# MODEL THEORY OF ABELIAN GROUPS

by

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Abstract

This thesis is mainly concerned with the following sort of question. Let A and C be Abelian groups with certain model-theoretic properties. What can be deduced about the model-theoretic properties of F(A,C) where F is some operation on the class of Abelian groups?

Since we never consider any other sort of group, we use the term <u>group</u> to mean Abelian group throughout. We give the following results:

in chapter 1, a characterisation of groups A such that for any group A', $A \equiv A'$ if and only if

$T(A) \equiv T(A')$ and $A/T(A) \equiv A'/T(A')$:

in chapter 2 we shew that the above characterisation also characterises groups A and C such that for any groups A' and C', $A \equiv A'$ and $C \equiv C'$ implies

$A \otimes C \equiv A' \otimes C'$:

in chapter 3 we shew that for any groups A, A', C, C'

$A \equiv_{\kappa\omega} A'$ and $C \equiv_{\kappa\omega} C'$ implies $Tor(A,C) \equiv_{\kappa\omega} Tor(A',C')$:

in chapter 4 we obtain some results about groups without elements of infinite height:

in chapters 5 and 6 we extend our investigations to Hom and Ext.

Finally in the appendix, we shew how to extend most of these results to modules over Dedekind domains.

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## Contents

Chapter 0

## Introduction

It has long been known that the elementary theory of a given Abelian group A can be determined by a set of algebraic invariants (see (7)). It has also been known for some time that the tensor product does not in general preserve elementary equivalence. It is therefore natural to seek an algebraic characterisation of those groups for which the tensor product does preserve elementary equivalence. This has been done successfully and as a bonus we get preservation of elementary embeddings. It is now natural to seek analagous results for Tor, Hom and Ext. We have complete results for Tor, and also for Hom(A,C) and Ext(A,C) where A is restricted to being a torsion group. We also consider infinitary properties of certain tosrion groups. For those again we have algebraic invariants as described in (1).

We use the notation of (3) together with the symbols $\equiv$, $\equiv_{\kappa\omega}$, $\prec$, $\prec_{\kappa\omega}$, denoting repectively elementary equivalence, $L_{\kappa\omega}$-equivalence, is elementarily embeddable in and is $L_{\kappa\omega}$-elementarily embeddable in.

Many of the results we need are found in (3); any page numbers given without any further qualification refer to (3).

Since we never consider any other sort of group, we use the term <u>group</u> to mean Abelian group throughout.

Following (2) we define the S-invariants of A (S for Szmielew) as follows:

for each prime p and natural numbor n:

$$U(p,n;A) = \dim (p^n A[p]/p^{n+1}A[p]) \text{ if this is finite}$$
$$= \omega \text{ otherwise}$$

$$Tf(p;A) = \lim_{n\to\infty} \dim (p^n A/p^{n+1}A) \text{ if this is finite}$$
$$= \omega \text{ otherwise}$$

$$D(p;A) = \lim_{n\to\infty} \dim (p^n A[p]) \text{ if this is finite}$$
$$= \omega \text{ otherwise}$$

$$Exp(A) = 0 \text{ if } A \text{ is of bounded order}$$
$$= \omega \text{ otherwise.}$$

We refer to (2) for proofs that for any groups A and A',

(i) $A \equiv A'$ if and only if A and A' have the same S-invariants.

(ii) $A \prec A'$ if and only if $A \equiv A'$ and A can be embedded in A' as a pure subgroup.

Definition

Let A be any group and let p be any prime. We say that A is of unbounded p-length if for infinitely many n, $U(p,n;A) \neq 0$; otherwise we say A is of bounded p-length. If A is of bounded p-length, we say that A is of p-length k, if k is the smallest natural number such that $U(p,l;A)$ equals zero for all $l \geq k$.

Lemma 0.1

If A is of unbounded p-length, then $Tf(p;A) = D(p;A) = Exp(A) = \infty$.

Proof

This proof is essentially to be found in (2).

The following sequence - where f is multiplication by p - is exact:

$$0 \to p^nA[p]/p^{n+1}A[p] \to p^nA/p^{n+1}A \xrightarrow{f} p^{n+1}A/p^{n+2}A \to 0.$$

Hence $\dim(p^nA/p^{n+1}A) = \dim(p^{n+1}A/p^{n+2}A) + U(p,n;A)$

By induction, for all $m \geq n$ we have

$$\dim(p^nA/p^{n+1}A) \geq \sum_{j=n}^{m} U(p,j;A)$$

Since A is of unbounded p-length, $\dim(p^nA/p^{n+1}A) = \infty$ for all n. Hence $Tf(p;A) = \infty$.

A similar argument yields $D(p;A) = \infty$, where we consider the exact sequence:

$$0 \to p^{n+1}A[p] \to p^nA[p] \to p^nA[p]/p^{n+1}A[p] \to 0.$$

Trivially $Exp(A) = \infty$. □

In our arguments we often consider p-basic subgroups and we give their definition and some of thir properties here.

Definition

Let A be any group, $B_p$ a subgroup of A. $B_p$ is called a p-basic subgroup of A if

(i) $B_p$ is a direct sum of cyclic p-groups and copies of Z.

(ii) $B_p$ is a p-pure subgroup of A.

(iii) $A/B_p$ is p-divisible.

Lemma 0.2

Let A be any group, p any prime. Then A has a p-basic subgroup; all p-basic subgroups of A are isomorphic; if D is a divisible group, then $A \oplus D$ has the same p-basic subgroups as A.

Proof

This is contained in (3) chapter VI. □

Lemma 0.3

If $B_p$ is a p-basic subgroup of A then for all $n \in \omega$
$p^nB_p[p]/p^{n+1}B_p[p] \cong p^nA[p]/p^{n+1}A[p]$ and
$p^nB_p/p^{n+1}B_p \cong p^nA/p^{n+1}A$. In particular,
$U(p,n;B_p) = U(p,n;A)$ and $Tf(p;B_p) = Tf(p;A)$.

Proof

This is a straightforward exercise; see pages 146 and 147. □

Lemma 0.4

If $B_p$ is a p-basic subgroup of A, then a p-basic subgroup of $A/T(A)$ is isomorphic to $B_p/T(B_p)$.

Proof sketch

One shews firstthat $B_p + T_p(A) / T_p(A)$ is a p-basic subgroup of $A/T_p(A)$, isomorphic to $B_p/T(B_p)$ (where $T_p(A)$ denotes the p-component of $T(A)$). One then shews that p-basic subgroups of $A/T_p(A)$ and $A/T(A)$, which are necessarily torsion-free, must be isomorphic. This uses the p-divisibility of $T(A)/T_p(A)$. □

Lemma 0.5

If for each prime p a group $B_p$ is given, which is a direct sum of cyclic p-groups and copies of Z, then there is a reduced group A, such that for all p, $B_p$ is isomorphic to a p-basic subgroup of A.

Proof

We construct $A_p$ from $B_p$ by putting $T(A_p) = T(B_p)$ and by replacing each copy of Z with a copy of $Q_p$. Then $A = \oplus A_p$ satisfies the conditions of the lemma. □

We have one more preliminary lemma.

Lemma 0.6

If $\{A_i : i \in I\}$ and $\{A_i' : i \in I\}$ are families of groups such that for all i, $A_i \equiv A_i'$, then the following are elementarily equivalent: $\oplus A_i$, $\prod A_i$, $\oplus A_i'$, $\prod A_i'$.

Proof

It is easy to obtain the S-invariants of $\oplus A_i$ etc., from those of the $A_i$. □

Chapter 1

## Pleasant Groups

Theorem 1.1

If $A$ and $A'$ are any groups such that $T(A) \cong T(A')$ and $A/T(A) \cong A'/T(A')$, then $A \cong A'$.

Proof

We shew that given the S-invariants of $T(A)$ and $A/T(A)$ we can determine those of $A$. We only consider the case where $A$ is not a torsion group.

It is clear that for any $p$ and any $n$, $U(p,n;A) = U(p,n;T(A))$ and $D(p;A) = D(p;T(A))$. Also, as $A$ is not torsion, $Exp(A) = \infty$. It remains to determine $Tf(p;A)$.

If $T(A)$ is of unbounded p-length, then so is $A$ and $Tf(p;A) = \infty$. Otherwise we prove (*); if $A$ is of bounded p-length, then $Tf(p;A) = Tf(p;A/T(A))$.

Say $T(A)$ and hence $A$ is of p-length $k$ and let $n \geq k$.

Then $p^n(A/T(A))/p^{n+1}(A/T(A))$

$\cong p^nA/p^nT(A) \ / \ p^{n+1}A/p^{n+1}T(A)$ by page 122, theorem 29.1 (a) and the purity of $T(A)$ in $A$.

$\cong p^nA/p^nT(A) \ / \ p^{n+1}A/p^nT(A)$, since $p^nT(A)$, being a torsion group with no p-component, is p-divisible.

$\cong p^nA/p^{n+1}A$ by the first isomorphism theorem. □

Theorem 1.2

If A and A' are any groups such that $A \equiv A'$, then $T(A) \equiv T(A')$.

Proof

We shew that given the S-invariants of A we can determine those of T(A). Clearly for any p and any n, $U(p,n;T(A)) = U(p,n;A)$ and $D(p;T(A)) = D(p;A)$. If A and hence T(A) is of unbounded p-length, then $Tf(p;T(A)) = \infty$. Otherwise T(A) is of bounded p-length and by (*), $Tf(p;T(A)) = Tf(p;T(A)/T(A)) = 0$.

Finally, if for any p, A and hence T(A) is of unbounded p-length, then $Exp(T(A)) = \infty$; if for infinitely many p, A has p-length greater than zero, then T(A) is not bounded and $Exp(T(A)) = \infty$; otherwise $Exp(T(A)) = 0$. □

Definition

We say a group A is pleasant if for all groups A', $A \equiv A'$ if and only if $T(A) \equiv T(A')$ and $A/T(A) \equiv A'/T(A')$.

By theorems 1.1 and 1.2 it is clear that a group A is pleasant if and only if for all groups A' such that $A \equiv A'$ we have $A/T(A) \equiv A'/T(A')$.

Theorem 1.3

Let A be a group satisfying:

(i) for all p A is of bounded p-length;

(ii) either T(A) is bounded or for some p $Tf(p;A) \neq 0$;

then A is pleasant.

Proof

If A is torsion and $A' \equiv A$, then by condition (ii) A' must be torsion as $Exp(A) = 0$. So we assume that A is not torsion and we determine the S-invariants of $A/T(A)$ from those of A.

Since $A/T(A)$ is a non-zero torsion-free group, we have for any p and any n $U(p,n;A/T(A)) = D(p;A/T(A)) = 0$ and $Exp(A/T(A)) = \omega$. Also by condition (i) we can apply (*) to obtain for any p, $Tf(p;A/T(A)) = Tf(p;A)$. □

Theorem 1.4

Let A be a group, failing to satisfy either condition (i) or condition (ii) of theorem 1.3. Then A is not pleasant.

Proof

Suppose A does not satisfy condition (i) and that for some prime q, A is of unbounded q-length; suppose further that $A_d$ is the maximal divisible subgroup of A.

For each prime p, pick a p-basic subgroup of A, $B_p$ say.

For each prime p, we define the group $B'_p$ as follows: for all $p \neq q$, $B'_p = B_p$; $B'_q = T(B_q)$ if $B_q$ is not torsion; otherwise $B'_q = Z \oplus B_q$. By lemma 0.5 there is a reduced group A' with p-basic subgroups isomorphic to $B'_p$.

It is not difficult to shew that $A' \oplus A_d \equiv A$. For example, for any p and any n, $U(p,n;A' \oplus A_d) = U(p,n;B'_p) = U(p,n;B_p) = U(p,n;A)$. Also $Tf(q;A' \oplus A_d) = \infty = Tf(q;A)$.

However, $Tf(q;A/T(A)) \neq 0$ if $B_q$ is not torsion, when $B'_q$ is torsion so that $Tf(q;A' \oplus A_d/T(A' \oplus A_d)) = 0$ and a similar argument covers the case where $B_q$ is torsion, so that $A' \oplus A_q/T(A' \oplus A_q) \not\cong A/T(A)$.

It now suffices to consider the case where A satisfies condition (i), but does not satisfy condition (ii). i.e. for all p $Tf(p;A) = 0$ and $T(A)$ is not bounded.

We shew that if A is torsion, then $A \cong A \oplus Q$ and if A is not torsion, then $A \cong T(A)$.

For any p and any n

$U(p,n;A) = U(p,n;T(A)) = U(p,n;A \oplus Q)$,

$D(p;A) = D(p;T(A)) = D(p;A \oplus Q)$,

$Tf(p;A \oplus Q) = Tf(p;A) = 0$, since Q is p-divisible,

$Tf(p;T(A)) = 0 = Tf(p;A)$ by the proof of theorem 1.2.

$Exp(A) = Exp(T(A)) = Exp(A \oplus Q)$, since $T(A)$ is not bounded. □

Chapter 2

## Pleasant Groups and Tensor Products

In this chapter we show that the pleasant groups are precisely those for which the tensor product preserves elementary equivalence. We refer to (3) extensively to obtain facts about tensor products.

Lemma 2.1

If A and A' are torsion-free groups and for each prime p, $B_p$ is a p-basic subgroup of A and $B'_p$ a p-basic subgroup of A'; then $A \equiv A'$ if and only if for all p $B_p \equiv B'_p$.

Proof

For any torsion-free group, any p and any n,
$U(p,n;G) = D(p;G) = 0$, $Exp(G) = \infty$, where G denotes the group.

If for all p $B_p \equiv B'_p$, then for all p,
$Tf(p;A) = Tf(p;B_p) = Tf(p;B'_p) = Tf(p;A')$
If $A \equiv A'$, then for all p the above equality holds. As $B_p$ is a direct sum of copies of Z, it is clear that for all primes q, $Tf(q;B_p) = Tf(p;B_p)$ depends only on the number of copies of Z and not on q. Thus the lemma is proved. □

Lemma 2.2

If A and C are torsion-free groups with p-basic subgroups $B_p$ and $D_p$ respectively, then $A \otimes C$ is torsion-free and a p-basic subgroup of $A \otimes C$ is isomorphic to $B_p \otimes D_p$.

Proof

Page 262 corollary 61.4. □

Lemma 2.3

If A and C are torsion-free groups, then for all A', C',

$A \equiv A'$ and $C \equiv C'$ $\Rightarrow$ $A \otimes C \equiv A' \otimes C'$.

Proof

Immediate from lemmas 2.1 and 2.2. □

Lemma 2.4

Let n be any cardinal number and C any group, then

$\oplus_n Z \otimes C \cong \oplus_n C$.

Proof

Page 255, (G) and (I). □

Lemma 2.5

Let A and C be p-groups, such that for all n $U(p,n;A) = a_n$ and $U(p,n;C) = c_n$.

Let $f(n) = a_n c_n + a_n . \sum_{n+1}^{\infty} c_r + c_n . \sum_{n+1}^{\infty} a_r$, where the conventions for $\infty$ are that $0.\infty = 0$ and if $n \neq 0$ then $n.\infty = \infty$.

Then (i) $U(p,n;A \otimes C) = f(n)$ for all $n \in \omega$.

(ii) $D(p;A \otimes C) = 0$ if A or C is bounded

$= \infty$ otherwise.

Proof

(i) follows from two applications of theorem 61.1 on page 261 together with (H) on page 255, the comment that $Z(p^r) \otimes Z(p^s) \cong Z(p^t)$, where $t = \min(r,s)$.

(ii) follows immediately from theorem 61.3 On page 262. □

Lemma 2.6

Let A and C be any groups; then

(i) $A \otimes C \,/\, T(A \otimes C) \cong A/T(A) \otimes C/T(C)$

(ii) $T(A \otimes C) \cong (T(A) \otimes T(C)) \oplus (T(A) \otimes C/T(C))$

$\oplus (T(C) \otimes A/T(A))$.

Proof

Page 263, theorem 61.5. □

Theorem 2.7

If A and C are pleasant, $A \equiv A'$, $C \equiv C'$, then

$A \otimes C \equiv A' \otimes C'$.

Proof

Since A and C are pleasant, we can determine the S-invariants of $A/T(A)$, $T(A)$, $C/T(C)$ and $T(C)$. We shew that from these we can determine the S-invariants of $A \otimes C \,/\, T(A \otimes C)$ and $T(A \otimes C)$ and hence by theorem 1.1 we obtain the S-invariants of $A \otimes C$ giving the required result.

By lemma 2.3 and lemma 2.6 (i) we obtain the S-invariants of $A \otimes C \,/\, T(A \otimes C)$. We complete the proof by determining those of $T(A) \otimes T(C)$ and $A/T(A) \otimes T(C)$ appealing to lemma 0.6. By lemma 2.5 and the fact that $T(A) \otimes T(C) \cong \oplus_p (T_p(A) \otimes T_p(C))$, where $T_p(\ )$ denotes the p-component of $T(\ )$, we have the S-invariants of $T(A) \otimes T(C)$. From page 262 we have $A/T(A) \otimes C \cong \oplus_p (B_p^* \otimes T_p(C))$ , where $B_p^*$ is a p-basic subgroup of $A/T(A)$. Applying lemma 2.4 the result is easy to obtain.□

Theorem 2.8

If A and C are pleasant, $A \prec A'$, $C \prec C'$, then $A \otimes C \prec A' \otimes C'$.

Proof

Since elementary equivalence is preserved it suffices to note that purity is preserved on taking the tensor product (page 259, theorem 60.4). □

Theorem 2.9

If A is not pleasant, then there are $A^*$ and C such that $A \equiv A^*$, but $A \otimes C \not\equiv A^* \otimes C$.

Proof

Suppose A is of unbounded q-length; let $A^* = A' \oplus A_d$ as constructed in the proof of theorem 1.4. Then one of $A \otimes Z(q^\infty)$, $A^* \otimes Z(q^\infty)$ is zero and the other is not. Suppose A is of bounded p-length for all primes p, but that condition (ii) of theorem 1.3 fails. Then if A is torsion $A \equiv A \oplus Q$; $A \otimes Q = 0$, but $(A \oplus Q) \otimes Q \neq 0$. If A is not torsion, then $A \equiv T(A)$; $A \otimes Q \neq 0$, but $T(A) \otimes Q = 0$. □

It is not the case that when A and C are pleasant that $A \otimes C$ is necessarily pleasant. In fact we have:

Lemma 2.10

If $A_1, \ldots, A_n$ are pleasant and $A_1 \otimes \ldots \otimes A_n$ is not, then

(i) for all p $A_1 \otimes \ldots \otimes A_n$ is of bounded p-length.

(ii) $A_1 \otimes \ldots \otimes A_n$ is not a torsion group.

Proof

(i) The p-length of $A_1 \otimes \dots \otimes A_n$ is the minimum of the p-lengths of $A_1, \dots, A_n$.

(ii) If any of $A_1, \dots, A_n$ is a torsion group it is easy to see that $A_1 \otimes \dots \otimes A_n$ must be pleasant. Hence $A_1/T(A_1), \dots, A_n/T(A_n)$ are all non-zero torsion-free groups. By an obvious extension of lemma 2.6 (i) we see that

$A_1 \otimes \dots \otimes A_n / T(A_1 \otimes \dots \otimes A_n)$ is nonzero. □

Theorem 2.11

If $A_1, \dots, A_n$ are pleasant and $A_1 \equiv A_1', \dots, A_n \equiv A_n'$, then $A_1 \otimes \dots \otimes A_n \equiv A_1' \otimes \dots \otimes A_n'$.

Proof

By induction: the theorem is true for n=2, by theorem 2.7 Suppose the theorem is true for n=k. We shew that the S-invariants of $A_1 \otimes \dots \otimes A_{k+1}$ are determined by those of $A_1 \otimes \dots \otimes A_k$ and $A_{k+1}$. If $A_1 \otimes \dots \otimes A_k$ is pleasant there is nothing to do. Suppose $A_1 \otimes \dots \otimes A_k$ is not pleasant. By (i) of lemma 2.10, $A_1 \otimes \dots \otimes A_k$ is of bounded p-length for all p and this makes it straightforward to determine $Tf(p;A_1 \otimes \dots \otimes A_{k+1})$ and $D(p;A_1 \otimes \dots \otimes A_{k+1})$.

If $A_{k+1}$ is a bounded group, then $Exp(A_1 \otimes \dots \otimes A_{k+1}) = 0$
Otherwise $Exp(A_1 \otimes \dots \otimes A_{k+1}) = \infty$,

for by pleasantness $A_{k+1}$ is not torsion and neither is $A_1 \otimes \dots \otimes A_k$ by (ii) of lemma 2.10. □

Theorem 2.12

If $A_1, \ldots, A_n$ are pleasant and $A_1 \prec A_1', \ldots, A_n \prec A_n'$,

then $A_1 \otimes \ldots \otimes A_n \prec A_n' \otimes \ldots \otimes A_n'$

Proof

As proof of theorem 2.8. □

Chapter 3

# Elementary and Infinitary Properties of Torsion Products

We begin with three lemmas that shew that in order to consider torsion products, we need only consider torsion groups.

Lemma 3.1

If A is any group, then for $\omega \leq \kappa \leq \infty$, we have if $A \equiv_{\kappa\omega} A'$, then $T(A) \equiv_{\kappa\omega} T(A')$.

Proof

For $\kappa = \omega$, this is theorem 1.2.

For $\kappa > \omega$, $T(A)$ is a definable subset of A. $\square$

Lemma 3.2

If A and C are any groups, then

$\text{Tor}(A,C) \cong \text{Tor}(T(A),T(C)) \cong \bigoplus_p \text{Tor}(T_p(A),T_p(C))$

Proof

Page 265, (B) and (F). $\square$

Lemma 3.3

For $\omega \leq \kappa \leq \infty$, if for all torsion groups A and C

$$A \equiv_{\kappa\omega} A' \text{ and } C \equiv_{\kappa\omega} C' \Rightarrow \text{Tor}(A,C) \equiv_{\kappa\omega} \text{Tor}(A',C'),$$

then for arbitrary groups A and C

$$A \equiv_{\kappa\omega} A' \text{ and } C \equiv_{\kappa\omega} C' \Rightarrow \text{Tor}(A,C) \equiv_{\kappa\omega} \text{Tor}(A',C').$$

Proof

Immediate. $\square$

Since we are only considering torsion groups, we can use the methods of (1) for infinitary languages.

Lemma 3.4

Let A, A' be torsion groups and $\omega \leq \kappa \leq \infty$, then $A \equiv_{\kappa\omega} A'$ if and only if for all p, $A_p \equiv_{\kappa\omega} A'_p$, where $A_p$ (resp. $A'_p$) denotes the p-component of A (resp. A').

Proof

For $\kappa = \omega$, this is obvious from what has gone before.
For $\kappa > \omega$, this is in (1) page 36 corollary 3.6. □

Lemma 3.5

Let A (resp. C) be a torsion group and $A_p$ (resp. $C_p$) its p-component. Then $Tor(A_p, C_p)$ is isomorphic to the p-component of Tor(A,C).

Proof

By lemma 3.2, it suffices to show that $Tor(A_p, C_p)$ is a p-group: this follows from (A) on page 265. □

Lemma 3.6

For $\omega \leq \kappa \leq \infty$, if for all p-groups A and C

$$A \equiv_{\kappa\omega} A' \text{ and } C \equiv_{\kappa\omega} C' \Rightarrow Tor(A,C) \equiv_{\kappa\omega} Tor(A',C'),$$

then for arbitrary groups A and C

$$A \equiv_{\kappa\omega} A' \text{ and } C \equiv_{\kappa\omega} C' \Rightarrow Tor(A,C) \equiv_{\kappa\omega} Tor(A',C')$$

Proof

Immediate. □

Definition

For any p-group A, we define for each ordinal $\alpha$,

$$f_\alpha(A) = \dim(p^\alpha A[p]/p^{\alpha+1}A[p]) \text{ if this is finite}$$
$$= \infty \text{ otherwise.}$$

$r_\alpha(A) = \dim(p^\alpha A[p])$ if this is finite

$= \infty$ otherwise.

$l(A)$ = the least ordinal $\beta$, such that for all $\alpha \geq \beta$,

$f_\alpha(A) = 0$.

<u>Lemma 3.7</u>

Let $\omega \leq \kappa \leq \alpha$; then

(i) if $A$, $A'$ are p-groups such that $A \equiv_{\kappa\omega} A'$, then for all $\alpha < \kappa$, $f_\alpha(A) = f_\alpha(A')$ and $r_\alpha(A) = r_\alpha(A')$.

(ii) If $A$, $A'$ are p-groups, such that for all $\alpha < \kappa$ $f_\alpha(A) = f_\alpha(A')$ and if $l(A) < \kappa$ implies $r_{l(A)}(A) = r_{l(A)}(A')$, then $A \equiv_{\kappa\omega} A'$.

<u>Proof</u>

If $\kappa = \omega$ this follows from what has gone before. If $\kappa > \omega$ then (i) is in (1) page 39 lemma 2.2 and (ii) is in (1) page 42 theorem 3.1. □

<u>Lemma 3.8</u>

If $A$ and $C$ are p-groups, then $r_\alpha \mathrm{Tor}(A,C) = r_\alpha(A).r_\alpha(C)$

$f_\alpha(\mathrm{Tor}(A,C)) = f_\alpha(A).f_\alpha(C) + f_\alpha(A).r_{\alpha+1}(C) + f_\alpha(C).r_{\alpha+1}(A)$

<u>Proof</u>

Page 273 theorem 64.4, (2) and (1). □

<u>Lemma 3.9</u>

For $\omega \leq \kappa \leq \alpha$ and for all p-groups $A$ and $C$ we have

$A \equiv_{\kappa\omega} A'$ and $C \equiv_{\kappa\omega} C' \Rightarrow \mathrm{Tor}(A,C) \equiv_{\kappa\omega} \mathrm{Tor}(A',C')$.

<u>Proof</u>

This is a straightforward application of lemmas 3.7 and 3.8. □

Theorem 3.10

For $\omega \leq \kappa \leq \infty$ and for any groups A and C we have

$A \equiv_{\kappa\omega} A'$ and $C \equiv_{\kappa\omega} C'$ $\Rightarrow$ $Tor(A,C) \equiv_{\kappa\omega} Tor(A',C')$.

Proof

This is immediate from lemma 3.6. □

Theorem 3.11

For $\omega \leq \kappa \leq \infty$ and for any groups A and C we have

$A \prec_{\kappa\omega} A'$ and $C \prec_{\kappa\omega} C'$ $\Rightarrow$ $Tor(A,C) \prec_{\kappa\omega} Tor(A',C')$.

Proof

For $\kappa = \omega$ it suffices to note that purity is preserved on taking the torsion product (page 270 theorem 63.2). For $\kappa > \omega$ this is a result of Hodges (see (4) page 20). □

# Chapter 4

## "Low" Groups

### Definition

We say an element x of a group A is of infinite height, if for all positive integers n, there is an element y of A such that x = ny.

The elements of infinite height in A form a subgroup $A^1$ of A called the first Ulm subgroup of A.

The quotient group $A/A^1 = A_0$ is called the 0th Ulm factor of A.

We prove a lemma about the 0th Ulm factor which will be useful in the next chapter and we then prove some results about torsion groups without elements of infinite height.

### Lemma 4.1

Given any group A, a p-basic subgroup of A is isomorphic to a p-basic subgroup of $A_0$.

### Proof

Let $B_p$ be a p-basic subgroup of A. Then $B_p \oplus A^1 / A^1$ is the required p-basic subgroup of $A_0$, for the sum $B_p + A^1$ is clearly a direct sum since elements of $B_p$ are of finite p-height (where p-height is defined analogously to height in the obvious manner) and elements of $A^1$ are of infinite p-height. It is straightforward to check that $B_p \oplus A^1 / A^1$ is p-pure in $A_0$ and that $A_0/(B_p \oplus A^1 / A^1)$ is p-divisible. □

Lemma 4.2

If $A$ is a reduced group, then $A \equiv A_0$.

Proof

$A_0$ is clearly reduced and it is easy to check that two reduced groups with isomorphic p-basic subgroups for all p are elementarily equivalent. □

We now turn our attention to p-groups and it is clear how our results extend to torsion groups. We generalize a well-known result of Abelian group theory, which appears as corollary 4.7.

Lemma 4.3

If $A$ is a p-group, then $A$ has no elements of infinite height if and only if $l(A) \leqslant \omega$ and $r_\omega(A) = 0$.

Proof

If $A$ has no element of infinite height, then $p^\omega A = 0$, whence $l(A) \leqslant \omega$ and $r_\omega(A) = 0$.

If $A$ has an element of infinite height, then $p^\omega A \neq 0$. If $p^{\omega+1}A = p^\omega A$, then $p^\omega A$ is divisible and $r_\omega(A) \neq 0$; if $p^{\omega+1}A \neq p^\omega A$, then $f_\omega(A) \neq 0$ and so $l(A) > \omega$. □

Lemma 4.4

If $A$, $A'$ are p-groups without elements of infinite height and $A \equiv A'$, then $A \equiv_{\infty\omega} A'$.

Proof

For all $\alpha < \omega$ $f_\alpha(A) = U(p,\alpha;A) = U(p,\alpha;A') = f_\alpha(A')$

For all $\alpha \geqslant \omega$ $f_\alpha(A) = 0 = f_\alpha(A')$

and $r_{l(A)}(A) = 0 = r_{l(A)}(A')$. □

Lemma 4.5

For every reduced p-group A there is a countable group B, which is a direct sum of cyclic groups, such that $A \equiv B$.

Proof

For all $n \in \omega$ if $U(p,n;A)$ is finite, put $k_n = U(p,n;A)$; otherwise put $k_n = \aleph_0$. Then let $B = \bigoplus_n [\bigoplus_{k_n} Z(p^{n+1})]$ □

Theorem 4.6

A p-group A is a group without elements of infinite height if and only if it is reduced and $L_{\infty\omega}$-equivalent to a countable group which is a direct sum of cyclic groups.

Proof

If A is $L_{\infty\omega}$-equivalent to a direct sum of cyclic groups, then $l(A) \leqslant \omega$; if in addition A is reduced, then $r_\omega(A) = 0$. If A has no element of infinite height, then clearly A is reduced. By lemma 4.5, there is a countable group B, which is a direct sum of cyclic groups such that $A \equiv B$. B clearly contains no element of infinite height and so by by lemma 4.4, $A \equiv_{\infty\omega} B$. □

Corollary 4.7

If A is a countable p-group without elements of infinite height, then A is a direct sum of cyclic groups.

Proof

This is a well-known property of $L_{\infty\omega}$; see e.g. (1). □

We close this chapter with a result about tensor products of p-groups. It is clear how one can extend the result to torsion groups.

<u>Lemma 4.8</u>

If A, A', C and C' are p-groups and $A \equiv A'$, $C \equiv C'$, then $A \otimes C \equiv A' \otimes C'$.

<u>Proof</u>

This is immediate from lemma 2.5. □

<u>Lemma 4.9</u>

If A and C are p-groups, then $A \otimes C$ is a direct sum of cyclic groups.

<u>Proof</u>

Page 262, theorem 61.3. □

<u>Theorem 4.10</u>

If A, A', C and C' are p-groups and $A \equiv A'$, $C \equiv C'$, then $A \otimes C \equiv_{\sigma\omega} A' \otimes C'$.

<u>Proof</u>

Immediate. □

# Chapter 5

## Some Remarkable Equivalences

In this chapter and the next, we extend our discussion to the functors Hom and Ext. We consider only the case where A is a torsion group in Hom(A,C) and Ext(A,C), since there is sufficient known in this case to give satisfactory results (see chapters VIII and IX of (3))

**Lemma 5.1**

Let A be a torsion group with p-components $A_p$ and let C be an arbitrary group. Then

(i) $\mathrm{Hom}(A,C) \cong \prod_p \mathrm{Hom}(A_p,C)$, where each $\mathrm{Hom}(A_p,C)$ is a reduced p-adic group.

(ii) $\mathrm{Ext}(A,C) \cong \prod_p \mathrm{Ext}(A_p,C)$, where each $\mathrm{Ext}(A_p,C)$ is a reduced p-adic group.

**Proof**

(i) is page 182 theorem 43.1 together with page 188 exercise 5 (straightforward exercise).

(ii) is page 222 theorem 52.2 together with page 237 lemma 55.3 and (I) page 223. □

In what follows we will obtain results for p-groups and it will be clear, using lemma 0.6 and 5.1, how to extend the results to torsion groups.

For the rest of the chapter we assume that A and C are reduced p-groups, with $U(p,n;A) = a_n$, $U(p,n;C) = c_n$ for each n.

As in lemma 2.5 we define

$$f(n) = a_n c_n + a_n \cdot \sum_{n+1}^{\infty} c_r + c_n \cdot \sum_{n+1}^{\infty} a_r .$$

We note that if A and C are of unbounded p-length, then

$f(n) = 0$ if $a_n = c_n = 0$

$= \infty$ otherwise;

if A (resp. C) is of unbounded p-length and C (resp. A) is of p-length k then,

$f(n) = 0$ if $a_n = c_n = 0$ or $n \geq k$

$= \infty$ otherwise.

Theorem 5.2

The invariants $(f(n))_{n \in \omega}$ determine $A \otimes C$ up to elementary equivalence.

Proof

This is the substance of lemma 2.5. □

Theorem 5.3

The invariants $(f(n))_{n \in \omega}$ determine Tor(A,C) up to elementary equivalence.

Proof

This is a straightforward consequence of the structure of Tor(A,C), using lemma 3.8. □

Theorem 5.4

The invariants $(f(n))_{n \in \omega}$ determine Hom(A,C) up to elementary equivalence.

Proof

Let B be a basic subgroup of A and D a basic subgroup of C. By the definition of final rank on page 150, since B (resp. C) (resp. D) is reduced fin r(B) (resp. fin r(C)) (resp. fin r(D)) is zero if B (resp. C) (resp. D) is bounded and is infinite otherwise. We use theorem 46.4 on page 197, putting the symbol $\infty$ in place of any infinite cardinal. Hom(A,C) is the pure-injective envelope of

$$* = \bigoplus_n \left[\bigoplus_{f(n)} Z(p^{n+1})\right] \oplus \left[\bigoplus_j J_p\right]$$

where j=0 if A or C is bounded and is infinite otherwise. Since Hom(A,C) is the pure-injective envelope of * it is not difficult to see that it is an elementary extension of * and in particular, that for all $n \in \omega$

$U(p,n;Hom(A,C)) = f(n)$.

If A and C are of unbounded p-length, then Hom(A,C) is of unbounded p-length and the invariants $(f(n))_{n \in \omega}$ determine Hom(A,C) up to elementary equivalence. If A or C is of bounded p-length, then * is bounded. A bounded group is its own pure-injective envelope, so Hom(A,C) is bounded and again the invariants $f(n)_{n \in \omega}$ determine Hom(A,C) up to elementary equivalence. □

Lemma 5.5

Let B be a basic subgroup of A and D a basic subgroup of C, then $Ext(A,C) \cong Ext(B,D) \oplus Ext(A/B,D)$.

Proof

By lemmas 4.2 and 5.1, $\mathrm{Ext}(A,C) \equiv \mathrm{Ext}(A,C)_0$

By page 244 theorem 57.2, $\mathrm{Ext}(A,C)_0 \cong \mathrm{Ext}(B,D) \oplus \mathrm{Ext}(A/B,D)_0$

By lemmas 4.2 and 5.1, $\mathrm{Ext}(A/B,D)_0 \equiv \mathrm{Ext}(A/B,D)$. □

Theorem 5.6

The invariants $(f(n))_{n\in\omega}$ determine $\mathrm{Ext}(A,C)$ up to elementary equivalence.

Proof

Let B be a basic subgroup of A and D a basic subgroup of C.

$$\text{Then } \mathrm{Ext}(B,D) = \mathrm{Ext}(\oplus_n \oplus_{a_n} Z(p^{n+1}), \oplus_m \oplus_{c_m} Z(p^{m+1}))$$

$$\cong \Pi_n \Pi_{a_n} [\mathrm{Ext}(Z(p^{n+1}), \oplus_m \oplus_{c_m} Z(p^{m+1}))]$$

$$\equiv \oplus_n \oplus_{a_n} [\mathrm{Ext}(Z(p^{n+1}), \oplus_m \oplus_{c_m} Z(p^{m+1}))$$

(by lemma 0.6)

$$\cong \oplus_n \oplus_{a_n} (\oplus_m \oplus_{c_m} Z(p^{m+1})/p^{n+1} \oplus_m \oplus_{c_m} Z(p^{m+1})$$

(by page 222 (D))

$$\cong \oplus_n \oplus_{a_n} (\bigoplus_{m=0}^{n-1} \oplus_{c_m} Z(p^{m+1}) \oplus \oplus_{\sum_n c_m} Z(p^{n+1}))$$

From here it is easy to see that

$$U(p,n;\mathrm{Ext}(B,D)) = a_n . \sum_n^{\infty} c_m + c_n . \sum_{n+1}^{\infty} a_m = f(n).$$

By page 237 lemma 55.1, $U(p,n;\mathrm{Ext}(A/B,D) = r(A/B).c_n$ if this is finite and is $\infty$ otherwise. This term can only be non-zero if A is of unbounded p-length, in which case $f(n) = \infty$ unless $a_n = c_n = 0$ or n is greater than the p-length of C. Thus if $f(n) \neq \infty$, then $c_n$ is necessarily 0. If $f(n) = \infty$, then $\infty = U(p,n;\mathrm{Ext}(B,D) \leqslant U(p,n;\mathrm{Ext}(A,C) = \infty$.

If A and C are of unbounded p-length, then so is Ext(A,C) and we have shewn that the invariants $(f(n))_{n\in\omega}$ determine Ext(A,C) up to elementary equivalence.

If A or C is of bounded p-length (and hence bounded since we are considering reduced p-groups), then Ext(A,C) is bounded (by page 223 (E)) and again the invariants $(f(n))_{n\in\omega}$ determine Ext(A,C) up to elementary equivalence. $\square$

We have shewn that for reduced p-groups A and C,

$$A \otimes C \equiv \mathrm{Tor}(A,C) \equiv \mathrm{Hom}(A,C) \equiv \mathrm{Ext}(A,C).$$

Chapter 6

## An Interesting Duality

In the previous chapter we restricted our attention to reduced p-groups and we now wish to relax some of that restriction. We first of all give two lemmas on homomorphism groups.

Lemma 6.1

If $A = \bigoplus_m Z(p^\infty)$, $C = \bigoplus_n Z(p^\infty)$ where m and n are any non-zero cardinals, then Hom(A,C) is a torsion-free p-adic group, where $Tf(p;Hom(A,C)) = mn$ if m and n are finite

$= \infty$ otherwise.

Proof

This is exercise 8, page 203 □

Lemma 6.2

If A is a p-group with $U(p,n;A) = a_n$ and $C = \bigoplus_k Z(p^\infty)$, where k is finite or infinity, then $U(p,n;Hom(A,C)) = ka_n$

Proof

Let B be a basic subgroup of A. Then the pure-exactness of $0 \to B \to A \to A/B \to 0$, together with the injectiveness of C, gives a pure-exact sequence:

$0 \to Hom(A/B,C) \to Hom(A,C) \to Hom(B,C) \to 0$,

by page 187 proposition 44.7 and page 186 proposition 44.5. Using theorem 29.1 page 122 a number of times we obtain

$U(p,n;Hom(A,C)) = U(p,n:Hom(B,C) + U(p,n;Hom(A/B,C)$

$= U(p,n;Hom(B,C))$

B is a direct sum of cyclic groups and from here it is straightforward to obtain the result. □

We now give a lemma which will be very useful in what follows: let A and C be p-adic groups with $U(p,n;A) = a_n$, $U(p,n;C) = c_n$. Then our definition of f(n) which we gave for p-groups is still valid.

Lemma 6.3

Suppose A and C are of unbounded p-length and s and t are either nonnegative integers or infinity. Then for each n

$f(n) + sa_n + tc_n = f(n)$.

Suppose A (resp. C) is of unbounded p-length and C (resp. A) is of p-length k and s, t are as above. Then for $n < k$

$f(n) + sa_n + tc_n = f(n)$.

Proof

If $a_n = c_n = 0$, then there is nothing to do; otherwise

$\infty = f(n) \leq f(n) + sa_n + tc_n = \infty$. □

The theorems we prove now will be about p-groups. It will be clear how to extend the results to torsion groups.

Theorem 6.4

Let A, A' be p-groups and C, C' arbitrary groups, such that $A \equiv A'$ and $C \equiv C'$. Then (a) $A \otimes C \equiv A' \otimes C'$ and (b) $Hom(A,C) \equiv Hom(A',C')$, if one of the following holds:

(i) A is of unbounded p-length,

(ii) $D(p;A) = 0$,

(iii) C is of bounded p-length.

Proof

For (a) we shew that the S-invariants of $A \otimes C$ are determined by those of A and C. Since $A \otimes C$ is a p-group it suffices to obtain $U(p,n;A \otimes C)$ for all $n \in \omega$ and if $A \otimes C$ is of bounded p-length to obtain $D(p;A \otimes C)$.

By page 261 theorem 61.1, if D is a p-basic subgroup of C, then $A \otimes C \cong A \otimes D$.

We assume $A = A_r \oplus \oplus_{a_d} Z(p^\infty)$, where $A_r$ is reduced;

$D = T(D) \oplus \oplus_{c_t} Z$.

If A is of bounded p-length, then $a_d = D(p;A)$;

if C is of bounded p-length, then $c_t = Tf(p;C)$.

$$\text{Now } U(p,n;A \otimes C) = U(p,n;A_r \otimes T(D)) + U(p,n;A_r \otimes \oplus_{c_t} Z)$$

$$= f(n) + a_n c_t.$$

If A is of unbounded p-length $U(p,n;A \otimes C) = f(n) + a_n Tf(p;C)$ since for n less than the p-length of C, if $a_n \neq 0$, then $f(n) = \infty$. The $U(p,n;A \otimes C)$, therefore, determine $A \otimes C$ up to elementary equivalence unless $Tf(p;C) = 0$, but then $D(p;A \otimes C) = D(p;\oplus_{a_d} Z(p^\infty) \otimes \oplus_{c_t} Z) = 0$.

Now suppose A is of bounded p-length and $D(p;A) = 0$. By a **similar** argument as above $U(p,n;A \otimes C) = f(n) + a_n Tf(p;C)$ and clearly $D(p;A \otimes C) = 0$, if C has unbounded p-length. If both A and C are of bounded p-length, it is easy to see how to obtain the S-invariants of $A \otimes C$.

For (b) we shew that the S-invariants of Hom(A,C) are determined by those of A and C. Since Hom(A,C) is a reduced p-adic group, it suffices to obtain $U(p,n;Hom(A,C))$ for all $n \in \omega$ and if Hom(A,C) is of bounded p-length, to obtain $Tf(p;Hom(A,C))$.

It is not difficult to see that we can assume that C is a p-group and that in particular:

$A = A_r \oplus \oplus_{a_d} Z(p^\infty)$ and $C = C_r \oplus \oplus_{c_d} Z(p^\infty)$, where $A_r$ and $C_r$ are reduced.

If A is of bounded p-length, then $a_d = D(p;A)$;

if C is of bounded p-length, then $c_d = D(p;C)$.

Now $U(p,n;Hom(A,C) = U(p,n;Hom(A_r,C_r) + U(p,n;Hom(A,\oplus_{c_d} Z(p^\infty)))$

$= f(n) + a_n c_d$ (by lemma 6.2).

By lemma 6.1, if Hom(A,C) is of bounded p-length, then

$Tf(p;Hom(A,C)) = Tf(p;Hom(\oplus_{a_d} Z(p^\infty),\oplus_{c_d} Z(p^\infty)) = a_d c_d$.

An exactly analagous argument to that used in the proof of (a) gives the S-invariants of Hom(A,C). □

Theorem 6.5

Let A be a p-group of bounded p-length such that $D(p;A) \neq 0$ and let C be a group of unbounded p-length. Then we can find a group C* such that $C \equiv C^*$, but $A \otimes C \not\cong A \otimes C^*$ and $Hom(A,C) \not\cong Hom(A,C^*)$.

Proof

For convenience we assume that C is p-adic. Let D be a p-basic subgroup of C. Let $D^* = T(D) \oplus \oplus_{c_t^*} Q_p$, where $c_t^* \neq c_t$ and let $C^* = D^* \oplus \oplus_{c_d^*} Z(p^\infty)$, where $c_d^* \neq c_d$.

Then $C \equiv C^*$, but $D(p;A \otimes C) = a_d c_t \neq a_d c_t^* = D(p;A \otimes C^*)$ and $Tf(p;\mathrm{Hom}(A,C)) = a_d c_d \neq a_d c_d^* = Tf(p;\mathrm{Hom}(A,C^*))$. □

Lemma 6.6

(i) If $A$ is a p-group and $C$ is an arbitrary group, then $\mathrm{Ext}(A,C) \equiv \mathrm{Ext}(A,D)$, where $D$ is a p-basic subgroup of $C$.

(ii) If $A$ is a p-group and $C = \oplus_k Z$, then

$\mathrm{Ext}(A,C) \cong \mathrm{Hom}(A,\oplus_k Z(p^\infty))$

(iii) If $C$ is a reduced p-group, then $\mathrm{Ext}(Z(p^\infty),C) \equiv C$.

Proof

(i) is lemma 4.2 together with page 246 exercise 1.

(ii) is page 224 theorem 52.3.

(iii) follows from page 237 lemma 55.1. □

Theorem 6.7

Let $A$, $A'$ be p-groups and $C$, $C'$ arbitrary groups such that $A \equiv A'$ and $C \equiv C'$. Then (a) $\mathrm{Tor}(A,C) \equiv \mathrm{Tor}(A',C')$ and (b) $\mathrm{Ext}(A,C) \equiv \mathrm{Ext}(A',C')$.

Proof

(a) clearly follows from the results of chapter 3.

For (b) we shew that the S-invariants of $\mathrm{Ext}(A,C)$ are determined by those of $A$ and $C$. Since $\mathrm{Ext}(A,C)$ is a reduced p-adic group it suffices to obtain $U(p,n;\mathrm{Ext}(A,C))$ for all $n \in \omega$ and if $\mathrm{Ext}(A,C)$ is of bounded p-length, to obtain $Tf(p;\mathrm{Ext}(A,C))$.

Let $D$ be a p-basic subgroup of $C$ and assume

$A = A_r \oplus \oplus_{a_d} Z(p^\infty)$ and $D = T(D) \oplus \oplus_{c_t} Z$, where $A_r$ is reduced.

If A is of bounded p-length, $a_d = D(p;A)$;

if C is of bounded p-length, $c_t = Tf(p;C)$.

$$U(p,n;Ext(A,C)) = U(p,n;Ext(A_r,T(D))) + U(p,n;Ext(A_r,\oplus_{c_t}Z)) + U(p,n;Ext(\oplus_{a_d}Z(p^\infty),T(D)))$$

$$= f(n) + a_n c_t + c_n a_d$$

It is easy to see, bearing in mind the properties of $f(n)$ that if either A or C is of unbounded p-length, then

$U(p,n;Ext(A,C)) = f(n) + a_n Tf(p;C) + c_n D(p;A)$.

Ext(A,C) can only be of bounded p-length in this case if either $Tf(p;C)$ or $D(p;A)$ is zero, but then

$Tf(p;Ext(A,C)) = Tf(p;Ext(\oplus_{a_d}Z(p^\infty),\oplus_{c_t}Z)) = a_d c_t = 0$.

If both A and C are of bounded p-length, it is easy to see how to obtain the S-invariants of Ext(A,C). □

Appendix

## Some Generalizations

Whenever one investigates properties of Abelian groups, the question always arises: for modules over what sort of rings do the properties still hold? We have essentially considered three different sorts of properties —— elementary properties of groups, infinitary properties of torsion groups and the way these two interrelate. We shew how to extend our results, with suitable modifications to modules over Dedekind rings. For the definition of a Dedekind ring and its simple properties, we refer to (2) page 161 and the references given there.

In (2) a set of invariants are defined which determine a module over a Dedekind ring up to elementary equivalence. Let R be a Dedekind ring. Breaking with tradition, we let A rather than M denote a module over R. We define the $S_R$-invariants as follows:

For each prime ideal P of R and each $n \in \omega$, we let

$U(P,n;A) = \dim(P^n A[P]/P^{n+1} A[P])$ if this is finite

$= \infty$ otherwise

$Tf(P;A) = \lim_{n\to\infty} \dim(P^n A/P^{n+1} A)$ if this is finite

$= \infty$ otherwise

$D(P;A) = \lim_{n\to\infty} \dim(P^n A[P])$ if this is finite

$= \infty$ otherwise.

Now for each prime ideal P of R such that R/P is finite, we put $U^*(P,n;A) = U(P,n;A)$, $Tf^*(P;A) = Tf(P;A)$ and $D^*(P;A) = D(P;A)$.

For each prime ideal P of R such that R/P is infinite, we put:

$$U^*(P,n;A) = 0, \text{ if } U(P,n;A) = 0$$
$$= \infty \text{ otherwise}$$
$$Tf^*(P;A) = 0, \text{ if } Tf(P;A) = 0$$
$$= \infty \text{ otherwise}$$
$$D^*(P;A) = 0, \text{ if } D(P;A) = 0$$
$$= \infty \text{ otherwise}$$

Finally we define

$$\mathrm{Exp}(A) = 0, \text{ if A is of bounded order}$$
$$= \infty \text{ otherwise.}$$

It is proved in (2) that the $S_R$-invariants, $U^*(P,n;A)$, $Tf^*(P;A)$, $D^*(P;A)$ and $\mathrm{Exp}(A)$ determine A up to elementary equivalence.

We assume a knowledge of what is meant by localization (see e.g. (5) page 36). It is noted in (2), that $Tf^*(P;A) = Tf^*(P;A_P)$, with similar results for the other $S_R$-invariants, where $A_P$, the localization of A to P, is a module over the principal ideal domain (in fact, discrete valuation domain) $R_P$. Since $R_P$ is a principal ideal domain, we can say, for example, $Tf^*(P;A_P) = Tf^*(p;A_P)$ for some non-zero $p \in P$.

Also, given an $R_p$-module $A_p$, we can define a basic submodule $B_p$ of $A_p$ which has analogous properties to those of a p-basic subgroup of a p-adic group, (not quite the same since a p-basic subgroup contains copies of Z rather than copies of $Q_p$ or $J_p$). This follows from (6) page 51, lemma 21, because $R_p$ is a discrete valuation ring.
We now assert that all our results on elementary properties of groups go over to modules over Dedekind domains. Where P is a prime ideal of a Dedekind ring R such that R/P is finite, the proofs are similar to the Abelian group case. Where R/P is infinite the proofs are even simpler, by the definition of the $S_R$-invariants.

It is clear immediately that the results of chapters 0 and 1 go over. For chapter 2, it suffices to note the following facts:

(i) The tensor product of torsion-free modules over a Dedekind ring is torsion-free (see page 274, notes), and

(ii) A torsion module A over a Dedekind ring R is isomorphic to the direct sum $\bigoplus_p A_p$, where the sum is taken over all prime ideals (see page 70, notes).

These two facts together with the preceding remarks serve to extend all our results on elementary properties of groups to modules over Dedekind rings.

We now turn our attention to infinitary properties of torsion groups. We note that it follows from the 'blanket assertion' on page 36 of (6), that Ulm's theorem holds for countably generated torsion modules over principal ideal domains. By our fact (ii) we can replace the words principal ideal domain by Dedekind ring. Sections 2 and 3 of (1) are a generalization of Ulm's theorem and the only place where their proofs come unstuck in the case of Dedekind domains is the question of definability in the languages $L_{\kappa\omega}$ Suppose the ring R that we are considering has cardinality $\lambda$. Then the results of sections 2 and 3 of (1) go over to torsion modules over R for the languages $L_{\kappa\omega}$, where $\kappa > \lambda$.

We note too that for $\kappa > \lambda$, T(A) is a definable subset of a module A and if A is a torsion module, then $A_p$ is a definable subset of A, in the language $L_{\kappa\omega}$. Lemma 3.8 also works for modules over Dedekind rings as can be straightforwardly checked and so our results of chapter 3 for infinitary languages go over, with the modification that we only consider languages $L_{\kappa\omega}$ with $\kappa > \lambda$.

The results interrelating elementary and infinitary properties do not go over in general, since by the definition of the $S_R$-invariants the conditions on two modules to be elementarily equivalent are weaker than in the Abelian group case, while the conditions for infinitary equivalence are as strong; thus, for example one might have two modules without elements of infinite height that are elementarily equivalent, but not $L_{\kappa\omega}$-equivalent.

The study of the elementary properties of Abelian groups suggests the following two questions, one specific and one more general. The specific question is how far do these results extend to modules over general rings; we have seen they extend as far as Dedekind rings. The other question is more in the nature of a program. Find a class of algebraic structures whose elementary theories can be determined by algebraic invariants. By purely algebraic means find out about the elementary properties of this class of structures.

One would clearly like to know something about the infinitary properties of torsion-free groups, however torsion-free groups - even countable ones - have notoriously pathological properties. Here is a result of Hodges (see (4)):

If A, A', C, C' are arbitrary groups such that

$A \equiv_{\infty\omega} A'$ and $C \equiv_{\infty\omega} C'$, then $A \otimes C \equiv_{\omega_1\omega} A' \otimes C'$.

It is an open question as to whether this can be improved in the obvious way.

One interesting fact brought out more by the extension to Dedekind rings than by the Abelian group case is how for sufficiently large $\kappa$ (in the case of a countable ring, all uncountable $\kappa$), the languages $L_{\kappa\omega}$ are in a sense similar and are very different from $L_{\omega\omega}$.

References

(1) J.Barwise and P.Eklof

Infinitary Properties of Abelian Torsion Groups

Annals of Math. Logic 1970

(2) P.Eklof and E.Fisher

The Elementary Theory of Abelian Groups

Annals of Math. Logic 1972

(3) L.Fuchs

Infinite Abelian Groups Volume I

Academic Press 1970

(4) W.Hodges

Infinitary Definability and Construction by Generators and Relations (Preprint)

(5) J.Knight

Commutative Algebra

Cambridge University Press 1970

(6) I.Kaplansky

Infinite Abelian Groups (revised edition)

Ann Arbor Press 1968

(7) W.Szmielew

Elementary Properties of Abelian Groups

Fund. Math. 41 1955

The above are all explicitly referred to in the text. The permutation from the natural ordering was due to a sad miscalculation while typing the text.