrow R)) \longrightarrow (Q \longrightarrow (P \longrightarrow R))$; then by using (2) we get

(4)..... $(P \longrightarrow \neg Q) \longrightarrow (Q \longrightarrow \neg P)$ which is interdeducible with that extra axiom of the minimal calculus which contains the negation-sign.(11) In this example (2) is a creative definition, i.e. not simply an abbreviation (typographical convenience) but it adds to the theoretical strength of the system.(12) Lukasiewicz called such a creative definition a hidden axiom, and held the view that one should avoid the use of creative definitions whenever possible:

> "In deductive systems the role of definitions seems to consist in allowing us to replace longer and more complicated expressions by shorter and simpler ones. Moreover, some

definitions can bring with them new, intuitively valuable insights. Under no circumstances, however, do definitions seem to be intended to give new properties to the undefined primitive concepts of the system. Primitive concepts should be characterized solely by axioms. If one takes this position, one should avoid the use of creative definitions whenever possible."(13)

Whether we accept Lukasiewicz's position or not, an explicit definition for the negation like (2) would not do, because it simply shifts the burden of the problem to another one, namely how do we define another connective ( $\longrightarrow$ ), or generally, how do we define the symbols occurring in the definiens.

2. What is a connective?

English language contains a number of words (phrases) by which we can make new sentences from others. A few examples are

(1)..... '...and...', '...or...', 'if...then...', 'neither...nor...', '...but...', '...if and only if...',

(2)..... 'it is not the case that...', 'it is not true that...',

(3)..... 'it is necessary that...', 'it is obligatory that...', 'it is known that...', 'it is believed that...'.

If we put sentences in place of the markers, new, more complex sentences are formed. These words (phrases) are called sentential connectives[14] (hereafter simply connectives). Logicians were always interested in these words (phrases) because the logical form of complex sentences to an important extent depends on them. Medieval logicians called such words syncategoremata in contrast to categorematic words.[15]

Connectives are classified under different aspects. A connective is called one-place (unary), two-place (binary), three-place connective according to, it needs one, two, three or three sentences to form a new sentence. The connectives in (1) are binary, in (2) and (3) are unary. A more important classification subdivides the connectives into truth-functional connectives and non-truth-functional connectives. A connective is truth-functional if the truth-values of the compound sentence formed by the connective depends only on the truth-value of the constituent sentences. Some uses of the connectives in (1) and (2) are truth-functional, in (3) are all non-truth-functional.[16] In the sequel we are only concerned with some uses of the connectives in (1) and (2).

It is important to remember that the definition of a connective must be relative to the language, in the sense that what has to be defined is 'connective of language L'. If L is a natural language,[17] then its connectives are words with meanings and some grammatical rules attached. If L is a formal language then it is to some extent arbitrary what you count as a connective, but in any case the connectives are specified by their shape and grammatical properties, without any meaning. In other words the signs of the formal language L receive meaning only if they are used for some purpose. In this respect L is like the language of an abstract arithmetical structure which may be interpreted in all sorts of different ways, and only in the interpretations do the symbols become meaningful.[18]

It should be noted that neither in mathematics nor in logic the use of a formal language implies formalism. The latter is a philosophical view according to which the signs of a formal language L are meaningless marks on paper devoid of any intended interpretation. J. Thomae describes this standpoint in regard to arithmetics with the following words: "For the formalist, arithmetic is a game with signs, which are called empty. That means they have no other content (in the calculating game) than they are assigned by their behaviour with respect to certain rules of combination (rules of the game)... The formal standpoint rids us of all metaphysical difficulties; this is the advantage it affords us."[19] Frege argued against this formalistic standpoint with considerable power.[20] In the discussion below when we talk about formal languages of logics, we, like Frege, do not intend to do this from a formalist standpoint.

What are signs of our formal language and what are they used for according to the intended interpretation? In our investigation L is a propositional language system. It may be defined as follows.

(4).... L is a triple $\langle V, C, S \rangle$ in which

a.) V is a set, at most denumerable, whose members are p, q, r,... (propositional variables)

b.) C is a set (disjunct from V) of 6 distinct elements $\neg$ (negation), & (conjunction), v (disjunction), $\longrightarrow$ (implication), (,) (brackets) as auxiliary signs to indicate the scope of the elements of C, which we call connectives.

c.) S (set of propositional formulae) is the smallest set including V and such that if P, Q are in S, so are $\neg P$, $P \& Q$, $P \vee Q$, $P \longrightarrow Q$. P, Q, R, ... etc. do not belong to the object language L; they are used to talk about propositional formulae of L.

According to the intended interpretation p,q,r... represent elementary (or atomic) propositions; the signs $\neg$, &, v, $\longrightarrow$ represent the connectives 'it is not the case...', '...and...', '...or..', 'if...then...' (or rather some special but basic uses of these connectives); P, Q, R are used to talk about propositions whether elementary or not. An elementary proposition may be briefly described as a meaningful string of words (or what they signify)[21] which is either true or false, and which does not contain any of the four connectives mentioned above. This description excludes not only such nonsensical expressions as 'quadruplicity drinks procrastination'[22] but also questions, commands, requests and some performatives which are apparently in the form of a declarative sentence.[23]

Propositional formulae are used for making true or false statements. But in the case of compound statements propositional formulae represent statements in such a way that the <u>logical form</u>[24] of the compound statement is made perspicuous. Let me illustrate this with an example. The propositional formula

(5)..... $((p \longrightarrow q) \& \neg q) \longrightarrow \neg p$

is built up from the signs of, p, q, $\neg$, &, and $\longrightarrow$ according to the syntactical rules (4) c.). Any use of these signs which does not agree with the syntactical rules of L is incorrect.[25] But the uses of these signs even if they are in complete agreement with the syntactical rules of L do not answer the question for what these signs are used according to the intended interpretation. The intention is that (5) represents all propositions which have the same logical form as (5). We may get an instance of these propositions if we put two elementary propositions, say *p and *q in place of p and q in (5). Then the propositional formula (5) becomes a proposition, say *P, with a certain meaning and a definite truth-value. The meaning of the compound proposition *P obviously depends on the meanings of *p, *q, on the connectives and on the manner *P is built up from all these. A theory of meaning has to specify this dependence.[26] According to such one theory we are provided with the meaning of *P if we know the necessary and sufficient conditions for the truth of *P. So it is hoped that the contribution which a connective makes in determining the truth-values of *P in all possible situations will clarify what is a connective. I shall return to this point and discuss it more fully in Section 4.

## 3. Can axioms and/or inference-rules define a particular connective?

In the previous section we made it clear that we use a formal language L according to an intended interpretation and the correct use of the signs of L must take into account this intention. But the major job of a logical system[27] (which pertains to L) is to characterize a deducibility relation $\vdash$, defined between sets of formulae and formulae of L. This is usually done by axioms and/or inference rules. The axioms specify what are the formulae which we accept as theorems, without proof, and the inference rules specify what transformations we may effect between formulae in the system.

If a formula P can be produced by an application of a finite sequence of these transformations to a set of formulae X, we say that P is deducible from X in the system, and denote it by $X \vdash_{Sys} P$. If P is such a formula into which any set of sentences (or the null set) can be transformed, we say that P is provable in the system. In symbols $\vdash_{Sys} P$, or alternatively $\emptyset \vdash_{Sys} P$.

(1).....A propositional logic is a logical system where the formal language to which the system pertains is a propositional language system(28) and on the relation $\vdash$ the following restrictions are placed:(29)

a.) If P is a member of X then $X \vdash P$.

b.) If $X \vdash P$ and $X \subseteq X'$ then $X' \vdash P$.

c.) If $X \vdash P$ and $X' \cup \{P\} \vdash Q$ then $X \cup X' \vdash Q$.

d.) If $X \vdash P$ and * is any substitution then $X^* \vdash P^*$. Here substitution is any mapping of the set of formulae into itself which satisfies $(P \& Q)^* = P^* \& Q^*$, $(P \vee Q)^* = P^* \vee Q^*$, $(P \longrightarrow Q)^* = P^* \longrightarrow Q^*$ and $(\neg P)^* = \neg P^*$; $X^*$ is $Q^*: Q \in X$ .

The restrictions on $\vdash$ are given implicitly in the definition of a propositional logic. But since we have fixed the formal language L by 2.(4) in every propositional logic investigated, and since the axioms can always be replaced by suitable inference-rules,(30) the difference in regard to $\vdash$ in the various logics (within the investigated family) can be pinned down in the difference of inference-rules which characterize the various logics. If, as it has been suggested, these inference rules can define a particular connective, then it is expected that the difference in meaning of a particular connective can also be pinned down by the difference in those rules which are characteristic for the various logics. Let me illustrate this with an example. In all of the propositional logics introduced in Chapter I, the following two inference rules hold

(2)..... If $X \vdash P, X \vdash Q$ then $X \vdash P \& Q$

(3)..... If $X \vdash P \& Q$ then $X \vdash P, X \vdash Q$

If we say that (2) and (3) define the connective & then we have to say also that in all our logics the meaning of & is exactly the same. On the other hand in the minimal logic ML only the first of the following inference rules hold, in contrast to the classical logic CL where both hold

(4)..... If $X, P \vdash Q$ ; $X, P \vdash \neg Q$ then $X \vdash \neg P$[31]

(5)..... If $X \vdash \neg\neg P$ then $X \vdash P$

If we say again that (4) defines $\neg$ in ML, and (4) and (5) defines $\neg$ in CL, then the difference in meaning between $\neg$ in ML and $\neg$ in CL can be pinned down to rule (5).

The idea of giving the full meaning of the logical connectives solely by inference rules have been attacked by A.H.Prior with subtle irony in his article 'The Runabout Inference Ticket'.[32] In it Prior shows that if we allow the introduction of connectives through inference rules without further assumptions, then some very unpleasant consequences follow: any statement may be inferred in an analytically valid way from any other. He does this by introducing a binary connective tonk with the rules

(6)..... from P infer P-tonk-Q;

(7)..... from P-tonk-Q infer Q.

He then uses tonk to derive 2+2=5 from 2+2=4. It is done in two steps:

(8)..... 2 + 2 = 4 by assumption;

(9)..... 2 + 2 = 4 - tonk - 2 + 2 = 5 by rule (6),

(10).....2 + 2 = 5 by rule (7).

If one asks what is the meaning of tonk, we can simply point to rules (6) and (7), and say that its meaning is completely given by them. There is nothing more to knowing the meaning of tonk than being able to perform these inferences, just as there is nothing more to knowing the meaning of the connective & than to be able to perform the inferences expressed in (2) and (3).

What are the philosophical implications of Prior's reductio ad absurdum? This is not an easy question to answer for Prior's paper is very concise and it is written in an ironical vein. But it is the most important question concerning Prior's paper. Unfortunately neither Stevenson in his paper 'Roundabout the Runabout Inference-Ticket'[(33)], nor Belnap in his "Tonk, Plonk and Plink'[(34)] address himself to this question.

Stevenson investigates what is wrong with the theory of analytic validity as Prior states it and how it should be correctly stated. He says 'The important difference between the theory of analytic validity as it should be stated and as Prior stated it lies in the fact that he gives the meanings of connectives in terms of permissive rules, whereas they should be stated in terms of truth-function statements in a meta-language.[35] Clearly Stevenson does not ask what Prior intended to show by his <u>reductio</u>, but presents a different theory.

Belnap, on the other hand, takes the position that it is quite legitimate to define the meanings of connectives in terms of the role they play in deduction if we have definite assumptions about the deducibility relation. These assumptions put certain constraints on the permissibility of the rules, and in turn, on the introduction of new connectives. The assumptions about the deducibility relation he characterized after Gentzen as follows:

(11)..... <u>Axiom</u> $A \vdash A$

(12)..... <u>Weakening</u> from $A_1, A_2, \ldots, A_n \vdash C$ to infer $A_1, A_2, \ldots, A_n B \vdash C$.

(13)..... <u>Permutation</u> from $A_1, \ldots, A_i, A_{i+1}, \ldots, A_n \vdash B$ to infer $A_1, \ldots, A_{i+1}, A_i, \ldots, A_n \vdash B$.

(14)..... <u>Contraction</u> from $A_1, \ldots, A_n, A_n \vdash B$ to infer $A_1, \ldots, A_n \vdash B$.

15)..... Transitivity from $A_1,\ldots,A_n \vdash B$, and $C_1,\ldots,C_n,B \vdash D$ to infer $A_1,\ldots,A_n,C_1,C_2,\ldots,C_n \vdash D$.

(16)..... Extension rules If a new connective, say plonk is introduced, then

a.) the notion of sentence is extended by introducing A-plonk-B as a sentence whenever A and B are sentences.

b.) we add some axioms of rules governing A-plonk-B as occurring either as one of the premises or as conclusion of a deducibility statement.

c.) the extension must be conservative, i.e. although the extension may have new deducibility statements, these new statements will all include plonk. (This condition states the demand for consistency.)

Belnap adds that 'the justification for unpacking the demand for consistency in terms of conservativeness is precisely our antecedent assumption that we already have all the universally valid deducibility statements not involving the new connectives.'(36) Prior's tonk is not conservative in the sense of (16)c.) since $A \vdash B$ holds for arbitrary A and B in the tonkitish system though it does not include tonk. Thus the introduction of tonk is inadmissible because it is inconsistent with antecedent assumptions of deducibility. Belnap points out that if we had allowed $A \vdash B$ initially, then there would have been no objection to tonk since then the extension would have been conservative. Also, there would have been no inconsistency had we omitted from our characterization the rule a Transitivity. Belnap concludes that 'one can define connectives in terms of deducibility but one bears the onus of proving at least consistency (existence) and if one wishes further to talk about the connective (instead of a connective) satisfying certain conditions, it is necessary to prove uniqueness as well. But it is not necessary to have an antecedent idea of the independent meaning of the connective.(37)

Some critical comments are in place here. Belnap is mistaken when he wants to unpack the demand for consistency in terms of conservativeness according to (16)c.). For this rule does not state the demand for consistency because non-conservative, consistent extensions of partial calculi do exist. It is rather a completeness claim which is not satisfied in the following extension of positive logic PL:(38)

(17)..... $(PL, F_1, F_2)$ in which $F_1$ and $F_2$ are the following axioms

(18)..... $F_1 = ((p \longrightarrow q) \& (p \longrightarrow \neg q)) \longrightarrow \neg p$

(19)..... $F_2 = (\neg\neg p \longrightarrow p)$

In extension (17) the newly introduced connective is the negation-sign $\neg$ . The new axioms about it are $F_1$ and $F_2$. It is well-known that this new extension is the classical logic CL. Nevertheless Belnap's rule of (16)c.) is not satisfied, for Peirce's law $((p \longrightarrow q) \longrightarrow p) \longrightarrow p$ is not provable in the unextended system PL, although it does not contain the negation-sign.

The extension (17), or rather its analogue can also be given purely via inference-rules without axioms, according to Gentzen's method of natural deduction.(39) In this method we draw inferences not only from theorems but also from hypotheses (assumptions) which may be introduced at any point of a proof-procedure. The rules of natural inference are usually divided into introduction and elimination rules. Hypotheses are discharged by means of the deduction rule. If we denote by $P \vdash Q$ that Q has been inferred from hypothesis P, and by $\vdash Q$ that Q has been proved (inferred from the empty set of hypotheses), then we can describe PL with the following rules[40]

(20)..... $\frac{P, Q}{P \& Q}$ , $\frac{P}{P \vee Q}$ , $\frac{Q}{P \vee Q}$ , $\frac{P \vdash Q}{P \rightarrow Q}$ as introduction rules

where the formulae below the line are inferred from the formulae or inferences above the line. The elimination rules are

(21)..... $\frac{P \& Q}{P}$ , $\frac{P \& Q}{Q}$ , $\frac{P, P \rightarrow Q}{Q}$ , $\frac{P \vee Q, P \rightarrow R, Q \rightarrow R}{R}$

If we add to (20) and (21) two more rules corresponding to (18) and (19)

(22)..... $\frac{P \vdash Q, P \vdash \neg Q}{\neg P}$

(23)..... $\frac{\neg \neg P}{P}$

then we get a system in which exactly the theorems of the classical logic CL are provable.[41] Hence Peirce's law is provable in it although it does not contain the negation-sign. This shows that the introduction of the connective $\neg$ is not conservative in Belnap's sense. Thus Belnap's demand for conservativeness is not really a demand for consistency.

But suppose Belnap had correctly unpacked the demand for consistency in the sense that it should contradict neither the structural assumptions from (11) to (15), nor the extension rules (16) a.) and b.), nor any theorem which is provable in the unextended system, even then, when all these demands are met, the philosopher would be unsatisfied. He would like to know what is the point of introducing a connective in such a way? Of what is existence proved by meeting the demands of consistency?[42] How and why does a proof of consistency establish existence? What are the grounds on which we lay down certain conditions for the deducibility relation? Why would an alternative set of assumptions not do? It seems that we cannot answer these questions unless we have some previous notion or understanding of the particular connective which we intend to define.[43] This is precisely the philosophical implication of Prior's reduction.

In conclusion we may say that the answer to our original question whether axioms or inference rules can define a particular connective must be negative. From this, however, we should not jump to the conclusion that every effort to characterize a particular connective by inference-rules is futile. I shall return to this point and will discuss it in the next chapter.

## 4. Do rules which give the truth-values of sentences containing the connective determine the meaning of the connective?

In logic books the connectives are usually introduced by definitions which give the truth-values of sentences in terms of the truth-values of their constituent sentences. Some authors add that these definitions give the full meanings of the connectives. Thus, for instance, Quine says

(1).....'The meanings of negation and conjunction are summed up in these laws: The negation of a true statement is false; the negation of a false statement is true; a conjunction of statements all of which are true is true; and a conjunction of statements not all of which are true is false.'(44)

(1) is problematical not so much because we may doubt whether it gives a correct account of the meanings of certain connectives of a natural language(45) but because it fails to explain the relation between the meaning of a word (connective) occurring in a statement and the statement's truth-value. For this it is necessary to supply a theory of meaning which makes this relationship clear; and this, in turn, raises questions of such fundamental character as when is a sentence true, and why is it that to each sentence we may ascribe one of exactly two possible truth-values, true or false.

Certainly (1), as it stands, is inadequate and thus the claim that it gives the full meaning of the connectives in question (negation, conjunction) is incorrect. But someone might say that there is a well-known theory of meaning, already outlined by Frege, which takes care of the indicated failure of (1). According to this theory we know the meaning of a statement if we know the conditions which must obtain for it to be true. And to know the meaning of a word in a statement is to know the contribution it makes to determining the truth-conditions of any statement in which it occurs.(46)

It is outside the scope of this discussion to explain the truth-condition theory of meaning in detail, and the controversies surrounding it. But it must be mentioned that the underlying assumption of the theory is that every sentence divides all possible state of affairs (possible situations, possible circumstances) into two disjoint and exhaustive[(47)] classes: into those in which the sentence could be used to make a correct assertion, and into those in which it could not. If a state of affairs of the first kind obtains then the resulting statement is true, and if the second then the statement is false.[(48)]

The question arises whether according to the truth-condition theory, the meaning of a statement is determined simply by what truth-values it takes in each possible situation. The answer is, of course, no. This can be easily seen by comparing the following two statements

(2)..... $5 + 5 = 2 + 8$

(3)..... A bachelor is an unmarried man.

Both (2) and (3) are such statements that they are true in each possible situation (they are logically true) but few would suggest that the meaning of (2) is the same as that of (3). Certainly Frege would not. What the truth condition theory of meaning does is to provide a set of rules (semantic rules) relating certain parts of the statement to the truth-value of the whole statement. By working out the truth-value of the statement from its parts according to the rules, the truth-conditions of the statement are made clear and its meaning is grasped. In this process the logical form of the statement plays an important role because it indicates the structure of the statement, which is relevant for finding the truth-conditions. Let me illustrate this with a simple example. Suppose in a geography class a teacher utters the sentence

(4)..... London is the capital of England and Paris is the capital of France.

The meaning of (4) obviously depends on the two constituent sentences 'London is the capital of England', 'Paris is the capital of France', and on the connective 'and' denoted respectively by *p, *q and 'and'. But the meaning of (4) depends on *p and *q in a different way from it does on 'and'. Within (4) *p and *q are elementary sentences; in certain situations they can be used to make a true or false statement. The conditions which have to be satisfied in order that *p and *q may be used to make true or false statements do not interest us at present because they have to do with the internal structure of the propositions *p and *q, and at present we are interested only how these propositions are combined. This is displayed by the symbolic notation of the logical form of (4):

(5)..... p & q

The symbol & is in quite a different sign-category of the formal language L from *p and *q. Unlike *p and *q (or any elementary propositions in place of p and q), the sign & is not meaningful in itself but it contributes to the meaning of (4), or rather of any proposition which we get by putting two elementary propositions in place of p and q of (5). The truth-table definition of conjunction may be understood as specifying this contribution within the truth-condition theory of meaning.

By understanding Quine's definition of (1) in this way and only in this way, i.e. within an appropriate theory of meaning we may say that (1) does indeed give one legitimate explanation of the meanings of negation and conjunction. In a similar way the truth-table definition of disjunction or implication does provide us with one legitimate account and a genuine insight into the meaning of that particular connective.

One may ask the question that whether the algebraic semantics as defined in Chapter I.4. can provide a legitimate account of the meanings of the connectives of the intermediate logics in a similar way, i.e. within an appropriate theory of meaning. One may try to do this in making a correspondence between 'true' and 'has a designated value', and between 'false' and 'has an undesignated value',[49] and the different designated values become different ways in which a sentence may be true, the different undesignated values become different ways in which a sentence may be false. To complete the semantic account of the language, one has to specify the conditions under which an elementary proposition has any of the various 'truth-values'. As Michael Dummett pointed out in a slightly different context[50] this position is cogent enough but it lacks credibility. The truth-condition theory of meaning has been devised for a classical outlook, and without a sufficient philosophical justification of the interpretation of the 'truth-values', algebraic semantics remains a purely formal exercise. This is not to deny the fact that algebraic semantics, and valuational systems as a whole are useful technical devices with considerable power, and retain their interest so long as intermediate logics retain their interest.[51]

I conclude this section by saying that the ordinary rules which give the truth-values of sentences containing the connectives do determine the meanings of the connectives (negation, conjunction, disjunction, implication) if they are understood within an appropriate theory of meaning, namely the theory of truth-conditions. But this theory presupposes a classical outlook. Furthermore, this theory, as Michael Dummett pointed out, 'faces formidable difficulties',[(52)] and a philosopher will not rest content until he has shown how it is possible to overcome them.

## 5. How does negation fare in the light of the discussion above?

Negation is not a thing; it is a logical connective relating sentences belonging to a language. Although it would be a mistake to seek a definition of negation, which would pin you down to one and only one thing, nevertheless it is quite legitimate to search for a clarification of this notion in the sense how negation is doing its job, what is its role and function. The main difficulty of elucidating the notion of negation is that negation is a family concept; one cannot give a clear content to it, unless one specifies the language in which it occurs and the logical system which pertains to it. If the language is a formal language, one also has to specify what the symbols of the formal language are used for according to the intended interpretation.

Axioms and/or inference-rules of a logical system can never give a full philosophical account of the meaning of negation for to know the meaning of a connective we need to know more than to perform certain rules. It is also necessary to know for what purpose those rules are used. This in turn involves the notions of correct and incorrect use, and when a proposition is true or false. Rules which give the truth-values of sentences containing negation will determine the meaning of negation only within an appropriate theory of meaning. The best candidate for an appropriate theory is the truth-condition theory of meaning but it has not been worked out in sufficient detail. It also presupposes a classical outlook with the underlying assumption of a bivalent language, i.e. the assumption that a proposition is either true or false in a given situation.

The Dutch philosopher G.Mannoury has drawn attention to the fact that even in a current language two forms of negation can be found which he called choice-negation (Greek οὐ) and exclusion-negation (Greek μή).(53) B.C. van Fraassen gave a precise formulation of Mannoury's distinction as follows(54)

(1)....Choice-negation: (not-A) is true (respectively,false) if and only if A is false (respectively,true);

(2).....Exclusion-negation: (not-A) is true if and only if A is not-true, and false otherwise.

The distinction between (1) and (2) is based on the possibility that 'A is false' is not identical with 'A is not true'. Hence in a bivalent language the distinction collapses. But in a non-bivalent language the distinction is important. If, for example, in Lukasiewicz's 3-valued logic the three values 1,½,0 are interpreted as 'true','neither true nor false', 'false' respectively, the negation-sign defined by Lukasiewicz is a choice-negation but not an exclusion-negation for $\neg 1 = 0$, $\neg 0 = 1$ and $\neg \frac{1}{2} = \frac{1}{2}$ , so $\neg \frac{1}{2} \neq 1$. On the other hand, in the 3-element chain-lattice $P_1$ of Fig. 3 (elements $1 > a > 0$ with interpretations 'true', 'neither true nor false', 'false' respectively) the negation-sign is defined as $\neg 1 = 0$, $\neg 0 = \neg a = 1$ is an exclusion-negation but not a choice-negation because (2) is satisfied by not (1).

It is an interesting fact that we find these two forms of negation in a natural language, which has no justification if language is bivalent.[(55)] The classical outlook is, however, well-established and hence I look at the intermediate logics between ML and CL as partial calculi of the classical logic in all philosophical seriousness, i.e. I regard them as calculi which pertain to the analysis of the structure of the classical logic. The negation-signs appearing in those calculi also pertain to the analysis of the classical negation-sign. Since within the family of the investigated intermediate logics each negation-sign is governed by the following inference-rule:

$$(3)\ldots\ldots \frac{X,P \vdash Q,\ X,P \vdash \neg Q}{X \vdash \neg P}$$

which expresses a basic partial requirement for the classical negation-sign, we legislate that whatever symbol, say $\neg$ , fulfills (3) is a negation-sign.

It must be realized, however, that some of the investigated logics, notably HL can be regarded as a formal statement about an intuitionist theory which has its own philosophical foundation. The intuitionist account of logical connectives and the philosophical basis of intuitionistic logic will be discussed in the next chapter. Here I only give an introductory note on how Heyting explains the notion of negations and on what are the major difficulties connected with it.

Heyting distinguishes the use of 'not' in mathematics from the use of 'not' in explanations which are not mathematical; and he remarks that in mathematical assertions no ambiguity can arise. A mathematical proposition p* always demands a mathematical construction with certain given properties; it can be asserted as soon as such a construction has been carried out. In this case one says that the construction proves the proposition p* and calls it a proof of p*. Heyting also denotes any construction which is intended by the proposi- with the same symbol p*. Then he gives the following definition

(4)..... ¬ p* can be asserted if and only if we possess a construction which from the supposition that a construction p* were carried out, leads to a contradiction.(56)

These words of Heyting are unclear. It could be taken to mean that to prove ¬ p* we have to have a construction for converting any proof of p* to a proof of absurdity. If this is what Heyting meant by (4), it would be difficult to see how one could prove ¬ p* at all for in fact there is no proof of p* and there could not be any. But from the example Heyting gives, namely the proof that $\sqrt{2}$ is not rational,(57) and also from other saying of his about negation,(58) it is clear that (4) means this: To prove ¬ p* we have to prove that supposing there were a proof of p*, then also there would be a proof of contradiction. So in (4) a genuine counter-factual or rather counter-possible is found, which does not imply the assertion of the antecedent, nor the consequent, only the connection between the two that is the assertion that if the antecedent were true then the consequent would also be true. One can assert 'if 3+4 = 8 then 1 = 2' without asserting either 3 + 4 = 8 or 1 = 2. The objection, however may be pressed that although we could assert 'the proof of p* leads to 1 = 2' without asserting p* or 1 = 2, if this is what constitutes the negation of p*, then the notion of negation is intuitively unclear. On this ground G.F.C.Griss objected to the use of negation in mathematics and tried to rebuild intuitionistic mathematics without negation.(59)

# CHAPTER V: INTUITIONISTIC ACCOUNT OF LOGICAL CONNECTIVES

> It must be remembered that no formal system can be proved to represent adequately any intuitionistic theory.
>
> (A. Heyting)

## 1. Heyting's understanding of the logical connectives.

In his book Intuitionism (HEYTING 1966) Heyting makes the preliminary remark that the intuitionistic logic he describes has only to do with mathematical propositions. Whether it admits any application outside mathematics does not concern him.(1) Before giving the axioms for his propositional logic, he wants to "fix, as firmly as possible, the meaning of the logical connectives" and he does this by giving necessary and sufficient conditions under which a complex expression can be asserted.(2) Then he gives the following rule

(1)..... P & Q can be asserted iff both P and Q can be asserted.

(2)..... P v Q can be asserted iff at least one of the propositions P, Q can be asserted.

(3)..... ¬P can be asserted iff we have a procedure by which from a construction for P a contradiction is derivable.

(4)..... $P \longrightarrow Q$ can be asserted iff we possess a construction R, which joined to a construction P, would automatically effect a construction, which proves Q. In other words, a proof of P, together with R would form a proof of Q.

He then proceeds to present the axiomatic system given in I.2.(1), and remarks that 'though the main differences between classical and intuitionistic logic are in the properties of negation, they do not coincide completely in their negationless formulae. The formula $(p \longrightarrow q) \vee (q \longrightarrow p)$ is a valid formula in classical logic, but it cannot be asserted in intuitionistic logic, as is clear from the definition. He adds that in the theory of negation, the principle of the excluded middle fails. $p \vee \neg p$ demands a general method to solve every problem, or, more explicitly, a general method which for any proposition P yields by specialization either a proof of P or a proof of $\neg P$. As we do not possess such a method of construction, we have no right to assert the principle.[3]

A number of comments are in place here. The way Heyting presents his system, and especially the fact that he thinks it necessary to fix the meaning of the connectives before giving the axioms, shows that here we face a system which cannot be regarded as purely formal. In a purely formal view any and every interpretation which satisfies the axioms and rules would be acceptable. In the intuitionistic view only one, which depends on the particular intuitionistic interpretation of constructivity, assertion and proof, is acceptable. No doubt some of the properties of these, say of intuitionistic assertion, are captured by the axiom system. For instance, one can prove, regardless of any interpretation that HL has the so-called disjunctive property, i.e.

(5)..... if $\vdash_{HL}$ P v Q, then either $\vdash_{HL}$ P or $\vdash_{HL}$ Q

Sometimes this fact is explained by saying that HL was constructed in such a way that it would make clear the meaning of v; but in (5) also the assertion sig $\vdash$ figures. Do we have in (5) a statement about v or $\vdash_{HL}$ , or about both? It seems clear from the writings of Heyting that the basic difference between an intuitionistic and classical understanding of an expression like P v Q goes deeper than merely a difference in meaning between an intuitionistic and classical 'v', and involves the very notion of assertion via the fundamental notion of 'construction'. According to Heyting a mathematical proposition P (and he considers only mathematical propositions) always demands a mathematical construction with certain properties; it can be asserted only if and as soon as such a construction has been carried out. We say in this case that the construction proves the proposition P and call is a proof of P. The very criterion for P to be a proposition is 'P has the form "I have effected a construction with the following properties..." '.[4]

From definition (3) of negation, it is clear that the notion of construction is involved. $\neg$ P can be asserted iff we possess a construction which proves that if there were a proof of P then there would be a proof of a contradiction. One might object that this definition is viciously circular for a contradiction has the form P & $\neg$P in which the notion of negation is already employed. Heyting himself poses this objection and answers it by saying that 'contradiction' must be taken as a primitive notion, and suggests that practically in all cases it can be brought into the form $1 \neq 2$.[5]

Similarly the definition (4) shows that in the definition of implication the intuitionistic conception of construction, and hence that of intuitionistic proof play an important part. The basic difficulty to compare the meanings of connectives in the intuitionistic logic and in the classical logic lies in the fact that a proposition P, without any connectives is already interpreted differently intuitionistically from classically.

## 2. The disadvantages and advantages of formal approach in describing the intuitionistic logic.

The great danger in looking at a formal description of intuitionistic logic is that its philosophical basis is ignored or disregarded. This happens, for instance, when in the literature classical and intuitionistic connectives are compared with respect to the 'strength'. Heyting himself calls the intuitionistic negation $\neg$ the strong mathematical negation,(6) but for K.Popper the classical negation is the strong one.(7) J. Lukasiewicz, after changing his mind, came to the conclusion that the classical connectives are weaker than the corresponding intuitonistic ones.(8) He says

> 'Between the classical functions C, K, and A and the intuitionistic functors F, T and O there exists a simple logical relation: all those classical functors are weaker than the corresponding intuitionistic ones. C is weaker than F because FFpqCpq holds in the intuitionistic system but its converse FCpqFpq is not provable in it. Similarly, the conjunctive functor K is weaker than T because the implication FTpqKpq is provable in the intuitionistic system, whereas its

> converse FKpqTpq is not provable. For the same reason A is weaker than O, as we can prove the thesis FOpqApq but not its converse FApqOpq."(9)

The trouble with Lukasiewicz's conclusion is not that there is any technical mistake in his argument. In his system FF pqCpq, FTpsCpq, FOpqApq are all provable formulae and their converses are not. It is also true that C, K, A have all the formal properties of the classical functors, and F, T, O those of the intuitionistic functors. Thus, for instance the strong principle of the excluded middle OpNp is unprovable, but the weak principle of excluded middle ApNp is provable in his system. His mistake is that he ignores the philosophical basis for his formulae. The functors F, T, O should not be called intuitionistic functors in a strict philosophical sense because the full philosophical assumptions concerning the intuitionistic conception of 'construction' and 'proof' are not met by his formal approach. For an intuitionist a hybrid formula like FFpqCpq would be inadmissible and unintelligible, just as Cpq is inadmissible and unintelligible if we take it with its classical interpretation.

In regard to the present question M. Dummett has the following comment to make:

> 'In some very vague intuitive sense one might say that the intuitionistic connective $\rightarrow$ was stronger than the classical $\rightarrow$ . This does not mean that the intuitionistic statement A $\rightarrow$ B is stronger than the classical A $\rightarrow$ B, for intuitively the antecedent of the intuitionistic conditional is also stronger. The classical

> antecedent is that A is true, irrespective of whether we can recognise it as such or not. Intuitionistically, this is unintelligible: the intuitionistic antecedent is that A is (intuitionistically) provable and this is a stronger assumption. We have to show that we could prove B on the supposition, not merely that A happens to be the case (an intuitionistically meaningless supposition) but that we have been given a proof of A. Hence intuitionistic $A \longrightarrow B$ and classical $A \longrightarrow B$ are in principle incomparable in respect of strength. We may sometimes have a classical proof of $A \longrightarrow B$ where we lack an intuitionistic one; but there is no reason why the converse should not sometimes hold too... Since $\neg$ is really a case of $\longrightarrow$, the same applies to intuitionistic negation. Classically, what we have to show to be absurd is the supposition that A should be true, irrespective of our knowledge; but intuitionistically, all that we have to show absurd is the supposition that we should have a proof of A.(10)

There is however a weak point in Dummett's argument. He says that in $A \longrightarrow B$ the classical antecedent (A is true, irrespective of whether we can recognise it as such or not) is a weaker supposition than the intuitionistic antecedent (A is intuitionistically provable). But on what ground does he assert this? Presumably on the ground that the proof of A is a way of recognising that A is true. In this case the provability of A is plainly a stronger supposition than the truth of A, because the first entails the second, and the proof of A contains an additional requirement, namely that the truth of A should be recognised in a certain way. But suppose someone

does not distinguish between 'A is true' and 'A is provable' but takes the epistemological position that when A is provable and only when A is provable can we state 'A is true', then there is no ground on which we can state that 'A is provable' is a stronger supposition than 'A is true'. Even if we take the strict intuitionist position that 'A is true irrespective of our knowledge' is a meaningless expression, it is very doubtful that we can state that this is a weaker supposition than 'A is intuitionistically provable'.
The reason for our doubt is that normally both suppositions should be meaningful before we can make any comparison of 'weaker' or 'stronger' between them.

There is also a paradoxical feature in Dummett's argument in that he says that the classical $A \longrightarrow B$ and the intuitionistic $A \longrightarrow B$ cannot be compared in respect of strength, and then he compares the classical antecedent A with the intuitionist antecedent A. But suppose A has the form $A' \longrightarrow B'$, i.e. the antecedent A is itself an implication. Then according to his argument, the intuitionist $A' \longrightarrow B'$ is a stronger supposition than the classical $A' \longrightarrow B'$, and thus, in fact, he compares an intuitionist implication with a classical one, which he may not do according to his own conclusion. This paradoxical feature of Dummett's argument may be avoided if he restricts his arguments to implications of the form $A \longrightarrow B$ where A and B are atomic formulae. But our previous considerations show that Dummett's arguments hold good only under certain epistemological views. If one takes the strict intuitionist position that 'A is true irrespective of our knowledge' is a meaningless expression, and thus the classical position is unintelligible, then this itself makes the comparison between the intuitionistic $A \longrightarrow B$ and the classical $A \longrightarrow B$ impossible in principle.

It must be also realized that no formal description of a system can represent fully an intuitionistic position because the basic intuitionistic epistemological assumptions seem to escape any formal approach. This is why Heyting warned us:

> 'It must be remembered that no formal system can be proved to represent adequately any intuitionistic theory. There always remains a residue of ambiguity in the interpretation of the signs, and it can never be proved with mathematical rigour that the system of axioms really embraces every valid method of proof.'(11)

The same point is made by E.Beth in a slightly different way:

> 'It should perhaps be emphasized once again that for an intuitionist, no formalisation can constitution a foundation for intuitionistic mathematics; it can give no more than a basically inadequate image of it. Hence the divergences existing between formalisations of classical mathematics are only of secondary importance; the main difference is between the attitudes adopted, by intuitionists and by adherents of classical mathematics in the interpretation of mathematical theories, whether formalised or not.'(12)

It would, however, be a mistake to think that the formal approach in describing the intuitionistic position is useless. It is invaluable as an exact method which could clear up certain ambiguities. This is why, when in 1930, Heyting presented his celebrated calculus which registered the principles of intuitionism, it was hailed as a great achievement and an important development within the intuitionistic school. As a consequence of Heyting's formalization certain differences between intuitionists came to light. Some complained that the axiom $p \longrightarrow (p \longrightarrow q)$ is intuitionistically not clear in his calculus, which gave the impetus in developing the minimal calculus as a reduced form of intuitionistic formalism. Even more importantly, certain features of the intuitionistic logic are captured by formal properties. For instance, the disjunction property in HL can be proved formally without any recourse to the intuitionistic interpretation of the connective 'or'. Thus the formal approach remains an indispensible tool for an intuitionist yet he knows that without the philosophical basis, his formalism is just an inadequate representation of his position.

## 3. Dummett on the philosophical basis of Intuitionistic Logic.

In a very interesting and thought-provoking paper, Professor M. Dummett discusses the question: 'What plausible rationale can there be for repudiating, within mathematical reasoning, the canons of classical logic in favour of those of intuitionistic logic?'(13)

M.Dummett poses this question not from an eclectic point of view, as if intuitionistic mathematics were an interesting and legitimate form of mathematics alongside classical mathematics. Rather he is concerned with the standpoint of the intuitionists themselves who repudiate classical reasoning as invalid on any legitimate construal of mathematical statements. Furthermore, he is concerned only with the most fundamental feature of intuitionistic mathematics, i.e. its underlying logic in which it differs from classical mathematics. In effect, he is solely concerned with the logical constants and the first order quantifiers. Thus he leaves aside some particular features of intuitionistic mathematics, such as the theory of free choice sequences, and he is not interested in the exegesis of the writings of intuitionistic mathematicians like Brouwer and Heyting.

He develops two kinds of arguments. The first is based on general semantical considerations and hence is virtually independent of any considerations relating specifically to the mathematical character of the statements. The second rests upon the special ontological status of mathematical statements, namely the thesis that mathematical statements do not relate to an objective mathematical reality existing independently of us. Since Michael Dummett comes to the conclusion that the second type of argument cannot stand up to a rigorous scrutiny, and since philosophically his first argument is far more interesting, I shall restrict my discussion to his general semantical argument. First I give a brief summary of his line of thought and then I make some critical comments.

The basic principle of Dummett's argument is that meaning is exhaustively determined by use. This Wittgensteinean principle is grounded in the thesis that the meaning of a (mathematical) statement consists solely in its role of being an instrument of communication, and hence it must be manifest and public in its use. If there were an ingredient in the meaning of a statement which is not observable (by its use), it would be irrevelant as an instrument of communication. Thus if two individuals agree completely about the use to be made of a statement, then they agree about its meaning. This does not mean that meaning and use (in general) are identical; it does not involve a holistic view of language as if we had to master the language as a whole, or rather the total use of a statement in the whole body of the language before we can know the meaning of an individual statement. The fact that we can learn a language from scratch, and that we can meaningfully criticise existing language-use suggests a molecular, as opposed to a holistic view of language: each statement must have a determinate individual content as regards its meaning. Thus there must be some central features of the use of a sentence other than its total use that constitute the meaning of a particular sentence. Nevertheless, the theory of meaning which M.Dummett advocates, requires that the central features of the use of a sentence, and the different features of the use in general, must be in harmony with each other. One principal aspect of the use of an utterance concerns the grounds on which the statement in question can be asserted, and another concerns its inferential consequences. Schematically M.Dummett speaks about the conditions for the utterance on the one hand, and all the consequences of it, on the other hand. He thinks that the harmony between these two main aspects issues in the demand that the addition of a new statement by which an assertion can be effected, to the language produces a conservative extension of the language, i.e. 'that it is not possible, by going via statements of this type as intermediaries, to deduce from premisses not of that type conclusions, also not of that type, which could not have been deduced before.'(14)

The central feature which determines the meaning of a sentence is various according to the different theories of meaning. It could be its truth-conditions, some particular method of its verification, the intention of the speaker, etc. According to the theory of meaning proposed by Dummett, the principle 'meaning is determined by use' is of a different character because, as was explained before, there are different features of the use of a sentence. What the principle does do is 'to restrict the selection of the features of sentences which is to be treated as central to the theory of meaning.'(15)

On a platonistic interpretation of a mathematical theory, the central notion is 'truth', i.e. 'a grasp of the meaning of a sentence of the language consists in a knowledge of what it is for that sentence to be true. On this view, in general, we are effectively incapable of deciding when the truth-conditions of a sentence do obtain when they obtain. Nevertheless, the grasp of the meaning of the sentence still consists in the knowledge what the condition is which has to obtain for the sentence to be true. This conception violates the principle that use exhaustively determines meaning because it is quite obscure in what the knowledge of the condition under when a sentence is true can consist, when that condition is not one which is always capable of being recognized as obtaining in actual linguistic practice. This for Dummett constitutes a reason for rejecting classical logic in favour of intuitionistic logic for mathematics.

It is, of course, impossible to do justice to Dummett's argument's argument in such a brief summary. One should read his whole paper to feel the full force of his reasoning. Nevertheless I would like to make some critical comments and indicate where his argument is weak and should be strengthened.

First, it must be said that there are several attractive traits in his argument: It takes the role of language as means of social communication very seriously. This is in sharp contrast to the writings of some intuitionists who undervalue such a role, and thus tend to solipsism. Furthermore, if the weak points in Dummett's argument can be eliminated, it contains the outline for a general theory of meaning since it is virtually independent of considerations which are special as regards mathematical statements. When this generalisation of the theory is accomplished, the central role of truth is replaced with that of a 'proof-procedure' which is not as narrow as that that has been proposed by logical positivists. In fact Dummett's argument is so broadly based that it can accommodate a platonistic ontology and is only incompatible with a platonistic meaning-theory. Finally, his insistence on a molecular view of language, in contrast to a holistic view, admits the criticism of the actual language-use and thus avoids of being to rigidly tied to actual language practice.

However, in spite of these attractive traits, there are some major weak points in his argument. The principle on which the whole body of his reasoning is based, namely that 'meaning exhaustively is determined by use' is very questionable.(16) Few would quarrel with the principle that in many situations 'meaning is elucidated by use', but to say that it is exhaustively determined by it, strikes us as an unworkable principle. For language-use, as P.F. Strawson has pointed out,(17) is a very fluid affair. We are continually putting words and sentences into new uses which are connected with, but not identical with their familiar uses. It is true that when we learn the meanings of certain expressions, we often learn them by seeing how the expressions are used in the language. But as D. Prawitz has observed,(18) this can be explained by a much weaker claim than Dummett's, namely that from their uses we only get some hints about their meanings. A sample of uses with which we are presented clarifies usually the meaning of an expression, about which we already have some notion. Without this preconceived meaning, a set of uses only enables us to form some hypotheses about the meaning. If this is the case, then one has to explain the fact that we agree rather well about the meaning. Prawitz things that this could perhaps be explained by reference to a genetic disposition to see certain kind of patterns and hence to form certain kind of hypotheses and theories upon seeing a few examples. Prawitz admits that this view would entail that we could never be sure that we knew the meaning of a sentence; a new unexpected use of it could show that we misunderstood the meaning, and would force us to revise our hypothesis. Presumably, by constant and careful revisions, our hypotheses about the meaning of a sentence will get a closer and closer fit, but it is very questionable that even the knowledge of the total use of a sentence as it exists at a certain period of time in a given community will provide us with the full meaning of the sentence. For there are cases where there seems to be no established usage, and our hypothesis may establish a (new) usage in this area where the practice seemed floating. Our hypothesis (or theory) about the meaning of a sentence may have a normative effect on its use, and Dummett does not take into account the complicated interrelation between them. In fact Dummett seems to be inconsistent when he says first that 'meaning is determined exhaustively by use' and then he asserts that what this principle does do is to restrict the selection of the feature of sentences which is to be treated as central to the theory of meaning. For such restriction cannot determine but only delimit the meanings of sentences. It will certainly exclude certain theories of meaning but it will not tell us which particular selection of features is to be treated as central within the imposed restriction. This is why there are different opinions among intuitionists themselves what rules we may use to establish the proof of a sentence.(19)

When Dummett speaks about the two main aspects of the use of a sentence, namely those which govern its conditions for asserting it, and those which govern the consequences which follow from it, it is not quite clear whether we have to know all the conditions and consequences of a sentence to know its meaning, or, alternatively only the principles or rules which govern these conditions and consequences. The first is very implausible for then we can hardly if ever know the meaning of a sentence without a holistic view of the language. If, however, the second alternative is the case, then again we do not exactly know which particular set of rules we have to select that govern those two main aspects of the use of the sentences. Here we seem to be back at the problems and impasse which we discussed in the previous chapter when we tried to characterise proof-theoretically the notion of logical connectives.[(20)]

To sum up my criticism of Dummett's argument. i. The thesis on which the whole edifice of Dummett's argument is based, namely that 'meaning is <u>exhaustively</u> determined by use" is rather questionable. ii. The way Dummett interprets this principle in regard to the two main aspects of the use of a sentence shows that use only delimits but does not determine the meaning.

It must be said that Dummett does not seem to commit himself fully to the argument he presents but says that a strong case can be made from it for repudiating the canons of classical logic. Although his argument deserves great attention, in no way can be regarded as conclusive.

# CHAPTER VI: COMMENTS ON THE RESULTS OF PART I

> "Much work in logic today has philosophical inspiration; much philosophy today uses logical tools or appeals to logical result."
>
> (B.C. van Fraassen)

In the previous chapter I discussed the intuitionistic account of the logical connectives and came to the conclusion that the reason put forward for the rejection of the canons of the classical logic in favour of the intuitionistic one is not persuasive. Thus I return now to the classical standpoint suggested at the end of Chapter IV, and make some comments on the results of Part I. According to that standpoint we regard ML and all intermediate logics between ML and CL as partial calculi in all philosophical seriousness, i.e. we regard them as calculi which pertain to the analysis of the structure of the classical logic. This enables us to look at the specific theorems of negation, and the negation-schemata in general, as indicating partial requirements for the use of the classical negation-sign.

## 1. The heuristic principle of the new results.

The extension-criteria results were based on the work of Helen Rasiowa (RASIOWA, 1974), who presented algebraic semantics for a large class of non-classical logics, including those in which I was interested. In Rasiowa's treatment, the semantics of ML and its extensions are implicative algebras (lattices) with certain complementation properties. ML is characterized by the class of non-degenerate implicative lattices of standard negation (complementation). I recall that an implicative lattice A is of standard negation if there is a distinguished fixed element $c_o$ of A, such that for any element a of A $\neg a = a \longrightarrow c_o$ where $\longrightarrow$ is the sign for the relative pseudo-complement. Thus every negation-schema which is derivable in ML is valid in each of those implicative lattices, and only negation-schemata which are derivable in ML are valid in all those implicative lattices. Let us designate this class of implicative lattices by $\left\{ A_i^{ML} \right\}$, i = 1, 2, 3, ... The extensions of ML in which I was interested were generated by adding negation-schemata as new axioms to the axioms of ML and leaving the rules of ML unchanged. In the semantics of those extensions the new axioms appeared as additional complementation properties which marked out subsets in $\left\{ A_i^{ML} \right\}$, i = 1,2,3,... and those subsets were characteristic in regard to the particular extensions of ML. So concerning the specific theorems of negations of those extensions with respect to ML, I was looking for implicative lattices of standard negation, in which the particular specific theorem of negation was invalid, but was valid in the subset of $\left\{ A_i^{ML} \right\}$ which was characteristic of the particular extension. Let me illustrate this in two simple examples.

Concerning the extension $Cal^7$ = (ML, $p \vee \neg p$) I was looking at implicative lattices of standard negation in which $p \vee \neg p$ was invalid. I found that $I_1$ (three-element chain) with $c_o = 0_{I_1}$ was such a lattice. From here it was only a short step to seek the proof of Theorem 1 of Chapter II, Section 1, in the form $(ML,F) \supseteq Cal^7$ iff F is invalid in $I_1$. The proof of the difficult, sufficiency part of this theorem depends on the completeness result of Rasiowa concerning zero-order theories of the minimal logic, i.e. the fact that if $(ML,F) \not\supseteq Cal^7$, then there is an implicative lattice A of standard negation in which F is valid but $(p \vee \neg\, p)$ is invalid. Hence F is valid in any sublattice of A, in particular in A' = $\{ 1_A, d = a \cup \neg a, c_o \}$. That A' is a three-element chain which is closed under the operation $\longrightarrow$ follows from the invalidity of $(p \vee \neg\, p)$ in A and from certain properties of the implicative lattices of standard negation. (See II.1.(1)). Hence whenever $(ML,F) \not\supseteq Cal^7$, F is valid in A'; and since A' is isomorphic with $I_1$, F is valid in $I_1$. This is exactly the sufficiency part of Theorem 1 in its contrapositive form.

Concerning HL = $(ML, \neg\, p \longrightarrow (p \longrightarrow q))$ I was looking at implicative lattices of standard negation in which $\neg\, p \longrightarrow (p \longrightarrow q)$ was invalid. I found that $M_o$ (two-element chain with $c_o = 1_{M_o}$) was such a lattice. From here it was easy to seek the proof of Theorem 3 of Chapter II, Section 1 in the form $(ML,F) \supseteq$ HL iff F is invalid in $M_o$. The proof of this theorem, however, was more difficult than the previous one. It did not go through without the help of Zorn's lemma, although I spent a lot of time in trying to prove Theorem 3 in the same simple way as Theorem 1 was proved.

The heuristic principle in regard to the other extension-criteria was similar as in the cases of Theorem 1 and Theorem 3.

## 2. Minimality conditions.

It is interesting to notice that certain minimality conditions obtain in regard to the implicative lattices occurring in the extension-criteria results. The three-element implicative lattice $I_1$, the two-element implicative lattice $M_o$, and the five-element implicative lattice $A_1$ (See Fig. 8) all contain a minimal number of elements in a certain sense. In order to formulate these minimality conditions in a precise way let me introduce the following definition:

(1)..... A non-degenerate implicative lattice A of standard negation, by definition, is <u>minimal</u> in a class of such lattices $S = \{\langle A_i, \trianglerighteq \rangle\}$ (See I.4.(15)) iff $A \in S$ and A contains a minimal number of elements.

We can now formulate the following conditions:

(2).....The implicative lattice $I_1$ in which $p \vee \neg p$ is invalid, is minimal in the class of S.

This can be justified by looking at the table of Fig. 2 and Fig. 3. On Fig. 2 we see the four different two-element implicative lattices of $I_o$, $M_o$, $S_o$, $P_o$. Out of these only two, namely $I_o$ and $M_o$ are of standard negation. But on both of these lattices $(p \vee \neg p)$ is valid. On Fig. 3 we see the twenty-seven different three-element implicative lattices. Out of these only three, namely $A_{11}$ (first two, first column), $A_{21}$ (second two, first column) and $A_{33}$ (third two, third column) are of standard negations with distinguished fixed elements $c_o = 1$, $c_o = a$, $c_o = $ ) respectively. But in both $A_{11}$ and $A_{21}$ $p \vee \neg p$ is valid and only in $A_{33}$ (i.e. $I_1$) is invalid. Evidently there are denumerably many other implicative lattices of standard negation in which $p \vee \neg p$ is invalid but they all contain more than three elements.

(3)..... The implicative lattice $M_o$ in which $\neg p \rightarrow (p \rightarrow q)$ is invalid is minimal in the class of S.

This statement can again be justified by looking at the table of Fig. 2.

(4)..... The implicative lattice $A_1$ in which $\neg p \vee \neg\neg p$ is invalid is minimal in the class of S.

Again this can be seen by examining the different non-degenerative implicative lattices of standard negation up to four elements and noticing that none is such that $\neg p \vee \neg\neg p$ is invalid in it. Similar minimality conditions are satisfied as regards the other extension-criteria results.

In Appendix I, I formulate some conjectures in which the minimality conditions appear as clues for finding further extension-criteria results. Unfortunately the existence of A* in Conjecture 2 is rather a daring supposition. As was mentioned earlier I could not even prove the existence of such a sublattice in trying to simplify Theorem 3 of Chapter II, Section 1.

## 3. The ordering of the negation-sign in the different infinite chains of logics.

In Chapter II, Section 2, it was proved that the following sequence of logics form infinite chains between CL and HL, Cal[7] and ML, HL and ML, CL and Cal[7] respectively

$$CL \supset (HL,F_1) \supset (HL,F_2) \supset \ldots \supset (HL,F_{n+1}) \supset \ldots \supset HL$$
$$Cal^7 \supset (ML,F_1) \supset (ML,F_2) \supset \ldots \supset (ML,F_{n+1}) \supset \ldots \supset ML$$
$$HL \supset (ML,F'_1) \supset (ML,F'_2) \supset \ldots \supset (ML,F_{n+1}) \supset \ldots \supset ML$$
$$CL \supset (Cal^7,F'_1) \supset (Cal^7,F'_2) \supset \ldots \supset (Cal^7,F'_{n+1}) \supset \ldots \supset Cal^7$$

where formulae $f_i$ and $F'_i$ are defined recursively as

$$F_1 = \neg p \vee \neg\neg p$$
$$F_{n+1} = ((p_{n+1} \longrightarrow F_n) \longrightarrow p_{n+1}) \longrightarrow p_{n+1}$$

$$F'_1 = \neg\neg(\neg\neg p_1 \longrightarrow p_1)$$
$$F'_{n+1} = ((p_{n+1} \longrightarrow F'_n) \longrightarrow p_{n+1}) \longrightarrow p_{n+1}$$

In each of the logics of these four infinite chains the symbol $\neg$ fulfills the requirement of IV.5.(3) and thus is a negation-sign. But, apart from ML, in each logic the symbol $\neg$ fulfills other additional proof-theoretic requirements. For instance, in CL it fulfills also

(1)..... $\dfrac{X \vdash \neg\neg P}{X \vdash P}$ where X is any set of formulae.

If we compare the semantics of ML with the semantics of CL, the additional proof-theoretic requirement corresponds to an additional complementation property. This determines a sub-class in the class of implicative lattices $\{A_i^{ML}\}$ of standard negation, namely the class of which single member is the two-element implicative lattice $I_o$. The validity of any formula in this sub-class corresponds to the derivability of the formula in CL. In general, if we are looking at the semantics of the logics which we get by extending ML in the way we described above, what the additional complementation properties are doing is nothing else than marking out different subclasses of $\{A_i^{ML}\}$ in which the validity corresponds to the derivability in the various logics. Furthermore these sub-classes are simply ordered by class-inclusion. This suggests that the negation-sign in $(ML, F_{n+1})$ indicates a semantically stronger notion, i.e. a more general concept than the negation-sign in $(ML, F_n)$. An obvious consequence of this statement is that if a negation-schema is provable in $(ML, F_{n+1})$, then the negation-schema of the same form is also provable in $(ML, F_n)$. The same can be said about the subsequent logics appearing in the other three infinite chains. Let us summarize these in the following four statements by abbreviating two subsequent logics in each of the four infinite chains with $L_n$ and $L_{n+1}$.

(2)..... Statement 1 There are denumerably many distinct logics $L_n$, n = 1,2,3,... which are strictly between CL and HL.[1] These logics diverge only in the vicinity of negation[2] and the negation-signs in them are simply ordered: the negation-sign in $L_{n+1}$ is stronger than in $L_n$.

(3)..... Statement 2 There are denumerably many distinct logics $L_n$, n = 1,2,3,... which are strictly between Cal[7] and ML. These logics diverge only in the vicinity of negation and the negation-signs in them are simply ordered: the negation-sign in $L_{n+1}$ is stronger than in $L_n$.

(4)..... Statement 3 There are denumerably many distinct logics $L_n$, n = 1,2,3,... which are strictly between HL and ML. These logics diverge only in the vicinity of negation and the negation-sign in them are simply ordered: the negation-sign is $L_{n+1}$ is stronger than in $L_n$.

(5)..... Statement 4 There are denumerably many distinct logics $L_n$, n = 1,2,3,... which are strictly between CL and $Cal^7$. These logics diverge only in the vicinity of negation and the negation-signs in them are simply ordered: the negation-sign in $L_{n+1}$ is stronger than in $L_n$.

It should be observed, however, that it would be a mistake to say that any two extensions of ML, say $L_1$ and $L_2$ are such that the negation-sign in one of them is stronger than in the other one. An obvious counter-example to such a statement is HL and $Cal^7$. One should also remember that we looked here at ML and HL, and other intuitionistic formalisms, essentially in a classical way. As has been noted before if we had taken the intuitionistic standpoint with its full philosophical commitment, the ordering of the negation-signs between the intuitionistic logic and the classical logic would have become, in principle, impossible.

## 4. Summary and conclusion.

It is rather difficult to draw definite conclusions from the discussion which is acceptable to all. The reason for this is that the outcome of several questions, such as can we order the different negation-signs occurring in the various investigated logics, are closely bound up with certain philosophical positions which are outside the realm of logic, and are neither provable nor disprovable in a strict sense. Taking an essentially classical position, I have argued that whenever two extensions of the minimal logic which appear in the four infinite chains, say $L_1$ and $L_2$ are such that $L_1 \supset L_2$, then the negation-sign in $L_2$ is stronger than in $L_1$.

# NOTES

## INTRODUCTION

(1) Theaetetus, 187. See, for instance, in Platonis Opera (ed.J.Burnet), Oxford, 1899-1906.

(2) Sophist, 237 ff. Translated by B.Jowett in The Dialogues of Plato, 2nd ed., Clarendon, Oxford, 1875, vol.IV. p.450.

(3) J.L. Austin, Truth in Proceedings of the Aristotelian Society, Supp.Vol.XXIV (1950). Reprinted in Philosophical Papers (ed. J.O. Urmson and G.J. Warnock), Clarendon, Oxford 1961, pp.96-97.

(4) (HEYTING, 1930).

(5) (JOHANSSON, 1936).

CHAPTER I (NOTES)

(1) Thus I do not distinguish between formulae and well-formed formulae.

(2) Theorems are derivable formulae in my terminology, hence they include the class of axioms if a calculus is given as an axiomatic system. Some logicians distinguish sharply between theorems and axioms and use the word thesis in the sense I use the word theorem.

(3) Since I investigate only propositional logics, I often say simply logic in place of propositional logic.

(4) These rules are often called the rule of uniform replacement and detachment respectively.

(5) A fairly comprehensive list of the different presentations of the classical and intuitionist logic and related systems can be found in (PRIOR, 1955) pp.301-313.

(6) The proof of Johansson that the negation-schema $\neg\neg(\neg\neg p \longrightarrow p)$ is underivable in his minimal logic can be somewhat simplified. (NEMESSZEGHY, E.A., 1976). See Appendix II.

(7) See, for instance, (HEYTING, 1930).

CHAPTER I (NOTES)

(8) Johansson recommended the study of this calculus as an interesting one. (JOHANSSON, 1936). p.129.

(9) In other words, if we leave out the only negation-schema as an axiom from the minimal calculus of Johansson.

(10) See, for instance, (RASIOWA,1974) p.234.

(11) See, for instance, (RASIOWA 1974) p.59 and p.254.

(12) See (SEGERBERG 1969) p.37.

(13) If $Cal_2$ = ($Cal_1$, $F_i$ i = 1,2,...) I speak about an infinite extension of $Cal_1$.

(14) In the literature the definition of intermediate logic is usually restricted to the definition given in (13).

(15) These are slight modifications of the definitions used by Trolestra (TROLESTRA 1965) p.143.

(16) See, for instance, (RESCHER 1969).

CHAPTER I (NOTES)

(17) For a new technical interpretation of three-valued logic see NEMESSZEGHY, G. & NEMESSZEGHY, E.A., On a technical interpretation of three-valued logic. See Appendix III.

(18) This statement will be discussed in Chapter III.

(19) The name "designated" seems to originate from Bernays, see (BERNAYS, 1926) p.316.

(20) See, for instance, (RASIOWA, 1963) p.58 ff.

(21) See, for instance, (RASIOWA, 1974) p.234 ff.

(22) See, for instance, (RASIOWA, 1974) p.60 ff. and (JOHANSSON, 1936) p.130.

(23) See also in (RASIOWA, 1974) p.254. It should be noted that Rasiowa calls a relatively pseudo-complemented lattice with standard negation <u>contrapositionally complemented lattice</u>.

(24) See, for instance, (TARSKI, 1938), (SKOLEM, 1958) or (TROLESTRA, 1965).

(25) We may observe that only the lattices $(I_o \times M_o)$, $(M_o \times M_o)$, $(M_o \times I_o)$, $(M_o \times I_1)$ are lattices with standard negation, distinguished element $c_o = \langle 0\ 1 \rangle$, $c_o = \langle 1\ 1 \rangle$, $c_o = \langle 1\ 0 \rangle$, $c_o = \langle 1\ 0 \rangle$, respectively.

CHAPTER II (NOTES)

(1) In the paper referred to Jankov did not give a proof.

(2) See (ANDERSON, 1969) p.259.

(3) See I.2.(4).

(4) See (ŁUKASIEWICZ, 1970) p.168. Another name for this formula is "consequentia mirabilis".

(5) See I.4.(35).

(6) The argument goes exactly as in the proof of THEOREM 1. See II.1.(1).

(7) This algebraic way of proving "sufficiency" has been suggested to me by my supervisor Dr.W.A. Hodges. The helpful comments of Prof.A.Horn are also gratefully acknowledged.

(8) See REMARK II.1.(3).

(9) See REMARK II.1.(3).

(10) I recall that by definition Calculus 1 strictly succeeds Calculus 2 iff Calculus 1 $\subset$ Calculus 2.

(11) By an ICD fragment of calculus Cal* McKay understands that part of Cal* which contains formulae only with implication, conjunction and disjunction but no negation.

CHAPTER II (NOTES)

(12) See below COROLLARIES (3), (4), (7), and (8).

(13) Here we used the fact that a lattice-homomorphism preserves the valuations on all subformulae of F. More precisely: h (V(F)) = (hV)F where h is a lattice homomorphism from A onto B, V(F) is any valuation of F in A, and (hV)F is the valuation of F in B determined by V(F) and the lattice-homomorphism h. See, for instance, (RASIOWA-SIKORSKI, 1963).

(14) See Basic step. Note also that $A_{k+1} = \Gamma A_k$, where $\Gamma$ is Jaskowski's operation.

(15) McKay proves that his intermediate logics have the same ICD fragment as HL in the following way: let A** be the ICDN algebra obtained by adding a zero-element to the ICD algebra A* of HL. The formula $F^{**} = \neg p_1 \vee \neg \neg p_1$ is valid in A** because for any valuation either $\neg p_1$ or $\neg \neg p_1$ takes the value $1_{A^{**}}$. Hence $(HL, \neg p_1 \vee \neg \neg p_1) \subseteq X (A^{**})$ where X (A**) is the logic characterised by A**. But if F is an ICD formula such that it is underivable in HL (invalid in A*) then it is underivable in X (A**) because for the same valuation for which $V (F) \neq 1_{A^*}$, $V (F) \neq 1_{A^{**}}$. Thus $(HL, \neg p_1 \vee \neg \neg p_1)$ has the same ICD fragment as HL.
Notice that McKay's proof can be applied to the result of THEOREM 1 because $(HL \neg p_1 \vee \neg \neg p_1) = (ML, \neg p \longrightarrow (p \longrightarrow q), \neg p \vee \neg \neg p) \supseteq$ (ML, F*, F**). Hence if F is an ICD formula which is underivable in (ML, F*), then F is underivable in X (A**). Thus (ML, F*, F**) has the same ICD fragment as (ML, F*).

CHAPTER II (NOTES)

(16) Here $F^{**} = \neg\neg(\neg\neg p \longrightarrow p)$, and let $A^{**}$ be the ICDN algebra obtained by adding a zero-element to the ICD agebra $A^{*}$ of $Cal^{7}$. The formula $F^{**}$ is valid in $A^{**}$ because $F^{**}$is a theorem of HL. The rest of the reasoning goes the same way as in the previous note.

CHAPTER III (NOTES)

(1) (Johansson 1936), pp. 121-126.

(2) (Johansson 1936), pp. 123-124.

(3) The sign F $\longleftrightarrow$ G is used as an abbreviation for (F $\longrightarrow$ G) & (G $\longrightarrow$ F).

(4) (K anger 1955), p. 100.

CHAPTER IV (NOTES)

(1) (Anscombe 1961) p.51. In asking this question Anscombe refers to Frege.

(2) It is not suggested that Miss Anscombe thought that negation was a thing; we suggest only that her writing could be easily misunderstood.

(3) For this example and the whole observation I am indebted to my supervisor Dr. W.A. Hodges.

(4) For a study of negation in English see E.S.Klima, Negation in English in (Fodor & Katz 1964) pp.246-323.

(5) See (Frege 1879) or (Dummett 1973) especially pp.XVII-XVIII; (Boole,1854). For a modern defence of the approach to language-study through formal languages see (Wiggins 1971).

(6) We may define a logical system after van Fraassen as follows (van Fraassen 1971) p.71:

A logical system is a triple $\langle \text{Syn}, \text{Th}, \vdash \rangle$ where

a) Syn is a syntactic system which comprises a vocabulary and a grammar. The vocabulary is usually given by a specification of its elements, and the grammar by certain rules (formation rules) which tell us how sentences are constructed from the elements of the vocabulary. A logical system is said to pertain to Syn.

b) $\vdash$ is a relation from sets of sentences of Syn to sentences of Syn (the consequence relation). This relation is usually specified by the so-called transformation rules. Note that these rules specify syntactic transformations, hence $\vdash$ should not be confused with a semantic entailment relation $\models$. Admittedly, it often happens that the philosophical motivation in constructing a logical system is that $\vdash$ should correspond to $\models$ under the intended interpretation. On this point see (Dummett 1973) p.433; on the distinction between $\vdash$ and $\models$ see (Smiley 1954); (Smiley 1976); (Beth 1955).

c) Th $= \{P : \emptyset \vdash P$ (The set of theorems)

(7) See (Rasiowa 1974).

(8) See (JOHANSSON 1936) pp.129-30.

(9) Some logicians call f a zero-order connective.

(10) I.2.(1) axiom XI.

(11) See, e.g. (HORN,1962) or (GOODSTEIN,1971) p.27.

(12) The expression 'creative definition' in this sense goes back to Lesniewski. See (RICKEY,1975) p.273. That the definition for the material implication in the system of Principia Mathematica is creative was first proved by my brother and myself.

(NEMESSZEGHY E.Z. & NEMESSZEGHY E.A.,1971); see other (NEMESSZEGHY, E.Z. & NEMESSZEGHY,E.A.1973). A critical comment on our result was published by V.F.Rickey; see (RICKEY,1975a). An answer to Rickey's criticism appeared in (NEMESSZEGHY E.A. & NEMESSZEGHY, E.Z.1977).

Although some logicians maintain that a definition should be called a proper definition only if it is non-creative, there is no compelling reason either to accept or to reject the methodological position that in a logical system only non-creative definitions are to be used. But it does seem to be reprehensible or at least inelegant to present a system in such a way that the logical status of a definition is obscured. If a definition is a disguised (hidden) axiom then it should be clearly stated that its logical status is that of a thesis. In particular, if a definition which contains the negation-sign is creative then the notion of the negation in the system is dependent on that creative definition.

(13) (LUKASIEWICZ,1970) p.277.

(14) Sentential connectives are also called sentence functors.

(15) See for instance (KNEALE,1968) pp.233-4.

(16) Not all uses of 'if...then...' are truth-functional. Even '...and...' may be used non-truth-functionally as indicating a chronological order. On the different varieties of modalities (temporal, alethic, deontic, epistemic, intentional) see (SNYDER,1971) pp.5-12.

CHAPTER IV (NOTES)

(17) Natural languages other than English have corresponding connectices of their own. See (DOHMANN,1959)

(18) On this point see the beginning of (BOOLE,1948).

(19) Quoted by Frege (FREGE,1903) ii.Pargr.88-89. On a more moderate formalism see (KNEALE,1962) pp.686-88.

(20) (FREGE,1903) ii.Pargr.86-137.

(21) I do not want to go into the controversy concerning the distinction between a statement and a proposition. On this see (LEMMON,1965).

(22) See (RUSSELL,1967) p.168.

(23) See (AUSTIN,1962).

(24) The logical form of a proposition depends on the aim of the logical analysis. In propositional logics, for instance we are uninterested in the internal logical structure of a proposition. We do not analyse a proposition in different constituent parts such as subjects, predicates and quantifiers. In a predicate calculus, however, all these concepts become important.

(25) For instance $\&P\longrightarrow$ , $vv\neg\ P\neg Q$ are meaningless expressions with respect to L. They may be compared to completely ungrammatical group of words such as 'this is no what'which is not a meaningful

string of words in English although there might be some secret language in which it is meaningful.

(26) Naturally it has to do much more. It must specify how elementary propositions are related to the 'world' (are true or false) and must give an account of synonymy and ambiguity, etc. On this point see (KEMPSON,1977)

(27) See note (6).

(28) See 2.(4)

(29) (DUMMETT,1973) p.430.

(30) On this point see (CARNAP,1959) pp.1-4 and 167-175.

(31) X,P is used as an abbreviation for $X \cup \{P\}$.

(32) (PRIOR,1960).

(33) (STEVENSON,1961) pp.124-28.

(34) (BELNAP,1962) pp.130-134.

(35) Ibid.p.127.

(36) Ibid.p.135.

(37) Ibid.p.137.

CHAPTER IV (NOTES)

(38) For the observation that Belnap's requirement of conservative extension is a completeness claim, I am indebted to Prof. B.van Fraassen. The example about the extension of PL is from me.

(39) (GENTZEN,1934)

(40) I have taken over the description of PL and CL from (GOODSTEIN, 1971) p.30. An elegant presentation of ML,HL and CL can be found in (PRAWITZ,1965) pp.20-1. See also (PRAWITZ,1971a). I am grateful to Prof.R.I.G. Hughes for drawing my attention to this article of Dag Prawitz.

(41) Mutatis mutandis my extension-criteria can be applied to inference-rules.

(42) For the formulation of this question I am grateful to Dr. W. Hart.

(43) This seems to be the position of I.Hacking. See (HACKING,1979) 'First, it is clear that these rules could not define the constants for a being that lacked all logical concepts. One must understand something like a conjunction to apply the conjunction rule...' (p.299).
See also Putnam's demonstration that the inference rules cannot fix the meaning of the connectives. (PUTNAM,1976)

(44) (QUINE,1974) p.10

(45) See beginning of Section 2.

(46) See (DUMMETT,1977) pp.67-137.

(47) Exhaustive in the sense that there are no possible relevant circumstances which are left out from the classification. On this point see (DUMMETT,1973) pp.417-419.

(48) 'To say something true is to say something correct, to say something false is to say something incorrect. Any workable account of assertion must recognize that an assertion is judged by objective standards of correctness, and that in making an assertion, a speaker lays claim, rightly or wrongly, to have satisfied those standards. It is from these primitive conceptions of the correctness or incorrectness of an assertion that the notion of truth and falsity take their origin.' (Dummett,1977) p.83. See also (HODGES,1977) pp.27-41.

(49) See (DUMMETT,1973) p.431 and (RESCHER,1969) pp.107f.

(50) (DUMMETT,1973) p.205.

(51) (DUMMETT,1973) p.432.

(52) (DUMMETT,1977) p.67.

(53) Mannoury's distinction is briefly discussed in (BETH,1968) p.631.

CHAPTER IV (NOTES)

(54) (VAN FRAASSEN,1969) p. 69

(55) One can press this further. In Greek οὐ and μή are (roughly speaking) indicative and subjunctive $\neg$. Polish distinguishes between indicative and subjunctive $\rightarrow$ viz. jeśli and gdyby. For this observation I am indebted to Dr. W.A.Hodges.

(56) (HEYTING,1966) p.102

(57) Ibid. p.102

(58) Ibid. p.18 " 'The proposition p is not true' or 'the proposition p is false' means 'if we suppose the truth of p, we are led to a contradiction'".

(59) (GRISS,1950)

CHAPTER V (NOTES)

(1) (HEYTING,1966) p.97

(2) (HEYTING,1966) p.98

(3) Ibid.pp.99-100

(4) Ibid.p.99

(5) Ibid.p.98

(6) Ibid.p.98

(7) (POPPER,1947) p.289, Footnote 20.

(8) (LUKASIEWICZ,1970) p.333

(9) Ibid.330

(10) (DUMMETT,1977) pp.16-17

(11) (HEYTING,1966) p.102

(12) (BETH,1968) pp.433-434. In view of these statements from leading intuitionists one has to say that completeness proofs from Heyting's calculus, e.g. by Kripke structures, have a rather limited value for an intuitionist. He may regard Kripke structures as a useful took, a technical device to obtain certain

results about an intuitionistic formal system, but he will not regard them in terms of which one can give an adequate account of his intuitionistic theory.

(13) (DUMMETT,1975). See also (DUMMETT,1973,1975a,1976) where similar questions are discussed.

(14) (DUMMETT,1975) p.12

(15) Ibid.p.14

(16) See, for example, (RYLE & FINDLAY,1961) reprinted in (PARKINSON,1968) pp.109-127.

(17) (STRAWSON,1974) p.230

(18) (PRAWITZ,1977) p.10

(19) Some find the principle $\neg p \longrightarrow (p \longrightarrow q)$ intuitionistically unclear and thus unacceptable.

(20) See IV.3.

CHAPTER VI (NOTES)

(1) This has been proved by C.C. McKay (McKAY,1968) and it is stated here only for the sake of completeness.

(2) I have taken over this expression from D.C.Makinson (MAKINSON, 1973) p.39. We may say that, by definition, two logics $L_1$ and $L_2$ diverge only in the vicinity of negation iff $L_1 \neq L_2$ and both have the same ICD fragment.

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NDJFL: Notre Dame Journal of Formal Logic.

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## APPENDIX I

I shall now formulate some conjectures in connection with the new results. Concerning the extension-criteria results I surmise the following:

CONJECTURE 1: (ML,F) $\supsetneq$ (ML,F*) iff F is invalid in A*, a minimal (non-degenerate) implicative lattice with standard negation, in which F* is invalid.

As was mentioned in the Discussion, all the extension-criteria results come under this conjecture, as well as Conjectures 3 and 4 below. It should also be observed that to prove the necessity part of Conjecture 1 is again easy, and the sufficiency part could also easily be proved if Conjecture 2 is correct.

CONJECTURE 2: If (ML,F) $\not\supsetneq$ (ML,F*) then there is an A* which is a minimal sublattice of any implicative lattice A with standard negation, in which F is valid and F* is invalid; moreover ' $\rightarrow$ ' is closed on the elements of A*.

Clearly, if F is valid in A then F is also valid in any sublattice of A, so it is valid in A*; and this is exactly the sufficiency part of Conjecture 1 in its contrapositive form.

CONJECTURE 3: For any $i = 1,2,3,...$

$(ML,F) \supseteq (ML,F_i)$ iff F is invalid in $A_i$ where $F_i$ and $A_i$ are defined as in II.2.(1).

Note that for $i = 1$, this has been already proved in III.3.(9).

CONJECTURE 4: For any $i = 1,2,3,...$

$(ML,F) \supseteq (ML,F_i')$ iff F is invalid in $A_i'$ where $F_i'$ and $A_i'$ are defined as in II.2.(5).

In the following conjectures I go beyond minimal logic ML and formulate some extension-criteria concerning the positive calculus PL.

CONJECTURE 5: $(PL,F) \supseteq ML$ iff F is invalid in $P_o$, $S_o$, and $P_1$ where $P_o$ and $S_o$ are defined as in Fig.2 of Chapter I, and $P_1$ is defined as in Fig.3 of Chapter I. (page 45)

Notice that if Conjecture 5 is correct then the following conjectures are also true:

CONJECTURE 6: $(PL,F) \supseteq HL$ iff F is invalid in $P_o$, $S_o$, $P_1$ and $M_o$.

CONJECTURE 7: $(PL,F) \supseteq CL$ iff F is invalid in $P_o$, $S_o$, $P_1$, $M_o$ and $I_1$.

CONJECTURE 8: $(PL,F) = CL$ iff F is a classical tautology and invalid in $P_o$, $S_o$, $P_1$, $M_o$ and $I_1$.

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# NOTE ON AN INDEPENDENCE PROOF OF JOHANSSON

E. A. NEMESSZEGHY

In [1], p. 124, I. Johansson proves that the propositional formula $\neg\neg(\neg\neg a \supset a)$ is underivable in his minimal logic. He establishes this result by the well-known matrix-method: he gives certain matrices in which all the axioms of the minimal logic are valid, the rules of the system preserve validity, but $\neg\neg(\neg\neg a \supset a)$ is invalid. The matrices he uses are $5 \times 5$ matrices, i.e., of 5 rows and 5 columns for the binary connectives. The purpose of this short note is to point out that there are simpler $3 \times 3$ matrices which do the same job. The matrices for the connectives are given below. The only designated value is 1.

| $\supset$ | 1 | 2 | 3 |
|---|---|---|---|
| *1 | 1 | 2 | 3 |
| 2 | 1 | 1 | 3 |
| 3 | 1 | 1 | 1 |

| $\wedge$ | 1 | 2 | 3 |
|---|---|---|---|
| *1 | 1 | 2 | 3 |
| 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 |

| $\vee$ | 1 | 2 | 3 |
|---|---|---|---|
| *1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 2 |
| 3 | 1 | 2 | 3 |

| $x$ | $\neg x$ |
|---|---|
| *1 | 2 |
| 2 | 1 |
| 3 | 1 |

It is easy to check that all the axioms of the minimal logic are valid in these matrices, and the rules of the system preserve validity; yet $\neg\neg(\neg\neg a \supset a)$ is invalid, for if the value of '$a$' is 3 then $\neg\neg(\neg\neg 3 \supset 3) = \neg\neg(2 \supset 3) = \neg\neg 3 = 2 \neq 1$.

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[1] Johansson, I., "Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus," *Compositio mathematica*, vol. 4 (1936), pp. 119-136.

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# APPENDIX III.

## ON A TECHNICAL INTERPRETATION OF THREE-VALUED LOGIC

Present-day computer-technology is, to a great extent, based on a technical interpretation of two-valued logic. In two-valued logic the propositional variables (say a,b,c.....) can have only two values (say "true" - "false", 1 - 0, "yes" - "no") and hence their technical interpretation demands physical elements with only two different states ("on" - "off"; "zero potential" - "non-zero potential" etc.). The simplicity of the technical interpretation of the two-valued logic is obviously a great convenience. Nevertheless in certain automatic control systems the use of three-valued logic offers considerable advantages in that the transfer of information can be speeded up.

The first three-valued logic has been invented by J Łukasiewicz (1). His ideas were motivated by certain considerations of modality, namely that statements expressing future-contingent events (that are possible but not necessary) are neither, strictly speaking, "true" or "false"; so they must possess a third value (say "neutral", "indifferent") which he designated by "$\frac{1}{2}$". He used two primitive functors (logical constants) "C" and "N", corresponding to implication and negation. They can be defined by the following value-table:

| | Cab | b: 1 | $\frac{1}{2}$ | 0 | Na |
|---|---|---|---|---|---|
| | 1 | 1 | $\frac{1}{2}$ | 0 | 0 |
| a | $\frac{1}{2}$ | 1 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ |
| | 0 | 1 | 1 | 1 | 1 |

Table 1

As one can read from this table, the values of Cab and Na can be expressed from the values of "a" and "b" in a simple way:

$$V(Cab) = \text{minimum } (1,\ 1 - V(a) + V(b))$$
$$V(Na) = 1 - V(a)$$

where V(P) means the value of P. He defined the other usual binary logical functions Aab (disjunction) and Kab (conjunction) in terms of 'C' and 'N':

Aab = CCabb

Kab = NANaNb

Eab = KCabCba

The values of Aab and Kab can again be expressed in a simple manner in terms of the values of 'a' and 'b':

$$V(Aab) = \text{maximum } (V(a),\ V(b))$$
$$V(Kab) = \text{minimum } (V(a),\ V(b))$$

The system of Łukasiewicz is not functionally complete. For instance, the following unary function Ta : $T1 = T\frac{1}{2} = T0 = \frac{1}{2}$ cannot be expressed in terms of the defined ones. However, Słupecki has shown that if we add the Ta function to the Łukasiewicz system then the supplemented system is functionally complete (2).

The technical interpretation of a three-valued logic must deal successfully with the following problems:

(a) It must choose three-state physical elements in accordance with the three possible values of the propositional variables, and make correspondences between them in a suitable way.

(b) It must show how these three-state elements may be combined to realise certain primitive functions. There must not be great technical difficulties in constructing such combinations of the elements.

(c) It must show how all functions can be expressed in terms of the chosen primitives. (This amounts to functional completeness.)

The rest of the paper is devoted to solving these problems.

(a) Let us consider an electrical network. The usual two alternative states (there is a potential on its output - there is no potential on its output) can be extended to a three-state system by distinguishing between positive and negative potentials. For example, we can ascribe the value 1 to the state of positive potential, the value of $\frac{1}{2}$ to that of negative potential, and the value of 0 to that of zero potential. The introduction of negative potential in addition to positive- and zero-potentials is not new, and in certain communication-systems is already used in practise. Analogously to electrical systems, pneumatic systems and mechanical systems can also realise three different states:

| | | |
|---|---|---|
| 1 : positive pressure | $\frac{1}{2}$ : negative pressure (vacuum) | 0 : zero pressure (ambient pressure) |

| | | |
|---|---|---|
| 1 : moves to right (rotates clockwise) | $\frac{1}{2}$ : moves to left (rotates anti-clockwise) | 0 : does not move |

Our first task is to find those basic fourterminal configurations that will help us to realise the suitable correspondences between the values 1, $\frac{1}{2}$, 0 and the potential-states described above. We present here four possible states of fourterminal network and name them 'break', 'cross-connection', 'reverse-connection', and 'through-connection'. (Fig. 1)

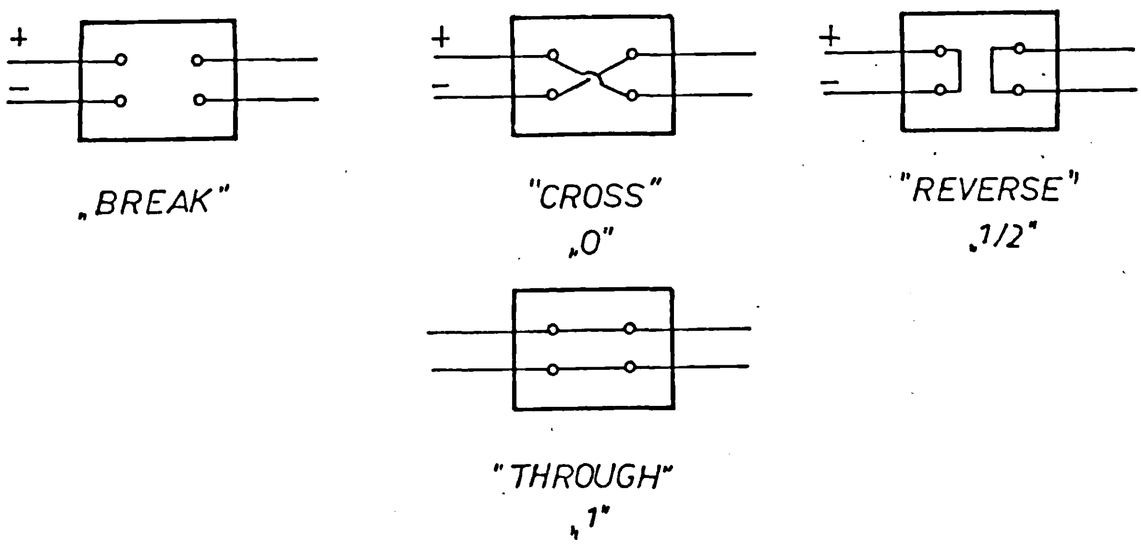



Figure 1.

In Table 2 we indicate four possible ways of ascribing (in columns $I/_1$, $I/_2$, $II/_1$, $II/_2$) some of these connections to the values 1, ½, 0.

| | $I/_1$ | $I/_2$ | $II/_1$ | $II/_2$ |
|---|---|---|---|---|
| '1' : | Through c. | Through c. | Through c. | o Through c. |
| "½" : | Cross c. | Cross c. | Reverse c. | Break |
| '0' : | Break | Reverse c. | Cross c. | Cross c. |

Table 2

The rationale of these ascriptions can be illustrated if we connect one pole-pair of these switches to a D.C. generator and the other pole-pair to a voltmeter. (Fig. 2)

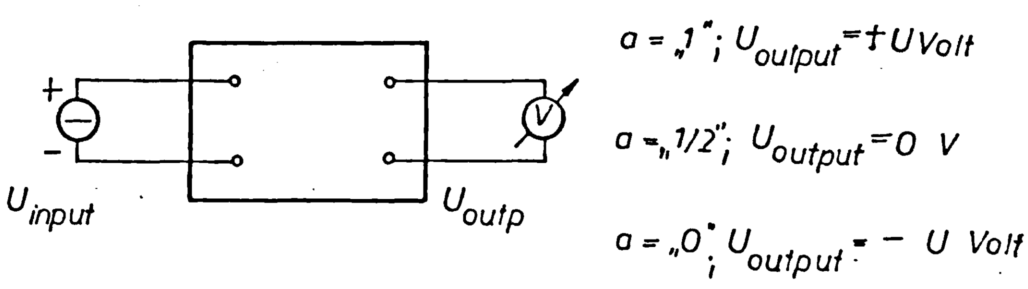
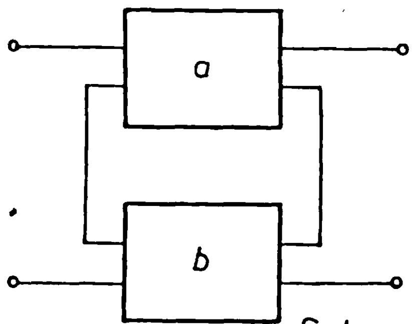


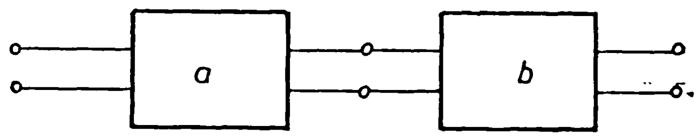

*Figure 2.*

If we substitute in 'a' the connections occurring in column $I/_1$, then the voltmeter will indicate positive, negative and zero potentials respectively. The same will happen if we substitute in 'a' the connections occurring in column $I/_2$. But if we put in 'a' the switches occurring in column $II/_1$ or $II/_2$ then the voltmeter will indicate in turn positive, zero and negative potentials. In the sequel we shall use only correspondence $II/_1$.

(b) For the interpretation of the logical constants our basic idea was to use those functions which are realised by the 'serial', and 'cascade' connections of our double-pole switches. These functions we designate by Sab, Lab respectively. Their switch-diagrams and value-tables are given in Fig. 3.




Sab




Lab

| Sab | 1 | ½ | 0 |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| ½ | 1 | ½ | 0 |
| 0 | 1 | 0 | ½ |

| Lab | 1 | ½ | 0 |
|---|---|---|---|
| 1 | 1 | ½ | 0 |
| ½ | ½ | ½ | ½ |
| 0 | 0 | ½ | 1 |

Fig. 3

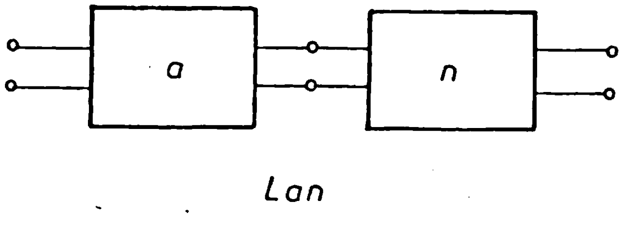
We observe that the matrices of Sab and Lab are symmetrical with respect to their main diagonals. So they define commutative functions with respect to their variables. ( Sab = Sba, Lab = Lba ) The matrix of Lab is also symmetrical with respect to its minor diagonal. (Lab = LNaNb where Na is the Łukasiewicz negation-operation.) It is also clear, from their definitions, that these functions are associative. (SSabc = SaSbc, LLabc = LaLbc).

We also need a fixed element (zero-order operation) which will be the 'cross-connection', and designate it by 'n'. We note that in the interpretation we are using ( see $II/_1$ in Table 2) this corresponds to the value '0'.

(c) We would like to show now that all the logical functions of the Łukasiewicz-Słupecki three-valued logic can be expressed in terms of the functions Sab, Lab and n. In view of the fact that the N, C, T system of the Łukasiewicz-Słupecki logic is functionally complete, it is sufficient to express only these three functions in terms of S, L, n.

The Łukasiewicz negation-function is expressed by Na = Lan, for L10 = 0, $L\frac{1}{2}0 = \frac{1}{2}$, L00 = 1. The value-table of these functions and their representative circuit can be seen in Fig. 4.

| a | Na | Lan |
|---|---|---|
| 1 | 0 | 0 |
| ½ | ½ | ½ |
| 0 | 1 | 1 |




Lan

Fig. 4

The Słupecki Ta function is given by Ta = Snn because Snn = SOO = ½
The table of these functions and their switch-diagram can be seen in Fig. 5.

| a | Ta | Snn |
|---|---|---|
| 1 | ½ | ½ |
| ½ | ½ | ½ |
| 0 | ½ | ½ |

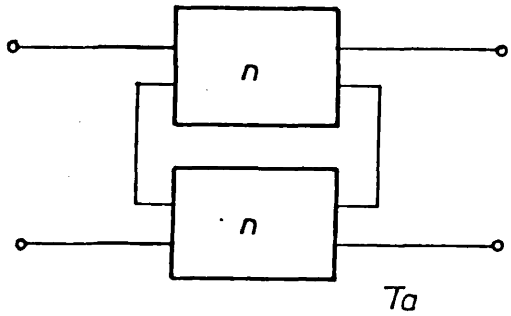



Fig.5.

Finally the Łukasiewicz Cab function is given by LSSLaanbSSLannb and its table and switch-diagram is in Fig. 6.

| a | b | n | Laa | SLaan | SSLaanb | Lan | SLann | SSLannb | LSSLaanbSSLannb | Cab |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 1 | 1 | 0 | ½ | 1 | 1 | 1 |
| 1 | ½ | 0 | 1 | 1 | 1 | 0 | ½ | ½ | ½ | ½ |
| 1 | 0 | 0 | 1 | 1 | 1 | 0 | ½ | 0 | 0 | 0 |
| ½ | 1 | 0 | ½ | 0 | 1 | ½ | 0 | 1 | 1 | 1 |
| ½ | ½ | 0 | ½ | 0 | 0 | ½ | 0 | 0 | 1 | 1 |
| ½ | 0 | 0 | ½ | 0 | ½ | ½ | 0 | ½ | ½ | ½ |
| 0 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | ½ | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

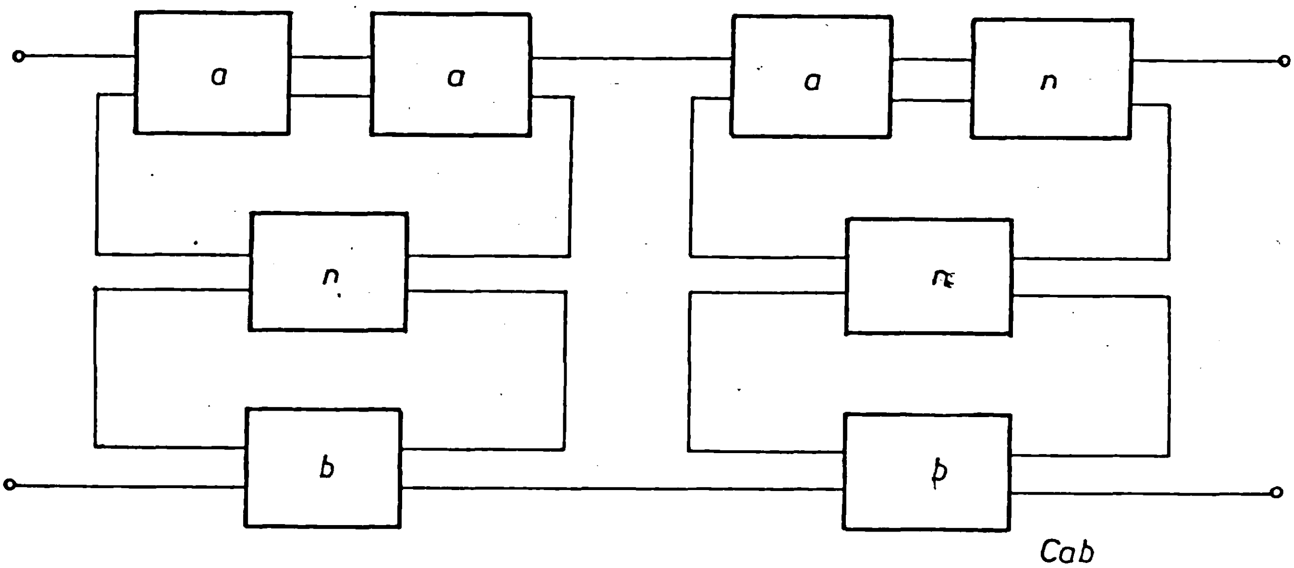



Figure 6.

Thus Sab, Lab, n do indeed constitute a functionally complete system.

We remark that in expressing Cab in terms of S,L, and n we did not use an arbitrary hit or miss trial procedure. Instead, we have worked out a method by which any two-place function can be constructed from S,L and n. The actual number of the different two-place functions in three-valued logic is $3^{3^2} = 19,683$. (This number compares with $2^{2^2} = 16$ different two-place functions in two-valued logic.)

It would take too long to give a precise description of our method. We only mention that it is based on the commutativity and associativity properties of the functions S, L, and the fact that we were able to express each of the 27 one-place functions in terms of S, L and n (Table 3), and use these in suitable combinations in place of the variables in the functions Sab and Lab.

Let us now illustrate our method in two simple examples:

Example 1: Consider two three-way switches ("a" and "b") whose three states are to mean the following:

"0" : motor is switched on and it rotates clockwise;
"½" : motor is switched off; it does not rotate;
"1" : motor is switched on and it rotates anti-clockwise.

Find an electrical network which ensures that in any possible state-combination of the switches "a" and "b" priority is given to the "higher" ( $1 > \frac{1}{2} > 0$ ) state. It is easy to see that to solve this problem we have to find the electrical network corresponding to the Lukasiewicz Aab function:

$$V(Aab) = \max V(a), V(b)$$

We get Aab in terms of S, L and n by certain transformations. We start from Sab by swapping the second and third rows in its value-table. This can be done, according to Table 3 by using $F_6$ = San, i.e. substituting San in place of "a" in the expression Sab: SSanb. The value-table of this function

| a | 1 | ½ | 0 | |
|---|---|---|---|---|
| | 1 | 1 | 1 | $F_1$ = Lnn |
| | 1 | ½ | 1 | $F_2$ = Laa |
| | 1 | 0 | 1 | $F_3$ = SLanSan |
| | 1 | 1 | ½ | $F_4$ = LSanSan |
| | 1 | ½ | ½ | $F_5$ = Saa |
| | 1 | 0 | ½ | $F_6$ = San |
| | 1 | 1 | 0 | $F_7$ = SaLSann |
| | 1 | ½ | 0 | $F_8$ = SSann |
| | 1 | 0 | 0 | $F_9$ = SSana |
| | ½ | 1 | 1 | $F_{10}$ = LSLSannnSLSannn |
| | ½ | ½ | 1 | $F_{11}$ = SLanLan |
| | ½ | 0 | 1 | $F_{12}$ = LnSLSannn |
| | ½ | 1 | ½ | $F_{13}$ = SLnSanLnSan |
| | ½ | ½ | ½ | $F_{14}$ = Snn |
| | ½ | 0 | ½ | $F_{15}$ = LnSLnSanLnSan |
| | ½ | 1 | 0 | $F_{16}$ = SnLnSan = Ln$F_{12}$ |
| | ½ | ½ | 0 | $F_{17}$ = LnSLanLan |
| | ½ | 0 | 0 | $F_{18}$ = LnLSLSannnSLSannn |
| | 0 | 1 | 1 | $F_{19}$ = LnSaSan |
| | 0 | ½ | 1 | $F_{20}$ = Lna |
| | 0 | 0 | 1 | $F_{21}$ = LnSaSan |
| | 0 | 1 | ½ | $F_{22}$ = LnSan |
| | 0 | ½ | ½ | $F_{23}$ = LnSaa |
| | 0 | 0 | ½ | $F_{24}$ = LnLSanSan |
| | 0 | 1 | 0 | $F_{25}$ = LnSLanSan |
| | 0 | ½ | 0 | $F_{26}$ = LnLaa |
| | 0 | 0 | 0 | $F_{27}$ = n |

Table 3

is given in Table 4. If we compare the value-table of SSanb with the value-table of Aab then we can see that we can get Aab if we chain-connect SSanb with Xab. (see Table 4).

| SSanb | 1 | ½ | 0 |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| ½ | 1 | 0 | ½ |
| 0 | 1 | ½ | 0 |

| Aab | 1 | ½ | 0 |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| ½ | 1 | ½ | ½ |
| 0 | 1 | ½ | 0 |

| Xab | 1 | ½ | 0 |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| ½ | 1 | ½ | 0 |
| 0 | 1 | 1 | 1 |

Table 4

But again according to Table 3 we can derive Xab from Sab if we substitute $F_2$ = Laa in place of "a" in Sab: SLaab. Thus the final result is LSSanbSLaab. The switch-diagram of this function can be seen in Fig. 7.

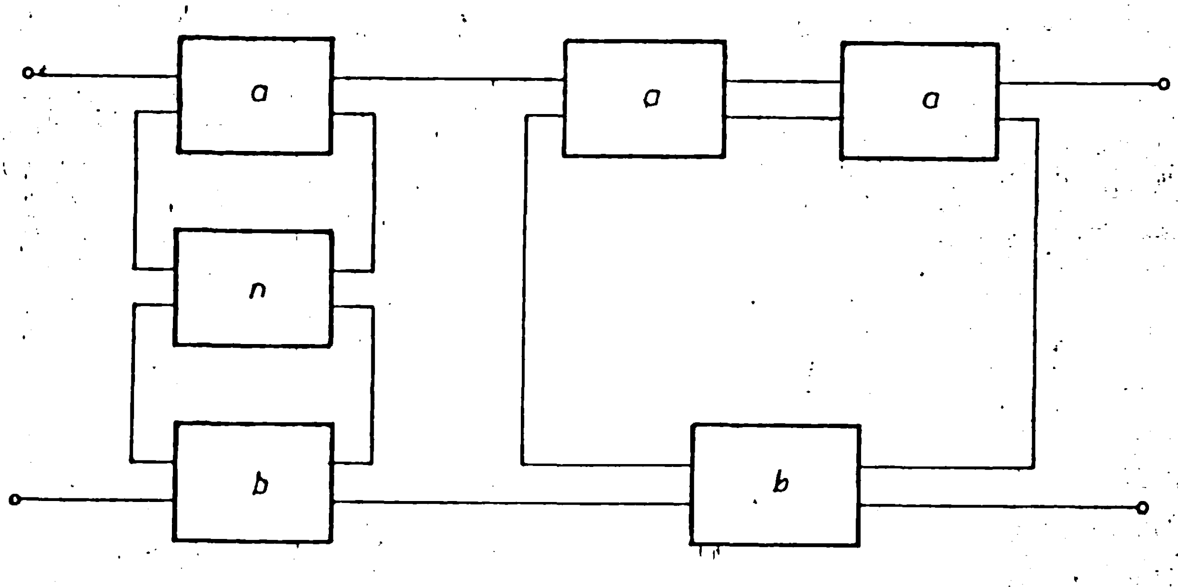



*Figure 7.*

We could have expressed Aab also by LSSanbSSLanab, or by LLSSabaSSabbSSabb but obviously these latter solutions are not as economical as the first one because they use more switches. The optimal solution from an economical point of view is the one which uses a minimal number of switches. At present we are working on this problem of minimalization.

Example 2. This is the well-known problem that presents itself in sleeping cars of trains. Suppose each compartment contains two berths each with a three-way switch:

0 : "dimmed light"

$\frac{1}{2}$ : "lights out"

1 : "full light"

Since the dimmed light does not disturb the sleep and provides a certain measure of safety, we would like to find an eletrical network that gives priority to the lower state ($0 \angle \frac{1}{2} \angle 1$ ) in any possible states of the switches. It is again easy to see that to solve this problem we have to find the electrical network corresponding to the Łukasiewicz Kab function: V(Kab) = min V(a), V (b). This time we shall not explain in detail how one can find this function in terms of L, S, n, because a solution can be found simply by the definition Kab = NANaNb and by some obvious simplification: Kab = LnLSSLannLnbSLaaLnb. The switch-diagram of this function is represented in Fig. 8.

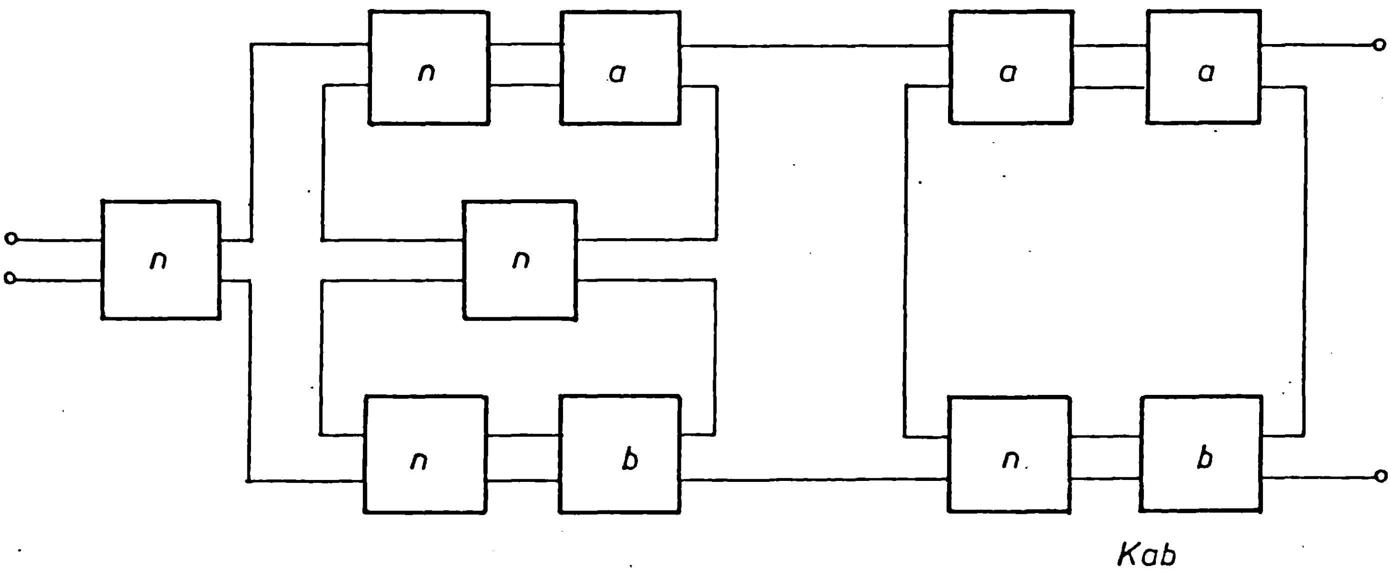



Figure 8.

The actual constructions of these and similar networks and their possible applications will be described in another paper. It should be only mentioned here that the switches 'a', 'b' can be mechanical, electromechanical or electronic switches which are either commercially available or can easily be built up from those which are. About the possible applications of these networks we mention testing for validity of certain propositional formulae, whose analogues are theses in certain non-classical systems. The technical interpretation here presented provides ways of testing for validity of propositional formulae not only in the Łukasiewicz three-valued logic but also in any three-valued logic that can be defined by a set of truth-tables. (This is a direct consequence of the constructive functional completeness of L,S,n.) Such systems have been presented, for example, by Bochwar (3), Kleene (4), and Post (5). In fact, our Lab function formally coincides with the equivalence-function of Kleene's system.

SUMMARY: In order to increase the transfer-velocity of information in certain automatic control systems, we have introduced a new technical interpretation of three-valued logic. We defined primitive functions Sab, Lab and n within this interpretation and proved that three-valued logic is functionally complete with respect to the defined primitives. Finally, we have illustrated in a few examples how certain functions can be expressed in terms of S, L, n. About the technical construction of electrical networks using this new interpretation and their possible applications we intend to publish another paper.

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## LIST OF SYMBOLS

| | |
|---|---|
| $a \in A$ | a is a member (element) of the set A |
| $A \subseteq A$ | A is a subset of the set A |
| $A \subset B$ | A is a proper subset of the set B |
| $a = b$ | a is identical with b |

Whenever a and b are elements of a lattice A:

| | |
|---|---|
| $a \leq b$ | a is not greater than b; $\leq$ is the lattice ordering; 40 |
| $a \cap b$ | greatest lower bound of a and b. |
| $a \cup b$ | least upper bound of a and b. |
| $1_A$ | the top element of A whenever it exists. |
| $0_A$ | the bottom element of A whenever it exists. |
| $a_0, c_0$ | fixed, distinguished elements in a lattice. |
| $a \rightarrow b$ | the pseudo-complement of a relative to b; 41 |
| $\neg a$ | complement of a; $\neg$ is a unary functor defined on the elements of A with values in A. |

Whenever A and B are lattices:

| | |
|---|---|
| $A \xrightarrow{h} B$ | the map h from A into B. |
| $A \times B$ | the direct product (Cartesian product) of A and B; 22 |
| $A^k$ | the direct product of A with itself k-times. |
| $\Gamma A$ | the result of Jaskowski's gamma functor applied to the lattice A; 38,49 |
| $\nabla$ | filter in a lattice; 41 |
| $\Delta$ | ideal in a lattice; 41 |

| | |
|---|---|
| $A/\nabla$ | the sublattice of A determined by the filter $\nabla$ in A;58 |
| $I_o$ | the two-element implicative lattice with distinguished element $c_o = 0_{I_o}$ ; see Fig.1; 45. |
| $M_o$ | the two-element implicative lattice with distinguished element $c_o = 1_{M_o}$ ; see Fig.2; 45. |
| $S_o$ | the two-element implicative lattice of Fig.2, 45. |
| $P_o$ | the two-element implicative lattice of Fig.2, 45 |
| ${}^{(3)}A_{ij}$ | the three-element implicative lattice in the ith row and the jth column of Fig.3, 46 |
| $I_1$ | ${}^{(3)}A_{33}$ of Fig.3, 46 |
| $P_1$ | ${}^{(3)}A_{31}$ of Fig.3, 46 |
| $A^*, A^{**}, \ldots$ | special lattices defined in the text. |
| $A', A'', \ldots$ | special lattices defined in the text. |
| $A_1, A_2, \ldots$ | special lattices defined in the text. |

| | |
|---|---|
| p,q,r,... | propositional variables, 25 |
| $p_1, p_2, \ldots$ | propositional variable, 25 |
| f | a fixed proposition (zero-order connective), usually interpreted as something false; 39 |

Whenever p and q are propositional variables:

| | |
|---|---|
| $\neg p$ | not-p; 25 |
| p & q | p and q; 25 |
| $p \vee q$ | p or q; 25 |
| $p \rightarrow q$ | p implies q; 25 |

| | |
|---|---|
| Np | $\neg p$ in Polish notation. |
| Kpq | p & q in Polish notation. |

Apq p v q in Polish notation.

Cpq p $\longrightarrow$ q in Polish notation.

Tpq Lukasiewcz's denotation for the intuitionistic conjuntion.

Opq Lukasiewicz's symbol for the intuitionistic disjuntion.

Fpq Lukasiewicz's symbol for the intuitionistic implication.

Sab the function defined in Appendix III; 180

Lab the function defined in Appendix III; 180

Ta the Slupecki function used in Appendix III; 182

Na Lukasiewicz's negation operation used in Appendix III;181

n the symbol for the zero element (cross-connection); 181

Note: The connectives should always be understood within the defined systems.

F,P,Q,R,... metavariables for any well-formed formulae; 25

$F^*, F^{**}, \ldots$ special well-formed formulae defined in the text.

$F', F'', \ldots$ special formulae defined in the text.

$F_1, F_2, \ldots$ any well-formed fixed formulae except in II.2 and II.3 where they designate special formulae; 63

$F_1', F_2', \ldots$ special formulae defined on 69

V(F) value-function of F; 35

$\models_M F$ F is valid in M; 35

CL classical propositional logic; 26

HL Heyting's propositional logic; 27

ML minimal logic of Johansson; 28

(ML,F) the propositional logic obtained by extending ML with the additional axiom F and leaving the rules of ML unchanged.

| | |
|---|---|
| $Cal^7$ | (ML, $p \vee \neg p$); 29 |
| $Cal_1$, $Cal_2$ | any propositional calculi |
| Cal*, Cal** | particular calculi defined in the text. |
| Cal', Cal'' | particular calculi defined in the text. |
| $\vdash_{Cal_1} F$ | F is provable (derivable) in $Cal_1$; 33 |
| PL | positive logic; 29 |