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TOPICS IN NON-COMMUTATIVE PROBABILITY THEORY WITH APPLICATIONS TO STATISTICAL MECHANICS

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THESIS SUBMITTED FOR THE PHD DEGREE

OF THE UNIVERSITY OF LONDON

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ABSTRACT

Chapter I contains a presentation of Non-Commutative Integration theory. The relation between Segal's [67] and Nelson's [46] definition of measurability is investigated, and a new proof of duality for non-commutative probability L_p spaces is given.

In chapter II, known results on isometries between Banach spaces of functions and operators are presented, and a new proof of the fact that unit-preserving isometries of abelian C^* algebras are $*$- isomorphisms is given. It is shown that unit-preserving $*$-isometries between non-commutative probability L_p spaces come from Jordan $*$-homomorphisms and several conclusions are drawn.

Chapter III is a presentation of Tomita-Takesaki theory. Possible generalizations are pointed out, and the Radon-Nikodym theorem is discussed.

In chapter IV the characterization of equilibrium in Quantum Statistical Mechanics by the KMS condition is investigated.

In chapter V, a class of Gibbs states w_β is defined on the algebra $\mathfrak{A}$ of the canonical commutation relations in infinitely many degrees of freedom. This is done by showing that for any $\beta > 0$ the second quantization H of a hamiltonian with positive polynomially bounded discrete spectrum defines a nuclear operator $\exp(-\beta H)$ from Fock space into $\mathcal{G}$, a generalization of Schwartz space for infinitely many variables.

This allows the construction of an "almost modular" Hiblert subalgebra $\tilde{\mathfrak{A}}$ of $\mathfrak{A}$ on which the modular automorphisms may be defined, and satisfy the KMS condition.

The final chapter contains a proof of a commutation theorem, namely that the commutant of $\mathfrak{A}$ in the GNS representation π_β induced by w_β is invariant under the modular automorphisms, and is isomorphic to its own commutant via an antiunitary involution of the GNS Hilbert space. This is done by showing that π_β is unitarily equivalent to left multiplication on Hilbert-Schmidt operators on Fock space, acting on a suitable tensor product of $\mathcal{G}$ with Fock space.

ACKNOWLEDGEMENTS

It gives me much pleasure to thank the following:

Professor R.F. Streater, who supervised this work.

Ms Ingeborg Koch for her collaboration.

Dr. Bernard Kay for helpful comments.

Ms Christine Mufti for her help with typing.

I would also like to express my gratitude to the University of London for financially supporting me during the conduct of this work.

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INTRODUCTION

In Quantum Field Theory and Statistical Mechanics one is often confronted with a complex involutive algebra and a positive linear functional on it, representing (the algebra generated by) the observables of a given physical system together with their expectation values.

It is therefore important to have a mathematical theory dealing with such systems in the abstract.

When the algebra in question is commutative, as in classical systems [#], then it is (isomorphic to) an algebra of measurable functions on a measure space, with the integral as the linear functional [##]. This observation is probably the origin of the term "Algebraic Integration Theory". [70]

\# And the algebra is sufficiently "well behaved", i.e. essentially may be realized as an algebra of operators acting on a Hilbert space, which are either bounded or generated by a self-adjoint operator.

\## This is essentially a combination of the GNS construction and the Gelfand theory. See SEGAL [70] where a very interesting review of the whole subject by one of its main authors is given, as it was in 1965, i.e. before Tomita-Takesaki theory showed that one can "integrate" with respect to a non-central linear functional.

When the algebra is non-commutative, as in quantum systems, the situation is less transparent. Here one would like to have a "Non-Commutative" Integration Theory, or, in the case of a bona fide linear functional # a "Non-Commutative" Probability Theory (Commutative Probability Theory being by now recognised to be the study of a commutative (real) abstract algebra, whose elements are interpreted as the "random variables", with a linear functional on it, whose value at an element of the algebra is interpreted as the expectation of the random variable). One would furthermore like to do this with algebras that do not have continuity properties, since one knows that observables in quantum theory are often unbounded. However, such a theory does not exist at present. Attempts have been made in this direction by GUDDER and HUDSON [22] but have only yielded partial results. I attempt in the second part of this thesis to develop such a "Non-Commutative Probability Theory" relevant to a class of examples of particular interest in Statistical Mechanics, where things are not entirely pathological.

As opposed to a function from the positive part of the algebra to $[0,+\infty]$, a situation relevant for example in the integration of continuous functions over a non-compact space.

When the given algebra (can be identified with) a Von Neumann algebra, then the theory is much better developed. Looking for the analogue of the integral as a linear functional on L^1 of a measure space, SEGAL [67] observed that the "dimension function" for a factor, constructed by MURRAY and VON NEUMANN [44], had the countable additivity property of a measure, and in addition had the property of being unitarily invariant, a feature which, in SEGAL'S [70] words, "compensates for the circumstance that the lattice of all projections is not Boolean" - a consequence of non-commutativity. Indeed the (generalized) Hölder inequality, and the consequent construction of "Non-Commutative L_p-spaces" (see chapter I) is invalid (for $p \neq 2$) without this additional feature, as observed by DIXMIER [14].

In any case, starting from a Von Neumann algebra with a unitarily invariant positive homogenous additive function from the positive elements of the algebra into $[0,+\infty]$ # - the "integral" - one develops a whole "Non-Commutative Integration Theory", parallel to the conventional one, with its Radon-Nikodym theorems, duality theorem, Riesz-Fischer, Lebesgue dominated convergence, Fubini theorems, etc., all

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This functional is defined either essentially by extension from the MURRAY-VON NEUMANN dimension function (SEGAL [67]), or directly axiomatically (DIXMIER [14], whose treatment we shall follow in part).

more or less direct analogues of the classical theorems with the same names. In this dissertation I shall describe the essential features of this theory, and make a small contribution to it, by proving some results in non-commutative probability theory, which are extensions of recent results in the conventional theory, and which, suitably extended, may throw some light on the structure of the non-commutative L_p-spaces and eventually Von Neumann algebras themselves [82].

Returning to the general theory, it was discovered that it could be used to prove several theorems in the theory of operator algebras, such as the commutation theorem for tensor products (see Chapter III, §1). However, the condition that the "integral" be unitarily invariant imposes restrictions on the algebra : the algebras for which such non-trivial (in a sense to be made precise later) "integrals" existed were seen to coincide with "semifinite" Von Neumann algebras; that is, those that did not have a type III component, in the terminology of MURRAY and VON NEUMANN [44]. However, it was soon shown that these by no means exhausted all Von Neumann algebras; not even those of direct physical relevance. In fact it became apparent [30] that type III factors were the general rule in the description of infinite systems in equilibrium at a finite non-zero temperature in Statistical mechanics.

On the Pure Mathematical side, one of the difficulties with such Von Neumann algebras was that they were not seen to have a "standard" form, which was indeed the case (modulo isomorphisms) for semifinite algebras (see Chapter III §1). As a result, for instance, the commutation theorem for tensor products remained an open question for such algebras for a long time. It was here that TOMITA's [84], [80] theory came in to show that any Von Neumann algebra with a faithful normal state # (whether unitarily invariant or not) could be put in a "standard form". This opened the way for a series of spectacular developments in the structure theory of Von Neumann algebras. From the point of view of Physics, the realization of the connection of TOMITA'S theory with the KMS condition in statistical mechanics [80], also opened the way for new developments in both fields (see Chapter IV).

Thus one now has a new kind of "Non-Commutative Integration" Theory, in which the objects to be integrated are again operators, as in the SEGAL-DIXMIER theory, but the "integral" is no longer required to be unitarily invariant.

In this dissertation I attempt an extension of this theory to a case where the operators to be integrated are no longer bounded, but still behave sufficiently well for the theory to go through. I construct the analogue of Tomita's "Generalized" and "Modular" Hilbert Algebras, and prove the corresponding commutation theorem using the latter object, as in TAKESAKI's [80] work.

The study of a general Von Neumann algebra can be reduced to the study of one with a faithful normal state.

A word about applications to Physics. The connection of Tomita-Takesaki theory with Statistical Mechanics has been made clear above. The specific example which we consider in chapters V and VI has obvious physical motivation. The SEGAL-DIXMIER Non-Commutative Integration theory, however, has a quite distinct, maybe unexpected, range of applications in Euclidean Fermion Quantum Field Theory.

This may be motivated by briefly looking into Euclidean Bose Quantum Field Theory. Here the fields are represented by commuting self-adjoint operators on a Hilbert space (see e.g. JAFFE [32]). Thus the algebra generated by them, together with the (Euclidean) vacuum state, becomes a commutative probability algebra. It can therefore be realized on a probability space, with the fields represented as random variables and the state as the probability measure. The fields then act as multiplication operators on L^2 over this probability space: this is referred to as the "wave picture" as the fields are diagonalized in this representation. This is contrasted with the Fock space situation, in which the number operator is diagonalized: the "particle picture". The fact that these two representations are unitarily equivalent is referred to as "wave-particle duality". (SEGAL [68])

By analogy with the Boson case, one thinks of Euclidean Fermi fields as elements of a non-commutative probability algebra. In fact, if one looks at representations of the Canonical Anticommutation Relations, indexed by a complex Hilbert space $\mathcal{H}$ with conjugation J, the relevant algebra is the "Clifford Algebra" $\mathcal{C}(\mathcal{H})$ over $\mathcal{H}$, acting on anti-symmetric Fock space $\Lambda(\mathcal{H})$ over $\mathcal{H}$. The vacuum state then turns out to be unitarily invariant on $\mathcal{C}(\mathcal{H})$, and the "L^2" space over $\mathcal{C}(\mathcal{H})$ with this state as the "integral" turns out to be isomorphic to Fock space itself, a fact expressing again the wave-particle duality (see SEGAL [69], GROSS [21], WILDE [87] etc.).

This thesis is divided into two parts : the first part deals with the development of the mathematical techniques, while the second part contains the applications to Quantum Statistical Mechanics. Chapter I is devoted to a description of the essential features of the theory of Von Neumann algebras and the DIXMIER-SEGAL Non-Commutative Integration Theory. Chapter II contains a study of the isometries of operator algebras and Non-Commutative L_p-spaces, culminating in a theorem relating certain isometries of non-commutative L_p-spaces to the relation between the algebraic structures of the underlying Von Neumann algebras. The first part of the thesis then concludes with Chapter III, in which the basic results of Tomita-Takesaki theory are described and possible generalizations of the theory investigated. The second part begins with Chapter IV, in which equilibrium states and time-translations in Quantum Statistical Mechanics are investigated, and special emphasis is laid on the KMS condition and its relation to Tomita-Takesaki theory.

The purpose of this chapter is twofold : on the one hand, to justify the use of the KMS condition as the defining property of equilibrium states of infinite systems, and on the other to physically motivate the study of Gibbs states of the canonical commutation relations undertaken in Chapters V and VI. In Chapter V, these states are defined, and the structure of the resulting probability algebra is studied; the special properties of an especially well behaved subalgebra are also described, and the KMS condition is shown to hold. Finally, in Chapter VI the main theorem of the second part of this thesis is proved, which constitutes a generalization of Tomita-Takesaki theory to the algebra of the CCR.

The main results of Chapter II, §3 have already appeared in [36], which is attached to this thesis. Some of the results in Chapters V and VI are joint work with Ingeborg KOCH. (See [37]).

P A R T   A

Chapter I

NON-COMMUTATIVE INTEGRATION

In this Chapter, I will describe the theory of "Non-Commutative Integration" originating in the papers of SEGAL [67] and DIXMIER [14]. This theory generalizes ordinary integration theory in the sense described in the introduction. As pointed out there, the objects to be integrated are no longer functions on a measure space, but rather operators in a Von Neumann algebra, and the "integral" is a function from positive operators to (extended) positive real numbers. In Non-Commutative Probability theory, with which we are principally concerned, the total "measure" of the space (i.e. the "integral" of the identity) is finite, and thus the "integral" defines a positive linear functional on the algebra.

Thus we must first discuss the basic facts about Von Neumann algebras and their linear forms. These facts are all taken, unless otherwise specified, from DIXMIER [16].

§1.1 C^* algebras

A C^* algebra is an involutive algebra with a norm making it a Banach space, with the properties

$$\|xy\| \leq \|x\|\|y\|$$
$$\|x^*x\| = \|x\|^2$$

Any C^* algebra is isometrically $*$-isomorphic to a concrete C^* algebra, that is, a uniformly closed $*$-subalgebra of the C^* algebra of all bounded operators on a Hilbert space (SAKAI [62] 1.16.6).

Any abelian unital C^* algebra is isometrically $*$-isomorphic to the C^* algebra $C(X)$ of all continuous functions on a compact Hausdorff space X, equipped with the supremum norm. In fact, we have:

Theorem (GELFAND-NAIMARK ; see [62] 1.2.1)

Let $\mathfrak{A}$ be an abelian unital C^* algebra, X the set of all non-zero homomorphisms (characters) $\phi: \mathfrak{A} \longrightarrow \mathbb{C}$. Equipped with the $\sigma(X,\mathfrak{A})$ topology +, i.e. the w^*-topology on X induced by the dual of $\mathfrak{A}$, X is a compact Hausdorff space, called the spectrum of $\mathfrak{A}$. Moreover, the mapping $x \longmapsto \hat{x}$ (the Gelfand Transform) given by

$$\hat{x}(\phi) = \phi(x) \qquad x \in \mathfrak{A}, \ \phi \in X$$

is an isometric $*$-isomorphism of $\mathfrak{A}$ onto $C(X)$.

§1.2. Von Neumann algebras

Consider first the algebra $B(\mathcal{H})$ of all bounded operators on a Hilbert space $\mathcal{H}$. Equipped with its norm or uniform topology, $B(\mathcal{H})$ is a C^* algebra. However there are other useful topologies on $B(\mathcal{H})$:

the strong topology (s) defined by the seminorms

$$(1.1) \qquad x \longmapsto \|x\zeta\| \qquad \zeta \in \mathcal{H}$$

the weak topology (w) given by the seminorms

$$(1.2) \qquad x \longmapsto |(\zeta, x\eta)| \qquad \zeta, \eta \in \mathcal{H}$$

+ that is, the weakest topology on X making all the maps $\phi \longmapsto \phi(x)$ $(x \in \mathfrak{A})$ continuous.

the ultraweak topology (uw) given by the seminorms

(1.3) $$x \longmapsto \left|\sum_{i=1}^{\infty}(\zeta_i, x\eta_i)\right| \qquad \zeta_i, \eta_i \in \mathcal{H}, \sum_{i=1}^{\infty}\|\zeta_i\|\,\|\eta_i\| < \infty$$

The weak topology is weaker than the strong and the ultraweak, which are both weaker than the uniform; but w and uw coincide on norm-bounded sets of $B(\mathcal{H})$.

A Von Neumann (VN) algebra $\mathcal{M}$ is, by definition, a *-subalgebra of $B(\mathcal{H})$ which is weakly closed.

By Von Neumann's density theorem, this is equivalent to $\mathcal{M}$ being strongly or ultraweakly closed. Furthermore, a VN algebra $\mathcal{M}$ always contains an identity, i.e. a largest projection p such that $px = xp = x$ for all $x \in \mathcal{M}$. Finally, Von Neumann's bicommutant theorem states that a *-subalgebra $\mathcal{M}$ of $B(\mathcal{H})$ containing the identity on $\mathcal{H}$ is a VN algebra iff $\mathcal{M} = \mathcal{M}''$ *.

Another density theorem which we shall need is Kaplansky's density theorem: If $\mathfrak{A} \subseteq \mathfrak{B} \subseteq B(\mathcal{H})$ are *-algebras, and if $\mathfrak{A}$ is strongly dense in $\mathfrak{B}$, then the unit ball of $\mathfrak{A}$ is strongly dense in the unit ball of $\mathfrak{B}$. (the converse is trivially satisfied).

Given any C^* algebra $\mathfrak{A}$, SHERMAN's Theorem ([62] 1.17.2) allows us to identify its double dual $\mathfrak{A}^{**}$ as a VN algebra, in which $\mathfrak{A}$ may be embedded as an uw dense *-subalgebra.

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* For any set $S \subseteq B(\mathcal{H})$, its commutant $S' = \{ x \in B(\mathcal{H}) : [x,S]=0\}$ where $[x,y] = xy-yx$

§ 1.3 Linear Forms

For $\zeta,\eta \in \mathcal{H}$, denote by $\omega_{\zeta\eta}$ the linear form on $B(\mathcal{H})$ given by

(1.4) $$\omega_{\zeta\eta}(x) = (\zeta, x\eta) \qquad x \in B(\mathcal{H})$$

and we abbreviate $\omega_{\zeta\zeta}$ to ω_ζ. $\omega_{\zeta\eta}$ is clearly w-continuous. Denote by $\mathcal{M}_\sim$ the set of all w-continuous linear forms on a VN algebra $\mathcal{M}$. Then $B(\mathcal{H})$ coincides with the linear span of $\{\omega_{\zeta\eta} : \zeta,\eta \in \mathcal{H}\}$, and $\mathcal{M}_\sim$ consists of all restrictions to $\mathcal{M}$ of elements of $B(\mathcal{H})_\sim$.

Let $\mathcal{M}_*$ denote the norm closure of $\mathcal{M}_\sim$ in $\mathcal{M}^*$. Then $\mathcal{M}_*$ consists of all uw-continuous linear forms on $\mathcal{M}$, which are the same as all restrictions to $\mathcal{M}$ of uw-continuous linear forms on $B(\mathcal{H})$. Each $\omega \in B(\mathcal{H})_*$ may be written:

(1.5) $$\omega = \sum_{i=1}^{\infty} \omega_{\zeta_i \eta_i} \quad , \zeta_i, \eta_i \in \mathcal{H} , \sum_{i=1}^{\infty} \|\zeta_i\| \, \|\eta_i\| < \infty$$

Finally, each $x \in \mathcal{M}$ defines a linear form ϕ_x on $\mathcal{M}_*$ by:

(1.6) $$\phi_x(\omega) = \omega(x) \qquad (\omega \in \mathcal{M}_*)$$

and the map

(1.7) $$\phi : \mathcal{M} \longrightarrow \mathcal{M}_*^* \qquad x \longmapsto \phi_x$$

is an isomorphism of $\mathcal{M}$ onto the (norm) dual of $\mathcal{M}_*$,and is isometric. Thus $\mathcal{M}_*$ is known as the predual of $\mathcal{M}$. In fact, the property of being the dual of a Banach space (which turns out to be unique) characterizes, up to isomorphism, VN algebras among all C* algebras ([62],1.16.7).

In the case of $B(\mathcal{H})$, the predual can be easily identified:

it is (isomorphic to) the space of all trace class operators ρ on $\mathcal{H}$, via the mapping which associates to each ρ the linear form

(1.8) $$\omega(x) = tr(\rho x) \qquad x \in B(\mathcal{H})$$

which is a Banach space isomorphism, when the space of all trace class operators is equipped with the trace norm :

$$\|\rho\|_1 = tr(|\rho|) \quad *$$

This is the first example of a "Radon-Nikodym " theorem: the "integral" is the trace, the objects to be integrated are elements of $B(\mathcal{H})$, and (1.8) states that any uw-continuous linear form ("absolutely continuous measure") corresponds to an "integrable" element of $B(\mathcal{H})$. We shall see later that this situation is quite general (see Thm.2.3, and Thm.4.4 of Ch.III)

For another example, let $\mathcal{H}$ be the Hilbert space of all (equivalence classes of) square-integrable functions on a measure space (Ω,μ). Let $\mathcal{M} \subset B(\mathcal{H})$ consist of all operators T_x with $x \in L^\infty(\Omega,\mu)$ defined by

(1.9) $$(T_x y)(\omega) = x(\omega)y(\omega) \quad , \; y \in L^2(\Omega,\mu)$$

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* A trace class operator is an operator such that $\|\sum_i |\rho|^{\frac{1}{2}}\zeta_i\|^2 < \infty$ for an orthonormal base $\{\zeta_i\}$ of $\mathcal{H}$, where $|\rho| = (\rho^*\rho)^{\frac{1}{2}}$. We then let $tr(|\rho|) = \sum_i (\zeta_i, |\rho|\zeta_i)$, which is independent of the base. tr then extends to a linear form on all trace class operators, and turns out to be positive and unitarily invariant i.e. $tr(u\rho^* u) = tr(\rho)$ for all trace class ρ & unitary u in $B(\mathcal{H})$

$\mathcal{M}$ is the multiplication algebra of $L^2(\Omega,\mu)$. Then one shows that $\mathcal{M}$ is a VN algebra, and its predual is isomorphic to $L^1(\Omega,\mu)$, via the mapping which sends $x \in L^1(\Omega,\mu)$ to the linear form

$$T_y \longmapsto \int_\Omega x(\omega)y(\omega)d\mu(\omega) \qquad (y \in L^1(\Omega,\mu))$$

which is automatically uw-continuous.

Note that this example exhausts all abelian VN algebras: Any abelian VN algebra is *-isomorphic to some $L^\infty(\Omega,\mu)$, and the isomorphism is uw-bi-continuous (SAKAI [62], 1.18.1).

§1.4. Positive linear forms and the GNS construction

Definition A linear form ω on a general *-algebra $\mathfrak{A}$ is said to be positive iff $\omega(x^*x)\geq 0$ for all $x\in\mathfrak{A}$ (for C* algebras, this is equivalent to $\omega(x)\geq 0$ for all $x\in\mathfrak{A}_+$) It is faithful if $\omega(x^*x)=0$ implies $x=0$. A positive linear form on a normed *-algebra $\mathfrak{A}$ is called a state iff it has norm 1 (as an element of the dual of $\mathfrak{A}$). A positive linear form (henceforth plf) on a general *-algebra with unit is called a state iff $\omega(1)=1$. These two definitions coincide on unital C* algebras ([15], 2.1.9)

We now define representations of general *-algebras by unbounded operators, as they will be useful in the sequel. These concepts, which are generalizations of the usual ones for bounded representations, are due to POWERS [49].

Definition Let $\mathfrak{A}$ be a unital *-algebra. A *-representation π of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$ with domain $D(\pi)$ is a linear map from $\mathfrak{A}$ into (possibly unbounded) operators on $\mathcal{H}$ with common dense domain $D(\pi)$ such that

(1.10) $\pi(x)D(\pi) \subseteq D(\pi)$ for all $x \in \mathfrak{A}$

(1.11) $\pi(x)\pi(y)\zeta = \pi(xy)\zeta$ for all $x, y \in \mathfrak{A}$, $\zeta \in D(\pi)$

(1.12) $(\zeta, \pi(x)\eta) = (\pi(x^*)\zeta, \eta)$ for all $x \in \mathfrak{A}$, $\zeta, \eta \in D(\pi)$

(i.e. $\pi(x^*) \subseteq \pi(x)^*$)

The induced topology on $D(\pi)$ is defined by the seminorms

(1.13) $\zeta \longmapsto \|\pi(x)\zeta\|$ $(x \in \mathfrak{A})$

A *-representation π is said to be closed iff $D(\pi)$ is complete in the induced topology.Any *-representation π has a (unique) closure $\bar{\pi}$ defined on the completion of $D(\pi)$ with respect to the induced topology; $\bar{\pi}$ extends π in the sense that $\bar{\pi}(x)\zeta = \pi(x)\zeta$ for all $\zeta \in D(\pi)$ and $x \in \mathfrak{A}$.

A vector $\zeta \in D(\pi)$ is called cyclic for $\mathfrak{A}$ iff $\pi(\mathfrak{A})\zeta$ is dense in $D(\pi)$ with respect to the Hilbert space topology; it is called strongly cyclic iff $\pi(\mathfrak{A})\zeta$ is dense in $D(\pi)$ with respect to the induced topology.
The (bounded) commutant $\pi'(\mathfrak{A})$ of a *-representation π is defined by : $x \in B(\mathcal{H})$ is in $\pi'(\mathfrak{A})$ iff,for each $y \in \mathfrak{A}$ and $\zeta, \eta \in D(\pi)$,we have

(1.14) $(\zeta, x\pi(y)\eta) = (\pi(y^*)\zeta, x\eta)$

The commutant is a weakly closed,*-invariant complex linear subspace of $B(\mathcal{H})$,but not necessarily an algebra.Furthermore, $\pi'(\mathfrak{A}) = \bar{\pi}'(\mathfrak{A})$.
A *-representation π is said to be irreducible iff $\pi'(\mathfrak{A})$ consists of multiples of the identity.

We now proceed to give a sketch of the GNS (GELFAND-NAIMARK-SEGAL) construction for unbounded representations, which is again a generalization of the GNS construction for C* algebras([15],2.4.4).

Given a plf ω on a unital *-algebra $\mathfrak{A}$, let $I=\{x\in\mathfrak{A}:\omega(x^*x)=0\}$ I is a left ideal of $\mathfrak{A}$. On the quotient vector space $D=\mathfrak{A}/I$ which consists of all equivalence classes

$$[x] = \{ y\in \mathfrak{A} : x-y \in I \} \quad (x\in\mathfrak{A})$$

we define

(1.15) $$([x],[y]) = \omega(x^*y)$$

This turns out to be a well-defined (positive definite) inner product on D. Let $\mathcal{H}_\omega$ denote the Hilbert space completion of D . The left regular representation of $\mathfrak{A}$ on itself factors to a well-defined *-representation π_ω of $\mathfrak{A}$ on $\mathcal{H}_\omega$ with domain D given by

$$\pi_\omega(x)[y] = [xy]$$

Let $\zeta_\omega = [1]\in D$. Then ζ_ω is a strongly cyclic for $\mathfrak{A}$ and

(1.16) $$\omega(x) = (\zeta_\omega, \pi_\omega(x)\zeta_\omega) \quad \text{for all } x\in\mathfrak{A}$$

Furthermore, the triple $(\mathcal{H}_\omega, \pi_\omega, \zeta_\omega)$, called the <u>GNS triple for $\mathfrak{A}$ determined by ω</u> , is uniquely determined, up to unitary equivalence, by (1.16) ; that is, if $(\mathcal{H},\pi,\zeta)$ is another such triple, the mapping U defined by $U\pi_\omega(x)\zeta_\omega = \pi(x)\zeta \quad (x\in\mathfrak{A})$, extends to a unitary from $\mathcal{H}_\omega$ onto $\mathcal{H}$, and maps the domains continuously onto one another (with respect to the induced topologies).

In the case where $\mathfrak{A}$ is a C* algebra, even without identity,

$\pi_\omega(x)$ defined above satisfies $\|\pi_\omega(x)\| \leq \|x\|$; and π_ω is a *-representation of $\mathfrak{A}$ on $\mathcal{H}_\omega$ in the usual sense.There always exists a cyclic vector (the two notions of cyclicity now coincide) $\zeta_\omega \in \mathcal{H}_\omega$ such that (1.16) is satisfied ([15],2.4.4).

Finally,we note that in case ω is faithful,we have I=0, and thus π_ω is faithful,in the sense that $\pi_\omega(x)\zeta = 0$ for all $\zeta \in D$ implies x = 0. In this case of course D = $\mathfrak{A}$.Furthermore, the cyclic vector ζ is also separating for $\pi_\omega(\mathfrak{A})$ *

Suppose now that $\{ \alpha_t : t \in G \}$ is a group of *-automorphisms of $\mathfrak{A}$,for which ζ_ω is invariant.Then

$$U_t[x] := [\alpha_t(x)] \qquad t \in G,\ x \in \mathfrak{A}$$

is well defined,and extends to a unitary representation of G on $\mathcal{H}_\omega$,such that

$$U_t \zeta_\omega = \zeta_\omega$$
$$U_t \pi_\omega(x) U_t^* = \pi_\omega(\alpha_t(x)) \qquad t \in G,\ x \in \mathfrak{A}$$

(in particular, $U_t D \subseteq D$ for all $t \in G$).We say $\{\alpha_t\}$ is unitarily implemented.

Finally,note that the representation π_ω is irreducible iff ω is pure , i.e. cannot be written as a convex combination of two distinct states.

Consider now the case of a plf ω on a VN algebra $\mathcal{M}$. Then ,as a consequence of positivity (a purely algebraic concept),$\omega \in \mathcal{M}^*$.Moreover,the ultraweak topology can also be characterised algebraically: a plf ω is in $\mathcal{M}_*$ iff it is

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* A vector $\zeta \in \mathcal{H}$ is said to be separating for a set S of operators iff $x \in S$, $x\zeta = 0$ implies x=0. If $\mathcal{M}$ is a VN algebra, ζ is separating for $\mathcal{M}$ iff it is cyclic for $\mathcal{M}'$.

normal, that is iff $\sup\omega(x_\lambda) = \omega(\sup x_\lambda)$ for each uniformly bounded increasing family $\{x_\lambda\} \subseteq \mathcal{M}_+$. For a normal plf ω, the GNS representation π_ω is continuous with respect to the uw topologies on $\mathcal{M}$ and $B(\mathcal{H}_\omega)$. Thus in particular $\pi_\omega(\mathcal{M})$ is a VN algebra.

It follows that the study of a VN algebra $\mathcal{M}$ with a faithfull normal state ω is the study of the VN algebra $\pi_\omega(\mathcal{M})$ with the cyclic and separating vector ζ_ω. These are the basic ingredients of Tomita-Takesaki theory. Before we describe that theory, however, let us first consider the DIXMIER-SEGAL ([14],[67]) Non-Commutative Integration theory, since it is both historically and logically prior to Tomita-Takesaki theory, of which it may be considered a special case.

§2. Non-Commutative Integration Theory

What follows is based on a course of seminars I gave at Bedford College. These seminars were in fact conceived as a preparation for a presentation of Tomita-Takesaki theory, and attempted to unify the treatments of SEGAL [67] and DIXMIER [14]. SEGAL constructed his non-commutative L_p spaces as spaces of (unbounded) operators "affiliated" to a VN algebra $\mathcal{M}$, while DIXMIER constructed them as abstract completions of a subset of $\mathcal{M}$ with respect to the "L_p norms". I shall follow NELSON [46] in constructing a larger set of "measurable" operators affiliated with $\mathcal{M}$, into which the L_p spaces, abstractly defined as in [14], are then embedded. Let us first consider the non-commutative extension of the concept of the integral.

§ 2.1. Traces

Definition Let $\mathfrak{A}$ be a C* algebra, $\mathfrak{A}_+$ its positive part. A trace on $\mathfrak{A}$ is a function $\tau : \mathfrak{A}_+ \longrightarrow [0,+\infty]$ with the properties

(i) Additivity: $x,y \in \mathfrak{A}_+ \Longrightarrow \tau(x+y) = \tau(x)+\tau(y)$

(ii) Homogeneity: $x \in \mathfrak{A}_+, \lambda \geq 0 \Longrightarrow \tau(\lambda x) = \lambda\tau(x)$

(iii) Unitary invariance: $x \in \mathfrak{A}_+, u \in \mathfrak{A}$ unitary $\Longrightarrow$
$\tau(u^* x u) = \tau(x)$

A trace τ is said to be :

faithful iff $x \in \mathfrak{A}_+$, $\tau(x) = 0$ implies $x = 0$

semifinite iff for each $x \in \mathfrak{A}_+$ ther exists a nonzero $y \in \mathfrak{A}_+$ such that $y \leq x$ and $\tau(y) < +\infty$

A trace τ on a VN algebra $\mathcal{M}$ is said to be normal iff for each uniformly bounded increasing family $\{x_\jmath\} \subseteq \mathcal{M}_+$, $\sup \tau(x_\jmath) = \tau(\sup x_\jmath)$

Theorem 2.1

Let $\mathcal{M}$ be a VN algebra, τ a trace on $\mathcal{M}$. Let $J_2 = \{x \in \mathcal{M} : \tau(x^* x) < \infty\}$ (the "τ-Hilbert-Schmidt operators"), and let J be the linear span of $\{xy : x,y \in J_2\}$. Then J, J_2 are two-sided *-ideals of $\mathcal{M}$, and $J \cap \mathcal{M}_+ = \{x \in \mathcal{M}_+ : \tau(x) < \infty\}$. There is a unique linear form $\hat{\tau}$ on J extending τ (i.e. $\hat{\tau}(x) = \tau(x)$ for $x \in J \cap \mathcal{M}_+$). $\hat{\tau}$ is central or tracial, that is $\hat{\tau}(xy) = \hat{\tau}(yx)$ for all $x,y \in J$. Finally, if τ is normal, then for all $x \in J$, the mapping

$$\omega_x : y \longmapsto \hat{\tau}(xy) \quad (y \in \mathcal{M})$$

is an uw continuous linear form on $\mathcal{M}$. ([62], 2.5)

The reasons why the notion of a trace is a proper extension of the notion of an integral have been explained in the Introduction. In closer analogy with measure theory, SEGAL[67] defines his "integral" (the gagé) initially on projections. It is required to be unitarily invariant and completely additive * . SEGAL([67],Thm. 10) then shows that the gage may be uniquely extended to a central plf on an ideal.

Definition A VN algebra $\mathcal{M}$ is called finite (respectively semifinite) iff for each $x \in \mathcal{M}_+$, $x \neq 0$, there exists a normal finite (resp. semifinite) trace τ on $\mathcal{M}$ such that $\tau(x) > 0$. It is called properly infinite (resp. purely infinite) iff there exists no nonzero finite (resp. semifinite) normal trace on $\mathcal{M}$.

Any abelian VN algebra is finite: any vector state ω_ζ is a finite normal trace. $B(\mathcal{H})$ is semifinite : tr is a normal faithful semifinite trace. A factor (i.e. a VN algebra $\mathcal{M}$ whose centre $\mathcal{M} \cap \mathcal{M}'$ is trivial) can be of at most one type: either finite (type I_n ; $n < \infty$ or II_1 in the MURRAY-VON NEUMANN classification [44]) or properly infinite semifinite (type I_∞ or II_∞) or purely infinite (type III)(see [16], I.6 Cor. 1 and 2 of prop. 8, and I.8.4). For a semifinite VN algebra $\mathcal{M}$, one can show that there exists a faithful normal semifinite trace on $\mathcal{M}$ ([16],I.6 prop.9(i)).

If $\mathcal{M}$ is a semifinite factor, this trace is shown to be unique up to normalization ([16],I.6, cor. of Thm.3). Thus for a factor, one may speak of trace class and Hilbert-Schmidt operators, independent of the trace. Example: $B(\mathcal{H})$.

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* That is, $\sum \tau(p_3) = \tau(\sum p_3)$ for any orthogonal family of projections $\{p_3\} \subset \mathcal{M}$. For an $\omega \in \mathcal{M}^*$, complete additivity is equivalent to uw continuity (see RINGROSE[56], Thm. 4.5).

§2.2. L_p Spaces

We now have our "measure" τ, which will be assumed a faithful, normal, semifinite trace in the sequel. As observed in the Introduction, the existence of such τ is a restriction on the type of VN algebra considered. What follows can only be done if such a non-trivial trace exists, i.e. if $\mathcal{M}$ is semifinite. If not, one has to resort to Tomita-Takesaki theory [+] (Ch.III).

We have seen that operators in $J \subseteq \mathcal{M}$ are "integrable" with respect to τ. In the case of $B(\mathcal{H})$, these are all: tr cannot be extended beyond the trace class operators, since they are already complete with respect to the trace-norm. In the case of a (conventional) probability space, we have the other extreme: all bounded measurable functions (elements of $L^\infty = \mathcal{M}$) are integrable, and there are more: we get the rest by completing L^∞ with respect to the L_1 norm. This suggests the

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[+] It is interesting to note that HAAGERUP [27] has recently constructed L_p spaces associated with an arbitrary VN algebra $\mathcal{M}$. These spaces consist of operators affiliated not with $\mathcal{M}$ itself, but with a larger VN algebra. Moreover, they not only consist exclusively of unbounded operators, but also $L_p \cap L_q = \{0\}$ for $p \neq q$ (compare the semifinite situation, where all the L_p spaces contain the common dense subspace J consisting of bounded operators). However, if $\mathcal{M}$ is semifinite, the L_p spaces of HAAGERUP turn out to be isometric and order-isomorphic to the ones constructed in this thesis. Note that these results in no way bypass Tomita-Takesaki theory, as HAAGERUP's construction strongly depends on recent results in that theory.

Definition Let $x\in J$, $1\leq p<\infty$. Define

(2.1) $$\|x\|_p := \tau(|x|^p)^{1/p} \qquad (|x| = (x^*x)^{\frac{1}{2}})$$

$$\|x\|_\infty := \|x\| = \text{the operator norm}$$

Denote by $L^p(\mathcal{M},\tau)$ the completion of J with respect to the $\|\cdot\|_p$-norm. $L^\infty(\mathcal{M},\tau) := \mathcal{M}$ (the uw closure of J).

Observe that, in case $\mathcal{M}$ is abelian, the above definition yields the ordinary L_p spaces. For if $\mathcal{M} = L^\infty(\Omega,\mu)$, τ defines a measure ν on Ω by $\nu(A) = \tau(\chi_A)$ where χ_A (the characteristic function of a measurable subset $A\subseteq\Omega$) is in $\mathcal{M}_+$. Since τ is faithful , ν is equivalent to μ , hence $\mathcal{M} = L^\infty(\Omega,\nu)$ (the latter space depending only on the measure class of ν). Now for a simple integrable function $f = \sum_{i=1}^{n} \lambda_i \chi_{A_i} \in \mathcal{M}$,

$$\tau(|f|^p) = \sum_i |\lambda_i|^p \tau(\chi_{A_i}) = \sum_i |\lambda_i|^p \nu(A_i) = \int |f|^p d\nu$$

which shows that $L^p(\Omega,\nu) = L^p(\mathcal{M},\tau)$

The L_p spaces are at the moment defined as abstract completions. We will later be able to identify their elements to (possibly unbounded) operators "affiliated" to $\mathcal{M}$. But already we can develop the analogues of some of the properties of classical L_p spaces.

Theorem 2.2

(i)(Hölder inequality) Let $1\leq p<\infty$, $q=p/p-1$ $(1/0=\infty)$, $x,y\in J$

Then

(2.2) $$|\tau(xy)| \leq \|xy\|_1 \leq \|x\|_p\|y\|_q$$

(ii) Let $x\in J$, $1<p<\infty$. Then

(2.3) $$\|x\|_p = \sup\{ |\tau(xy)| : y\in\mathcal{M},\ \|y\|_q = 1 \}$$

(2.4) $$\|x\|_1 = \sup\{ |\tau(xy)| : y\in\mathcal{M},\ \|y\|_\infty \leq 1 \}$$

Both sups are attained.

(iii) $\|\cdot\|_p$ is a norm on J (Minkowski inequality). Hence $L^p(\mathcal{M},\tau)$ is a Banach space.

Theorem 2.3 ("Radon-Nikodym")

$L^1(\mathcal{M},\tau)$ is (isometrically isomorphic to) $\mathcal{M}_*$.

We have seen (Thm.2.1) that for $x\in J$, $\omega_x(y) = \tau(xy)$ $(y\in\mathcal{M})$ defines an element of $\mathcal{M}_*$. But by (2.4), the mapping

$$(J, \|\ \|_1) \longrightarrow (\mathcal{M}_*, \|\ \|)$$
$$x \longmapsto \omega_x$$

is isometric. One shows that it has dense range in $\mathcal{M}_*$, and hence extends to the required isomorphism onto.

The justification of the name "Radon-Nikodym" is seen if we interpret ω_x to be the "indefinite integral" corresponding to x (i.e. $\omega_x(\cdot) = $ "$\int_{\cdot} x d\tau$ "). Thus the theorem says that every $\omega \in\mathcal{M}_*$ has a "Radon-Nikodym derivative" in $L^1(\mathcal{M},\tau)$ with respect to τ . In fact SEGAL ([67], Thm. 14) shows that in case τ is not assumed faithful, ω has to vanish on the projections on which τ vanishes (absolute continuity).

§ 2.3 Measurable operators

So far the L_p spaces have been defined as abstract completions of the ideal J; thus, for instance, one cannot decide whether two elements of distinct L_p spaces "actually" coincide. The purpose of this section is to identify elements of all L_p spaces with (possibly unbounded) operators . This will bring the theory into closer analogy with classical integration theory, where each $f\in L_p$ can be thought of as an unbounded multiplication operator on the Hilbert space L_2 .

This identification will be made in three steps: Firstly, a new topology will be defined on $\mathcal{M}$, in imitation of the topology of convergence in measure defined for measurable functions on a measure space. Then the completion of $\mathcal{M}$ with respect to this topology will be identified with unbounded operators (the "measurable" operators). Finally, the L_p spaces will be continuously embedded in this completion.

This treatment is due to NELSON [46].

§2.3.1. The measure topology

Let (Ω,μ) be a measure space. A net $\{f_r\}$ of measurable functions on Ω is said to converge to the measurable function f in measure iff, given $\epsilon > 0$ and $\delta > 0$, we have, for large enough r,

$$\mu\{\omega\in\Omega : |f_r(\omega) - f(\omega)|\geq\epsilon\} < \delta$$

We may reformulate this as follows: denoting the characteristic function of the set

$$\{\omega\in\Omega : |f_r(\omega) - f(\omega)| < \epsilon\}$$

by p_r (a projection in $L^\infty(\Omega,\mu)$), we see that $f_r \to f$ in measure iff given $\epsilon>0$ and $\delta>0$ there exist projections $p_r\in L^\infty(\Omega,\mu)$ such that for large enough r,

$$\|p_r(f_r-f)\|_\infty<\epsilon \quad \text{and} \quad \int p_r^\perp d\mu < \delta$$

where $p_r^\perp = 1-p_r$. Motivated by this, we define:

Definition Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a VN algebra, τ a faithful normal semifinite trace on $\mathcal{M}$. Given $\epsilon>0$ and $\delta>0$, consider

(2.5) $N(\epsilon,\delta) = \{ x\in\mathcal{M} : \exists p\in\mathcal{M} \text{ proj. such that } \|xp\|\leq\epsilon \text{ and } \tau(p^\perp)\leq\delta \}$

(2.6) $O(\epsilon,\delta) = \{ \zeta\in\mathcal{H} : \exists p\in\mathcal{M} \text{ proj. such that } \|p\zeta\|\leq\epsilon \text{ and } \tau(p^\perp)\leq\delta \}$

The measure topology on $\mathcal{M}$(respectively $\mathcal{H}$) is the vector space topology which has $\{ N(\epsilon,\delta) : \epsilon,\delta > 0\}$ (respectively $\{ O(\epsilon,\delta) : \epsilon,\delta > 0\}$) as a base of neighbourhoods of zero. $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{H}}$ denote the completions of $\mathcal{M}$ and $\mathcal{H}$ with respect to these topologies.

The idea of these definitions is to break up the spaces into two parts, one on which things "behave well" and one whose "measure" is small. Thus one shows [46] that given $x \in \tilde{\mathcal{M}}$ and $\epsilon>0$, there exists a projection $p \in \mathcal{M}$ such that $xp \in \mathcal{M}$ and $\tau(p^{\perp}) \leq \epsilon$.

NELSON [46] now shows that $\tilde{\mathcal{H}}$ (resp. $\tilde{\mathcal{M}}$) is a well defined Hausdorff topological vector space (resp. $*$-algebra) and that the identity representation of $\mathcal{M}$ extends to a continuous $*$-representation of $\tilde{\mathcal{M}}$ on $\tilde{\mathcal{H}}$.

§ 2.3.2. Measurable operators

I now wish to identify elements of $\tilde{\mathcal{M}}$ with (possibly unbounded) operators on $\mathcal{H}$, which are affiliated to $\mathcal{M}$ in the following sense:

Definition An operator A on $\mathcal{H}$ (not necessarily bounded or everywhere defined) is said to be affiliated to $\mathcal{M}$ iff for each unitary $u \in \mathcal{M}'$, $Au = uA$ (in particular, $uD(A) \subset D(A)$)
We say $A \eta \mathcal{M}$.
Clearly $A \in B(\mathcal{H})$ and $A \eta \mathcal{M}$ implies $A \in \mathcal{M}$.

Definition If $x \in \tilde{\mathcal{M}}$ and $\zeta \in \mathcal{H}$,then $x\zeta \in \tilde{\mathcal{H}}$. If $x\zeta$ happens to lie in the (measure dense) subset $\mathcal{H} \subseteq \tilde{\mathcal{H}}$,we say $\zeta \in D(T_x)$, and we define the operator T_x on $\mathcal{H}$ with domain $D(T_x)$ by

(2.7) $$T_x \zeta = x\zeta$$

Theorem 2.4

For each $x \in \tilde{\mathcal{M}}$, T_x is a closed, densely defined operator affiliated with $\mathcal{M}$, and the mapping $x \mapsto T_x$ is an injective *-homomorphism of $\tilde{\mathcal{M}}$ into dense closed operators on $\mathcal{H}$, equipped with strong addition and multiplication, (SEGAL [67]) i.e.

$$T_x^* = T_{x^*} \tag{2.8}$$

$$\overline{T_x + T_y} = T_{x+y} \tag{2.9}$$

$$\overline{T_x T_y} = T_{xy} \tag{2.10}$$

where $\bar{A}$ is the closure of A.

Definition A measurable operator on $\mathcal{H}$ is a T_x with $x \in \tilde{\mathcal{M}}$.

This definition, due to NELSON [46], is not standard. SEGAL's [67] original definition may be paraphrased as follows:

Definition A closed densely defined operator $A \eta \mathcal{M}$ is said to be SEGAL-measurable iff there exists a sequence of projections $p_n \in \mathcal{M}$ such that $p_n \mathcal{H} \subset D(A)$, $p_n^\perp$ is finite *, and $p_n^\perp \downarrow 0$

SEGAL's definition is more natural, in the sense that, in the abelian case, it covers exactly the set of all (multipli-

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* A projection $p \in \mathcal{M}$ is said to be finite with respect to $\mathcal{M}$ iff, whenever $u \in \mathcal{M}$ is such that $u^*u = p$ and $uu^* \leqslant p$, then $uu^* = p$. That is, p is finite iff it cannot dominate any projection which is unitarily equivalent to it via an element of $\mathcal{M}$.

cations by) almost everywhere finite * measurable functions (modulo null functions, of course) (SEGAL [67], Thm. 2).

On the relation between the two definitions, we have the

Proposition 2.5

(i) For all $x \in \widetilde{\mathcal{M}}$, T_x is SEGAL-measurable.

(ii) Suppose τ is finite. Then the two definitions of measurability coincide, and yield all closed, densely defined $T \eta \mathcal{M}$.

(iii) Suppose, in addition, that $\mathcal{M}$ is abelian (so that $\mathcal{M} = L^{\infty}(\Omega,\mu)$ with $\mu(\Omega) = \tau(1) < \infty$. Then $\widetilde{\mathcal{M}}$ consists of all (equivalence classes of) measurable a.e. finite functions on Ω.

Proof (i) For each $n \in N$ we may find (see § 2.3.1) a projection $p_n \in \mathcal{M}$ such that $xp_n \in \mathcal{M}$ and $\tau(p_n^{\perp}) \leq (\frac{1}{2})^n$

Letting $q_n = \bigwedge_{k \geq n}^{\infty} p_k$ (the projection onto the intersection of the range spaces of p_k, $k \geq n$), we have $\tau(q_n^{\perp}) \leq \sum_{k=n}^{\infty} \tau(p_k^{\perp}) \longrightarrow 0$, $xq_n = xp_n q_n \in \mathcal{M}$, and $q_n^{\perp} \downarrow 0$, for if $q \leq q_n^{\perp}$ for all n, then $\tau(q) \leq \tau(q_n^{\perp})$ so that $\tau(q) = 0$ and hence $q = 0$.

Clearly $q_n \mathcal{H} \subseteq D(T_x)$ and $xq_n = T_x q_n$.

Finally each $q_n^{\perp}$ is finite, for if $u_n \in \mathcal{M}$ is such that $u_n^* u_n = q_n^{\perp}$ and $u_n u_n^* \leq q_n^{\perp}$, then $\tau(u_n u_n^*) = \tau(u_n^* u_n) = \tau(q_n^{\perp}) < \infty$, so that $\tau(q_n^{\perp} - u_n u_n^*) = 0$, or $q_n^{\perp} = u_n u_n^*$.

The sequence (q_n) therefore satisfies the requirements of SEGAL's definition, and hence T_x is SEGAL-measurable.

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*SEGAL [67] fails to mention this restriction. However if, for a measurable function f on (Ω,μ), $\mu\{\omega \in \Omega : |f(\omega)| = \infty\} > 0$, then the characteristic function of any subset of this set of finite positive measure is orthogonal to $D(T_f)$; thus T_f cannot be densely defined.

(ii) If τ is finite, NELSON [46] shows that every closed, densely defined operator affiliated to $\mathcal{M}$ is a T_x ,with $x\in\tilde{\mathcal{M}}$ Thus every SEGAL-measurable operator is measurable in our definition. In this case the other requirements of SEGAL's definition are automatically satisfied by any increasing sequence of projections $p_n \in \mathcal{M}$ such that $p_n\mathcal{H} \subseteq D(T_x)$ and $p_n \longrightarrow 1$, since $\tau(p_n^\perp)<\infty$ and thus $p_n^\perp$ must be finite,as observed in part (i) for $q_n^\perp$.

(iii) If $x\in\tilde{\mathcal{M}}$, there exists a sequence $(x_s)\subseteq\mathcal{M}$ such that $x_s \longrightarrow x$ in measure.But x_s correspond to measurable functions in $L_\infty(\Omega,\mu)$ which are Cauchy in measure (in the conventional sense).Therefore (HALMOS [28],Thm.22.E) they must converge in measure to a measurable function f .Since the two notions of convergence in measure coincide for the abelian case,and since the measure topology is Hausdorff,so that limits are unique,it is clear that x must correspond to f , i.e. $T_x=T_f$.

Note that we have not so far used the assumption that τ is finite,so that in fact we have shown that in general $\tilde{\mathcal{M}}$ consists of (not necessarily all) measurable functions on Ω.

Conversely,let f be an almost everywhere finite measurable function on Ω ,and let $\epsilon>0$ be given.Putting $A_n = \{\omega\in\Omega : |f(\omega)|\leq n\}$,we see that $A_n^c \downarrow \cap_n A_n^c = A =$ $=\{\omega\in\Omega : |f(\omega)| = +\infty\}$. Thus $\mu(A) = 0$, and hence $\lim_n\mu(A_n^c) = 0$ (it is here that the finiteness of μ comes in).Choose $n_0\in N$ such that $\mu(A_{n_0}^c)\leq\epsilon$, and put $p_n = \chi_{A_n}$. For all $n\geq n_0$,we have $\tau(p_n^\perp) = \mu(A_n^c)\leq\epsilon$,and

$$\|(f - fp_n)p_n\| = \|fp_n^\perp p_n\| = 0 < \epsilon$$

This shows that $fp_n \longrightarrow f$ in measure, and since

$$\|fp_n\| = \sup\{|f(\omega)| \ : \ \omega\in A_n\} = n < \infty$$

i.e. $fp_n \in \mathcal{M}$, it follows that $f \in \widetilde{\mathcal{M}}$.

QED

Observe that,contrary to Segal's definition,our definition of measurability does not include "very unbounded" measurable functions,such as $f(\omega) = \omega$ on $(\mathbb{R},d\omega)$.This is because for all $n\in\mathbb{N}$, $\{\omega\in R : |f(\omega)|>n\}$ has infinite measure, so that we cannot approximate this function <u>in measure</u> by bounded ones: the set on which our approximation is "bad" always has infinite measure.

The advantage of NELSON's definition is that it provides us with a class of operators with desirable topological properties ,while at the same time being a minimal extension of $\mathcal{M}$ which contains all the "interesting" operators (the elements of all L_p spaces,as we shall see below.

We close this subsection with an observation about Non-Commutative Probability spaces,since we are mostly concerned with them in the applications of the theory in this thesis.

<u>Proposition 2.6</u>

If τ is a tracial state,(sothat $J = \mathcal{M}$),the measure topology on $\mathcal{M}$ is given by the metric

(2.11) $$\rho(x) := \tau(|x|(1 + |x|)^{-1}) \qquad (|x|=(x^*x)^{\frac{1}{2}} ,x\in\mathcal{M})$$

<u>Proof</u>. STINESPRING[74] Theorem 5.1.

§ 2.3.3. p-Integrable Operators

We are now ready to embed the $L_p(\mathcal{M},\tau)$ spaces into $\tilde{\mathcal{M}}$, thus showing, in view of Thm. 2.4, that they all consist of (possibly unbounded) operators affiliated to $\mathcal{M}$.

SEGAL [67] proceeds in the opposite direction: First he defines a notion of "pointwise almost everywhere convergence" of operators; then he calls a (SEGAL)-measurable operator <u>integrable</u> iff it is the pointwise a.e. limit of operators in J; its <u>integral</u> is then the limit of the traces of these operators. Then it is a non-trivial problem to show that this integral is unique, and that integrable operators form a Banach space (Non-Commutative Riesz-Fischer Theorem).

Through the use of the measure topology of NELSON, we will be able to bypass the counter-intuitive concept of "pointwise a.e. convergence" (there are no points in a non-commutative situation!) by identifying our L_p spaces with subspaces of $\tilde{\mathcal{M}}$.

<u>Theorem 2.7.</u>

The topology induced on J by the measure topology of $\mathcal{M}$ is weaker than that induced by the L_p norm $(1 \leq p \leq \infty)$. Thus the identity extends to a continuous linear mapping

$$(L_p(\mathcal{M},\tau)\ ,\ \|\ \|_p) \longrightarrow (\tilde{\mathcal{M}},\ \text{measure top.})$$

This mapping is in fact injective, so that we may identify each $L_p(\mathcal{M},\tau)$ with a subspace of $\tilde{\mathcal{M}}$. Thus each $x \in L_p(\mathcal{M},\tau)$ uniquely defines a closed, densely defined operator $T_x \eta \mathcal{M}$.

<u>Proof</u> NELSON [46], Thm. 5

Corollary 2.8 (NELSON [46])

Let $1 \leq p < \infty$. $x \in \tilde{\mathcal{M}}$ is in $L_p(\mathcal{M},\tau)$ iff

$$\int_0^\infty t^p d\tau(e_t) < \infty \quad , \text{ where } \quad T_{|x|} = \int_0^\infty t\, de_t$$

If so, then

$$\|x\|_p = \left(\int_0^\infty t^p d\tau(e_t)\right)^{1/p}$$

Having set up a non-commutative integration theory, it is natural to attempt to extend to it various results valid in ordinary Integration theory. Many such results have been proved; see, for example, SEGAL [67], STINESPRING [74], KUNZE [42] and YEADON [91]. We collect here some results which will be useful in the sequel.

Lemma 2.9

Let $x \in \mathcal{M}$, $y \in J$, $1 \leq p \leq \infty$. Then

(2.12) $$\|xy\|_p \leq \|x\|_\infty \|y\|_p \quad , \quad \|yx\|_p \leq \|y\|_p \|x\|_\infty$$

Thus each $x \in \mathcal{M}$ defines bounded operators of left and right multiplication on each $L_p(\mathcal{M},\tau)$.

Proof The case $p = \infty$ is obvious. The case $p=1$ is (2.2)

For $1 < p < \infty$, we have

$$\begin{aligned}
\|xy\|_p &= \sup\{ |\tau(xyz)| \;:\; z \in J,\ \|z\|_q \leq 1 \} \quad (q = p/p-1) \quad (2.3)\\
&\leq \|x\|_\infty \sup\{ \|yz\|_1 \;: z \in J\ , \|z\|_q \leq 1 \} \quad (2.2)\\
&\leq \|x\|_\infty \|y\|_p \sup\{ \|z\|_q \;:\; z \in J,\ \|z\|_q \leq 1 \} \quad (2.2)\\
&= \|x\|_\infty \|y\|_p
\end{aligned}$$

$$\begin{aligned}
\|yx\|_p &= \sup\{ |\tau(yxz)| \;:\; z \in J,\ \|z\|_q \leq 1 \}\\
&= \sup\{ |\tau(zyx)| \;:\; z \in J,\ \|z\|_q \leq 1 \}\\
&\leq \sup\{ \|zy\|_1 \;:\; z \in J,\ \|z\|_q \leq 1 \} \|x\|_\infty \leq \|y\|_p \|x\|_\infty
\end{aligned}$$

Lemma 2.10

Suppose $\tau(1) = 1$.

(i) If $1 \leq p \leq q \leq \infty$,then

(2.13) $$\|x\|_p \leq \|x\|_q \quad \text{for all } x \in \mathcal{M}$$

so that

(2.14) $$\mathcal{M} = L_\infty(\mathcal{M},\tau) \subseteq L_q(\mathcal{M},\tau) \subseteq L_p(\mathcal{M},\tau) \subseteq L_1(\mathcal{M},\tau) = \mathcal{M}_*$$

(ii) Let $x \in \cap\{ L_p(\mathcal{M},\tau) : 1 \leq p < \infty\}$ .Then

(2.15) $$\|x\|_\infty = \sup \|x\|_p = \lim \|x\|_p$$

in the sense that $\sup \|x\|_p < \infty$ iff $x \in \mathcal{M}$,and if so (2.15) holds.

Proof (i)It is enough to assume $p<\infty$,for otherwise(2.13) follows from (2.12) with $y=1$.For $x \in \mathcal{M}$ and $p<q$,let $r = q/p > 1$, $r' = r/r-1$. Then:

$$\|x\|_p^p = \tau(|x|^p) = \tau(|x|^p 1) \leq \||x|^p\|_r \|1\|_{r'} \qquad (2.2)$$
$$= \tau(|x|^{pr})^{1/r} = \tau(|x|^q)^{p/q} = \|x\|_q^p$$

These inequalities extend to $x \in L_q(\mathcal{M},\tau)$ by continuity,and hence (2.14) follows.

(ii) Since $\|x\|_p = \||x|\|_p$,it is enough to consider x self-adjoint.Thus letting $\mathcal{N} \subseteq \mathcal{M}$ be the abelian VN algebra generated by x and 1,and writing $\mathcal{N} = L_\infty(\Omega,\mu)$ with $\tau(f) = \int f(\omega)d\mu(\omega)$,the problem is reduced to the corresponding commutative one ,which is well known.I include a proof for completeness.

Let $M = \sup \|x\|_p$, $\epsilon > 0$.It is enough to assume $M < \infty$, for otherwise (2.15) is obvious from (2.13), and so $x \notin \mathcal{M}$.

Let $A = \{\omega\in\Omega : |x(\omega)| \geq M+\epsilon\}$, $p\in[1,\infty)$, We have

$$M^p \geq \|x\|_p^p \equiv \int_\Omega |x(\omega)|^p d\mu(\omega) \geq \int_A |x(\omega)|^p d\mu(\omega) > (M+\epsilon)^p \mu(A)$$

Thus $\mu(A) < (M/M+\epsilon)^p$ for all p, and hence $\mu(A) = 0$.
Therefore for almost all $\omega\in\Omega$, $|x(\omega)| < M+\epsilon$, so that $x\in L_\infty(\Omega,\mu)$ and

$$\sup \|x\|_p \leq \|x\|_\infty \leq \sup \|x\|_p + \epsilon$$

for all $\epsilon > 0$, which also proves (2.15). Conversely if $x\in \mathcal{M}$ then $M \leq \|x\|_\infty < \infty$ by (2.13), and so (2.15) is again valid.

QED

A useful technique in functional analysis is interpolation of linear operators between different L_p spaces. A classic result here is the RIESZ-THORIN Theorem (see e.g. ZYGMUND [93], Thm.1.11, page 95). This theorem has been extended to the non-commutative case by KUNZE([42] ,Corollary 3.1).

<u>Theorem 2.11</u>

Let $1 \leq p_1, p_2, q_1, q_2, \leq \infty$. Let $t\in(0,1)$ and define p, q by

$$p^{-1} = (1-t)p_1^{-1} + tp_2^{-1}$$

$$q^{-1} = (1-t)q_1^{-1} + tq_2^{-1} \qquad (0^{-1} = \infty \ , \infty^{-1} = 0)$$

For i=1,2, let $\mathcal{M}_i$ be a VN algèbra, τ_i a normal faithful semi-finite trace on $\mathcal{M}_i$ with ideal of definition J_i. Let $T : J_1 \longrightarrow J_2$ be a linear mapping such that

$$\|Tx\|_{q_i} \leq M_i \|x\|_{p_i} \qquad (i=1,2) \text{ for all } x\in J_1$$

Then

$$\|Tx\|_q \leq M_1^{1-t} M_2^t \|x\|_p \qquad \text{for all } x\in J_1$$

If $p < \infty$ then T extends to an operator $L_p(\mathcal{M}_1,\tau_1) \to L_q(\mathcal{M}_2,\tau_2)$ with the same bound.

An interesting question is whether there is any relation between the topology induced on J by the L_p norm and the various topologies of $\mathcal{M}$. In general there is not much one can say. The following result, a special case of Prop.7 of DIXMIER [14], is concerned with what happens in the non-commutative probability case.

Proposition 2.12

Let $p \in [1, \infty)$, and suppose $\tau(1) = 1$ (so that $J = \mathcal{M}$). On the unit ball of $\mathcal{M}$, the topology induced by the L_p norm coincides with the strong topology.

Proof (i) Let $\{x_s\} \subset \mathcal{M}$ be a net such that $\|x_s\|_\infty \leq 1$ and $\|x_s\|_p \to 0$. I claim that $x_s^* x_s \to 0$ ultraweakly. To see this, let $\omega \in \mathcal{M}_*$. By Thm.2.3, there is a unique $y \in L_1(\mathcal{M}, \tau)$ such that $\omega = \omega_y$. Let $y' \in J$ be such that $\|y - y'\|_1 < \epsilon$. We then have:

$$
\begin{aligned}
|\omega_y(x_s^* x_s)| = |\tau(x_s^* x_s y)| &\leq |\tau(x_s^* x_s y')| + \\
&\quad + |\tau(x_s^* x_s (y - y'))| \\
&\leq \|x_s^* x_s\|_p \|y'\|_q + \|x_s^* x_s\|_\infty \|y - y'\|_1 \\
&\leq \|x_s^*\|_\infty \|x_s\|_p \|y'\|_q + \|x_s\|_\infty^2 \|y - y'\|_1 \\
&< \quad \epsilon \quad \|y'\|_q + \quad \epsilon
\end{aligned}
$$

for all large enough s. This proves the claim.

Thus $x_s^* x_s \to 0$ (uw), hence weakly (see §1.2). Therefore $x_s \to 0$ strongly, for if $\zeta \in \mathcal{H}$, $\|x_s \zeta\|^2 = (\zeta, x_s^* x_s \zeta) \to 0$.

This shows that the topology induced by the L_p norm is finer than the strong topology on the unit ball of $\mathcal{M}$.

(ii) For the converse, suppose that the net $\{x_s\} \subseteq \mathcal{M}$ converges to zero strongly, while $\|x_s\|_\infty \leq 1$. Then $|x_s|^p \longrightarrow 0$ strongly* hence weakly, hence ultraweakly (since $\|x_s\|_\infty \leq 1$). Therefore

$$\|x_s\|_p^p = \tau(|x_s|^p) \longrightarrow 0 \qquad (\tau \in \mathcal{M}_*)$$

which shows that the strong topology is finer than the L_p topology on the unit ball of $\mathcal{M}$.

QED

Another interesting problem is that of duality. As is well known, the dual of $L_p(\Omega,\mu)$ is $L_q(\Omega,\mu)$, where $1 \leq p < \infty$ and $q = p/p-1$ $(1/0 = \infty)$. We already know that $(L_1(\mathcal{M},\tau))^* = (\mathcal{M}_*)^* = \mathcal{M} = L_\infty(\mathcal{M},\tau)$ (Thm.2.3). The case $p>1$ was first proved by DIXMIER [14] in the abstract setting. A more direct proof was given by YEADON [91] in SEGAL's setting. Both proofs are rather more complicated than the corresponding ones in the abelian case. In the case where τ is finite, however, a simpler proof may be given, which is more intuitive, in the sense of being closer to one of the standard proofs in ordinary integration theory (see SEGAL and KUNZE [71], Thm.6.1) I would like to close the presentation of (SEGAL-DIXMIER) Non-Commutative Integration Theory with this proof.

---

* for $\||x_s|^p\zeta\|^2 = \int_0^1 t^{2p} d\|e_t^s\zeta\|^2 \leq \int_0^1 t^2 d\|e_t^s\zeta\|^2 = \||x_s|\zeta\|^2 =$

$$= (|x_s|\zeta, |x_s|\zeta) = (\zeta, x_s^* x_s \zeta) = \|x_s\zeta\|^2 \longrightarrow 0$$

for each $\zeta \in \mathcal{H}$, where by the spectral theorem

$$|x_s| = \int_0^1 t\, de_t^s$$

Theorem 2.13

If $1 \leq p < \infty$, $q = p/p-1$ $(1/o = \infty)$ then $L_q(\mathcal{M},\tau)$ is (isometrically isomorphic to) the Banach space dual of $L_p(\mathcal{M},\tau)$.

Proof For the general case ,see DIXMIER [14], Thm.7 or YEADON [91],Thm.4.4. I shall prove the Theorem for the case $\tau(1) = 1$. The case p=1 is Thm.2.3.Therefore assume that p>1.

(i) Let $x \in \mathcal{M}$ .Define ,as in Thm.2.1 the linear form ω_x by

$$\omega_x(y) = \tau(xy) \quad (y \in \mathcal{M})$$

One then has,if $\|\circ\|_{p'}$ denotes the dual norm on $(L_p(\mathcal{M},\tau))^*$,

$$\begin{aligned} \|\omega_x\|_{p'} &= \sup\{|\omega_x(y)| : y \in \mathcal{M},\ \|y\|_p = 1\} \\ &= \sup\{|\tau(xy)| : y \in \mathcal{M},\ \|y\|_p = 1\} \\ &= \|x\|_q \end{aligned}$$

by (2.3).Therefore the mapping $x \longmapsto \omega_x$ extends to a linear isometry from $L_q(\mathcal{M},\tau)$ into $(L_p(\mathcal{M},\tau))^*$.
It remains to show that it is onto.

(ii)Let $F \in (L_p(\mathcal{M},\tau))^*$. I first claim that F restricted to $\mathcal{M}$ is ultraweakly continuous, so that $F|_{\mathcal{M}} \in \mathcal{M}_*$.

To see this,first observe that $F|_{\mathcal{M}}$ is norm-continuous, for the norm topology on $\mathcal{M}$ is stronger than the L_p topology by Lemma 2.10(i).Thus,in view of [56],Thm.4.5 (see the foot-note in § 2.1) it is enough to show that $F|_{\mathcal{M}}$ is completely additive,i.e. that given any family of orthogonal projections $\{p_i : i \in I\} \subset \mathcal{M}$,we have

$$\sum_{i\in I} F(p_i) = F(\sum_{i\in I} p_i)$$

where the (possibly uncountable) sum on the left hand side is defined to be the limit of the net of all partial (finite) sums, directed by inclusion of the index sets. $\sum_{i\in I} p_i$ is the limit of the increasing net of projections $\{q_J : J\subseteq I \text{ finite}\}$ where $q_J := \sum_{j\in J} p_j$, in the strong topology. But on projections, it is easy to see that the strong, weak and ultraweak topologies coincide, and they also coincide with the L_p topology by Prop. 2.12, since projections are contained in the unit ball of $\mathcal{M}$. Therefore

$$\left| F(\sum_{i\in I} p_i) - \sum_{j\in J} F(p_j)\right| = \left|F(\sum_{i\in I} p_i) - F(q_J)\right|$$
$$\leq \|F\|_{p'} \|\sum_{i\in I} p_i - q_J\|_p \longrightarrow 0$$

which proves the claim.

Thus by the Radon-Nikodym Thm. 2.3 there exists a unique $x\in L_1(\mathcal{M},\tau)$ such that $F = \omega_x$, that is

(2.16) $$F(y) = \tau(xy) \qquad \text{for all } y\in\mathcal{M}$$

(iii) To conclude the proof, we must show that $x\in L_q(\mathcal{M},\tau)$. Since $x\in\tilde{\mathcal{M}}$, writing $T_x = u\int_0^\infty t\,de_t$ (see Thm. 2.4), it is enough to show that $\int_0^\infty t^q d\tau(e_t) < \infty$ (Cor. 2.8).

Let $x_n = u\int_0^n t\,de_t = xe_n$. Then $x_n\in\mathcal{M}$, and hence by part (i), x_n defines a unique $F_n\in(L_p(\mathcal{M},\tau))^*$. For $y\in\mathcal{M}$,

$$|F_n(y)| = |\tau(xe_ny)| = |F(e_ny)| \leq \|F\|_{p'} \|e_ny\|_p \leq \|F\|_{p'} \|y\|_p$$

so that

$$\|x_n\|_q = \|F_n\|_{p'} \leq \|F\|_{p'}$$

Now let $f(t) := t^q \quad t \geq 0$

and
$$f_n(t) := \begin{cases} t^q & 0 \leq t \leq n \\ 0 & t > n \end{cases}$$

We have
$$\int_0^\infty f_n(t)d\tau(e_t) = \int_0^n t^q d\tau(e_t) = \|x_n\|_q^q \leq \|F\|_p^q,$$

for all $n \in \mathbb{N}$, where we have used Cor. 2.8.

Since $0 \leq f_n(t) \uparrow f(t)$ for all $t \geq 0$, the monotone convergence theorem implies that

$$\int_0^\infty f_n(t)d\tau(e_t) \uparrow \int_0^\infty f(t)d\tau(e_t) = \int_0^\infty t^q d\tau(e_t)$$

and therefore
$$\int_0^\infty t^q d\tau(e_t) \leq \|F\|_p^q,$$

Thus by Cor. 2.8 ,
$$x \in L_q(\mathcal{M}, \tau)$$

and
$$\|x\|_q \leq \|F\|_p,$$

Since, by (2.16), the linear functionals F and ω_x (which are now both in $(L_p(\mathcal{M},\tau))^*$ by part (i) since $x \in L_q(\mathcal{M},\tau)$) agree on the dense set $\mathcal{M}$ in $L_p(\mathcal{M},\tau)$, they must agree everywhere, and the proof is complete.

QED

Chapter II
ISOMETRIES

This Chapter is devoted to the study of Isometries between various Banach spaces of operators.The aim is to discover how much of the algebraic structure is transported by these isometries.

I shall describe various aspects of this problem,starting with the abelian case,where the results are known.The final paragraph is devoted to a partial extension of some of these results.Specifically,the aim is to find sufficient conditions for an isometry between two Non-Commutative L_p spaces to preserve the algebraic structure of the underlying VN algebras.

The solution of this problem is of interest in the classification theory of Banach spaces,as it shows that certain Non-Commutative L_p spaces can be distinguished,as Banach spaces, from classical L_p spaces. More precisely, if $L_p(\mathcal{M},\tau)$ is isomorphic, as a Banach space, to $L_p(\Omega,\mu)$, then it will follow that $\mathcal{M}$ is isomorphic, as an algebra, to $L_\infty(\Omega,\mu)$, and therefore is abelian.

§1. A classic result in the abelian case is the following:

THEOREM 1.1 (BANACH [7], STONE [75])

Let X , Y be compact Hausdorff spaces, ϕ: X $\longrightarrow$ Y a bijective homeomorphism.Then the mapping

$$T : C(X) \longrightarrow C(Y)$$

given by

$$(Tf)(y) = f(\phi^{-1}(y)) \quad (y \in Y , f \in C(X))$$

is a linear isometry of $C(X)$ onto $C(Y)$, and $T(1) = 1$. Conversely, let T be a linear isometry of $C(X)$ onto $C(Y)$. Then there exists a bijective homeomorphism ϕ of X onto Y and a unitary $u \in C(Y)$ such that

$$(Tf)(y) = u(\phi^{-1}(y))\, f(\phi^{-1}(y)) \qquad (y \in Y,\ f \in C(X))$$

In particular, all unit preserving isometries are induced by homeomorphisms.

I now wish to translate this theorem into an internal characterization of isometries between abelian C^* algebras. First of all, we know (Chapter I, §1.1) that $C(X)$ is the most general abelian C^* algebra. It is further known (and not too difficult to see, given the Gelfand theory and the fact that the topology of a compact Hausdorff space is the weak topology determined by its continuous functions) that there exists a one-to-one correspondence between homeomorphisms of compact Hausdorff spaces and $*$-isomorphisms of their function algebras. (see e.g. SIMMONS [73], Thm. 74D). Thus Thm.1.1 gives the

THEOREM 1.1'

Let T be a bijective linear isometry between two unital abelian C^* algebras $\mathfrak{A}$ and $\mathfrak{B}$. Then there exists a $*$-isomorphism S of $\mathfrak{A}$ onto $\mathfrak{B}$ such that, for all $f \in \mathfrak{A}$,

$$Tf = u\, Sf$$

where $u = T(1) \in \mathfrak{B}$ is unitary.

I include a new proof of this result, which is much quicker than the original one (but of course uses modern machinery)

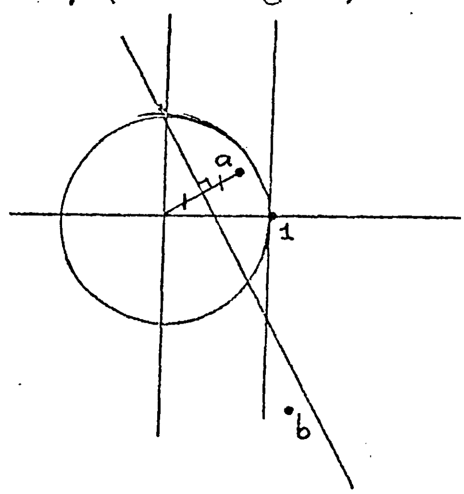
Let $T(1) = u$. Then u is extreme in the unit sphere of $\mathfrak{B}$ (since 1 is extreme in the unit sphere of $\mathfrak{A}$) hence is unitary (KADISON [33] , Thm.1).Let

$$S : \mathfrak{A} \longrightarrow \mathfrak{B}$$
$$f \longmapsto u^*Tf$$

Then S is isometric and $S(1) = 1$.

I claim that S is positivity preserving.

Suppose not, and let $f \in \mathfrak{A}$, $0 \leq f \leq 1$ be such that Sf is not nonnegative. Writing $\mathfrak{A} = C(X)$, $\mathfrak{B} = C(Y)$, this means that there exists $y \in Y$ such that $(Sf)(y) := a \notin [0,+\infty]$. Note that $|a| = |(Sf)(y)| \leq \|Sf\|_\infty = \|f\|_\infty \leq 1$ .Therefore there exists $b \in \mathbb{C}$ with $\mathrm{Re}\, b \geq 1$ and $|b| < |b-a|$ (see diagram)



But then

$$\|b - f\|_\infty \leq |b| < |b - a| =$$
$$= |b - (Sf)(y)| =$$
$$= |S(b-f)(y)| \leq$$
$$\leq \|S(b-f)\|_\infty = \|b - f\|_\infty$$

a contradiction.

This shows that the dual map

$$S^* : \mathfrak{B}^* \longrightarrow \mathfrak{A}^*$$

given by

$$(S^*\phi)(f) = \phi(Sf) \qquad (f \in \mathfrak{A} \ , \ \phi \in \mathfrak{B}^*)$$

is isometric, positivity preserving, and $(S^*\phi)(1) = \phi(1)$. Therefore S^* sends states onto states, and, being isometric, extreme elements of the unit ball of $\mathfrak{B}^*$ onto extreme elements of the unit ball of $\mathfrak{A}^*$.

Therefore S^* maps the pure states of $\mathfrak{B}$ onto the pure states of $\mathfrak{A}$. But the pure states of C(X) are precisely the evaluation functionals+ δ_x (x e X) given by

$$\delta_x(f) = f(x) \qquad (f \in C(X))$$

Now for f,g e $\mathfrak{A}$ and y e Y, we have, letting $\delta_x = S^*\delta_y$,

$$(S(fg))(y) = \delta_y(S(fg)) = (S^*\delta_y)(fg) = \delta_x(fg) = f(x)g(x) =$$

$$= (\delta_x f)(\delta_x g) = (S^*\delta_y)(f)(S^*\delta_y)(g) =$$
$$= (\delta_y(Sf))(\delta_y(Sg)) = (Sf)(y)(Sg)(y)$$

so that

$$S(fg) = (Sf)(Sg)$$

This shows that S is multiplicative, and it is clearly a $*$-map, being positivity preserving.

QED

+ We can see this as follows: A state on C(X) is a regular probability measure μ (Riesz representation Thm.; see e.g. HALMOS [28] Thm. 56D). If μ is pure, and its support contains two distinct points, then it must contain two disjoint non-empty open sets U, V . Thus $0 < \mu(U) < 1$. But then defining

$$\nu_1(A) = \mu(U \cap A)/\mu(U) \ , \ \nu_2(A) = \mu(U^c \cap A)/\mu(U^c)$$

(where $U^c = X \setminus U$) we get two probability measures (states) ν_1, ν_2 which are distinct and such that

$$\mu = \mu(U)\nu_1 + \mu(U^c)\nu_2$$

This contradicts the assumption that μ is pure. Therefore the support of μ contains exactly one point x , and so $\mu = \delta_x$.

I now wish to discuss isometries of classical L_p spaces. I shall only describe some recent results, which will be extended to the non-commutative situation in §3.

Let (Ω_i, μ_i) be two (classical) probability spaces. Let $\mathcal{U} \subseteq L_\infty(\Omega_1, \mu_1)$ be a subalgebra containing constants. The following result was proved by FORELLI ([19], Prop.2):

Theorem 1.2

Let $T : \mathcal{U} \longrightarrow L_\infty(\Omega_2, \mu_2)$

be a linear mapping such that $T(1) = 1$.

If there exists $p \in [1,\infty]$, $p \neq 2$, such that T preserves the L_p norm, then T is a homomorphism.

This Theorem was extended by SCHNEIDER ([65], Thm.B) for the case $p>2$ as follows:

Theorem 1.3

Let $T : \mathcal{U} \longrightarrow L_p(\Omega_2, \mu_2)$

be a linear L_p-isometry such that $T(1) = 1$.

Then T is a homomorphism, $\|Tf\|_\infty = \|f\|_\infty$ and $\|Tf\|_2 = \|f\|_2$ for all $f \in \mathcal{U}$.

The crucial difference is that the fact that T sends bounded functions into bounded functions is not assumed, but is a conclusion of the theorem.

The proof of this result depends on the following

Proposition 1.4

Let $2 < p < \infty$, $f_i \in L_p(\Omega_i, \mu_i)$ $(i=1,2)$. If there exists $r > 0$ such that, whenever $z \in \mathbb{C}$ and $|z| < r$, we have:

$$\|1 + zf_1\|_p = \|1 + zf_2\|_p$$

then

(a) $\|f_1\|_2 = \|f_2\|_2$

(b) $\|f_1\|_4 = \|f_2\|_4$

<u>Proof</u> (a) is proved in FORELLI ([19],Prop.1) and (b) in SCHNEIDER ([65] ,Thm.A). I sketch the proof of (b),since it is basic to the non-commutative extension.

First note that since $p > 2$, $\|f_j\|_2 < \infty$ .If $\|f_1\|_4 = \|f_2\|_4 = \infty$ there is nothing to prove.Assume therefore that $\|f_1\|_4 < \infty$.

Consider

$$(1.1) \qquad \frac{1}{2\pi}\int_0^{2\pi} |1 + ze^{ix}|^p dx - \frac{p^2}{4}|z|^2 - 1$$

Expanding $(1 + ze^{ix})^{p/2}$ as a power series,convergent for $|z| < 1$,we find that (1.1) equals

$$(1.2) \qquad \binom{p/2}{2}^2 |z|^4 + \sum_{k=3}^{\infty} \binom{p/2}{k}^2 |z|^{2k}$$

where $\binom{p/2}{k}$ is the usual binomial coefficient.This shows

$$(1.3) \quad F_j(r,\omega) := r^{-4}\left(\frac{1}{2\pi}\int_0^{2\pi} |1 + rf_j(\omega)e^{ix}|^p dx - \frac{p^2}{4}|rf_j(\omega)|^2 - 1\right)$$

converges to $\binom{p/2}{2}^2 |f_j(\omega)|^4$ as $r \searrow 0$ for (almost) all $\omega \in \Omega_j$.

Now SCHNEIDER [65] shows that (1.1) is non-negative for all $z \in \mathbb{C}$, hence Fatou's Lemma (SEGAL & KUNZE [71] ,Cor.3.4.2) is applicable to (1.3) and gives

$$(1.4) \qquad \binom{p/2}{2}^2 \int_{\Omega_2} |f_2(\omega)|^4 d\mu_2(\omega) \leq \liminf_{r \to 0} \int_{\Omega_2} F_2(r,\omega)\, d\mu_2(\omega)$$

Using (1.2),one now finds that

$$(1.5) \qquad F_j(r,\omega) \leq A|f_j(\omega)|^4 + Br^{p-4}|f_j(\omega)|^p \quad \text{if } p \geq 4$$

and

(1.6) $F_j(r,\omega) \leq C\ |f_j(\omega)|^4$ if $2 < p \leq 4$

where A,B,C are constants.We notice that,for j=1, the right hand side of both (1.5) and (1.6) are integrable,and hence the Lebesgue Dominated Convergence theorem (SEGAL & KUNZE [71] Cor.3.4.5) may be applied to (1.3) with j=1 to give

(1.7) $\lim_{r\to 0}\int_{\Omega_1} F_1(r,\omega)d\mu_1(\omega) = (\frac{p/2}{2})^2\int_{\Omega_1}|f_1(\omega)|^4 d\mu_1(\omega) < \infty$

However

(1.8) $\int_{\Omega_1} F_1(r,\omega)d\mu_1(\omega) =$

$$= r^{-4}\left(\frac{1}{2\pi}\int_0^{2\pi}\|1 + re^{ix}f_1\|_p^p\, dx - \frac{p^2}{4}r^2\|f_1\|_2^2 - 1\right) =$$
$$= r^{-4}\left(\frac{1}{2\pi}\int_0^{2\pi}\|1 + re^{ix}f_2\|_p^p\, dx - \frac{p^2}{4}r^2\|f_2\|_2^2 - 1\right) =$$
$$= \int_{\Omega_2} F_2(r,\omega)d\mu_2(\omega)$$

where we have used the assumption of the Proposition,part (a) ,and Fubini's Theorem ([71] ,Thm.3.4) to interchange $\int_{\Omega_j}$ with $\int_0^{2\pi}$.

Combining now (1.4),(1.7) &(1.8) ,we find:

$$(\frac{p/2}{2})^2\int_{\Omega_2}|f_2(\omega)|^4 d\mu_2(\omega) \leq \liminf_{r\to 0}\int_{\Omega_2} F_2(r,\omega)d\mu_2(\omega) =$$
$$= \liminf_{r\to 0}\int_{\Omega_1} F_1(r,\omega)d\mu_1(\omega) =$$
$$= (\frac{p/2}{2})^2\int_{\Omega_1}|f_1(\omega)|^4 d\mu_1(\omega)$$

Thus

$$\|f_2\|_4 \leq \|f_1\|_4 < +\infty$$

so that we may now repeat the argument with f_1 and f_2 interchanged to get the required equality.

The proof of part (a) is very similar, the basic difference being that one replaces (1.1) by

$$\frac{1}{2\pi}\int_0^{2\pi}|1 + ze^{ix}|^p dx \quad - 1$$

QED

Another extension of this result is due to YOUNG [92] :

Theorem 1.5

Suppose $\mathcal{U}$ is a $*$-subalgebra of $L^\infty(\Omega_1,\mu_1)$, and that either $1\leq p\leq q<2$ or $2<q\leq p<\infty$. Let

$$T : \mathcal{U} \longrightarrow L_q(\Omega_2,\mu_2)$$

be a linear mapping such that $\|Tf\|_q = \|f\|_p$ for all $f \in \mathcal{U}$ and $T(1) = 1$. Then T is a $*$-homomorphism.

His proof is completely different from those of FORELLI and SCHNEIDER, and relies on the fact that T may be extended to the weak closure of $\mathcal{U}$ ([92] , Lemma 1), which allows him to conclude that this extension maps characteristic functions of measurable disjoint sets to characteristic functions of measurable disjoint sets, and therefore it (and hence also T) is a homomorphism.

§2.1. The non-commutative analogue of these problems have a long history. The first version, which was discussed before the appearance of Non-Commutative Integration theory, was the following : If two C^* algebras are isomorphic as Banach spaces, how much of the algebraic structure is transferred by the isometry? We have seen that this problem is completely solved in the abelian case by Thm.1.1'. A non-commutative extension of this result is due to KADISON [33]. First we need the definition :

Definition Let $\mathfrak{A}$, $\mathfrak{B}$ be $*$-algebras. A Jordan $*$-homomorphism $T : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a linear mapping which preserves the Jordan product and the involution, that is

$$T(x^2) = (T(x))^2$$

and

$$T(x^*) = (T(x))^* \qquad \text{for all } x \in \mathfrak{A}.$$

Theorem 2.1

A linear bijection T between two unital C^* algebras is isometric iff it is a Jordan $*$-homomorphism composed with left multiplication by a fixed unitary element, namely T(1) .

Proof The fact that a Jordan $*$-homomorphism is isometric was proved by KADISON ([33], Thm.5) and STØRMER ([76], Cor.3.5). The converse assertion is Thm.7 of KADISON [33] .

The structure of Jordan $*$-homomorphisms between C^* algebras has a particularly simple form. We first have a Lemma:

Lemma 2.2

Let T be a Jordan $*$-homomorphism from a C^* algebra $\mathfrak{A}$ into a VN algebra $\mathcal{M}$. Then T has an uw continuous extension

$$T^{**}: \mathfrak{A} \longrightarrow \mathcal{M}$$

which is also a Jordan $*$-homomorphism.

A word of explanation is needed here. Firstly, if $\mathfrak{A}$ is given the w*-topology (i.e. the $\sigma(\mathfrak{A}, \mathfrak{A}^*)$ topology, and $\mathcal{M}$ is given the uw topology, then by duality we get a map

$$T^*: \mathcal{M}_* \longrightarrow \mathfrak{A}^*$$

since T is continuous with respect to these topologies.

The dual of this map is

$$T^{**}: \mathcal{M} \longrightarrow \mathfrak{A}^{**}$$

since $(\mathcal{M}_*)^* = \mathcal{M}$ (Chapter I,§1.3).STØRMER[76](Lemma 3.1) now shows that T^{**} is a Jordan $*$-homomorphism and extends T, using Sherman's thm.(Chapter I,§1.2).

Using this Lemma,one now proves:

Theorem 2.3

Let $T: \mathfrak{A} \longrightarrow \mathfrak{B}$ be a Jordan $*$-homomorphism between two C^* algebras.Then T is the sum of a $*$-homomorphism and a $*$-antihomomorphism in the following sense:There exist orthogonal central projections $p,q \in T(\mathfrak{A})''$ (we assume,as we may (Ch.I,§1.1), that $\mathfrak{B}$ is faithfully represented on a Hilbert space) such that $p+q=1$ and

$$T_1: x \longmapsto T(x)p$$

is a $*$-homomorphism,while

$$T_2: x \longmapsto T(x)q$$

is a $*$-antihomomorphism,and $T = T_1 + T_2$ as linear maps.

Proof STØRMER[76],Thm.3.3,extending KADISON[33],Thm.10 .

Corollary 2.4

Suppose in addition that $T(\mathfrak{A})''$ is a factor.Then T is either a $*$-homomorphism or a $*$-antihomomorphism.

§2.2. In the previous section we analysed completely the structure of isometries of C^* algebras. We now continue our extension of results in classical analysis by considering isometries of Non-Commutative L_p spaces.

The only result, to my knowledge, in the semifinite case is due to BROISE ([10], Thm. 1 & Prop. 1) :

<u>Theorem 2.5</u>

Let $(\mathcal{M}_i, \tau_i)$ (i=1,2) be semifinite VN algebras with faithful normal semifinite traces τ_i.

(i) Let

$$T : L_2(\mathcal{M}_1, \tau_1) \longrightarrow L_2(\mathcal{M}_2, \tau_2)$$

be an isometric bijection (i.e. a unitary), which is positivity preserving (i.e. Tf is a positive operator affiliated to $\mathcal{M}_2$ whenever f is a positive operator in $L_2(\mathcal{M}_1, \tau_1)$). Then there exists a unique Jordan $*$-isomorphism S of $\mathcal{M}_1$ onto $\mathcal{M}_2$ and a positive self-adjoint operater $z \eta \mathcal{M}_2 \cap \mathcal{M}_2'$ such that

$$T(f) = zS(f) \qquad \text{for all } f \in \mathcal{M}_1 \cap L_2(\mathcal{M}_1, \tau_1)$$

in particular, if $\mathcal{M}_1$ or $\mathcal{M}_2$ is a factor, then z is a scalar multiple of the identity and S is either a $*$-homomorphism or a $*$-antihomomorphism.

(ii) Conversely, let

$$S : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$$

be a Jordan $*$-isomorphism onto. Then there is a unique faithful normal semifinite trace τ on $\mathcal{M}_2$ such that

$$\tau(g) = \tau_1(S^{-1}g) \qquad (g \in \mathcal{M}_2^+)$$

$$\tau(S(f^*)S(f)) = \tau_1(f^*f) \qquad (f \in \mathcal{M}_1)$$

$$S(\mathcal{M}_1 \cap L_2(\mathcal{M}_1,\tau_1)) = \mathcal{M}_2 \cap L_2(\mathcal{M}_2,\tau))$$

There exists a unique unitary bijection :

$$T' : L_2(\mathcal{M}_1,\tau_1) \longrightarrow L_2(\mathcal{M}_2,\tau_2)$$

which is positivity preserving and coincides with S on $\mathcal{M}_1 \cap L_2(\mathcal{M}_1,\tau_1)$ .Finally, there exists a unique positive self-adjoint operator $z \,\eta\, \mathcal{M}_2 \cap \mathcal{M}_2'$ such that

$$\begin{aligned} T \;:\; L_2(\mathcal{M}_1,\tau_1) &\longmapsto L_2(\mathcal{M}_2,\tau_2) \\ x &\longmapsto zT'(x) \end{aligned}$$

is a unitary positivity preserving bijection.

From now on, we restrict ourselves to the finite case. To the end of this section, $\mathcal{M}$ will be a (finite) VN algebra, and $\tau \in \mathcal{M}_*$ a faithful tracial state.The first result,due to RUSSO [61] (Theorem 1) is a partial extension of Thm.1.5 to the non-commutative situation (for p=q=1) obtained basically by duality (see Chapter I,Thm.2.3) from KADISON's Thm.2.1.

Theorem 2.6

Let

$$T: L_1(\mathcal{M},\tau) \longmapsto L_1(\mathcal{M},\tau)$$

be an isometric bijection.Then there exist: a Jordan $*$-automorphism S of $\mathcal{M}$, a positive self-adjoint operator $z \,\eta\, \mathcal{M} \cap \mathcal{M}'$ and a unitary $u \in \mathcal{M}$,such that

$$T(x) = S(x)z^2u \qquad (x \in \mathcal{M})$$

In particular,if $\mathcal{M}$ is a factor and T(1)=1,then $T|_{\mathcal{M}}$ is either a $*$-automorphism or a $*$-antiautomorphism.

The next result is a partial extension of Thm.1.2.

Theorem 2.7

Let T be a linear bijection of $\mathcal{M}$ which preserves the L_p norm for some $p \in [1,+\infty)$. The following are equivalent :

(i) T is a Jordan $*$-automorphism.

(ii) T is positivity preserving and $T(1) = 1$.

(iii) T sends projections to projections.

(iv) T maps the self-adjoint part of the unit ball into itself and $T(1) = 1$.

Corollary 2.8

Suppose, in Thm.2.7, that $\mathcal{M}$ is a factor. Then:

(i) T is either a $*$-automorphism or a $*$-antiautomorphism.

(ii) In case p = 1 or 2 , the condition T(1)=1 may be dropped from condition (ii) of the theorem.

Proofs RUSSO[61], Theorem 2, Corollary 1 and 2.

The necessity of assuming positivity preservation is an interesting question. We shall see (§3 ,Thm.3.1) that this assumption is redundant in case $p > 2$. Russo [61] gives a counterexample to the assertion that any unit preserving $*$-linear L_2 isometry of a finite factor is necessarily either a $*$-automorphism or a $*$-antiautomorphism. We shall see in the next section that the assumption of positivity preservation is essential in case p=2 , even in the abelian case.

§3. In the previous sections, I described known results in the problem of isometries. I shall now embark on the extension of Theorem 1.3 to the non-commutative situation. The main results of this section are improvements of results that have appeared in [36].

Throughout this section, let $\mathcal{M}_i$ (i=1,2) be two VN algebras, $\tau_i \in \mathcal{M}_{i*}$ faithful tracial states. I wish to prove the following Theorem:

<u>Theorem 3.1</u>

Let $\mathcal{U}$ be a unital $*$-subalgebra of $\mathcal{M}_1$. For some $p \in (2,\infty)$ let

$$T : \mathcal{U} \longrightarrow L_p(\mathcal{M}_2, \tau_2)$$

be a $*$-linear mapping such that $T(1) = 1$. Suppose that

$$\|Tx\|_p = \|x\|_p \qquad \text{for every normal } x \in \mathcal{U}.$$

Then T is a Jordan $*$-homomorphism.

Note that I need to assume that T preserves the involution and the identity. In order to be able to distinguish $L_p(\mathcal{M},\tau)$, as a Banach space, from classical L_p spaces (see the introductory remarks to this Chapter), at least in the finite case, one would need to prove that these two assumptions are redundant. The assumption that T is a $*$-map enters only through the non-commutative extension of Prop.1.4 (see Prop. 3.3). In a complete extension of this result, it seems plausible that this assumption need not be made. An attempt to prove such an extension, using known non-commutative analogues of the Theorems of Fatou, Fubini and Lebesgue Dominated Convergence (see the proof of Prop.1.4), was inconclusive. Thus this is still an open problem. In order to get rid of the assumption that $T(1) = 1$, one would need to study a geometrical property of 1 in the unit ball of $L_p(\mathcal{M}_1, \tau_1)$ (in the C^* algebra case, (Thm.2.1) the extremity of 1 is used to show that $T(1)$ is in fact a unitary) [82] . Again this is an open problem.

YOUNG [92] observed that one might as well assume the domain of T to be a VN algebra. Specifically, he proved the

Lemma 3.2

With the assumptions of Thm.3.1, T has a unique extension

$$T_e : \mathcal{U}^- \longrightarrow L_p(\mathcal{M}_2, \tau_2)$$

(where $\mathcal{U}^-$ denotes the strong closure of $\mathcal{U}$) which is also an L_p isometry on normal elements.

Proof Let T_1 denote the restriction of T to the unit ball $\mathcal{U}_1$ of $\mathcal{U}$. Now on the unit ball of $\mathcal{M}_1$ (hence also on $\mathcal{U}_1$)) the L_p topology coincides with the strong topology (Chapter I, Prop.2.12). Therefore

$$T_1 : (\mathcal{U}_1, s) \longrightarrow (L_p(\mathcal{M}_2, \tau_2), \|\cdot\|_p)$$

is continuous, and hence may be extended to the strong closure $\mathcal{U}_1^-$ of $\mathcal{U}_1$, the resulting extension $\overline{T}_1$ still being an L_p isometry by continuity.

By Kaplansky's density Theorem (Chapter I, §1.2) $\mathcal{U}_1^-$ coincides with the unit ball of the strong closure $\mathcal{U}^-$ of $\mathcal{U}$. Thus for all $x \in \mathcal{U}^-$, $(x/\|x\|_\infty) \in \mathcal{U}_1^-$, so we may define

$$T_e x = \|x\|_\infty \overline{T}_1(x/\|x\|_\infty)$$

and T_e is clearly an L_p isometry.

QED

The main tool in the proof of Thm.1.3 is Prop.1.4. But the statement and conclusion of this Proposition only involves one element from each $L_p(\Omega_i,\mu_i)$.This observation allows one to prove:

Proposition 3.3

Let $p \in (2,\infty)$, $x_i \in L_p(\mathcal{M}_i,\tau_i)$,T_{x_i} normal ($T_{x_i}\eta\mathcal{M}_i$ - see Chapter I,Thm.2.8)

If there exists $r>0$ such that,whenever $z \in \mathbb{C}$ is such that $|z| < r$,we have

$$\|1 + zx_1\|_p = \|1 + zx_2\|_p$$

then

$$(a) \quad \|x_1\|_2 = \|x_2\|_2$$

and

$$(b) \quad \|x_1\|_4 = \|x_2\|_4 .$$

Proof Since $T_{x_i}\eta\,\mathcal{M}_i$,the spectral projections of T_{x_i} (that is, the projections e^i_λ such that $T_{x_i} = \int_{\mathbb{C}} \lambda \, de^i_\lambda$ by the spectral theorem (see e.g. RUDIN [59] ,Thm.13.33)) belong to $\mathcal{M}_i$ .Denote by $\mathcal{N}_i \subseteq \mathcal{M}_i$ the abelian VN algebra generated by these projections. We may write $\mathcal{N}_i = L_\infty(\Omega_i,\mu_i)$ with $\int_{\Omega_i} x_i(\omega)d\mu_i(\omega) = \tau_i(x_i)$ for all $x_i \in \mathcal{N}_i$,and thus $L_p(\mathcal{N}_i,\tau_i)= L_p(\Omega_i,\mu_i)$ (Chapter I,§2.2).The identity mapping obviously extends to an isometric embedding $L_p(\mathcal{N}_i,\tau_i) \subseteq L_p(\mathcal{M}_i,\tau_i)$.

Thus

$$x_i \in L_p(\Omega_i,\mu_i) \subseteq L_p(\mathcal{M}_i,\tau_i)$$

and therefore the problem is reduced to the abelian case. Thus the result follows by Prop.1.4.

QED

Proof of Thm.3.1

(i) Let $x \in \mathcal{U}$ be self-adjoint, $z \in \mathbb{C}$. Since $T(1 + zx) = 1 + zTx$ and since Tx is self-adjoint, we have :

$$\|1 + zx\|_p = \|1 + zTx\|_p$$

Thus Prop.3.3(b) shows that

$$\|1 + zx\|_4 = \|1 + zTx\|_4 < \infty \qquad \text{,since } x \in \mathcal{M}_1 \subseteq L_4$$

Now

$$|1 + zx|^4 = \sum_{j,k=0}^{2} \binom{2}{j}\binom{2}{k} \bar{z}^j z^k x^j x^k .$$

and so

$$\|1 + zx\|_4^4 = \sum_{j,k=0}^{2} \binom{2}{j}\binom{2}{k} \bar{z}^j z^k \tau_1(x^j x^k)$$

Similarly

$$\|1 + zTx\|_4^4 = \sum_{j,k=0}^{2} \binom{2}{j}\binom{2}{k} \bar{z}^j z^k \tau_2((Tx)^j (Tx)^k)$$

Therefore

(3.1) $$\tau_1(x^j x^k) = \tau_2((Tx)^j (Tx)^k) \qquad j,k = 0,1,2$$

(ii)Putting j=1,k=2 in (3.1) gives

$$\tau_1(x^3) = \tau_2((Tx)^3)$$

Replacing x by $x+ay$, $x,y \in \mathcal{M}_1$ self-adjoint, $a \in \mathbb{R}$, expanding and comparing terms in a^2, one finds

$$\tau_2(Tx(Ty)^2 + TxTyTx + (Ty)^2 Tx) = \tau_1(xy^2 + xyx + y^2 x)$$

or,in view of the centrality of the traces,

(3.2) $$\tau_2(Tx(Ty)^2) = \tau_1(xy^2)$$

On the other hand,polarization of part (a) of Prop.3.3 gives

$$\tau_2(TxTy) = \tau_1(xy)$$

since x and y are self-adjoint.Replacing y by y^2 above,and comparing the result with (3.2),one finds

$$\tau_2(Tx(Ty)^2) = \tau_2(TxT(y^2))$$

and, replacing x by y^2, one gets

$$\tau_2(T(y^2)(Ty)^2) = \tau_2((T(y^2))^2) \tag{3.3}$$

But (3.1) gives, for j=k=2

$$\tau_2((Ty)^4) = \tau_1(y^4)$$

while (3.2) with $x=y^2$ becomes

$$\tau_2(T(y^2)(Ty)^2) = \tau_1(y^4)$$

hence

$$\tau_2((Ty)^4) = \tau_2(T(y^2)(Ty)^2) \tag{3.4}$$

Therefore

$$\|(Ty)^2-T(y^2)\|_2^2 = \tau_2((Ty)^4-(Ty)^2T(y^2) - T(y^2)(Ty)^2+(T(y^2))^2) = 0$$

by (3.3) and (3.4), and thus

$$(Ty)^2 = T(y^2) \qquad \text{for all self-adjoint } y \in \mathcal{U}$$

(iii) Now let $x \in \mathcal{U}$ be arbitrary, and write $x=x_1+ix_2$, with x_1, x_2 self-adjoint. Since x_1+x_2 is self-adjoint, part (ii) gives

$$(Tx_1 + Tx_2)^2 = T((x_1+x_2)^2)$$

Expanding and using part (ii) again, one finds

$$Tx_1Tx_2 + Tx_2Tx_1 = T(x_1x_2 + x_2x_1)$$

Therefore

$$T(x^2) = T((x_1+ix_2)^2) = T(x_1^2 - x_2^2 + i(x_1x_2 + x_2x_1))$$
$$= (Tx_1)^2-(Tx_2)^2-i(Tx_1Tx_2 + Tx_2Tx_1) = (Tx)^2$$

QED

Several conclusions may now be drawn from this result. But first we need some simple Lemmas.

Lemma 3.4

Let $\mathfrak{A}$ be a $*$-algebra, ω a state on $\mathfrak{A}$. Define, for $n \in \mathbb{N}$

$$\|x\|_{2n} := \omega(\ (x^*x)^n)^{1/2n} \qquad (x \in \mathfrak{A}\ )$$

Then

(i) $\|x^*x\|_{2^n} = \|x\|^2_{2^{n+1}}$

(ii) $\|x^2\|_{2^n} = \|x\|^2_{2^{n+1}}$, if x is normal.

(iii) $\|x^*\|_{2n} = \|x\|_{2n}$, if ω is tracial.

(iv) $\|xy\|_2 \leq \|x\|_4\|y\|_4$, if ω is tracial.

Proof (i) $\|x^*x\|^{2^n}_{2^n} = \omega((x^*x)^{2^n}) = \|x\|^{2^{n+1}}_{2^{n+1}}$

(ii) $\|x^2\|^{2^n}_{2^n} = \omega((x^*x^*xx)^{2^{n-1}}) = \omega((x^*xx^*x)^{2^{n-1}}) = \omega((x^*x)^{2^n})$

$$= \|x\|^{2^{n+1}}_{2^{n+1}}$$

(iii) $\|x\|^{2n}_{2n} = \omega((x^*x)^n) = \omega((xx^*)^n) = \|x^*\|^{2n}_{2n}$

(iv) $\|xy\|^2_2 = \omega(y^*x^*xy) = \omega(x^*xyy^*) = \|x^*x\|_2\|y\ y^*\|_2 = \|x\|^2_4\|y\|^2_4$

QED

It is interesting to note that one may prove, without any continuity assumptions whatever, that in case ω is tracial and faithful, $\|\cdot\|_{2n}$ are all increasing norms [13], [78]. This points to yet another direction in which one might try to develop a Non-commutative Integration theory. My attempts in this direction have so far not yielded very significant results, and are therefore not included in this thesis.

Lemma 3.5

Let $T : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a Jordan homomorphism between associative algebras. Then

(i) $T(xyx) = TxTyTx$

(ii) $T((xy)^n+(yx)^n) = (TxTy)^n+(TyTx)^n$

(iii) $T((xy)^n(yx)^n) = (TxTy)^n(TyTx)^n$

(iv) $T((xy)^n x) = (TxTy)^n Tx$

for all $x,y \in \mathfrak{A}$, $n \in \mathbb{N}$.

Proof (i) HERSTEIN [29] ,Lemma 3.1.

(ii) KADISON [33] ,Lemma 6.

(iii) & (iv) follow from (i) by induction on n.

Theorem 3.6

With the assumptions of Thm.3.1, we have:

(i) T is isometric with respect to the L_2 norm.

(ii) For all $x \in \mathcal{U}$ and $q \geqslant 2$, $\|Tx\|_q \leq \|x\|_q$ .Equality holds if $q=2n$, $n \in \mathbb{N}$.

(iii) $T(\mathcal{U}) \subseteq \mathcal{M}_2$ and $\|Tx\|_\infty = \|x\|_\infty$ for all $x \in \mathcal{U}$.

(iv) T is positivity preserving.

Proof (i) Let $x=x_1+ix_2 \in \mathcal{U}$ be arbitrary, with x_j self-adjoint. As in the proof of Thm.3.1, part (a) of Prop.3.3 implies that

$$\|Tx_j\|_2 = \|x_j\|_2 \qquad (j=1,2)$$

But then

$$\|Tx\|_2^2 = \|Tx_1\|_2^2 + \|Tx_2\|_2^2 + i\tau_2(Tx_1Tx_2) - i\tau_2(Tx_2Tx_1) =$$
$$= \|Tx_1\|_2^2 + \|Tx_2\|_2^2 = \|x_1\|_2^2 + \|x_2\|_2^2 = \|x\|_2^2$$

(ii)&(iii) For $x \in \mathcal{U}$, $n \in \mathbb{N}$, consider

$$\|Tx\|_{2n}^{2n} = \tau_2((Tx^*Tx)^n) = \tau_2(Tx(Tx^*Tx)^{n-1}Tx^*) =$$

$= (Tx^*, (Tx^*Tx)^{n-1}Tx^*) = (Tx^*, T((x^*x)^{n-1}x^*))$ (Lemma 3.5(iv))

$= (x^*, (x^*x)^{n-1}x^*)$ by part (i)

$= \tau_1(x(x^*x)^{n-1}x^*) = \tau_1((x^*x)^n) = \|x\|_{2n}^{2n}$

Therefore

$$\|Tx\|_{2n} = \|x\|_{2n} \qquad \text{for all } n \in \mathbb{N}$$

and letting $n \to \infty$, we have, by Lemma 2.10 of ChapterI,

$$\|Tx\|_\infty = \|x\|_\infty$$

We may now apply the RIESZ-THORIN-KUNZE Theorem (Chapter I, Thm.2.11) with $p_1=q_1=2$, $p_2=q_2=\infty$, $p=q$ to conclude that

$$\|Tx\|_q \leq \|x\|_q \quad \text{for all } x \in \mathcal{U} \text{ and } q > 2.$$

(iv) In view of part (ii), there is no loss of generality in assuming $\mathcal{U}$ to be norm-closed in $\mathcal{M}_1$.

If $x \in \mathcal{U}$ is positive, there is a unique $y \in \mathcal{U}$ such that $y^2=x$ and $y \geq 0$. Now $Tx = T(y^2) = (Ty)^2$ is positive since Ty is self-adjoint.

This completes the proof.

Having now shown that $T(\mathcal{U}) \subseteq \mathcal{M}_2$ and that T is a Jordan $*$-homomorphism, we may apply the results of KADISON and STØRMER on the structure of such mappings between operator algebras. The following result thus follows from Thm.2.3.

<u>Theorem 3.7</u>

Under the assumptions of Thm.3.1, there exists a projection $p \in \mathcal{M}_2$, in the centre of $T(\mathcal{U})''$, such that the mapping

$$T_1 : x \longmapsto T(x)p$$

(respectively $T_2 : x \longmapsto T(x)p^{\perp}$)

is a $*$-homomorphism (resp. a $*$-antihomomorphism) of $\mathcal{U}$ into $\mathcal{M}_2$, and $T = T_1 + T_2$ as linear maps.

We close our study of isometries of L_p spaces, which began with the abelian case, with a result at the other end of the commutativity spectrum.

Theorem 3.8

Let $\mathcal{M}_i$ $(i=1,2)$ be two VN algebras, $\tau_i \in \mathcal{M}_{i*}$ faithful tracial states. Let $p \in (2,\infty]$ and

$$T : L_p(\mathcal{M}_1,\tau_1) \longrightarrow L_p(\mathcal{M}_2,\tau_2)$$

be a $*$-linear isometric bijection preserving the identity. If either $\mathcal{M}_1$ or $\mathcal{M}_2$ is a factor, then T is an isometric, uw-bicontinuous $*$-isomorphism or $*$-antiisomorphism of $\mathcal{M}_1$ onto $\mathcal{M}_2$.

Proof Both $T|_{\mathcal{M}_1}$ and $T^{-1}|_{\mathcal{M}_2}$ satisfy the assumptions of Thm.3.1. Therefore they are Jordan $*$-isomorphisms, inverses of each other, and hence in particular $T(\mathcal{M}_1) = \mathcal{M}_2$, $T^{-1}(\mathcal{M}_2)=\mathcal{M}_1$

That T preserves the operator norm follows from Thm.3.6(ii) T is therefore uw-bicontinuous by Lemma 2.2 applied to T and T^{-1} (or indeed by Lemma 3.2, since both T and T^{-1} map the unit ball into the unit ball).

Let $p \in \mathcal{M}_2$ be the central projection guaranteed by Thm.3.7 Then $q:= T^{-1}(p)$ is a projection in $\mathcal{M}_1$. Clearly $T_2(q)=T(q)p^{\perp}$ $=pp^{\perp} = 0$ in the notation of Thm.3.7. For $x \in \mathcal{M}_1$, we have

$$T(xq - qx) = T_1(xq - qx) + T_2(xq - qx) =$$
$$= T_1xT_1q - T_1qT_1x + T_2qT_2x - T_2xT_2q =$$
$$=(T_1x)p - p(T_1x) = 0$$

since p is central. Thus $qx - xq = 0$ (T is injective), and so q is also central. Hence if either $\mathcal{M}_1$ or $\mathcal{M}_2$ is a factor, both p and q will be either 0 or 1, so that $T = T_1$ or T_2.

QED

<u>Counterexample</u>

This is to show that Thm.3.8 fails for p=2 even in the abelian case. It will follow that the stronger assumption of positivity preservation (which follows automatically in case p>2 -see Thm.3.6(iv)) is actually essential (see Thm.2.5)

Let

$$H_n(t) = (2^n n! \sqrt{\pi})^{-1/2} (-1)^n \, e^{t^2} \frac{d^n}{dt^n} \, e^{-t^2} \qquad (t \in \mathbb{R})$$

be the Hermite polynomials: It is well known (see e.g. OKIKIOLU [47], 3.10.9) that $\{H_n : n=0,1,\ldots\}$ is an orthonormal base of $L_2(\mathbb{R}, e^{-t^2}dt)$. They are also clearly real-valued, and $H_o = \pi^{-1/4}$.

For

$$f = \sum_{n=0}^{\infty} c_n H_n \in L_2(\mathbb{R}, e^{-t^2}dt)$$

define

$$Tf = c_o + c_2H_1 + c_1H_2 + \sum_{n=3}^{\infty} c_n H_n$$

Clearly T is unitary, and $*$-linear, for if $f=f^*$, so that $c_n \in \mathbb{R}$, then $Tf = Tf^*$. T obviously preserves the identity, but is not a homomorphism; for instance,

$$(TH_1).(TH_2) = H_2H_1$$

while

$$T(H_1H_2) \neq H_2H_1$$

Chapter III

TOMITA-TAKESAKI THEORY

According to the point of view adopted in the introduction, non-commutative probability theory is concerned with the study of an involutive algebra and a state on it. More generally, non-commutative integration is concerned with an involutive algebra and a (not necessarily finite) positive real valued function on its positive part (a weight - see § 2). If this state or weight happens to be unitarily invariant, i.e. a trace, then we have the DIXMIER-SEGAL non-commutative integration theory, with which we were concerned in the previous chapters.

In both cases we may construct the GNS Hilbert space of the algebra induced by the trace or weight and represent the algebra on that Hilbert space. In order to study the most general situation, it is important to realise the special properties of the GNS representation induced by a trace. It turns out, in fact, that one can canonically associate with such a situation an object which has the properties both of a Hilbert space and of an involutive algebra, with left and right multiplications being continuous in the Hilbert space topology. This is called a Hilbert Algebra (§1). In the general case, when the weight is no longer unitarily invariant, the corresponding canonical object is a left Hilbert Algebra (§2), in which multiplication is continuous only on the left.

Tomita-Takesaki theory is concerned with generalizing the properties of Hilbert Algebras to left Hilbert Algebras. One is then able to put an arbitrary VN algebra in a <u>standard form</u>, i.e. to represent it faithfully as the VN algebra generated by the left regular representation of a left Hilbert algebra. Thus the GNS Hilbert space induced by an arbitrary weight is the analogue of the space $L^2(\mathcal{M},\tau)$ induced by a trace, which we studied in Chapter I.

We therefore begin our study with Hilbert algebras. We are interested in them not only because we later want to generalise them to left Hilbert Algebras, but also because they arise in our study of the algebra of the CCR in Chapter VI. Thus we need some of their properties which we describe in § 1.

§1. Hilbert Algebras

<u>Ex. 1.1</u> Let $\mathfrak{A}$ be a unital *-algebra, τ a state on $\mathfrak{A}$ (not necessarily faithful). We assume, in addition, that τ is <u>tracial</u> in the sense that $\tau(xy) = \tau(yx)$ for all $x,y\in\mathfrak{A}$. We let $I = \{ x\in\mathfrak{A}: \tau(x^*x) = 0 \}$. As in chapter I, §1.4, I is a left ideal. However, the centrality of τ now ensures that I is in fact a two-sided *-ideal of $\mathfrak{A}$*. Thus the

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* <u>Proof</u> $\tau((x^*)^*(x^*)) = \tau(xx^*) = \tau(x^*x)$

Thus $x^*\in I$ iff $x\in I$

Also $\tau((xy)^*(xy))^2 = \tau(y^*x^*xy)^2 = \tau(yy^*x^*x)^2 \leqslant$

$\leqslant \tau((yy^*x^*)^*(yy^*x^*))\tau(x^*x)$

Thus $x\in I \Longrightarrow xy\in I$

quotient space $\mathcal{B} = \mathfrak{A}/I$ is a *-algebra, and τ canonically defines a tracial faithful state on $\mathcal{B}$.

We may thus * restrict ourselves to the case where τ is faithful on $\mathfrak{A}$. As in Chapter I, § 1.3, We may define the GNS representation π_τ on $\mathcal{H}_\tau = \overline{\mathfrak{A}}$, with domain $\mathfrak{A}$. But note that, due again to the centrality of τ, the <u>right</u> regular (anti)-representation $\rho_\tau(x)y = yx$ of $\mathfrak{A}$ on itself is also a *-(anti)-representation.

Observe also that $\rho_\tau(x)$ and $\pi_\tau(y)$ commute for all $x,y \in \mathfrak{A}$.

<u>Ex. 1.2</u> Now let $\mathcal{M}$ be a VN algebra, τ a normal trace on $\mathcal{M}$ (not necessarily semifinite or faithful), with ideal of definition J. (see Chapter I, Thm. 2.1). Since τ is now a tracial state on J, we may apply the above construction to (J, τ), and we may assume, as above, that τ is faithful. In this case both $\pi_\tau(x)$ and $\rho_\tau(x)$ are bounded on J, ** and hence extend to bounded operators on $\mathcal{H}_\tau$. Note that J_2 (see Chapter I, Thm. 2.1) is also dense in $\mathcal{H}_\tau$, and hence $\mathcal{H}_\tau = L^2(\mathcal{N},\tau)$ in the sense of Chapter I, § 2.2, where $\mathcal{N}$ is the uw closure of J_2 (so that $L^\infty(\mathcal{N},\tau) = \mathcal{N}$).

We see that J_2, equipped with the inner product $(x,y) = \tau(x^*y)$, has all the properties of what is known as a Hilbert Algebra :

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* replacing, if necessary, $\mathfrak{A}$ by $\mathcal{B}$.

** e.g. $\|\rho_\tau(x)y\|_2^2 = \|yx\|_2^2 = \tau(x^*y^*yx) = \tau(yxx^*y^*)$
$\leq \tau(yy^*)\|xx^*\|_\infty^2 = \|y\|_2^2\|x\|_\infty^2$

Definition. A Hilbert Algebra $(\mathfrak{A}, *, (.,.))$ is a *-algebra with a (positive-definite) inner product satisfying :

(i) $(xz,y) = (z,x^*y)$

(ii) $y \longmapsto xy$ is continuous on $\mathfrak{A}$ for all $x \in \mathfrak{A}$

(iii) $\mathfrak{A}^2$ = the linear span of $\{ xy : x, y \in \mathfrak{A}\}$ is dense in $\mathfrak{A}$

(iv) $(x,y) = (\overline{x^*,y^*})$

Note that our first example also satisfies these properties, except for continuity of multiplication.

The crucial property is (iv). It ensures that $x \mapsto x^*$ extends to an antilinear isometry J on $\mathcal{H} = \overline{\mathfrak{A}}$, such that $J^2 = I$. It is more convenient to view J as a linear isometry (hence a unitary) between $\mathcal{H}$ and its opposed Hilbert space $\overline{\mathcal{H}}$, defined as follows.

Definition. The opposed Hilbert space $\overline{\mathcal{H}}$ of a Hilbert space $\mathcal{H}$ consists of the same elements as $\mathcal{H}$ as a set, denoted now by $\overline{\zeta}$ instead of ζ, but is equipped with the operations

$$a\overline{\zeta} + b\overline{\eta} = \overline{(\overline{a}\zeta + \overline{b}\eta)}$$
$$(\overline{\zeta}, \overline{\eta})_{\overline{\mathcal{H}}} = \overline{(\zeta, \eta)}_{\mathcal{H}}$$

(iv) also ensures that there is complete symmetry between left and right multiplication. Thus we may define a (bounded) *-representation π (respectively, *-anti-representation ρ) of $\mathfrak{A}$ on $\mathcal{H}$ as above.

We let $\mathcal{L}(\mathfrak{A}) = \{\pi(x) : x \in \mathfrak{A}\}^-$: the left VN algebra of $\mathfrak{A}$

and $\mathcal{R}(\mathfrak{A}) = \{\rho(x) : x \in \mathfrak{A}\}^-$: the right VN algebra of $\mathfrak{A}$

bars denoting weak closures in $\mathcal{B}(\mathcal{H})$. (iii) ensures that $\mathcal{L}(\mathfrak{A})$ and $\mathcal{R}(\mathfrak{A})$ are indeed VN algebras, i.e. contain the identity.

Then we have the following fundamental theorem:

Theorem 1.1

(Commutation Theorem for Hilbert Algebras) let $(\mathfrak{A},*,(\ ,\ ))$ be a Hilbert algebra, with $\mathcal{L}(\mathfrak{A})$, $\mathcal{R}(\mathfrak{A})$, J as above. Then $\mathcal{L}(\mathfrak{A})$ and $\mathcal{R}(\mathfrak{A})$ are commutants of each other, and

$$J\mathcal{L}(\mathfrak{A})J = \mathcal{R}(\mathfrak{A}) \tag{1.1}$$

Proof. $\mathcal{L}(\mathfrak{A})' = \mathcal{R}(\mathfrak{A})$ is proved, in a more general setting, by DIXMIER ([16], I. 5.2. Thm.1). The second relation follows from the formula

$$J\pi(x)J = \rho(x^*) \tag{1.2}$$

which is immediate from the definitions.

In Ex. 1.2, one finds (DIXMIER [16], I. 6.2 Thm.2) that the mapping $\overline{\pi_\tau}$ given by

$$\overline{\pi_\tau}(x)y = xy \qquad (x\in\mathcal{N},\ y\in J_2)$$

extends to an uw bi-continuous *-isomorphism of $\mathcal{N} = L^\infty(\mathcal{N},\tau)$ onto $\mathcal{L}(J_2)$. It is important to realize that, unless τ is semifinite, $\mathcal{M} \neq \mathcal{N}$. Although we may extend $\overline{\pi_\tau}$ to a representation of $\mathcal{M}$ onto $\mathcal{L}(J_2)$, this is no longer faithful, because $\overline{\pi_\tau}(x) = 0$ only implies that x annihilates the subspace $\overline{J_2\mathcal{H}} = \mathcal{N}\mathcal{H}$.

Ex. 1.3. As a particular case of the previous example we have the Hilbert algebra of all Hilbert-Schmidt operators on a Hilbert space $\mathcal{H}$. We may identify this with the Hilbert space tensor product $\mathcal{H} \hat{\otimes} \overline{\mathcal{H}}$,* by considering each element $\zeta \otimes \bar{\eta}$ ($\zeta \in \mathcal{H}$, $\bar{\eta} \in \overline{\mathcal{H}}$) as the rank one operator on $\mathcal{H}$ given by

$$(1.3) \qquad (\zeta \otimes \bar{\eta})(\xi) = (\eta, \xi)\zeta \qquad : \quad \xi \in \mathcal{H}$$
$$= (|\zeta\rangle\langle\eta|)|\xi\rangle$$

We observe that the Hilbert Schmidt norm $\|x\|_2 = tr(x^*x)^{\frac{1}{2}}$ (see Chapter I, §1.3) coincides, on finite rank operators, with the norm of $\mathcal{H} \hat{\otimes} \overline{\mathcal{H}}$, and that the latter operators are dense in the Hilbert Schmidt ones. In the notation of Chapter I, we may therefore write $\mathcal{H} \hat{\otimes} \overline{\mathcal{H}} = L^2(B(\mathcal{H}), tr)$.

Note that this Hilbert algebra is already complete in the Hilbert space norm, i.e. $J_2 = L^2$ in this case. Furthermore, the left VN algebra $\mathcal{L}(\mathcal{H} \hat{\otimes} \overline{\mathcal{H}})$ is isomorphic to $B(\mathcal{H})$, and hence is a factor.

Ex. 1.4. As a final example, let G be a locally compact group, with left Haar measure dx, and modular function Δ (defined as the Radon-Nikodym derivative of right Haar measure with respect to dx). Let $\mathfrak{A}(G)$ be the set of all con-

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* defined as the completion of the algebraic tensor product $\mathcal{H} \otimes \overline{\mathcal{H}}$ with respect to the unique inner product (.,.) satisfying

$$(\zeta \otimes \bar{\eta}, \zeta' \otimes \bar{\eta}') = (\zeta, \zeta')(\eta', \eta)$$

-tinuous functions of compact support. Equip $\mathfrak{A}(G)$ with the convolution product, the involution $f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}$, and the inner product induced by $L^2(G)$. One may check that $\mathfrak{A}(G)$ satisfies properties (i), (ii) and (iii) of the definition of a Hilbert algebra. It only satisfies property (iv) <u>if</u> G is unimodular, so that $\Delta(x) = 1$. In this case, $\mathfrak{A}(G)$ is a Hilbert algebra, and its completion is $L^2(G)$. Moreover, one shows (DIXMIER [15] 13.10.2) that $\mathcal{L}(\mathfrak{A}(G))$ coincides with the VN algebra generated by $\{ U(x) : x \in G \}$, where

$$(U(x)f)(y) = f(x^{-1}y) \qquad : f \in L^2(G),\ y \in G \tag{1.4}$$

is the left regular representation of G on $L^2(G)$.

The special properties of $\mathcal{L}(\mathfrak{A})$ expressed in Thm. 1.1 motivate the following

<u>Definition</u>. A VN algebra is said to be <u>standard</u> (in the sense of DIXMIER) iff it is the left VN algebra of a Hilbert algebra.

Note that the property of being standard is invariant under unitary equivalence, (for the Hilbert algebra structure is transported by the unitary) but not under *-isomorphism. Note also that a standard VN algebra satisfies

$$J\mathcal{M}J = \mathcal{M}' \tag{1.5}$$

$$JxJ = x^* \qquad \text{for all } x \in \mathcal{M} \cap \mathcal{M}' \tag{1.6}$$

J being an involution of the underlying Hilbert space. Later we shall generalize the notion of a standard VN algebra based on these two properties.

We have seen above, that if $\mathcal{M}$ is a semifinite VN algebra then it is *-isomorphic to the left VN algebra of a Hilbert algebra. We also have a converse:

<u>Theorem 1.2</u>

(i) let $\mathfrak{A}$ be a Hilbert algebra. Then the definition:

(1.7) $$\tau(x) = \|\zeta\|^2 \quad \text{if} \quad \pi(\zeta) = x^{\frac{1}{2}} \qquad : x \in \mathcal{L}(\mathfrak{A})_+ \ , \ \zeta \in \mathfrak{A}$$

gives a normal semifinite faithful trace on $\mathcal{L}(\mathfrak{A})$.
Thus $\mathcal{L}(\mathfrak{A})$ is semifinite.

(ii) let $\mathcal{M}$ be a semifinite VN algebra, τ a faithful normal trace on $\mathcal{M}$. With J_2 as in Chapter I, Thm. 2.1, $\mathcal{M}$ is *-isomorphic to $\mathcal{L}(J_2)$. Moreover, the trace defined on $\mathcal{L}(J_2)$ as in (i) (transported to $\mathcal{M}$ via this isomorphism) coincides with τ.

<u>Proof</u> DIXMIER [16] I.6.2 Theorems 1 and 2.

Therefore, Hilbert algebras allow us to study the structure of <u>semifinite</u> VN algebras. For example, one may use them to prove the famous commutation theorem for tensor products, namely that, if $\mathcal{M}_1, \mathcal{M}_2$ are semifinite VN algebras we have:

$$(\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2)' = \mathcal{M}_1' \hat{\otimes} \mathcal{M}_2'$$

where $\mathcal{M}_1 \hat{\otimes} \mathcal{M}_2$ denotes the weak closure of the algebraic tensor product $\mathcal{M}_1 \otimes \mathcal{M}_2$, in $B(\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2)$ (where $\mathcal{M}_i$ acts on $\mathcal{H}_i$)

This theorem is proved by defining a Hilbert algebra structure on the algebraic tensor product $\mathfrak{A}_1 \otimes \mathfrak{A}_2$ of two Hilbert algebras in the obvious way, and then showing, using

Thm. 1.1, that

$$\mathcal{L}(\mathfrak{A}_1)\hat{\otimes}\mathcal{L}(\mathfrak{A}_2) = \mathcal{L}(\mathfrak{A}_1\otimes\mathfrak{A}_2)$$

and therefore

$$(\mathcal{L}(\mathfrak{A}_1)\hat{\otimes}\mathcal{L}(\mathfrak{A}_2))' = \mathcal{R}(\mathfrak{A}_1\otimes\mathfrak{A}_2) = \mathcal{R}(\mathfrak{A}_1)\hat{\otimes}\mathcal{R}(\mathfrak{A}_2)$$
$$= \mathcal{L}(\mathfrak{A}_1)'\hat{\otimes}\mathcal{L}(\mathfrak{A}_2)'$$

However, one knows that semifinite VN algebras do not exhaust all VN algebras, not even the ones useful in Mathematical Physics (see the introduction, and [30]). For general VN algebras, and in particular for those arising from non-unimodular groups, problems such as the commutation theorem for tensor products remained open for a long time. They were not solved until TOMITA [84] introduced the concept of a left Hilbert algebra.

§ 2. Left Hilbert Algebras

Ex. 2.1 Let us consider our example 1.4. We have already observed that, if G is not unimodular, then property (iv) of a left Hilbert algebra is not satisfied. In fact the involution (which we now denote by $\#$) is not even continuous. However, we observe that, if we define :

$$f^{\flat}(x) = \overline{f(x^{-1})}, \text{ then}$$

$$(f^{\#},g) = \overline{(f,g^{\flat})} \quad \text{for all } f,g\in\mathfrak{A}(G)$$

Thus the operator $f\mapsto f^{\#}$, considered as a linear operator $L^2(G)\longrightarrow\overline{L^2(G)}$ (see § 1), densely defined on $\mathfrak{A}(G)$, has an adjoint $f\mapsto f^{\flat}$: $\overline{L^2(G)}\longrightarrow L^2(G)$, again densely defined on $\mathfrak{A}(G)$. Therefore it is closable. This suggests that the following concept would be a useful generalisation of the concept of a Hilbert Algebra :

Definition. A (TOMITA) Left Hilbert algebra $(\mathfrak{A}, \#, (.,.))$ is an involutive algebra with an inner product satisfying :

(i) $(xy,z) = (y,x^{\#}z)$

(ii) $y \mapsto xy$ is continuous on $\mathfrak{A}$ for all $x \in \mathfrak{A}$

(iii) $\mathfrak{A}^2$ is dense in $\mathfrak{A}$

(iv) $x \mapsto x^{\#}$ is closable, as a densely defined linear map: $\overline{\mathfrak{A}} = \mathfrak{H} \longrightarrow \overline{\mathfrak{H}}$.

Note that we no more have a symmetry between left and right multiplication, as the latter is not even continuous. As in § 1, we may define $\mathcal{L}(\mathfrak{A})$, but not $\mathcal{R}(\mathfrak{A})$. Thus we need the dual concept of a right Hilbert algebra $(\mathfrak{A}, \flat, (.,.))$ which satisfies:

(i) $(xy,z) = (x,zy^{\flat})$

(ii) $y \mapsto yx$ is continuous on $\mathfrak{A}$ for all $x \in \mathfrak{A}$

(iii) $\mathfrak{A}^2$ is dense in $\mathfrak{A}$

(iv) $x \mapsto x^{\flat}$ is closable.

For a right Hilbert algebra we define $\mathcal{R}(\mathfrak{A}) = \overline{\{\rho(x) : x \in \mathfrak{A}\}}$ where $\rho(x)y = yx$ is bounded. Observe that, in Ex. 2.1, $(\mathfrak{A}(G), \#, (.,.))$ is a left Hilbert algebra, and $(\mathfrak{A}(G), \flat, (.,.))$ is a right Hilbert algebra.

Guided by this definition, and by our example 1.1, we see that an adequate generalization of the concept of a left Hilbert algebra, in the case of absence of continuity properties, is the following :

Definition. A closable probability algebra $(\mathfrak{A}, \#, (.,.))$ is a unital involutive algebra with the properties

(i) $(xy,z) = (y,x^{\#}z)$

(iv) $x \mapsto x^{\#}$ is closable.

Note that (iii) is trivially satisfied, due to the existence of a unit. Furthermore, this allows us to define a state

$$\omega(x) = (1,x) \quad (x \in \mathfrak{A})$$

on any closable probability algebra. Conversely, we may equivalently define a probability algebra $(\mathfrak{A}, \#, \omega)$ (that is a unital involutive algebra with a faithful state ω on it) to be closable iff the involution induces a closable mapping in the GNS Hilbert space associated to ω (see GUDDER and HUDSON [22], to whom this concept is due).

We observe that the concept of a closable probability algebra is more general than that of a left Hilbert algebra, since left multiplication is no longer required to be continuous, but it is less general, in that the algebra is always assumed to contain an identity. This assumption is a necessary replacemant of property (iii) in the definition of a left Hilbert algebra, since continuity properties are no longer present.

The very interesting problem now arising is whether one can extend the results of Tomita-Takesaki theory to a closable probability algebra $\mathfrak{A}$. GUDDER and HUDSON ([22], §5) were able to construct a right Hilbert algebra $\mathfrak{A}'$ associated to $\mathfrak{A}$, assuming that the GNS representation π_ω of $\mathfrak{A}$ is essentially self-adjoint in the sense of POWERS [49].* This is already quite a strong restriction, and we shall see later that they need to impose further restrictions in order to get a commutation theorem. In Chapters V and VI, we will be studying a class of examples of closable probability algebras to which Tomita-Takesaki theory can be extended.

Ex. 2.2 We now consider, as in example 1.2, a VN algebra $\mathcal{M} \subseteq B(\mathcal{H})$ with a normal faithful state ω. We have seen in Chapter I, §1.4, that we may as well suppose that ω is a vector state, i.e. that $\mathcal{M}$ has a cyclic and separating vector ζ_0. If we now let $\mathfrak{A} = \mathcal{M}\zeta_0$, we may give $\mathfrak{A}$ the structure of a left Hilbert algebra by letting

$$(x\zeta_0)(y\zeta_0) = xy\zeta_0$$

$$(x\zeta_0)^{\#} = x^*\zeta_0$$

$$((x\zeta_0),(y\zeta_0))_{\mathfrak{A}} = (x\zeta_0,\ y\zeta_0)_{\mathcal{H}}$$

Clearly $\mathfrak{A}$ satisfies (i), (ii) and (iii). To prove (iv), let $x \in \mathcal{M}$, $y \in \mathcal{M}'$. We have :

$$((x\zeta_0)^{\#},\ y\zeta_0) = (x^*\zeta_0,\ y\zeta_0) = (y^*x^*\zeta_0,\ \zeta_0) = (x^*y^*\zeta_0,\ \zeta_0)$$
$$= (y^*\zeta_0,\ x\zeta_0)$$

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* That is, provided that $D(\pi_\omega(x)^*) \subseteq D(\overline{\pi_\omega})$ = the domain of the closure of π_ω (see Chapter I, § 1.4), for all $x \in \mathfrak{A}$.

Thus $\#$ is closable, containing the dense set $\mathfrak{A}' := \mathcal{M}'\zeta_o$ in the domain of its adjoint. In fact, one may show that $(\mathfrak{A}', \flat, (\cdot,\cdot))$ is a right Hilbert algebra with the product $(x\zeta_o)(y\zeta_o) = yx\zeta_o$ and the involution $(x\zeta_o)^\flat = x^*\zeta_o$ $(x \in \mathcal{M}')$. Moreover, $\#$ and $\flat$ have closures, denoted by S and F respectively, which are adjoints of each other. Finally, we find $\mathcal{L}(\mathfrak{A}) = \mathcal{M}$, $\mathcal{R}(\mathfrak{A}') = \mathcal{M}'$. (see TAKESAKI [81], § 2). We observe that both $\mathfrak{A}$ and $\mathfrak{A}'$ have ζ_o as a unit.

Let us now return to the general situation of a left Hilbert algebra. We denote by S and F the closure and the adjoint of $x \mapsto x^\#$, with domains D(S), D(F) $\subseteq \mathcal{H}$, respectively. Let $\mathfrak{A}'$ be the set of $y \in D(F)$ such that the mapping

$$x \longmapsto \pi(x)y \qquad (x \in \mathfrak{A})$$

(i.e. "right multiplication " by y) extends to a bounded operator on $\mathcal{H}$, denoted by $\rho(y)$. We have the

<u>Proposition 2.2</u> Equipped with the product $xy = \rho(y)x$, the involution $x^\flat = Fx$, and the scalar product of $\mathcal{H}$, $\mathfrak{A}'$ becomes a right Hilbert algebra with completion $\mathcal{H}$, and ρ is a $\flat$-anti-representation of $\mathfrak{A}'$ on $\mathcal{H}$.
(TAKESAKI [80], Lemmas 3.2, 3.3.)

If one repeats the same procedure, one arrives at a left Hilbert algebra $\mathfrak{A}''$, containing $\mathfrak{A}$, with the involution S. One says that a left Hilbert algebra is <u>full</u> (<u>achevée</u>) iff $\mathfrak{A} = \mathfrak{A}''$. Each left Hilbert algebra is contained in a minimal full left Hilbert algebra, namely $\mathfrak{A}''$. One can now show the following <u>commutation theorem</u>.

Theorem 2.3.

Let $\mathfrak{A}$ be a left Hilbert algebra, $\mathfrak{A}'$ the associated right Hilbert algebra (see above). Then we have $\mathcal{L}(\mathfrak{A})' = \mathcal{R}(\mathfrak{A}')$

(TAKESAKI [80], Thm. 3.1.)

In particular, this shows that $\mathcal{L}(\mathfrak{A}'') = \mathcal{R}(\mathfrak{A}''')' = \mathcal{R}(\mathfrak{A}')' = \mathcal{L}(\mathfrak{A})$. Thus $\mathfrak{A}$ and $\mathfrak{A}''$ generate the same VN algebra .

We observe that, although this theorem is an important one, it is not as complete as the corresponding theorem for Hilbert Algebras, since two different algebras intervene in its statement. In particular, it does not give us information on the relation between $\mathcal{L}(\mathfrak{A})$ and its own commutant , and thus does not allow us, for example, to prove the commutation theorem for tensor products.

What we need is a commutation theorem involving only $\mathfrak{A}$ (and not $\mathfrak{A}'$), with left and right multiplications by elements of $\mathfrak{A}$. But we have seen these latter are not continuous. There are two solutions of this problem : One is to look for an algebra $\tilde{\mathfrak{A}}$ inside $\mathfrak{A}$ whose right multiplications are also continuous. This was the method originally used by TAKESAKI [80], and is rather involved technically (see §3). The other method, due to RIEFFEL and VAN DAELE [55], is based on the following observation : If $\mathfrak{A}$ is a left Hilbert algebra, we let K be the real closed subspace of $\mathcal{H}$ generated by $\{ x^{\#}x : x \in \mathfrak{A} \}$. Then the condition that $x \longmapsto x^{\#}$ be closable turns out to be equivalent to $K \cap iK = \{0\}$. This allows the development of the theory without any reference to unbounded operators such as S and F, or the algebra $\tilde{\mathfrak{A}}$. I shall describe the original method in more detail, since the algebra $\tilde{\mathfrak{A}}$ happens to arise in a very natural manner in our examples (cf.Chapter V). I would like to observe, however, that the method of RIEFFEL and VAN DAELE seems to be the most appropriate one to generalize to the case of closable probability algebras. I hope that this problem will be investigated further.

Returning, for a minute, to the subject of closable probability algebras, we have seen that GUDDER and HUDSON [22] were able to construct under certain conditions, a right Hilbert Algebra $\mathfrak{A}'$. But they have no way of asserting that this algebra has any elements at all. However, if they assume that $\mathfrak{A}'$ is also dense in $\mathcal{H}$, then they are able to prove the analogue of Thm. 2.3, namely, that $\mathcal{R}(\mathfrak{A}') =$ $= \pi_\omega(\mathfrak{A})'$ (cf. Chapter I §1.4 for the definition of the commutant of an unbounded representation).

I would like to close this section with a result analogous to Thm. 1.2 and its converse. First we need a definition:

Definition. Let $\mathcal{M}$ be a VN algebra. A weight ϕ on $\mathcal{M}$ is a map

$$\phi : \mathcal{M}_+ \longrightarrow [\, 0, +\infty\,]$$

which is additive, i.e.

$\phi(x + y) = \phi(x) + \phi(y)$ for all $x, y \in \mathcal{M}_+$

and positive homogenuous i.e.

$$\phi(\lambda x) = \lambda\phi(x) \qquad \text{for all } x \in \mathcal{M}_+,\ \lambda \geq 0$$

(where $0(+\infty) = +\infty$)

Let $\mathcal{N}_\phi = \{ x \in \mathcal{M} : \phi(x^*x) < \infty \}$, $\mathcal{M}_\phi = \mathcal{N}_\phi^* \mathcal{N}_\phi$

Then $\mathcal{N}_\phi$ is a left ideal and $\mathcal{M}_\phi$ a $*$-subalgebra of $\mathcal{M}$, whose positive part is the set $\{ x \in \mathcal{M}_+,\ \phi(x) < +\infty \}$. ϕ extends to a plf on $\mathcal{M}_\phi$.

A weight ϕ is said to be

<u>faithful</u> iff $\phi(x^*x) = 0$ implies $x=0$

<u>semifinite</u> iff $\mathcal{M}_\phi$ (equivalently $\mathcal{N}_\phi$) is uw dense in $\mathcal{M}$

<u>normal</u> iff there exists a set $\{ \omega_s \}$ of normal plf's on $\mathcal{M}$ such that $\phi(x) = \sup \omega_s(x)$ for all $x \in \mathcal{M}_+$.

We see that the notion of a weight is a generalization of the notion of a trace, where unitary invariance is no longer assumed. (cf. Chapter I, §2.1)

We then have the following results, due to COMBES [11]:

<u>Theorem 2.4</u>

(i) Let $\mathfrak{A}$ be a full left Hilbert algebra. For $x \in \mathcal{L}(\mathfrak{A})_+$ define

$$\phi(x) = \begin{cases} \|\xi\|^2 & \text{if } x = \pi(\xi)^*\pi(\xi) \text{ for some } \xi \in \mathfrak{A} \\ +\infty & \text{otherwise} \end{cases}$$

then ϕ is a normal semifinite faithful weight on $\mathcal{L}(\mathfrak{A})$.

(ii) Let $\mathcal{M}$ be a VN algebra, ϕ a normal semifinite faithful weight on $\mathcal{M}$. We equip $\mathcal{N}_\phi$ with the scalar product $(x,y) = \phi(x^*y)$. Since $\mathcal{N}_\phi$ is a left ideal of $\mathcal{M}$, the GNS representation (Chapter I, § 1.4) of $\mathcal{M}_\phi$ induced by ϕ can be extended to a representation π_ϕ of $\mathcal{M}$, given by $\pi_\phi(x)y = xy$ $(x\in\mathcal{M},\ y\in\mathcal{N}_\phi)$. Equipping $\mathfrak{A} = \mathcal{N}_\phi \cap \mathcal{N}_\phi^*$ with the product and involution inherited from $\mathcal{M}$ and the above scalar product, $\mathfrak{A}$ becomes a full Hilbert algebra such that $\mathcal{L}(\mathfrak{A}) = \pi_\phi(\mathcal{M})$. Moreover the weight defined on $\mathcal{L}(\mathfrak{A})$ as in (i) (transported to $\mathcal{M}$ via π_ϕ) coincides with ϕ.

Just as Thm. 1.2 and its converse ensured that $\mathcal{L}(\mathfrak{A})$, for $\mathfrak{A}$ a Hilbert algebra, exhausted, up to *-isomorphism, all semifinite VN algebras, Thm. 2.4 shows that $\mathcal{L}(\mathfrak{A})$, with $\mathfrak{A}$ a left Hilbert algebra, exhausts all (arbitrary) VN algebras (up to *-isomorphism). This is because any VN algebra can be equipped with a faithful normal semifinite weight. This also follows from the following result:

Theorem 2.5

An arbitrary VN algebra $\mathcal{M}$ is isomorphic, as a VN algebra, to the left VN algebra of a full left Hilbert algebra. (TAKESAKI [80], Thm. 12.2)

§3. Modular Hilbert Algebras.

We have seen in the previous section that Thm. 2.3 is not an adequate commutation theorem; what is needed is a theorem involving left and right multiplications by elements of $\mathfrak{A}$ itself. Hence we are looking for a subset of $\mathfrak{A}$ for which right multiplications are also continuous.

For example 2.1., $\mathfrak{A}(G)$ itself will do, since right multiplications are in fact continuous. This is because $\mathfrak{A}(G)$ is both a right and a left Hilbert algebra, a fact essentially due to the existence of the modular function $x \longmapsto \Delta(x)$. But this is just an accident, and is certainly not the case for example 2.2.

In the general case, we have constructed two linear maps $S : \mathcal{H} \longrightarrow \overline{\mathcal{H}}$, $F : \overline{\mathcal{H}} \longrightarrow \mathcal{H}$, adjoints of each other. Writing $\Delta = FS$, one easily sees that Δ is a positive self adjoint non-singular operator $\mathcal{H} \longrightarrow \mathcal{H}$. Moreover, the polar decomposition $S = J|FS|^{\frac{1}{2}} = J\Delta^{\frac{1}{2}}$ furnishes us with a partial isometry $J : \mathcal{H} \longrightarrow \overline{\mathcal{H}}$ which one easily proves to be an isometry onto, such that $J^2 = 1$. The operator Δ , called the modular operator plays a fundamental role in the theory. One could say that Δ arises from the difference between (x,x) and $(x^{\#},x^{\#})$ (which is zero in the case of a Hilbert algebra, where $S = J$ and $\Delta = 1$.) In fact, we have

$$(x^{\#},x^{\#}) = (SX,Sx) = (FSx,x) = (\Delta x,x)$$

Let us illustrate this with the case of example 2.1. We see that $(\Delta f)(x) = (FSf)(x) = (Fg)(x) = \overline{g(x^{-1})}$ where $g(x) = (Sf)(x) = \Delta(x^{-1})\overline{f(x^{-1})}$.

$$\therefore \quad (\Delta f)(x) = \Delta(x)f(x)$$

Moreover, one checks that elements of $\mathfrak{A}(G)$ form a dense set of analytic vectors for Δ, and hence we may define, for each $z \in C$ and $f \in \mathfrak{A}(G)$

$$\Delta(z)f = \Delta^{z} f \in \mathfrak{A}(G)$$

Thus we have defined a group, called the modular automorphism group, of automorphisms of $\mathfrak{A}(G)$ which has the following properties :

(v) $(\Delta(z)f)^{\#} = \Delta(-\bar{z})f^{\#}$ $z\in\mathbb{C}$, $f\in\mathfrak{A}(G)$

(vi) $(\Delta(z)f,g) = (f,\Delta(\bar{z})g)$ $z\in\mathbb{C}$, $f,g\in\mathfrak{A}(G)$

(vii) $(\Delta(1)f,g) = (g^{\#},f^{\#})$

(viii) $z \longmapsto (f,\Delta(z)g)$ is entire for all $f,g\in\mathfrak{A}(G)$

(ix) $\mathfrak{A}_t = \{ (1 + \Delta(t))f : f\in\mathfrak{A}(G)\}$ is dense in $\mathfrak{A}(G)$ for all $t\in\mathbb{R}$

These observations help to motivate the following definition:

Definition. A Modular Hilbert Algebra is a left Hilbert Algebra $(\mathfrak{A}, \#, (.,.))$ equipped with a one parameter complex automorphism group $\{ \Delta(z) : z\in C \}$ satisfying properties (v) - (ix) above.

We note that property (iv) of the definition of a left Hilbert algebra is now redundant, because it follows from (vii) that $f\mapsto f^{\#}$ has an adjoint $f\mapsto\Delta(1)f^{\#}$, with dense domain $\mathfrak{A}$. One can show that, defining Δ = FS as above, $\overline{\Delta(z)} = \Delta^z$ $(z\in\mathbb{C})$ (the bar denoting closure). The polar decomposition of S (see above) gives a unitary involution $J : \mathcal{H}\to\mathcal{H}$, which moreover, due to property (vii), leaves $\mathfrak{A}$ invariant. This ensures, as in the case of Hilbert algebras, that right multiplication is also continuous. Thus the right Hilbert algebra $\mathfrak{A}'$ defined in §2. now contains $\mathfrak{A}$. One is therefore able to prove, without much difficulty, the following.

Theorem 3.1.

Commutation theorem for modular Hilbert Algebras.

Let $(\mathfrak{A}, \#, (.,.), \{\Delta(z)\})$ be a modular Hilbert algebra.

We have :

$$\mathcal{L}(\mathfrak{A})' = \mathcal{R}(\mathfrak{A})$$

$$J\,\mathcal{L}(\mathfrak{A})J = \mathcal{R}(\mathfrak{A})$$

(together with their dual relations of course)
(TAKESAKI [80], Thm. 4.1)

Nowadays modular Hilbert Algebras are a bit out of fashion. This is due to the fact that they are generally accepted to be unnecessary for the development of the theory of left Hilbert algebras, as simpler methods have been discovered. (see [55] and [86]). It does not seem possible to apply these newer methods in a natural way, to this particular problem, as they rely heavily on the fact that the representations π and ρ are bounded. It must be emphasized, however, that for the general case of a closable probability algebra (cf. § 2) these new methods, especially those of [55], seem to be precisely the ones one should try to generalize.

The real importance of modular Hilbert algebras lies in the fact that every left Hilbert algebra has one of them conveniently placed inside it. This is the most difficult part of TOMITA's work (see [80], § 5 - 10). We formulate this as a theorem.

Theorem 3.2. (The Fundamental Theorem of TOMITA)

For every left Hilbert algebra $\mathfrak{A}$, there exists a dense involutive subalgebra $\mathfrak{B} \subseteq \mathfrak{A}$, and a modular automorphism group $\{ \Delta(z) : z \in C \}$ of $\mathfrak{B}$, making $\mathfrak{B}$ into a modular Hilbert algebra such that $\mathfrak{B}'' = \mathfrak{A}''$ (see § 2)

In particular, therefore, $\mathcal{L}(\mathcal{B}) = \mathcal{L}(\mathfrak{A})$ (TAKESAKI [80], Thm. 10.1)

Combining this with Thm. 3.1, we therefore get :

Theorem 3.3

For each left Hilbert algebra $\mathfrak{A}$, there exists an anti-unitary involution J of $\mathcal{H} = \overline{\mathfrak{A}}$, such that

$$J\mathcal{L}(\mathfrak{A})J = \mathcal{L}(\mathfrak{A})'$$

and $\quad J x J = x^* \quad$ for all $x \in \mathcal{L}(\mathfrak{A}) \cap \mathcal{L}(\mathfrak{A})'$

We now define :

Definition A VN algebra $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ is said to be standard in the sense of Takesaki [81] iff there exists an antiunitary involution J of $\mathcal{H}$ such that

$$J\mathcal{M}J = \mathcal{M}'$$

and $\quad J x J = x^* \quad$ for all $x \in \mathcal{M} \cap \mathcal{M}'$.

(Compare the definition of standard in the sense of Dixmier in §1)

Since every VN algebra is isomorphic to an $\mathcal{L}(\mathfrak{A})$ (§2, Thm. 2.5), Thm. 3.3 now gives us the :

Theorem 3.4

Every VN algebra $\mathcal{M}$ has a faithful standard representation in the sense of Takesaki (that is, a faithful representation π such that $\pi(\mathcal{M})$ is standard in the above sense).

We sketch the construction of the algebra $\mathcal{B}$ in Thm. 3.2: We have seen how to construct the modular operator Δ, which is a positive self-adjoint non-singular operator. Thus Δ has a logarithm. One now defines $\mathfrak{B}$ to be the *-sub-algebra of $\mathfrak{A}$ generated by elements of the form $f(\log \Delta)x$, with $x \in \mathfrak{A}$ and $f \in ((0,+\infty)) = \infty$- differentiable functions of compact support contained in $(0,+\infty)$. It is a highly non-trivial problem, of course, to verify that $\mathfrak{B}$ has the required properties.

§4. The modular automorphisms and the KMS condition.

§4.1. The importance of Tomita-Takesaki theory came from the realization, due to Takesaki [80], that the modular automorphism group defined for a modular Hilbert algebra satisfied the KMS-condition of Quantum Statistical Mechanics, first formulated in the algebraic framework by HAAG, HUGENHOLTZ and WINNINK [26]. This has had tremendous implications for the development of both Quantum Statistical Mechanics (see Chapter IV) and Tomita-Takesaki theory itself. We formulate this observation in the form of two propositions:

Proposition 4.1. Let $\mathfrak{A}$ be a left Hilbert algebra. As in Thm. 3.2 construct the modular automorphism group $\{\Delta(\alpha): \in C\}$. Then $\{ \Delta(it) : t \in R \}$ (see §3) extends to a unitary group on $\mathcal{H} = \overline{\mathfrak{A}}$, which leaves $\mathfrak{A}$ invariant, and acts as an automorphism group of $\mathfrak{A}$. Moreover, for all $\zeta \in \mathfrak{A}$

$$\pi(\Delta^{it}\zeta) = \Delta^{it}\pi(\zeta)\Delta^{-it} \qquad (t \in \mathbb{R}) \tag{4.1}$$

Thus $\{ \Delta^{it} : t \in \mathbb{R} \}$ induces a one-parameter automorphism group $\{\sigma^t\}$ of the left VN algebra $\mathcal{L}(\mathfrak{A})$.
(TAKESAKI [80], Cor.9.1)

Theorem 4.2

Let $\mathcal{M}$ be a VN algebra, ϕ a faithful normal semifinite weight on $\mathcal{M}$ (see §2). With the notations of Thm. 2.4 (ii), the modular automorphism group $\{ \Delta(it) \; ; \; t\in R \}$ of $\mathfrak{A}$ induces a strongly continuous one-parameter automorphism group σ_t of $\mathcal{M}$ such that

(4.2) $$\pi_\phi(\sigma_t(x)) = \Delta^{it}\pi_\phi(x)\Delta^{-it} \qquad (t\in R, \; x\in\mathcal{M})$$

The weight ϕ is a KMS weight with respect to σ_t, in the following sense :

(i) $\sigma_t(x)\in\mathfrak{M}_\phi$ iff $x\in\mathfrak{M}_\phi$ and

(4.3) $$\phi(\sigma_t(x)) = \phi(x)$$

(ii) for all $x,y\in\mathfrak{A} = \mathfrak{N}_\phi\cap\mathfrak{N}_\phi^*$, there exists a function F, defined, bounded and continuous on the strip $\{ z\in C : 0 \leq \operatorname{Im} z \leq 1 \}$ and analytic in the interior, such that

(4.4) $$F(t) = \phi(\sigma_t(x)y), \quad F(t+i) = \phi(y\sigma_t(x)) \qquad t\in R$$

Moreover, σ_t is the only strongly continuous one-parameter automorphism group of $\mathcal{M}$ satisfying (4.3) and (4.4)

Finally, ϕ is a trace (i.e. is unitarily invariant) iff $\sigma_t(x) = x$ for all $x\in\mathcal{M}_+$ (which implies that $\Delta = 1$, $F = S = J$ and $\mathfrak{A}$ is a Hilbert algebra).
(COMBES [11], §4)

It is interesting to observe that when one is given the modular automorphism group σ_t of $\mathcal{M}$ corresponding to a normal faithful semifinite weight ϕ, then it is easy to construct a maximal modular Hilbert algebra $\tilde{\mathfrak{A}}$ inside the left Hilbert algebra $\mathfrak{A} = \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$ (see Thm. 2.4(ii)). One considers the algebra $\mathcal{M}_0$ of analytic elements of $\mathcal{M}$, namely those $x \in \mathcal{M}$ for which $t \mapsto \sigma_t(x)$ extends to a (necessarily unique) entire $\mathcal{M}$-valued function $z \mapsto \sigma_z(x)$.

For each $x \in \mathcal{M}$ and $s>0$, define x_s by the Bochner integral :

$$x_s = (s/\pi)^{\frac{1}{2}} \int_{\mathbb{R}} (\exp(-st^2))\sigma_t(x)dt \tag{4.5}$$

Then $x \in \mathcal{M}_0$, and in fact

$$\sigma_z(x_s) = (s/\pi)^{\frac{1}{2}} \int_{\mathbb{R}} (\exp(-s(t-z)^2))\sigma_t(x)dt$$

But as $s \to \infty$, $x_s \to x$ ultraweakly. Thus $\mathcal{M}_0$ is uw dense in $\mathcal{M}$.

If one now lets $\tilde{\mathfrak{A}} := \mathfrak{A} \cap \mathcal{M}_0$ one sees that $\tilde{\mathfrak{A}}$, equipped with the left Hilbert algebra structure of $\mathfrak{A}$ (see Thm. 2.4(ii)) and the automorphism group $\{ \sigma_z : z \in \mathbb{C} \}$ is a modular Hilbert algebra, containing any other modular Hilbert algebra $\mathfrak{B} \subseteq \mathfrak{A}$, and $\mathcal{L}(\tilde{\mathfrak{A}}) = \overline{\mathcal{M}_0}^{uw} = \mathcal{M}$.
(PEDERSEN and TAKESAKI [48], §3)

§4.2 To further justify the claim that Tomita-Takesaki theory is a Non-Commutative Integration theory, I shall give a brief account of the Radon-Nikodym theorem for arbitrary VN algebras.

We have seen (Chapter I, Thm. 2.3) that for a semifinite VN algebra $\mathcal{M}$ equipped with a (faithful, normal, semifinite) trace τ, each $\omega \in \mathcal{M}_*$ has a "Radon-Nikodym derivative" with respect to τ; namely, we constructed a unique $h \in L_1(\mathcal{M},\tau)$

such that

(4.6) $$\omega(x) = \tau(hx) \qquad \text{for all} \quad x\in\mathcal{M}.$$

In cases where unitary invariance is absent, one cannot hope for a similar theorem to hold for any two weights on $\mathcal{M}$. However, one does have a positive result, if the weights in question are required to commute, in a sense to be made precise below.

First we define the <u>centralizer</u> $\mathcal{M}^{\sigma}$ of a faithful normal semifinite weight ϕ with modular automorphism group $\{\sigma_t\}$ to be the set of fixed points of σ, i.e.

(4.7) $$\mathcal{M}^{\sigma} = \{ h\in\mathcal{M} : \sigma_t(h) = h, \quad t\in\mathbb{R} \}$$

This definition is justified by the

<u>Lemma 4.3</u> $h\in\mathcal{M}^{\sigma}$ iff $h\mathfrak{M}_\phi \subseteq \mathfrak{M}_\phi$ $\mathfrak{M}_\phi h \subseteq \mathfrak{M}_\phi$ and $\phi(xh) = \phi(hx)$ for each $x\in\mathfrak{M}_\phi$ (see §2).
(PEDERSEN and TAKESAKI [48] Thm. 3.6)

Now we say that a normal semifinite weight ψ <u>commutes</u> with ϕ in case ψ is σ-invariant in the sense of eqn.(4.3).

This definition is justified by observing that, in case $\mathcal{M}$ is semifinite with a (faithful normal semifinite) trace τ on $\mathcal{M}$, and $\phi,\psi\in\mathcal{M}_*$ are states, we may write :

$$\phi(x) = \tau(hx), \qquad \psi(x) = \tau(kx) \qquad (x\in\mathcal{M})$$

where $h,k\in L_1(\mathcal{M},\tau)$ are the Radon-Nikodym derivatives of ϕ (respectively ψ) with respect to τ. The modular automorphism groups are given by

$$\sigma_t^{\phi}(x) = h^{it}xh^{-it} \quad , \quad \sigma_t^{\psi}(x) = k^{it}xk^{-it} \qquad (x\in\mathcal{M},\ t\in\mathbb{R})$$

since σ^{ϕ}(respectively σ^{ψ}) satisfies the KMS condition with respect to ϕ (respectively ψ) (Thm. 4.2). We see that ϕ commutes with ψ in the above sense precisely when (the spectral projections of) h commute with (those of) k.

Theorem 4.4 (Radon-Nikodym)

Let $k\eta\mathcal{M}^{\sigma}$ (see Chapter I, § 2.3.2) be a positive self-adjoint operator. Define

(4.8) $$\phi_K(x) := \phi(k^{\frac{1}{2}}x\ k^{\frac{1}{2}}) \qquad (x\in\mathcal{M}_+)$$

Then ϕ_K is a normal semifinite weight on $\mathcal{M}$, commuting with ϕ.

Conversely, if ψ is a normal semifinite weight on $\mathcal{M}$ commuting with ϕ, then there exists a unique positive self-adjoint operator $k\eta\mathcal{M}^{\sigma}$ (the Radon Nikodym derivative of ψ with respect to ϕ) such that $\psi = \phi_K$. Moreover, if ψ is faithful, the modular automorphism group σ^{ψ} of ψ is given by

$$\sigma_t^{\psi}(x) = k^{it}\sigma_t(x)k^{-it} \qquad (x\in\mathcal{M},\ t\in\mathbb{R})$$

(PEDERSEN and TAKESAKI [48], Theorems 4.6 and 5.12)

To justify the need for the extra assumption that ψ should commute with ϕ, consider the case where ϕ is a (normal, faithful, semifinite) trace on $\mathcal{M}$ (which is always true if $\mathcal{M}$ is abelian). Then $\mathcal{M}^{\sigma} = \mathcal{M}$ since σ^{ϕ} is trivial (Thm. 4.2). Thus any (normal semifinite) weight ψ on $\mathcal{M}$

commutes with ϕ. In particular, if $\psi \in \mathcal{M}_*$ is a plf, we recover Thm.I.2.3, since $\phi(k) = \psi(1) < \infty$ (by (4.8) with $x=1$) implies that $k \in L_1(\mathcal{M}, \phi)$.

Just as we could characterize, in Thm. 4.2, all (strongly continuous) automorphism groups with respect to which a given (normal, faithful, semifinite) weight ϕ satisfies the KMS condition (there is exactly one, namely σ^{ϕ} !) so we can characterize all normal semifinite weights that satisfy the KMS condition with respect to a given (strongly continuous) automorphism group of $\mathcal{M}$. There may exist none! (PEDERSEN and TAKESAKI [48] Cor. 7.5). If, however, there exists one, then we know all the others :

Proposition 4.5 Let ϕ be a faithful normal semifinite weight on $\mathcal{M}$, $\{\sigma_t\}$ the corresponding modular automorphism group. If ψ is a normal semifinite weight on $\mathcal{M}$ satisfying the KMS condition ((4.3) and (4.4)) with respect to $\{\sigma_t\}$, then $\psi = \phi_K$ with a unique $k \eta \mathcal{M} \cap \mathcal{M}'$. Conversely, if $k \eta \mathcal{M} \cap \mathcal{M}'$ is positive and self-adjoint, then ϕ_K satisfies the KMS condition with respect to $\{\sigma_t\}$.
(PEDERSEN and TAKESAKI [48], Cor. 4.7 and Thm. 5.4)

Suppose that $\mathcal{M}$ is semifinite, and ϕ is a (faithful normal semifinite) trace on $\mathcal{M}$. Each normal semifinite weight ψ on $\mathcal{M}$ is then a ϕ_K, and $\sigma_t^{\psi}(x) = k^{it} x k^{-it}$ by Thm. 4.4 since σ_t^{ϕ} is trivial. Since $k \eta \mathcal{M}^{\sigma} = \mathcal{M}$, $k^{it} \in \mathcal{M}$, and is unitary, so that σ_t^{ψ} is inner. Conversely, if the modular automorphism group σ_t^ψ of a faithful normal semifinite weight ψ on $\mathcal{M}$ is inner, say $\sigma_t^\psi(x) = k^{it} x k^{-it}$, then k is positive and self-adjoint and non-singular by Stone's theorem, and $k^{it} \in \mathcal{M}$ implies that $k \eta \mathcal{M}$. By Thm. 4.4, we now see that $\phi = \psi_{k^{-1}}$, is a normal faithful (since k^{-1} is non-singular) semifinite weight on $\mathcal{M}$. Moreover, we have :

$$\sigma_t^\phi(x) = (k^{-1})^{it} \sigma_t^\psi(x) (k^{-1})^{-it} = x$$

for all $x \in \mathcal{M}$, so that, by Thm. 4.2, ψ is a trace, and hence $\mathcal{M}$ is semifinite. Thus we have shown the

Theorem 4.6

A VN algebra $\mathcal{M}$ is semifinite iff the modular automorphism group of any normal semifinite faithful weight on $\mathcal{M}$ is inner.

This theorem, with "state" replacing "weight" (thus applicable only to VN algebras that possess faithful normal states) is due to TAKESAKI ([80], Thm. 14.2) where the proof takes about ten pages. The above proof is due to PEDERSEN and TAKESAKI ([48], Thm. 7.4).

§ 4.3 For applications to Mathematical Physics, (see Chapter IV) one often needs to consider a state ω of a C*-algebra $\mathfrak{A}$ satisfying the KMS condition with respect to a given automorphism group $\{\sigma_t\}$ of $\mathfrak{A}$. One then has :

Theorem 4.7

In the situation above, the left kernel $\{ x \in \mathfrak{A} : \omega(x^*x) = 0 \}$ of ω is a two-sided *-ideal of $\mathfrak{A}$, and coincides with the kernel of the GNS representation π_ω *. If ζ_ω is the corresponding cyclic vector, then it is cyclic and separating for the VN algebra $\mathcal{M} = \pi_\omega(\mathfrak{A})''$.

Moreover, σ_t is the modular automorphism group corresponding to ω, in the sense that

$$\pi_\omega(\sigma_t(x)) = \Delta^{it}\pi_\omega(x)\Delta^{-it} \qquad (x\in\mathfrak{A},\ t\in\mathbb{R}).$$

This completes our discussion of the most basic features of TOMITA-TAKESAKI theory. I hope that I have indicated the directions in which one might attempt to generalize this theory, and justified the point of view put forward in the Introduction, namely that TOMITA-TAKESAKI theory should be regarded as a form of Non-Commutative Integration Theory. Clearly, however, TOMITA-TAKESAKI theory is much more than this, as its applications and developments, which I have not had the opportunity to go into, have shown (see CONNES [12], and references quoted there).

This chapter also completes the first part of this thesis, which was concerned with a discussion of Non-Commutative Integration Theory. In the second part, I shall first describe some applications of Tomita-Takesaki theory in Quantum Statistical Mechanics, and justify its relevance, and then analyse a class of examples, motivated from Quantum Statistical Mechanics, where the theory can be generalized.

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* This is also the case when ω is tracial – see §1.

P A R T B

Chapter IV

EQUILIBRIUM STATES AND TIME TRANSLATIONS IN QUANTUM STATISTICAL MECHANICS

The purpose of this chapter is to provide an introduction and physical motivation to the material of chapters V and VI, by describing the algebraic approach to Equilibrium States and Time Translations in Quantum Statistical Mechanics, as developed initially by HAAG, HUGENHOLTZ and WINNINK [26] (henceforth HHW) and subsequently by other authors.

§ 1. The state of a physical system is totally specified by the expectation values of all observables in that state; when the observables are taken to form the self-adjoint part of a complex involutive algebra $\mathfrak{A}$, there corresponds to each (physical) state of the system a (mathematical) state ω of the algebra (i.e. a positive normalised linear form on $\mathfrak{A}$) via the interpretation that the expectation of an observable $x \in \mathfrak{A}$ in that state is given by $\omega(x)$.

In Quantum Statistical Mechanics of finite systems (say a gas enclosed in an isolated container), the observables are (bounded) operators on a Hilbert space $\mathcal{H}$, and the equilibrium state ω_β of a system at a temperature $T = 1/k\beta$ ($\beta>0$, k=Boltzman's constant) in the canonical ensemble is defined by the formula

$$\omega_\beta(x) = tr(\exp(-\beta H)x) \;/\; tr(\exp(-\beta H)) \tag{1.1}$$

where H is the Hamiltonian of the system. This corresponds to assigning to each eigenstate $\zeta_i \in \mathcal{H}$ of the Hamiltonian a probability $c_i = \exp(-\beta E_i) / \Sigma \exp(-\beta E_i)$ (where E_i is the energy of ζ_i, $H\zeta_i = E_i \zeta_i$), so that the equilibrium state is the mixture $\rho = \Sigma c_i \zeta_i$.* In the grand-canonical ensemble, the equilibrium state is given by (1.1), where H is now replaced by $H' = H - \mu N$, μ being the chemical potential and N the number operator.

The aim of statistical mechanics is to derive the macroscopic or thermodynamic properties of large systems from the equations of motion of the individual particles. The thermodynamic functions, such as specific heat, are never significantly dependent on the volume or shape of the object being measured in any experimental situation. Now in finite models, these thermodynamic functions usually do depend on the volume or shape of the object; this dependence only disappears when the volume and number of particles is allowed to increase to infinity, while their ratio (the density) remains finite. This process is called taking the thermodynamic limit, and the observed independence of volume or shape of the thermodynamic functions provides justification for the claim that the experimentalist is faced with a situation

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* One may ensure that the Hamiltonian has discrete spectrum and that $\exp(-\beta H)$ is trace class, by choosing suitable boundary conditions on the walls of the container.

actually close to that described by the thermodynamic limit.

Now a good model for a thermodynamic system should account for the characteristic properties of such a system, such as phase transitions or transport phenomena. Typically, phase transitions manifest themselves experimentally by abrupt changes in certain thermodynamic functions (such as specific heat). To exhibit this most dramtically in a mathematical model, and simultaneously to be able to characterise it in a sharp, well defined manner, one defines a phase transition to be a discontinuity in the relevant thermodynamic function. Now in finite models actual discontinuities never occur; at best very steep gradients ("smooth phase transitions") may be present. Of course the experimentalist cannot distinguish between a steep gradient and an actual discontuity (due to the limited accuracy of his measuring apparatus), but the definition of a phase transition as a discontinuity is certainly consistent with the experimental situation, and is also dictated by mathematical convenience.

For these reasons, among others, it is necessary to consider states of infinite systems. The first problem now presenting itself is that in most cases (1.1) does not make sense any more. The usual way around this is to consider the infinite system as the limit, in an appropriate sense, of finite subsystems; the equilibrium state is then defined to be the limit of the "local Gibbs states" given by (1.1) (provided this limit exists).

Specifically, let Γ * be the physical space of the system, and let L be the set of all bounded regions $\Lambda \subseteq \Gamma$. For each $\Lambda \in L$, one defines $\mathfrak{A}(\Lambda)$ to be the C*-algebra generated by the bounded observables for the region Λ. One assumes that the property of isotony holds :
For $\Lambda_1 \subseteq \Lambda_2 \in L$, there exists an injective *-homomorphism $i_{21} : \mathfrak{A}(\Lambda_1) \to \mathfrak{A}(\Lambda_2)$, preserving the identity, and such that $i_{32} i_{21} = i_{31}$ whenever $\Lambda_1 \subseteq \Lambda_2 \subseteq \Lambda_3 \in L$. (This is the mathematical expression of the fact that the observables for the region Λ_1 may be considered as observables for the region Λ_2).

It is then shown (SAKAI [62], 1.23.2), that there exists a C*-algebra $\mathfrak{A}$, the C*-inductive limit of $\{ \mathfrak{A}(\Lambda) : \Lambda \in L \}$, such that we may identify the $\mathfrak{A}(\Lambda)$'s, up to *- isomorphism, with nested ** subalgebras of $\mathfrak{A}$, and $\mathfrak{A}_L = \cup \{ \mathfrak{A}(\Lambda) : \Lambda \in L \}$ is (norm) dense in $\mathfrak{A}$. Physically, self-adjoint elements of $\mathfrak{A}_L$ correspond to observables that can be measured by experiments in some bounded region $\Lambda \in L$ (local observables) while self-adjoint elements of $\mathfrak{A}$ can be "approximately" so measured (quasi-local observables).

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* Γ will be $\mathbb{R}^d$ with its usual topology for continuous systems, or $\mathbb{Z}^d$ with the discrete topology for lattice systems, d being the dimension of the physical space.

** i.e. $\Lambda_1 \subseteq \Lambda_2 \in L$ implies $\mathfrak{A}(\Lambda_1) \subseteq \mathfrak{A}(\Lambda_2)$

We now turn to the definition of the equilibrium state and dynamics of the infinite system. For each region $\Lambda \epsilon L$, we assume that the "local time development" is well defined as an automorphism group $\{ \alpha_t^\Lambda : t \epsilon R \}$ of $\mathfrak{A}(\Lambda)$. Suppose that each $\mathfrak{A}(\Lambda)$ is faithfully represented on a Hilbert space $\mathfrak{H}(\Lambda)$, on which the local time development α_t^Λ is unitarily implemented. Letting H_Λ be the corresponding Hamiltonian, acting on $\mathfrak{H}(\Lambda)$, we thus have:

(1.2) $$\alpha_t^\Lambda(x) = \exp(itH_\Lambda)x \exp(-itH_\Lambda) \qquad (x \epsilon \mathfrak{A}(\Lambda),\ t \epsilon R)$$

Again one may ensure that $\exp(-\beta H_\Lambda)$ is trace class for $\beta > 0$, so that one may define the "<u>local Gibbs state</u>" ω_β^Λ on $\mathfrak{A}(\Lambda)$ by (1.1).

By the isotony condition, it is clear that if $x \epsilon \mathfrak{A}_L$, for example $x \epsilon \mathfrak{A}(\Lambda_0)$ for some $\Lambda_0 \epsilon L$, $\omega_\beta^\Lambda(x)$ is defined for all $\Lambda \epsilon L$ such that $\Lambda \supseteq \Lambda_0$. Physically, $\omega_\beta^\Lambda(x)$ yields information on the behaviour or particles in the subregion Λ_0 of Λ. Thus we expect $\omega_\beta^\Lambda(x)$ to become independent of Λ as the boundary of Λ recedes to infinity. Indeed, in many models (see e.g. [57]) it can be shown that

(1.3) $$\omega_\beta(x) := \lim_\Lambda \omega_\beta^\Lambda(x)$$

exists for all $x \epsilon \mathfrak{A}_L$ for a suitable interpretation of the limiting procedure (see RUELLE [60]). The existence of this limit is in fact the first assumption of HHW [26]. ω_β then defines a continuous positive linear form on $\mathfrak{A}_L$ and thus extends to a state on $\mathfrak{A}$, the <u>limit (or global)</u>

gibbs state.

Turning now to the definition of "global dynamics", we first observe that, for all $x \in \mathfrak{A}_L$, say $x \in \mathfrak{A}(\Lambda_0)$, the isotony condition ensures that $x \in \mathfrak{A}(\Lambda)$ and so $\alpha_t^\Lambda(x) \in \mathfrak{A}(\Lambda)$ for all $\Lambda \supseteq \Lambda_0$, $\Lambda \in L$. Thus it is meaningful to consider $\lim_{\Lambda \supseteq \Lambda_0} \alpha_t^\Lambda(x)$.

The existence of "global" time translations is a considerably more difficult problem than that of the global gibbs state. HHW [26] require that $\lim_{\Lambda} \alpha_t^\Lambda(x)$ exists for all $x \in \mathfrak{A}_L$, $t \in \mathbb{R}$, the limit being taken in an appropriate sense in the norm topology. Then one may define an automorphism group α_t of $\mathfrak{A}_L$, by :

(1.4) $$\alpha_t(x) = \lim_{\Lambda} \alpha_t^\Lambda(x) \qquad x \in \mathfrak{A}_L, \ t \in \mathbb{R}$$

and extend α_t by continuity to all of $\mathfrak{A}$.

This second assumption of HHW has been shown (STREATER [77], ROBINSON [58]) to be justified for a large class of lattice systems. However, it has been shown to be invalid (DUBIN and SEWELL [17], henceforth DS) in the case of the BCS model (BARDEEN, COOPER and SCHRIEFFER [8]), and the ideal Bose gas model, although HHW's main conclusions have been shown (ARAKI and WOODS [6], see § 3) to hold in the latter case. For this reason, DS [17] have weakened this assumption, replacing both assumptions by the following :

(a) for all $x_1 \dots x_n \in \mathfrak{A}_L$ $\quad t_1 \dots t_n \in \mathbb{R}$, $n \in \mathbb{N}$, $\quad \lim_{\Lambda} \omega_\beta^\Lambda(\alpha_{t_1}^\Lambda(x_1) \dots \alpha_{t_n}^\Lambda(x_n))$ exists

(b) for all $x_1 \dots x_n, y_1 \dots y_m \in \mathfrak{A}_L$, $t_1 \dots t_n$, $s_1 \dots s_m \in \mathbb{R}$, $n,m \in \mathbb{N}$

$\lim_{\Lambda} \lim_{\Lambda'} \omega_\beta^\Lambda(\alpha_t^\Lambda(x_1) \dots \alpha_t^\Lambda(x_n)\, \alpha_s^{\Lambda'}(y_1) \dots \alpha_s^{\Lambda'}(y_m))$ exists and equals :

$$\lim_{\Lambda} \omega_\beta^\Lambda(\alpha_t^\Lambda(x_1) \dots \alpha_t^\Lambda(x_n)\, \alpha_s^\Lambda(y_1) \dots \alpha_s^\Lambda(y_m))$$

Note that (a) implies the existence of $\lim_{\Lambda} \omega_\beta^\Lambda(x)$ for all $x \in \mathfrak{A}_L$; moreover, DS [17] show that their assumptions are also satisfied by the ideal Bose Gas model, and by the BCS model, at least in the strong coupling case.

§ 2. Before we present the main results of HHW and DS, let us explain what is meant by the KMS condition. This was first formulated by KUBO [41], MARTIN and SCHWINGER [43] as a boundary condition for "thermodynamic Green's functions"; in the algebraic formulation, it was first used by HHW, and has since become an essential tool in statistical mechanics, especially since the realization of its intimate connection with the Left Hilbert Algebras of Tomita (TAKESAKI [80]; see Chapter III of this dissertation).

Consider first a finite system, where the Gibbs state ω_β is defined by (1.1). ω_β has the following two properties:

(a) faithfulness $\omega_\beta(x^*x) = 0 \implies x = 0 \quad (x \in \mathfrak{A})$

(b) KMS condition let $x,y \in \mathfrak{A}$. Then there exists a function f, defined, bounded and continuous in the strip $\{ z \in C : 0 \leq \mathrm{Im} z \leq \beta \}$ and analytic in the interior, such that

$$(2.1) \qquad \omega_\beta(\alpha_t(x)y) = f(t)$$
$$\omega_\beta(y\alpha_t(x)) = f(t + i\beta) \qquad (t\in\mathbb{R})$$

(For the simple proof, see WINNINK [89], p.238)

Putting $y = 1$ in (2.1), one obtains, using the Schwartz reflection principle with respect to the lines $\text{Im}z = n\beta$, $n\in\mathbb{Z}$, an entire function $f(z)$, which is periodic with period $i\beta$, and also uniformly bounded (by $\|x\|$). Therefore it must be constant, which shows that :

$$\omega_\beta(\alpha_t(x)) = f(t) = f(0) = \omega_\beta(x) \qquad (x\in\mathfrak{A})$$

Thus a KMS state is time-invariant (WINNINK [88]).

From this WINNINK [88] now concludes, in the case of a finite system, that the Gibbs state is the unique normal KMS state with respect to α_t for the inverse temperature β, if one assumes $\mathfrak{A} = \mathfrak{B}(\mathfrak{H})$.

Thus in this case, the KMS condition characterizes the equilibrium state for inverse temperature $\beta > 0$.

§ 2.1 Clearly, if one can show, for a specific model, that a KMS state exists, and that it is unique, then that state must describe equilibrium. This has been done in special cases. An example is the one dimensional lattice system considered by ARAKI [3], which we now describe.

For lattice systems, the local algebras $\mathfrak{A}(\Lambda)$ can be taken to be $\mathfrak{B}(\mathfrak{H}_\Lambda)$, $\mathfrak{H}_\Lambda$ being finite dimensional. (For details see HUGENHOLTZ [31]). For each $\Lambda\in L$, we are given a potential $\phi(\Lambda)\in\mathfrak{A}(\Lambda)$ and the "local time development" α_t^Λ is given by

(2.2) $$\alpha_t^{\Lambda}(x) = \exp(iU(\Lambda)t)\ x\ \exp(-iU(\Lambda)t) \qquad x\in \mathfrak{A}_L,\ t\in\mathbb{R}$$

where

(2.3) $$U(\Lambda) = \sum_{\Lambda'\subseteq\Lambda} \phi(\Lambda')$$

In the one-dimensional case ($\Gamma=\mathbb{Z}$) ARAKI [3] requires the following two conditions :

(i) The potential is tempered :

(2.4) $$\sup_{n\in\mathbb{Z}} \sum_{\Lambda} \{ (\exp\ (\alpha N(\Lambda)))\|\phi(\Lambda)\| \ :\ n\epsilon\Lambda\} < \infty \qquad (\alpha>0)$$

(ii) The "surface energy"

(2.5) $$w(\Lambda) \equiv \sum_{\Lambda'} \{ \ \phi(\Lambda')\ :\ \Lambda'\epsilon L,\ \Lambda'\cap\Lambda \neq \phi,\ \Lambda'\cap\Lambda^c \neq \phi\ \} \qquad (\ \Lambda^c = \Gamma\setminus\Lambda)$$

is bounded in norm independently of $\Lambda\epsilon L$.

It is then shown that the limit (1.4) exists by virtue of (i), and defines the time development α_t of the infinite system. Under these conditions, ARAKI [3] shows that there exists exactly one KMS state for α_t at each fixed $\beta>0$.

It may be of interest to note that such existence and uniqueness theorems have been formulated in the abstract:

Let $\mathfrak{A}$ be a UHF algebra*, and δ a normal *-derivation

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* A UHF algebra is a C*-algebra $\mathcal{U}$ with identity 1 which contains an increasing sequence of C*-subalgebras $\mathcal{B}_n \ni 1$ which are (isomorphic to) full matrix algebras, such that $\bigcup_n \mathcal{B}_n$ is (norm) dense in $\mathcal{U}$. Each $\mathcal{B}_n$ has a unique tracial state τ_n , hence $\mathcal{U}$ has a unique tracial state τ such that $\tau\big|_{\mathcal{B}_n} = \tau_n$.

of $\mathfrak{A}^*$. Write $D(\delta) = \bigcup_n \mathfrak{A}_n$, $\{\mathfrak{A}_n\}$ being an increasing sequence of matrix algebras. Let $h_n = h_n^* \in \mathfrak{A}$ be such that $\delta(x) = i[h_n, x]$ for all $x \in \mathfrak{A}_n$ ([63], Thm.2). Let $P_n : \mathfrak{A} \longrightarrow \mathfrak{A}_n$ be the canonical conditional expectation of $\mathfrak{A}$ onto $\mathfrak{A}_n$ **. If $\sup_n \|P_n h_n - h_n\| < \infty$ the closure of δ generates a one-parameter automorphism group ρ_t of $\mathfrak{A}$ satisfying:

(2.6) $$\rho_t(x) = \lim_n \exp(iP_n h_n t)x \exp(-iP_n h_n t) \qquad x \in D(\delta), \quad t \in \mathbb{R}$$

(see KISHIMOTO [38]). We say ρ_t is <u>approximately inner</u>.

Under these circumstances, there exists one (POWERS and SAKAI [50]) and only one (KISHIMOTO [39]) KMS state of $\mathfrak{A}$ with respect to ρ_t (for $\beta=1$).

Note that $P_n h_n - h_n$ corresponds to the surface term $w(\Lambda)$ for the case of a one dimensional lattice system.

Coming back to the general case of an infinite system, we note that although ω_β cannot now be written in the form (1.1), it still satisfies the KMS condition, both under the

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* A <u>normal *-derivation</u> is a linear map $\delta : D(\delta) \longrightarrow \mathfrak{A}$, where $D(\delta)$ is a dense *-subalgebra of $\mathfrak{A}$, with the properties:

(i) $\delta(x^*) = \delta(x)^* \qquad (x \in D(\delta))$

(ii) $\delta(xy) = x\delta(y) + \delta(x)y \qquad (x, y \in D(\delta))$

(iii) $D(\delta) = \bigcup_n \mathfrak{A}_n$, each $\mathfrak{A}_n$ being (isomorphic to) a full matrix algebra. A normal *-derivation is always closable ([51], Thm. 1).

** That is, the mapping defined by the property $\tau((P_n x)y) = \tau(xy)$, for all $x \in \mathfrak{A}$, $y \in \mathfrak{A}_n$.

assumptions of HHW and under those of DS ([26], [17]). Thus, and this was the crucial observation of HHW, the KMS condition may be used as an algebraic requirement on the equilibrium states of infinite systems. However it is ultimately desirable to define equilibrium in terms of simple physical requirements. Several such requirements have been studied by different authors :

§2.2. HAAG, KASTLER and TRYCH-POHLMEYER [25] require that an equilibrium state be :

(a) Stationary in time

(b) Stable under local perturbations of the dynamics

(c) Relatively pure.

These requirements correspond to the following mathematical assumptions on the observable algebra $\mathfrak{A}$, the time development α_t (which we now require to exist as an automorphism group of $\mathfrak{A}$) and the equilibrium state ω :

(a) $\omega(\alpha_t(x)) = \omega(x)$ for all $x \in \mathfrak{A}$, $t \in \mathbb{R}$

(b) Given $h = h^* \in \mathfrak{A}$, one may construct the perturbed automorphism group $\alpha_t^{(h)}$ which is related to the unperturbed one by the following property :

$$(2.7) \qquad i\frac{d}{dt}\,\alpha_t^{(h)}(x)\Big|_{t=0} = i\frac{d}{dt}\,\alpha_t(x)\Big|_{t=0} + [h,x]$$

for all $x \in \mathfrak{A}$ such that one of the above derivatives exists *.

* Roughly speaking, if $\alpha_t(x) = (\exp(itH))x(\exp(-itH))$ then $\alpha_t^{(h)}(x) = (\exp(it(H+h)))x\,(\exp(-it(H+h)))$

The state ω is now required to be folium stable for inner (quasi-local) perturbations in the following sense:

There is a map $\lambda \longmapsto \omega^{(\lambda h)}$ from a zero neighbourhood in $\mathbb{R}$ to the state space of $\mathfrak{A}$ such that $\omega^{(oh)} = \omega$ and

(i) $\omega^{(\lambda h)}(\alpha_t^{(h)}(x)) = \omega^{(\lambda h)}(x)$ for all $x \in \mathfrak{A}$, $t \in \mathbb{R}$

(ii) $\left.\frac{d\omega^{(\lambda h)}}{d\lambda}\right|_{\lambda=0} \equiv \omega_1^{(h)}$ exists in the weak sense.

(iii) for all $h = h^* \in \mathfrak{A}$, $\omega_1^{(h)}$ is a normal form in the GNS representation defined by ω * : we say $\omega_1^{(h)}$ lies in the normal folium of ω.

This formulates the physical requirement that when the time evolution is perturbed by λh (λ= small coupling constant) the new stationary state $\omega^{(\lambda h)}$ is only "slightly" different from the old one, in the sense of having a perturbation expansion $\omega^{(\lambda h)} = \omega + \lambda\omega_1 + \dots\dots$

(c) ω has L^1- decrease of correlation in time in the following sense :

There exists a norm dense, self-adjoint subset $\mathcal{S} \subseteq \mathfrak{A}$ such that for each $x_1 \dots x_n \in \mathcal{S}$ with $n \leqslant 6$ there exist $C, \delta > 0$ such that

$$| \omega_n^T (\alpha_{t_1}(x_1), \dots \alpha_{t_n}(x_n)) | \leqslant C(1 + \sup_{i,j} |t_i - t_j|)^{1-\delta} \qquad (2.8)$$

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* see Chapter I, § 1.4.

where ω_n^T is the truncated n-point function.*

This technical condition is a strengthening of the property of <u>weak clustering</u> (i.e. vanishing in the mean of the truncated two point functions) which is known to be equivalent to extremal invariance for asymptotically abelian systems (see KASTLER [34], Chapter II). Extremal invariance, i.e. the requirement that the equilibrium state be extremal among all time-invariant states, is the least specialized version of relative purity ([25], Section III) : the equilibrium state is pure, relative to all stationary states.

Under the above conditions (the last one having been slightly weakened by KASTLER and BRATELLI [35]), it is shown ([25]) that any ω is either a KMS state or that the spectrum of the energy (i.e. of the generator of time-translations in the GNS representation defined by ω) is one-sided, a situation corresponding to the ground state, which is to be interpreted as the zero temperature state. (see STREATER and WIGHTMAN [79])

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* By definition: $0 = \omega_0^T$

$$\omega(x_1) = \omega_1^T(x_1)$$

$$\omega(x_1 x_2) = \omega_2^T(x_1, x_2) + \omega_1^T(x_1)\omega_1^T(x_2)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\omega(x_1 x_2 \ldots\ldots x_n) = \sum \omega_n^T(x_{k_1^1}, x_{k_2^1}, \ldots x_{k_{n_1}^1}) \ldots \omega_n^T(x_{k_1^s}, x_{k_2^s}, \ldots x_{k_{n_s}^s})$$

where the summation extends over all partitions of $\{1 \ldots n\}$ into $\{k_1^1 \ldots k_{n_1}^1\}\{k_1^2 \ldots k_{n_2}^2\} \ldots \{k_1^s \ldots k_{n_s}^s\}$, $(n_1 + \ldots + n_s = n)$ with $k_1^r < k_2^r < \ldots < k_n^r$ and $k_1^1 < k_1^2 < \ldots k_1^s$.

§2.3. In a recent paper, PUSZ and WORONOWICZ [52] have formulated the physical requirements for equilibrium in a different way, which allows them to dispense with the requirement of relative purity and time invariance of the equilibrium state. Their requirement, termed passivity, is inspired by the second law of thermodynamics. Namely, if an isolated system in equilibrium is perturbed by changing the external conditions (e.g. acting on it with an external force) for a finite time interval $[0,T]$, after which it is unperturbed ..., then a non-negative amount of energy must be transmitted to it in the interval $[0,T]$; for thermodynamic systems in equilibrium cannot perform work in cyclic processes.

This requirement is formulated mathematically as follows: One considers a C*-dynamical system, i.e. a C*-algebra $\mathfrak{A}$ with a str. continuous automorphism group $\{\alpha_t : t \in R\}$, generated by an (unbounded) derivation δ (see §2.1). The system is perturbed in a time interval $[0,T]$ by a local change in the dynamics, i.e. its perturbed time evolution $\alpha_t^{(h)}$ is generated by a derivation δ_t with $D(\delta_t) = D(\delta)$ given by

(2.9) $$\delta_t(x) = \delta(x) + i[h_t, x]$$

where $t \longmapsto h_t$ is a continuous mapping of $\mathbb{R}$ into the self-adjoint part of $\mathfrak{A}$ (= quasi-local perturbation) which is zero outside $(0,T)$, and differentiable in norm inside that interval. (smooth changes). The energy given to the system in a state ω of $\mathfrak{A}$ in the interval $[0,T]$ is seen to equal

(2.10) $$L^{(h)}(\omega) = \int_0^T \omega(\alpha_t^{(h)}(\frac{dh_t}{dt}))dt$$

We then define a state ω to be passive iff $L^{(h)}(\omega) \geq 0$ for any local smooth change in the dynamics in the above sense. One proves that a passive state is necessarily time imvariant ([52], Thm. 1.1) and that all KMS-states or ground states are passive ([52], Thm. 1.2). One also shows, under the additional assumption of relative purity defined in terms of weak clustering (see § 2.2), KMS-states and ground states exhaust all passive states ([52], Thm. 1.3).

To relax the assumption of relative purity, one has to assume a stronger condition than passivity.* This is called complete passivity and is the requirement that, if one considers, for arbitrary $n \in \mathbb{N}$, n uncorrelated copies of the system, all in the same state, then the resulting state on the enlarged system should be passive. More precisely, one calls a state ω of a C*-algebra $\mathfrak{A}$ completely passive, iff for all $n \in \mathbb{N}$, the state $\overset{n}{\otimes}\omega$ on $\overset{n}{\otimes}\mathfrak{A}$ is passive, relative to the time evolution $\overset{n}{\otimes}\alpha_t$. It can then be shown that a completely passive state of a C*-dynamical system is either β - KMS for some $\beta \geq 0$, or a ground state. ([52], Thm. 1.4).

* That simple passivity is not sufficient is seen from the fact that a convex combination of two KMS states at different temperatures cannot be KMS, but is clearly passive.

§ 2.4 Another physical requirement for equilibrium is Global Thermodynamic Stability, formulated by ARAKI [2]* for a quantum lattice as follows :

The algebras are as in § 2.1, and the local time development α_t^Λ is given by (2.2) and (2.3), where the conditions on the potential are now the following :

(i) Translational covariance : $\phi(\Lambda + a) = \tau_a \phi(\Lambda)$ $(\Lambda \in L,\ a \in \Gamma)$, where $\{\tau_a\}$ are the space translation automorphisms of $\mathfrak{A}$ and $\tau_a \mathfrak{A}(\Lambda) = \mathfrak{A}(\Lambda + a)$ (see [31], p.152)

(ii) Finite body interaction : $\exists\ N_0 \in \mathbb{N}$ such that $\phi(\Lambda) = 0$ if the number of points $N(\Lambda)$ in Λ is larger than N_0.

(iii) Relatively short range: $\|\phi\| = \sum_{\Lambda \ni 0} N(\Lambda)^{-1} \|\phi(\Lambda)\| < \infty$

Under these conditions, one then shows [2],that the limit (1.4) exists, and gives rise to an automorphism group α_t of $\mathfrak{A}$.

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* ARAKI [2] refers to this as the variational principle.

A translationally invariant state ω of $\mathfrak{A}^*$, is said to satisfy GTS iff:

(2.11) $$s(\omega) - \beta\omega(A) = \sup_{\omega'} [s(\omega') - \beta\omega'(A)], \quad A = \sum_{\Lambda \ni 0} N(\Lambda)^{-1}\phi(\Lambda)$$

the sup being taken over all translationally invariant states ω'.

Here the mean entropy $s(\omega)$ is given by the limit, in a suitable sense

(2.12) $$s(\omega) = \lim_{\Lambda} N(\Lambda)^{-1}\omega(- \log \rho_\Lambda)$$

where ρ_Λ is given by

(2.13) $$\omega(x) = \mathrm{tr}\ (\rho_\Lambda x) \ : \ x \epsilon\, \mathfrak{A}(\Lambda)$$

ARAKI [2] then shows that translationally invariant states satisfy GTS iff they satisfy KMS with respect to α_t .

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* $\omega \circ \tau_a = \omega$ for all $a \epsilon \Gamma$. We note that (2.11) expresses the minimisation of the mean free energy functional $\omega(A) - (1/\beta)s(\omega)$. The existence of the limit (2.12) follows from the subadditivity property of the entropy for a finite region, together with translation invariance.

§ 2.5 In the absence of translation invariance, GTS cannot be formulated at all, since the mean entropy functional (2.11) is not well defined. In this setting, ARAKI and SEWELL [5] have formulated the physical requirement of Local Thermodynamic Stability (LTS). This means that the "local free energy" for a bounded region $\Lambda \epsilon L$, i.e. the free energy corresponding to the open system consisting of particles in Λ interacting both with one another and with the outside, should be minimal for variations in the state which leave it unchanged outside Λ. Thus we are interested here, not in global variations of mean quantities, but in local variations.

For a quantum lattice system (see § 2.1 and [31]) ARAKI and SEWELL define the conditional entropy $\tilde{s}_\Lambda(\omega)$ and the conditional free energy $\tilde{F}_\Lambda(\omega)$ induced by a state ω for the region $\Lambda \epsilon L$ as follows :

$$(2.14) \qquad \begin{cases} \tilde{s}_\Lambda(\omega) = \lim\limits_{\Lambda'} [\, \omega(-\log \rho_{\Lambda'}) - \omega(-\log \rho_{\Lambda'-\Lambda})] \\ \tilde{F}_\Lambda(\omega) = \omega(H_\Lambda) - (1/\beta)\tilde{s}_\Lambda(\omega) \end{cases}$$

Here $\Lambda, \Lambda' \epsilon L$, $\rho_\Lambda \epsilon \mathfrak{A}(\Lambda)$ is the density matrix defined by (2.13), and $H_\Lambda \epsilon \mathfrak{A}$ represents the energy of interaction of particles in Λ with one another and with particles outside.*

A state ω of $\mathfrak{A}$ is then said to satisfy LTS for a

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* Restrictions are imposed on these interactions to ensure that $H_\Lambda \epsilon \mathfrak{A}$.

temperature $1/\beta$ (>o) iff, for each $\Lambda\epsilon L$,

(2.15) $$\tilde{F}_\Lambda(\omega) \leq \tilde{F}_\Lambda(\omega')$$

for each state ω' such that $\omega'|_{\Lambda^c} = \omega|_{\Lambda^c}$

The time evolution of the system is assumed to be given by a automorphism group α_t of $\mathfrak{A}$, generated by an unbounded derivation δ (see §2.1), which is the closure of its restriction to $\mathfrak{A}_L$ and such that :

$$\delta(x) = i[\,H_\Lambda, x\,] \qquad (\,x\epsilon\mathfrak{A}(\Lambda),\ \Lambda\epsilon L)$$

Under these conditions, it is shown ([5], [72]) that a state ω of $\mathfrak{A}$ satisfies LTS iff it satisfies KMS with respect to α_t.

Thus in general one may say that, for quantum lattice systems with suitably tempered interactions, LTS⇔KMS, and for translationally invariant states, LTS⇔KMS⇔GTS.

The above examples provide sufficient justification, in our view, for using the KMS condition as the defining property of equilibrium in an infinite system, since it is shown to be equivalent to the Gibbs condition in a finite system, and also to correspond to simple physical requirements under fairly general conditions, in an infinite system.

§ 3. We now proceed to examine the consequences of the KMS condition for the structure of the representation of the algebra of observables induced by a KMS state.

The existence of inequivalent * representations of the algebra of observables is characteristic of systems with infinitely many degrees of freedom, both in Statistical Mechanics and in Quantum Field Theory, and was one of the main arguments in favour of the algebraic approach, ever since the pioneering work of SEGAL [66] and HAAG [24].

A pure state on the algebra of observables gives rise to an irreducible GNS representation, and vice-versa [Chapter I, § 1.4]. Thus a reducible GNS representation implies that the state is not a pure state, but a mixture.

Consider now the "limit Gibbs state" ω_β on the algebra of quasilocal observables $\mathfrak{A}$, given by (1.2). Under the assumptions of HHW [26], ω_β is invariant under the time evolution α_t given by (1.4) (see § 2.0). The corresponding GNS triple $(\mathcal{H}_\beta, \pi_\beta, \zeta_\beta)$ has the following properties [26]:

(i) The spectrum of the infinitesimal generator of the time translations U_t is symmetric about the origin (in contrast to the ground state, where it is one-sided).

(ii) ζ_β is not only cyclic, but also separating for $\pi_\beta(\mathfrak{A})''$. Thus it is also cyclic (and separating) for $\pi_\beta(\mathfrak{A})'$.**

* i.e. not related by a unitary mapping between the underlying Hilbert spaces.

** see Chapter I, § 1.4.

(iii) Not only is $\pi_\beta(\mathfrak{A})$ reducible, but it is in one-to-one correspondence with its commutant. More precicely, there exists an antilinear, antiunitary operator J on $\mathcal{H}_\beta$ such that $J^2 = I$, commuting with the time evolution U_t, and such that

(3.1) $$J\pi_\beta(\mathfrak{A})''J = \pi_\beta(\mathfrak{A})'$$

Thus the commutant $\pi_\beta(\mathfrak{A})'$ provides another representation of $\mathfrak{A}$ on $\mathcal{H}_\beta$, which is (anti) unitarily equivalent to π_β.

We have seen (Chapter III, Thm. 4.7) that, as a result of Tomita-Takesaki theory, this structure is exhibited by the GNS representation of <u>any</u> C*-algebra induced by a KMS state.

These properties were previously discovered by ARAKI and WOODS [6]. They were investigating representations of the CCR which describe a non-relativistic infinite free Bose gas of uniform density. They used the bounded form of the CCR, defined as follows:

Given a real inner product space of test functions T, a <u>cyclic representation of the CCR</u> is a map from T into unitary operators U(f), V(g) on a Hilbert space with the properties

(3.2) (i) $$\begin{cases} U(f)\,V(g) = V(g)\,U(f)\exp(-i(f,g)) \\ U(f)\,U(g) = U(f+g) \\ V(f)\,V(g) = V(f+g) \end{cases}$$

(ii) $\mathbb{R} \ni \lambda \longmapsto U(\lambda f), V(\lambda f)$ is weakly continuous.

(iii) There exists a cyclic vector ζ for the Von Neumann algebra $\mathfrak{A}$ generated by $\{U(f), V(g) : f, g \in \mathbf{T}\}$.

It is known (see e.g. [18], section III.1.c., where more details can be found) that a cyclic representation of the CCR is determined, up to unitary equivalence, by the functional:

$$E(f,g) = \langle \zeta, U(f)V(g)\zeta \rangle \qquad (f,g \in T) \tag{3.3}$$

ARAKI and WOODS [6] determine the functional $E(f,g)$ pertinent to a free infinite Bose gas whose density is given as a continuous integrable non-zero positive function in momentum space, by first considering the situation of a discrete density distribution enclosed in a box, then letting the box become infinitely large, and finally letting the distribution tend to a continuous one.

On constructing the representation of the CCR determined by this functional, one indeed gets a reducible representation, whose commutant furnishes another, equivalent representation of the CCR [6]. Note also the remarkable fact that there appears nowhere in ARAKI and WOODS' treatment a time evolution, let alone a KMS state. This is because there is exactly one time evolution satisfying the KMS condition with respect to some given state; this is constructed out of the algebra itself (see Chapter III, Thm. 4.2).

Another problem [17] is that the free time evolution does not give rise to an automorphism group of $\mathfrak{A}$. One may define a local time development as in (1.2), with $\mathcal{H}_\Lambda$ the second quantization of the one-particle free hamiltonian for a bounded region, but the limit (1.4) does not exist. However, assumptions (a) and (b) of DS (see §1) are satisfied. As observed previously, this allows the definition of the limit Gibbs state ω_β. Using assumption (a), DS are able to construct a Hilbert space $\mathcal{H}$, a representation π of in $B(\mathcal{H})$, a unit vector $\zeta \in \mathcal{H}$ and a unitary group $U_t \in B(\mathcal{H})$ representing time development such that $U_t\zeta = \zeta$; this is done by a method parallel to the Wightman reconstruction theorem [79]. Then using property(b), they are able to show that this representation is in fact unitarily equivalent to the GNS representation $(\mathcal{H}_\beta, \pi_\beta, \zeta_\beta)$ corresponding to ω_β. They further show that

(3.4) $$\tau_t(x) \equiv U_t x U_t^{-1} \qquad t \in \mathbb{R}, \quad x \in \pi_\beta(\mathfrak{A})''$$

defines a strongly continuous one parameter automorphism group of $\pi_\beta(\mathfrak{A})''$, though <u>not of $\mathfrak{A}$ itself</u>, which is the limit of the "local time development", in the sense that

(3.5) $$\lim_{\Lambda} \pi_\beta(\alpha_t^\Lambda(x)) = U_t\, \pi_\beta(x) U_t^{-1} \qquad x \in \mathfrak{A}_L, \; t \in \mathbb{R}$$

the limit being taken in the <u>weak</u> operator topology.

In this setting, they are able to show that ω_β satisfies the KMS condition with respect to τ_t *, and that $\pi_\beta(\mathfrak{A})''$

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* Modified in the obvious way to allow for the fact that $\tau_t \notin \mathrm{Aut}\,\mathfrak{A}$

has the structure described by HHW.

This concludes our general discussion of the present status of the KMS condition for equilibrium states in Quantum Statistical Mechanics, and the structure of the associated representations of the observable algebras. We have seen (Chapter III) that a faithful normal state ω on a Von Neumann algebra $\mathcal{M}$ uniquely defines an automorphism group α_t of $\mathcal{M}$ and satisfies the KMS condition with respect to α_t. This "kind of miracle", in ARAKI's [4] words, has been the starting point for many developments both in Statistical Mechanics (some of which have been described in this Chapter) and in the structure theory of Von Neumann algebras, thus providing yet another link between Mathematical Physics and Pure Mathematics.

CHAPTER V

GIBBS STATES ON THE ALGEBRA OF THE CANONICAL COMMUTATION RELATIONS

We now embark on our extension of Tomita-Takesaki theory to an algebra of unbounded operators.As described in Chapter III,the general problem of a closable probability algebra was attempted by GUDDER & HUDSON [22] (see Chapter III,§2 &3),but their analysis only yielded partial results.

Our examples are motivated by Quantum Statistical Mechanics.We will take as our algebra the one generated by the Canonical Commutation Relations (CCR) in their unbounded form. On this algebra,we will construct a class of Gibbs states (see Chapter IV),and a corresponding class of time evolutions and study the properties ofthe GNS representations induced by these states.In the final Chapter,we will prove a commutation theorem,analogous to Theorem 3.3 of Chapter III.

The main problem we have to face is,of course,the unbounded character of our representation.For this reason,the domain of the representation must be chosen carefully,and it is with an analysis of this domain that we begin our investigation. On this domain,our representation will be continuous,but we can no longer hope to have a Hilbert space structure.However, it is still possible to prove that the GNS representation induced by our Gibbs state is a closable probability algebra (see Chapter III,§2),and,what is more important,that one may construct an "almost" Modular Hilbert subalgebra (Chapter III §3)with the same commutant.This latter algebra is the essential tool that will enable us to prove our commutation theorem in the last Chapter.

It is probably possible to prove the commutation theorem by studying the associated Weyl form of the CCR(see Chapter IV,§3).It has to be stressed,however,that the whole purpose of this construction is to develop a method for extending Tomita-Takesaki theory to algebras of unbounded operators. It is in this sense that the present work is new,and may prove useful in dealing with more general examples.Part of the results of this Chapter are taken from joint work with Ingeborg Koch (see [37]).

Let us begin by describing the relevant features of the domain on which our representation of the CCR is to be defined. The results of §1.1 & §1.2 are taken, unless otherwise specified,from THUE POULSEN [83].

§1.1 Spaces of type S^n

Consider Quantum Mechanics of a single particle.Here we are concerned with the conjugate pair of observables q and p (position and conjugate momentum) which satisfy the CCR:

(1.1) $$[p,q] = -i$$

The traditionally most important realisation of these observables is by self-adjoint operators on $L^2(\mathbb{R})$ given by:

(1.2) $$(pf)(t)= -i(df/dt)(t)$$
$$(qf)(t)= tf(t)$$

defined on a suitable common dense domain in $L^2(\mathbb{R})$.It is well known (see e.g.EMCH [18] III.1.b Thm.1)that there is no realisation of (1.1) by bounded self-adjoint operators.Therefore any realisation of (1.1) on a Hilbert space immediately creates problems due to the unbounded character of the operators involved.

KRISTENSEN, MEJLBO & THUE POULSEN [40] proposed the following alternative approach to the study of the representations of (1.1): Instead of requiring the representation space to be a Hilbert space, they require the operators to be everywhere defined and continuous, and then study the possible representation spaces, which will be locally convex spaces, equipped with a continuous scalar product, with respect to which p & q are required to be symmetric (the question of self-adjointness does not arise, since we do not have a Hilbert space structure; this is one of the main advantages of this approach). We formulate the requirements placed on the representation space as a definition for the case of n particles :

Definition:

A space of type S^n is a locally convex Hausdorff space S with the following properties:

(1.3) There exists a scalar product (i.e. a sesquilinear positive definite form) $(.,.)$ on S such that the corresponding norm $\|\cdot\|$ is continuous on S.

(1.4) There exist continuous linear mappings b_j, b_j^+ (j=1,2,...n) on S such that

$$(f, b_j g) = (b_j^+ f, g)$$

and

(1.5) $$[b_j, b_k] = [b_j^+, b_k^+] = 0$$

(1.6) $$[b_j, b_k^+] = \delta_{jk} \qquad j,k=1,\dots n$$

(1.7) There exists a cyclic element e_o in S such that $\|e_o\| = 1$, $b_j e_o = o$ for $j = 1,\dots n$ and

(1.8) If $\mathcal{R}$ denotes the algebra generated by $\{b_j, b_j^+ : j=1,\dots n\}$ then $\mathcal{R}e_o$ is dence in S.

Here b_j and b_k^+ are the usual raising and lowering operators of Quantum Mechanics, given by:

$$b_j = 2^{-1/2}(p_j - iq_j) \tag{1.9}$$
$$b_k^+ = 2^{-1/2}(p_k + iq_k)$$

Proposition 1.1

For $x \in \mathcal{R}$, $f \in S$, define:

$$\|f\|_x = \|xf\| \tag{1.10}$$

The locally convex topology defined on S by the seminorms $\|\cdot\|_x$ is the weakest locally convex topology on S with respect to which S is a space of type S^n, and is equivalently given by the sequence of norms $\|\cdot\|_r$ ($r \in \mathbb{N}$) given by:

$$\|f\|_r^2 = (f, b^r f) \tag{1.11}$$

where

$$b = \sum_{j=1}^{n} b_j b_j^+ \tag{1.12}$$

moreover, we have:

$$\|f\|_{r+1}^2 \geq n\|f\|_r^2$$

The most important fact about spaces of type S^n is that they can be continuously embedded, in a way preserving the properties of the definition, in a maximal space of type S^n, which is unique (up to an isomorphism preserving the properties of the definition).This space is nothing but $S(\mathbb{R}^n)$, i.e. the space of all infinitely differentiable functions on $\mathbb{R}^n$ which decrease at infinity faster than any polynomial.We make $S(\mathbb{R}^n)$ into a space of type S^n by equiping it with its usual topology, the scalar product inherited from $L^2(\mathbb{R}^n)$, and :

(1.13) $$e_o(t) = \pi^{-n/4}\exp\left(-\frac{1}{2}\sum_{j=1}^{n} t_j^2\right) \qquad : t=(t_1,\dots t_n) \in \mathbb{R}^n$$

(1.14) $$(q_j f)(t) = t_j f(t)$$

(1.15) $$(p_j f)(t) = -i(\partial/\partial t_j) f\,(t) \qquad : j=1,\dots n$$

The usual topology of $S(\mathbb{R}^n)$ is the same as that defined in Prop.1.1.The following <u>sequence representation</u> of $S(\mathbb{R}^n)$ will be very useful:

<u>Proposition 1.2</u>

Let s^n be the space of all multisequences $c=(c_\nu)$ $\nu \in \mathbb{N}^n$ such that the sums:

(1.16) $$\sum_{\nu\in\mathbb{N}^n} (|\nu|+n)^r |c_\nu|^2 = \|c\|_r^2 < \infty \qquad r \in \mathbb{N}$$

Equip s^n with the topology defined by the norms $\|\cdot\|_r$, $r \in \mathbb{N}$, the scalar product

(1.17) $$(c,c') = \sum_{\nu\in\mathbb{N}^n} \bar{c}_\nu c'_\nu \, .$$

and the structure of a space of type S^n given by:

(1.18) $$(e_o)_\nu = \begin{cases} 1 \, , & \text{for } \nu = (0,0,\dots 0) \\ 0 \, , & \text{otherwise} \end{cases}$$

(1.19) $$(b_j c)_{\nu_1\dots\nu_n} = (\nu_j+1)^{1/2} c_{\nu_1\dots(\nu_j+1)\dots\nu_n}$$

(1.20) $$(b_j^+ c)_{\nu_1\dots\nu_n} = \begin{cases} \nu_j^{1/2} \, c_{\nu_1\dots(\nu_j-1)\dots\nu_n} & \text{if } \nu_j > 0 \\ 0 & \text{if } \nu_j = 0 \end{cases}$$

Then the mapping:

$$(1.21)\qquad J : s^n \longrightarrow S(\mathbb{R}^n)$$
$$c \longmapsto \sum_{\nu\in\mathbb{N}^n} c_\nu e_\nu$$

where the Hermite elements e_ν , $\nu\in\mathbb{N}^n$ are given by:

$$(1.22)\qquad e_{\nu_1\nu_2\ldots\nu_n} = (\nu_1!\,\nu_2!\ldots\nu_n!)^{-1/2}\, b_1^{+\nu_1} b_2^{+\nu_2}\ldots b_n^{+\nu_n} e_0$$

is a linear topological isomorphism of s^n onto $S(\mathbb{R}^n)$, preserving the properties of the definition of a space of type S^n.

Note that the Hermite elements algebraically span the dense subspace $\mathcal{K}e_0$ of $S(\mathbb{R}^n)$ (see (1.8)). Moreover, they form an orthonormal base in $L^2(\mathbb{R}^n)$, the completion of $S(\mathbb{R}^n)$ in the $\|\cdot\|_0$ norm.

Henceforth we shall have to deal exclusively with maximal spaces, and therefore we shall write S^n for $S(\mathbb{R}^n)$ and S for $S(\mathbb{R})$. We introduce a conjugation in S, denoted by a bar, given by the usual complex conjugation. One checks that

$$(1.23)\qquad \overline{\sum_n c_n e_n} = \sum_n (-1)^n\, \bar{c}_n e_n$$

so that, in the sequence representation,

$$(1.24)\qquad \overline{(c_n)} = ((-1)^n\, \bar{c}_n)$$

We will also use the fact that S^n (and hence also s^n) is a countably Hilbert space, and in fact a nuclear space.

Definition (GELFAND & VILENKIN [20])

A countably Hilbert space E is a Fréchet space (i.e. a complete metrizable locally convex space) whose topology can be defined by a sequence of increasing norms $\|\cdot\|_r$ ($r\in\mathbb{N}$) each coming from an inner product $(.,.)_r$.

We see that, if E_r denotes the Hilbert space completion of

$(E,\|\cdot\|_r)$, we have:

$$E=\bigcap_r E_r = \varprojlim E_r$$

where $\varprojlim E_r$ denotes the <u>projective limit</u> of the spaces E_r, defined as follows:

<u>Definition</u> (SCHAEFFER [64])

Let $\{F_i,\ i\in I\}$ be a family of locally convex spaces indexed by a directed set I. Suppose that, for $i\leq j$, there exists a continuous linear mapping:

$$g_{ji}: F_j \longrightarrow F_i$$

Let F be the subspace of the cartesian product of the F_i given by:

$$F=\{(x_i) : x_i=g_{ji}(x_j) \text{ for } i\leq j\}$$

Equipped with the <u>projective topology</u> with respect to the mappings $f_i: F \longrightarrow F_i$, the restrictions of the canonical projections onto F_i, (i.e. the weakest topology on F making each f_i continuous) F is called the <u>projective limit</u> of the spaces F_i with respect to the mappings g_{ji}.

In the case of a countably Hilbert space E, the identity:

$$(E,\|\cdot\|_p) \longrightarrow (E,\|\cdot\|_r)$$

extends to a continuous linear mapping:

$$g_{pr}: E_p \longrightarrow E_r$$

for $p\geq r$, since the norms are increasing.

<u>Definition</u> (TREVES[85])

Let E be a locally convex Hausdorff space, with its topology given by an increasing set of seminorms $\|\cdot\|_r$. We let E_r be the completion of the normed space $(E/\ker\|\cdot\|_r, \|\cdot\|_r)$. For $p\geq r$,

$\ker\|.\|_p \subseteq \ker\|.\|_r$,and so the identity factors to a continuous linear mapping:

$$g_{pr}: (E/\ker\|.\|_p) \longrightarrow (E/\ker\|.\|_r)$$

which extends to the completions E_p and E_r.
E is said to be nuclear iff for all r there exists $p \geq r$ such that g_{pr} is a nuclear mapping.

The following characterization of a nuclear mapping can be taken as the definition for our purposes:

Definition (TREVES [85] Prop.47.2)
A continuous linear mapping u between two locally convex Hausdorff spaces E and F is said to be nuclear iff we may write

$$u(x) = \sum_{i=1}^{\infty} c_i x_i'(x) y_i$$

where:

$(c_i) \in l_1(\mathbb{N})$;

(x_i') is an equicontinuous sequence in the dual E' of E (i.e. a sequence such that given $\varepsilon > 0$, there exists a 0-neighbourhood U of E such that $|x_i'(x)| < \varepsilon$ for all $x \in U$ and all i.)

(y_i) is contained in a convex, balanced infracomplete bounded subset of F (a convex set B such that $x \in B$, $|c| \leq 1 \Longrightarrow cx \in B$ -balanced ,and such that the subspace of F spanned by B is complete when equipped with the norm $\|x\| = \inf\{c > 0: x \in cB\}$ This is satisfied when B itself is complete in the topology of F (TREVES [85] Lemma 36.1)-infracomplete)

One should note that in case E and F are Hibert spaces, the above definition is easily seen to be equivalent to the definition of a trace class operator u from E to F.

Thus for a countably Hilbert space E, the definition of nuclearity reduces to the following requirement : for all r, there exists $p \geq r$ such that the mapping $g_{pr}: E_p \to E_r$ is trace class.

In the case of s^n, if we denote by s^n_r the Hilbert space completion of s^n with respect to the $\| \cdot \|_r$ norm (given by (1.16)), then we see that the identity extends to a nuclear mapping $s^n_{r+n+1} \longrightarrow s^n_r$. For, letting

$$f_v = (|v|+n)^{-r-n-1} e_v \quad v \in \mathbb{N}^n$$

where $e_v \in s^n$ is the multisequence given by (1.22), we have, if $p=r+n+1$:

$$\sum_{v \in \mathbb{N}^n} \|g_{pr} f_v\|_r = \sum_{v \in \mathbb{N}^n} \|f_v\|_r = \sum_{v \in \mathbb{N}^n} (|v|+n)^{-p} \|e_v\|_r =$$

$$= \sum_{v \in \mathbb{N}^n} (|v|+n)^{-p} (|v|+n)^r = \sum_{v \in \mathbb{N}^n} (|v|+n)^{-n-1} < \infty$$

(This proof is a simplification of Thm.51.5 of TREVES [85]).

§1.2. Spaces of type $\mathfrak{S}$.

Having finished with the case of a finite number of particles, we now attack an infinite number of them. Here we need to represent the canonical pair of field operators and conjugate momenta, indexed by some test function space.

Again in a Hilbert space representation one is faced with problems due to the necessarily unbounded character of the representation, namely the problem of choosing a suitable common dense domain on which these (infinitely many) operators are all (essentially) self-adjoint.

We once again adopt the point of view of §1.1, that is we require the operators to be everywhere defined and continuous and look for a suitable space to represent them on. As we are interested in cyclic representations, we shall assume the existence of a cyclic "vacuum".

Definition.

A space of type $\mathfrak{G}$ is a locally convex space $\mathcal{G}$ with the properties:

(1.25) There exists a scalar product $(.,.)$ such that the corresponding norm is continuous.

(1.26) There exist continuous linear mappings:

$$a, a^{+}: S \longrightarrow L(\mathcal{G}) \quad *$$

such that

(1.27) $$(a(\bar{f})F, G) = (F, a^{+}(f)G) \qquad f \in S,\ F, G \in \mathcal{G}$$

(1.28) $$[a(\bar{f}), a(\bar{g})] = [a^{+}(f), a^{+}(g)] = 0$$

(1.29) $$[a(\bar{f}), a^{+}(g)] = (f, g)$$

(1.28), (1.29): CCR

(1.30) There exists a vacuum element $\Psi_0 \in \mathcal{G}$ such that $\|\Psi_0\| = 1$

---

* $L(\mathcal{G})$ denotes the space of all continuous linear mappings on $\mathcal{G}$. It is equipped with the topology of uniform convergence on bounded sets, which is generated by the seminorms:

$$\|T\|_{r,B} = \sup\{\|TF\|_r : F \in B\}$$

where $\|\cdot\|_r$ are the seminorms defining the topology of $\mathcal{G}$, and B runs over all bounded sets of $\mathcal{G}$, i.e. sets in which all the seminorms are bounded. This topology is the natural generalisation of the norm topology on $L(\mathcal{B})$, where $\mathcal{B}$ is a normed space.

and $a(\bar{f})\Psi_0 = 0$ for all $f \in S$.

(1.31) If $\mathfrak{A}$ denotes the algebra generated by $\{a(\bar{f}), a^+(f) : f \in S\}$ then $\mathfrak{A}\Psi_0$ is dense in $\mathcal{G}$.

(1.32) To every symmetric operator $k \in \mathcal{R}$ (cf.(1.8)) there exists a symmetric operator $K \in L(\mathcal{G})$,the <u>second quantisation</u> of k, satisfying:

(1.33) $$[K, a^+(f)] = a^+(kf) \qquad , f \in S$$

We are aiming at realisations of a space of type $\mathcal{G}$ which have the same relationship to Fock space (over $L^2(\mathbb{R})$) as $S(\mathbb{R})$ has to $L^2(\mathbb{R})$.Roughly speaking, our space will be an infinite direct sum of n-fold symmetrized tensor products of $S(\mathbb{R})$.

Firstly,we define on the n-fold algebraic tensor product $S^{\otimes n}$ of S,a <u>symmetrization</u> operator:

(1.34) $$\mathrm{sym}(f_1 \otimes f_2 \ldots \otimes f_n) = (n!)^{-1} \sum_{p \in P_n} f_{p(1)} \otimes f_{p(2)} \otimes \ldots \otimes f_{p(n)}$$

(where P(n) is the symmetric group) and extend by linearity. We denote its range by $S_+^{\otimes n}$,the <u>symmetric part</u> of $S^{\otimes n}$.We find that sym is continuous in the topology of S^n (into which $S^{\otimes n}$ may be continuously embedded as a dense subspace),and hence defines a continuous linear mapping of S^n,whose range we denote by S_+^n. We may correspondingly define the symmetric part s_+^n of s^n.

For any operator $k \in L(S)$, we define

(1.35) $$k^{(n)} = k \otimes 1 \otimes 1 \otimes \ldots \otimes 1 + 1 \otimes k \otimes 1 \otimes \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes k$$

It is clear that $k^{(n)} \in L(S^n)$,and that it commutes with sym. It therefore leaves $S_+^{\otimes n}$ and S_+^n invariant.

A maximal orthonormal set in S_+^n may be constructed as follows (see GUICHARDET [23] §2.1): Starting from a maximal orthonormal set $\{e_n\}$ in S(for example, the Hermite elements) we write

(1.36) $$e(n_o,n_1,\dots) = (n!(\prod_k n_k!)^{-1})^{1/2}\mathrm{sym}(e_o^n \otimes e_1^n \otimes \dots)$$

where $\sum_k n_k = n$ (and hence there are only finitely many non-zero n_k's)

Let now $\mathcal{G}$ be a space of type $\mathfrak{S}$. We define a map :

(1.37) $$a^{+\otimes n}: S^{\otimes n} \longrightarrow L(\mathcal{G})$$

$$f_1 \otimes f_2 \otimes \dots \otimes f_n \longmapsto a^+(f_1)a^+(f_2)\dots a^+(f_n)$$

from (1.28) we see that $a^{+\otimes n} = a^{+\otimes n}\mathrm{sym}$.

Proposition 1.3

The mapping $\quad \Psi_n : S_+^{\otimes n} \longrightarrow \mathcal{G}$

$\qquad\qquad\qquad f \longmapsto (n!)^{-1}a^{+\otimes n}(f)\,\Psi_o$

is isometric with respect to the norm $\|\cdot\|_b$. The ranges of Ψ_n and Ψ_m are orthogonal with respect to the scalar product for $m \neq n$, and their direct sum equals $\mathfrak{A}\,\Psi_o$, hence is dense in $\mathcal{G}$. Therefore a maximal orthonormal set in $\mathcal{G}$ is given by:

(1.38) $$\left\{\Psi_n(e(n_o,n_1,\dots)) \;:\; \sum_k n_k = n\ ,\ n \in \mathbb{N}\right\}$$

Proposition 1.4

Every symmetric $k \in L(S)$ has a unique second quantization $K \in L(\mathcal{G})$ satisfying (1.33) and

$$K\,\Psi_o = 0$$

This is characterized by

(1.39) $$K\Psi_n(f) = \Psi_n(k^{(n)}f) \quad , \; f \in S_+^{\otimes n} \; , \; n > 0$$

where $k^{(n)}$ is given by (1.35). All second quantizations of k differ from K by a scalar multiple of the identity.

Prop.1.3 shows that the algebraic direct sum $\bigoplus_n S_+^{\otimes n}$ (with $S_+^{\otimes 0} = \mathbb{C}$) is linearly embedded as a dense subspace of any space of type $\mathfrak{S}$. Furthermore, we may equip this space with the following structure:

The scalar product

$$(F,G) = \sum_n (f_n, g_n) \tag{1.40}$$

where $F = (f_n)$, $G = (g_n)$ and the scalar products on the right hand side are inherited from $L^2(\mathbb{R}^n)$.

The vacuum element $\Psi_0 = (1,0,0,\ldots)$

The operators $a(\bar{f}), a^+(f)$, $f \in S$ given by:

$$a^+(f)\, g = (n+1)^{1/2} \mathrm{sym}(f \otimes g) \quad g \in S_+^{\otimes n} \tag{1.41}$$

$$a(\bar{f})\,\mathrm{sym}(f_1 \otimes \ldots \otimes f_n) = \begin{cases} (n)^{-1/2} \sum_{j=1}^{n} (f, f_j) f_1 \otimes \ldots f_{j-1} \otimes f_{j+1} \otimes \ldots f_n, & n>0 \\ 0, & n = 0 \end{cases} \tag{1.42}$$

The <u>locally convex direct sum topology</u> (i.e. the finest locally convex topology making all the canonical embeddings $S_+^{\otimes n} \longrightarrow \bigoplus_n S_+^{\otimes n}$ continuous).

Then $\bigoplus_n S_+^{\otimes n}$ becomes a <u>minimal</u> space of type $\mathfrak{S}$, in the sense that it can be continuously embedded in any space of type $\mathfrak{S}$, in a way preserving the $\mathfrak{S}$-space structure.

On an arbitrary space $\mathcal{G}$ of type $\mathfrak{S}$, we define the topology τ generated by the seminorms $\|\cdot\|_x$, where

$$\|F\|_x = \|xF\|_0 \qquad (F \in \mathcal{G})$$

and x is in the algebra generated by $\mathfrak{A}$ and all second quatizations of symmetric operators in $\mathcal{R}$ (see (1.8)''). We have:

Proposition 1.5

The topology τ is given by an increasing sequence of seminorms $\|\cdot\|_r$ given by:

(1.43) $$\|F\|_r^2 = (F, B^r F) \qquad F \in \mathcal{G}, \; r \in \mathbb{N}$$

where B is the second quantization of $bb^+ \in L(S)$ (see Prop.1.1) given by Prop.1.4.

Clearly, the topology τ is weaker than the original topology of the space $\mathcal{G}$; in particular, it is weaker than the direct sum topology on $\bigoplus_n S_+^{\otimes n}$. Thus the τ-completion of $\bigoplus_n S_+^{\otimes n}$ contains as a dense subspace any space of type $\mathcal{G}$ (up to linear topological isomorphism). In fact, since the restriction of $\|\cdot\|_r$ to $S_+^{\otimes n}$ is just that given by (1.11) (this follows from (1.39)), this completion contains the completion of $S_+^{\otimes n}$ with respect to this topology, namely S_+^n. Furthermore, one may show that the operators

$$a, a^+ : S \longmapsto L(\bigoplus_n S_+^{\otimes n})$$

are continuous when the right hand side is equipped with the topology of uniform convergence on τ-bounded sets, which shows that this completion is in fact a space of type $\mathcal{G}$. Therefore we have:

Theorem 1.6

Let $\mathcal{G}$ be the completion of the algebraic direct sum of S_+^n, with respect tothe topology defined by the norms:

(1.44) $$\|F\|_r^2 = \sum_{n=0}^{\infty} (f_n, b^{(n)r} f_n) \quad , \; F = (f_n) \; , \; r \in \mathbb{N}$$

where

$$b^{(n)} = \sum_{i=1}^{n} (1/2)\;(p_i^2 + q_i^2 + 1) \quad , n > 0$$
$$b^{(o)} = 1$$

(The formula for $b^{(n)}$ follows easily from (1.6),(1.9) and (1.12)) .Then $\mathcal{G}$ is a maximal space of type $\mathfrak{S}$ in the sence explained above.These norms are all increasing,and $\mathcal{G}$ is a countably Hilbertspace (hence a Fréchet space).Moreover,if $\mathcal{G}_r$ denotes the completion of $\mathcal{G}$ with respect to $\|\cdot\|_r$, $\mathcal{G}_r$ is a Hilbert space, and $\mathcal{G}_r = \bigoplus_n S^n_{+r}$ (Hilbert space direct sum - see §1.1). Thus $\mathcal{G}_o = \mathcal{H}_o$ is just the usual Fock space over $L^2(R)$.Finally,

$$\mathcal{G} = \bigcap_r \mathcal{G}_r = \varprojlim \mathcal{G}_r \qquad \text{(see §1.1)}$$

Notice the difference between (1.43) and (1.44):in the second equation,we have put $b^{(o)} = 1$ to ensure that $\|\cdot\|_r$ is in fact a norm,whereas in (1.43) $b^{(o)} = 0$ (see Prop.1.4) However,both sets of seminorms actually define the same topology.

In conclusion,one should observe that the space $\bigoplus_n S_+^{\otimes n}$, completed in the locally convex direct sum topology (see above) and equipped with the structure of an involutive algebra,has been studied extensively by BORCHERS [9] and other authors.We will not need this extra structure for our treatment of the CCR.

§2. Gibbs states on the CCR.

Having now constructed our domain,we turn to the definition of the representation.Slightly changing our point of view, we consider the algebra $\mathfrak{A}$ of the CCR as an (abstract) involutive algebra, and its realization as continuous linear operators as a representation on $\mathcal{G}$.

Definition

The algebra $\mathfrak{A}$ of the Canonical Commutation Relations is the algebra of all polynomials in $a(\bar{f})$ and $a^+(f)$, $f \in S$, subject to (1.28) and (1.29) (the CCR) . We give $\mathfrak{A}$ an involutive algebra structure by requiring

$$(2.1)\qquad \begin{aligned} a(\bar{f})^{\#} &= a^+(f) \\ a^+(f)^{\#} &= a(\bar{f}) \end{aligned}$$

Note that a and a^+ are complex linear from S to $\mathfrak{A}$!

We consider (1.41) and (1.42) as defining a #-representation π_0 of $\mathfrak{A}$ on $\mathcal{H}_0$ with domain $\mathcal{G}$ (see Chapter I, §1.4) If $\{ e_r, r \in \mathbb{N} \}$ denotes any maximal orthonormal set in S (for example, the Hermite elements) , we easily find that

$$(2.2)\qquad \pi_0(a^+(e_r))\Psi_n(e(n_0,..n_r,..)) = (n_r+1)^{1/2}\Psi_{n+1}(e(n_0,..n_r+1..))$$

$$\pi_0(a(\bar{e}_r))\Psi_n(e(n_0,..n_r,..)) = \begin{cases} (n_r)^{1/2}\Psi_{n-1}(e(n_0,..n_r-1..)) & n_r>0 \\ 0 & n_r=0 \end{cases}$$

Since the mappings

$$\pi_0 \circ a^+ , \quad \pi_0 \circ a \quad : \quad S \longrightarrow L(\mathcal{G})$$

are continuous (because $\mathcal{G}$ is a space of type $\mathfrak{S}$), we see that (2.2) is sufficient to define the representation π_0.

We now turn to the definition of our class of Gibbs states. Let w be a non-negative function on the integers such that there exists a positive integer t such that

$$w(k) \le (k+1)^t \qquad \text{for every } k \in \mathbb{N}.$$

Define the operator h on S as follows:

$$he_k = w(k)e_k \qquad (k \in \mathbb{N})$$

If $f = \sum_{k=1}^{N} a_k e_k \in S$ and $m \in \mathbb{N}$, we have:

$$\|hf\|_m^2 = \| \sum_{k=1}^{N} a_k w(k) e_k \|_m^2 = \sum_{k=1}^{N} |a_k|^2 \, w(k)^2 (k+1)^m$$
$$\leq \sum_{k=1}^{N} |a_k|^2 (k+1)^{m+2t} = \|f\|_{m+2t}^2$$

since $(e_k, e_j)_m = (j+1)^m (e_k, e_j)_o = (j+1)^m \delta_{kj}$

Thus h is continuous with respect to the topology of S, and hence extends to an $h \in L(S)$. Denote by $H \in L(\mathcal{G})$ its second quantization (Prop.1.4)

Now we find that

$$\text{(2.3)} \qquad H\Psi_n(e(n_o,n_1,\ldots)) = (\sum_k w(k) n_k)\, \Psi_n(e(n_o,n_1,\ldots)).$$

We are now able to define, for $x \in \mathcal{O}\!\mathcal{L}$ and $\beta > 0$:

$$\text{(2.4)} \quad w_\beta(x) = \frac{\sum_{n=0}^{\infty} \sum_{J_n} (\Psi_n(e(n_o,n_1,\ldots)), (\exp(-\beta \sum_k w(k) n_k)) \pi_o(x) \Psi_n(e(n_o,\ldots)))}{\sum_{n=0}^{\infty} \sum_{J_n} \exp(-\beta \sum_k w(k) n_k)}$$

where $J_n = \{(n_o, n_1, \ldots) \;:\; \sum_k n_k = n \}$.

Note that
$$w_\beta(x) = \text{"} tr((\exp(-\beta H))\pi_o(x)) \,/\, tr(\exp(-\beta H)) \text{"}$$

THEOREM 2.1

For $x \in \mathcal{O}\!\mathcal{L}$, $\beta > 0$, $w_\beta(x)$ is well defined whenever there exists $\varepsilon > 0$, no matter how small, such that $w(k) \geq (k+\varepsilon)$. w_β is then a faithful state (i.e. a normalised positive definite Hermitian form) on $\mathcal{O}\!\mathcal{L}$.

The proof will follow from the following lemmas.

We first define:

(2.5) $S_n = \sum_{J_n} \exp(-\beta \sum_k kn_k)$ $\quad n \in \mathbb{N}$

(2.6) $S_n^v = \sum_{J_n^v} \exp(-\beta \sum_k kn_k)$ $\quad n, v \in \mathbb{N}$ where

(2.7) $J_n^v = \{(n_o, n_1, \ldots) \in J_n \; : \; n_i = o, \; o \leq i \leq v-1 \}$

(2.8) $S_o^v = 1$ $\quad v \in \mathbb{N}$

<u>Lemma 2.2</u>

$$S_n^v = \sum_{m=v}^{\infty} \sum_{k=1}^{n} (\exp(-km\beta)) S_{n-k}^{m+1} \qquad (n \geq 1, \quad v \geq o)$$

<u>Proof</u> The first non-zero index, namely n_v can take values $n, \ldots, 1, 0$. If $n_v = k$, $0 < k \leq n$, then its contribution to the sum is

$$\sum_{J_n^v \cap \{n_v = k\}} \exp(-\beta \sum_i in_i) = (\exp(-\beta vk)) \sum_{J_{n-k}^v \cap \{n_v = o\}} \exp(-\beta \sum_i in_i) = (\text{change } n_v \text{ to } n_v - k)$$

$$= (\exp(-\beta vk)) \sum_{J_{n-k}^{v+1}} \exp(-\beta \sum_i in_i) = (\exp(-\beta vk)) S_{n-k}^{v+1}$$

Therefore contribution for $n_v > 0$ is

$$\sum_{k=1}^{n} (\exp(-\beta vk)) S_{n-k}^{v+1}.$$

If $n_v = 0$, consider n_{v+1}. By the same argument, the contribution from $n_{v+1} > 0$ is:

$$\sum_{k=1}^{n} (\exp(-\beta(v+1)k)) S_{n-k}^{v+2}.$$

So the total sum is the sum of the contributions for all $n_m > 0$, $m = v, v+1, \ldots,$ which gives the required result. QED.

<u>Corr 2.3</u> $\quad S_n = \sum_{m=0}^{\infty} \sum_{k=1}^{n} (\exp(-\beta mk)) S_{n-k}^{m+1}$.

<u>Lemma 2.4</u>

$$S_n^v = (\exp(-nv\beta)) \prod_{k=1}^{n} (1 - \exp(-k\beta))^{-1} \qquad (n \geq 1)$$

where empty products are defined to be 1.

<u>Proof</u> By induction: $S_1^v = \sum_{m=v}^{\infty} (\exp(-m\beta)) = \dfrac{\exp(-v\beta)}{1 - \exp(-\beta)}$

Suppose true up to n-1 and consider

$$S_n^v = \sum_{m=v}^{\infty} \sum_{k=1}^{n} (\exp(-km\beta)) S_{n-k}^{m+1}$$

$$= \sum_{m=v}^{\infty} \Big(\sum_{k=1}^{n-1} (\exp(-km\beta))(\exp(-(n-k)(m+1)\beta)) \prod_{r=1}^{n-k} (1 - (\exp(-r\beta))^{-1} + \exp(-nm\beta) \Big)$$

$$= \sum_{m=v}^{\infty} (\exp(-nm\beta))(1 + \sum_{k'=1}^{n-1} \exp(-k'\beta) \prod_{r=1}^{k'} (1 - (\exp(-r\beta))^{-1})$$

($k'=n-k$, where we have used the induction hypothesis and the fact that $S_0^{m+1}=1$).

$$= \sum_{m=v}^{\infty} (\exp(-nm\beta)) \prod_{r=1}^{n} (1 - (\exp(-r\beta))^{-1} \qquad \text{as required} \qquad \text{QED.}$$

<u>Proposition 2.5</u>

For all $\beta > 0$, $\exp(-\beta H)$ is a trace class positive operator, that is

$$(2.9) \qquad \sum_{n=0}^{\infty} \sum_{J_n} (\exp(-\beta \sum_k w(k) n_k)) < \infty$$

<u>Proof</u>

$$\sum_{n=0}^{\infty} \sum_{J_n} \exp(-\beta \sum_k w(k) n_k) \leq \sum_{n=0}^{\infty} \sum_{J_n} \exp(-\beta \sum_k (k+\varepsilon) n_k)$$

$$\sum_{n=0}^{\infty} (\exp(-\beta n \varepsilon)) \sum_{J_n} \exp(-\beta \sum_k k n_k) = \sum_{n=0}^{\infty} (\exp(-\beta n \varepsilon) \; S_n =$$

$$\sum_{n=0}^{\infty} (\exp(-\beta n \varepsilon) \prod_{k=1}^{n} (1-\exp(-k\beta))^{-1}$$

by Lemma 2.4. Now the ratio test shows that this sum is finite, for the ratio of the (n+1)th term to the nth

$$\exp(-\beta\varepsilon)(1-\exp(-\beta(n+1)))^{-1} \longrightarrow \exp(-\beta\varepsilon) < 1$$

as n tends to infinity.

QED

Lemma 2.6

Let

$$\Phi_n = (\exp(-(\beta/4)\sum_k w(k)n_k))\,\Psi_n(e(n_o,n_1,\ldots))$$
$$= (\exp(-\beta H/4))\Psi_n$$

The closed convex balanced hull of the Φ_n's is a τ-bounded subset of $\mathcal{G}$.

Proof It is clearly sufficient to show that, given $m \in \mathbb{N}$,

$$\sup\{\|\Phi_n\|_m : (n_o,n_1,\ldots) \in J_n ,\ n \in \mathbb{N}\}$$

is finite. We have

$$\|\Phi_n\|_m^2 = (\exp((-\beta/2)\sum_k w(k)n_k))\,\|\Psi_n\|_m^2 =$$
$$\leq (\exp((-\beta/2)\sum_k (k+\varepsilon)n_k))\ (\sum_k (k+1)n_k)^m$$

(using (1.44)) which are clearly bounded, uniformly in $n,n_o,n_1\ldots$

QED

THEOREM 2.7

$$\exp(-\beta H) : (\mathcal{H}_o, \|\cdot\|_o) \longrightarrow (\mathcal{G}, \tau)$$

is a nuclear mapping (cf. §1.1)

Proof(i) We first show that exp(-ßH) maps $\mathcal{H}_o$ continuously into $\mathcal{G}$. Consider $\mathcal{O}\Psi_o$ equipped with the topology inherited from $\mathcal{H}_o$ (respectively $\mathcal{G}$) as a dense linear subset of $\mathcal{H}_o$ (resp. $\mathcal{G}$). The unit ball of the first space consists of finite convex combinations of $\{\Psi_n : (n_o,n_1,\ldots) \in J_n ,\ n \in \mathbb{N}\}$. By Lemma 2.6, this is sent to a τ-bounded set of the second space by exp(-ßH). Thus exp(-ßH) maps bounded sets to bounded sets. But both spaces are metrizable, and therefore this shows that exp(-ßH) is continuous (TREVES [85], Prop.14.8). Hence it extends to a continuos linear mapping of the completions $\mathcal{H}_o$ & $\mathcal{G}$.

(ii) For any $F \in \mathcal{H}_0$, we have:

$$(\exp(-\beta H))F = \sum_{n=0}^{\infty}\sum_{J_n}(\exp(-\beta\sum_k w(k)n_k))(\Psi_n,F)\Psi_n$$
$$= \sum_{n=0}^{\infty}\sum_{J_n}(\exp((-3\beta/4)\sum_k w(k)n_k))(\Psi_n,F)\Phi_n$$

Now $\sum_{n=0}^{\infty}\sum_{J_n}(\exp((-3\beta/4)\sum_k w(k)n_k)) < \infty$ by Prop.2.5, the Φ_n lie in a convex balanced complete bounded set in $\mathcal{G}$ by Lemma 2.6 and the $\Psi_n \in \mathcal{H}_0 = \mathcal{H}_0'$ are uniformly bounded, hence equicontinuous Therefore (see §1.1) $\exp(-\beta H)$ is nuclear.

QED

Corollary 2.8

(i) "$|tr((\exp(-\beta H))\pi_0(x))|$" $< \infty$ for all $x \in \mathcal{O}\!\mathcal{l}$.

(ii) $\sum_{n=0}^{\infty}\sum_{J_n}(\exp(-\beta\sum_k w(k)n_k))(\sum_k(k+1)n_k)^m < \infty$ for all $m \in \mathbb{N}$.

Proof Let $A \in L(\mathcal{G})$ denote either $\pi_0(x)$ ($x \in \mathcal{O}\!\mathcal{l}$) or B^m (see (1.43)). Then $A \cdot \exp(-\beta H)$ is the composite

$$\mathcal{H}_0 \xrightarrow{\exp(-\beta H)} (\mathcal{G},\tau) \xrightarrow{A} (\mathcal{G},\tau) \longrightarrow \mathcal{H}_0$$

the last arrow denoting the identity mapping, which is continuous, since τ is finer than the $\|\cdot\|_0$ topology. Since the nuclear mappings are a bimodule over the continuous mappings (TREVES [85] Prop. 47.1), this composite mapping is nuclear, i.e. trace class, as the spaces involved are Hilbert spaces. The first assertion of the Corollary is now immediate, while the second follows from

$$B^m\Psi_n = (\sum_k(k+1)n_k)^m\Psi_n$$

QED

We can now complete the proof of theorem 2.1. Prop.2.5 and Cor.2.8 (i) show that $w_\beta(x)$ is well defined for all $x \in \mathcal{O}\!\mathcal{l}$ and $w_\beta(1) = 1$. Simple calculations show that

$w_\beta(x^\#) = \overline{w_\beta(x)}$ and

$w_\beta(x^\# x) \geq 0$

Finally, if $w_\beta(x^\# x) = 0$, then

$\| \pi_o(x)\Psi_n(e(n_o,n_1,\ldots))\| = 0$ for all $n, n_o, n_1, \ldots$

which, in view of the fact that such vectors are total in $\mathcal{G}$, implies that $x=0$.

Therefore w_β is a well defined state on $\mathfrak{A}$.

§3. Having constructed our state w_β, we shall now show that $(\mathfrak{A}, \#, w_\beta)$ is a closable probability algebra. The final aim, as explained in the introduction of this Chapter, is to prove a commutation theorem for $\mathfrak{A}$, analogous to Thm.3.3 of Chapter III. Following TOMITA's original method (see Chapter III §3), we shall construct an "almost" modular Hilbert subalgebra of $\mathfrak{A}$, equipped with a modular automorphism group. We shall prove that w_β is a KMS state with respect to this group (see Chapter III, §4)

Definition

An Almost Modular Hilbert Algebra is an involutive algebra with an inner product and a complex one-parameter automorphism group satisfying all the properties of the definition of a modular Hilbert algebra (see Chapter III, §3) except for continuity of multiplication.

We define the subalgebra $\tilde{\mathfrak{A}}$ of $\mathfrak{A}$ to be the one (algebraically) generated by $\{ a(\bar{e}_r), a^+(e_r) : r \in \mathbb{N} \}$.

Proposition 3.1

$\tilde{\mathfrak{A}}$ is dense in $\mathfrak{A}$ in the topology inherited from $L(\mathcal{G})$ via π_o.

<u>Proof</u> (i) We observe that the map

$$S\times S \longrightarrow (\mathfrak{A}, \text{ topology inherited from } L(\mathcal{G}))$$

$$(f,g) \longmapsto a^{\natural}(f)a^{\natural}(g) \quad (\text{where } a^{\natural} \text{ stands for } a$$

or a^{+}) is separately continuous.

(ii) Since S is a Fréchet space, the above map is jointly continuous (TREVES [85], Corr of thm 34.1).

(iii) For $f = \sum_{r=0}^{\infty} c_r e_r$, $g = \sum_{r=0}^{\infty} d_r e_r \in S$

$a^{\natural}(f_N)a^{\natural}(g_M) \in \tilde{\mathfrak{A}}$ approximates $a^{\natural}(f)a^{\natural}(g) \in \mathfrak{A}$, where

$f_N = \sum_{r=0}^{N} c_r e_r$, $g_M = \sum_{r=0}^{M} d_r e_r$. It now follows by induction that

$x = a^{+}(f_1)\ldots a^{+}(f_n)a(\overline{g}_1)\ldots a(\overline{g}_m)$ can be approximated by elements of $\tilde{\mathfrak{A}}$ in the topology inherited from $L(\mathcal{G})$.

Q.E.D.

<u>THEOREM 3.2</u>

Let $z \in \mathbb{C}$, $f \in \mathcal{R}e_0 \subset S$ (See (1.8)) .Define

$$(3.1)\quad \begin{cases} \Delta(z)\ a^{+}(f) = a^{+}((\exp(-\beta z h))f) \\ \Delta(z)\ a(\overline{f}) = a((\exp(\beta z h))\overline{f}) \\ \Delta(z)\ 1 = 1 \end{cases}$$

Then $\Delta(z)$ is well defined on $\tilde{\mathfrak{A}}$, and $\Delta(.)$ is a one-parameter complex automorphism group of $\tilde{\mathfrak{A}}$. Equipped with the inner product induced by w_β and the modular automorphism group $\{\Delta(z) : z \in \mathbb{C}\}$, $\tilde{\mathfrak{A}}$ is an Almost Modular Hilbert algebra. That is , we have:

(3.2) $(xy,u) = (y,x^{\#}u)$

(3.3) $\tilde{\mathfrak{A}}^2$ is dense in $\mathfrak{A}$

(3.4) $(\Delta(z)x)^{\#} = \Delta(-\overline{z})x^{\#}$

(3.5) $(\Delta(z)x,y) = (x,\Delta(\overline{z})y)$

(3.6) $(\Delta(1)x,y) = (y^{\#},x^{\#})$

(3.7) $z \longmapsto (x,\Delta(z)y)$ is entire

(3.8) For each $t \in \mathbb{R}$, $\tilde{\mathfrak{A}}_t = \{(1+\Delta(t))x : x \in \tilde{\mathfrak{A}}\}$ is dense in $\tilde{\mathfrak{A}}$.

Proof It is clear that $\Delta(z)$ is well defined on $\tilde{\mathfrak{A}}$ by (3.1) (where $(\exp(\beta zh))e_r := (\exp(\beta w(r)z))e_r$). Observe that $\Delta(z)$ preserves the CCR (1.28 & 1.29) on $\tilde{\mathfrak{A}}$, hence defines a unique automorphism on $\tilde{\mathfrak{A}}$. It is easy to check (3.2) and (3.4), while (3.3) is trivial, since $1 \in \tilde{\mathfrak{A}}$.

To prove (3.5), I first claim that

(3.9) $\pi_0(\Delta(z)x)\Psi_n = (\exp(-\beta zH))\,\pi_0(x)(\exp\beta zH)\Psi_n \qquad (x \in \tilde{\mathfrak{A}})$

where $(\exp\beta zH)\Psi_n := (\exp(\beta z\sum_k w(k)n_k))\Psi_n$, and therefore the right hand side of (3.9) is well defined, since $\pi_0(x)\Psi_n$ is a finite linear combination of Ψ_n's. Since both sides of (3.9) preserve the CCR, it is sufficient to verify equality on generating elements of $\tilde{\mathfrak{A}}$. We have

$(\exp(-\beta zH))\,\pi_0(a(\bar{e}_r))(\exp z\beta H)\Psi_n =$

$(\exp(-\beta zH))\,\pi_0(a(\bar{e}_r))(\exp(z\beta\sum_k w(k)n_k))\Psi_n(e(n_0,\ldots n_r,\ldots)) =$

$(\exp(-\beta zH))(n_r)^{1/2}\,(\exp(z\beta\sum_k w(k)n_k))\Psi_{n-1}(e(n_0,\ldots n_r-1,\ldots)) =$

$(\exp(-\beta z(\sum_k w(k)n_k - w(r))))(n_r)^{1/2}(\exp(z\beta\sum_k w(k)n_k))\Psi_{n-1}(e(n_0,\ldots n_r-1\ldots)) =$

$(\exp\beta zw(r))\,(n_r)^{1/2}\Psi_{n-1}(e(n_0,\ldots n_r-1,\ldots)) =$

$(\exp\beta zw(r))\,\pi_0(a(\bar{e}_r))\Psi_n(e(n_0,\ldots n_r,\ldots)) = \pi_0(\Delta(z)a(\bar{e}_r))\Psi_n$

if $n_r > 0$, and if $n_r = 0$ then both sides are equal to 0.

A similar calculation shows that (3.9) is valid with $x = a^+(e_r)$ and therefore it is valid for all $x \in \tilde{\mathfrak{A}}$.

Using (3.9), we see that

$(\pi_o(\Delta(z)x)\Psi_n, \pi_o(y)\Psi_n) =$

$((\exp(-\beta zH))\pi_o(x)(\exp(\beta zH))\Psi_n, \pi_o(y)\Psi_n) =$

$(\exp(\beta\bar{z}\sum_k w(k)n_k))((\exp(-\beta zH))\pi_o(x)\Psi_n, \pi_o(y)\Psi_n) =$

$((\exp(-\beta zH))\pi_o(x)\Psi_n, \pi_o(y)\ (\exp(\beta\bar{z}H))\Psi_n) =$

$(\pi_o(x)\Psi_n, (\exp(-\beta\bar{z}H))\pi_o(y)\ (\exp(\beta\bar{z}H))\Psi_n) = (\pi_o(x)\Psi_n, \pi_o(\Delta(\bar{z})y)\Psi_n)$

and therefore (3.5) follows using (2.4) and the definition of the inner product on $\tilde{\mathfrak{A}}$.

To prove (3.6), let $x \in \underline{\mathfrak{A}}$ and $r \in \mathbb{N}$. Consider

$\operatorname{tr}(\exp(-\beta H))\ (\Delta(1)a(\bar{e}_r), x) =$

$$\sum_{n=0}^{\infty}\sum_{J_n}(\exp(-\beta\sum_k w(k)n_k))(\pi_o(\Delta(1)a(\bar{e}_r))\Psi_n, \pi_o(x)\Psi_n) =$$

$$\sum_n\sum_{J_n}(\exp(-\beta\sum_k w(k)n_k))(\exp\beta w(r))\ (\pi_o(a(\bar{e}_r))\Psi_n, \pi_o(x)\Psi_n) =$$

$$\sum_n\sum_{J_n}(\exp(-\beta(\sum_k w(k)n_k - w(r))))(n_r)^{1/2}(\Psi_{n-1}(e(\dots n_r-1\dots)), \pi_o(x)\Psi_n) =$$

$$\sum_n\sum_{J_n}(\exp(-\beta(\sum_k w(k)n_k)))(n_r+1)^{1/2}(\Psi_n(e(\dots n_r\dots)), \pi_o(x)\Psi_{n+1}(e(\dots n_r+1\dots))) =$$

(changing n_r to m_r+1, since $n_r=0$ does not contribute to the sum)

$$\sum_n\sum_{J_n}(\exp(-\beta\sum_k w(k)m_k))\ (n_r+1)^{1/2}(\pi_o(x^{\#})\Psi_n, \Psi_{n+1}(e(\dots n_r+1\dots))) =$$

$$\sum_n\sum_{J_n}(\exp(-\beta\sum_k w(k)n_k))\ (\pi_o(x^{\#})\Psi_n, \pi_o(a^+(e_r))\Psi_n) =$$

$\operatorname{tr}(\exp(-\beta H))\ (x^{\#}, a(\bar{e}_r)^{\#})$.

Therefore

$$(\Delta(1)a(\bar{e}_r), x) = (x^{\#}, a(\bar{e}_r)^{\#})$$

and similarly

$$(\Delta(1)a^+(e_r), x) = (x^{\#}, a^+(e_r)^{\#})$$

and the general case follows from these, as $\Delta(1)$ is an automorphism of $\tilde{\mathfrak{A}}$.

(3.7) follows from the calculation

$(x,\Delta(z)a^+(e_{j1})\ldots a^+(e_{jm})a(\bar{e}_{k1})\ldots a(\bar{e}_{kn})) =$

$(x,\prod_{r=1}^{m}\Delta(z)a^+(e_{jr})\prod_{s=1}^{n}\Delta(z)a(\bar{e}_{ks})) =$

$\prod_{r=1}^{m}(\exp(-\beta zw(jr)))\prod_{s=1}^{n}(\exp(\beta zw(ks)))\ (x,\prod_{r=1}^{m}a^+(e_{jr})\prod_{s=1}^{n}a(\bar{e}_{ks})) =$

$(\exp(-\beta z(\sum_{r=1}^{m}w(jr)-\sum_{s=1}^{n}w(ks))))\ (x,\prod_{r=1}^{m}a^+(e_{jr})\prod_{s=1}^{n}a(\bar{e}_{ks}))$

Thus for all $x \in \underline{\mathfrak{A}}$, $y \in \tilde{\mathfrak{A}}$ the function

$$z \longmapsto (x,\Delta(z)y)$$

is entire, since every $y \in \tilde{\mathfrak{A}}$ may be expressed as a finite linear combination of elements of the form

(3.10) $$y = a^+(e_{j1})\ldots a^+(e_{jm})a(\bar{e}_{k1})\ldots a(\bar{e}_{kn})$$

Finally, let $y \in \tilde{\mathfrak{A}}$ be given by (3.10) .Putting

$$x = (1+\exp(-\beta t(\sum_{r=1}^{m}w(jr)-\sum_{s=1}^{n}w(ks))))^{-1}y \in \tilde{\mathfrak{A}} \qquad (t \in \mathbb{R})$$

we see that

$$y = (1+\Delta(t))x \in \tilde{\mathfrak{A}}_t$$

This shows that in fact $\tilde{\mathfrak{A}}_t = \mathfrak{A}$,so that (3.8) holds. This completes the proof of the Theorem.

<u>Lemma 3.3</u>

$x \longmapsto w_\beta(x^{\#}x)^{1/2}$ is a continuous norm on $\mathfrak{A}$ equipped with the topology inherited from $L(\mathcal{G})$.Hence the (pre-)Hilbert space topology on $\mathfrak{A}$ defined by w_β is weaker than the one inherited from $L(\mathcal{G})$.

<u>Proof</u> Let (x_i) be a net in $\mathfrak{A}$ such that $\pi_o(x_i) \longrightarrow 0$ in the topology of $L(\mathcal{G})$. Thus $\pi_o(x_i) \longrightarrow 0$ uniformly on any τ-bounded set of $\mathcal{G}$ .Therefore, if Φ_n is as in Lemma 2.6, for each positive ε and each sufficiently large i ,we have :

$$\|\pi_\circ(x_i)\varphi_n\|^2 < \varepsilon \qquad \text{for all } n, n_o, n_1, \ldots$$

Now

$$(\operatorname{tr}(\exp(-\beta H)))\ \|x_i\|^2 = (\operatorname{tr}(\exp(-\beta H)))\ w_\beta(x_i^\# x_i) =$$

$$= \sum_n \sum_{J_n} (\exp(-\beta \sum_k w(k)n_k))\ \|\pi_\circ(x_i)\Psi_n\|^2 =$$

$$= \sum_n \sum_{J_n} (\exp((-\beta/2)\sum_k w(k)n_k))\ \|\pi_\circ(x_i)\varphi_n\|^2 <$$

$$< \sum_n \sum_{J_n} (\exp((-\beta/2)\sum_k w(k)n_k))\ \varepsilon$$

$$= (\operatorname{tr}(\exp(-\beta H/2)))\ \varepsilon$$

Thus $\|x_i\| \longrightarrow 0$.

QED

Lemma 3.4

(i) For $\operatorname{Re} z \leq 0$, $\exp(\beta zh) \in L(S)$ is well defined, and $z \longmapsto (\exp(\beta zh))f$ is analytic for all $f \in S$.

(ii) Define

$$(3.11)\ \begin{cases} \Delta(z)a^+(f) = a^+((\exp(-\beta zh))f) & \operatorname{Re} z \geq 0,\ f \in S \\ \Delta(z)a(\bar{f}) = a((\exp\beta zh)\bar{f}) & \operatorname{Re} z \leq 0,\ f \in S \end{cases}$$

Then (3.4), (3.5) and (3.6) are valid for all $x, y \in \mathfrak{A}$, $z \in \mathbb{C}$ for which they are meaningful. In particular, $\{\sigma_t = \Delta(it) : t \in \mathbb{R}\}$ is a group of isometric automorphisms of $\mathfrak{A}$.

Proof (i) We first observe that the map

$$z \longmapsto (\exp\beta zh)e_r = (\exp\beta zw(r))e_r$$

is analytic into S. If $f \in S$ is arbitrary, $f = \sum_{r=0}^{\infty} c_r e_r$,

$$\|\sum_{r=N}^{M} c_r(\exp\beta zh)e_r\|_m^2 = \sum_{r=N}^{M} |c_r|^2 (r+1)^{2m} \exp 2\beta \operatorname{Re} z w(r) \underset{N, M \to \infty}{\longmapsto} 0$$

uniformly in z in the strip $\operatorname{Re} z \leq 0$.

Hence $\sum_{r=0}^{\infty} c_r(\exp\beta zh)e_r := (\exp\beta zh)f$ converges in S inside this strip, and defines an analytic function of z for $\operatorname{Re} z < 0$, which is continuous on the boundary.

(ii) Part (i) shows that the definitions are meaningful, and define analytic functions into $(\mathfrak{A}, \| \ \|)$. This is because of (i) and because

$$S \longrightarrow L(\mathcal{G})$$

$$f \longmapsto \pi_0(a(\bar{f})) \qquad \text{resp.} \qquad f \longmapsto \pi_0(a^+(f))$$

is continuous and hence

$$S \longrightarrow (\mathfrak{A}, \| \ \|)$$

$$f \longmapsto a(\bar{f}) \qquad \text{resp.} \qquad f \longmapsto a^+(f)$$

is continuous by Lemma 3.3.

It is immediate from the definitions that

$$(\Delta(z)a(\bar{f}))^{\#} = \Delta(-\bar{z})a^+(f) \qquad \text{for Re} z \leq 0 \text{, and}$$

$$(\Delta(z)a^+(f))^{\#} = \Delta(-\bar{z})a(\bar{f}) \qquad \text{for Re} z \geq 0$$

It is also clear that, for each $z \in \mathbb{C}$, $\Delta(z)$ preserves the commutation relations between elements of $\mathfrak{A}$ in its domain. This already shows that $\{ \sigma_t = \Delta(it) \ : t \in \mathbb{R} \}$ is an automorphism group of $\mathfrak{A}$, since it is everywhere defined. It also shows that (3.4) follows from the above equalities, whenever both sides of the equation make sence.

We can show, for example, that

$$(\Delta(z)a(\bar{f}), a(\bar{g})) = (a(\bar{f}), \Delta(\bar{z})a(\bar{g})) \qquad \text{for Re} z \leq 0$$

by using the definition of the Lemma and the fact that $f, g \in S$ can be approximated, in the topology of S, by finite linear combinations of e_r's. The general case now follows from the observations of the previous paragraph.

We have seen in the proof of Thm.3.2 that (3.6) is in fact valid for all $x \in \mathfrak{A}$ and $y \in \widetilde{\mathfrak{A}}$. If we approximate a general

$y = a^+(f_1)\ldots a^+(f_n) \in D(\Delta(1))$ by the corresponding element y' of $\tilde{\mathfrak{A}}$ as in Prop.3.1, we see that $y^{\#} = a(\bar{f}_n)\ldots a(\bar{f}_1)$ is approximated by $y'^{\#}$, and $\Delta(1)y = a^+((\exp(-\beta h))f_1)\ldots a^+((\exp(-\beta h))f_n)$ is approximated by $\Delta(1)y'$, since $\exp(-\beta h) \in L(S)$. The validity of (3.6) is thus established.

QED

THEOREM 3.5

w_β is invariant under the automorphisms σ_t of $\mathfrak{A}$, and satisfies the KMS-condition wrt σ_t on $\mathfrak{A}$. That is, for every $x,y \in \mathfrak{A}$, there is a function $F_{x,y}$ continuous and uniformly bounded on the strip $\{z \in \mathbb{C} : 0 \leq \operatorname{Im} z \leq 1\}$, and analytic inside this strip, such that

$$F_{x,y}(t) = w_\beta(\sigma_t(x)y)$$

$$F_{x,y}(t+i) = w_\beta(y\sigma_t(x)).$$

Proof Invariance follows from the calculation

$$w_\beta(\sigma_t(x)) = (1,\sigma_t(x)) = (\sigma_{-t}(1), x) = (1,x) = w_\beta(x).$$

For $x = a^+(f_1)\ldots a^+(f_n)a(\bar{g}_1)\ldots a(\bar{g}_m) \in \mathfrak{A}$, $y \in \mathfrak{A}$, define

$$F_{x,y}(z) = w_\beta(\Delta(iz)(a(\bar{g}_1)\ldots a(\bar{g}_m))y\Delta(iz+1)(a^+(f_1)\ldots a^+(f_n)))$$

It is clear from Lemma 3.4 that $F_{x,y}$ is well defined in the strip $0 \leq \operatorname{Im} z \leq 1$, that it is continuous, and analytic in the interior. Hence it must attain its maximum modulus on the boundary. For real t, we have:

$$F_{x,y}(t) = ((\Delta(it)(a(\bar{g}_1)\ldots a(\bar{g}_m))y)^{\#}, \Delta(1)\Delta(it)(a^+(f_1)\ldots a^+(f_n)))$$

$$= ((\Delta(it)(a^+(f_1)\ldots a^+(f_n)))^{\#}, \Delta(it)(a(\bar{g}_1)\ldots a(\bar{g}_m))y)$$

$=w_\beta((\Delta(it)x)y)$

and

$$F_{x,y}(t+i)=((\Delta(it-1)(a(\bar{g}_1)\ldots a(\bar{g}_m)))^{\#},y\Delta(it)(a^+(f_1)\ldots a^+(f_n)))$$

$$=((y\Delta(it)(a^+(f_1)\ldots a^+(f_n)))^{\#},\Delta(1)\Delta(it-1)(a(\bar{g}_1)\ldots a(\bar{g}_m)))$$

$$=(y^{\#},\Delta(it)(a^+(f_1)\ldots a^+(f_n))\Delta(it)(a(\bar{g}_1)\ldots a(\bar{g}_m)))$$

$$=w_\beta(y\Delta(it)x)$$

as required. Finally, the inequalities

$$|F_{x,y}(t)| = |(\sigma_t(x)^{\#},y)|=|(\sigma_t(x^{\#}),y)|\leq \|x^{\#}\|.\|y\|$$

and

$$|F_{x,y}(t+i)|=|(y^{\#},\sigma_t(x))|\leq \|y^{\#}\|.\|x\| \quad \text{(since } \sigma_t \text{ is unitary)}$$

show that $F_{x,y}$ is uniformly bounded in the strip.

This concludes the proof.

Definition

For $x\in\tilde{\mathfrak{A}}$ define $\flat$ by $x^\flat := \Delta(1)x^{\#}$

and $*$ by $x^* := \Delta(\frac{1}{2})x^{\#}$.

Observe that $\Delta(z)$ is defined for all z on both $a^+(f)$ and $a(\bar{f})\in\tilde{\mathfrak{A}}$ as there are no problems of convergence, and hence the above definitions make sense. Formula (3.6) shows that $*$ is isometric on $\tilde{\mathfrak{A}}$, hence extends to a conjugate linear isometry J of $\mathcal{H}$, the completion of $(\tilde{\mathfrak{A}},\|\ \|)$ (which is also the completion of $(\mathfrak{A},\|\ \|)$ by prop 3.1, and lemma 3.3.

Note that $\flat$ and $*$ are both involutions on $\tilde{\mathfrak{A}}$, called the adjoint and unitary involutions resp.. Note also that w_β is a faithful state on the involutive algebra $(\tilde{\mathfrak{A}},\flat)$.

The following result is now immediate

THEOREM 3.6

$(\mathfrak{A}, \#, w_\beta)$ is a closable probability algebra (see Chapter III, §2).

Proof For $x \in \mathfrak{A}$, $y \in \tilde{\mathfrak{A}}$, we have, by (3.6)

$$(x, y^\flat) = (x, \Delta(1)y^\#) = (y, x^\#)$$

Thus the mapping

$$\mathfrak{A} \ni x \longmapsto (y, x^\#)$$

is continuous for all $y \in \tilde{\mathfrak{A}}$, and hence the involution has a densely defined adjoint, i.e. is closable.

QED

This conclues our study of the probability algebra $(\mathfrak{A}, \#, w_\beta)$ and the almost modular Hilbert algebra $(\tilde{\mathfrak{A}}, \#, w_\beta, \Delta(z))$. In the final Chapter, we will use this latter algebra to prove our commutation theorem.

CHAPTER VI

TOMITA TAKESAKI THEORY ON THE ALGEBRA OF THE CCR

The aim of this Chapter is to prove the analogue of Theorem 3.3 of Chapter III for the algebra of the CCR. Specifically, if we denote by π_β the closure of the GNS representation induced by w_β (see Chapter I, §1.4), we would like to prove that $\pi_\beta(\mathfrak{A})$ is, in some sense, isomorphic to its commutant. Now $\pi_\beta(\mathfrak{A})$ consists of unbounded operators, and thus not a VN algebra. However we shall show that its commutant $\pi_\beta(\mathfrak{A})'$ is a VN algebra, and

$$J\pi_\beta(\mathfrak{A})'J = \pi_\beta(\mathfrak{A})''$$

where J is the (anti)unitary involution defined at the end of Chapter V.

We will follow the original method of TOMITA, by first proving our commutation theorem for the almost modular Hilbert algebra $\tilde{\mathfrak{A}}$ constructed in Chapter V. For this algebra, the right regular representation is easily seen to extend to a closed (unbounded) b-antirepresentation ρ_β of $\tilde{\mathfrak{A}}$ on $\mathcal{H}$.

Specifically, we equip $\tilde{\mathfrak{A}}$ with the <u>left</u> (respectively <u>right</u>) <u>induced topology</u> λ (resp. δ) given by the seminorms

$$\{ {}_x\|\cdot\| , x \in \tilde{\mathfrak{A}} \} \qquad (\text{resp.} \{ \|\cdot\|_x , x \in \tilde{\mathfrak{A}} \})$$

given by

$${}_x\|y\| = \|\pi_\beta(x)y\| = \|xy\| \ (\text{resp. } \|y\|_x = \|\rho_\beta(x)y\| = \|yx\|)$$

The domains of the representations π_β and ρ_β will be

$D(\pi_\beta) = \overline{(\tilde{\mathfrak{A}},\lambda)}$ and $D(\rho_\beta) = \overline{(\tilde{\mathfrak{A}},\delta)}$ (the bars denoting completions).

In order to prove that the commutants of these representations are commutants of each other(analogously to Thm. 3.1 of Chapter III),we need to put these representations in a different form,in which the structure of their commutants can be studied.

Thus the program for this Chapter is first to construct a new representation of $\mathfrak{A}$ (and hence of $\tilde{\mathfrak{A}}$) for which the commutant theorem can be proved, and then to show that this representation is unitarily equivalent (see Chapter I,§1.4) to π_β. It is interesting to note that this method is similar to the method used by HHW[26] in their original paper ; the important difference is,of course,the unbounded character of our representation,which this time creates unexpected problems.Some of the results of this Chapter are taken from joint work with Ingeborg Koch (see[37]).The same problem in the one dimensional case was treated by GUDDER & HUDSON [22] .

The new representation we will define is essentially left multiplication by $\pi_0(x)$,acting on the Hilbert space of all Hilbert-Schmidt operators on Fock space.This representation being unbounded,again the problem of choosing a suitable domain arises . We begin with a study of this domain .

§1.1 Let $\mathcal{K}$ be a separable Hilbert space. The algebraic tensor product $s^n \otimes \overline{\mathcal{K}}$ may be identified with the set of all finite rank operators from $\mathcal{K}$ into s^n. We equip this space with the topology given by all the norms $\{ p_m^n , m \in \mathbb{N} \}$, where p_m^n is the norm on the Hilbert space $s_m^n \hat{\otimes} \overline{\mathcal{K}}$ of all Hilbert-Schmidt operators from $\mathcal{K}$ into the completion s_m^n of s^n in the $\| \|_m$ norm (see Chapter V, §1.1). We let $s^n \hat{\otimes} \overline{\mathcal{K}}$ be the completion of $s^n \otimes \overline{\mathcal{K}}$ in this topology. It is clear that $s^n \hat{\otimes} \overline{\mathcal{K}}$ is a countably Hilbert space, and that

$$s^n \hat{\otimes} \overline{\mathcal{K}} = \bigcap_m s_m^n \hat{\otimes} \overline{\mathcal{K}} = \underleftarrow{\lim} \, s_m^n \hat{\otimes} \overline{\mathcal{K}}$$

Note that, for $T \in s^n \hat{\otimes} \overline{\mathcal{K}}$, $p_0(T)$ is the usual Hilbert-Schmidt norm. Thus $s^n \hat{\otimes} \overline{\mathcal{K}}$ may be embedded as a dense subspace in $l^2(\mathbb{N}^n) \hat{\otimes} \overline{\mathcal{K}}$. Clearly, under this embedding, each $T \in s^n \hat{\otimes} \overline{\mathcal{K}}$ has range in each s_m^n, hence in s^n. Moreover T is continuous as a mapping from $\mathcal{K}$ into s_m^n for every m. Therefore

$$T : (\mathcal{K}, \| \|) \longrightarrow (s^n, \tau_n)$$

is continuous, where τ_n denotes the usual topology of s^n.

Conversely,

<u>Proposition 1.1</u>

Every bounded operator $S : \mathcal{K} \longrightarrow l^2(\mathbb{N}^n)$ with range in s^n is in $s^n \hat{\otimes} \overline{\mathcal{K}}$.

<u>Proof</u> (based on a comment of WORONOWICZ [90])

We show S has closed graph in $(\mathcal{K}, \| \|) \times (s^n, \tau_n)$

If $\|x_n - x\| \longrightarrow 0$, and $\|Sx_n - y\|_m \longrightarrow 0$ for each $m \in \mathbb{N}$, then in particular $\|Sx_n - y\|_0 \longrightarrow 0$ and hence $Sx = y$, because S, being bounded, has closed graph in $(\mathcal{K}, \| \|) \times (l^2(\mathbb{N}^n), \| \|_0)$.

Since both $\mathcal{K}$ and s^n are complete and metrizable, the closed graph theorem (TREVES [85], Cor.4 of Thm.17.1) implies that

$$S : (\mathcal{K}, \| \|) \longrightarrow (s^n, \tau_n)$$

is continuous.

But s^n is a nuclear space (Chapter V, §1.1) and hence, given $m \in \mathbb{N}$, there exists $r \in \mathbb{N}$ such that the identity ι on s^n extends to a nuclear operator from s^n_r to s^n_m. Thus

$$S : (\mathcal{K}, \| \|) \longrightarrow (s^n, \| \|_r) \xrightarrow{\ \iota\ } (s^n_m, \| \|_m)$$

is nuclear, hence Hilbert-Schmidt. Hence $S \in s^n_m \hat{\otimes} \overline{\mathcal{K}}$ for all $m \in \mathbb{N}$, and therefore $S \in s^n \hat{\otimes} \overline{\mathcal{K}}$.

QED

Proposition 1.2

Consider the algebraic direct sum $\bigoplus_{v \in \mathbb{N}^n} \mathcal{K}$ (i.e. the set of all mappings $\mathbb{N}^n \ni v \longmapsto a_v \in \mathcal{K}$ which are 0 almost everywhere, with pointwise linear operations), equipped with the topology defined by the norms $\{ \||\cdot\||_m , m \in \mathbb{N} \}$, where

$$\||(a_v)\||_m^2 = \sum_{v \in \mathbb{N}^n} (|v| + n)^m \|a_v\|^2$$

The mapping

$$\bigoplus_{v \in \mathbb{N}^n} \mathcal{K} \longrightarrow s^n \otimes \overline{\mathcal{K}}$$

$$(a_v) \longmapsto \sum_{v \in \mathbb{N}^n} e_v \otimes \bar{a}_v \qquad (e_v)_{v'} = \begin{cases} 1 & , v' = v \\ 0 & , v' \neq v \end{cases}$$

extends to a linear topological isomorphism from the completion $s^n(\mathcal{K})$ of $\bigoplus_{v \in \mathbb{N}^n} \mathcal{K}$ onto $s^n \hat{\otimes} \overline{\mathcal{K}}$.

Proof Let $m \in \mathbb{N}$. We have

$$\||(a_v)\||_m^2 = \sum_{v \in \mathbb{N}^n} (|v| + n)^m \|a_v\|^2 = \sum_{v, v'} (|v| + n)^m (e_v, e_{v'})_0 (\bar{a}_v, \bar{a}_{v'})$$

$$= \sum_{v, v'} (e_v, e_{v'})_m (\bar{a}_v, \bar{a}_{v'}) = p^n_m \Big(\sum_{v \in \mathbb{N}^n} e_v \otimes \bar{a}_v \Big)^2$$

Therefore the mapping preserves each m-norm, and thus it is bicontinuous and injective. To show that it is surjective, it is enough to show that its range contains the rank one operators. Now each $c \in s^n$ may be written

$$c = \sum_{v \in \mathbb{N}^n} c_v e_v$$

the sum converging in the topology τ_n. Thus for all $a \in \mathcal{K}$ the finite sums

$$\sum_v c_v e_v \otimes \bar{a}$$

converge to $c \otimes a$ in the topology of $s^n \hat{\otimes} \overline{\mathcal{K}}$, and they are the images of $(\bar{c}_v a) \in s^n(\mathcal{K})$.

QED

The space $s^n(\mathcal{K})$ may be termed the sequence representation of $s^n \hat{\otimes} \overline{\mathcal{K}}$, in the same way as s^n is called the sequence representation of $S(\mathbb{R}^n)$. But $s^n \hat{\otimes} \overline{\mathcal{K}}$ also has a function representation. We see this as follows:

By Prop.1.1, every $T \in s^n \hat{\otimes} \overline{\mathcal{K}}$ gives rise to a bounded operator from $\mathcal{K}$ into s^n; thus for all $a \in \mathcal{K}$, $Ta \in s^n = S(\mathbb{R}^n)$ (supressing the isomorphism). For each $t \in \mathbb{R}^m$, the mapping

$$a \longmapsto (Ta)(t)$$

is continuous on $\mathcal{K}$, since $T: \mathcal{K} \longrightarrow S(\mathbb{R}^n)$ is continuous and τ_n is finer than the topology of pointwise convergence on S^n. Therefore by the Riesz representation theorem (REED & SIMON [53], Thm.II.4) there is a unique $b(t) \in \mathcal{K}$ such that

$$(b(t), a) = (Ta)(t) \qquad a \in \mathcal{K}$$

Thus each $T \in s^n \hat{\otimes} \overline{\mathcal{K}}$ uniquely defines $b \in S(\mathbb{R}^n, \mathcal{K})$, the space of all functions $b: \mathbb{R}^n \longrightarrow \mathcal{K}$ such that for each $a \in \mathcal{K}$, the function $t \longmapsto (b(t), a)$ is in $S(\mathbb{R}^n)$.

Conversely let $b \in S(\mathbb{R}^n, \mathcal{K})$. Define

$$T : \mathcal{K} \longrightarrow S(\mathbb{R}^n)$$

by

$$(Ta)(t) = (b(t), a) \qquad a \in \mathcal{K}\ ,\ t \in \mathbb{R}^n$$

First observe that the weak differentiability of $t \mapsto b(t)$ implies strong differentibility. In fact, if $t^i_r \xrightarrow[r\to\infty]{} t^i \in \mathbb{R}$, the sequence

$$b^i_r = (t^i_r - t^i)^{-1}(b(t_r) - b(t)) \in \mathcal{K}$$

converges weakly to $(\partial b/\partial t^i) \in \mathcal{K}$ (where $t = (t^1, \ldots t^n) \in \mathbb{R}^n$ and $t_r = (t^1, \ldots t^i_r \ldots t^n) \in \mathbb{R}^n$). Therefore, by the uniform boundedness principle (REED & SIMON [53] Thm.III.9), it converges strongly. This shows that, if $m \in \mathbb{N}$, we have

$$\|Ta\|^2_m = \int_{\mathbb{R}} |P_m(t^i, \partial_i)(b(t), a)|^2 dt =$$
$$= \int_{\mathbb{R}} |(P_m(t^i, \partial_i) b(t), a)|^2 dt \leq$$
$$\leq \|a\|^2 \int_{\mathbb{R}} \|P_m(t^i, \partial_i) b(t)\|^2 dt$$

where $P_m(t^i, \partial_i)$ is the diff.operator defining the m-norm.

Hence $$T : \mathcal{K} \longrightarrow (S(\mathbb{R}^n), \tau_n)$$

is continuous. Thus we have proved:

Proposition 1.3

$S^n \hat{\otimes} \overline{\mathcal{K}}$ is isomorphic to $S(\mathbb{R}^n, \mathcal{K})$.

§1.2 We may now define the domain of our new representation. We let $\mathcal{G} \hat{\otimes} \overline{\mathcal{K}}$ be the completion of the algebraic tensor product in the topology given by the norms $\{p_m\ ,\ m \in \mathbb{N}\}$ where p_m is the norm on the Hilbert space $\mathcal{G}_m \hat{\otimes} \overline{\mathcal{K}}$ of Hilbert-Schmidt operators from $\mathcal{K}$ into $\mathcal{G}_m$ (see Chapter V, Thm.1.6). Again we see that $\mathcal{G} \hat{\otimes} \mathcal{K}$ is countably

Hilbert, and that

$$\mathcal{G} \hat{\otimes} \bar{\mathcal{K}} = \bigcap_m \mathcal{G}_m \otimes \bar{\mathcal{K}} = \varprojlim \mathcal{G}_m \hat{\otimes} \bar{\mathcal{K}}$$

We also see, as in §1.1, the following

Proposition1.4

$\mathcal{G} \hat{\otimes} \bar{\mathcal{K}}$ may be embedded, as a dence subspace, in $\mathcal{K}_0 \hat{\otimes} \bar{\mathcal{K}}$.
In this embedding, elements of $\mathcal{G} \hat{\otimes} \bar{\mathcal{K}}$ have range contained in $\mathcal{G}$, and are continuous from $(\mathcal{K}, \| \|)$ into $(\mathcal{G}, \tau)$.

Note that since the space $\mathcal{G}$ is not nuclear, we do not have a converse of this statement as in Prop.1.1.

Since $\mathcal{G}_m \hat{\otimes} \bar{\mathcal{K}} = \bigoplus_n s^n_{+m} \hat{\otimes} \bar{\mathcal{K}}$ (Hilbert space direct sum) it is clear that each $T \in \mathcal{G} \hat{\otimes} \bar{\mathcal{K}}$ restricts to $T_n \in s^n_{+m} \hat{\otimes} \bar{\mathcal{K}}$ and $p_m(T)^2 = \sum_{n=0}^{\infty} p^n_m(T_n)^2$

We also have a sequence representation:

THEOREM 1.5

For each $n \in \mathbb{N}$, let $s^n_+(\mathcal{K})$ denote the image of $s^n_+ \hat{\otimes} \mathcal{K}$ under the isomorphism between $s^n(\mathcal{K})$ and $s^n \hat{\otimes} \mathcal{K}$ given by Prop.1.2. Equip the algebraic direct sum $\bigoplus_n s^n_+(\mathcal{K})$ with the norms $|||(a^n)|||^2_m = \sum_{n=0}^{\infty} |||a^n|||^2_m$ (we use the convention $s^0_+(\mathcal{K}) = \mathcal{K}$ and $|||a^0|||_m = \|a^0\|_{\mathcal{K}}$ for all $m \in \mathbb{N}$))

Then the direct sum of the isomorphisms given by Prop.1.2 extends to a linear topological isomorphism of the completion $\mathcal{G}(\mathcal{K})$ of $\bigoplus_n s^n_+(\mathcal{K})$ onto $\mathcal{G} \hat{\otimes} \bar{\mathcal{K}}$.

Proof Since, for all $m \in \mathbb{N}$, $s^n_{+m}(\mathcal{K})$ is isomorphic, as a hilbert space, to $s^n_{+m} \hat{\otimes} \bar{\mathcal{K}}$, it follows that the Hilbert space direct sum $\mathcal{G}_m(\mathcal{K}) := \bigoplus_n s^n_{+m}(\mathcal{K})$ is isomorphic to $\mathcal{G}_m \hat{\otimes} \bar{\mathcal{K}}$, the isomorphism being the direct sum of the isomorphisms between the summands.

However, $\mathcal{G}(\mathcal{K}) = \bigcap_m \mathcal{G}_m(\mathcal{K}) = \varprojlim \mathcal{G}_m(\mathcal{K})$, since all three spaces contain densely the algebraic direct sum $\bigoplus_n s_+^n(\mathcal{K})$, and their topologies are easily seen to coincide on $\bigoplus_n s_+^n(\mathcal{K})$. But it is clear from the definition (or see SCHAEFFER [64] II.5.2) that the projective limits of isomorphic spaces are themselves isomorphic.

QED

This concludes our study of the space $\mathcal{G}\hat{\otimes}\overline{\mathcal{K}}$.

§2. We are now ready to represent $\mathcal{O}\!\mathcal{l}$ as unbounded operators on the Hilbert space $\mathcal{H}_o\hat{\otimes}\mathcal{K}$.

Prop. 2.1

(i) Let $x \in \mathcal{O}\!\mathcal{l}$, $T \in \mathcal{G}\hat{\otimes}\overline{\mathcal{K}}$. Then $\pi_o(x)T \in \mathcal{G}\hat{\otimes}\overline{\mathcal{K}}$, and the map $\pi(x) : \mathcal{G}\hat{\otimes}\overline{\mathcal{K}} \longrightarrow \mathcal{G}\hat{\otimes}\overline{\mathcal{K}}$

$$T \longmapsto \pi_o(x)T$$

is continuous wrt $\{p_m\}$.

(ii) π is a #-representation of $\mathcal{O}\!\mathcal{l}$ on $\mathcal{H}_o\hat{\otimes}\overline{\mathcal{K}}$.

Proof $\pi_o(x) \in L(\mathcal{G})$, hence for each $m \in \mathbb{N}$ there exists a K and an $r \in \mathbb{N}$ such that

$$\|\pi_o(x)F\|_m \leq K\,\|F\|_r \qquad \text{for each } F \in \mathcal{G}$$

Now let $T = \sum_{i=1}^{N} F_i \otimes \bar{f}_i \in \mathcal{G}\otimes\overline{\mathcal{K}}$, ($\{f_i\}$ is an o.n. base of $\mathcal{K}$). Then $p_m(\pi_o(x)T)^2 = \sum_{i=1}^{N} \|\pi_o(x)F_i\|_m^2 \leq K^2 \sum_{i=1}^{N}\|F_i\|_r^2$

$$= K^2 p_r(T)^2.$$

So $\pi_o(x)T \in \mathcal{G}\hat{\otimes}\overline{\mathcal{K}}$ and $\pi(x)$ is continuous.

Part (ii) is now immediate.

QED

The following result is well known (see, e.g. REED & SIMON[54] Thm.X.41(a))

Lemma 2.2

Let $q(f) := 2^{-1/2}(a(\bar{f})+a^+(f))$, $p(f) := -i2^{-1/2}(a^+(f)-a(\bar{f}))$, $f \in S$

Then each Ψ_n is an analytic vector for $\pi_0(q(f))$ and $\pi_0(p(f))$.

Proposition 2.3

$\pi(q(f))$ and $\pi(p(f))$ are essentially self-adjoint on $\mathcal{G}\hat{\otimes}\bar{\mathcal{K}}$.

Proof Let $a \in \mathcal{K}$ .Noting that $\pi(x)(F\otimes\bar{a}) = (\pi_0(x)F)\otimes\bar{a}$,we have:

$$p_0(\pi(q(f))^k(\Psi_n\otimes\bar{a})) = p_0((\pi_0(q(f))^k\Psi_n)\otimes\bar{a}) = \|\pi_0(q(f))^k\Psi_n\|_0 \ \|a\|_{\mathcal{K}}$$

and therefore

$$\sum_{k=0}^{\infty}\frac{t^k}{k!}\, p_0(\pi(q(f))^k(\Psi_n\otimes\bar{a})) = \|a\|\sum_{k=0}^{\infty}\frac{t^k}{k!}\,\|\pi_0(q(f))^k\Psi_n\|_0 < \infty$$

by Lemma 2.2.Hence $\Psi_n\otimes\bar{a}$ is an analytic vector for $\pi(q(f))$ (similarly for $\pi(p(f))$).Since such vectors are contained in $\mathcal{G}\hat{\otimes}\bar{\mathcal{K}}$ and are total in $\mathcal{H}_0\hat{\otimes}\bar{\mathcal{K}}$,the result follows by NELSON's Thm. [45].

QED

The next result is known in different contexts (see, e.g. REED & SIMON [54] ,Lemma 1 ,p.232 or STREATER & WIGHTMAN [79] ,Thm.4.5)

Lemma 2.4

$\pi_0(\tilde{\mathfrak{A}})$ is irreducible, i.e. $T \in \pi_0(\tilde{\mathfrak{A}})'$ implies $T = tI$ ($t \in \mathbb{C}$)

Proof For $r \in \mathbb{N}$, define $N(e_r) = a^+(e_r)a(\bar{e}_r) \in \tilde{\mathfrak{A}}$. Clearly $\pi_0(N(e_r))\Psi_n(e(\ldots n_r\ldots)) = n_r\Psi_n(e(\ldots n_r\ldots))$. Thus we have

$$n_r(\Psi_n, T\Psi_0) = (\pi_0(N(e_r))\Psi_n, T\Psi_0) = (\Psi_n, T\pi_0(N(e_r))\Psi_0) = 0$$

since $T \in \pi_0'(\tilde{\mathfrak{A}})$ and $\pi_0(N(e_r))\Psi_0 = 0$. Thus $T\Psi_0$ is orthogonal to all S_+^n , $n>0$,hence $T\Psi_0 = t\Psi_0$, for some $t \in \mathbb{C}$. For $x \in \tilde{\mathfrak{A}}, F \in \mathcal{G}$,

$$(F, T\pi_0(x)\Psi_0) = (\pi_0(x^{\#})F, T\Psi_0) = t(\pi_0(x^{\#})F, \Psi_0) = t(F, \pi_0(x)\Psi_0).$$

Hence $T\pi_0(x)\Psi_0 = t\,\pi_0(x)\Psi_0$. But Ψ_0 is cyclic for $\pi_0(\mathfrak{A})$, (Chapter V, (1.31)), hence also for $\pi_0(\tilde{\mathfrak{A}})$. To see this, let $\varepsilon > 0$, and $F \in \mathcal{H}_0$ be arbitrary. There exists $x \in \mathfrak{A}$, such that

$$\| F - \pi_0(x)\Psi_0 \| < \varepsilon/2 .$$

By Chapter V, Prop. 3.1& Lemma 3.3, there exists $x' \in \tilde{\mathfrak{A}}$ s.t.

$$\|\pi_0(x - x')\Psi_0\|^2 \leq \operatorname{tr}(\exp(-\beta H))\ \ w_\beta((x-x')^*(x-x')) < \varepsilon^2/4 .$$

Hence $\| F - \pi_0(x')\Psi_0 \| < \varepsilon$.

Since T is bounded, we now see that $T = tI$. QED.

Corr 2.5

$\pi_0(\mathfrak{A})$ is irreducible, since $\{tI\} = \pi_0'(\tilde{\mathfrak{A}}) \supseteq \pi_0'(\mathfrak{A})$.

THEOREM 2.6

The commutant π' of π consists of all post-multiplications by bounded operators on $\mathcal{K}$. That is,

$$\pi' = \left\{ C \in \mathcal{B}(\mathcal{H}_0 \hat{\otimes} \overline{\mathcal{K}}) \ : \ \exists\, C_1 \in \mathcal{B}(\mathcal{K}) \ \text{ s.t. } C(T) = TC_1 \quad \forall T \in \mathcal{H}_0 \hat{\otimes} \overline{\mathcal{K}} \right\}$$

Proof Let $C_1 \in \mathcal{B}(\mathcal{K})$. Define

$C \in \mathcal{B}(\mathcal{H}_0 \hat{\otimes} \overline{\mathcal{K}})$ by $C(T) := T.C_1 \quad (T \in \mathcal{H}_0 \hat{\otimes} \overline{\mathcal{K}})$.

Since each $\mathcal{G}_m \hat{\otimes} \overline{\mathcal{K}}$ is a right $\mathcal{B}(\mathcal{K})$-module, it is clear that C leaves $\mathcal{G} \hat{\otimes} \overline{\mathcal{K}}$ invariant. An application of the definition of the commutant (Chapt. I §1.4) now shows that $C \in \pi'$.

Conversely, let $C \in \pi'$. Fix $a, a' \in \mathcal{K}$, and consider $B_{a,a'}$ defined on $\mathcal{H}_0 \times \mathcal{H}_0$ by:

$$B_{a,a'}(F,F') := (F \otimes \overline{a},\ C(F' \otimes \overline{a}'))_0$$

since $|B_{a,a'}(F,F')| \leq (\|C\| \|a\|_{\mathcal{K}} \|a'\|_{\mathcal{K}}) \|F\|_0 \|F'\|_0$, (+)

$B_{a,a'}$ is a bounded sesqui-linear form and thus there exists a unique $K(a,a') \in \mathcal{B}(\mathcal{H}_0)$ such that:

$$B_{a,a'}(F,F') = (F,\ K(a,a')F') \quad \text{for all } F,\ F' \in \mathcal{H}_0$$

Now I claim that $K(a,a') \in \pi_0(\tilde{\mathcal{A}})'$.

To see this, let $G, G' \in \mathcal{G}$, $x \in \tilde{\mathcal{A}}$ and consider:

$$(G, K(a,a')\pi_0(x)G') = B_{a,a'}(G, \pi_0(x)G')$$

$$= (G \otimes \bar{a}, C((\pi_0(x)G') \otimes \bar{a}'))_0 = (G \otimes \bar{a}, C\pi(x)(G' \otimes \bar{a}'))_0$$

$$= (\pi(x^{\#})(G \otimes \bar{a}), C(G' \otimes \bar{a}'))_0 = B_{a,a'}(\pi_0(x^{\#})G, G')$$

$$= (\pi_0(x^{\#})G, K(a,a')G') .$$

This proves the claim, and hence by Lemma 2.4 there exists $k(\bar{a},\bar{a}') \in \mathbb{C}$ such that $K(a,a') = k(\bar{a},\bar{a}')I$.

Thus for $F, F' \in \mathcal{H}_0$, we have

$$k(\bar{a},\bar{a}') = \frac{(F, K(a,a')F')}{(F, F')} = \frac{B_{a,a'}(F, F')}{(F, F')}$$

$$= \frac{(F \otimes \bar{a}, C(F' \otimes \bar{a}'))_0}{(F, F')} \qquad (++)$$

This shows that the mapping

$$\overline{\mathcal{K}} \times \overline{\mathcal{K}} \longrightarrow \mathbb{C}$$
$$(\bar{a}, \bar{a}') \longmapsto k(\bar{a},\bar{a}')$$

is sesquilinear and is bounded since

$$|k(\bar{a},\bar{a}')| = \|K(a,a')\|_{\mathcal{B}(\mathcal{H}_0)} \leq \|C\| \|\bar{a}\|_{\overline{\mathcal{K}}} \|\bar{a}'\|_{\overline{\mathcal{K}}}$$

by (+). Hence there exists a unique $C_1 \in \mathcal{B}(\overline{\mathcal{K}})$ such that $k(\bar{a},\bar{a}') = (\bar{a}, C_1^{*}\bar{a}')_{\overline{\mathcal{K}}}$.

Then (++) gives

$$(F \otimes \bar{a}, C(F' \otimes \bar{a}'))_0 = k(\bar{a},\bar{a}') (F,F')$$

$$= (\bar{a}, C_1^{*}\bar{a}')_{\overline{\mathcal{K}}} (F,F') = (F \otimes \bar{a}, F' \otimes C_1^{*}\bar{a}')_0$$

$$= (F \otimes \bar{a}, (F' \otimes \bar{a}')C_1)_0$$

Since rank-one operators are total in $\mathcal{H}_0 \hat{\otimes} \overline{\mathcal{K}}$, we have $C(F \otimes \bar{a}) = (F \otimes \bar{a})C_1$ and thus $C(T) = T.C_1$ for each $T \in \mathcal{H}_0 \hat{\otimes} \overline{\mathcal{K}}$.

QED

(This proof is a generalization of GUDDER & HUDSON (22 Lemma 24).

§3. For the rest of this chapter, we assume $\mathcal{K} = \mathcal{H}_o$.

Thus the representation π constructed in the previous section is a representation of $\mathfrak{A}$ on the Hilbert-Schmidt oparators on Fock space. Its commutant $\pi(\mathfrak{A})' = \pi(\tilde{\mathfrak{A}})'$ (since the proof of Thm.2.6 only depended on the irreducibility of π_0) has a particularly simple form: it is the right VN algebra of the Hilbert algebra $\mathcal{H}_o \hat{\otimes} \mathcal{H}_o$ (see Chapter III, §1.). Thus if we can prove that π is unitarily equivalent to π_β, we are almost finished. This will be done by using the uniqueness of the GNS construction (see Chapter I, §1.4) Thus we need to construct a unit vector $\Omega_\beta \in \mathcal{G} \hat{\otimes} \mathcal{H}_o$, such that

(3.1) $$(1, \pi_\beta(x)1) = w_\beta(x) = (\Omega_\beta, \pi(x)\Omega_\beta)$$

for all $x \in \mathfrak{A}$, and which is strongly cyclic for π . It is this last point, surprisingly, which is the most difficult. Even in the one-dimensional case (see GUDDER & HUDSON [22], Lemma 27, although our proof is simpler) the proof is rather technical. Our proof is inspired from the proof that coherent vectors are total in Fock space.

It is easy to guess what the cyclic vector Ω_β should look like. By inspection of (3.1) and the definition of w_β , we get the

<u>Definition</u> $T = \exp(-\beta H/2)$, $\Omega_\beta = T/(\mathrm{tr}(\exp(-\beta H)))^{1/2}$

<u>THEOREM 3.1</u>

Ω_β is strongly cyclic for $\pi(\tilde{\mathfrak{A}})$ on $\mathcal{G} \hat{\otimes} \mathcal{H}_o$.

<u>Proof</u> First $\Omega_\beta \in \mathcal{G} \hat{\otimes} \mathcal{H}_o$, since $p_m(T)^2 = \mathrm{tr}((\exp(-\beta H))B^m) < \infty$ (see Chapter V, Cor.2.8(ii)). We show that, if $\mathcal{C}$ is the closure of $\pi(\tilde{\mathfrak{A}})\Omega_\beta$ in the topology of $\mathcal{G} \hat{\otimes} \mathcal{H}_o$, then $\mathcal{C} = \mathcal{G} \hat{\otimes} \mathcal{H}_o$.

This will show that $\pi(\tilde{\mathfrak{A}})\Omega_\beta$ is dense in $\mathcal{G}\hat{\otimes}\overline{\mathcal{H}}_o$ in the topology defined by the (continuous) seminorms $\{\|\pi(x).\| : x\in\tilde{\mathfrak{A}}\}$, because, by prop 2.1 (i) the latter topology is weaker, hence gives rise to larger closures.

Define, for $x\in\tilde{\mathfrak{A}}$, $r\in\mathbb{N}$, and $\vec{z}=(z_o,z_1,..z_r)\in\mathbb{C}^{r+1}$ with $\operatorname{Im}z_j > -\beta\varepsilon/4$

$$T_r(x,\vec{z}) := \sum_{n=0}^{\infty}\sum_{\mathcal{J}_n}(\exp(-\tfrac{\beta}{2}\sum_{k=0}^{\infty}w(k)n_k))(\exp(i\sum_{j=0}^{r}n_jz_j))\pi_o(x)\Psi_n\otimes\overline{\Psi}_n \, .$$

The idea of the proof is the following:

We first prove $T_r(x,\vec{\theta}) = $ "$\pi(x(\exp i\sum_{j=0}\theta_j N(e_j)))T$" is in $\mathcal{C}$ $\forall\theta_j\in[0,2\pi]$ where $N(e_j)$ is as in the proof of Lemma 2.4. Then we show that

$$\left[\frac{1}{2\pi}\right]^{r+1}\int_{[0,2\pi]^{r+1}}(\exp(-i\sum_{j=0}^{r}k_j\theta_j))\,T_r(x,\vec{\theta})\,d\vec{\theta} \qquad \forall r,k_j\in\mathbb{N}$$

exists in the weak sense, and hence is in $\mathcal{C}$. Next we show that this integral approximates, for large enough r,

$$\pi_o(x)\Psi_k(e(k_o,k_1,..))\otimes\overline{\Psi}_k(e(k_o,k_1,...))$$

and the result follows from this, since such vectors are clearly total in $\mathcal{G}\hat{\otimes}\overline{\mathcal{H}}_o$.

(i) To show that $T_r(x,\vec{\theta})\in\mathcal{C}$ for $\vec{\theta}=(\theta_o,\theta_1,...\theta_r)\in[0,2\pi]^{r+1}$

(a) Let

$$F_{N,q}(x,\vec{z}) = \sum_{n=0}^{N}\sum_{\substack{n_o,..,n_q\\(\Sigma n_k=n)}}(\exp(-\tfrac{\beta}{2}\sum_k w(k)n_k))(\exp(i\sum_{j=0}^{r}n_jz_j))\pi_o(x)\Psi_n\otimes\overline{\Psi}_n$$

with $N,q\in\mathbb{N}$ and $\operatorname{Im}z_j > -\beta\varepsilon/4$.

These are (separately) analytic functions of z into $\mathcal{G}\hat{\otimes}\overline{\mathcal{H}}_o$, being finite rank operators with analytic coefficients.

Noting that $\beta\sum_k w(k)n_k + 2\sum_{j=0}^{r}\operatorname{Im}z_jn_j \geq$

$$\geq \beta\sum_{k=0}^{\infty}w(k)n_k - \sum_{j=0}^{r}(\beta\varepsilon/2)n_j \geq \beta\sum_{k=0}^{\infty}(w(k)-\tfrac{\varepsilon}{2})n_k$$

we have:

$p_m(T_r(x,\vec{z}) - F_{N,q}(x,\vec{z}))^2$

$$= \Big(\sum_{n=0}^{\infty} \sum_{J_n} - \sum_{n=0}^{N} \sum_{\substack{n_0,\dots,n_q \\ (\Sigma n_k = n)}}\Big) p_m\Big((\exp(\tfrac{-\beta}{2}\sum_k w(k)n_k))(\exp i\sum_{j=0}^{r} n_j z_j)\pi_o(x)\Psi_n \otimes \overline{\Psi}_n\Big)^2$$

$$\leq \Big(\sum_{n=0}^{\infty} \sum_{J_n} - \sum_{n=0}^{N} \sum_{\substack{n_0,\dots,n_q \\ (\Sigma n_k = n)}}\Big)(\exp(-\beta \sum_k (w(k) - \tfrac{\varepsilon}{2})n_k)) \|\pi_o(x)\Psi_n\|_m^2 \longrightarrow 0$$

as $N,q \longrightarrow \infty$, uniformly in z in the region $\mathrm{Im} z_j > -\beta\varepsilon/4$, since $\sum_{n=0}^{\infty} \sum_{J_n} (\exp(-\beta \sum_k w(k)n_k)) \|\pi_o(x)\Psi_n\|_m^2 = \mathrm{tr}((\exp(-\beta H))\pi_o(x) B^m \pi_o(x))$ is finite by Cor.2.8(i)&(ii) of Chapter V, where we may replace $w(k)$ by $w'(k) := w(k) - \frac{\varepsilon}{2}$, since $w'(k) = k + \frac{\varepsilon}{2}$.

Thus we have shown that $\lim_{N,q\to\infty} F_{N,q}(x,\vec{z}) = T_r(x,\vec{z})$ uniformly in $\vec{z}$ in the region $\mathrm{Im} z_j > -\beta\varepsilon/4$ in the topology of $\mathcal{G} \hat{\otimes} \mathcal{H}_o$.
Thus $T_r(x,\vec{z})$ is (separately) analytic in $\vec{z}$ in the above region.

(b) Now let $S \in (\mathcal{G} \hat{\otimes} \mathcal{H}_o)^*$, the topological dual of $\mathcal{G} \hat{\otimes} \mathcal{H}_o$; be such that $S(\pi(x)\Omega_\beta) = 0$ for each $x \in \widetilde{\mathcal{O}}$.
We shall show that $S(T_r(x,\vec{\theta})) = 0$ for each $\vec{\theta} \in [0,2\pi]^{r+1}$.
This will prove, by the Hahn-Banach thm, that $T_r(x,\vec{\theta}) \in \mathcal{E}$.

Define $f(\vec{z}) := S(T_r(x,\vec{z}))$. This is analytic in the same region.
For $k_0, k_1, \dots k_r \in \mathbb{N}$, consider:

$$\partial_0^{k_0} \partial_1^{k_1} \dots \partial_r^{k_r} F_{N,q}(x,\vec{z}) \qquad \text{where} \quad \partial_j^{k_j} := \Big(\frac{\partial}{\partial z_j}\Big)^{k_j}$$

$$= \sum_{n=0}^{N} \sum_{\substack{n_0,\dots,n_q \\ (\Sigma n_k = n)}} (\exp(\tfrac{-\beta}{2}\sum_k w(k)n_k))(in_0)^{k_0}(in_1)^{k_1} \dots (in_r)^{k_r}(\exp i\sum_{j=0}^{r} n_j z_j)\pi_o(x)\Psi_n \otimes \overline{\Psi}_n$$

$$= \sum_{n=0}^{N} \sum_{\substack{n_0,\dots,n_q \\ (\Sigma n_k = n)}} (\exp(\tfrac{-\beta}{2}\sum_k w(k)n_k + i\sum_{j=0}^{r} n_j z_j))\pi_o(x(iN(e_0))^{k_0} \dots (iN(e_r))^{k_r})\Psi_n \otimes \overline{\Psi}_n$$

$$\longrightarrow T_r(x(iN(e_0))^{k_0} \dots (iN(e_r))^{k_r}, \vec{z}) \quad , \quad \text{as in (a)}$$

where $N(e_j)$ is as defined in the proof of Lemma 2.4.

Therefore

$$\lim_{N,q\to\infty} S(\partial_0^{k_0}\ldots\partial_r^{k_r}F_{N,q}(x,\vec{z})) = S(T_r(x(iN(e_0))^{k_0}\ldots(iN(e_r))^{k_r}, \vec{z})$$

and hence

$$\partial_0^{k_0}\ldots\partial_r^{k_r}f(\vec{\theta}) = S(T_r(x(iN(e_0))^{k_0}\ldots(iN(e_r))^{k_r},\vec{\theta}))$$

as can be seen by repeated application of the Dom. Convergence thm. Thus

$$\partial_0^{k_0}\ldots\partial_r^{k_r}f(\vec{\theta})|_{\vec{\theta}=0} = S(\pi(x(iN(e_0))^{k_0}\ldots(iN(e_r))^{k_r})T) = 0,$$

since $x(iN(e_0))^{k_0}\ldots(iN(e_r))^{k_r} \in \widetilde{\mathfrak{A}}$, and $T_r(y,\vec{0}) = \pi(y)T$

Hence the (r+1)-fold Taylor expansion for f, which converges for $\vec{\theta} \in [0,2\pi]^{r+1}$, since f is analytic, gives:

$$f(\vec{\theta}) = \sum_{k=0}^{\infty}(k!)^{-1}(\sum_{j=0}^{r}\theta_j\frac{\partial}{\partial\varphi_j})^k f(\vec{\varphi})|_{\vec{\varphi}=0} = 0$$

i.e. $S(T_r(x,\vec{\theta})) = 0$, which shows that $T_r(x,\vec{\theta}) \in \mathscr{C}$.

(ii) The function

$$[0,2\pi]^{r+1} \ni \vec{\theta} \longmapsto (\exp(-i\sum_{j=0}^{r}k_j\theta_j))\, T_r(x,\vec{\theta}) \in \mathscr{C}$$

is continuous, hence integrable in the weak sense (RUDIN [59] 3.27), and its integral $T_r^{\vec{k}}(x) \in \mathscr{C}$, ($\vec{k} = (k_0,k_1,\ldots k_r)$). We find:

$$T_r^{\vec{k}}(x) = \left[\frac{1}{2\pi}\right]^{r+1}\int_{[0,2\pi]^{r+1}}\exp(-i\sum_{j=0}^{r}k_j\theta_j))\, T_r(x,\vec{\theta})d\vec{\theta}$$

$$= \sum_{n=0}^{\infty}\sum_{J_n}(\exp(\frac{-\beta}{2}\sum_k w(k)n_k))\left[\frac{1}{2\pi}\right]^{r+1}\int_{[0,2\pi]^{r+1}}(\exp i\sum_{j=0}^{r}(n_j-k_j)\theta_j)\pi_0(x)\Psi_n\otimes\overline{\Psi}_n d\vec{\theta}$$

$$=\sum_{n=0}^{\infty}\sum_{J_n}(\exp(\frac{-\beta}{2}\sum_k w(k)n_k))\prod_{j=0}^{r}\delta(k_j,n_j)\,\pi_0(x)\Psi_n\otimes\overline{\Psi}_n$$

$$= (\exp(\frac{-\beta}{2}\sum_{j=0}^{r}w(j)k_j))\pi_0(x)\Psi_k(e(k_0,\ldots k_r,0,\ldots))\otimes\overline{\Psi}_k(e(k_0,\ldots k_r,0..))$$

$$+\sum_{n=k+1}^{\infty}\sum_{I_n^r}(\exp(\frac{-\beta}{2}\sum_i w(i)n_i))\pi_0(x)\Psi_n\otimes\overline{\Psi}_n$$

where $I_n^r = \{(n_0,n_1,\ldots) : \sum_i n_i=n \text{ and } n_j=k_j \text{ for } 0\le j\le r\}$

and $\sum_{j=0}^{r} k_j = k$.

Now, keeping $r, k_0, k_1, \ldots k_r \in \mathbb{N}$ fixed, define for each $r' \in \mathbb{N}$, $r' \geq r$ the string $\vec{k}' = (k_0, \ldots k_r, k_{r+1}, \ldots k_{r'})$ such that $k_j = 0$ for $r < j \leq r'$

It follows, as above, that

$$T_{r'}^{\vec{k}'}(x) := \left[\frac{1}{2\pi}\right]^{r'+1} \int_{[0,2\pi]^{r'+1}} (\exp(-i \sum_{j=0}^{r'} k_j \theta_j))\, T_{r'}(x, \vec{\theta})\, d\vec{\theta}$$

$$= (\exp(-\frac{\beta}{2} \sum_{j=0}^{r'} w(j) k_j))\, \pi_0(x) \Psi_k(e(k_0, \ldots k_r, 0, \ldots)) \otimes \overline{\Psi}_k(e(k_0, \ldots k_r, 0.)$$

$$+ \sum_{n=k+1}^{\infty} \sum_{I_n^{r'}} (\exp(-\frac{\beta}{2} \sum_i w(i) n_i))\, \pi_0(x) \Psi_n \otimes \overline{\Psi}_n$$

Notice that the first term remains unchanged for any $r' \geq r$, since $k_j = 0$ for $r < j \leq r'$.

(iii) We claim that the second term goes to zero, as $r' \longrightarrow \infty$. That is, given $\varepsilon > 0$ and $m \in \mathbb{N}$, there exists r' such that, whenever $r'' \geq r'$, the p_m-norm of the second term is less than ε. This will show that

$$p_m(T_{r''}^{\vec{k}''}(x) - (\exp(-\frac{\beta}{2} \sum_{j=0}^{r''} w(j) k_j))\, \pi_0(x) \Psi_k \otimes \overline{\Psi}_k) < \varepsilon$$

for each $r'' \geq r'$ by the above formula, and therefore, since $T_{r''}^{\vec{k}''}(x) \in \mathcal{C}$ for each r'', that

$$\pi_0(x) \Psi_k \otimes \overline{\Psi}_k \in \mathcal{C}$$

$(\exp(-\frac{\beta}{2} \sum_{j=0}^{r''} w(j) k_j))$ being a scalar, independent of r''.

To prove the claim, let $E_{r'}$ be the orthogonal projection onto the subspace of Fock space spanned by $\{\Psi_n(e(n_0, \ldots)) : (n_0, \ldots) \in I_n^{r'},\ n \in \mathbb{N}\}$. Observe that

$$E_{r'} \Psi_k(e(k_0, \ldots k_r, 0, 0, \ldots)) = \Psi_k(e(k_0, \ldots k_r, 0, 0, \ldots))$$

for all $r' \geq r$, while $E_{r'} \longrightarrow 0$ strongly on the orthogonal complement of $\Psi_k(e(k_0,\ldots k_r,0,0,\ldots))$. We can see this as follows: If $E_{r'} \not\longrightarrow 0$, there must exist Ψ_n such that $E_{r'}\Psi_n \not\longrightarrow 0$. But this means that $E_{r'}\Psi_n = \Psi_n$ for all large enough r', since the $I_n^{r'}$, and hence also the $E_{r'}$, are monotonically decreasing. Now $E_{r'}\Psi_n = \Psi_n$ implies $n_j = k_j$ for $0 \leq j \leq r'$ by definition of $E_{r'}$, and hence, since r' is arbitrarily large, $n_j = k_j$ for all j. Thus

$$\Psi_n(e(n_0,\ldots)) = \Psi_k(e(k_0,\ldots k_r,0,0\ldots))$$

Now let $T_k = \exp(-\beta H)F_k$, where F_k is the projection onto the orthogonal complement of the subspace spanned by $\{\Psi_n, 0 \leq n \leq k\}$. T_k is a nuclear operator by Thm.2.7 of Chapter V.

For $m \in \mathbb{N}$, $x \in \tilde{\mathfrak{A}}$, and $G \in \mathcal{G}$, consider

$$\|B^m \pi_0(x) T_k E_{r'} G\|_0^2 = \|\pi_0(x) T_k E_{r'} G\|_{2m}^2 \leq C^2 \|T_k E_{r'} G\|_{2p}^2$$

where B^m is as in Chapter V, (1.43), $p \in \mathbb{N}$, C is a constant, and the last inequality follows from $\pi_0(x) \in L(\mathcal{G})$

$$= C^2\|B^p T_k E_{r'} G\|_0^2 \leq C^2\|B^p T_k\|^2\|G\|_0^2$$

since $B^p T_k$ is bounded, being trace class (Chapter V, Cor.2.8(ii))

Thus

$$\|B^m \pi_0(x) T_k E_{r'}\| \leq C\|B^p T_k\|$$

and hence the sequence $(B^m \pi_0(x) T_k E_{r'})_{r'}$ is a norm-bounded sequence, converging to 0 strongly, hence ultrastrongly. Thus we have:

$$p_{2m}\Big(\sum_{n=k+1}^{\infty}\sum_{I_n^{r'}}(\exp(\tfrac{-\beta}{2}\sum_i w(i)n_i))\pi_0(x)\Psi_n\otimes\bar{\Psi}_n\Big)^2 =$$

$$\sum_{n=k+1}^{\infty}\sum_{I_n^{r'}}(\exp(-\beta\sum_i w(i)n_i))\|\pi_0(x)\Psi_n\|_{2m}^2 =$$

$$\sum_{n=k+1}\sum_{I_n^{r'}}(\exp(-\beta\sum_i w(i)n_i))\|B^m\pi_0(x)\Psi_n\|_0^2 =$$

$$\sum_{n=0}^{\infty}\sum_{J_n}\|B^m \pi_0(x)T_k E_{r'}\Psi_n\|_0^2 \longrightarrow 0 \quad \text{as } r' \longrightarrow \infty$$

by the definition of the ultrastrong topology.

(iv) We have now shown that, for each $r, k_0, \ldots k_r \in \mathbb{N}$, and for each $x \in \mathfrak{A}$,

$$\pi_0(x)\Psi_k(e(k_0,\ldots k_r,0,0,\ldots))\otimes\overline{\Psi}_k(e(k_0,\ldots k_r,0,0,\ldots)) \in \mathcal{C}$$

To complete the proof, let $F \in \mathcal{G}$ be given. Since $\{\pi_0(x)\Psi_0 : x \in \tilde{\mathfrak{A}}\}$ is dense in $(\mathcal{G}, \tau)$ (this follows from the definition of $\mathcal{G}$ together with the density of $\tilde{\mathfrak{A}}$ in $\mathfrak{A}$ in the topology inherited from $L(\mathcal{G})$ (Chapter V, Prop.3.1)), given $\epsilon > 0$ and $m \in \mathbb{N}$, we can find $x \in \tilde{\mathfrak{A}}$ such that

$$\|F - \pi_0(x)\Psi_0\|_m < \epsilon$$

Now given $k_0, k_1, \ldots k_r \in \mathbb{N}$, let $y \in \tilde{\mathfrak{A}}$ be such that

$$\pi_0(y)\Psi_k(e(k_0,\ldots k_r,0,0,\ldots)) = \Psi_0$$

Then

$$p_m(F\otimes\overline{\Psi}_k - \pi_0(xy)\Psi_k\otimes\overline{\Psi}_k) = \| F - \pi_0(xy)\Psi_k\|_m =$$

$$= \|F - \pi_0(x)\Psi_0\|_m < \epsilon$$

which shows that $F\otimes\overline{\Psi}_k(e(k_0,\ldots k_r,0,0,\ldots)) \in \mathcal{C}$

This concludes the proof, since such elements are clearly total in $\mathcal{G}\hat{\otimes}\overline{\mathcal{H}_0}$.

QED!!

§4. We are at last able to put all the pieces together and prove the commutation theorem. We will first prove this theorem for the almost modular Hilbert algebra $\tilde{\mathfrak{A}}$, and then extend it to $\mathfrak{A}$. Thus we first restrict attention to $\tilde{\mathfrak{A}}$.

Theorem 3.1 and equation (3.1) show that $(\mathcal{H}, \pi_\beta, 1)$ and $(\mathcal{H}_0 \hat{\otimes} \overline{\mathcal{H}}_0, \pi, \Omega_\beta)$ are two GNS triples for $(\tilde{\mathfrak{A}}, w_\beta)$. Therefore they must be equivalent, that is, there exists a unitary

$$U : \mathcal{H} \xrightarrow{\text{onto}} \mathcal{H}_0 \hat{\otimes} \overline{\mathcal{H}}_0$$

such that

$$U \pi_\beta(x) 1 = \pi(x) \Omega_\beta$$

for all $x \in \tilde{\mathfrak{A}}$, and U maps $D(\pi_\beta)$ continuously onto $D(\bar{\pi})$ with respect to the induced topologies (see Chapter I, §1.4).

We now define the b-antirepresentation ρ of $\tilde{\mathfrak{A}}$ on $UD(\rho_\beta)$ by

$$\rho(x) := U \rho_\beta(x) U^{-1}$$

(see the introduction to this Chapter) and the (anti-)unitary involution J_1 on $\mathcal{H}_0 \hat{\otimes} \overline{\mathcal{H}}_0$ by

$$J_1 := UJU^{-1}$$

Lemma 4.1 (GUDDER & HUDSON [22] Lemma 21)

J maps $D(\pi_\beta)$ continuously onto $D(\rho_\beta)$ and vice versa. Hence J_1 maps $D(\bar{\pi})$ continuously onto $D(\bar{\rho})$. For $x \in \tilde{\mathfrak{A}}$,

$$\pi_\beta(x) = J \rho_\beta(x^*) J$$

Proof For $x, y \in \tilde{\mathfrak{A}}$,

$$_y\|x^*\| = \|yx^*\| = \|J(yx^*)\| = \|xy^*\| = \|x\|_{y^*}$$

It follows that J is a topological isomorphism from $(\tilde{\mathfrak{A}}, \lambda)$ onto $(\tilde{\mathfrak{A}}, \delta)$, and hence extends to the completions.

For $x, y \in \tilde{\mathfrak{A}}$, we have

$$J\rho_\beta(x^*)Jy = J\rho_\beta(x^*)y^* = J(y^*x^*) = xy = \pi_\beta(x)y$$

Thus $J\rho_\beta(x^*)Jy = \pi_\beta(x)y$ for all $y \in D(\pi_\beta)$, by the continuity of the operators involved. This proves the last claim.

QED

Lemma 4.2

For all $T \in \mathcal{H}_0 \hat{\otimes} \overline{\mathcal{H}_0}$, $J_1 T = T^*$, the operator adjoint of T.

Proof Since $T \rightarrow T^*$ and J_1 are antilinear isometries of $\mathcal{H}_0 \hat{\otimes} \overline{\mathcal{H}_0}$, it is sufficient to prove the claim on a dense subset. Thus let $T = \pi(x)\Omega_\beta$ for some $x \in \tilde{\mathfrak{A}}$. We have

$$J_1 T = UJU^{-1}T = UJx = Ux^* = \pi(x^*)\Omega_\beta$$

Thus we must show that $(\pi(x)\Omega_\beta)^* = \pi(x^*)\Omega_\beta$ for all $x \in \tilde{\mathfrak{A}}$.

Let $F \in \mathcal{G}$, $r \in \mathbb{N}$. We have

$$\operatorname{tr}(\exp -\beta H)\cdot(\pi(a^+(e_r))\Omega_\beta)F =$$

$$= \sum_{n=0}^{\infty} \sum_{J_n} (\exp(\tfrac{-\beta}{2} \sum_k w(k)n_k))\ (\Psi_n, F)\pi_0(a^+(e_r))\Psi_n$$

$$= \sum_{n=0}^{\infty} \sum_{J_n} (\exp(-\tfrac{\beta}{2} \sum_k w(k)n_k))\ (n_r + 1)^{1/2}(\Psi_n, F)\Psi_{n+1}(e(n_0, \ldots n_r+1, \ldots))$$

$$= \sum_{n=0}^{\infty} \sum_{J_n} (\exp(-\tfrac{\beta}{2}(\sum_k w(k)n_k - w(r)))(\pi_0(a(\bar{e}_r)\Psi_n, F)\Psi_n \qquad \text{(changing } n_r \text{ to } n_r - 1\text{)}$$

$$= (\exp\tfrac{\beta}{2}w(r)) \sum_{n=0}^{\infty} \sum_{J_n} (\exp(-\tfrac{\beta}{2} \sum_k w(k)n_k))(\Psi_n, \pi_0(a^+(e_r))\, F)\Psi_n$$

Thus,

$$(\pi(a^+(e_r))\Omega_\beta)F = (\exp\tfrac{\beta}{2}w(r))\Omega_\beta \pi_0(a^+(e_r))\,F = \Omega_\beta \pi_0(\Delta(\tfrac{-1}{2})a^+(e_r))\,F$$

Similarly, we find

$$(\pi(a(\bar{e}_r))\Omega_\beta)\ F = (\exp\tfrac{-\beta}{2}w(r))\Omega_\beta\pi_0(a(\bar{e}_r))\ F = \Omega_\beta\pi_0(\Delta(-\tfrac{1}{2})a(\bar{e}_r))\ F$$

Thus we see that in general

$(\pi(x)\Omega_\beta)F = \Omega_\beta\pi_0(\Delta(-1/2)x)F$

Hence, for each $F,G \in \mathcal{G}$,

$(\pi(x)\Omega_\beta F,G) = (\Omega_\beta(\pi_0(\Delta(-1/2)x))F,G) = (\pi_0(\Delta(-1/2)x)F,\Omega_\beta G) =$

$(F,\pi_0((\Delta(-1/2)x)^{\#})\Omega_\beta G) = (F,\pi_0(x^*)\Omega_\beta G)$

since $(\Delta(-1/2)x)^{\#} = \Delta(1/2)x = x^*$ (Chapter V,(3.4)) , and thus the claim follows, since $\pi(x)\Omega_\beta$ is a bounded operator.

QED

We can now prove the analogue of Thm.3.1 of Chapter III.

THEOREM 4.3

The commutants $\pi_\beta(\tilde{\mathfrak{A}})'$ and $\rho_\beta(\tilde{\mathfrak{A}})'$ are Von Neumann algebras, commutants of each other. $\rho_\beta(\tilde{\mathfrak{A}})'$ is standard in the sence of DIXMIER (see Chapter III,§1) and a factor.Finally, J induces a spatial isomorphism of either one of them onto the other,that is

$$J\pi_\beta(\tilde{\mathfrak{A}})'J = \rho_\beta(\tilde{\mathfrak{A}})'$$
$$J\rho_\beta(\tilde{\mathfrak{A}})'J = \pi_\beta(\tilde{\mathfrak{A}})'$$

Proof(i) Let $C \in \pi(\tilde{\mathfrak{A}})'$, $F,G \in D(\rho)$, $x \in \tilde{\mathfrak{A}}$ .We have

$$(J_1CJ_1\rho(x)F,G)_o = (J_1C\pi(x^*)J_1F,G)_o \qquad \text{(Lemma 4.1)}$$
$$= (J_1G,C\pi(x^*)J_1F)_o = (\pi(x^{*\#})J_1G,CJ_1F)_o$$
$$= (J_1\rho(x^{*\#*})G,CJ_1F)_o = (J_1\rho(x^{b})G,CJ_1F)_o$$

since $x^{*\#} = (\Delta(1/2)x^{\#})^{\#} = \Delta(-1/2)x$, so that

$$x^{*\#*} = \Delta(1/2)(\Delta(-1/2)x)^{\#} = \Delta(1/2)\Delta(1/2)x =$$
$$= \Delta(1)x = x^{b}$$

Thus

$$(J_1CJ_1\rho(x)F,G)_0 = (J_1CJ_1F,\rho(x^b)G)$$

i.e. $J_1CJ_1 \in \rho(\tilde{\mathfrak{A}})'$. Thus $J_1\pi(\tilde{\mathfrak{A}})'J_1 = \rho(\tilde{\mathfrak{A}})'$, and the equalities of the Theorem follow in the same way.

(ii) By Thm.2.6, we know that $\pi(\tilde{\mathfrak{A}})' = \mathcal{R}(\mathcal{U})$, where $\mathcal{U}$ is the Hilbert algebra $\mathcal{H}_0\hat{\otimes}\overline{\mathcal{H}_0}$ (see Chapter III, §1). Thus each $C \in \pi(\tilde{\mathfrak{A}})'$ is of the form $C(T) = TC_1$ $(T \in \mathcal{U})$ for a unique $C_1 \in \mathcal{B}(\mathcal{H}_0)$. Hence

$$J_1CJ_1(T) = J_1C(T^*) = J_1(T^*C_1) = C_1^*T$$

using Lemma 4.2. Combining this with (i) we see that

$$\rho(\tilde{\mathfrak{A}})' = \mathcal{L}(\mathcal{U})$$

the left algebra of $\mathcal{U}$. But the commutant theorem for Hilbert algebras (Chapter III thm.1.1) shows that these latter VN algebras are commutants of each other. Therefore

$$\pi(\tilde{\mathfrak{A}})' = \rho(\tilde{\mathfrak{A}})''$$

(iii) It is clear that

$$U\pi_B(\tilde{\mathfrak{A}})'U^{-1} = \bar{\pi}(\tilde{\mathfrak{A}})'$$

and

$$U\rho_B(\tilde{\mathfrak{A}})'U^{-1} = \rho(\tilde{\mathfrak{A}})'$$

Thus by (ii) $\rho_B(\tilde{\mathfrak{A}})'$ is a standard factor, being unitarily equivalent to the standard factor $\mathcal{L}(\mathcal{U})$ (see Chapter III, Ex.1.3) and $\pi_B(\tilde{\mathfrak{A}})' = \rho_B(\tilde{\mathfrak{A}})''$. This concludes the proof.

QED

Our main theorem is now trivial to prove:

THEOREM 4.4

The commutant $\pi_{\mathbb{B}}(\mathfrak{A})'$ is a Von Neumann algebra, and it is standard in the sense of DIXMIER . In particular,

$$J\,\pi_{\mathbb{B}}(\mathfrak{A})'J = \pi_{\mathbb{B}}(\mathfrak{A})''$$

Moreover,

$$\Delta^{it}\pi_{\mathbb{B}}(\mathfrak{A})'\Delta^{-it} = \pi_{\mathbb{B}}(\mathfrak{A})' \qquad \text{for all } t \in \mathbb{R}$$

Proof Except for the last assertion, everything follows from Thm.4.3, since

$$\begin{aligned}\pi_{\mathbb{B}}(\mathfrak{A})' &= U^{-1}\pi(\mathfrak{A})'U = \\ &= U^{-1}\pi(\tilde{\mathfrak{A}})'U = \pi_{\mathbb{B}}(\tilde{\mathfrak{A}})'\end{aligned}$$

To check the last equality, let $C \in \pi_{\mathbb{B}}(\mathfrak{A})'$, $x,y,z \in \mathfrak{A}$ and $t \in \mathbb{R}$

We have

$$(\Delta^{it}C\,\Delta^{-it}\pi_{\mathbb{B}}(x)y,z) =$$

$$(C\Delta^{-it}(xy),\Delta^{-it}z) = (C(\Delta^{-it}x)(\Delta^{-it}y),\Delta^{-it}z) =$$

$$(C\pi_{\mathbb{B}}(\Delta^{-it}x)\Delta^{-it}y,\Delta^{-it}z) =$$

$$(C\Delta^{-it}y,\pi_{\mathbb{B}}((\Delta^{-it}x)^{\#})\Delta^{-it}z) =$$

$$(C\Delta^{-it}y,\pi_{\mathbb{B}}(\Delta^{-it}x^{\#})\Delta^{-it}z) = \qquad \text{(Chapter V, eq.(3.4))}$$

$$(C\Delta^{-it}y,(\Delta^{-it}x^{\#})(\Delta^{-it}z)) =$$

$$(C\Delta^{-it}y,\Delta^{-it}(x^{\#}z)) = (\Delta^{it}C\Delta^{-it}y,\pi_{\mathbb{B}}(x^{\#})z)$$

that is $\Delta^{it}C\Delta^{-it} \in \pi_{\mathbb{B}}(\mathfrak{A})'$.

This concludes the proof.

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Can. J. Math., Vol. XXVIII, No. 6, 1976, pp. 1180–1186

ISOMETRIC MAPPINGS OF NON-COMMUTATIVE L_p SPACES

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If the L_p spaces of two measure spaces are "the same", to what extent can we identify the underlying measure spaces? This question has been partially answered by Schneider [7] (extending results of Forelli [2]). He proves that a linear isometry between the L_p spaces of two finite measure spaces is in fact an (isometric) homomorphism between the corresponding L_∞ spaces, if it preserves the identity.

Kadison [4] and later Russo [10], have considered what might be called non-commutative analogues of the above problem. Their point of view is different from ours, however, since their "measure spaces" are already in bijective correspondence by assumption, and their goal is to determine how much of the algebraic structure is transferred by this bijection.

In this paper, we attempt to extend Schneider's result to the non-commutative case, thus strengthening Theorem 2 of Russo [10]. Specifically, we consider two finite Von Neumann algebras $\mathscr{A}_1, \mathscr{A}_2$ with faithful traces m_1, m_2, and a *-linear map T from a *-subalgebra $\mathscr{U}$ of $\mathscr{A}_1$ to L_p $(\mathscr{A}_2, m_2)$ for some $p > 2$, which preserves the identity and the L_p-norm (see Segal [8] for the relevant definitions). We prove that T must be a Jordan homomorphism, and must preserve the operator norm (and thus, by the Riesz-Thorin-Kunze theorem [5], all L_q-norms for $q > 2$). In the absence of commutativity, we cannot conclude that T is an associative homomorphism without some extra assumptions. In fact, if $\mathscr{A}_2$ is a factor, then we can show that T must be either an (associative) homomorphism or an antihomomorphism.

The results of this paper are similar to well known results of Kadison [4]. However, our hypotheses are weaker, in that he considers the mapping T to be an isometric bijection between $\mathscr{A}_1$ and $\mathscr{A}_2$. Furthermore, his results are not applicable to our problem (but see Corollary 2.1 (ii)), because we need to prove first that T is a Jordan homomorphism (using a method entirely different from Kadison's) in order to be able to conclude that it preserves the operator norm. A similar relation exists between our results and results of B. Russo [10]. We note that our Theorem 2 is stronger, since, starting from weaker assumptions (namely, that T maps a *-subalgebra $\mathscr{U}$ of $\mathscr{A}_1$ *into* L_p $(\mathscr{A}_2, m_2)$ rather than $\mathscr{A}_1$ onto itself, and that T is *-linear, rather than positivity preserving) we are able to get stronger conclusions (namely, positivity preservation, and

Received August 13, 1975 and in revised form, March 22, 1976.

preservation of all L_p-norms, for p in $[2, \infty])$. Finally, we note that M. Broise [9] has obtained partial results in the semi-finite case.

Throughout this paper, we let $\mathscr{A}_1, \mathscr{A}_2$ be two finite Von Neumann algebras. Thus there exist faithful, central, normal states m_i on $\mathscr{A}_i$ $(i = 1, 2)$. Then if $\mathscr{A}_i$ acts on the Hilbert space $\mathscr{H}_i$, $(\mathscr{H}_i, \mathscr{A}_i, m_i)(i = 1, 2)$ are finite regular gage spaces in the sense of Segal [8].

We need a technical result constituting an extension to the present, non-commutative case, of results of Schneider [7] and Forelli [2];

THEOREM 1. *Let* $0 < p < \infty$, $f_i \in L_p(\mathscr{H}_i, \mathscr{A}_i, m_i)$ $(i = 1, 2)$ f_i *normal. Suppose that there is a positive constant* A, *such that, whenever* $z \in \mathbf{C}$ *is such that* $|z| < A$, *we have*

$$||1 + zf_1||_{L_p(m_1)} = ||1 + zf_2||_{L_p(m_2)}.$$

Then

(a) $||f_1||_{L_2(m_1)} = ||f_2||_{L_2(m_2)}$
(b) *If* $p > 2$, *then* $||f_1||_{L_4(m_1)} = ||f_2||_{L_4(m_2)}$.

Proof. Let $\mathscr{B}_i \subseteq \mathscr{A}_i$ be the Von Neumann algebra generated by the spectral projections of f_i (that is by the projections $e_\lambda{}^i$ such that $f_i = \int_{\mathbf{C}} \lambda de_\lambda{}^i$. Since f_i may be identified with a closed densely defined operator acting on $\mathscr{H}_i$ (this is because the gages are finite; see [6, Theorems 4 and 5]), it follows that $e_\lambda{}^i \in \mathscr{A}_i$).

Then $(\mathscr{H}_i, \mathscr{B}_i, m_i|_{\mathscr{B}_i})$ is a commutative finite regular gage space. It is therefore [8, pp. 402–3] algebraically equivalent to the gage space built on a finite measure space $(\mathscr{X}_i, \sigma_i)$. Since f_i is measurable with respect to $\mathscr{B}_i$ [8, Definition 2.1] it follows by [8, Theorem 2] that f_i corresponds, under the above equivalence, to a measurable function φ_i on $(\mathscr{X}_i, \sigma_i)$.

We now apply the commutative theorem of Forelli-Schneider to the functions φ_i on the measure spaces $(\mathscr{X}_i, \sigma_i)$. Note that, if $z \in \mathbf{C}$ is such that $|z| < A$,

$$\begin{aligned} ||1 + z\varphi_1||_{L_p(\sigma_1)} &= \left[\int |1 + z\varphi_1(x)|^p d\sigma_1(x) \right]^{1/p} \\ &= [m_1(|1 + zf_1|^p)]^{1/p} \quad \text{by the above equivalence} \\ &= [m_2(|1 + zf_2|^p)]^{1/p} = \left[\int |1 + z\varphi_2(x)|^p d\sigma_2(x) \right]^{1/p} \\ &= ||1 + z\varphi_2||_{L_p(\sigma_2)} < \infty \quad \text{since } f_i \in L_p(\mathscr{H}_i, \mathscr{A}_i, m_i). \end{aligned}$$

Thus the hypotheses of [2, Proposition 1] and [7, Theorem A] are satisfied, and so we conclude

(a) $||\varphi_1||_{L_2(\sigma_1)} = ||\varphi_2||_{L_2(\sigma_2)}$

and

(b) If $p > 2$, then $||\varphi_1||_{L_4(\sigma_1)} = ||\varphi_2||_{L_4(\sigma_2)}$.

The desired conclusion now follows from the fact that if $0 < q < \infty$,

$$||f_i||^q_{L_q(m_i)} = m_i(|f_i|^q) = \int |\varphi_i(x)|^q d\sigma_i(x) = ||\varphi_i||^q_{L_q(\sigma_i)}.$$

THEOREM 2. *Let $\mathscr{U} \subseteq \mathscr{A}_1$ be a unital *-subalgebra. For some p in $(2, \infty)$, let*

$$T\colon \mathscr{U} \to L_p(\mathscr{A}_2, m_2)$$

*be a *-linear map such that $T(1) = 1$. Suppose that*

$$||Tf||_{L_p(m_2)} = ||f||_{L_p(m_1)} \quad \textit{for every normal } f \in \mathscr{U}.$$

Then T is a Jordan homomorphism, that is,

$$T(fg + gf) = TfTg + TgTf, \quad f, g \in \mathscr{U}.$$

Remark 1. Young [**12**] has shown, based on the coincidence of the L_p topology and the strong topology on the unit ball of $\mathscr{A}_1$ (Dixmier [3]) that T admits an extension T_e to the weak closure $\mathscr{U}^-$ of $\mathscr{U}$, which is also an L_p-isometry. By Corollary 2.1 (see below) $T_e(\mathscr{U}^-) \subseteq \mathscr{A}_2$. By Dixmier's result, T_e will be ultraweakly continuous at 0, hence everywhere in $\mathscr{A}_1$. This provides a quicker, if indirect, proof of Lemma 3.1 of Størmer [**11**].

Remark 2. Russo [**10**] provides an example showing that our assumptions are too weak for the case $p = 2$. In this case, the stronger assumptions of his Theorem 2 are essential.

Proof. (i) Let $f \in \mathscr{U}$ be self-adjoint, and $z \in \mathbf{C}$. Since $T(1 + zf) = 1 + zTf$, we have (since Tf is also self-adjoint)

$$||1 + zf||_{L_p(m_1)} = ||1 + zTf||_{L_p(m_2)}.$$

Thus Theorem 1 (b) shows

$$||1 + zf||_{L_4(m_1)} = ||1 + zTf||_{L_4(m_2)} < \infty, \quad \text{since } f \in \mathscr{A}_1 \subseteq L_4(m_1).$$

Now

$$|1 + zf|^4 = \sum_{j,k=0}^{2} \binom{2}{j}\binom{2}{k} \bar{z}^j z^k f^j f^k$$

and so

$$||1 + zf||_4^4 = \sum_{j,k=0}^{2} \binom{2}{j}\binom{2}{k} \bar{z}^j z^k m_1(f^j f^k).$$

Similarly

$$||1 + zTf||_4^4 = \sum_{j,k=0}^{2} \binom{2}{j}\binom{2}{k} \bar{z}^j z^k m_2((Tf)^j (Tf)^k).$$

Therefore

$$(1) \quad m_1(f^j f^k) = m_2((Tf)^j (Tf)^k), \quad j, k = 0, 1, 2.$$

(ii) Putting $j = 1$, $k = 2$, in (1) yields

$$m_2((Tf)^3) = m_1(f^3).$$

Replacing f with $f + ag$, a real, f, g self-adjoint, expanding and comparing terms in a^2, we find

$$m_2(Tf(Tg)^2 + TgTfTg + (Tg)^2Tf) = m_1(fg^2 + gfg + g^2f)$$

or, in view of the centrality of the traces

(2) $m_2(Tf(Tg)^2) = m_1(fg^2)$.

On the other hand, putting $j = k = 1$ in (1) yields

$$m_2((Tf)^2) = m_1(f^2)$$

which, upon "linearization" and use of centrality as above, yields

$$m_2(TfTg) = m_1(fg).$$

Replacing g by g^2 above, and comparing the result with (2) we find

$$m_2(Tf(Tg)^2) = m_2(TfT(g^2))$$

and, replacing f by g^2, we get

(3) $m_2(T(g^2)(Tg)^2) = m_2((T(g^2))^2)$.

Finally, if we put $j = k = 2$ *in* (1), we find

$$m_2((Tg)^4) = m_1(g^4)$$

while (2) with $f = g^2$ becomes

$$m_2(T(g^2)(Tg)^2) = m_1(g^4)$$

hence

(4) $m_2((Tg)^4) = m_2(T(g^2)(Tg)^2)$.

Therefore

$$||(Tg)^2 - T(g^2)||_2^2 = m_2((Tg)^4 - (Tg)^2T(g^2) - T(g^2)(Tg)^2 + (T(g^2))^2) = 0$$

by (3) and (4), and so $(Tg)^2 = T(g^2)$ for every self-adjoint g in $\mathcal{U}$.

(iii) Now let $f \in \mathcal{U}$ be arbitrary, and write $f = f_1 + if_2$ with $f_1, f_2 \in \mathcal{U}$ self-adjoint. Since $f_1 + f_2$ is self-adjoint, part (ii) yields

$$T((f_1 + f_2)^2) = (T(f_1 + f_2))^2 = (Tf_1 + Tf_2)^2.$$

That is,

$$T(f_1^2 + f_2^2 + f_1f_2 + f_2f_1) = T(f_1^2) + T(f_2^2) + T(f_1f_2 + f_2f_1)$$
$$= (Tf_1)^2 + (Tf_2)^2 + (Tf_1Tf_2 + Tf_2Tf_1).$$

Thus

$$T(f_1f_2 + f_2f_1) = Tf_1Tf_2 + Tf_2Tf_1.$$

Therefore,

$$\begin{aligned} T(f^2) = T((f_1 + if_2)^2) &= T(f_1^2 - f_2^2 + i(f_1f_2 + f_2f_1)) \\ &= (Tf_1)^2 - (Tf_2)^2 + i(Tf_1Tf_2 + Tf_2Tf_1) \\ &= (Tf_1 + iTf_2)^2 = (Tf)^2. \end{aligned}$$

Finally, if $f, g \in \mathscr{U}$ are arbitrary, we have

$$\begin{aligned} T((f + g)^2) &= T(f^2 + g^2 + fg + gf) = (Tf)^2 + (Tg)^2 + T(fg + gf) \\ &= (T(f + g))^2 = (Tf)^2 + (Tg)^2 + TfTg + TgTf. \end{aligned}$$

Therefore,

$$TfTg + TgTf = T(fg + gf).$$

COROLLARY 2.1. (i) *If $f \in \mathscr{U}$ is self-adjoint,* $||Tf||_\infty = ||f||_\infty$.

(ii) *For every $f \in \mathscr{U}$, $||Tf||_\infty = ||f||_\infty$. Hence $T(\mathscr{U}) \subseteq \mathscr{A}_2$.*

(iii) *T is positivity preserving.*

Proof. (i) Let $l \in \mathbf{N}$. We have

$$\begin{aligned} ||Tf||^{2l}_{L_{2l}(m_2)} &= m_2(|Tf|^{2l}) = m_2((Tf^*)^l(Tf)^l) \\ &= m_2((T(f^l))^*(T(f^l))) \quad \text{by Theorem 2} \\ &= ||T(f^l)||^2_{L_2(m_2)} = ||f^l||^2_{L_2(m_1)} \quad \text{by Theorem 1 (a)} \\ &= m_1(f^{*l}f^l) = m_1(|f|^{2l}) = ||f||^{2l}_{L_{2l}(m_1)}. \end{aligned}$$

Thus

$$||Tf||_{L_{2l}(m_2)} = ||f||_{L_{2l}(m_1)}.$$

The result follows by letting l tend to infinity.

(ii) If $f \in \mathscr{U}$ is arbitrary, write $f = f_1 + if_2$ with f_k self-adjoint. Since $f_1 = \frac{1}{2}(f + f^*)$, $f_2 = (1/2i)(f - f^*)$, it follows that $||f_k||_\infty \leqq ||f||_\infty$. Therefore

$$||Tf||_\infty \leqq ||Tf_1||_\infty + ||Tf_2||_\infty = ||f_1||_\infty + ||f_2||_\infty \leqq 2||f||_\infty.$$

This shows that $T(\mathscr{U}) \subseteq \mathscr{A}_2$.

Now the proof of Kadison [**4**, Theorem 5] is applicable, and shows that T is actually isometric. (Although he assumes T to be a bijection, the argument proving that T is isometric does not depend on this, but only on the fact that T is a Jordan homomorphism and that it is isometric on self-adjoint elements, which follows from part (i) of the corollary.)

(iii) In view of part (ii), there is no loss of generality in assuming $\mathscr{U}$ to be uniformly closed.

If $f \in \mathscr{U}$ is positive, there is a unique $g \in \mathscr{U}$ such that $g^2 = f$ and $g \geqq 0$. Now $T(f) = T(g^2) = (Tg)^2$ is positive since Tg is self-adjoint. This completes the proof.

THEOREM 3 *There exists an orthogonal central projection* $p \in \mathscr{A}_2$ *such that the map*

$$T_1 : f \rightarrow T(f)p$$
$$(\textit{respectively } T_2 : f \rightarrow T(f)(1 - p))$$

*is a *-homomorphism (respectively a *-anti-homomorphism) and* $T = T_1 + T_2$ *as linear maps.*

Proof. We have shown that the image of $\mathscr{U}$ under T consists of bounded operators. Therefore, the extension T_e of T to $\mathscr{U}^-$ (cf. the first remark following Theorem 2) satisfies the hypotheses of Theorem 3.3 of Størmer [**11**].

COROLLARY 3.1. *Suppose that, in addition to the assumptions of Theorem 2,* $\mathscr{A}_2$ *is a factor. Then* T *must be either an (associative) homomorphism or an antihomomorphism.*

Proof. As T has now been proved to be a Jordan homomorphism from $\mathscr{U}$ into $\mathscr{A}_2$, this is an immediate consequence of Theorem 3.

Remark. It is not possible, without some extra assumption, to exclude one or the other possibility. For example, the identity mapping is an homomorphism of any factor onto itself, and it clearly preserves the L_p-norm. As an example of an antihomomorphism consider the mapping T defined as follows: $\mathscr{A}_1$ is a factor on a Hilbert space $\mathscr{H}$; v is an antilinear antiunitary operator from $\mathscr{H}$ to some Hilbert space $\mathscr{K}$; we put $T(f) = v^{-1}f^*v$, $(f \in \mathscr{A}_1)$. Then $T(\mathscr{A}_1)$ is a factor and T preserves the L_p-norm for every $p > 1$. (This example is due to Dixmier [**1**]).

Acknowledgements. I would like to thank Professor R. F. Streater for suggesting this problem to me. My thanks are also due the referee who pointed out Reference [**10**] to me, and an error in the original version of this paper.

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