AUTOMORPHISMS OF BOOLEAN-VALUED MODELS OF SET-THEORY

by

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ABSTRACT

This thesis is concerned with models $\mathcal{M}$ of ZF that admit automorphisms of order greater than 1.

We obtain such models using Boolean-valued models.

Starting with a fixed ω-non-standard countable $\mathcal{M}$, and considering the algebra $\mathcal{B} \in M$ whose universe is $B = RO(X^I)$ $(X, I \in M)$, we construct

- a normal filter Γ of subgroups of a group of automorphisms of $Aut(\mathcal{B})$,
- the Γ-stable subalgebra $\mathcal{B}^\Gamma$ of $\mathcal{B}$,
- an automorphism π of the replica $\underset{\sim}{\mathcal{B}}^\Gamma$ of $\mathcal{B}^\Gamma$ and
- an ultrafilter U that in a natural sense is generic in $\underset{\sim}{\mathcal{B}}^\Gamma$,

so that π induces an automorphism of $\mathcal{M}^\Gamma/U$.

Part of the construction is quite general and applies to any $B = RO(X^I)$. (Chapters I-IV.)

In Chapter I, by simulating the construction of $B = RO(X^I)$ outside the model, we obtain a Boolean-algebra that is isomorphic to $\underset{\sim}{\mathcal{B}}$.

In Chapter II we list some known connections between generic ultrafilters and models of ZF which hold when $\mathcal{M}$ is non-standard and $\mathcal{B}$ is replaced by $\underset{\sim}{\mathcal{B}}$.

We introduce the concept of $\mathcal{M}$-standardness.

In Chapter III the concepts of 'extendability', of 'almost-genericity' and of 'locally-expressible' permutations and automorphisms are introduced.

A generalised version of the "$\hat{x}$'s": $\hat{x}_b = \{<\hat{y}_b, b>: y \in x\}$, is given $(x \in M, b \in B)$. Some of their properties are examined.

It is shown that the condition $\pi[U] = U$ (*) is necessary and sufficient in order to induce automorphisms in $\mathcal{M}^\Gamma/U$, and that extendability constitutes a sufficient condition in order to obtain π, U satisfying (*). Such π, U are constructed simultaneously.

In Chapter IV we construct automorphisms of two symmetric

Boolean-valued submodels of $\mathcal{M}^B$ via locally expressible permutations π ($\notin$ M) of the extension of I.

If π is locally-expressible, formulae of the form $\phi(\tilde{\pi}x_1,\ldots,\tilde{\pi}x_n)$, $(x_1,\ldots,x_n \in M\ ,\ \pi \notin M)$, can be considered as formulae of the language of M .

In Chapter V, we consider the $\mathcal{M}^\Gamma$'s introduced previously with $B = RO(2^{\omega\times\omega\times(\kappa+1)})$, κ an ω-non-standard number in $\mathcal{M}$. Results from earlier chapters lead in each case to automorphisms $\tilde{\pi}$ of $\mathcal{M}^\Gamma$ and generic ultrafilters U, so that $\tilde{\pi}$ induces an automorphism of $\mathcal{M}^\Gamma/U$.

To Sebastian, my son.

CONTENTS

Introduction and remarks on notation and basic assumptions.

Remarks on notation and basic assumptions.

We shall assume familiarity with the notations, concepts and known results concerning Boolean-valued models of set theory, for which we follow J. Bell [3], with the following slight differences in notation.

- $\mathcal{M} = \langle M,E \rangle$ will denote a model of set theory with universe M, and E the membership relation interpreted in $\mathcal{M}$. Similarly with $\mathcal{M}' = \langle M',E' \rangle$.
- $\mathcal{B} = \langle B, \vee, \wedge, *, 0_B, 1_B \rangle$ will denote a Boolean algebra whose universe is B. $\vee$, $\wedge$, $*$, 0_B and 1_B will have the usual meanings.

With this notation, the $\mathcal{B}$-extension of $\mathcal{M}$ should be denoted by $\mathcal{M}^{\mathcal{B}}$; however, for the sake of simplicity, it will be denoted by $\mathcal{M}^B$.

Accordingly, M^B will be used to denote the universe of $\mathcal{M}^B$.

Let Γ be a normal filter of subgroups of a group of automorphisms of $\mathcal{B}$. Then M^Γ denotes the class of all elements of M^B whose stabilizers (definition 26, p.22) hereditarily belong to Γ, and $\mathcal{M}^\Gamma$ denotes the (Boolean-valued symmetric) submodel of $\mathcal{M}^B$ with universe M^Γ.

As in [3], by 'formula' or 'sentence' we mean, respectively, a formula or sentence of the language $\mathcal{L}$ of set theory; that is, a first order language with equality and the binary predicate symbol $\in$.

We recall that the language of $\mathcal{M}^B$, $\mathcal{L}^B_M$, is the first order language obtained from $\mathcal{L}$ by adding a name for each element of M^B, while the language of $\mathcal{M}^\Gamma$, $\mathcal{L}^\Gamma_M$, is the sublanguage of $\mathcal{L}^B_M$ obtained by removing all names not denoting elements of M^Γ.

For convenience we identify each element of M^B with its name in $\mathcal{L}^B_M$.

If ϕ is a sentence of $\mathcal{L}^B_M$, its Boolean truth value (or, simply, its Boolean value) in $\mathcal{M}^B$ will be denoted by $\|\phi\|$. If ϕ is a sentence of

$\mathcal{L}^{\Gamma}_{M}$, its Boolean truth value in $\mathcal{M}^{\Gamma}$ will also be denoted by $||\phi||$. No confusion or ambiguity will arise from these abuses of notations.

We shall also assume familiarity with axiomatic set theory and forcing. The references include J. Bell and M. Machover [4], P. Cohen [5] and G. Takeuti and W.M. Zaring [14], [15].

Introduction.

In 1974 Cohen [6] showed how to construct a model of ZF admitting an automorphism of order 2 using the notion of forcing in a non-standard model of set theory (cf. also, Anapolitanos [1]).

The existence of such models directly implies the independence of the Axiom of Choice. (Cohen [6], p.326 and Anapolitanos [1], p.31.)

Observing that no standard model of ZF has non-trivial automorphisms, Cohen [6] starts with a countable non-standard model $\mathcal{M}$, and by means of a combinatorial technique, obtains a complete sequence P of forcing conditions together with a rank preserving permutation π of the generic elements, of order 2 and such that for each $p \in P$ and for any generic x,y

$$(x\delta y) \in p \leftrightarrow (\pi x \, \delta \, \pi y) \in p, \text{ with } \delta \in \{\varepsilon, \notin\}. \qquad (*)$$

The construction of these π and P is the key point in the method, (Cohen [6], pp.327-328).

π induces a permutation $\tilde{\pi}$ of the resulting model $\mathcal{M}'$ in a natural way.

Cohen's combinatorial technique can be generalised to obtain a model with an automorphism of any given order.

It is to be observed that $\tilde{\pi}$ does not belong to $\mathcal{M}'$, and is not even a definable class in $\mathcal{M}'$. (This is also true of π.) For, if ϕ is a formula in two variables of the language of ZF such that

$$\tilde{\pi}(x) = y \leftrightarrow \phi(x,y), \text{ for every } x,y,$$

then one proves that ZF $\vdash \neg(\exists x)(\exists y)(\phi(x,y) \wedge x \neq y)$ (induction on rank).

It is also to be observed that in this particular problem, the interest lies in the permutation π rather than in the generic elements themselves.

These two observations are to be kept in mind when stating the Boolean-valued approach to the problem.

The aim of this thesis is to construct a model of ZF with an automorphism $\tilde{\pi}$ of any given order $N > 1$ (and therefore, with automorphisms $\tilde{\pi}_1,\ldots,\tilde{\pi}_\alpha$, of any given orders $N_1,\ldots,N_\alpha$, $\alpha \in \omega$), within the framework of the Boolean-valued models.

The natural Boolean-valued counterpart to the problem is as follows:

given an ω-non-standard model $\mathcal{M}$ of ZF, to find sets $I,X \in M$, and to construct the following

- (inside $\mathcal{M}$), the complete Boolean algebra $\mathcal{B}$ of the regular open sets $RO(X^I)$, where X^I is the product topological space with the discrete topology on X.
- (inside $\mathcal{M}$), a normal filter Γ of subgroups of a group of automorphisms of $\mathcal{B}$,
- (inside $\mathcal{M}$), the Γ-stable subalgebra $\mathcal{B}^\Gamma$ of $\mathcal{B}$ (definition 31),
- (outside $\mathcal{M}$), an automorphism π of the replica $\underset{\sim}{\mathcal{B}}^\Gamma$ of $\mathcal{B}^\Gamma$, (see definition 1 and corollary 3), via a permutation of the extension of I, and
- (outside $\mathcal{M}$), a generic ultrafilter U in $\mathcal{B}^\Gamma$, so that π induces an automorphism of $\mathcal{M}^\Gamma/U$, of order $N > 1$.

The necessity for considering $\mathcal{B}^\Gamma$ instead of $\mathcal{B}$ will become apparent in III.1.

As we are working with a non-standard model, Mostowski's collapsing lemma cannot be applied here in order to obtain $\mathcal{M}^\Gamma[U]$. Therefore the process of the construction must stop with $\mathcal{M}^\Gamma/U$.

Another version of a Boolean-valued approach to the problem is discussed in chapter III.1, (p.26).

As in the forcing version, $\tilde{\pi}$ (or π) is a non-definable class in $\mathcal{M}^{\Gamma}/U$.

On the other hand, as $\mathcal{M}$ is non-standard, $\mathcal{B}$ does not have to be a Boolean algebra and, indeed, not even a structure in the sense of the universe of sets (Y. Suzuki and G. Wilmers 13 , p.11).

These comments force us to focus attention on automorphisms of $\tilde{\mathcal{B}}$ rather than of $\mathcal{B}$.

Let $\mathrm{Form}_n(\mathcal{L})$ be the class of formulae of $\mathcal{L}$ in n variables. Although $\tilde{\pi}$ is non-definable in $\mathcal{M}$, since we will be interested in computing their Boolean values, we will need to regard expressions of the form $\phi(\tilde{\pi}x_1,\ldots,\tilde{\pi}x_n)$, with $\phi \in \mathrm{Form}_n(\mathcal{L})$, and $x_1,\ldots,x_n \in M^{\Gamma}$, as sentences of the language of $\mathcal{M}^{\Gamma}$. The concept of 'locally expressible' automorphism is introduced for this purpose.

That $\tilde{\pi}$ is locally expressible means that for each $x_1,\ldots,x_n \in M^{\Gamma}$, there is in M an automorphism σ whose effect on $x_1,\ldots,x_n$ is that of $\tilde{\pi}$; thus, σ 'represents' $\tilde{\pi}$ inside M for these particular $x_1,\ldots,x_n$.

It is found (proposition 49, p.45) that the condition $\pi[U] = U$ is necessary and sufficient for π to induce an automorphism of $\mathcal{M}^{\Gamma}/U$. This suggests a construction that keeps synchronised control on π and U.

An additional complication is introduced by the requirement that π must respect the forcing conditions in the sense of (*) above. At this point, Cohen's combinatorial technique plays an essential role in the construction (definition 67 and proposition 68, pp. 64,65).

The concept of 'extendability' (definition 39, p.32) constitutes a sufficient condition in order to get π, U as above, (proposition 44, p.35). It is also necessary in the weak sense of proposition 43 (p.36). Two different definitions of $\mathcal{M}^{\Gamma}$ lead to two different constructions.

The work is set up in quite a general frame when going from $\mathcal{M}$ to

$\mathcal{M}^B$ and $\mathcal{M}^\Gamma$, (chapters I-IV), and we need to particularise only in chapter V, where we take specific X, I, Γ.

Due to the combinatorics involved in this work we will have to make abundant use of sub and superindices. This makes it inconvenient to use the standard notations, '$t^{(\mathcal{M})}$' and '$\phi^{(\mathcal{M})}$' to represent the term 't' and the formula 'ϕ' as relativized to $\mathcal{M}$. Instead, we will refer to set-theoretical concepts relativized to $\mathcal{M}$ by means of the expressions 'in the sense of $\mathcal{M}$', 'in $\mathcal{M}$', or by means of the prefix '$\mathcal{M}$__', where '__' represents the concept referred to in each case. However, for the sake of clarity we will not adhere rigorously to the application of this: sometimes the difference between work carried out inside and outside $\mathcal{M}$ will not be made explicit; we are confident that the distinction will become clear from the context.

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I

The replica of a Boolean algebra

1. Definition

(i) Let $\mathfrak{A}$ be an $\mathfrak{M}$-structure. I.e., for some A, R_i, f_j, c_k, where the i's, j's and k's take values from given $\mathfrak{M}$-sets of subindices, we have

$\mathfrak{M} \models (\mathfrak{A} = \langle A, R_i, f_j, c_k \rangle_{i,j,k}$

is a structure whose universe, relations, functions and distinguished elements are A, R_i, f_j, and c_k, respectively).

Define

$$\underset{\sim}{A} = \{x \in M: \mathfrak{M} \models x \in A\}$$

$$\underset{\sim}{R} = \{\langle x_1, \ldots, x_n \rangle: \mathfrak{M} \models \langle x_1, \ldots, x_n \rangle \in R\}$$

$$\underset{\sim}{f} = \{\langle x_1, \ldots, x_n, y \rangle: \mathfrak{M} \models y = f\langle x_1, \ldots, x_n \rangle\}$$

$$\underset{\sim}{\mathfrak{A}} = \langle \underset{\sim}{A}, \underset{\sim}{R}_i, \underset{\sim}{f}_j, c_k \rangle_{i,j,k}.$$

$\underset{\sim}{\mathfrak{A}}$ is called the replica of $\mathfrak{A}$, and it is a V-structure, (a structure in the sense of the universe of sets).

(ii) For each $x \in M$, the extension of x is

$$x_E = \{y \in M: \mathfrak{M} \models y \in x\}.$$

2. Proposition. Let $\phi(v_1, \ldots, v_n)$ be a formula of the language of $\underset{\sim}{\mathfrak{A}}$, and $a_1, \ldots, a_n \in \underset{\sim}{A}$.

Then

$$\mathcal{M} \models \mathrm{Sat}(\mathfrak{A}, \ulcorner\phi\urcorner, \langle a_1,\ldots,a_n\rangle) \leftrightarrow \underset{\sim}{\mathfrak{A}} \models \phi[a_1,\ldots,a_n]$$

Proof. Cf. G. Wilmers [16], p.viii. □

3. Corollary. Let $\mathcal{B}$ be an $\mathcal{M}$-(cBa), (complete Boolean algebra).

Then

$$\underset{\sim}{\mathcal{B}} = \langle B, \underset{\sim}{\wedge}, \underset{\sim}{\vee}, (\cdot)\underset{\sim}{*}, 0_B, 1_B\rangle$$

is a (V-) Boolean algebra. □

From now onwards, $\mathcal{M}$ and $\mathcal{B}$ will have the meanings described above, unless otherwise stated.

4. Definition

$$\underset{\sim}{P}(\underset{\sim}{B}) = \{x_E : \mathcal{M} \models x \subseteq B\}.$$

We have $\underset{\sim}{P}(\underset{\sim}{B}) \subseteq P(\underset{\sim}{B})$, but, since the notion of power set is not absolute, the equality is not expected to hold.

5. Definition. Let X, x, y $\in$ M be such that

$$\mathcal{M} \models X \subseteq B \quad (\text{i.e. } X_E \in \underset{\sim}{P}(\underset{\sim}{B}))$$

$$\mathcal{M} \models x, y \in B \quad (\text{i.e. } x, y \in \underset{\sim}{B} = B_E).$$

Define

(i) $\underset{\sim}{\vee}X_E = y \leftrightarrow \mathcal{M} \models y = \vee X.$

(ii) $\underset{\sim}{\wedge}X_E = y \leftrightarrow \mathcal{M} \models y = \wedge X.$

(iii) $x \underset{\sim}{\leq} y \leftrightarrow \mathcal{M} \models x \leq y.$

6. Proposition. Let x, y $\in \underset{\sim}{B} = B_E$, and let A $\in$ M be such that

$A_E \in \underset{\sim}{P}(\underset{\sim}{B})$.

Then

(i) $x \underset{\sim}{\leq} y \leftrightarrow x \underset{\sim}{\wedge} y = x \leftrightarrow x \underset{\sim}{\vee} y = y$.

(ii) $\underset{\sim}{\vee} A_E = \text{Sup } A_E$ (with respect to $\underset{\sim}{\leq}$).

(iii) $\underset{\sim}{\wedge} A_E = \text{Inf } A_E$ (with respect to $\underset{\sim}{\leq}$). □

Although $\underset{\sim}{\mathcal{B}}$ is not complete, it is $\underset{\sim}{P}(\underset{\sim}{B})$-complete. More suggestively, we can say that $\underset{\sim}{\mathcal{B}}$ is $\mathcal{M}$-complete.

We recall that if $\langle P, \leq \rangle$ is a poset and $\mathcal{B}$ is a complete Boolean algebra in the universe of sets, then (cf. J. Bell [3], Ch.2)

(a) two elements $p, q \in P$ are said to be <u>compatible</u>, 'Comp(p,q)', if there is $r \in P$ such that $r \leq p$ and $r \leq q$,

(b) $Q \subseteq P$ is <u>compatible</u>, 'Comp(Q)', if any two elements of Q are compatible,

(c) $D \subseteq P$ is <u>dense</u> if $(\forall x \in P)(\exists y \in D)(y \leq x)$,

(d) $A \subseteq B$ is <u>dense</u> if $0 \notin A$ and for each $0 \neq b \in B$ there is $a \in A$ such that $a \leq b$,

(e) P is said to be <u>refined</u> if

$$(\forall p, q \in P)[q \nleq p \rightarrow (\exists p' \leq q) \neg \text{Comp}(p, p')],$$

(f) P is refined iff it is order-isomorphic to a dense subset of a cBa,

(g) if e is an order-isomorphism of P onto a dense subset of B we say that $\langle \mathcal{B}, e \rangle$ is a <u>Boolean completion</u> of P and that P is a <u>basis</u> for $\mathcal{B}$,

(h) if $\langle \mathcal{B}, e \rangle$ and $\langle \mathcal{B}', e' \rangle$ are Boolean completions of P, then there is an isomorphism between $\mathcal{B}$ and $\mathcal{B}'$ which interchanges $e[P]$ and $e'[P]$.

We will freely make use of these statements as relativized to $\mathcal{M}$.

In $\mathcal{M}$, let X,I be sets, and let $B = RO(X^I)$.

Define (in $\mathcal{M}$),

(i) $C(I,X) = \{p: (\text{dom } p \subseteq I) \wedge (\text{ran } p \subseteq X) \wedge \text{Fin}(\text{dom } p)\}$. Put $C(I,X) = P$.

(ii) $p \leq q \leftrightarrow p \supseteq q$, for $p,q \in P$.

(iii) $N[P] = \{f \in X^I: p \subseteq f\}$, for $p \in P$,

We know that P is refined, that each N[P] is clopen and, therefore, a regular open set of the topological space X^I, when X is assigned the discrete topology. Also, $\langle RO(X^I), N\rangle$ is a Boolean completion of $C(I,X)$, and the latter is a basis for $RO(X^I)$.

Let us consider the replica $\underset{\sim}{\mathcal{B}}$ of this $\mathcal{M}$-(c B a).

Notice that the definition of $\underset{\sim}{\mathcal{B}}$ given in 1, ignores the process of construction of $\mathcal{B}$ inside $\mathcal{M}$.

We are interested in producing a replica, outside $\mathcal{M}$, of <u>the construction</u> of $B = RO(X^I)$ in $\mathcal{M}$.

This process will lead to a Boolean algebra $\underline{\mathcal{B}}$ that is isomorphic to $\underset{\sim}{\mathcal{B}}$.

7. <u>Definition</u>. Let $\mathcal{B}, X, I, P \in M$ be such that

$\mathcal{M} \models$ ($\mathcal{B}$ is the cBa of the regular open sets of the topological space X^I), and

$\mathcal{M} \models P = C(I,X)$.

For any $A, f, x, y, p \in M$ such that

$\mathcal{M} \models p \in P$,

$\mathcal{M} \models f \in X^I$,

$\mathcal{M} \models x, y \in RO(X^I)$ and

$\mathcal{M} \models A \subseteq B$,

define

(i) $\underline{p} = \{\langle i,x\rangle : \mathcal{M} \models \langle i,x\rangle \in p\}$.

(ii) $\underline{P} = \{\underline{p} \quad : \mathcal{M} \models p \in P\}$.

(iii) $\underline{f} = \{\langle i,x\rangle : \mathcal{M} \models \langle i,x\rangle \in f\}$.

(iv) $\underline{N[p]} = \{\underline{f} : \mathcal{M} \models f \in N[p]\}$.

(v) $\underline{(X^I)} = \{\underline{f} : \mathcal{M} \models f \in X^I\}$.

(vi) $\underline{x} = \{\underline{f} : \mathcal{M} \models f \in x\}$.

(vii) $\underline{B} = \{\underline{x} : \mathcal{M} \models x \in RO(X^I)\}$.

(viii) $0_{\underline{B}} = \{\underline{f} : \mathcal{M} \models f \in 0_B\} = 0$.

(ix) $1_{\underline{B}} = \{\underline{f} : \mathcal{M} \models f \in 1_B\} = \underline{(X^I)}$.

(x) $\underline{x} \underline{\wedge} \underline{y} = \{\underline{f} : \mathcal{M} \models f \in x \wedge y\}$.

(xi) $\underline{x} \underline{\vee} \underline{y} = \{\underline{f} : \mathcal{M} \models f \in x \vee y\}$.

(xii) $\underline{x}^{\underline{*}} = \{\underline{f} : \mathcal{M} \models f \in x^*\}$.

(xiii) $\underline{\mathcal{B}} = \langle \underline{B}, \underline{\wedge}, \underline{\vee}, (\cdot)^{\underline{*}}, 0_{\underline{B}}, 1_{\underline{B}}\rangle$.

(xiv) $\underline{x} \underline{\leq} \underline{y} \longleftrightarrow \mathcal{M} \models x \leq y$.

(xv) $\underline{A} = \{\underline{x} : \mathcal{M} \models x \in A\}$.

(xvi) $\underline{\vee A} = \{\underline{f} : \mathcal{M} \models f \in \vee A\}$.

(xvii) $\underline{\wedge A} = \{\underline{f} : \mathcal{M} \models f \in \wedge A\}$.

8. Proposition. (Hypotheses and notations as in definition 7)

(1) $\underline{P}$ is dense in $\underline{B}$. (2) For every x, y, A we have

(i) $\underline{x} \underline{\wedge} \underline{y} = \underline{x \wedge y}$

(ii) $\underline{x} \underline{\vee} \underline{y} = \underline{x \vee y}$

(iii) $\underline{x}^{\underline{*}} = \underline{(x^*)}$

(iv) $\underline{x} \underline{\leq} \underline{y} \leftrightarrow \underline{x} \underline{\wedge} \underline{y} = \underline{x} \leftrightarrow \underline{x} \underline{\vee} \underline{y} = \underline{y}$

(v) $\underline{\vee A} = \underline{\vee A} = \text{Sup } \underline{A}$ (under "$\underline{\leq}$").

(vi) $\underline{\wedge A} = \underline{\wedge A} = \text{Inf } \underline{A}$ (under "$\underline{\leq}$").

(vii) $\underline{\mathcal{B}}$ is a Boolean algebra.

(viii) $\underline{\mathcal{B}}$ and $\mathcal{B}$ are $\mathcal{M}$-complete isomorphic.

Proof. The proofs are all trivial, and the isomorphism in (viii) is given by

$$\underline{x} \longmapsto x. \qquad \square$$

Obviously we will refer to $\underline{x}$ only when $x \in M$. Sometimes it will be convenient to write '$(x)_{-}$' for '$\underline{x}$' and at times we will use expressions like 'let $\underline{x}$ be such that...' to mean 'let x be such that $\underline{x}$ is such that...', for short. The same applies to x_E.

Proposition 8 gives us considerable freedom to 'jump' into and out of $\mathcal{M}$ while considering processes involving $\underline{\mathcal{B}}$. As everything belonging to $\underline{\mathcal{B}}$ replicates a related element of M, often we will loosely make no distinction between the $\underline{x}$'s and the x's. Instead, if some element $v \in V$ replicates an element of M, we will say that v is <u>expressible</u> in $\mathcal{M}$.

Proposition 8 will allow us to induce automorphisms of $\underset{\sim}{\mathcal{B}}^{\Gamma}$ by means of permutations of I_E. This, in turn, will induce automorphisms of $\mathcal{M}^{\Gamma}$. We know that this is always possible when the permutations of I_E involved are expressible in $\mathcal{M}$.

We shall see that by means of a certain kind of permutations of I_E which are not expressible in $\mathcal{M}$, it is still possible to induce, in a natural way, automorphisms of $\mathcal{M}^{\Gamma}$.

Typical examples of permutations of I_E which are not expressible in $\mathcal{M}$ are the ones whose definitions involve the notions of finiteness or standardness. Also, the ones obtained by means of ultrafilters.

9. <u>Definition</u>. A <u>filter</u> in $\mathcal{B}$ is a non-empty subset F of $\underset{\sim}{B}$ such that for every $x,y \in \underset{\sim}{B}$

(i) $(x \in F) \wedge (x \underset{\sim}{\leq} y \in \underset{\sim}{B}) \rightarrow y \in F$.

(ii) $(x,y \in F) \wedge (z = x \underset{\sim}{\wedge} y) \rightarrow z \in F$.

(iii) $0_B \notin F$.

If, in addition, F satisfies

(iv) $x \in F$ or $x^{\underset{\sim}{*}} \in F$, for every $x \in \underset{\sim}{B}$, then F is called an <u>ultrafilter</u> in $\mathcal{B}$.

10. Proposition

F is an ultrafilter in $\mathcal{B}$ $\leftrightarrow$ F is a $\subseteq$-maximal filter in $\mathcal{B}$. $\square$

11. Definition

(i) Let S be such that

$$x \in S \rightarrow \mathcal{M} \models x \in P(B).$$

Let U be an ultrafilter in $\mathcal{B}$. Then U is said to be S-complete iff for every $x \in S$

$$\underset{\sim}{V} x_E \in U \rightarrow x_E \cap U \neq 0$$

(ii) U is said to be M-generic, or simply generic iff it is $\underset{\sim}{P}(\underset{\sim}{B})$-complete. (i.e. iff for every x such that

$$\mathcal{M} \models x \subseteq B,$$

we have $\underset{\sim}{V} x_E \in U \rightarrow x_E \cap U \neq 0$).

II

Generic ultrafilters and models of set theory

In this chapter we list some known connections between generic ultrafilters and models of Set-theory which hold, mutatis mutandis, when we consider $\underset{\sim}{\mathcal{B}}$ instead of $\mathcal{B}$.

With variations, the proofs go along similar lines to the usual ones.

As the ground models $\mathcal{M}$ we have in mind are non-standard, Mostowski's Collapsing Lemma does not apply here. Hence, our final step in the process that leads to the supermodel of $\mathcal{M}$, will have to be $\mathcal{M}^B/U$ instead of $\mathcal{M}[U]$.

We recall that Boolean values are assigned to the sentences of $\mathcal{L}^B_M$ as follows.

Let σ,τ be sentences of $\mathcal{L}^B_M$, $\phi(u)$ be a formula of $\mathcal{L}^B_M$ and $x,y \in M^B$. Then

$$||\sigma \wedge \tau|| = ||\sigma|| \wedge ||\tau||,$$

$$||\neg\sigma|| = ||\sigma||^*,$$

$$||\exists u\phi(u)|| = \bigvee_{x\in M^B} ||\phi(x)||,$$

$$||x=y|| = \bigwedge_{z\in \mathrm{dom}\, y} (y(z) \rightarrow ||z \in x||) \wedge \bigwedge_{z\in \mathrm{dom}\, x} (x(z) \rightarrow ||z\in y||),$$

$$||x \in y|| = \bigvee_{z\in \mathrm{dom}\, y} (y(z) \wedge ||z=x||).$$

Cf. remark in p. 24.

12. Definition. Let U be an ultrafilter in $\mathcal{B}$ and let $x,y \in M^B$. Then

(i) $x \sim_u y \leftrightarrow ||x=y|| \in U$.

(ii) $x^u = \{y \in M^B: x \sim_u y\}$.

(iii) $x^u E^u y^u \leftrightarrow ||x \in y|| \in U$.

(iv) $\mathcal{M}^B/U = \langle\{x^u: x \in M^B\}, E^u\rangle$.

13. <u>Proposition</u>. Let $\phi(v_1,\ldots,v_n)$ be a formula with no quantifiers, U be any ultrafilter in $\mathcal{B}$, and $x_1,\ldots,x_n$ an $\mathcal{M}$-(finite sequence) of elements of M^B. Then, if ϕ has standard length

$$\mathcal{M}^B/U \models \phi[x_1^u,\ldots,x_n^u] \leftrightarrow ||\phi(x_1,\ldots,x_n)|| \in U. \quad \square$$

14. <u>Proposition</u>. (Same hypotheses)

(i) If $\phi(v_1,\ldots,v_n)$ is a formula of the form $\exists a\,\psi$, where ψ has no quantifiers, then

$$\mathcal{M}^B/U \models \phi[x_1^u,\ldots,x_n^u] \rightarrow ||\phi(x_1,\ldots,x_n)|| \in U.$$

(ii) If $\phi(v_1,\ldots,v_n)$ is a formula of the form $\forall a\,\psi$, where ψ has no quantifiers,

$$||\phi(x_1,\ldots,x_n)|| \in U \rightarrow \mathcal{M}^B/U \models \phi[x_1^u,\ldots,x_n^u]. \quad \square$$

15. <u>Proposition</u>. (Same hypotheses)

(i) If $\phi(v_1,\ldots,v_n)$ is a Σ_1-formula, then

$$\mathcal{M}^B/U \models \phi[x_1^u,\ldots,x_n^u] \rightarrow ||\phi(x_1,\ldots,x_n)|| \in U.$$

(ii) If $\phi(v_1,\ldots,v_n)$ is a Π_1-formula, then

$$||\phi(x_1,\ldots,x_n)|| \in U \rightarrow \mathcal{M}^B/U \models \phi[x_1^u,\ldots,x_n^u]. \quad \square$$

16. <u>Proposition</u>. (Same hypotheses)

If U is $\mathcal{M}$-generic and $\phi(v_1,\ldots,v_n)$ is any formula of standard length, then $\mathcal{M}^B/U \models \phi[x_1^u,\ldots,x_n^u] \leftrightarrow ||\phi(x_1,\ldots,x_n)|| \in U$,

for any $\mathcal{M}$-finite $\mathcal{M}$-sequence $x_1,\ldots,x_n$ of elements of M^B. $\square$

17. <u>Corollary</u>. If U is M-generic, and σ is any sentence of standard length, then

$$\mathcal{M}^B \models \sigma \rightarrow \mathcal{M}^B/U \models \sigma.$$

In particular,

$$\mathcal{M}^B \models ZF \rightarrow \mathcal{M}^B/U \models ZF.$$

$$\mathcal{M}^B \models ZFC \rightarrow \mathcal{M}^B/U \models ZFC. \qquad \square$$

We know that

$$\mathcal{M} \models \mathrm{Ord}(\alpha) \rightarrow \mathcal{M}^B \models \mathrm{Ord}[\hat{\alpha}]$$

$$\rightarrow \mathcal{M}^B/U \models \mathrm{Ord}[\hat{\alpha}^u].$$

18. <u>Definition</u>

(i) Let $x \in \mathcal{M}^B/U$ be such that

$$\mathcal{M}^B/U \models \mathrm{Ord}[x].$$

Then we say that x is an <u>$\mathcal{M}$-standard ordinal</u> of $\mathcal{M}^B/U$ if there exists an $\mathcal{M}$-ordinal α such that

$$\mathcal{M}^B/U \models x = \hat{\alpha}^u.$$

(ii) An element $y \in \mathcal{M}^B/U$ is said to be <u>$\mathcal{M}$-standard</u> if its rank in $\mathcal{M}^B/U$ is an $\mathcal{M}$-standard ordinal of $\mathcal{M}^B/U$.

(iii) $\mathcal{M}$ and $\mathcal{M}^B/U$ are said <u>to have the same ordinals</u> if all the ordinals in $\mathcal{M}^B/U$ are $\mathcal{M}$-standard.

(Particularly in non-standard cases, this notion can be described by saying that $\mathcal{M}$ and $\mathcal{M}^B/U$ 'have the same degree of non-standardness').

Now, let $x \in M^B$.

Let $S = \{y\colon (\exists x \in M^B)(\mathcal{M} \models y = \{\|x = \hat{\alpha}\|\colon \mathrm{Ord}(\alpha)\})\}$.

(Then $x \in S \rightarrow \mathcal{M} \models x \subseteq B$.)

19. <u>Definition</u>. Let $A \in M$.

Then A is called an <u>M-partition of unity</u> in $\mathcal{B}$ if

$$\mathfrak{M} \models A \subseteq B,$$

$$\mathfrak{M} \models (\forall a,b \in A)(a \neq b \rightarrow a \wedge b = 0),$$

and $$\mathfrak{M} \models \vee A = 1_B.$$

20. <u>Proposition</u>. The following conditions are equivalent if $\mathfrak{M} \models ZFC$

(i) U is M-generic

(ii) For any M-partition of unity A in B,

$$A_E \cap U \neq 0.$$ □

21. <u>Proposition</u>. Let $\mathfrak{M} \models ZF$.

Consider the following conditions.

(i) U is S-complete.

(ii) $\mathfrak{M}$ and $\mathfrak{M}^B/U$ have the same ordinals.

(iii) U is M-generic.

Then

(iii) → (ii) → (i).

If, in addition, $\mathfrak{M} \models ZFC$, then

(iii) ↔ (ii) ↔ (i). □

22. <u>Proposition</u>. For any ultrafilter U, the function given by

$$x \longmapsto \hat{x}^u,$$

is an embedding of $\langle M,E\rangle$ into $\langle M^B/U,E^u\rangle$. I.e., $\mathfrak{M}$ is isomorphic to a submodel $\mathfrak{M}'$ of $\mathfrak{M}^B/U$. □

Putting

$$M' = \{\hat{x}^u : x \in M\} \subseteq M^B/U$$

and $\mathfrak{M}' = \langle M', E^U/M' \rangle$,

we have $\mathfrak{M} \cong \mathfrak{M}' \subseteq \mathfrak{M}^B/U$.

23. Definition

$$A = \{\langle \hat{x}, x \rangle : x \in B\}$$

Then we have $A \in M^B$, and

24. Corollary. For any ultrafilter U, and any model of ZF (ZFC), $\mathfrak{M}^B/U$ is a model of ZF (ZFC) that includes $\mathfrak{M}'$ and contains A^u. □

Now, let G be a subgroup of Aut($\underset{\sim}{\mathcal{B}}$), (the automorphism group of $\underset{\sim}{\mathcal{B}}$), and let Γ be a normal filter of subgroups of G, (i.e., Γ is a filter, and $\pi \in G$ and $H \in \Gamma \to \pi H \pi^{-1} \in \Gamma$).

25. Definition. Let $x \in M^B$.

Define, in the usual way,

(i) $\mathrm{stab}(x) = \{\sigma \in G: \tilde{\sigma}x = x\}$, (the stabilizer of x).

(ii) $M^\Gamma_\alpha = \{x: \mathrm{Fun}(x) \wedge \mathrm{ran}(x) \subseteq B \wedge \mathrm{stab}(x) \in \Gamma \wedge (\exists \xi < \alpha)(\mathrm{dom}(x) \subseteq M^\Gamma_\xi)$,

where α is an ordinal of M. Put

$$M^\Gamma = \{x : \exists \alpha (x \in M^\Gamma_\alpha)\}.$$

Here we recall the definition of the Boolean value in $\mathfrak{M}^\Gamma$ of a sentence of $\mathcal{L}^\Gamma_M$:

For $x,y \in \mathfrak{M}^\Gamma$, $||x \in y||^\Gamma$ and $||x = y||^\Gamma$ are defined as in page 18, and for sentences σ,τ of $\mathcal{L}^\Gamma_M$, and any formula $\phi(u)$ of $\mathcal{L}^\Gamma_M$,

$$\|\sigma \wedge \tau\|^{\Gamma} = \|\sigma\|^{\Gamma} \wedge \|\tau\|^{\Gamma},$$

$$\|\neg\sigma\|^{\Gamma} = \|\sigma\|,$$

$$\|\exists u\phi(u)\|^{\Gamma} = \bigvee_{x \in M^{\Gamma}} \|\phi(x)\|^{\Gamma}.$$

26. Proposition.

$$\mathcal{M} \models ZF \rightarrow \mathcal{M}^{\Gamma} \models ZF.$$

(In particular, $\mathcal{M} \models ZFC \rightarrow \mathcal{M} \models ZF$). □

27. Definition. Let U be a generic ultrafilter in $\mathcal{B}$.

Put $M^{\Gamma}/U = \{x^{u} : x \in M^{\Gamma}\}$.

Then $M^{\Gamma}/U \subseteq M^{B}/U$.

Set $\mathcal{M}^{\Gamma}/U = \langle M^{\Gamma}/U, E^{u}/(M^{\Gamma}/U)\rangle$.

28. Proposition. Let U be a generic ultrafilter in $\mathcal{B}$, $\phi(v_1,\ldots,v_n)$ be a formula of standard length and $x_1,\ldots,x_n$ be a finite sequence of elements of M^{Γ}.

Then

(i) $\mathcal{M}^{\Gamma}/U \models \phi[x_1^{u},\ldots,x_n^{u}] \leftrightarrow \|\phi(x_1,\ldots,x_n)\| \in U.$

(ii) For any sentence σ

$$\mathcal{M}^{\Gamma} \models \sigma \rightarrow \mathcal{M}^{\Gamma}/U \models \sigma.$$

In particular

$$\mathcal{M}^{\Gamma} \models ZF \rightarrow \mathcal{M}^{\Gamma}/U \models ZF.$$ □

Finally in this chapter, we observe that if U is M-generic, ,

$\mathfrak{M}^B/U$ and $\mathfrak{M}^\Gamma/U$ have the same ordinals, since

(i) $\hat{\alpha} \in M^\Gamma$ for every $\mathfrak{M}$-ordinal α, and

(ii) if $x \in M^B$ is such that $\mathfrak{M}^B/U \models \mathrm{Ord}[x^u]$,

then
$$\|\mathrm{Ord}(x)\| = \bigvee_{\alpha \in \mathrm{ORD}(\mathfrak{M})} \|x = \hat{\alpha}\| \in U.$$

Thus $\|x = \hat{\alpha}\| \in U$ for some $\mathfrak{M}$-ordinal α.

(iii) The same as in (ii), with $x \in M^\Gamma$ instead of $x \in M^B$.

N.B. Regarding the definition of $\|\phi\|$ (p.18), one replicates this process outside $\mathfrak{M}$ in a natural way.

Let us write $\|\phi\|_\sim$ for the Boolean-value of ϕ as computed in M. We define a $\underset{\sim}{\mathcal{B}}$-structure with universe $M^B = \{x \in M: \quad x \in V^B\}$ as follows.

For $x, y \in M^B$, put

$$\|x = y\|^{\underset{\sim}{B}} = \|x = y\|_\sim \quad , \quad \|x \in y\|^{\underset{\sim}{B}} = \|x \in y\|_\sim ,$$

and if ϕ is a standard sentence of the language of M^B, define $\|\phi\|^B$ recursively as follows

$$\|\neg\psi\|^{\underset{\sim}{B}} = (\|\psi\|^{\underset{\sim}{B}})^*$$

$$\|\psi_1 \vee \psi_2\|^{\underset{\sim}{B}} = \|\psi_1\|^{\underset{\sim}{B}} \vee \|\psi_2\|^{\underset{\sim}{B}}$$

$$\|\exists x\, \psi(x)\|^{\underset{\sim}{B}} = \bigvee_{x \in M^B} \|\psi(x)\|^{\underset{\sim}{B}}$$

Then for any standard formula ϕ and $x_1, \ldots, x_n \in M^B$, with n finite, we have $\|\phi\|^{\underset{\sim}{B}} = \|\phi\|_\sim$

Since $1_{\underset{\sim}{B}} = 1_B$, M^B is a $\underset{\sim}{\mathcal{B}}$-structure for which the axioms of ZF are Boolean-valid.

III

The stable subalgebra $\mathcal{B}^{\Gamma}$.

Locally expressible automorphisms of $\mathcal{B}^{\Gamma}$, towards automorphisms of $\mathcal{M}^{\Gamma}$ and $\mathcal{M}^{\Gamma}/U$, where $B = RO(X^{I})$.

Let $\mathcal{M}$ be a model of ZF and $\mathcal{B}$ be an $\mathcal{M}$-(cBa).

29. Definition. For each $x \in M$ and $b \in B$, let

$$\hat{x}_b = \{\langle \hat{y}_b, b\rangle : y \in x\}$$

30. Proposition. For every $y, x \in M$ and every $b, c \in B$

(i) $y \in x \rightarrow \|\hat{y}_b \in \hat{x}_c\| = b \wedge c.$

(ii) $y \notin x \rightarrow \|\hat{y}_b \in \hat{x}_c\| = 0.$

(iii) $y = x \rightarrow \|\hat{y}_b = \hat{x}_c\| = (b^* \vee c) \wedge (b \vee c^*).$

(iv) $y \neq x \rightarrow \|\hat{y}_b = \hat{x}_c\| = (b \vee c)^*.$

Proof. Induction on rank. The induction hypothesis being for all z with $\mathrm{rank}(z) < \mathrm{rank}(x)$,

(a) $\forall y(z \in y \rightarrow \|\hat{z}_b \in \hat{y}_c\| = b \wedge c).$

(b) $\forall y(z \notin y \rightarrow \|\hat{z}_b \in \hat{y}_c\| = 0).$

(c) $\forall y(z = y \rightarrow \|\hat{z}_b = \hat{y}_c\| = (b^* \vee c) \wedge (b \vee c^*)).$

(d) $\forall y(z \neq y \rightarrow \|\hat{z}_b = \hat{y}_c\| = (b \vee c)^*).$

31. Definition. Let $b \in B$, G,Γ as in definition 26.

(i) $\mathrm{stab}(b) = \{\sigma \in G : \sigma(b) = b\}$.

(ii) $B^{\Gamma} = \{b \in B : \mathrm{stab}(b) \in \Gamma\}$.

32. Proposition

(i) $\sigma\hat{x}_b = \hat{x}_{\sigma b}$ for any $b \in B$, $x \in M$, $\sigma \in G$; hence, $\mathrm{stab}(\hat{x}_b) = \mathrm{stab}(b)$.

(ii) Let $b \in B$. Then there are $u,v,u',v' \in M^B$ such that $b = \|u = v\|$ and $b = \|u' \in v'\|$.

(iii) Let $b \in B^{\Gamma}$. Then there are $u,v,u',v' \in M^{\Gamma}$ such that $b = \|u = v\|$ and $b = \|u' \in v'\|$.

Proof. (i) Straightforward induction on rank.

Let $b \in B$, $w,x,y,z \in M$ with $x \neq x$ and $y \in z$. We have

(ii) $\|\hat{w}_{b*} = \hat{x}_{b*}\| = b = \|\hat{y}_b \in \hat{z}_b\|$ (1)

(iii) If $b \in B^{\Gamma}$, using (i) one sees that $\hat{w}_{b*}, \hat{x}_{b*}, \hat{y}_b, \hat{z}_b \in M^{\Gamma}$ and (iii) follows from (i) above. □

33. Proposition. For every $b \in B$ and $\sigma \in G$

(i) $\mathrm{stab}(b)$ is a subgroup of G.

(ii) $\mathrm{stab}(\sigma b) = \sigma\,\mathrm{stab}(b)\sigma^{-1}$. Hence $\sigma[B^{\Gamma}] = B^{\Gamma}$.

(iii) B^{Γ} is a subalgebra of B.

(iv) If $B = RO(X^I)$, then $N[p] \in B^{\Gamma}$ for every $p \in P$.

(v) Put $B^{\mathrm{Sent}} = \{\|\phi\| : \phi \in \mathrm{Sent}(\mathcal{L}^{\Gamma})\}$. Then $B^{\Gamma} = B^{\mathrm{Sent}}$.

(vi) $x(y) \in B^{\Gamma}$ for every $x \in M$ and $y \in \mathrm{dom}\ x$. That is $M^{\Gamma} \subseteq M^{(B^{\Gamma})}$.

Proof. Straightforward. Use 32(iii) to show that $B^{\Gamma} \subseteq B^{\mathrm{Sent}}$ in (v). To show (vi) observe that $\mathrm{stab}(x(y)) \supseteq \mathrm{stab}(x) \cap \mathrm{stab}(y) \in \Gamma$. □

Finally, the following properties of the $\hat{x}_b$'s can be easily verified.

Let $b \in B$ and $x,y \in M$ with $x \neq y$.

(i) $\|\hat{x}_b = \tilde{\pi}\hat{x}_b\| \in U$ for every ultrafilter U with $\pi[U] = U$.

(ii) $b \in U \leftrightarrow \|\hat{y}_b = \tilde{\pi}\hat{x}_b\| \notin U$.

(iii) If $y \in x$, then $b \in U \leftrightarrow \|\hat{y}_b \in \tilde{\pi}\hat{x}_b\| \in U$.

III.1

Locally expressible permutations and automorphisms of $\underset{\sim}{\mathcal{B}}^{\Gamma}$.

We recall that permutations of I lead to automorphisms of $B = RO(X^I)$ as follows.

Let $\sigma: J \xrightarrow{1-1} \sigma[J] \subseteq I$; $\eta: \phi[J] \to (\eta \circ \sigma)[J] \subseteq I$ and $f: J \to X$. Define $\sigma^* f = \{\langle i, f(\sigma^{-1}(i))\rangle : i \in \sigma[J]\}$. Then one easily proves the following

(1) $\sigma^*: X^J \to X^{\sigma[J]}$. (2) $(Id/J)^* = Id/X^J$.

(3) $\eta^* \circ \sigma^* = (\eta \circ \sigma)^*$ (4) $\sigma^* \circ (\sigma^{-1})^* = Id/X^J$

and

$(\sigma^*)^{-1} = (\sigma^{-1})^*$. If $\sigma,\eta \in I!$ and $p \in C(I,X)$ then

(5) $\sigma/\text{dom } p = \eta/\text{dom } p \to \sigma^*(p) = \eta^*(p) \wedge \sigma^*[N[p]] = \eta^*[N[p]]$.

(6) $\sigma^*[N[p]] = N[(\sigma/\text{dom } p)^*(p)]$

(7) $\sigma^*: X^I \to X^I$ is a homeomorphism.

Finally, defining (8) $\sigma^{**}(b) = \sigma^*[b]$ for $b \in B$, we have

(9) $\sigma^{**} \in \text{Aut}(\;)$ and for any $\eta,\sigma \in I!$

(10) $(\eta \circ \sigma)^{**} = \eta^{**} \circ \sigma^{**}$,

(11) $(\sigma^{-1})^{**} \circ \sigma^{**} = \text{Id}/B$ and $(\sigma^{**})^{-1} = (\sigma^{-1})^{**}$.

As usual, we identify σ,σ^* and σ^{**} and write σ for any of them.

Now let $B = RO(X^I)$ and let G be the group of permutations of I. G is identified as a subgroup of $\text{Aut}(\mathcal{B})$. For $i \in I$, let $G_i = \{\sigma \in G: \sigma(i) = i\}$; then G_i is a subgroup of G. Put $G_J = \bigcap_{i\in J} G_i$ for each finite subset J of I and let Γ be the filter of subgroups of G generated by the G_i's; i.e.

$$\Gamma = \{L: L \in \text{Subgr}(G) \wedge (\exists \text{finite } J \subseteq I)(G_J \subseteq L).$$

It is easy to show that Γ is normal.

If $b \in B$ and $\text{stab}(b) \supseteq G_J$, J is called a <u>support</u> of b.

It is straightforward that if J is a support of b and $\sigma_1,\sigma_2 \in G$, then $\sigma_1/J = \sigma_2/J \to \sigma_1(b) = \sigma_2(b)$.

Throughout, B,G,G_J,Γ will have these meanings unless otherwise stated.

34. <u>Definition</u>. Let $\pi \in (I_E)!$. π is said to be <u>finitely locally expressible in $\mathfrak{M}$</u> or, simply, <u>locally expressible</u> if for every $\mathfrak{M}$-finite

$J \subseteq I$ there are $\sigma, \eta \in M$ such that $\models (\mathrm{dom}\sigma = J) \wedge (\mathrm{dom}\eta = J)$,

$$\pi/J_E = \sigma_- \quad \text{and} \quad \pi^{-1}/J_E = \eta_-.$$

Clearly, π is locally expressible iff there are $\sigma, \eta \in M$ such that $\models \sigma, \eta \in I!$, $\pi/J_E = (\sigma/J)_-$ and $\pi^{-1}/J_E = (\eta/J)_-$.

Let us denote by $\mathcal{E}$ the set of locally expressible permuations of I_E.

35. Proposition.

(i) $\mathcal{E}$ is a subgroup of $(I_E)!$ and $\mathcal{E} \supseteq \{\sigma_- : \sigma \in G\}$.

(ii) Let $\pi \in \mathcal{E}$, $b \in B^\Gamma$, $\sigma_1, \sigma_2 \in G$ and J_1, J_2 be $\mathfrak{M}$-finite supports of b such that $(\sigma_1/J_1)_- = \pi/(J_1)_E$ and $(\sigma_2/J_2)_- = \pi/(J_2)_E$. Then $\sigma_1(b) = \sigma_2(b)$.

Proof. (i) Straightforward.

(ii) Put $J = J_1 \cup J_2$. Then J is M-finite and there is $\sigma \in G$ such that $(\sigma/J)_- = \pi/J_E$. It follows that $(\sigma/J_1)_- = \pi/J_E = (\sigma_1/J_1)_-$ and $(\sigma/J_2)_- = \pi/(J_2)_E = (\sigma_2/J_2)_-$. Then $\sigma_1(b) = \sigma(b) = \sigma_2(b)$. □

36. Definition. Let $\pi \in \mathcal{E}$, $p \in P$ and $b \in B^\Gamma$, with $\mathrm{stab}(b) \supseteq G_J$, where J is $\mathfrak{M}$-finite. By definition 34 there are $\pi_p, \pi_b \in M$ such that $\pi_p, \pi_b \in I!$ (in $\mathfrak{M}$), $(\pi_p/\mathrm{dom}\ p)_- = \pi/(\mathrm{dom}\ p)_E$ and $(\pi_b/J)_- = \pi/J_E$.
Define

(i) $\pi^*(p) = (\pi_p/\mathrm{dom})^*(b)$. (ii) $\pi^{**}(b) = \pi_b^{**}(b)$.

From (5) at the beginning of III.1 and proposition 35, this definition is sound and it is easy to verify that for any $\pi \in \mathcal{E}$ we have (1) π^* is an order-isomorphism of P_E, (2) if $p \in P$ (in $\mathfrak{M}$), then $\pi^{**}(N[p]) = N[\pi^*(p)]$, (3) $\pi^{**} \in \mathrm{Aut}(\mathfrak{B}^\Gamma)$ and (4) if π has order N, so have π^* and π^{**}.

If $\sigma \in \mathrm{Aut}(\mathcal{B}^\Gamma)$, we say that σ is locally expressible if $\sigma = \pi^{**}$ for some $\pi \in \mathcal{E}$. We identify π and π^{**} and write π for both.

Observe that any $\pi \in \mathcal{E}$ will not, in general, induce an automorphism of the whole of $\mathcal{B}$, thus we must restrict ourselves to $\mathcal{B}^\Gamma$. On the other hand, as $\mathcal{B}^\Gamma$ is not complete, we do not attempt to work with $M^{\mathbf{B}^\Gamma}$. Instead, proposition 33(v) suggests that M^Γ will be suitable for our purposes. Finally, if U is an $\mathcal{M}$-generic ultrafilter in $\mathcal{B}^\Gamma$, $\mathcal{M}$ and $\mathcal{M}^\Gamma/U$ have the same ordinals; as standard models of ZF do not have non-trivial automorphisms (well known) one sees that $\mathcal{M}$ and $\mathcal{M}^\Gamma/U$ will have to be non-standard.

We will not need that AC hold in the ground model. Therefore we start with a non-standard model $\mathcal{M} = \langle M,E \rangle$ of ZF. (We do not yet require that M be countable.)

Alternatively, we could consider a standard model $\mathcal{M}$ and a non-generic ultrafilter U. However this leads to great difficulties, e.g. (cf. proposition 21 and end of chapter II) we would loose control over the ordinals of $\mathcal{M}^\Gamma/U$ and we will not pursue this alternative further.

Anapolitanos [2] gives a proof based on a <u>standard</u> model. He first establishes a necessary and sufficient syntactic condition for a theory T to have models admitting an automorphism of order N. More specifically, there is a certain class of sentences S_T of the language of T such that T admits an automorphism of order N iff $T \cup S_T$ is consistent. Then, working with a countable standard model, he proceeds to show that $ZF \cup S_{ZF}$ is consistent, by showing that for each $s \in S_{ZF}$, $||s||^\Gamma = 1$, where B and Γ are similar to the ones in chapter V. The proof uses a technique that resembles that of Cohen [6].

III.2

Construction of a locally expressible automorphsim π of $\mathcal{M}^\Gamma$ where $B = RO(X^I)$, together with a generic ultrafilter U such that $\pi[U] = U$.

Let $\langle\Sigma,\leqslant_\Sigma\rangle$, $\langle P,\leqslant_P\rangle \in M$ be $\mathcal{M}$-posets and let $H \subseteq \Sigma_E \times P_E$.

We do not require that H be definable in $\mathcal{M}$.

Define

$$\sigma \leqslant_E \nu \leftrightarrow \mathcal{M} \models \sigma \leqslant_\Sigma \nu \quad \text{for} \quad \sigma,\nu \in \Sigma_E,$$

$$p \leqslant_E q \leftrightarrow \mathcal{M} \models p \leqslant_p \nu \quad \text{for} \quad p,q \in P_E$$

and

$$\langle\sigma,p\rangle \leqslant_H \langle\nu,q\rangle \leftrightarrow (\sigma \leqslant_E \nu) \wedge (p \leqslant_E q) \quad \text{for} \quad \langle\sigma,p\rangle, \langle\nu,q\rangle \in H.$$

H will have this meaning until otherwise stated.

Clearly $\langle\Sigma,\leqslant_E\rangle$, $\langle P,\leqslant_E\rangle$ and $\langle H,\leqslant_H\rangle$ are posets.

We write '$\leqslant$' indistinctly for '$\leqslant_\Sigma$', '$\leqslant_P$', '$\leqslant_E$' and '$\leqslant_H$'.

37. Definition. (Notations as above.)

Let $Q \subseteq P_E$.

Define the $\leqslant$-closure of Q as

$$Q^{\leqslant} = \{p \in P_E\colon (\exists q \in Q)(q \leqslant p)\}.$$

38. Definition. (Notations as above.)

If $G \subseteq \Sigma_E \times P_E$

let $G_\Sigma = proj_0(G)$ and $G_P = proj_1(G)$.

39. Definition. (Notations as above).

Let $\langle\sigma,p\rangle \in H$ and $H \subseteq \Sigma_E \times P_E$.

(i) $\langle\sigma,p\rangle$ is extendable in H or simply extendable, if

$$(\forall q \in P_E)(q \leq p \rightarrow (\exists\langle\sigma_1,p_1\rangle \in H)((\langle\sigma_1,p_1\rangle \leq \langle\sigma,p\rangle) \wedge (p_1 \leq q \leq p))).$$

(ii) H is extendable if all its elements are extendable.

(iii) For each -dense $D \subseteq P$, let

$$Y_D = \{\langle\sigma,p\rangle \in H: (\exists d \in D_E)(p \leq d)\} \text{ and}$$

$$\mathcal{Y} = \{Y_D: \models (D \subseteq P \text{ is dense})\}.$$

Let $G \subseteq H$. G is said to be H-almost-generic. (H-ag) if Comp(G), $G^{\leq} = G$ and G is $\mathcal{Y}$-complete.

We recall that

(a) $G \subseteq P_E$ is $\mathcal{M}$-generic in P, or simply, generic if

(i) $G^{\leq} = G$,

(ii) (in $\mathcal{M}$) Comp(G) and

(iii) G intersects every dense subset of P (in $\mathcal{M}$). More precisely if $\models (D \subseteq P \text{ is dense})$, then $G \cap D_E \neq \emptyset$.

(b) If G is a generic subset of P_E and $\langle P,\leq\rangle$ is a basis for $\mathcal{B}$ in $\mathcal{M}$, then $U = \{x \in B_E: (\exists y \in G)(y \leq x)\}$ is a generic ultrafilter in $\mathcal{B}$ called the generic ultrafilter associated to G. We have $G = U \cap P_E$.

Since $P \subseteq B^\Gamma$, $G \subseteq B^\Gamma$. Put $F = U \cap B^\Gamma$; then F is an ultrafilter in B^{Γ} that respects all 'sups' in B^{Γ}. Conversely, if F is an ultrafilter in B^{Γ} that respects all 'sups' in B^{Γ}, $F^{\leqslant}$ (the $\leqslant$-closure of F in B) is a generic ultrafilter in B. Thus the notion of genericity naturally restricts to B^{Γ}.

40. Definition. An ultrafilter F in B^{Γ} is said to be generic if $F^{\leqslant}$ is generic in B.

41. Proposition. Let H be extendable. Then

(i) Y_D is dense in H for each $\mathcal{M}$-dense $D \subseteq P$

and if $\mathbb{M}$ is countable

(ii) For every $\langle\sigma,p\rangle \in H$ there exists an H-almost generic $G \subseteq H$ that contains $\langle\sigma,p\rangle$;

(iii) for such G, $G_P^{\leqslant}$ is a generic subset of P_E.

Proof. (i) Let $\langle\eta,q\rangle \in H$ and $D \subseteq P$ be dense. Then there exists $q_1 \in D$ with $q_1 \leqslant q$. As H is extendable there is $\langle\sigma,p\rangle \in H$ with $\langle\sigma,p\rangle \leqslant \langle\eta,q\rangle$ and $p \leqslant q_1 \leqslant q$. Now, since $p \leqslant q_1 \in D$, $\langle\sigma,p\rangle \in Y_D$. Thus Y_D is dense in H.

(ii) is direct consequence of Rasiowa-Sikorski's Lemma.

(iii) Let G be as in (i). The compatibility and closure of $G_P^{\leqslant}$ under $\leqslant$ are obvious. Now, let D be a dense subset of P in the sense of $\mathcal{M}$. We will show that $G_P^{\leqslant} \cap D_E \neq 0$.

As Y_D is dense in H (from (i)) and G is H-ag, we have $G \cap Y_D \neq 0$. Thus, let $\langle\sigma,p\rangle \in G \cap Y_D$. Then $p \in G_P$ and $p \leqslant d$ for some $d \in D_E$. Thus $d \in D_E$ and $d \in G_P^{\leqslant}$. That is $d \in G_P^{\leqslant} \cap D_E \neq 0$. □

Now, we recall the comments made on page 16.

Let $I, X \in M$.

Also, let (in $\mathcal{M}$)

$$P = C(I,X) = \{p: (\text{dom } p \subseteq I) \wedge (\text{ran } p \subseteq X) \wedge \text{Fin}(\text{dom } p))\}.$$

As it is customary, we identify P with N[P], and call its elements 'forcing conditions'. (For N[P], see page 14.)

Finally, let (in $\mathcal{M}$),

$$\Sigma = \{\sigma: (\exists p \in P)(\sigma \in (\text{dom } p)!)\}.$$

Then P and Σ are $\mathcal{M}$-posets ordered in $\mathcal{M}$ by '$\supseteq$'. We write '$\leq$' for '$\supseteq$'.

I, X, P and Σ will have these meanings until otherwise stated.

Remark.

Clearly we have that $\underline{\Sigma}$ and $\underline{P}$ are order-isomorphic to Σ_E and P_E respectively. In this sense Σ_E and $\underline{\Sigma}$ are interchangeable, as are P_E and $\underline{P}$. Therefore H can be considered as a subset of $\underline{\Sigma} \times \underline{P}$. In future we will make free use of this without further reference. In particular, the definitions given at the beginning of this chapter, together with proposition 40 apply when $\underline{\Sigma}$ and $\underline{P}$ are substituted for Σ_E and P_E respectively.

42. Proposition

(i) Let F be a generic subset of $\underline{P}$. Then

$$\bigcup F\colon I_E \to X_E.$$

(ii) Let $H \subseteq \underline{\Sigma} \times \underline{P}$ and suppose that H is extendable.

Let G be an H-ag subset of H. Then

(a) $\bigcup G_P\colon I_E \to X_E$

(b) $\bigcup G_\Sigma \in I_E!$ and $\bigcup G_\Sigma$ is locally expressible.

Proof.

(i) Since F is compatible, $\bigcup F$ is a function.

Obviously, $\mathrm{dom}(\bigcup F) \subseteq I_E$. Suppose that for some $i \in I_E$

$$i \notin \mathrm{dom}(\ F).$$

i.e. $i \notin \mathrm{dom}\ p$ for every $p \in F$.

Let x be a fixed element of X_E. Put $p^x = p \cup \{\langle i,x\rangle\}$ for each $p \in F$, and $F^x = \{p^x\colon p \in F\}$.

Then $F \cup F^x$ is compatible, contradicting the maximality of F.

(ii) (a) Observe that

$$\bigcup G_P = \bigcup G_P^{\leq},$$

and apply (i).

(b) Since G_Σ is a compatible set of permutations of subsets of I_E, $\cup G_\Sigma$ is a permutation. Clearly, $\cup G_\Sigma$ is locally expressible.

We have

$$\mathrm{dom}(\cup G_\Sigma) \subseteq I_E.$$

Let $i \in I_E$.

Then, by (a), $i \in \mathrm{dom}\ p$ for some $p \in G_P$.

Hence $(\exists \sigma \in G_\Sigma)(<\sigma,p> \in G)$.

Whence $i \in \mathrm{dom}\sigma \subseteq G_\Sigma$.

43. Proposition. If no element of H is extendable, then for no H-almost generic $G \subseteq H$ we will have that $G_P^{\leq}$ is a generic subset of P.

Proof. Let G be an H-ag subset of H and $p \in P$. Put $D = P \setminus G_P^{\leq}$.

If $p \notin D$, then $p \in G_P^{\leq}$ and there is $<\nu,q> \in G$ such that $q \leq p$.

As $<\nu,q>$ is not extendable, there is $d \in P$ such that $d \leq q$ and $d \notin G_P^{\leq}$. Hence $d \leq p$ and $d \in D$. Whence D is dense in P. Since $D \cap G_P = 0$, $G_P^{\leq}$ is not a generic subset of P. □

44. Proposition. Let H be extendable and G be an H-ag subset of H.

Put $\pi = \cup G_\Sigma$

(i) $\pi[G_P^{\leq}] = G_P^{\leq}$

(ii) If U is the generic ultrafilter in $\mathcal{B}^{\Gamma}$ associated to $G_P^{\leq}$, then $\pi[U] = U$.

<u>Proof</u>. Straightforward. $\square$

IV

Construction of automorphisms of symmetric submodels of $\mathcal{M}^B$, where $B = RO(X^I)$, via locally expressible permutations of I_E

In this chapter we consider two different definitions of M^Γ, and show how to induce automorphisms in M^Γ in each case, via locally expressible permutations of I_E.

This will lead to two constructions of models of ZF with an automorphism of order N. These are dealt with in two separate parts of this chapter (IV.1 and IV.2).

Throughout, definitions 7 and 34, together with proposition 8, must be kept in mind.

We recall that if π is an $\mathcal{M}$-automorphism of $\mathcal{B}$, π induces an automorphism $\tilde{\pi}$ of $\mathcal{M}^B$, given by $\tilde{\pi}x = \{<\tilde{\pi}\ y,\ \pi(x(y))\ :\ y \in \text{dom } x\}$, for $x \in M^B$. If $\pi \in G$ this definition naturally restricts to $\mathcal{M}^\Gamma$. We shall assume aquaintance with the properties of $\tilde{\pi}$. (Cf. J. Bell [3], Theorem 3.2, p.63).

IV.1

First construction

In this part we work with $\mathcal{M}^{\Gamma}$ as given in definition 25.

45. Proposition. (In $\mathcal{M}$.)

Let $x, x_1, \ldots, x_n$ be a finite sequence of elements of M^B and let $J \subseteq I$.

We have

(i) If $\mathrm{stab}(x) \supseteq G_J$ and $\sigma_1, \sigma_2 \in I!$ are such that

$$\sigma_1/J = \sigma_2/J,$$

then $$\tilde{\sigma}_1 x = \tilde{\sigma}_2 x.$$

(ii) If for any $\sigma_1, \sigma_2 \in I!$ we have

$$\sigma_1/J = \sigma_2/J \rightarrow \tilde{\sigma}_1 x = \tilde{\sigma}_2 x,$$

then $$\mathrm{stab}(x) \supseteq G_J.$$

(iii) For any $\sigma \in I!$,

$$\mathrm{stab}(x) \supseteq G_J \leftrightarrow \mathrm{stab}(\tilde{\sigma} x) \supseteq G_{\sigma[J]}.$$

(iv) If J is a common support of $x_1, \ldots, x_n$, and $\sigma_1, \sigma_2 \in I!$ are such that

$$\sigma_1/J = \sigma_2/J,$$

then for any formula $\phi(v_1, \ldots, v_n)$, we have

$$\sigma_1 \| \phi(x_1, \ldots, x_n) \| = \sigma_2 \| \phi(x_1, \ldots, x_n) \| .$$

Proof. Assume the hypotheses in each case.

(i) As $\sigma_1/J = \sigma_2/J$,

$$(\sigma_1^{-1} \circ \sigma_2)/J = \mathrm{Id}/J.$$

Thus $\sigma_1^{-1} \circ \sigma_2 \in G_J$

and $(\sigma_1^{-1} \circ \sigma_2)^{\sim} \in \mathrm{stab}(x)$.

As $(\sigma_1^{-1} \circ \sigma_2)^{\sim} = \tilde{\sigma}_1^{-1} \circ \tilde{\sigma}_2$, we have

$$\tilde{\sigma}_1^{-1}(\tilde{\sigma}_2 x) = (\tilde{\sigma}_1^{-1} \circ \tilde{\sigma}_2) x = (\sigma_1^{-1} \circ \sigma_2)^{\sim} x = x.$$

Hence $\tilde{\sigma}_2 x = \tilde{\sigma}_1 x$.

(ii) Let $\sigma \in G_J$.

Then $\sigma / J = Id / J$

and $\tilde{\sigma} x = \tilde{Id} / x = x$, by hypothesis.

(iii) ($\rightarrow$) Suppose that $stab(x) \supseteq G_J$.

Let $\rho \in G_{\sigma[J]}$.

Then $\rho(i) = i$ for every $i \in \sigma[J]$,

and $\rho(\sigma(j)) = \sigma(j)$ for every $j \in J$

so $(\sigma^{-1} \circ \rho \circ \sigma)(j) = j$ for every $j \in J$.

Therefore $(\sigma^{-1} \circ \rho \circ \sigma)^{\sim} = \tilde{\sigma}^{-1} \circ \tilde{\rho} \circ \tilde{\sigma} \in stab(x)$.

i.e. $(\tilde{\sigma}^{-1} \circ \tilde{\rho} \circ \tilde{\sigma}) x = x$.

Thus $\tilde{\rho}(\tilde{\sigma} x) = (\tilde{\rho} \circ \tilde{\sigma}) x = x$.

That is $\rho \in stab(\tilde{\sigma} x)$.

($\leftarrow$) On the other hand, suppose that

$$stab(\tilde{\sigma} x) \supseteq G_{\sigma[J]}.$$

Then, using the first part of the proof,

$$stab(\sigma x) \supseteq G_{\sigma[J]} \rightarrow stab((\tilde{\sigma}^{-1} \circ \sigma) x) \supseteq G_{\sigma^{-1} \circ \sigma[J]}$$

$$\rightarrow stab(x) \supseteq G_J.$$

(iv) Utilizing (i), we have

$$\sigma_1 \| \phi(x_1, \ldots, x_n) \| = \| \phi(\tilde{\sigma}_1 x_1, \ldots, \tilde{\sigma}_1 x_n) \|$$

$$= \| \phi(\tilde{\sigma}_2 x_1, \ldots, \tilde{\sigma}_2 x_n)$$

$$= \sigma_2 \| \phi(x_1, \ldots, x_n) \| . \qquad \square$$

Proposition 45 motivates definition 46, below. In particular, 45(i) asserts that (in $\mathcal{M}$) the movements performed on elements of M^Γ depend only on the movements performed on their supports.

Throughout the rest of IV.1, let $B = RO(X^I)$, and let π be a locally expressible permutation of I_E such that

$$\pi^{(N)} = Id/I_E \neq \pi^{(i)}, \quad i = 1, \ldots, N-1$$

with $N \in \omega$, $N \neq 0$.

46. Definition

(i) For each $x \in M^\Gamma$, let

$$S_x = \{J_E \subseteq I_E : \mathcal{M} \models ((J \text{ is finite}) \wedge (J \subseteq I) \wedge (stab(x) \supseteq G_J))\}.$$

(ii) Let

$\{(J_x)_E : x \in M^\Gamma\}$ be a selection of $\{S_x : x \in M^\Gamma\}$.

(iii) Put

$$(K_x)_E = \bigcup_{m=1}^{N} \pi^{(m)}[(J_x)_E], \text{ for every } x \in M^\Gamma.$$

Therefore $\{(K_x)_E : x \in M^\Gamma\}$ is a selection of $\{S_x : x \in M^\Gamma\}$, and for every $x \in M^\Gamma$, we have $\pi^{(m)}[(K_x)_E] = (K_x)_E$ for $m \in \mathbb{Z}$.

(iv) Let $x \in M^\Gamma$. Set

$$(\pi_x)_- = \{<i, \pi(i)> : i \in (K_x)_E\} \cup Id/(I_E \setminus (K_x)_E)$$

and $\tilde{\pi}x = \tilde{\pi}_x(x)$.

Clearly, $\tilde{\pi}$ is a map of M^Γ to itself.

(v) Let $y_1, \ldots, y_n$ be an $\mathcal{M}$-(finite sequence) of elements of M^Γ. Set

$$(\pi_{y_1,\ldots,y_n})_- = \{<i, \pi(i)> : i \in \bigcup_{i=1}^{n} (K_{y_i})_E\} \cup Id/(I_E \setminus \bigcup_{i=1}^{n} (K_{y_i})_E).$$

Remarks

(i) As π is locally expressible in $\mathcal{M}$, π_x and $\tilde{\pi}_x$ belong to M for each $x \in M^\Gamma$.

(ii) π_x and $\tilde{\pi}_x$ depend on x.

(iii) The definition of $(\pi_x)_-$ on $I_E \setminus (K_x)_E$ is merely conventional; it has been adopted by simplicity.

If $\{f_x : x \in M^\Gamma\}$ is any family of permutations of I_E which are expressible in $\mathcal{M}$ and such that

$$f_x^{(N)} = Id \restriction I_E \quad \text{and} \quad f_x \restriction (K_x)_E = \pi \restriction (K_x)_E$$

for every $x \in M^\Gamma$, the definition of $(\pi_x)_-$ on $I_E \setminus (K_x)_E$ as

$$(\pi_x)_-(i) = f_x(i),$$

would do just as well as that of 46(iv).

In future we will not distinguish between $(\pi_x)_-$, $(K_x)_E$, etc. and π_x, K_x, etc., respectively.

47. Proposition. For any $\mathcal{M}$-(finite sequence) $y_1,\ldots,y_n,z$ of elements of M^Γ, and for any formula $\phi(v_1,\ldots,v_n)$, we have

(i) $\pi_{y_1,\ldots,y_n}{}^{(N)} = Id \restriction I_E$.

(ii) $\tilde{\pi}_{y_1,\ldots,y_n}(y_i) = \tilde{\pi}_{y_i}(y_i)$, $1 \leq i \leq n$.

(iii) $\pi_{y_1,\ldots,y_n} \|\phi(y_1,\ldots,y_n)\| = \pi_{y_1,\ldots,y_n,z} \|\phi(y_1,\ldots,y_n)\|$.

(iv) $\pi \|\phi(y_1,\ldots,y_n)\| = \pi_{y_1,\ldots,y_n} \|\phi(y_1,\ldots,y_n)\|$.

Proof. Assume the hypotheses.

(i) Trivial.

(ii) This is a direct application of proposition 45(i).

(iii) $\pi_{y_1,\ldots,y_n}\|\phi(y_1,\ldots,y_n)\| = \|\phi(\tilde{\pi}_{y_1,\ldots,y_n}(y_1),\ldots,\tilde{\pi}_{y_1,\ldots,y_n}(y_n)\|$

$$= \|\phi(\tilde{\pi}_{y_1,\ldots,y_n,z}(y_1),\ldots,\breve{\pi}_{y_1,\ldots,y_n,z}(y_n))\|$$

$$= \pi_{y_1,\ldots,y_n,z}\|\phi(y_1,\ldots,y_n)\|,$$

using (ii) in the second line.

(iv) If $\pi\|\phi(y_1,\ldots,y_n)\| \nleq \pi_{y_1,\ldots,y_n}\|\phi(y_1,\ldots,y_n)\|$, then

$$\pi\|\phi(y_1,\ldots,y_n)\| \wedge \pi_{y_1,\ldots,y_n}\|\phi(y_1,\ldots,y_n)\|^* \neq 0,$$

and for some $p \in P$ (in $\mathcal{M}$),

$$p \leq \pi\|\phi(y_1,\ldots,y_n)\| \wedge \pi_{y_1,\ldots,y_n}\|\phi(y_1,\ldots,y_n)\|^*. \quad (1)$$

By (iii), there is no loss of generality in assuming

$$\bigcup_{i=1}^{n} K_{y_i}'' \supseteq \text{dom } p.$$

Then we have

$$\pi^{(m)}p = \pi_{y_1,\ldots,y_n}^{(m)}(p);\ m \in \omega. \quad (2)$$

Since $\pi^{(N)} = \text{Id}/I_E$, (1) gives

$$\pi^{(N-1)}p \leq \|\phi(y_1,\ldots,y_n)\| \wedge (\pi^{(N-1)} \circ \pi_{y_1,\ldots,y_n})\|\phi(y_1,\ldots,y_n)\|^*.$$

Thus $\pi^{(N-1)}p \leq \|\phi(y_1,\ldots,y_n)\|$. (3)

Also, using (1), (2) and (i), we have

$$\pi^{(N-1)}p \leq (\pi_{y_1,\ldots,y_n}^{(N-1)} \circ \pi)\|\phi(y_1,\ldots,y_n)\| \wedge \|\phi(y_1,\ldots,y_n)\|^*,$$

which gives

$$\pi^{(N-1)}p \leq \|\phi(y_1,\ldots,y_n)\|^*,$$

contradicting (3).

Hence

$$\pi\|\phi(y_1,\ldots,y_n)\| \leq \pi_{y_1,\ldots,y_n}\|\phi(y_1,\ldots,y_n)\|.$$

Similarly, we prove

$$\pi_{y_1,\ldots,y_n}\|\phi(y_1,\ldots,y_n)\| \leq \pi\|\phi(y_1,\ldots,y_n)\|. \qquad \Box$$

48. Proposition.

(i) $\tilde{\pi}$ is one-one.

(ii) $\tilde{\pi}$ is onto.

(iii) $\tilde{\pi}^{(N)} = \widetilde{Id}$.

(iv) $\tilde{\pi} \neq \widetilde{Id}$.

(v) $\pi\|\phi(y_1,\ldots,y_n)\| = \|\phi(\tilde{\pi}y_1,\ldots,\tilde{\pi}y_n)\|$.

Proof

(i) Let $x,y \in M^\Gamma$.

Then, (proposition 45(i)), we have

$$\tilde{\pi}x = \tilde{\pi}_x(x) = \tilde{\pi}_{x,y}(x), \text{ and}$$

$$\tilde{\pi}y = \tilde{\pi}_y(y) = \tilde{\pi}_{x,y}(y).$$

As $\tilde{\pi}_{x,y}$ is one-one,

$$x \neq y \to \tilde{\pi}x = \tilde{\pi}_{x,y}(x) \neq \tilde{\pi}_{x,y}(y) = \tilde{\pi}(y).$$

(ii) Let $x \in M^\Gamma$.

Let $z \in M^\Gamma$ be such that

$$\tilde{\pi}_x(z) = x.$$

Then stab $(\tilde{\pi}_x(z)) \supseteq G_{K_x}$,

and stab(z) $\supseteq G_{\pi^{-1}[K_x]} = G_{K_x}$ (proposition 45(iii)).

Thus $\tilde{\pi}_z(z) = \tilde{\pi}_x(z) = x$, (proposition 45(i)),

and $x = \tilde{\pi}z$.

(iii) Trivial.

$$\tilde{\pi}_x^{(N)}(x) = (\pi_x^{(N)})^{\sim} x = \tilde{Id}\ x = x.$$

(iv) Trivial.

For example, let $i \neq j = \pi(i)$, and let $a \in X$.

Put $p = \{<i,a>\}$ and $q = \pi p$.

Then $q = \pi p = \{<j,a>\}$ and we have, (proposition 35(i))

$$\tilde{\pi}\hat{x}_p = \hat{x}_{\pi p} = \hat{x}_q \neq \hat{x}_p.$$

(v) $\pi\|\phi(y_1,\ldots,y_n)\| = \pi_{y_1,\ldots,y_n}\|\phi(y_1,\ldots,y_n)\|$ (proposition 47(iii))

$$= \|\phi(\tilde{\pi}_{y_1,\ldots,y_n}(y_1),\ldots,\tilde{\pi}_{y_1,\ldots,y_n}(y_n))\|.$$

As, (proposition 47(ii)),

$$\tilde{\pi}_{y_1,\ldots,y_n}(y_i) = \tilde{\pi}_{y_i}(y_i), \qquad 1 \leqslant i \leqslant n,$$

then

$$\pi\|\phi(y_1,\ldots,y_n)\| = \|\phi(\tilde{\pi}_{y_1}(y_1),\ldots,\tilde{\pi}_{y_n}(y_n))\|$$

$$= \|\phi(\tilde{\pi}y_1,\ldots,\tilde{\pi}y_n) \quad \text{(definition 46(iv))}. \ \square$$

π induces in a natural way an automorphism of $\mathcal{M}^{\Gamma}/U$ via $x^u \longrightarrow (\tilde{\pi}x)^u$. We show next that such a definition is possible if and only if $\pi[U] = U$.

49. <u>Proposition</u>. The following conditions are equivalent

(i) $\pi[U] = U$.

(ii) $x^u = y^u \rightarrow (\tilde{\pi}x)^u = (\tilde{\pi}y)^u$ for every $x, y \in M^{\Gamma}$.

<u>Proof</u>. (i) $\rightarrow$ (ii) trivial.

(ii) $\rightarrow$ (i). Assume (ii), that is $\|x = y\| \in U \rightarrow \|\tilde{\pi}x = \tilde{\pi}y\| \in U$ for every $x,y \in M^{\Gamma}$.

Let $a \in U$. By proposition 32(iii) there are $x,y \in M^\Gamma$ such that $a = \|x = y\| \in U$. It follows that

$$\pi(a) = \pi\|x = y\| = \|\tilde{\pi}x = \tilde{\pi}y\| \in U.$$

Therefore $U \subseteq \pi[U]$ and $U = \pi[U]$, since $\pi[U]$ is also an ultrafilter. □

Our task would be completed if we could obtain x,G and $\langle\sigma,p\rangle$ satisfying the conditions (I) of proposition 50, below. This is done in chapter V, while propositions 50-2 show that such a task is impossible if H is definable in $\mathcal{M}$. We recall the meanings of H,P and Σ (pp. 31,34).

50. <u>Proposition</u>. Let H be definable in $\mathcal{M}$ and extendable. Also, let $x \in M$.

Let $\psi(y)$ be the formula of $\mathcal{L}_B^\Gamma$

$$\exists\alpha(y \in M_\alpha^\Gamma \wedge (\exists\langle\sigma,p\rangle \in H)((\mathrm{stab}(y) \supseteq G_{\mathrm{dom}\ p}) \wedge$$
$$(\forall\rho \in I!)(\rho/\mathrm{dom}\ p = \sigma/\mathrm{dom}\ p \rightarrow \mathrm{Comp}(p,\|y = \tilde{\rho}y\|)))).$$

Then the following conditions are equivalent.

(I) $x \in M^\Gamma$ and there are G, $\langle\sigma,p\rangle$ such that

(i) G is an H-almost generic subset of H,

(ii) $\langle\sigma,p\rangle \in G$ and

(iii) $\mathrm{stab}(x) \supseteq G_{\mathrm{dom}\ p} \wedge \|x \neq \tilde{\pi}x\| \in U$, where

$$U = G_p^{\leq} \quad \text{and} \quad \pi = \cup G_\Sigma.$$

(II) $\models \psi[x]$.

<u>Proof</u>. (I) → (II). Suppose (I); then, trivially we have

$$\models \exists\alpha(x \in M_\alpha^\Gamma \wedge (\exists\langle\sigma,p\rangle \in H)(\mathrm{stab}(x) \supseteq G_{\mathrm{dom}\ p})).$$

Suppose that

$$\mathcal{M} \models \rho \in I! \wedge \rho/\!\!\upharpoonright \text{dom } p = \sigma/\!\!\upharpoonright \text{dom } p.$$

We show that

$$\mathcal{M} \models \text{Comp}(p, \|x \neq \rho x\|).$$

For, since $\text{stab}(x) \supseteq G_{\text{dom } p}$,

$$\tilde{\rho}x = \tilde{\pi}x \quad \text{and}$$

$$\|x \neq \tilde{\pi}x\| = \|x \neq \tilde{\rho}x\| \in U.$$

As $p \in U$, then

$$\mathcal{M} \models \text{Comp}(p, \|x \neq px\|).$$

(II) → (I).

Assume (II).

Let $x \in M$ be such that $\mathcal{M} \models \psi[x]$.

Then, (working in V with the replicas of the notions in $\mathcal{M}$), for some $\langle \nu, q\rangle \in H$ we have

$$\text{stab}(x) \supseteq G_{\text{dom } q} \quad \text{and}$$

$$\rho/\!\!\upharpoonright \text{dom } q = \nu/\!\!\upharpoonright \text{dom } q \rightarrow \text{Comp}(q, \|x \neq \tilde{\rho}x\|)$$

for every $\rho \in I!$, (more precisely: for every $\underline{\rho} \in (I_E)!$, that is the replica of some ρ such that $\mathcal{M} \models \rho \in I!$).

Let $\rho \in I!$ be such that

$$\rho/\!\!\upharpoonright \text{dom } q = \nu/\!\!\upharpoonright \text{dom } q.$$

Then $\text{Comp}(q, \|x \neq \tilde{\rho}x\|)$.

Thus, let $q' \leq q \wedge \|x \neq \tilde{\rho}x\|$.

Since H is extendable, there is

$$\langle \sigma, p\rangle \in H \quad \text{such that}$$

$$\langle \sigma, p\rangle \leq \langle \nu, q\rangle \quad \text{and}$$

$$p \leq q' \lesssim q.$$

As
$$\mathrm{stab}(x) \supseteq G_{\mathrm{dom}\ q},$$

$$\mathrm{stab}(x) \supseteq G_{\mathrm{dom}\ p}.$$

Again, as H is extendable, we use propositions 41,2 to obtain

(i) an H-ag subset G of H such that

(ii) $\langle \sigma, p\rangle \in G$

$U = G_P^{\leq}$ is generic in $\mathcal{B}^{\Gamma}$ and

$\pi = \cup G_{\Sigma} \supseteq \sigma$.

As $\tilde{\pi}x = \tilde{\rho}x$ and $p \in U$ and $p \lesssim q' \lesssim q \wedge \|x \neq \tilde{\rho}x\|$, we conclude

(iii) $\|x \neq \tilde{\pi}x\| \in U$. □

51. Proposition. Let H, $\psi(x)$ be as in proposition 50.

Then

$$\mathcal{M} \models (\forall x \in M^{\Gamma})(\neg\psi(x)).$$

Proof. Let $\mathcal{M} \models x \in M^{\Gamma}$ and suppose that

$$\mathcal{M} \models (\forall y \in \mathrm{dom}\ x)(\neg\psi(y)) \qquad (1)$$

If $\mathcal{M} \models \psi[x]$,

then, by proposition 50, there are

(i) an H-ag $G \subseteq H$,

(ii) $\langle \sigma, p\rangle \in H$ such that $\langle \sigma, p\rangle \in G$, and

(iii) $\mathrm{stab}(x) \supseteq \mathrm{dom}\ p$ and $\|x \neq \tilde{\pi}x\| \in U$, where $U = G_P^{\leq}$ and $\pi = \cup G_{\Sigma}$.

Claim

$\|y \neq \tilde{\pi}y\| \in U$ for some $y \in \mathrm{dom}\ x$.

For, since

$$\|x \neq \tilde{\pi}x\| = \bigvee_{y \in \mathrm{dom}\, x} [x(y) \wedge \bigwedge_{z \in \mathrm{dom}\, x} (\pi(x(z))^* \vee \|\tilde{\pi}z \neq y\|)] \vee$$

$$\bigvee_{y \in \mathrm{dom}\, x} [\pi(x(y)) \wedge \bigwedge_{z \in \mathrm{dom}\, x} (x(z)^* \vee \|z \neq \tilde{\pi}y\|)],$$

then, if $\|x \neq \tilde{\pi}x\| \in U$, by the genericity of U, we have that for some $y \in \mathrm{dom}\, x$

$$x(y) \wedge \bigwedge_{z \in \mathrm{dom}\, x} (\pi(x(z))^* \vee \|\pi z \neq y\|) \in U, \text{ or}$$

$$\pi(x(y)) \wedge \bigwedge_{z \in \mathrm{dom}\, x} (x(z)^* \vee \|z \neq \tilde{\pi}y\|) \in U.$$

In particular

$$x(y) \wedge (\pi(x(y))^* \vee \|\pi y \neq y\|) \in U \quad \text{or}$$

$$\pi(x(y)) \wedge (x(y))^* \vee \|y \neq \pi y\|) \in U.$$

As $x(y) \in U \leftrightarrow \pi(x(y)) \in U$, $(\pi[U] = U)$, then $x(y) \wedge \pi(x(y))^* = 0 = \pi(x(y)) \wedge x(y)^*$, and $\|y \neq \tilde{\pi}y\| \in U$. This proves the claim.

Going back to the proof, let

$\mathrm{stab}(y) \supseteq G_J$ and let

$q \leq p \wedge \|y \neq \tilde{\pi}y\|$ be such that

$\mathrm{dom}\, q \supseteq J \cup \mathrm{dom}\, p$.

As H is extendable, there exists

$<\sigma_1, p_1> \in H$ such that

$<\sigma_1, p_1> \leq <\sigma, p>$ and $p_1 \leq q \leq p$.

Again, as H is extendable, there is

(i) a generic $G' \subseteq H$, such that

(ii) $<\sigma_1, p_1> \in G'$.

Let $U' = G'^{\leq}_P$.

As dom $p_1 \supseteq$ dom q,

$$\text{dom } p_1 \supseteq J \tag{2}$$

As $p_1 \in G'$, $p_1 \in U'$. (3)

Now, (2) and (3) give

(iii) $\text{stab}(y) \supseteq G_{\text{dom } p_1}$ and

$\|y \neq \tilde{\pi}y\| \in U'$, where

$U' = G_P'^{\leq}$ and $\pi = \cup G_\Sigma$.

Proposition 50 gives

$$\mathcal{M} \models \psi[y].$$

Then $\mathcal{M} \models (\exists y \in \text{dom } x)\psi(y)$, contradicting (1). We conclude

$$\mathcal{M} \models \neg\psi[x].$$

Therefore

$$\mathcal{M} \models (\forall y \in \text{dom } x)(\neg\psi(y) \to \neg\psi(x)).$$

Finally, the induction principle for $\mathcal{M}^\Gamma$ implies

$$\mathcal{M} \models (\forall x \in M^\Gamma)\ \neg\psi(x). \qquad \square$$

<u>Remark</u>. The proof of proposition 51 can be given (apart from the mention of G) entirely inside $\mathcal{M}$ by induction on 'd $\in$ dom x', by 'reflecting' inside $\mathcal{M}$ the conditions (I) of proposition 50.

52. <u>Proposition</u>. If H is definable in $\mathcal{M}$ and extendable then there is no $x \in M^\Gamma$ that meets the conditions (I) of proposition 50.

<u>Proof</u>. Immediate, from proposition 51.

Proposition 52 can be generalised as follows.

53. <u>Proposition</u>. Let $H' \subseteq \Sigma \times P$ be definable in $\mathfrak{M}$ and extendable.

Let H be a dense extendable subclass of H'. (We make no assumption as to whether H is definable in $\mathfrak{M}$.)

Then there are no $x \in M^{\Gamma}$, $\langle\sigma,p\rangle \in H$ and $G \subseteq H$ such that

(i) G is H-ag in H,

(ii) $\langle\sigma,p\rangle \in G$ and

(iii) $stab(x) \supseteq G_{dom\ p} \wedge ||x = \tilde{\pi}x|| \in U$, where

$$U = G_P^{\leq} \quad \text{and} \quad \pi = \cup G_\Sigma .$$

<u>Proof</u>. Assume the hypotheses and let G be an H-ag subset of H.

<u>Claim</u>. $G^{\leq}$ is an H'-ag subset of H'.

Proof of claim.

Let $D \subseteq P$ be dense,

put $\qquad Y_D = \{\langle\sigma,p\rangle \in H' \colon (\exists d \in D_E)(p \leq d)\}$

and $\qquad Y = Y_D \cap H.$

Then $\qquad Y = \{\langle\sigma,p\rangle \in H \colon (\exists d \in D_E)(p \leq d)\}.$

As G is H-ag, then $G \cap Y \neq 0$.

Thus $G^{\leq} \cap Y_D \neq 0$.

Now, if $x \in M^{\Gamma}$, $\langle\sigma,p\rangle \in H$ and $G \subseteq H$ met the conditions (i)-(iii) above, then x, $\langle\sigma,p\rangle$ and $G^{\leq}$ would satisfy condition (I) of proposition 50, with $G^{\leq}$ in the place of G, which is impossible (proposition 52).

□

IV.2

Second construction

In this part of chapter IV, we work with $\mathcal{M}^{\Gamma}$ defined as follows.

(i) $\mathrm{STAB}(x) = \{\sigma \in G : \|\tilde{\sigma}x = x\| = 1\}$, for $x \in M^{B}$.

(ii) $\mathcal{M}^{\Gamma}$ as in definition 26, but with 'STAB' instead of 'stab'.

54. Proposition. All the contents from proposition 27 to the end of chapter III, continue to apply when 'stab' is replaced by 'STAB'. □

55. Proposition (Cf. proposition 45).

Let $x, x_1, \ldots, x_n$ be an $\mathcal{M}$-(finite sequence) of elements of M^{B}, and let, (in $\mathcal{M}$), $J \subseteq I$.

We have, (in $\mathcal{M}$),

(i) If $\mathrm{STAB}(x) \supseteq G_J$, and $\sigma_1, \sigma_2 \in I!$ are such that

$$\sigma_1 \restriction J = \sigma_2 \restriction J,$$

then $\|\tilde{\sigma}_1 x = \tilde{\sigma}_2 x\| = 1$.

(ii) If for any $\sigma_1, \sigma_2 \in I!$, we have

$$\sigma_1 \restriction J = \sigma_2 \restriction J \to \|\tilde{\sigma}_1 x = \tilde{\sigma}_2 x\| = 1,$$

then $\quad \mathrm{STAB}(x) \supseteq G_J$.

(iii) For any $\sigma \in I!$,

$$\mathrm{STAB}(x) \supseteq G_J \leftrightarrow \mathrm{STAB}(\tilde{\sigma}x) \supseteq G_{\sigma[J]}.$$

(iv) If J is a common support of $x_1, \ldots, x_n$, and if $\sigma_1, \sigma_2 \in I!$ are such that

$$\sigma_1 \restriction J = \sigma_2 \restriction J$$

then, for any formula $\phi(v_1,\ldots,v_n)$, we have

$$\sigma_1\|\phi(x_1,\ldots,x_n)\| = \sigma_2\|\phi(x_1,\ldots,x_n)\|.$$

Proof. Assume the hypotheses in each case.

(i) We have

$$(\sigma_2^{-1} \circ \sigma_1)/\hat{J} = Id/\hat{J}.$$

i.e. $\sigma_2^{-1} \circ \sigma_1 \in G_J$

Therefore $\|(\tilde{\sigma}_2^{-1} \circ \tilde{\sigma}_1)x = x\| = 1$, and

$$\sigma_2\|(\tilde{\sigma}_2^{-1} \circ \tilde{\sigma}_1)x = x\| = \|(\tilde{\sigma}_2 \circ \tilde{\sigma}_2^{-1} \circ \tilde{\sigma}_1)x = x\| = \|\tilde{\sigma}_1 x = x\| = 1.$$

(ii) Let $\sigma \in G_J$.

Then $\sigma/\hat{J} = Id/\hat{J}$

and $\|\tilde{\sigma}x = \tilde{Id}\ x\| = 1 = \|\tilde{\sigma}x = x\|$ by hypothesis.

Thus $\sigma \in STAB(x)$.

(iii) ($\rightarrow$) Suppose that $STAB(x) \supseteq G_J$.

Let $\rho \in G_{\sigma[J]}$.

Then $\rho(i) = i$ for every $i \in \sigma[J]$,

and $\rho(\sigma(j)) = \sigma(j)$ for every $j \in J$.

So $(\sigma^{-1} \circ \rho \circ \sigma)(j) = j$ for every $j \in J$

Therefore $(\sigma^{-1} \circ \rho \circ \sigma)^{\sim} = \tilde{\sigma}^{-1} \circ \tilde{\rho} \circ \tilde{\sigma} \in STAB(x)$.

I.e., $\|(\tilde{\sigma}^{-1} \circ \tilde{\rho} \circ \tilde{\sigma})x = x\| = 1$.

Thus $\|\tilde{\rho}(\tilde{\sigma}x) = \tilde{\sigma}x\| = \|(\tilde{\rho} \circ \tilde{\sigma})x = \tilde{\sigma}x\| = 1$.

That is $\rho \in STAB(\tilde{\sigma}x)$.

($\leftarrow$) On the other hand, suppose that

$$STAB(\tilde{\sigma}x) \supseteq G_{\sigma[J]}.$$

Using the first part of the proof, we have

$$STAB(\tilde{\sigma}x) \supseteq G_{\sigma[J]} \rightarrow STAB((\tilde{\sigma}^{-1} \circ \tilde{\sigma})x) \supseteq G_{\sigma^{-1} \circ \sigma[J]} = G_J$$
$$\rightarrow STAB(x) \supseteq G_J.$$

(iv) Applying (i), we have

$$\sigma_1\|\phi(x_1,\ldots,x_n)\| = \|\phi(\tilde{\sigma}_1 x_1,\ldots,\tilde{\sigma}_1 x_n)\|$$
$$= \|\phi(\tilde{\sigma}_2 x_1,\ldots,\tilde{\sigma}_2 x_n)\|$$
$$= \tilde{\sigma}_2\|\phi(x_1,\ldots,x_n)\|. \qquad \square$$

Our definition of an automorphism $\tilde{\pi}$ of M^{Γ} via a locally expressible bijection π of I_E, will rely, as in IV.1, on the way in which π acts on the elements of a support of x, for each $x \in M^{\Gamma}$.

However, from the definition of STAB(x), one sees that the image of an element $x \in M^{\Gamma}$ under $\tilde{\pi}$, unlike the situation in IV.1, will depend on the support of x one chooses: In general, two different supports will produce two different images of x. (However, according to 55(i), those images are identical in $\mathcal{M}^{\Gamma}$).

The fact that, in general, there is a vast availability of supports for a given element of M^{Γ}, makes it difficult to control their properties so that the final $\tilde{\pi}$ is one-one and onto.

This forces us to go through a process of selection of the supports available, so that their variety is reduced to a controllable minumum.

That is the aim of definition 56, below.

Throughout the rest of IV, let π be a locally expressible permutation of I_E such that

$$\pi^{(N)} = Id\restriction I_E \neq \pi,\ N \in \omega,\ N \neq 0.$$

56. Definition. Let

(i) $S_x = \{J_E\colon \mathcal{M} \models J \text{ is a finite support of } x\}$, for $x \in M^{\Gamma}$.

(ii) $\{x_i\colon i \in \omega\}$ be an enumeration of the elements of M^{Γ}, (with $i \neq j \rightarrow x_i \neq x_j$).

(iii) $<(J_i)_E\colon i \in \omega>$ be an increasing selection of pairwise different elements of $\{S_{x_i} : i \in \omega\}$. $(i < j \to (J_i)_E \subsetneqq (J_j)_E)$.

(iv) $(K_i)_E = \bigcup_{m=1}^{N} \pi^{(m)}[(J_i)_E]$, for each $i \in \omega$.

(v) $D_0 = \{x \in M^\Gamma : (K_0)_E \in S_x\}$.

$D_{n+1} = \{x \in M^\Gamma : (\bigwedge_{i=0}^{n} (K_i)_E \notin S_x) \wedge ((K_{n+1})_E \in S_x)\}$, $n \in \omega$.

57. Proposition.

(i) $\pi^{(n)}[(K_i)_E] = (K_i)_E$ for every $i \in \omega$ and $n \in \mathbb{Z}$.

(ii) $\mathrm{STAB}(\tilde{\rho} x_i) \supseteq G_{K_i}$, for every $i \in \omega$,

and every $\mathcal{M}$-permutation ρ of I such that

$$\rho \restriction K_i = \pi \restriction K_i.$$

(iii) $m \neq n \to D_m \cap D_n = 0$ for every $m,n \in \omega$.

(iv) For every $m,n \in \omega$, with $m < n$,

$$x_m \notin D_n.$$

(v) For every $n \in \omega$, there is (a unique) $m \leq n$ such that

$$x_n \in D_m.$$

(Therefore, for each $x \in M^\Gamma$, there is a unique $m \in \omega$ such that $x \in D_m$).

Proof.

(i) Definition 56 (iv).

(ii) Proposition 55 (iii).

(iii) First, we trivially have $0 \neq n \to D_0 \cap D_n = 0$ for every $n > 0$.

Now, let $m,n > 0$.

With no loss of generality, assume that $m < n$.

Then, if $x \in D_m \cap D_n$, $(x \in M^\Gamma)$,

$$[\bigwedge_{i=0}^{m-1} ((K_i)_E \notin S_x) \wedge ((K_m)_E \in S_x)] \wedge [\bigwedge_{i=0}^{n-1} ((K_i)_E \notin S_x) \wedge ((K_n)_E \in S_x)] .$$

But this implies $((K_m)_E \in S_x) \wedge ((K_m)_E \notin S_x)$, a contradiction.

(iv) Direct from definition 56.

(v) (a) $x_0 \in D_0$.

(b) Suppose, first, that $x_n \in D_n$. Then

$$\bigwedge_{i=0}^{n-1} ((K_i)_E \notin S_{x_n})$$

Therefore $x_n \notin D_m$ for every $m < n$, and (v) follows in this case.

Now, suppose that $x_n \notin D_n$. Then

$$\neg[\bigwedge_{i=0}^{n-1} ((K_i)_E \notin S_{x_n}) \wedge ((K_n)_E \in S_{x_n})].$$

Equivalently,

$$\bigvee_{i=0}^{n-1} ((K_i)_E \in S_{x_n}) \vee ((K_n)_E \notin S_{x_n}).$$

If $K_0 \in S_{x_n}$ then $x_n \in D_0$, and $x_n \notin D_m$ for any $m > 0$, (from (iii)).

If $K_0 \notin S_{x_n}$, let m be the first $\mathcal{M}$-(natural less or equal than) n, such that $(K_m)_E \in S_{x_n}$.

Then $x_n \in D_m$ and $x_n \notin D_s$ for any $s > m$. □

58. Definition. (Cf. definition 46).

Let $x \in M^\Gamma$.

(i) (Cf. proposition 57(v)). Let $s(x)$ = the unique $i \in \omega$ such that $x \in D_i$. (Thus $x \in D_{s(x)}$ for every $x \in M^\Gamma$).

Also, set

(ii) $(\pi_x)_- = \{\langle i,\pi(i)\rangle : i \in (K_{s(x)})_E\} \cup Id\restriction(I_E \setminus (K_{s(x)})_E)$, and

(iii) $\tilde{\pi}x = \tilde{\pi}_x(x)$.

Clearly, $\tilde{\pi}$ is a map of M^Γ to itself.

Finally,

(iv) Let $y_1,\ldots,y_n$ be an $\mathcal{M}$-(finite sequence) of elements of M^Γ. Set

$$(\pi_{y_1,\ldots,y_n})_- = \{<i,\pi(i)>: i \in \bigcup_{i=1}^{n} (K_{s(y_i)})_E\} \cup Id/(I_E \setminus \bigcup_{i=1}^{n} (K_{s(y_i)})_E)$$

<u>Remarks</u>

(i) Definition 58 can be given in place of definition 46; in which case, all the results of IV.1 continue to hold. However, the converse is not true.

(ii) As π is locally expressible in $\mathcal{M}$, π_x and $\tilde{\pi}_x$ belong to M, for each $x \in M^\Gamma$.

(iii) Other things being equal, π_x depends on x.

(iv) π depends on the enumeration of the elements of M^Γ, (definition 56(ii)), and on the selection of the supports of the elements of M^Γ, (definition 56, (iii), (iv)).

(v) The definition of $(\pi_x)_-$ on $I_E\setminus(K_{s(x)})_E$ is merely conventional; it has been adopted by simplicity.

If $\{f_x: x \in M^\Gamma\}$ is any family of permutations of I_E which are expressible in $\mathcal{M}$ and such that for every $x \in M^\Gamma$

$$f_x^{(N)} = Id/I_E \quad \text{and} \quad f_x/(K_{s(x)})_E = \pi/(K_{s(x)})_E,$$

the definition of $(\pi_x)_-$ on $I_E/(K_{s(x)})_E$ as

$$(\pi_x)_-(i) = f_x(i),$$

would do just as well as the one given in 58(ii).

In future we will not distinguish either between $(\pi_x)_-$ and π_x; $(\pi_{y_1,\ldots,y_n})_-$ and $\pi_{y_1,\ldots,y_n}$, or $(K_x)_E$ and K_x.

59. <u>Proposition</u>. For any $\mathcal{M}$-(finite sequence) $y_1,\ldots,y_n,z$ of elements of M^Γ, and for any formula $\phi(v_1,\ldots,v_n)$, we have

(i) $\pi^{(N)}_{y_1,\ldots,y_n} = \mathrm{Id}/\!\!\!\wedge I_E$

(ii) $\|\pi_{y_1,\ldots,y_n}(y_i) = \pi_{y_i}(y_i)\| = 1, \quad 1 \leq i \leq n.$

(iii) $\pi_{y_1,\ldots,y_n}\|\phi(y_1,\ldots,y_n)\| = \pi_{y_1,\ldots,y_n,z}\|\phi(y_1,\ldots,y_n)\|.$

(iv) $\pi\|\phi(y_1,\ldots,y_n)\| = \pi_{y_1,\ldots,y_n}\|\phi(y_1,\ldots,y_n)\|.$

<u>Proof</u>. Assume the hypotheses.

(i) Trivial.

(ii) This is a direct application of proposition 55(i).

(iii) Suppose that

$$\pi\|\phi(y_1,\ldots,y_n)\| \nleq \pi_{y_1,\ldots,y_n}\|\phi(y_1,\ldots,y_n)\|.$$

Then

$$\pi\|\phi(y_1,\ldots,y_n)\| \wedge \pi_{y_1,\ldots,y_n}\|\phi(y_1,\ldots,y_n)\|^* \neq 0,$$

and for some $p \in P$ (in $\mathcal{M}$) we have

$$p \leq \pi\|\phi(y_1,\ldots,y_n)\| \wedge \pi_{y_1,\ldots,y_n}\|\phi(y_1,\ldots,y_n)\|^*. \tag{1}$$

By (iii), there is no loss of generality in assuming that

$$\bigcup_{i=1}^{n} K_{s(y_i)} \supseteq \mathrm{dom}\ p.$$

Then we have

$$\pi^{(m)}p = \pi^{(m)}_{y_1,\ldots,y_n}(p), \text{ for } m \in \omega. \tag{2}$$

Since $\pi^{(N)} = \mathrm{Id}/\!\!\!\wedge I_E$, (1) gives

$$\pi^{(N-1)}p \leq \|\phi(y_1,\ldots,y_n)\| \wedge (\pi^{(N-1)} \circ \pi_{y_1,\ldots,y_n})\|\phi(y_1,\ldots,y_n)\|^*,$$

and $\pi^{(N-1)}p \leq \|\phi(y_1,\ldots,y_n)\|.$ (3)

Also, using (1), (2) and (i),

$$\pi^{(N-1)}_{y_1,\ldots,y_n}(p) \leq (\pi^{(N-1)}_{y_1,\ldots,y_n} \circ \pi)\|\phi(y_1,\ldots,y_n)\| \wedge \|\phi(y_1,\ldots,y_n)\|^*.$$

Thus, (using (2)),

$$\pi^{(N-1)}p \leq \|\phi(y_1,\ldots,y_n)\|^*,$$

Contradicting (3).

We conclude that

$$\pi\|\phi(y_1,\ldots,y_n)\| \leq \pi_{y_1,\ldots,y_n}\|\phi(y_1,\ldots,y_n)\|.$$

Similarly, we prove

$$\pi_{y_1,\ldots,y_n}\|\phi(y_1,\ldots,y_n)\| \leq \pi\|\phi(y_1,\ldots,y_n)\|,$$

and (iv) follows. □

60. Proposition.

(i) $\tilde{\pi}x \in D_{s(x)}$, for every $x \in M^\Gamma$.

(ii) $\tilde{\pi}$ is one-one.

(iii) $\tilde{\pi}$ is onto.

(iv) $\tilde{\pi}^{(N)} = \tilde{Id}$.

(v) $\tilde{\pi} \neq Id$.

For any formula $\phi(v_1,\ldots,v_n)$, and for any $\mathcal{M}$-(finite sequence) $y_1,\ldots,y_n$ of elements of M^Γ,

(vi) $\pi\|\phi(y_1,\ldots,y_n)\| = \|\phi(\tilde{\pi}y_1,\ldots,\tilde{\pi}y_n)\|$.

Proof.

(i) Let $x \in M^\Gamma$.

(a) If $s(x) = 0$, then $STAB(x) \supseteq G_{K_0}$ and (proposition 55(iii)), $STAB(\tilde{\pi}_x(x)) \supseteq G_{\pi_x[K_0]} = G_{K_0}$.

Thus $x \in D_0$.

(b) If $s(x) > 0$,

$$\bigwedge_{i=0}^{s(x)-1} (STAB(x) \not\supseteq G_{K_i}) \wedge (STAB(x) \supseteq G_{K_{s(x)}}).$$

Therefore, (proposition 55(iii)),

$$\bigwedge_{i=0}^{s(x)-1} (\mathrm{STAB}(\tilde{\pi}_x(x)) \not\supseteq G_{\pi_x[K_i]}) \wedge (\mathrm{STAB}(\tilde{\pi}_x(x)) \supseteq G_{\pi_x[K_{s(x)}]}).$$

As $\langle K_i : i \in \omega\rangle$ is increasing, (definition 56(iv)),

$$\pi_x[K_i] = K_i \qquad 1 \leq i \leq s(x).$$

Thus

$$\bigwedge_{i=0}^{s(x)-1} (\mathrm{STAB}(\pi_x(x)) \not\supseteq G_{K_i}) \wedge (\mathrm{STAB}(\pi_x(x)) \supseteq G_{K_{s(x)}}).$$

This is $\tilde{\pi}x = \tilde{\pi}_x(x) \in D_{s(x)}$.

(ii) Let $x,y \in M^\Gamma$, with $x \neq y$.

(a) Suppose $s(x) = s(y)$.

Then $\pi_x = \pi_y$, (definition 58(ii)),

and $\tilde{\pi}x = \tilde{\pi}_x(x) = \tilde{\pi}_y(x) \neq \tilde{\pi}_y(y) = \tilde{\pi}y$.

(b) Suppose $s(x) \neq s(y)$.

Then $D_{s(x)} \cap D_{s(y)} = 0$, (proposition 57(iii)).

As $\tilde{\pi}x \in D_{s(x)}$ and $\tilde{\pi}y \in D_{s(y)}$, (by (i)), we have $\tilde{\pi}x \neq \tilde{\pi}y$.

(iii) Let $x \in M^\Gamma$.

Put $y = (\pi_x^{-1})^{\sim}x$.

Using proposition 55(iii), one proves as in (i) that

$$y \in D_{s(x)}.$$

Hence $\pi_y = \pi_x$

and $\tilde{\pi}y = \tilde{\pi}_y(y) = \tilde{\pi}_x(y) = x$.

(iv) Trivial. Observe that if $\tilde{\pi}x = y$, $\pi_x = \pi_y$.

(v) Trivial. For example, let $i \neq j = \pi(i)$, and let $a \in X$.

Put $p = \{\langle i,a\rangle\}$ and $q = \{\langle j,a\rangle\}$.

Then $q = \pi p$, and we have, (proposition 35(i)),

$$\tilde{\pi}\hat{x}_p = \hat{x}_{\pi p} = \hat{x}_q \neq \hat{x}_p.$$

(vi) We have

$$\pi\|\phi(y_1,\ldots,y_n)\| = \pi_{y_1,\ldots,y_n}\|\phi(y_1,\ldots,y_n)\| \quad \text{(proposition 59(iv))}$$

$$= \|\phi(\tilde{\pi}_{y_1,\ldots,y_n}(y_1),\ldots,\tilde{\pi}_{y_1,\ldots,y_n}(y_n))\|.$$

As $\|\tilde{\pi}_{y_1,\ldots,y_n}(y_i) = \tilde{\pi}_{y_i}(y_i)\| = 1$, $1 \leq i \leq n$, (proposition 59 (ii)), we have, (definition 58(iii)),

$$\pi\|\phi(y_1,\ldots,y_n)\| = \|\phi(\tilde{\pi}_{y_1}(y_1),\ldots,\tilde{\pi}_{y_n}(y_n))\|$$

$$= \|\phi(\tilde{\pi}y_1,\ldots,\tilde{\pi}y_n)\|. \qquad \square$$

61. Proposition. Propositions 49-53 continue to hold when 'stab' is replaced by 'STAB'. $\square$

V

Models of ZF with automorphisms of order N

In this chapter, in order to obtain the desired models, we select specific X, I, H and G in order to obtain x, π and U such that $\|x \neq \tilde{\pi}x\| \in U$.

Obviously for such an x, if $x \in M_{\alpha}^{\Gamma}$, α must be a non-standard ordinal of $\mathcal{M}$, while propositions 52 and 53 tell us that H must not be definable in $\mathcal{M}$ and not be a dense subclass of an H' that is extendable and definable in $\mathcal{M}$. This will be avoided by an appropriate choice of $G \in \mathrm{Subgr}(I!)$. The rest will be accounted for, on the one hand, along the lines of IV.1, and on the other hand, along the lines of IV.2. (For the first, cf. propositions/definitions 39,40,41,43,44 and 45-8. For the second, cf. propositions/definitions 39,40,41,43,44 and 55-60).

V.1

Preliminary material

Let $\mathcal{M} = \langle M,E \rangle$ be a countable ω-non-standard model of ZF.

Let κ be a non-standard $\mathcal{M}$-(natural number).

κ will be kept fixed throughout this chapter.

Let (in $\mathcal{M}$)

$I = \omega \times \omega \times (\kappa+1)$.

$X = 2$.

$P = C(I,2)$

$B = RO(X^I) = RO(2^I)$.

In the sequel, $\mathcal{M}$, $\mathcal{B}$, I, X, P and κ will have the meanings stated above.

62. <u>Definition</u>. (In $\mathcal{M}$).

Let $\langle\rho_0,\ldots,\rho_\kappa\rangle \in (\omega!)^{\kappa+1}$

$\langle\rho_0,\ldots,\rho_\kappa\rangle$ induces a permutation

$\rho = \langle\rho_0,\ldots,\rho_\kappa\rangle^{\dagger}$ of I, given by

$$\rho\langle i,j,0\rangle = \langle\rho_0,\ldots,\rho_\kappa\rangle^{\dagger}(\langle i,j,0\rangle) = \langle i,\rho_0(j),0\rangle; \quad i,j \in \omega$$

$$\rho\langle i,j,h\rangle = \langle\rho_0,\ldots,\rho_\kappa\rangle^{\dagger}(\langle i,j,h\rangle) = \langle\rho_{h-1}(i),\rho_h(j),h\rangle; \; i,j \in \omega; \; h \leq \kappa.$$

To simplify notations we will write ρ for any of the $\rho_0,\ldots,\rho_\kappa,\rho$, unless the distinction between them needs to be made explicit.

Following definition 62, we trivially have

63. <u>Proposition</u>

$$\rho_i^{(N)} = \mathrm{Id}/\omega, \; 0 \leq i \leq \kappa \rightarrow \rho^{(N)} = \mathrm{Id}/I_E. \qquad \square$$

64. <u>Definition</u>. (In $\mathcal{M}$. Cf. definition 62).

(i) Let $A = \{\langle\rho_0,\ldots,\rho_\kappa\rangle: (\rho_i \in \omega!) \wedge (\rho_i^{(N)} = \mathrm{Id}/\omega), \; 0 \leq i \leq \kappa\}$.

(ii) Let $G = \{x^{\dagger}: x \in A\}$

$= \{\rho \in I!: \langle\rho_0,\ldots,\rho_\kappa\rangle \in A\}$

(iii) For each $\langle i,j,h\rangle \in I$

put $G_{j,h} = \{\rho \in G: \rho_h(j) = j\}$,

and $G_{\langle i,j,h\rangle} = \{\rho \in G: \rho\langle i,j,h\rangle = \langle i,j,h\rangle\}$.

Clearly we have $G_{\langle i,j,k\rangle} = G_{i,h-1} \cap G_{j,h}$ and

$G_{\langle i,j,h\rangle} \subseteq G_{j,h}$ for every $i \in \omega$.

(iv) For every finite

$$J = \{\langle j_1, h_1\rangle, \ldots, \langle j_n, h_n\rangle\} \subseteq \omega \times (\kappa+1), \text{ set}$$

$$G_J = \bigcap_{m=1}^{n} G_{j_m, h_m}.$$

(v) For every finite $K \subseteq I$, put

$$G_K = \bigcap_{\eta \in K} G_\eta.$$

65. Definition. (In $\mathfrak{M}$)

$$\Gamma = \{L : L \in \mathrm{Subgr}(G) \wedge (\exists \text{ finite } J \subseteq \omega \times (\kappa+1))(G_J \subseteq L)\}.$$

66. Proposition.

(i) G is a subgroup of I!

(ii) Γ is a normal filter of subgroups of G. □

67. Definition.

(i) (In $\mathfrak{M}$). Let $p \in P$. We say that p is full if for every m, n, i, j $\in \omega$, and $h \in \kappa+1$, we have

$$(\langle i,j,h\rangle \in \mathrm{dom}\ p) \wedge (\langle n,m,h\rangle \in \mathrm{dom}\ p \rightarrow (\langle i,m,h\rangle \in \mathrm{dom}\ p) \wedge (\langle n,j,h\rangle \in \mathrm{dom}\ p).$$

(ii) (In $\mathfrak{M}$).

$$\Sigma = \{\sigma : (\exists \rho \in G)(\exists p \in P)(\sigma = \rho \restriction \mathrm{dom}\ p \wedge \sigma \in (\mathrm{dom}\ p)!\}.$$

(iii)

$$H = \{\langle\sigma, p\rangle \in \Sigma \times P : (p \text{ is full}) \wedge (\sigma p = p) \wedge (\forall i \in I)(\sigma(i) \neq i \rightarrow \mathrm{proj}_2(i) \text{ is non-standard})\}.$$

Remark. Observe that if $\mathfrak{M}$ is standard,

$$\langle\sigma, p\rangle \in H \rightarrow \sigma = \mathrm{Id} \restriction \mathrm{dom}\ p.$$

Hence $\mathcal{M}$ is standard $\leftrightarrow H \in M$.

In the proof of 68, below, the following definition will be used. Let (in $\mathcal{M}$), $r \in C(I,2)$, $X \subseteq \mathrm{dom}\ r$ and $\delta \in \{0,1\}$.

Define

$$r(X/\delta\rangle = r'$$

as the forcing condition that results from r when r(Y) is made to be δ for every $Y \in X$ and everything else remains unchanged. I.e., for every $\langle x,y,z\rangle \in \mathrm{dom}\ r$,

$$r'\langle x,y,z\rangle = \delta \qquad , \text{ if } \langle x,y,z\rangle \in X$$

$$r'\langle x,y,z\rangle = r\langle x,y,z\rangle, \text{ otherwise.}$$

68. Proposition. (Cohen [6]).

H is extendable.

Proof. Let $\langle\sigma,p\rangle \in H$, and let $q \leqslant p$.

We prove by induction on the number of extra elements of q, that there exists $\langle\rho,r\rangle$ such that

$$\langle\rho,r\rangle \in H,\ \langle\rho,r\rangle \leqslant \langle\sigma,p\rangle, \text{ and } r \leqslant q \leqslant p. \qquad (*)$$

(I). Let

$$q = p \cup \{\langle\langle a_0,b_0,j\rangle,\delta\rangle\}, \text{ where } \delta \in \{0,1\}.$$

Let, (see definition 63),

$$\mathrm{dom}\ \sigma_{j-2} = \{\underline{h}_i : 0 \leqslant i \leqslant s\},$$

$$\mathrm{dom}\ \sigma_{j-1} = \{\bar{h}_i : 0 \leqslant i \leqslant t\},$$

$$\mathrm{dom}\ \sigma_j = \{\underline{k}_i : 0 \leqslant i \leqslant u\} \text{ and}$$

$$\mathrm{dom}\ \sigma_{j+1} = \{\bar{k}_i : 0 \leqslant i \leqslant v\},$$

where s,t,u,v are $\mathcal{M}$-(natural numbers).

Let $a_1,\ldots,a_N$ be pairwise different $\mathcal{M}$-(natural numbers) such that

$$a_i \notin \operatorname{dom} \sigma_{j-1}, \qquad 0 \leq i \leq N.$$

Similarly, let $b_1,\ldots,b_N$ be pairwise different $\mathcal{M}$-(natural numbers) such that

$$b_i \notin \operatorname{dom} \sigma_j, \qquad 0 \leq i \leq N.$$

Observing that the case

$$a_0 \in \operatorname{dom} \sigma_{j-1}, \quad b_0 \in \operatorname{dom} \sigma_j$$

is ruled out, since $\langle\langle a_0,b_0,j\rangle,\delta\rangle \notin p$ and p is full, the following cases are possible.

(i) $a_0 \in \operatorname{dom} \sigma_{j-1}$, $b_0 \notin \operatorname{dom} \sigma_j$.

(ii) $a_0 \notin \operatorname{dom} \sigma_{j-1}$, $b_0 \in \operatorname{dom} \sigma_j$.

(iii) $a_0 \notin \operatorname{dom} \sigma_{j-1}$, $b_0 \notin \operatorname{dom} \sigma_j$.

Case (i)

First, we extend q to be full:

Put

$$q' = p \cup \bigcup_{i=0}^{u} \{\langle\langle \underline{k}_i,b_0,j\rangle,\delta\rangle\} \cup \bigcup_{i=0}^{v} \{\langle\langle b_0,\bar{k}_i,j\rangle,\delta\rangle\}$$

Then q' is full and q' $\leq$ q.

Define

$$\rho = \sigma \cup \{\langle b_0,b_1\rangle,\ldots,\langle b_{N-1},b_N\rangle,\langle b_N,b_0\rangle\}, \text{ and}$$

$$r = q' \cup \bigcup_{m=1}^{N} \bigcup_{i=0}^{u} \{\langle\langle \rho^{(m)}(\underline{k}_i),\rho^{(m)}(b_0),j\rangle,\delta\rangle\}$$
$$\cup \bigcup_{m=1}^{N} \bigcup_{i=0}^{v} \{\langle\langle \rho^{(m)}(b_0),\rho^{(m)}(k_i),j+1\rangle,\delta\rangle\}.$$

Then one verifies that $\langle\rho,r\rangle$ satisfies (*).

Case (ii)

Symmetric to (i).

Case (iii)

Again, we extend q to be full:

Put

$$q' = p \cup \bigcup_{i=0}^{s} \{\langle\langle \underline{h}_i, a_0, j-2\rangle, \delta\rangle\} \cup \bigcup_{i=0}^{t} \{\langle\langle a_0, \bar{h}_i, j-1\rangle, \delta\rangle\}$$

$$\cup \bigcup_{i=0}^{u} \{\langle\langle \underline{k}_i, b_0, j\rangle, \delta\rangle\} \cup \bigcup_{i=0}^{v} \{\langle\langle b_0, \bar{k}_i, j+1\rangle, \delta\rangle\}$$

Then q' is full and $q' \leq q$.

Define

$$\rho = \sigma \cup \{\langle a_0,a_1\rangle,\ldots,\langle a_{N-1},a_N\rangle,\langle a_N,a_0\rangle\}$$

$$\cup \{\langle b_0,b_1\rangle,\ldots,\langle b_{N-1},b_N\rangle,\langle b_N,b_0\rangle\}, \text{ and}$$

$$r = q' \cup \bigcup_{m=1}^{N} \bigcup_{i=1}^{s} \{\langle\langle \rho^{(m)}(\underline{h}_i), \rho^{(m)}(a_0), j-2\rangle, \delta\rangle\}$$

$$\cup \bigcup_{m=1}^{N} \bigcup_{i=1}^{t} \{\langle\langle \rho^{(m)}(a_0), \rho^{(m)}(\bar{h}_i), j-1\rangle, \delta\rangle\}$$

$$\cup \bigcup_{m=1}^{N} \bigcup_{i=1}^{u} \{\langle\langle \rho^{(m)}(\underline{k}_i), \rho^{(m)}(b_0), j\rangle, \delta\rangle\}$$

$$\cup \bigcup_{m=1}^{N} \bigcup_{i=1}^{v} \{\langle\langle \rho^{(m)}(b_0), \rho^{(m)}(\bar{k}_i), j+1\rangle, \delta\rangle\}.$$

Again, one verifies that $\langle\rho,r\rangle$ satisfies (*).

II. Let

$$q = p \cup \bigcup_{i=0}^{n+1} \{\langle\langle a_i,b_i,j_i\rangle,\delta_i\rangle\}, \text{ where } \delta_i \in \{0,1\},\ 0 \leq i \leq n+1.$$

For the sake of legibility, put $j_{n+1} = j$ and $\delta_{n+1} = \delta$.

Let

$$q' = p \cup \bigcup_{i=0}^{n} \{\langle\langle a_i,b_i,j_i\rangle,\delta_i\rangle\},$$

and assume (induction hypothesis) that there exists $\langle\rho',r'\rangle$ such that

$\langle\rho',r'\rangle \in H$, $\langle\rho',r'\rangle \leq \langle\sigma,p\rangle$ and $r' \leq q' \leq p$.

(A) If $\langle a_{n+1},b_{n+1},j\rangle \in \text{dom } r'$, then either

(a) $\langle\langle a_{n+1},b_{n+1},j\rangle,\delta\rangle \in r'$, or

(b) $\langle\langle a_{n+1},b_{n+1},j\rangle,\delta\rangle \notin r'$.

If (a) is the case, then $\langle\rho,r\rangle = \langle\rho',r'\rangle \in H$ satisfies (*).

If (b) is the case, then

$\langle\langle a_{n+1},b_{n+1},j\rangle,1-\delta\rangle \in r'$. Furthermore

$\langle\langle\rho^{(m)}(a_{n+1}),\rho^{(m)}(b_{n+1}),j\rangle,1-\delta\rangle \in r'$, for $m = 0,1,\ldots,N$.

Put

$X = \{\langle\langle\rho^{(m)}(a_{n+1}),\rho^{(m)}(b_{n+1}),j\rangle,1-\delta\rangle : 0 \leq m \leq N\}$, and

$r'' = r(X/\delta)$.

Then, one verifies that $\langle\rho',r''\rangle \in H$, and that $\langle\rho,r\rangle = \langle\rho',r''\rangle$ satisfies (*).

(B) If $\langle a_{n+1},b_{n+1},j\rangle \notin \text{dom } r'$, then case I applies with $\langle\rho',r'\rangle$ instead of $\langle\sigma,p\rangle$, and a_{n+1},b_{n+1} in place of a,b, respectively.

This completes the proof of 69. □

Remarks.

(i) Observe that if the requirement that the p's be full were omitted in 67(iii), the resulting H would not be extendable.

(ii) Let $\langle\rho_0,\ldots,\rho_{\kappa+1}\rangle \in (\omega!)^{\kappa+2}$. Then $\langle\rho_0,\ldots,\rho_{\kappa+1}\rangle$ induces a permutation $\rho = \langle\rho_0,\ldots,\rho_{\kappa+1}\rangle^{\dagger}$ of I given by

$$\rho\langle i,j,h\rangle = \langle\rho_h(i),\rho_{h+1}(j),h\rangle, \quad 0 \leq h \leq \kappa.$$

Put $B = \{\langle\rho_0,\ldots,\rho_{\kappa+1}\rangle \in (\omega!)^{\kappa+2} : \rho_i^{(N)} = \text{Id}\restriction\omega,\ 0 \leq i \leq \kappa+1\}$.

Let $G' = \{g(x) : x \in B\}$ and define $G'_{j,h}$, $G'_{\langle i,j,h\rangle}$, G'_J, G'_K and Γ'

as in 64(iii), (iv), (v) and 65, respectively, replacing G' for G and Γ' for Γ. Then it is easy to see that proposition 66 holds for G' and Γ'. Also define Σ' as in 67 with G' in the place of G. Finally, set

$$H' = \{\langle\sigma,p\rangle \in \Sigma' \times P: (p \text{ is full}) \wedge (\sigma p = p)\}.$$

We trivially have that H is a dense subclass of H' and also that H' is definable in $\mathcal{M}$. In addition, one proves as in 68 that H' is extendable. Hence, proposition 53 applies here. Whence, no solution can be achieved with G' defined as above.

69. Definition. (In $\mathcal{M}$).

For every $\langle j,h\rangle \in \omega \times (\kappa+1)$, set

$$u_{j,0} = \{\langle\hat{i},\{\langle\langle i,j,0\rangle,1\rangle\}\rangle: i \in \omega\}, \text{ and}$$

$$u_{j,h} = \{\langle u_{i,h-1},\{\langle\langle i,j,h\rangle,1\rangle\}\rangle: i \in \omega\},\ h > 0.$$

70. Proposition. (In $\mathcal{M}$).

(i) For every $\rho \in G$ and every $\langle j,h\rangle \in \omega \times (\kappa+1)$,

$$\tilde{\rho}u_{j,h} = u_{\rho(j),h}.$$

(More precisely, $\tilde{\rho}u_{j,h} = u_{\rho_h(j),h}$. Cf. remark after definition 62).

(ii) $i \neq j \rightarrow \|u_{i,h} \neq u_{j,h}\| \neq 0;\ i,j \in \omega$. (Cf. remark (iii), p.76)

Proof.

(i)

(a) For h = 0, we have

$$\begin{aligned}\text{dom}(\tilde{\rho}u_{j,0}) &= \tilde{\rho}[\text{dom } u_{j,0}]\\ &= \tilde{\rho}[\{\hat{i}: i \in \omega\}]\\ &= \{\tilde{\rho}\hat{i}: i \in \omega\}\\ &= \{\hat{i}: i \in \omega\}\\ &= \text{dom}(u_{\rho(j),0}).\end{aligned}$$

Also,

$$
\begin{aligned}
(\tilde{\rho}u_{j,0})(\hat{i}) &= (\tilde{\rho}u_{j,0})(\tilde{\rho}\hat{i}) \\
&= \rho(u_{j,0}(\hat{i})) \\
&= \rho\{\langle\langle i,j,0\rangle,1\rangle\} \\
&= \{\langle\rho\langle i,j,0\rangle,1\rangle\} \\
&= \{\langle\langle i,\rho_0(j),0\rangle,1\rangle\} \\
&= u_{\rho_0(j),0}(\hat{i}) \\
&= u_{\rho(j),0}(\hat{i}).
\end{aligned}
$$

Thus $\tilde{\rho}u_{j,0} = u_{\rho(j),0}$.

(b) For $h > 0$, suppose that the assertion is true for $h-1$.

We have

$$
\begin{aligned}
\mathrm{dom}(\tilde{\rho}u_{j,h}) &= \tilde{\rho}[\mathrm{dom}\ u_{j,h}] \\
&= \tilde{\rho}[\{u_{i,h-1}: i \in \omega\}] \\
&= \{\tilde{\rho}u_{i,h-1}: i \in \omega\} \\
&= \{u_{\rho_{h-1}(i),h-1}: i \in \omega\} \quad \text{(Induction hypothesis)} \\
&= \{u_{i,h-1}: i \in \omega\} \quad (\text{since } \rho_{h-1} \in \omega!) \\
&= \mathrm{dom}(u_{\rho_h(j),h})
\end{aligned}
$$

Also

$$
\begin{aligned}
(\tilde{\rho}u_{j,h})(u_{\rho_{h-1}(i),h-1}) &= (\tilde{\rho}u_{j,h})(\tilde{\rho}u_{i,h-1}) \quad \text{(Induction hypothesis)} \\
&= \rho(u_{j,h}(u_{i,h-1})) \\
&= \rho\{\langle\langle i,j,h\rangle,1\rangle\} \\
&= \{\langle\rho\langle i,j,h\rangle,1\}\\
&= \{\langle\langle\rho_{h-1}(i),\rho_h(j),h\rangle,1\rangle\} \\
&= u_{\rho_h(j),h}(u_{\rho_{h-1}(i),h-1}).
\end{aligned}
$$

Thus $\tilde{\rho}u_{j,h} = u_{\rho(j),h}$.

(ii) One sees that for $i \neq j$,

$$\|u_{i,h} = u_{j,h}\| \neq 1.$$

For example, let $f \in 2^I$ be such that

$f\langle m,j,h\rangle = 0$ for every $m \in \omega$, and

$f\langle n,i,h\rangle = 1$ for every $n \in \omega$.

This is possible, since $i \neq j$.

Then $f \notin \|u_{i,h} = u_{j,h}\|$. □

At this point, we diverge in two directions. One along the lines of IV.1; the other along the lines of IV.2.

V.2

First construction

(Cf. IV.1).

Throughout V.2 we apply all the contents of sections I, II, III and IV.1, to the particular case described in V.1.

71. Proposition (In $\mathcal{M}$).

For every $j \in \omega$ and $h \lesssim \kappa$,

$$\mathrm{stab}(u_{j,h}) \supseteq G_{j,h}.$$

Therefore, for every $j \in \omega$ and $h \lesssim \kappa$,

$$u_{j,h} \in M^{\Gamma} \text{ and}$$

$$\mathrm{stab}(u_{j,h}) \supseteq G_{\langle n,j,h\rangle}, \text{ for every } n \in \omega.$$

Proof. Immediate, after proposition 70(i). □

72. Construction. Part of the construction involves work performed

inside $\mathcal{M}$. The rest is carried out outside $\mathcal{M}$. We indicate that the work is being performed outside $\mathcal{M}$ by putting it between the signs '(⊥)' and '(⊤)'. If no sign is used, that means that we are staying in $\mathcal{M}$.

Definition 7 and remarks after proposition 8 must be kept in mind.

Let $h \in \kappa+1$ (⊥) be non-standard (⊤), and let $i,j \in \omega$ be such that $i \neq j$.

(⊥) Let $\rho^{\cdot} \in G$ be such that (⊤) $\rho^{\cdot}_h(i) = j$.

Then, (proposition 70(i)),

$$\tilde{\rho}^{\cdot} u_{i,h} = u_{\rho^{\cdot}_h(i),h} = u_{\rho^{\cdot}(i),h}$$

Also, (proposition 71), for every $n \in \omega$

$$\mathrm{stab}(u_{i,h}) \supseteq G_{<n,i,h>} \text{ and}$$

$$\mathrm{stab}(u_{\rho^{\cdot}(i),h}) \supseteq G_{<n,\rho^{\cdot}(i),h>}.$$

Furthermore, (proposition 70(ii)),

$$\|u_{i,h} \neq u_{\rho^{\cdot}(i),h}\| \neq 0.$$

Thus $\|u_{i,h} \neq \tilde{\rho}^{\cdot} u_{i,h}\| \neq 0$.

Therefore, for some $q \in P$

$$q \leq \|u_{i,h} \neq \tilde{\rho}^{\cdot} u_{i,h}\|.$$

Since q is finite, let $n \in \omega$ be such that

$$<n,i,h> \notin \mathrm{dom}\ q$$

and $$<n,j,h> \notin \mathrm{dom}\ q.$$

There is no loss of generality in assuming that $\rho^{\cdot}_{h-1}(n) = n$.

Let $p = \{<<n,i,h>,1>,<<n,j,h>,1>\}$.

Then $\mathrm{stab}(u_{i,h}) \supseteq G_{\mathrm{dom}\ p}$

and $\mathrm{stab}(u_{\rho^{\cdot}(i),h}) \supseteq G_{\mathrm{dom}\ p}$

Put $u_{i,h} = x$ and $u_{\rho^\bullet(i),h} = y$.

(⊥) In definition 46(ii), arrange things so that

$$J_x = \{<n,i,h>\} \text{ and } J_y = \{<n,j,h>\} \quad (\top).$$

Then, using $\rho^\bullet$ instead of π in definition 46,

$$K_x = K_y = \text{dom } p$$

Now, define $\sigma \in (\text{dom } p)!$ as

$$\sigma\{<<n,i,h>,1>\} = \{<<n,j,h>,1>\}$$

$$\sigma\{<<n,j,h>,1>\} = \{<<n,i,h>,1>\}$$

Then: Comp(p,q),

$$\sigma = \rho^\bullet \restriction \text{dom } p$$

and (⊥) $<\sigma,p> \in H$ (⊤).

As $p \wedge q \leqslant p$,

(⊥) There exists, (proposition 68),

$$<\sigma_1,p_1> \in H \quad (\top)$$

such that $<\sigma_1,p_1> \leqslant <\sigma,p>$

and $p_1 \leqslant p \wedge q \leqslant p$.

(⊥) Let $\rho \in G$ be such that (⊤)

$$\rho \restriction \text{dom } p_1 = \sigma_1, \text{ (always possible)}.$$

Then

$$\tilde{\rho}^\bullet u_{i,h} = u_{\rho^\bullet(i),h} = u_{\rho(i),h} = \tilde{\rho} u_{i,h}.$$

Now, (⊥) let $\underset{\cdot}{G}$ be a generic subset of H such that $<\sigma_1,p_1> \in \underset{\cdot}{G}$.

Then $\underset{\cdot}{G}_P^{\leqslant}$ is a generic subset of P, (proposition 40), and

$$p_1 \in \underset{\cdot}{G}_P^{\leqslant}.$$

Let U be the generic ultrafilter associated with $G^{<}_{\cdot P}$.

Then we have $p_1 \in U$ (T)

and

$$p_1 \leq p \wedge q \leq q \leq \|u_{i,h} \neq \tilde{\rho}^{\cdot} u_{i,h}\| = \|u_{i,h} \neq \tilde{\rho}^{\cdot} u_{i,h}\|.$$

Therefore

$$(\perp) \quad \|u_{i,h} \neq \tilde{\rho} u_{i,h}\| \in U$$

Finally, put

$$\pi = \cup G_{\cdot\Sigma}.$$

Then, (proposition 45(i) and definition 46(iv)),

$$\tilde{\rho} u_{i,h} = \tilde{\pi} u_{i,h}.$$

Thus $\quad \|u_{i,h} \neq \tilde{\pi} u_{i,h}\| \in U.$ (1)

By proposition 44(v), we know that

$$\pi[U] = U.$$

Finally, proposition 36 allows us to define an automorphism of $\mathcal{M}^{\Gamma}/U$ via

$$x^u \to (\tilde{\pi} x)^u.$$

By proposition 63, it is obvious that such automorphism has order N, while (1), above, shows that it is a non-trivial automorphism of $\mathcal{M}^{\Gamma}/U$. (T). □

Remark. All the work in V.2 can be performed with $I = \omega \times \omega \times \omega$ instead of $\omega \times \omega \times (\kappa+1)$. We have preferred the latter only in order to follow more closely Cohen's work. (Cohen [6]).

The same applies to V.3.

V.3

Second construction

Throughout V.3 we apply all the contents of chapters I, II, III and IV.2 to the particular case described in V.1.

73. Proposition. For every $j \in \omega$, and $h \leq \kappa$,

$$STAB(u_{j,h}) \supseteq G_{j,h}.$$

Therefore $u_{j,h} \in M^{\Gamma}$, and

$$STAB(u_{j,h}) \supseteq G_{<n,j,h>} \text{ for every } n \in \omega. \qquad \square$$

74. Construction. Follow the same steps prescribed in construction 72, with the following changes, (in this order):

(a) Instead of 'stab', use 'STAB'.

(b) Instead of definition 46, use definition 56.

(c) In definition 56(ii), put

$$x_0 = u_{i,h} \text{ and } x_1 = u_{j,h}.$$

Then, instead of J_x and J_y (in 72), put

$$J_0 = \{<n,i,h>\} \text{ and } J_1 = J_0 \cup \{<n,j,h>\},$$

respectively.

(d) Instead of proposition 45(i), apply proposition 55(i). $\square$

Remarks. (i) The construction described in 74 can be given, mutatis mutandis, instead of the one in 72, V.2. The converse does not apply. (Cf. remark (i), after definition 58.)

(ii) Although the M^{Γ}'s of V.1 and V.2 are different, it is not clear whether their quotients over U essentially differ.

(iii) In proposition 70(ii), we actually have

$$j \neq k \rightarrow \| u_{j,k} = u_{k,h} \| = 0, \text{ for } h \in \omega,$$

which we prove by induction on h.

(1) First, it is easy to see that

$$\| \hat{i} \in u_{k,0} \| = u_{k,0}(\hat{i}), \text{ for } k \in \omega.$$

Then we have

$$\| u_{j,0} = u_{k,0} \| \leq \bigwedge_{i \in \omega} (u_{j,0} \Rightarrow \| \hat{i} \in u_{k,0} \|)$$

$$= \bigwedge_{i \in \omega} (u_{j,0}(\hat{i}) \Rightarrow u_{k,0}(\hat{i}))$$

Now, suppose that $\| u_{j,0} = u_{k,0} \| \neq 0$. Then $p \leq \| u_{j,0} = u_{k,0} \|$, for some $p \in P$. Let $\ell \in \omega$ be such that $\langle \ell,j,0 \rangle, \langle \ell,k,0 \rangle \notin \text{dom } p$, (always possible), and put $q = p \cup \{\langle\langle \ell,j,0\rangle,1\rangle, \langle\langle \ell,k,0\rangle,0\rangle\}$. Then $q \leq p \leq \bigwedge_{i \in \omega} (u_{j,0}(\hat{i}) \Rightarrow u_{k,0}(\hat{i}))$.

In particular,

$$q \leq (u_{j,0}(\hat{\ell}) \Rightarrow u_{k,0}(\hat{\ell}))$$

$$= \{\langle\langle \ell,j,0\rangle,0\rangle\} \vee \{\langle\langle \ell,k,0\rangle,1\rangle\},$$

a contradiction.

(2) Using the induction hypothesis, it is easy to see that

$$\| u_{i,h} \in u_{j,h+1} \| = u_{j,h+1}(u_{i,h}).$$

Then we have

$$\|u_{j,h+1} = u_{k,h+1}\| \leq \bigwedge_{i\in\omega} [u_{j,h+1}(u_{i,h}) \Rightarrow \|u_{i,h} \in u_{k,h+1}\|]$$

$$= \bigwedge_{i\in\omega} [u_{j,h+1}(u_{i,h}) \Rightarrow u_{k,h+1}(u_{i,h})]$$

An argument similar to that of the end of (1) shows that the last expression is 0. □

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