MODELS OF NUMBER THEORY

by

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Abstract.

After introducing basic notation and results in chapter one, we begin studying the model theory of the Peano axioms, P, proper in the second chapter where we give a proof of Rabin's theorem :- that P is not axiomatizable by any consistent set of Σ_n sentences for any $n \in \omega$, and also answer a question of Gaifman raised in [7] p. 141.

Another problem, from the same article, is partially answered in chapter three, where we show every countable non-standard model, M, of P has an elementary equivalent end extension solving a Diophantine equation with coefficients in M, that was not solvable in M.

In chapter four we investigate substructures of countable non-standard models of P, and show that every such model M, contains $2^{\aleph_0}$ substructures all isomorphic to M. Other related results are also proved.

Chapter five contains theorems on indescernibles and omitting certain types in models of P.

Chapter six is concerned with the following problem : 'If $M \models P$, the set $\$(M)$, of elementary substructures of M, is lattice ordered by inclusion. Which lattices are of the form $\$(M)$ for some $M \models P$?'. We show that the pentagon lattice is of this form (answering a question suggested in [7] p. 280) and produce a class of non-modular lattices all of whose members are not of the form $\$(M)$ for

any $M \equiv N$, the standard model of P.

Elementary cofinal extensions of models of P are also investigated in this chapter.

Finally, chapter seven concludes the thesis by posing some open problems suggested by the preceding text.

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Contents.

Chapter One Introduction and Notation.

1.1 Introduction.

There are structures which cannot be distinguished from the natural number system by first order logical properties of addition and multiplication, but which are otherwise very different. Such structures are known as non-standard models of arithmetic and are the objects of investigation in this thesis.

All first order statements true of the natural numbers that we shall need in proving results about these models can in fact be deduced from a suitable first order formulation of the well-known Peano axioms, P, ([11]) and hence to obtain more generality we shall for the most part only assume our models satisfy these axioms.

In chapter two we state the Peano axioms and use model theoretic methods to investigate various equivalent and non-equivalent versions of them.

Chapter three deals with the solvability of certain Diophantine equations with coefficients possibly in a non-standard model rather than just in N, the natural number system, while chapters four and five develop further the model theory of P.

In chapter six we regard a model of arithmetic merely as a universal algebra and investigate the possible arrangements of elementary substructures and extensions of it. Finally, in chapter seven we make some concluding remarks and suggest some open problems connected with the preceding work.

1.2 Notation

We shall assume familiarity with general mathematical logic and model theory throughout (as developed in e.g. [1]). In particular we shall use the following logical symbols :

$\wedge$ - 'and' ; $\vee$ - 'or' ; $\rightarrow$ - 'implies' ; $\neg$ - 'not' ; $\exists$ - 'there exists' ; $\exists !$ - 'there exists a unique' ; $\forall$ - 'for all' .

Other symbols used are :

$\vdash$ - '(proof theoretically) entails' ; $\models$ - 'is a model of' ; $\subset$ - 'is a substructure of' or 'is a subset of', depending on the context ; $\simeq$ - 'is isomorphic to' ; $\equiv$ - 'is elementarily equivalent to' ; $\preccurlyeq$ - 'is an elementary substructure of' ; $\cap$ - intersection (of sets) ; $\cup$ - union (of sets).

If M is an L-structure for some first order language L, Th(M) denotes the set of all sentences of L true in M.

The vector symbol $\vec{x}$ will denote a sequence $x_0, x_1, \ldots$ of arbitrary finite length unless we specifically mention the length.

Structures will usually be identified with their domains where no confusion can arise. Thus if M is a structure we write $a \in M$ for a is an element of the domain of M, and $\overline{\overline{M}}$ for the cardinality of the domain of M, etc. Also, if $a_0, \ldots, a_{n-1} \in M$ and $\phi(x_0, \ldots, x_{n-1})$ is a formula of L(M), the language of M, with the variables $x_0,\ldots,x_{n-1}$ free, we write $M \models \phi(a_0,\ldots,a_{n-1})$, where we should properly write $M \models \phi(x_0,\ldots,x_{n-1})[a_0,\ldots,a_{n-1}]$.

Finally, ω will always denote the set of natural numbers - i.e. the first transfinite ordinal, and m, n will be reserved for representing elements of ω.

Other notations and conventions will be introduced in the sequel as we need them.

As we have already mentioned, we shall require some theorems, known to be true in N, to be provable from P. Such proofs of most of the theorems we need will usually be very easy ; although there are two exceptions.

The first is Matijasevič's theorem (3.1.1.); and that the usual proof [9], can be converted to one from P has been checked by A. Pridor ([7] footnote p. 133).

The second is some form of an enumeration theorem (e.g. 3.2.4.) of Σ_n predicates (see def. 2.3.2.). Since usual proofs in Th(N) of such theorems only require a certain elementary coding defined by induction, they need only a little extra formalism to be rigorous proofs from P, and we leave the details to the reader.

Chapter 2. P and related systems.

2.1. The Peano axioms and their basic model theory.

Let L be the first-order predicate language having as non-logical symbols two 2-place function symbols, + (addition) and · (multiplication); and one 1-place function symbol, ' (successor); and one 0-place function symbol 0 (zero).

This thesis is concerned with the model theory of the following axiom system, denoted by P, formulated in L :

P.1 $(\forall x)(x' \neq 0)$,

P.2 $(\forall x)(\forall y)(x' = y' \rightarrow x = y)$,

P.3 (i) $(\forall x)(x + 0 = x)$,

(ii) $(\forall x)(\forall y)(x + y' = (x + y)')$,

P.4 (i) $(\forall x)(x \cdot 0 = 0)$,

(ii) $(\forall x)(\forall y)(x \cdot y' = x \cdot y + x)$,

P.5$_\phi$ $((\phi(0) \wedge (\forall y)(\phi(y) \rightarrow \phi(y'))) \rightarrow (\forall y)\phi(y))$,

where $\phi(y)$ is any formula of L having just the variable y free.

This chapter is concerned with various equivalent and non-equivalent reformulations of these axioms. We first, however, introduce some basic well-known facts about the model theory of P.

N will denote the <u>standard model of P</u>, i.e. the L-structure $\langle \omega, +, \cdot, ', 0 \rangle$, where the operations mentioned are just the ordinary addition, multiplication and successor functions on the set of natural numbers, ω.

Any model, M, of P, not isomorphic to N will be called <u>non-standard</u>.

If $M \models P$, a subset A of M will be called <u>definable</u> (<u>in M</u>) if there is a formula $\phi(x)$ in L, with just x free, s. th.

$$a \in A \quad \text{iff} \quad M \models \phi(a).$$

An element a, of M, will be called definable if $\{a\}$ is definable (in M), and M is <u>pointwise-definable</u> if a is definable for all $a \in M$.

A subset A of M is an <u>initial segment of M</u> if $a \in A$, $b \in M$ and $M \models b \leq a \Rightarrow b \in A$, where $(x \leq y)$ is the formula in L defined by

$$x \leq y \quad \text{iff} \quad (\exists z)(x + z = y).$$

We further define:

$$x < y \quad \text{iff} \quad x \leq y \wedge x \neq y.$$

It is easy to show that if $M \models P$, there is a unique embedding $e: N \rightarrow M$ s. th. $e[N]$ is an initial segment of M, and we always identify N (or ω) with this initial segment, and call any element of M-N <u>non-standard</u> or <u>infinite</u>.

The following results are easily proved.

<u>Theorem 2.1.1.</u>

(i) If $M \equiv M'$ and M and M' are pointwise-definable models of P, then $M \simeq M'$.

(ii) if $M \models P$, there is an $M' \preccurlyeq M$ s. th. M' is pointwise- definable, and (by (i)) M' is unique with these properties.

The reason for our current interest in pointwise definable models is to prove a syntactic result about P, namely

<u>Theorem 2.1.2.</u> (Friedman).

Let P' consist of the axioms P.1, P.2, P.3, P.4 and for each formula $\phi(\vec{x},y)$ of L having just the variables $\vec{x}$, y free,

P.5_ϕ $\quad (\forall\vec{x})((\phi(\vec{x},0) \wedge (\forall y)(\phi(\vec{x},y) \rightarrow \phi(\vec{x},y'))) \rightarrow (\forall y)\phi(\vec{x},y))$.

Then P and P' are deductively equivalent.

<u>Proof.</u>

Clearly $P' \vdash P$.

Suppose $M \models P$. It is sufficient to show $M \models P'$. Using thm. 2.1.1., let $M' \preccurlyeq M$ where M' is pointwise definable.

Let $\phi(\vec{x},y)$ be any formula of L having just the variables $\vec{x}$, y free. It is sufficient to show

$$M' \models P.5_\phi.$$

Suppose $\vec{a} \subset M'$. Say $\vec{a} = \langle a_0,\dots,a_{n-1}\rangle$.

Choose fmls. $\phi_0(y),\dots,\phi_{n-1}(y)$ of L s. th.

$$M' \models \phi_i(a_i) \wedge (\exists! y)\phi_i(y) \qquad 0 \leqslant i \leqslant n-1,$$

which is possible since M' is pointwise definable.

Let $\psi(z)$ be the formula:

$$(\exists y_0,\dots,y_{n-1})(\bigwedge_{i=0}^{n-1} \phi_i(y_i) \wedge \phi(\vec{y},z)).$$

Then since $M' \preccurlyeq M \models P$ we have $M' \models P.5_\psi$, from which it follows that

$$M' \models (\phi(\vec{a},0) \wedge (\forall y)(\phi(\vec{a},y) \rightarrow \phi(\vec{a},y')) \rightarrow (\forall y)\phi(\vec{a},y)).$$

But $\vec{a}$ was an arbitrary n-tuple from M'. Hence $M' \models P.5_\phi$ as required.

□

Perhaps the most well-known variant of P are the well-ordering axioms, W.O.. It is easy to show that P can prove the formula $(x < y)$ defines a total ordering, but W.O. states it is, in a certain sence, a well-ordering. More precisely the axioms of W.O. are:

P.1, P.2, P.3, P.4, together with

WO_ϕ : $\quad ((\exists y)\phi(y) \rightarrow (\exists y)(\phi(y) \wedge (\forall z)(z < y \rightarrow \neg\phi(z))))$, where $\phi(y)$ is any formula of L having just the variable y free.

It is easy to show :

Theorem 2.1.3.

$$P \vdash W.O.$$

However, we have :

Theorem 2.1.4.

$$W.O. \nvdash P.$$

Proof.

Consider the L-structure $\langle \omega^{\omega}, \oplus, \odot, \textcircled{\prime}, \emptyset \rangle = M$, where $\oplus$, $\odot$, $\textcircled{\prime}$ are just ordinal addition, multiplcation and successor respectively, restricted to the ordinal ω^{ω}.

That $M \models W.O.$ is clear. However, ordinal addition is not commutative, whereas the sentence $(\forall x)(\forall y)(x + y = y + x)$ can easily be proved in P. The theorem now follows.

□

The proof of theorem 2.1.4. exhibits a very simple sentence of L which is provable in P but not in W.O., and thus one might think that W.O. is very much weaker than P.

The gap between P and W.O., however, can easily be bridged.

Let Q denote the (Robinson) sentence :

$$(\forall x)(x \neq 0 \rightarrow (\exists y)(y' = x)).$$

Then it is easy to verify :

Theorem 2.1.5.

P and W.O. $\cup$ {Q} are deductively equivalent.

We can now define W.O.' in an analagous way to P' and we leave the reader to check that W.O.' ∪ {Q} and W.O. ∪ {Q} are deductively equivalent.

2.2 Overspill.

The well-ordering axioms imply a very important model-theoretic result which we shall be using throughout this and the following two chapters. It is the so-called overspill lemma, and has many forms, the most easily stated of which is :

Theorem 2.2.1. (Robinson).

Suppose M is a non-standard model of P, $\phi(\vec{x},y)$ any formula of L, and $\vec{a} \subset M$. Suppose further that for all infinite $b \in M$, $M \models (\exists x)(x < b \wedge \phi(\vec{a},x))$. Then $M \models \phi(\vec{a},n)$ for some $n \in \omega$.

Proof.

Since $M \models W.O.'$, there must be some $\leqslant$-least element, n of M s. th. $M \models \phi(\vec{a},n)$ and such an n cannot be infinite by the theorem hypothesis.

□

There are other variations of 2.2.1. that we shall use in the sequel and we shall just refer to them as 'overspill'. The most common will be- 'if $M \models (\forall x)(\exists ! y)\phi(x,y)$, so that we may write $g(x) = y$ for $\phi(x,y)$, and if g takes only finite values for finite arguements, and takes arbitrary large finite values, then g takes arbitrary small infinite values for arbitrary small infinite arguements.'

This can be proved easily using 2.2.1., and we leave the details to the reader.

2.3 Finite axiomatizability.

In [16] R. Ryll-Nardzewski proves that if S is any finite consistent set of sentences of L, then $S \nvdash P$. Rabin, in [13], proves a more general result, but we need some definitions before we can state it.

Def. 2.3.1.

The set B, of bounded formulae of L, is the smallest set s. th. :

(i) Every atomic formula of L is in B.

(ii) If ϕ, $\psi \in B$, then so are $\phi \wedge \psi$, $\neg\phi$, and $\phi \vee \psi$.

(iii) If $\phi \in B$, then $(\forall x)(x < y \rightarrow \phi) \in B$ and $(\exists x)(x < y \wedge \phi) \in B$, where x, y do not occur bound in ϕ.

We write $(\forall x{<}y)\phi$ and $(\exists x{<}y)\phi$ for $(\forall x)(x{<}y \rightarrow \phi)$ and $(\exists x)(x{<}y \wedge \phi)$ respectively, from now on.

Def. 2.3.2.

The sets Σ_n, Π_n of formulae of L are defined by induction on $n \in \omega$:

(i) $\Sigma_0 = \Pi_0 = B$.

(ii) $\Sigma_{n+1} = \{(\exists\vec{x})\phi \ : \ \phi \in \Pi_n$, no member of $\vec{x}$ bound in $\phi\}$.

$\Pi_{n+1} = \{(\forall\vec{x})\phi \ : \ \phi \in \Sigma_n$, no member of $\vec{x}$ bound in $\phi\}$.

The following theorem is well-known.

Theorem 2.3.3.

If $n \geq 1$ and $\phi, \psi \in \Sigma_n$ (Π_n), there are formulae $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5 \in \Sigma_n$ (Π_n), and $\psi_6 \in \Pi_n$ (Σ_n) s. th. :

(i) $P \vdash \psi_1 \leftrightarrow (\phi \wedge \psi)$,

(ii) $P \vdash \psi_2 \leftrightarrow (\phi \vee \psi)$,

(iii) $P \vdash \psi_3 \leftrightarrow (\exists x)\phi$ $(P \vdash \psi_3 \leftrightarrow (\forall x)\phi)$,

(iv) $P \vdash \psi_4 \leftrightarrow (\exists x<y)\phi$,

(v) $P \vdash \psi_5 \leftrightarrow (\forall x<y)\phi$,

(vi) $P \vdash \psi_6 \leftrightarrow \neg\phi$,

where neither x nor y occur bound in ϕ.

Now Rabin's theorem asserts that if $n \in \omega$ and S is any consistent set of sentences s. th. $S \subset \Sigma_n$, then $S \nvdash P$. We give here another proof of Rabin's theorem while answering, en route, a problem raised by Gaifman in [7]. Gaifman asked whether a certain semantic property of L-structures forced them to be models of P. We make this more precise now.

Def. 2.3.4.

Let T be any extension of P in L. A fml. $\phi(\vec{x},y)$ is said to be <u>T-functional</u> if :

(i) $T \vdash (\forall \vec{x})(\exists! y)\phi(\vec{x},y)$,

and n-T-functional if we also have :

(ii)$_n$ $\exists \psi(\vec{x},y) \in \Sigma_n$ s. th. $T \vdash (\forall \vec{x},y)(\psi(\vec{x},y) \leftrightarrow \phi(\vec{x},y))$.

Def. 2.3.5.

Let $M \models P$ and $M' \subset M$. We say that

(i) M' is <u>n-functionally closed in M</u> if whenever $\phi(\vec{x},y)$ is n-Th(M)-functional, $\vec{a} \subset M'$ and

$M \models \phi(\vec{a}, b)$ for $b \in M$, then $b \in M'$,

and (ii) <u>$M' \preccurlyeq_n M$</u> if $\forall \phi(\vec{x}) \in \Sigma_n$, $\forall \vec{a} \subset M'$, $M \models \phi(\vec{a})$ iff $M' \models \phi(\vec{a})$.

Now Gaifman's problem is this : 'Is there an $n \in \omega$, s. th. whenever $M \models P$ and M' is an n-functionally closed initial segment of M, then $M' \models P$?'.

We prove the following :

<u>Theorem 2.3.6.</u>

Let M be a non-standard model of P and $n \in \omega$. Then there is an initial segment, I, of M s. th.

(i) $I \preccurlyeq_n M$.

(ii) I is n+1-functionally closed in M.

(iii) $I \not\models P$.

<u>Proof.</u>

Let b be an infinite element of M. The domain of I is the set $\{a \in M : M \models (\exists y)(\phi(b,y) \wedge y \geq a)$, for some n+1-Th(M)-functional formula, $\phi(x,y)$, of L with just x, y free. $\}$.

+, $\cdot$, and ' are defined on I as those functions induced by M. Clearly I is an initial segment of M. To show I is n+1-functionally closed in M, suppose $\vec{a} = \langle a_0, \ldots, a_{m-1} \rangle \subset I$, $\phi(\vec{x}, y)$ is any n+1-Th(M)-functional formula, and that $M \models \phi(\vec{a}, c)$ where $c \in M$.

We must show $c \in I$.

Now by def. of I, there are n+1-Th(M)-functional $\psi_0(x,y), \ldots, \psi_{m-1}(x,y)$ s. th.

$$M \models \bigwedge_{i=0}^{m-1} (\exists y)(\psi_i(b,y) \wedge y \geq a_i).$$

We define $\psi(x,y)$ s. th.

$$\text{"}\psi(x,y) \text{ iff } y = \max_{\substack{t_i \leq z_i \\ \text{s.th. } \psi_i(x,z_i) \\ 0 \leq i \leq m-1}} \{s : \phi(\vec{t},s)\}\text{"}.$$

More precisely ψ is defined by :

$$\psi(x,y) \text{ iff } (\exists z_0,\ldots,z_{m-1})(\bigwedge_{i=0}^{m-1}\psi_i(x,z_i) \wedge (\forall t_0 \leqslant z_0)\ldots$$
$$(\forall t_{m-1} \leqslant z_{m-1})(\exists s)(\phi(\vec{t},s) \wedge y \geqslant s) \wedge (\exists t'_0 \leqslant z_0)\ldots(\exists t'_{m-1} \leqslant z_{m-1})$$
$$(\exists s')(\phi(\vec{t}',s') \wedge s' = y)).$$

That $\psi(x,y)$ satisfies (i) of def. 2.3.4. is easily checked when $T = Th(M)$, and $2.3.4.(ii)_{n+1}$ follows from theorem 2.3.3. Hence $\psi(x,y)$ is n+1-Th(M)-functional.

So for some $d \in I$, $M \models \psi(\dot{b},\dot{d})$.

Also by the def. of ψ, $M \models d \geqslant c$.

Therefore $c \in I$ as required, by the def. of I.

We now show $I \preccurlyeq_n M$.

We prove by induction on m, that if $0 \leqslant m$ and $m \leqslant n$, then

(*)..... $\phi(\vec{x}) \in \Sigma_m$, $\vec{a} \subset I \Rightarrow (M \models \phi(\vec{a}) \Rightarrow I \models \phi(\vec{a}))$.

For $m = 0$, (*) follows from the fact that I is an initial segment of M and classical preservation theorems (see e.g. [3]).

Suppose (*) is true for some $m < n$, $\phi'(\vec{x}) \in \Sigma_{m+1}$, $\vec{a} \subset I$ and $M \models \phi'(\vec{a})$.

We must show $I \models \phi'(\vec{a})$.

Now $\phi'(\vec{x})$ can be supposed to be of the form $(\exists y)\phi(\vec{x},y)$, where $\phi(\vec{x},y) \in \Pi_m$, by using some standard Σ_0 coding of finite sequences (see e.g. [7] for details), and employing theorem 2.3.3.

Define $\psi(\vec{x},y)$ by :

$\psi(\vec{x},y) \Leftrightarrow_{df.} ((\exists z)\phi(\vec{x},z) \wedge \phi(\vec{x},y) \wedge (\forall t<y)\neg\phi(\vec{x},t)) \vee$

$\vee (\neg(\exists z)\phi(\vec{x},z) \wedge y = 0)$.

Again it is easy to check that $\psi(\vec{x},y)$ is an m+2-Th(M)-functional formula.

But $m+2 \leq n+1$, and so $M \models \psi(\vec{a},c) \Rightarrow c \in I$, since I is n+1-functionally closed in M and $\vec{a} \subset I$. Also, $M \models (\forall \vec{x})((\exists y)\phi(\vec{x},y) \rightarrow (\exists y)(\psi(\vec{x},y) \wedge \phi(\vec{x},y)))$, from the def. of ψ; and $M \models (\exists y)\phi(\vec{a},y)$, by supposition. Hence $\exists c \in I$ s. th. $M \models \phi(\vec{a},c)$.

But $\phi \in \Pi_m$, so, by the inductive hypothesis, $I \models \phi(\vec{a},c)$. Therefore $I \models (\exists y)\phi(\vec{a},y)$; i.e. $I \models \phi'(\vec{a})$, as required.

Now to prove $I \not\models P$.

Let $S(x,y,z)$ be a formula of L s. th.

$\forall \phi(y,z) \in \Sigma_{n+1}$ $\exists m \in \omega$ s. th. $P \vdash (\forall y,z)(S(m,y,z) \leftrightarrow \phi(y,z))$.

(See chapter one for details about the existence of such an S).

Now suppose $I \models P$.

Then I claim that, for all infinite $c \in I$,

(**)..... $I \models (\forall x)(\exists y<c)(\exists z)(S(y,b,z) \wedge z \geq x \wedge (\forall t)(S(y,b,t) \rightarrow$

$\rightarrow t = z))$.

For let c be an infinite element of I, and $x_0 \in I$.

Then there is an n+1-Th(M)-functional $\psi(x,y)$ and $d \in M$ s. th. $M \models \psi(b,d) \wedge d \geq x_0$. In fact $d \in I$, clearly.

Also there is a $\chi(x,y) \in \Sigma_{n+1}$ and $m \in \omega$ s.th.

$$M \models (\forall x,y)(\psi(x,y) \leftrightarrow \chi(x,y)),$$

and $$M \vdash (\forall x,y)(S(m,x,y) \leftrightarrow \chi(x,y)). \quad .....(1)$$

It follows that :

$M \models S(m,b,d) \wedge d \geq x_0 \wedge (\forall t)(S(m,b,t) \rightarrow t = d)$........(2).

Now from (1), we can find $\chi^*(u,x,y) \in \Pi_n$ s.th.

$P \vdash (\forall x,y)(S(m,x,y,) \leftrightarrow (\exists u)\chi^*(u,x,y))$ ....(3).

Define $\theta(x,y)$ by :

$$\theta(x,y) \Leftrightarrow_{df.} (\exists u,y)(z = 2^u 3^y \wedge \chi^*(u,x,y)) \wedge (\forall u,y<z)(2^u 3^y < z \rightarrow \neg\chi^*(u,x,y)).$$

Then $\theta(x,y)$ is $n+1$-Th(M)-functional, and, by the similar property of χ, it follows that there is some $e \in I$ s. th. $M \models \theta(b, 2^e 3^d)$.

Thus $M \models \chi^*(e,b,d)$, by the def. of θ.

But $\chi^* \in \Pi_n$, and $I \preccurlyeq_n M$; so $I \models \chi^*(e,b,d)$.

Hence $I \models (\exists u)\chi^*(u,b,d)$.

Therefore by (3), $I \models S(m,b,d)$ since $I \models P$.

Hence from (2) and the fact that $I \preccurlyeq_n M$,

$I \models d \geq x_0$, and $I \models (\forall t)(S(m,b,t) \rightarrow t = d)$.

Also, since c is infinite and $m \in \omega$, $I \models m<c$.

Putting all this together, and observing that x_0 was any member of I gives (**).

Now, by overspill, (**) must hold for some finite c, and this is clearly impossible because it implies I has a $\leq$-largest element, whereas I, being a model of P, cannot have. Hence $I \not\models P$.

□

We now have the following immediate consequences of theorem 2.3.6.

<u>Corollary 2.3.7.</u>

Gaifman's problem (on p.16) has a negative answer.

<u>Corollary 2.3.8.</u>

If T is any consistent set of Σ_n sentences, then $T \not\vdash P$.

Chapter 3 Diophantine Equations over Models of P, and Related Topics.

3.1 Introduction to the problem.

In [12] Rabin shows that if M is any non-standard model of P, there is a diophantine equation, with coefficients in M, which is unsolvable in M, but solvable in some extension, M', of M so that M' ≡ M. In the light of Matijasevic's theorem ([9]) however, (which was not known when Rabin proved his theorem), Rabin's result is rather easily proved using the existence of a (Post) simple set. One now naturally asks - 'What sort of extension of M can M' be ?'.

Gaifman has shown ([7]) that M' can always be chosen to be an end extension of an elementary cofinal extension of M, and asks whether it could in fact be chosen to be an end extension of M. In this chapter we prove that it can when M is countable, and would like to take this opportunity of thanking A. Macintyre for first suggesting this problem to the author and for pointing out that Friedman's theorem (3.2.7.) might be helpful in its solution.

We now state Matijasevic's theorem which will be required in the proof.

Theorem 3.1.1.

Let $\phi(\vec{x})$ be any Σ_1 formula. Then there is a Σ_1 formula $\psi(\vec{x})$ in prenex normal form, all of whose quantifiers are existential, s. th.

$$P \vdash (\forall \vec{x})(\phi(\vec{x}) \leftrightarrow \psi(\vec{x})).$$

(See chapter one for further comments on this result).

3.2 Construction of non-$\leq_n$ extensions.

We now take a non-standard model, M, of P which will remain fixed throughout this chapter.

Def. 3.2.1.

A formula $\phi(\vec{x},y)$, of L, is said to be uniform in y, if

(i) $M \models (\forall \vec{x})((\exists y)\phi(\vec{x},y) \rightarrow (\exists! y)\phi(\vec{x},y))$.

Thus uniform formulae define partial functions in M.

If ϕ satisfies, in additon to (i),

(ii) $\phi(\vec{x},y) \in \Sigma_n$,

then we write $\phi \in \Sigma_n(\vec{x} \rightarrow y)$.

Lemma 3.2.2. (Uniformisation).

Suppose $n \geq 1$, and $\phi(\vec{x},y) \in \Sigma_n$. then there is a formula $\phi^*(\vec{x},y) \in \Sigma_n(\vec{x} \rightarrow y)$ s. th.

(i) $M \models (\forall \vec{x},y)(\phi^*(\vec{x},y) \rightarrow \phi(\vec{x},y))$.

(ii) $M \models (\forall \vec{x})((\exists y)\phi(\vec{x},y) \rightarrow (\exists! y)\phi^*(\vec{x},y))$.

Proof.

Suppose $\phi(\vec{x},y) = (\exists z)\phi'(\vec{x},y,z)$ where $\phi' \in \Pi_{n-1}$. Let $(s = \langle u,v \rangle)$ be a formula in B s. th. $\lambda u,v{:}\langle u,v \rangle$ is a pairing function.

Let $\psi(\vec{x},y,z) \iff_{df.} \phi'(\vec{x},y,z) \wedge (\exists t)(t = \langle y,z \rangle \wedge$ $\wedge (\forall s<t)((\exists y',z')(s = \langle y',z' \rangle \wedge \neg\phi'(\vec{x},y',z'))))$.

Now put $\phi^*(\vec{x},y) \iff_{df.} (\exists z)\psi(\vec{x},y,z)$.

It is easy to check that ϕ^* has the required properties.

□

Def. 3.2.3.

A formula $\phi(x)$ having just one free variable is called n-simple iff :

(i) $\phi(x) \in \Sigma_n$.

(ii) $M \models (\forall x)(\exists y>x)\neg\phi(y)$.

(iii) If $\psi(x) \in \Sigma_n$, and $M \models (\forall x)(\exists y>x)\psi(y)$, then $M \models (\exists y)(\psi(y) \wedge \phi(y))$.

To prove the existence of an n-simple formula we introduce a full form of the enumeration theorem. (We only required a weak form in theorem 2.3.6.).

Thus we assume the following :

Lemma 3.2.4. (essentially Kleene [8]).

If n, m ⩾ 1, there is a formula $T_{n,m}(t,x_0,\ldots,x_{m-1})$ of L in m+1 free variables s.th.

(i) $T_{n,m} \in \Sigma_n$.

(ii) $\forall\, \psi(x_0,\ldots,x_{m-1}) \in \Sigma_n$, $\exists k \in \omega$ s. th. $M \models (\forall \vec{x})(T_{n,m}(k,\vec{x}) \leftrightarrow \psi(\vec{x}))$.

It will be convenient to use set-theoretic notation from now on. In particular we shall write $\vec{x} \in W_t^{n,m}$ for $T_{n,m}(t,\vec{x})$, and if A is a definable subset of M 'A infinite' means $(\forall x)(\exists y>x)(y \in A)$ -i.e. A is unbounded in M or A is M-infinite. We shall also identify formulae of m free variables with the sets they define in M, and use finite intersection $(\bigcap_{i\leqslant x})$ and union $(\bigcup_{i\leqslant x})$ signs etc. It will be clear that such 'formulae' can be naturally translated back into proper expressions in L.

Lemma 3.2.5. (Post).

If n ⩾ 1, there is an n-simple formula.

Proof.

Let $\psi(x,y) \Leftrightarrow_{df.} y \in w_x^{n,1} \wedge y > 2x$.

Let $\psi^*(x,y)$ be the uniformisation of $\psi(x,y)$ for y given by lemma 3.2.2.

Then $\theta(y) \Leftrightarrow_{df.} (\exists x)\psi^*(x,y)$ is n-simple. (See [15] p.106 for the easy details).

□

Lemma 3.2.6.

Let ϕ be an n-simple formula, where $n \geq 1$. then there are elements $a, b \in M$ s. th.

(i) $M \models \neg\phi(a)$ and $M \models \phi(b)$,

and (ii) $\forall \psi \in \Sigma_n$ having just one free variable, $M \models \psi(a) \Rightarrow M \models \psi(b)$.

Proof.

We define (in P), sets $R_0, R_1, \ldots$ s. th. $(y \in R_x) \in L$, by induction as follows :

$R_0 = \{x : \phi(x) \wedge x \in w_0^{n,1}\}$ if this is infinite,

$\{x : \phi(x) \wedge x \notin w_0^{n,1}\}$ otherwise.

$R_{x+1} = R_x \cap w_{x+1}^{n,1}$ if this is infinite,

$R_x \cap Cw_{x+1}^{n+1}$ otherwise.

(CA = complement of A).

This can be shown to be a good definition in P, and the following results follow from the induction schema in P - which is true in M.

$M \models (\forall x)(R_x$ is infinite) ....(1).

$M \models (\forall x)(R_x \subset \{x : \neg\phi(x)\} \wedge R_{x+1} \subset R_x)$.......(2).

$M \models (\forall x)(R_x \subset w_x^{n,1} \vee R_x \subset Cw_x^{n,1})$ ....(3).

Let S_ϕ be $\{x \in M : M \models \phi(x)\}$.

Then since S_ϕ is an 'n-simple set' we have :

$\forall p \in \omega \quad M \models (w_p^{n,1}$ infinite$) \rightarrow (\exists z)(z \in w_p^{n,1} \wedge z \in S_\phi)$.
Therefore by overspill, for some infinite $\beta \in M$ we have :

$$M \models (\forall s<\beta)((w_s^{n,1} \text{ infinite}) \rightarrow (\exists z)(z \in w_s^{n,1} \wedge z \in S_\phi) \ldots (*).$$

Now put $g(x) =_{df.} \mu y: w_y^{n,1} = \cap\{w_i^{n,1} : i<x \wedge A(i)\}$ where $A(i) \Leftrightarrow_{df.} R_i \subset w_i^{n,1}$.

Since the conjunction of finitely many Σ_n formulae is equivalent in P (and so in Th(M)) to a Σ_n formula (by theorem 2.3.3.), g takes only finite values for finite arguements and can be supposed to take arbitrarily large finite values. It now follows from an overspill arguement that we can find an infinite $\alpha \in M$ s. th. $g(\alpha)$ is infinite and α, $g(\alpha) < \beta$.

Now by (2), $M \models R_\gamma \subset w_{g(\alpha)}^{n,1}$, where γ is the largest member of $M < \alpha$ s. th. $R_\gamma \subset w_\gamma^{n,1}$. ($\gamma$ must exist, cleary).

Hence by (1), $M \models (w_{g(\alpha)}^{n,1}$ is infinite$)$.

So by (*), $M \models (\exists z)(z \in w_{g(\alpha)}^{n,1} \wedge z \in S_\phi)$.

Let b be such a z, and let a be any element of $R_{g(\alpha)}$ ($\neq \emptyset$ by (1)).

Then (i) of the theorem is satisfied by such an a and b by their choice and (2).

For (ii) suppose $\psi \in \Sigma_n$ and $M \models \psi(a)$.

This can be written as $M \models a \in w_k^{n,1}$ for some $k \in \omega$.

We show $M \models R_k \subset w_k^{n,1}$.

Suppose this false. Then by (3) :

$$M \models R_k \subset Cw_k^{n,1} ,$$

But $\kappa < \gamma$, so it follows from (2) that :

$$M \models R_{g(\alpha)} \subset R_k .$$

Hence $M \models a \in Cw_k^{n,1}$ by def. of a, – a contradiction.

Thus we have $M \models k < \gamma \wedge R_k \subset w_k^{n,1}$.

Therefore $M \models w_{g(\alpha)}^{n,1} \subset w_k^{n,1}$ by def. of g.

So $M \models b \in w_k^{n,1}$ by def. of b.

i.e. $M \models \psi(b)$ – hence (ii).

□

Now to complete the proof of the result mentioned in section 3.1. we require a generalisation of a theorem of Friedman [4], which is :

<u>Theorem 3.2.7.</u>

Every non-standard countable model of P is isomorphic to a proper initial segment of itself.

The generalisition, which is obtained by an easy modification of Friedman's proof, is :

<u>Theorem 3.2.8.</u>

Let $n \geq 1$ and $a, b \in M$ be s. th. for all formulae $\psi(x) \in \Sigma_n$ with just x free, $M \models \psi(a) \Rightarrow M \models \psi(b)$. Then there is a proper initial segment $I \subset M$ s. th.

(i) there is an isomorphism $e: M \rightarrow I$,

(ii) $b \in I$ and $e(a) = b$,

and (iii) $I \preccurlyeq_{n-1} M$.

(In fact, this result is a trivial corollary of 4.1.10., proved in the next chapter).

We can now prove the main result of this chapter :

Theorem 3.2.9.

If $n \geq 1$, M contains a proper initial segment I, s. th.

(i) $I \simeq M$,

and (ii) $I \preccurlyeq_{n-1} M$ but $I \not\preccurlyeq_n M$

Proof.

Let $n \geq 1$. Choose a, b, ϕ with the properties stated in lemma 3.2.6. and I with the properties in theorem 3.2.8. with this a and b.

Then $I \simeq M$ and $I \preccurlyeq_{n-1} M$.

Now $M \models \neg\phi(a)$, therefore $I \models \neg\phi(e(a))$, since e is an isomorphism from M to I.

Also $M \models \phi(b)$, i.e. $M \models \phi(e(a))$.

Thus $e(a) = b \in I$, $\phi(x) \in \Sigma_n$ and $M \models \phi(e(a))$, but $I \models \neg\phi(e(a))$. This shows $I \not\preccurlyeq_n M$ and completes the proof.

□

Corollary 3.2.10.

There is an end extension M' of M s. th. $M' \simeq M$, and s. th. M' solves a Diophantine equation with coefficients in M, that is not solvable in M.

Proof.

By theorem 3.2.9., with $n = 1$, M may be regarded as a non-$\preccurlyeq_1$ end extension of itself. The corollary now follows from theorem 3.1.1.

□

Chapter 4 Substructures of Models of P.

4.1. The number of substructures.

M will again be a countable non-standard model of P fixed until further notice.

Theorem 3.2.7. tells that M contains infinitely many substructures all isomorphic to M. Clearly there can be at most $2^{\aleph_0}$ such substructures and this section is devoted to proving that there are exactly $2^{\aleph_0}$.

We require some definitions and lemmas.

Def. 4.1.2.

If S_1, S_2 are subsets of M, $S_1 < S_2$ iff $M \models a < b \ \forall a \in S_1, \forall b \in S_2$. If $a \in M$, $S_1 < a$ iff $S_1 < \{a\}$.

Def. 4.1.3.

$2 = p_0$, $3 = p_1, \ldots, p_\alpha, \ldots$ $\alpha \in M$ is the enumeration of the primes of M in increasing order (this is definable in M), and $\exp_t(x)$ is the exponent of p_t in the prime factorisation of x (which is also definable in M).

Lemma 4.1.4.

Let $\alpha \in M$ be infinite. Then there is an initial segment, I, of M, s. th. I contains infinite elements of M, $I < \alpha$ and $\forall k \in \omega$,

$$\alpha_0, \ldots, \alpha_{k-1} \in I \implies 2^{\alpha_0} 3^{\alpha_1} \ldots p_{k-1}^{\alpha_{k-1}} \in I.$$

Proof.

Define the function $F(x,y)$ by induction as follows :

$$F(0,y) = p_y$$
$$F(x+1,y) = p_y^{F(x,y)}.$$

Choose $b \in M$, infinite, s. th. $F(b,b) < \alpha$. (This is possible by overspill).

Then $I = \{a \in M : \exists k \in \omega,\ M \models a < F(k,b)\}$ can easily be shown to satisfy the lemma conclusions.

□

<u>Def, 4.1.5.</u>

For $\vec{b}$, $a \subset M$, we write $\underline{\vec{b} \to_n a}$ iff there is a fml. $\phi(\vec{x},y) \in \Sigma_n(\vec{x} \to y)$, s. th. $M \models \phi(\vec{b},a)$.

$\vec{b} \not\to_n a$ means not$(\vec{b} \to_n a)$.

<u>Def. 4.1.6.</u>

If $S \subset M$, we write $C^n(S)$ iff :

(i) $\vec{a} \subset S$, $\phi(\vec{x},y) \in \Sigma_n$ and $M \models (\exists y)\phi(\vec{a},y)$ imply $\exists b \in S$, $M \models \phi(\vec{a},b)$,

and (ii) there is an infinite $a \in M$ s. th. $\forall x \in M$, $x \leqslant a \Rightarrow x \in S$.

We describe (i) by saying S is Σ_n-closed (in M).

<u>Lemma 4.1.7.</u>

Suppose $n \geqslant 1$, $S \subset M$, $C^n(S)$, $\vec{b} \subset S$ and $a \in M$ is s. th. $\vec{b} \not\to_n a$. then there is an $S_a \subset S$, s. th. $C^n(S_a)$, $\vec{b} \subset S_a$ and $a \notin S_a$.

<u>Proof.</u>

Let $\sigma^t_{n,r}$ denote the fml. with one free variable t, $(n.r \in \omega)$:

$(\forall\vec{x})((\exists y)(<\vec{x},y> \in w_t^{n,r+1}) \to (\exists! y)(<\vec{x},y> \in w_t^{n,r+1}))$,

where $\vec{x} = <x_0,\ldots,x_{r-1}>$.

We first show that it is not the case that for all infinite $\beta \in M$, $\exists\alpha \in M$ with $\alpha < \beta$, s. th. $<\vec{b},\alpha> \to_n a$.

For suppose it was.

Then for all infinite $\beta \in M$,

$M \models (\exists k,\alpha<\beta)(<\vec{b},\alpha,a> \in w_k^{n,m+2} \wedge \sigma_{n,m+1}^k)$,

where $\vec{b} = <b_0,\ldots,b_{m-1}>$.

Hence there must be some finite β s. th. this fml. holds, i.e. there are natural numbers s, t s. th.

$M \models (<\vec{b},s,a> \in w_t^{n,m+2} \wedge \sigma_{n,m+1}^t)$.

Let $\psi(x_0,\ldots,x_{m-1},y) \Leftrightarrow_{df.} <x_0,\ldots,x_{m-1},s,y> \in w_t^{n,m+2}$.

Then $\psi \in \Sigma_n$, since s, $t \in \omega$, and $\psi(\vec{x},y)$ is uniform in y since $M \models \sigma_{n,m+1}^t$.

Thus $\psi(\vec{x},y) \in \Sigma_n(\vec{x}\to y)$ and $M \models \psi(\vec{b},a)$, which contradicts $\vec{b} \not\to_n a$.

It now follows from this contradiction that there is an infinite $c \in M$ s. th. $\alpha<c \Rightarrow <\vec{b},\alpha> \not\to_n a$.

Now choose $I \subset S$ s. th. I is an initial segment of M containing infinite elements, $I < c$ and $\forall k \in \omega$ $\forall \alpha_0,\ldots,\alpha_{k-1} \in I$, $2^{\alpha_0}\cdots p_{k-1}^{\alpha_{k-1}} \in I$. This is possible by lemma 4.1.4., $C^n(S)$, and the def. of c.

Now put $S_a = \{i \in S : M \models \psi(\vec{b},\alpha,i)$ for some $\alpha \in I$ and some $\psi(x_0,\ldots,x_m,y) \in \Sigma_n(\vec{x}\to y).\}$.

Clearly $S_a \subset S$, $\vec{b} \subset S_a$, $a \notin S_a$ and (ii) of def. 4.1.6. are satisfied. To show (i) of def. 4.1.6. suppose $\vec{t} = <t_0,\ldots,t_{k-1}> \subset S_a$, $\phi(\vec{y},z) \in \Sigma_n$ and $M \models (\exists z)\phi(\vec{t},z)$.

Then for some $\alpha_0,\ldots,\alpha_{k-1} \in I$ and

$\psi_0(x_0,\ldots,x_{m-1},x_m,y),\ldots,\psi_{k-1}(x_0,\ldots,x_{m-1},x_m,y) \in \Sigma_n(\vec{x}\to y)$,

we have :

$$\forall z \in M,\quad M\models\phi(\vec{t},z) \iff M\models(\exists z_0,\ldots,z_{k-1})(\bigwedge_{i=0}^{k-1}\psi_i(\vec{b},\alpha_i,z_i)\wedge$$
$$\wedge\ \phi(z_0,\ldots,z_{k-1},z)).$$

$$\iff M\models(\exists z_0,\ldots,z_{k-1})(\exists u_0,\ldots,u_{k-1})$$
$$(\bigwedge_{i=0}^{k-1}(u_i=\exp_i(\alpha)\wedge\psi_i(\vec{b},u_i,z_i))\wedge\phi(z_0,\ldots,z_{k-1},z)),$$

where $\alpha = 2^{\alpha_0}\cdots p_{k-1}^{\alpha_{k-1}} \in I$.

The result now follows by uniformising the formula :

$$\psi(x_0,\ldots,x_m,y) =_{df.} (\exists z_0,\ldots,z_{k-1})(\exists u_0,\ldots,u_{k-1})$$
$$(\bigwedge_{i=0}^{k-1}(u_i=\exp_i(x_m)\wedge\psi_i(x_0,\ldots,x_{m-1},u_i,z_i))\quad\wedge$$
$$\wedge\ \phi(z_0,\ldots,z_{k-1},y))$$

for y, observing that $\psi\in\Sigma_n$, and using the above bi-implications with the fact that $M\models\psi(\vec{b},\alpha,d) \Rightarrow$ $\Rightarrow d\in S_a$.

□

<u>Lemma 4.1.8.</u>

Suppose $n\in\omega$ and $S\subset M$ satisfies $C^{n+1}(S)$. Suppose further that $\vec{a}\subset M$ and $\vec{b}\subset S$ are m-termed sequences s. th. $\forall\phi(\vec{x})\in\Sigma_{n+1}$, $M\models\phi(\vec{a})\Rightarrow M\models\phi(\vec{b})$. Then :

(i) for any $\alpha\in M$, $\exists\beta\in S$ s. th. $\forall\phi(\vec{x},y)$ $\in\Sigma_{n+1}$ $M\models\phi(\vec{a},\alpha)\Rightarrow M\models\phi(\vec{b},\beta)$. Further, if $\vec{a}\not\to_{n+1}\alpha$, we may choose β s. th. $\vec{b}\not\to_{n+1}\beta$.

(ii) For any $\beta\in M$ s. th. $\beta<\max.\{b_0,\ldots,b_{m-1}\}$ (in M), $\exists\alpha\in M$ s. th. $\forall\phi(\vec{x},y)\in\Sigma_{n+1}$, $M\models\phi(\vec{a},\alpha)\Rightarrow$ $\Rightarrow M\models\phi(\vec{b},\beta)$,

Proof.

(i) Suppose $\vec{a} \to_{n+1} \alpha$. Then there is a formula $\phi_0 \in \Sigma_{n+1}(\vec{x} \to y)$ s. th. $M \models \phi_0(\vec{a},\alpha)$. Thus $M \models (\exists y)\phi_0(\vec{b},y)$, and so by the lemma hypothesis we have $M \models (\exists y)\phi_0(\vec{b},y)$. In fact, since $\phi_0 \in \Sigma_{n+1}(\vec{x} \to y)$, we must have $M \models (\exists ! y)\phi_0(\vec{b},y)$ and so we can choose β uniquely s. th. $M \models \phi_0(\vec{b},\beta)$. It is now easy to verify that $\phi(\vec{x},y) \in \Sigma_{n+1}$ and $M \models \phi(\vec{a},\alpha)$ imply $M \models \phi(\vec{b},\beta)$.

Now suppose $\vec{a} \not\to_{n+1} \alpha$. It follows from this fairly easily that there are infinitely many $u \in M$ (though not necessarily M-infinitely many $u \in M$) s. th. $M \models \phi(\vec{a},u)$, whenever $\phi(\vec{x},y) \in \Sigma_{n+1}$ and $M \models \phi(\vec{a},\alpha)$.

Hence for all $p \in \omega$:

$$M \models (\exists x)(\forall t,t'<p)((t \neq t' \to \exp_t(x) \neq \exp_{t'}(x)) \wedge$$
$$\wedge \; <\vec{a},\exp_t(x)> \in w^{n+1,m+1}_{g(p)}), \qquad \ldots\ldots.(*)$$

where $g(x) =_{df.} \mu y: w^{n+1,m+1}_y = \cap\{w^{n+1,m+1}_i : i<x \wedge A(i)\}$,

and $A(s) \Leftrightarrow_{df.} <\vec{a},\alpha> \in w^{n+1,m+1}_s$.

Arguing as in the proof of 3.2.6., $g(p)$ is finite for finite p and takes arbitrary large finite values. Also by 2.3.3., the formula in (*) is Σ_{n+1} $(z = \exp_y(x) \in \Sigma_1)$ and hence by the lemma hypotheses we have, for all $p \in \omega$:

$$M \models (\exists x)(\forall t,t'<p)((t \neq t' \to \exp_t(x) \neq \exp_{t'}(x)) \wedge$$
$$\wedge \; <\vec{b},\exp_t(x)> \in w^{n+1,m+1}_{g(p)}). \qquad \ldots\ldots.(**).$$

(Perhaps we should point out here that the definition of g depends on α and is probably not even Σ_{n+1}. However, this does not affect the above deduction since we are only asserting (**) when p, and hence $g(p)$, is finite and therefore the manner in which we define g is irrelevant).

Now using (**), $C^{n+1}(S)$ and overspill we can find a $p_o \in M$ s. th. both p_o and $g(p_o)$ are infinite members of an initial segment of M included in S, and s. th. :

$$M \models (\exists x)(\forall t,t' \leqslant p_o)((t \neq t' \rightarrow \exp_t(x) \neq \exp_{t'}(x)) \wedge$$
$$\wedge \langle \vec{b}, \exp_t(x)\rangle \in w^{n+1,m+1}_{g(p_o)}).$$

But S is Σ_{n+1}-closed in M (because $C^{n+1}(S)$), and so there is some $\beta' \in S$ s. th.

$$M \models (\forall t,t' \leqslant p_o)((t \neq t' \rightarrow \exp_t(\beta') \neq \exp_{t'}(\beta')) \wedge$$
$$\wedge \langle \vec{b}, \exp_t(\beta')\rangle \in w^{n+1,m+1}_{g(p_o)}).$$

But $t, \beta' \in S \Rightarrow \exp_t(\beta') \in S$, and $t \leqslant p_o \Rightarrow t \in S$; so it follows that there are infinitely many $u \in M$ (again, not necessarily M-infinitely many u) s. th. :

$$M \models \langle \vec{b}, u\rangle \in w^{n+1,m+1}_{g(p_o)} \wedge (\exists t \leqslant p_o)(u = \exp_t(\beta')), \qquad \ldots\ldots(***)$$

and any such u must be in S.

We now show that any u satisfying (***) has the property : $\forall \phi(\vec{x},y) \in \Sigma_{n+1}$, $M \models \phi(\vec{a},\alpha) \Rightarrow M \models \phi(\vec{b},u)$. For suppose $\phi(\vec{x},y) \in \Sigma_{n+1}$ and $M \models \phi(\vec{a},\alpha)$.

We can express this as $M \models \langle \vec{a},\alpha\rangle \in w^{n+1,m+1}_{p'}$, for some suitably chosen $p' \in \omega$.

Thus $M \models A(p') \wedge p' < p_o$, and it follows from the defs. of A and g that $M \models w^{n+1,m+1}_{g(p_o)} \subset w^{n+1,m+1}_{p'}$,

Therefore, by (***), $M \models \langle \vec{b},u\rangle \in w^{n+1,m+1}_{p'}$, i.e. $M \models \phi(\vec{b},u)$ as required.

To complete the proof of (i), it suffices to find a u satisfying (***) and $\vec{b} \not\rightarrow_{n+1} u$.

That we can do this follows from the

following general claim :

If $A(\vec{x},y)$ is any formula of L, and $\vec{c}$, $\vec{s} \subset M$ are s. th. there infinitely many $u \in M$ s. th. $M \models A(\vec{c},u)$, then there is $u_0 \in M$, s. th. $M \models A(\vec{c},u_0)$ and $\vec{s} \not\to_{n+1} u_0$.

Proof of claim.

Suppose it false. Suppose $\vec{s} = \langle s_0,\ldots,s_{l-1}\rangle$.

Then for all infinite $\gamma \in M$,
$M \models (\forall x)(A(\vec{c},x) \to (\exists k<\gamma)(\langle\vec{s},x\rangle \in w_k^{n+1,l+1} \wedge$
$(\exists! z)(\langle\vec{s},z\rangle \in w_k^{n+1,l+1}))$.

Now this formula must hold for some finite γ. But then there would be infinitely many x's satisfying $A(\vec{c},x)$ (by the claim hypothesis) and only finitely many satisfying the right hand side of the implication - a contradiction that proves the claim, and completes the proof of (i).

(ii) We first note that the lemma hypotheses are equivalent to :

$$\forall \phi(\vec{x}) \in \Pi_{n+1},\quad M \models \phi(\vec{b}) \Rightarrow M \models \phi(\vec{a}).$$

Now suppose $M \models \max\{b_0,\ldots,b_{m-1}\} = b_k$.

Let $A(s) \iff_{df.} \langle\vec{b},\beta\rangle \notin w_s^{n+1,m+1}$,

and $g(x) =_{df.} \mu y\colon w_y^{n+1,m+1} = \cup\{w_i^{n+1,m+1} : i<x \wedge A(i)\}$.

It follows from these definitions that :
$\forall p \in \omega,\ M \models (\exists x<b_k)(\langle\vec{b},x\rangle \notin w_{g(p)}^{n+1,m+1})$, since $x = \beta$ satisfies this formula.

Now using lemma 2.3.3., this formula is (equivalent in P and therefore in Th(M) to) a Π_{n+1} formula, and thus by the above comment we have :

$\forall p \in \omega,\ M \models (\exists x < a_k)(\langle \vec{a}, x\rangle \notin w^{n+1,m+1}_{g(p)})$.

The remainder of the proof is now similar to that of (i) and we leave it to the reader.

□

We now have sufficient lemmata to prove :

Theorem 4.1.9.

$\forall n \in \omega$, there is a set H of substructures of M s. th. :

(i) $\overline{\overline{H}} = 2^{\aleph_0}$.

(ii) $M' \in H \Rightarrow M' \simeq M$.

(iii) $M' \in H \Rightarrow M' \preccurlyeq_n M$ and $M' \not\preccurlyeq_{n+1} M$.

Proof.

Choose a, b ∈ M as given by lemma 3.2.6. with n replaced by n+1, and let $a_0, a_1, \ldots, a_k, \ldots$ $(k \in \omega)$ be an enumeration of M s. th. $a = a_0$.

We construct a tree $\langle T, \leqslant_T\rangle$ s. th. $\forall m \in \omega$:-

(1) Every node has either one or two immediate successors - nodes of the same level having the same number of successors (the least element of T being at the $0^{th.}$ level).

(2) Each node of T is a pair $\langle c, S\rangle$ s. th. $S \subset M$, $C^{n+1}(S)$ and $c \in S$.

(3) $\langle c', S'\rangle \leqslant_T \langle c, S\rangle \Rightarrow c' \in S$ and $S' \supset S$.

(4) If $\langle c', S'\rangle$ and $\langle c, S\rangle$ are $\leqslant_T$-incomparable and have a common $\leqslant_T$-immediate predecessor, then either $c' \notin S$ or $c \notin S'$.

$(5)_m$ If $\langle c_0, S_0\rangle \leqslant_T \langle c_1, S_1\rangle \leqslant_T \ldots \leqslant_T \langle c_{m-1}, S_{m-1}\rangle$ where $\langle c_i, S_i\rangle$ is of level i $(0 \leqslant i \leqslant m-1)$, and if

$\phi \in \Sigma_{n+1}$ has only m free variables, then

$$M \models \phi(a_0,\dots,a_{m-1}) \Rightarrow M \models \phi(c_0,\dots,c_{m-1}) .$$

Firstly, the least element of the tree is $\langle b,M \rangle$. The conditions are easily verified for $m = 1$.

Now suppose T has been defined up to, and including, level $m-1$ $(m \geq 1)$ s. th. conditions $(1)-(5)_m$ hold for all nodes defined so far.

Let us pick any branch, say $\langle c_0,S_0 \rangle \leq_T \leq_T \langle c_1,S_1 \rangle \leq_T \dots \leq_T \langle c_{m-1},S_{m-1} \rangle$ thus defined. We construct the immediate successor(s) to this branch by cases :

Case 1. $\vec{a} = \langle a_0,\dots,a_{m-1} \rangle \longrightarrow_{n+1} a_m$.

Our inductive hypotheses imply the conditions of lemma 4.1.8. are satisfied with $\vec{b} = \vec{c}$, $S = S_{m-1}$.

Applying (i) of this lemma we obtain $\beta \in S_{m-1}$ s. th. $M \models \phi(\vec{a},a_m) \Rightarrow M \models \phi(\vec{c},\beta)$ $\forall \phi \in \Sigma_{n+1}$, and we let $\langle \beta,S_{m-1} \rangle$ be the one and only immediate successor of $\langle c_{m-1},S_{m-1} \rangle$. The conditions $(1)-(5)_{m+1}$ are now clearly satisfied for the branch $\langle c_0,S_0 \rangle \leq_T \leq_T \dots \leq_T \langle c_{m-1},S_{m-1} \rangle \leq_T \langle c_m,S_m \rangle$ where $c_m = \beta$ and $S_m = S_{m-1}$.

Case 2. $\vec{a} \not\rightarrow_{n+1} a_m$.

Again we use lemma 4.1.8. to obtain $\beta \in S_{m-1}$ so that $\langle \beta,S_{m-1} \rangle$ is one immediate successor of $\langle c_{m-1},S_{m-1} \rangle$ but s. th. $\langle c_0,\dots,c_{m-1} \rangle \not\rightarrow_{n+1} \beta$. Now we can apply lemma 4.1.7. with $\vec{b} = \vec{c}$, $a = \beta$, $n = n+1$

and $S = S_{m-1}$ to get $S_a \subset S$ s. th. $C^{n+1}(S_a)$, $\vec{c} \subset S_a$ and $\beta \notin S_a$. Now use lemma 4.1.8. again with $S = S_a$, $\vec{b} = \vec{c}$, $\alpha = a_m$ which gives us a $\beta' \in S_a$ s. th. $M \models \phi(\vec{a}..,a_m) \Rightarrow M \models \phi(\vec{c},\beta')$, $\forall \phi \in \Sigma_{n+1}$.

Let $<\beta',S_a>$ be an immediate successor of $<c_{m-1},S_{m-1}>$ incomparable with $<\beta,S_{m-1}>$.

The conditions $(1)-(5)_{m+1}$ are again clearly satisfied by our construction, condition (1) following from the fact that whether we added one or two successors to any node depended only on a property of our original enumeration of M and not on which branch we extended at any given level.

The construction of T is now completed by induction.

Now for each branch, B, of T let

$$M_B = \{c \in M: \exists S \subset M, <c,S> \in B\},$$

and e_B be the mapping $M \rightarrow M_B$ taking a_k to the element c, of M_B, s. th. $\exists S \subset M$, $<c,S>$ is of level k. e_B induces in the obvious way, definitions of + and $\cdot$ in M_B, so that $M_B \simeq M$.

To show $M_B \preccurlyeq_n M$, suppose $\vec{c} \subset M_B$, $\phi(\vec{x}) \in \Sigma_n$ and $M \models \phi(\vec{c})$. Then for some $\vec{b} \subset M$, $e(\vec{b}) = \vec{c}$, so $M \models \phi(e(\vec{b}))$. But $\phi \in \Sigma_n \Rightarrow \phi \in \Pi_{n+1}$, so by by condition $(5)_m$ (for some $m \in \omega$), and the def. of e, $M \models \phi(\vec{b})$. But e is an isomorphism from M to M_B, so $M_B \models \phi(e(\vec{b}))$, i.e. $M_B \models \phi(\vec{c})$ as required.

That $M_B \not\preccurlyeq_{n+1} M$ follows immediately from our initial choice of a and b.

Now suppose $B \neq B'$ are branches of T. It follows from (2) and (3) that $M_B \subset S$ $\forall S$ s. th. $\exists c \in M$, $\langle c,S\rangle \in B$, and similarly for B'. Hence from (4) we have $M_B \neq M_{B'}$.

The theorem is thus proven if we can show T has $2^{\aleph_0}$ branches. However, if this were not the case we would have, by (1), a level m, s. th. every node of level $\geq m$ has only one successor. Hence case (2) in our proof would hold only finitely often which implies $\forall a \in M$, $\langle a_0,\ldots,a_{m-1}\rangle \rightarrow_{n+1} a$, contradicting the claim proved on p. 33, with $A(\vec{x},y) = (y = y)$ and $\vec{s} = \langle a_0,\ldots,a_{m-1}\rangle$.

Theorem 4.1.9. is now proved, where

$$H = \{M_B : B \text{ a branch of } T.\}.$$

□

A natural question generalising theorem 4.1.9. would be to ask whether H can consist only of initial segments of M. Unfortunately I can only prove this when M satisfies certain conditions, but can show that any non-standard M is elementarily equivalent to $2^{\aleph_0}$ initial segments of itself. This I now do.

<u>Lemma 4.1.10.</u>

$\forall n \in \omega$, there a set E_M of embeddings of M into itself s. th. :

(i) $\overline{\overline{E}}_M = 2^{\aleph_0}$.

(ii) $\forall e \in E_M$, e[M] is an initial segment of M.

(iii) $\forall e \in E_M$ $e[M] \preccurlyeq_n M$ and $e[M] \not\preccurlyeq_{n+1} M$.

Proof.

Let T be the tree of height ω which is (completely) defined by : every node in T of level k, where $k \equiv 3 \pmod 4$ (the least element of T being at the $0^{th.}$ level), has precisely two immediate successors, and every other node has precisely one immediate successor.

We take two copies, $\langle T_D, \leqslant_D \rangle$ (the domain tree), and $\langle T_I, \leqslant_I \rangle$ (the image tree), of T and let f be the natural isomorphism from T_D to T_I. The idea of the proof is to associate one element of M to each node of T_D and one to each node of T_I, s. th. given any branch, B_D, of T_D, every element of M is associated with some node in B_D; and given any branch, B_I, of T_I, the set J, of of elements of M associated with some node of B_I forms an initial segment of M. Further, the map $f^* : M \to I$ which takes the element of M associated with the node ν of B_D to the element of J associated with the node $f(\nu)$ of T_I will be an isomorphism from M to the initial segment, J, of M.

We now describe the construction in more detail. The first few steps of it are illustrated in fig. (i) on p. 42.

To avoid clumsiness of expression we identify nodes of the trees T_D and T_I with the elements we have associated with them, and hence f^* with $f \restriction B_D$, etc.

Our inductive assumption is :

$(C)_m$ If $c_0 \leqslant_D c_1 \leqslant_D \cdots \leqslant_D c_{m-1}$ are elements of T_D where c_i is of level i $(0 \leqslant i \leqslant m-1)$ and $\phi(x_0,\ldots,x_{m-1}) \in \Sigma_{n+1}$, then

$$M \models \phi(c_0,\ldots,c_{m-1}) \implies M \models \phi(f(c_0),\ldots,f(c_{m-1})).$$

Now choose $a, b \in \mathbb{M}$ as given by lemma 3.2.6. with n replaced by $n+1$, and let $a_0, a_1, \ldots, a_k, \ldots$ $(k \in \omega)$ be an enumeration of M s. th. $a = a_0$.

We associate a_0 with the least element of T_D and b with the least element of T_I. $(C)_1$ is easily verified.

Now suppose elements of $\mathbb{M}$ have been associated with every node of T_D and T_I of level $\leqslant m-1$, s. th. $(C)_m$ holds $(m > 1)$.

<u>Case 1.</u> $m \equiv 2 \pmod 4$.

Let us pick any sub-branch, say $c_0 \leqslant_D c_1 \leqslant_D \leqslant_D \cdots \leqslant_D c_{m-1}$ of T_D of length m.

We only have to find one successor to c_{m-1}, and we let it be a_k where $k = \frac{m-2}{4} + 1$. $f(a_k)$ can now be defined so that $(C)_{m+1}$ is satisfied by using lemma 4.1.8.(i) with $\vec{a} = \langle c_0,\ldots,c_{m-1}\rangle$, $S = M$, $\vec{b} = \langle f(c_0),\ldots,f(c_{m-1})\rangle$ and $\alpha = a_k$. This construction is repeated for all sub-branches of length m with which elements of M have so far been associated. Note that every node of level m in T_D has a_k associated with it.

Case 2. $m \equiv 0 \pmod 4$.

Then $m-1 \equiv 3 \pmod 4$ so we must find elements of M to associate with the two immediate successor nodes of nodes of level $m-1$ in T_D. Let $\vec{d}$ consist of all elements of M so far associated with nodes of T_D arranged in a finite sequence. By the claim on p. 33 we can find $c \in M$ s. th. $\vec{d} \not\rightarrow_{n+1} c$. We associate c with every node of T_D of level m. Now if $c_0 \leqslant_D \cdots \leqslant_D c_{m-1}$ is any sub-branch of T_D of length m we certainly have $\langle c_0, \ldots, c_{m-1} \rangle \not\rightarrow_{n+1} c$, since $\langle c_0, \ldots, c_{m-1} \rangle \subset \vec{d}$. Hence by 4.1.8.(i) (using the inductive hypothesis) $\exists \beta \in M$ s. th. $\forall \phi \in \Sigma_{n+1}$,

$$(*) \ldots\ldots \quad M \models \phi(c_0, \ldots, c_{m-1}, c) \Rightarrow M \models \phi(f(c_0), \ldots, f(c_{m-1}), \beta),$$

and $\langle f(c_0), \ldots, f(c_{m-1}) \rangle \not\rightarrow_{n+1} \beta$.

Using a similar technique to that in the proof of the preceding theorem we can also find $\beta' \in M$, $\beta' \neq \beta$, s. th. (*) holds with β' replacing β. We associate β with one successor node of $f(c_{m-1})$ in T_I and β' with the other. After repeating this construction for each possible $\langle c_0, \ldots, c_{m-1} \rangle$, $(C)_{m+1}$ is easily checked.

Case 3. m odd.

Here we first extend each sub-branch of T_I, so suppose $f(b_0) \leqslant_I \dots \leqslant_I f(b_{m-1})$ is such a branch of length m of T_I. Let β be the element of M with the property $M \models (\beta < \max\{f(b_0),\dots,f(b_{m-1})\} \wedge$ $\wedge\, \beta \notin \{f(b_0),\dots,f(b_{m-1})\})$ that occurs first in our enumeration of M, and associate β with the node in T_I immediately succeding $f(b_{m-1})$. Lemma 4.1.8.(ii) now provides us with an $\alpha \in M$ that can be associated with the node in T_D immediately succeding b_{m-1} so that $(C)_{m+1}$ holds.

Now for each branch, B, of T let e_B be the map that takes the set B_D of elements associated with the copy of B in T_D to the corresponding set, B_I, in T_I, in the natural way.

The domain of e_B is M, since if $k \in \omega$, then a_k is the element of B_D occuring at the (4k-2)th. level if $k \geqslant 1$, whereas a_0 occurs at the $0^{\text{th.}}$ level. Of course e_B is a function since if a_k occurs at two different levels in B_D the corresponding elements in B_I must be equal because $(x = y)$ is a Σ_1 formula and hence preserved by e_B. That the range of e_B is an initial segment of M follows easily by case (3) of the construction, and that e_B is an isomorphism onto this initial segment follows using the same arguement as in the proof of the preceding theorem, as do the facts that $e_B[M] \preccurlyeq_n M$ and $e_B[M] \not\preccurlyeq_{n+1} M$.

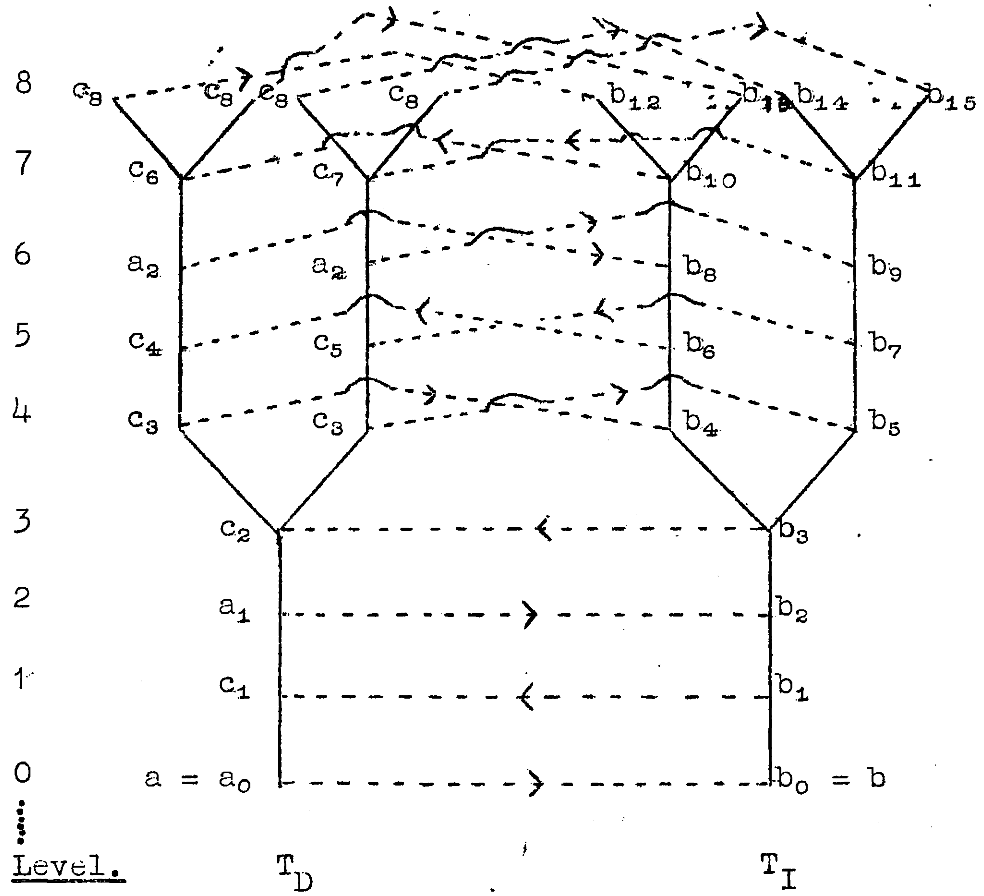


fig. (i).

$b_1 = a_k$ where $k = \mu i \epsilon \omega$: $a_i < b$.

$b_3 = a_k$ where $k = \mu i \epsilon \omega$: $a_i < \max\{b_0, b_1, b_2\} \wedge a_i \notin \{b_0, b_1, b_2\}$.

$b_6 = a_k$ where $k = \mu i \epsilon \omega$: $a_i < \max\{b_0, \ldots, b_4\} \wedge a_i \notin \{b_0, \ldots, b_4\}$.

$b_7 = a_k$ where $k = \mu i \epsilon \omega$: $a_i < \max\{b_0, b_1, b_2, b_3, b_5\} \wedge$

$\wedge\ a_i \notin \{b_0, b_1, b_2, b_3, b_5\}$.

$b_{10} = a_i$ where $k = \mu i \epsilon \omega$: $a_i < \max\{b_0, \ldots, b_4, b_6, b_8\} \wedge$

$\wedge\ a_i \notin \{b_0, \ldots, b_4, b_6, b_8\}$.

$b_{11} = a_i$ where $k = \mu i \epsilon \omega$: $a_i < \max\{b_0, b_1, b_2, b_3, b_5, b_7, b_9\} \wedge$

$\wedge\ a_i \notin \{b_0, b_1, b_2, b_3, b_5, b_7, b_9\}$.

$c_3 =$ some $c \epsilon M$ s. th. $\langle a_0, c_1, a_1, c_2 \rangle \not\to_{n+1} c$.

$c_8 =$ some $c \epsilon M$ s. th. $\langle a_0, c_1, a_1, c_2, c_3, c_4, a_2, c_6, c_5, c_7 \rangle \not\to_{n+1} c$.

The theorem is now proved if we can show that $B \neq B'$ implies $e_B \neq e_{B'}$. But this is immediate from case (2) of our construction since we have ensured that $e_B(c) \neq e_{B'}(c)$ where c is the element of M associated with all nodes of the level at which B and B' first differ.

□

Of course lemma 4.1.10. only provides us with $2^{\aleph_0}$ embeddings onto initial segments and does not guarantee these initial segments are all distinct. To obtain this one would require a combination of the techniques of 4.1.10. and 4.1.9., which boils down to proving a stronger version of lemma 4.1.7. (with an adapted definition of $\vec{b} \rightarrow_n a$, on the lines of $\exists \phi \in \Sigma_n$ s. th. $M \models (\exists x)(\phi(\vec{b},x) \wedge x \geqslant a)$), the truth of which seems doubtful. However, we have the following :

<u>Theorem 4.1.11.</u>

Let M be a non-standard countable model of P, which is rigid (i.e. has no non-trivial automorphisms), and let E_M be the set of embeddings given by lemma 4.1.10. Then $\forall e, j \in E_M$, $e \neq j \Rightarrow$ $\Rightarrow e[M] \neq e[M]$.

Hence M is isomorphic to $2^{\aleph_0}$ initial segments of itself (which can be chosen to be $\leqslant_n$- but not $\leqslant_{n+1}$- substructures).

<u>Proof.</u>

Suppose $e, j \in E_M$, $e \neq j$ and $e[M] = j[m] = I$.

Then $j \cdot e^{-1}$ is a non-trivial automorphism of I. But $I \simeq M$ (e.g. by e), and so M has a non-trivial automorphism – a contradiction. Thus $e[M] \neq j[M]$.

□

We now show that any non-standard M (countable or not) is elementarily equivalent to at least $2^{\aleph_0}$ initial segments of itself. We require two known results, both due to Gaifman, namely :

Lemma 4.1.12. (see [5]).

Every non-standard model of P contains a countable non-standard elementary substructure which is rigid.

Lemma 4.1.13. (see [7]).

Suppose M is a non-standard model of P, $M_1 \subset M$ and $M_1 \equiv M$.

Let $M_1{*}M = \{a \in M : M \models a \leq b$ for some $b \in M_1\}$, and define + and · on $M_1{*}M$ as those functions induced from M.

Then $M_1{*}M$ is an initial segment of M and $M_1 \preccurlyeq M_1{*}M$. A fortiori $M_1{*}M \equiv M$.

We can now prove :

Theorem 4.1.14.

Let M be any non-standard model of P. Then there is a set H of initial segments of M s. th. :

(i) $\overline{\overline{H}} = 2^{\aleph_0}$.

(ii) $M' \in H \Rightarrow M \equiv M'$.

Proof.

Let M_1 be a countable non-standard rigid elementary substructure of M whose existence is given by lemma 4.1.12. Then $H = \{e[M_1]^*M : e \in E_{M_1}\}$ has, by theorem 4.1.11. and lemma 4.1.13., the required properties, where E_{M_1} is the set of embeddings given by lemma 4.1.10.

□

Chapter 5 Further Applications of the Method.

5.1. Introduction.

The preceding results have been proved using variations of a certain technique - namely using a function enumerating Σ_n -sets and then looking at a non-standard stage of the enumeration. This method was first used by Ryll-Nardzewski [16] and Rabin [12], although, as we have already said the use is unnecessary in the latter.

This chapter is devoted to proving two results about models of P using the same method, and I should repeat Rabin's comment (in [12]) here - that the method should still have many more interesting applications.

5.2. On omitting types in models of P.

We first introduce some well-known concepts from general model theory.

Def. 5.2.1.

Let L* be any first order language and S any set of formulae from L*. Then τ is called an S-type iff $\tau \subset S$ and every formula in τ has just the variable x free.

If $\mathfrak{A}$ is an L*- structure, we say $\mathfrak{A}$ realises τ iff $\exists a \in \mathfrak{A}$ s. th. $\mathfrak{A} \models \phi(a)$ $\forall \phi(x) \in \tau$; and $\mathfrak{A}$ omits τ if $\mathfrak{A}$ does not realise τ.

Theorem 5.2.2.

Let $n \in \omega$ and $M \models P$ be non-standard and τ any Σ_n-type (in L). Then if τ is omitted in M it is omitted in every elementary end extension of M.

Proof.

Suppose $M^* \succcurlyeq M$, M^* an end extension of M and that M^* realises τ.

Choose $a \in M^*$ s. th. $M^* \models \phi(a)\ \forall \phi \in \tau$.

Define $B(x,y,z)$, $A(x,y) \in L$ by :

$$A(x,y) \Leftrightarrow_{df.} x \in w_y^{n,1}$$

$$B(x,y,z) \Leftrightarrow_{df.} z = \prod_{\substack{w<y \\ A(x,y)}} p_w \quad \text{(where } \Pi\emptyset = 0).$$

Then it is easy to show that $P \vdash (\forall x,y)(\exists! z)B(x,y,z)$ and so we write $B(x,y) = z$ for $B(x,y,z)$.

Now let c be an infinite element of M.

Then $M^* \models (\prod_{\substack{u<c \\ A(a,u)}} p_u \leqslant \prod_{u<c} p_u \leqslant c!)$.

Now $M^* \succcurlyeq M$, hence if $c \in M$ $c!$ denotes the same element in M as it does in M^*, and since M^* is an end extension of M we must have, by the above, $\prod_{\substack{u<c \\ A(a,u)}} p_u \in M$, i.e. $B(a,c) \in M$.

Let $d = B(a,c)$.

Now we may suppose $\tau \subset \{(x \in w_m^{n,1}) : m \in \omega$ s.th. $M \models p_m | d \}$......(*), for if $\phi(x) \in \tau$, then $\phi(x) \in \Sigma_n$ and $M^* \models \phi(a)$. Hence $\exists m \in \omega$ s. th.

$$M^* \models (\forall x)(x \in w_m^{n,1} \leftrightarrow \phi(x)).$$

So $M^* \models a \in w_m^{n,1}$. And therefore $M^* \models A(a,m)$ and $M^* \models m < c$. Hence $M^* \models p_m | B(a,c) = d$, from which it follows that $M \models p_m | d$ since $M \preccurlyeq M^*$, and (*) is justified.

Also, $\forall m \in \omega$, $M^* \models (\forall k<m)(p_k | d \rightarrow a \in w_k^{n,1})$ (by def. of d).

Thus, $\forall m \in \omega$, $M^* \models (\exists x)(\forall k<m)(p_k | d \rightarrow x \in w_k^{n,1})$.

So, $\forall m \in \omega$, $M \models (\exists x)(\forall k<m)(p_k | d \rightarrow x \in w_k^{n,1})$,

since $M \preccurlyeq M^*$.

It now follows from overspill that for some infinite $e \in M$, $M \models (\exists x)(\forall k<e)(p_k | d \rightarrow x \in w_k^{n,1})$.

Say $M \models (\forall k<e)(p_k | d \rightarrow a' \in w_k^{n,1})$, where $a' \in M$.

But this, together with (*) implies a' realises τ; so M realises τ, and theorem 5.2.2. is established by contradiction.

□

Theorem 5.2.2. can be strengthened to allow finitely many (constants representing) elements of M to occur amongst formulae of τ, and also to replacing 'elementary end extension' by '$\preccurlyeq_n$ - end extension'. We leave the proof, which is similar to the above, to the reader.

5.3. On indescernibles in models of P.

Theorem 5.3.1.

Let $n \in \omega$ and M be any non-standard model of P. Then there is a set $S \subset M$, s. th. $\overline{\overline{S}} = \overline{\overline{M}}$ and $\forall m \geq 1$, $\forall \phi(x_0,\ldots,x_{m-1}) \in \Sigma_n$, and $\forall \vec{b}$, $\vec{a} \in [S]^m$ $= \{<t_0,\ldots,t_{m-1}> : t_i \in S,\ 0 \leq i < m \text{ and } M \models t_0 < \ldots < t_{m-1}\}$,

$$M \models \phi(a_0,\ldots,a_{m-1}) \iff M \models \phi(b_0,\ldots,b_{m-1}).$$

(In the jargon of model theory this says that every non-standard model, M, of P contains a set, of the same cardinality as M, which is indescernible for Σ_n formulae.)

Proof.

Firstly we fix a Σ_1 coding of finite sequences of M, so that any definable subset of M may be regarded as a set of t-tuples for any $t \in M$.

Now let A, B, D be any definable (without parameters) subsets of M and $t \in M$.

Then we write $D \to (A,B)^t$ iff either $[D]^t \subset A \cap [B]^t$ or $[D]^t \subset CA \cap [B]^t$, where A is regarded as a set of t-tuples.

Now Ramsey's theorem [14] asserts that if $M = N$, A, B are infinite and $t \in N$, then there is an infinite D s. th. $D \to (A,B)^t$.

Checking the proof of Ramsey's theorem one sees that it can be proved in P, and that D can be obtained uniformly from A, B and t; and hence using the methods of [6] it is easy, but somewhat tedious, to check that the following informal definition of the predicate $x \in R_y$ (in L) can be made a sound one in P.

Firstly we can suppose our coding of ordered pairs, $x = \langle y,z \rangle$, say, has the property that y and z are both finite iff $\langle y,z \rangle$ is finite.

We now define $x \in R_y$ by induction :

$$\begin{cases} R_0 = M \\ R_{y+1} = \text{An infinite set } D \subset R_y \text{ s. th. if} \end{cases}$$

$y = \langle s,t \rangle$ then $D \to (w_s^{n,1}, R_y)^t$.

We clearly have $M \models (\forall x,y)(x < y \to R_x \supset R_y) \ldots (*)$.

Now let a be an infinite element of M.

We claim $S = R_a$ has the property required in the theorem.

For suppose $\phi(x_0,\ldots,x_{m-1}) \in \Sigma_n$. Then the fml. ψ defined by :

$$\psi(x) \iff (\exists x_0,\ldots,x_{m-1})(x = \langle x_0,\ldots,x_{m-1}\rangle \wedge$$
$$\wedge \quad \phi(x_0,\ldots,x_{m-1})),$$

is Σ_n and has just x free.

Hence for some $s \in \omega$, $M \models (\forall x)(\psi(x) \leftrightarrow x \in w_s^{n,1})$.

Suppose $y_0 = \langle s,m\rangle$. Then $y_0 \in \omega$ by our assumption on the pairing function.

Thus $y_0 + 1 < a$, so $R_{y_0+1} \supset R_a = S$ by (*).

But $R_{y_0+1} \longrightarrow (w_s^{n,1}, R_{y_0})^m$; hence, $[R_{y_0+1}]^m \subset w_s^{n,1} \cap [R_y]^m$, or $[R_{y_0+1}]^m \subset Cw_s^{n,1} \cap [R_{y_0}]^m$.

A fortiori, $[S]^m \subset w_s^{n,1} \cap [M]^m$ or $[S]^m \subset Cw_s^{n,1} \cap [M]^m$.

It follows that S is indescernible for $\phi(x_0,\ldots,x_{m-1})$.

To show $\bar{\bar{S}} = \bar{\bar{M}}$, we merely note that S is a definable (with parameters) subset of M and $M \models$ 'S is infinite'. Hence there is a one-one definable mapping from S onto M. But clearly this mapping must have these same properties in the real world, and so $\bar{\bar{M}} = \bar{\bar{S}}$.

□

Chapter 6. On the Lattices of Elementary Substructures of Models of P.

6.1. The problem and preliminary results.

Throughout sections 6.1.-6.4. of this chapter we fix an arbitrary complete, consistent theory T extending P in L, and let M be the pointwise definable model of T. This is justified by theorem 2.1.1. from which it also follows that M has no proper elementary substructures. Our current aim is to investigate by how much an arbitrary model M^*, of T, can fail to be pointwise definable, and the above comment suggests the following

Def. 6.1.1.

If $M^* \models T$, $\$(M^*)$ denotes the set $\{M' : M' \preccurlyeq M\}$ partially ordered by $\preccurlyeq$.

Now it follows from previous results that if $\phi(\vec{x},y) \in L$, and $T \vdash (\forall \vec{x})(\exists y)\phi(\vec{x},y)$, then there is a total T-functional formula $\phi_0(\vec{x},y)$ s. th. $T \vdash (\forall \vec{x})(\forall y)(\phi_0(\vec{x},y) \rightarrow \phi(\vec{x},y))$. Hence if we add to **L** a function symbol F_{ϕ_0}, for each total T-functional formula ϕ_0, and add to T all axioms $(\forall \vec{x},y)(\phi_0(\vec{x},y) \leftrightarrow F_{\phi_0}(\vec{x}) = y)$ the resulting system will be a conservative extension of T. It follows that the partial order on $\$(M^*)$ is in fact a lattice order, where, for M_1, $M_2 \preccurlyeq M^*$, the domain of $M_1 \wedge M_2$ (the infimum of M_1 and M_2 in $\$(M^*)$) is just the intersection of the domains of M_1 and M_2, and the domain of $M_1 \vee M_2$ (the supremum of M_1 and M_2 in $\$(M^*)$) is the subset of M^* generated from the union of the domains of M_1 and M_2 by all the F_{ϕ_0}'s.

Our basic problem can now be stated as - 'which lattices are isomorphic to $\$(M^*)$ for some $M^* \models T$?'.

(This situation is analagous to one in recursion theory, where the non-recursiveness of a set A of natural numbers is measured as the upper-semi lattice of sets recursive in A. The two representation problems, however, are technically quite different.)

Most of the positive results concerning the above problem are contained in the following three theorems :

<u>Theorem 6.1.2.</u> (Gaifman)

There is a model M* of T s. th. $\$(M^*) \simeq \langle \omega_1, \epsilon \rangle$.

<u>Theorem 6.1.3.</u> (Gaifman)

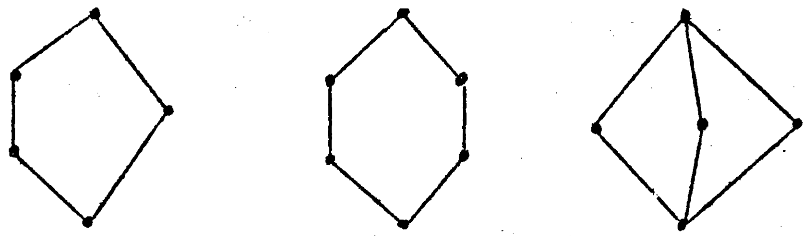
For every set A, there is a model M* of T s. th. $\$(M^*) \simeq \langle P(A), \subset \rangle$, where P(A) denotes the set of all subsets of A.

<u>Theorem 6.1.4.</u> (Paris)

If L is any complete, distributive, ω-compactly generated lattice, there is a model M* of T s. th. $\$(M^*) \simeq L$.

Proofs of the above results can be found in [10].

In view of 6.1.4., we shall restrict our attention to non-distributive lattices and answer a question raised in [10], by showing that the five-element non-modular lattice P_5, is of the form $\$(M^*)$ for some $M^* \models T$, and we shall also produce a class of lattices no member of which is of the form $\$(M^*)$ for any $M^* \models T$ when $T = Th(N)$. The simplest member of this class is the hexagon lattice, H, below.



P_5 H C_5

fig. (ii).

6.2. Construction of simple extensions of M.

Def. 6.2.1.

(i) We denote by S_n the set of all n-place total T-function symbols (in our extended language). We do not distinguish between these symbols and their interpretations in models of T, and hence we identify S_0 with (the domain of) M.

(ii) If $M^* \models T$ and A is any subset of M^*, $M^*[A]$ denotes the elementary substructure of M^* generated from A in M^* by $\bigcup_{n \in \omega} S_n$. If $a_1, \ldots, a_n \in M^*$, we write $M^*[a_1, \ldots, a_n]$ for $M^*[\{a_1, \ldots, a_n\}]$; and call M^* simple if $M^* = M^*[a]$ for some $a \in M^*$.

Now let $\mathbb{B}$ be the Boolean algebra of M-definable sets and U any ultrafilter over $\mathbb{B}$. (See e.g. [1] for definitions of these classical concepts).

We define an equivalence relation $\sim_U$ on S_1 by :

$$f \sim_U g \iff \{x \in M : M \models f(x) = g(x)\} \in U,$$

and set $f^U = \{ g \in S_1 : f \sim_U g \}$ and $M_U = \{ f^U : f \in S_1 \}$.

We turn M_U into an L-structure by defining :

$$f^U + g^U = h^U \iff \{ x \in M : M \models f(x) + g(x) = h(x) \} \in U,$$

and $f^U \cdot g^U = h^U \iff \{ x \in M : M \models f(x) \cdot g(x) = h(x) \} \in U.$

That + and · so defined are functions on M_U, and that $\sim_U$ is a congruence relation for these functions is easily verified, as is the following theorem, which is a 'definable analogue' of Los' theorem on ultrapowers (see[1]).

<u>Theorem 6.2.2.</u>

If $\phi(x_0,\ldots,x_{n-1}) \in L$ and $f_0,\ldots,f_{n-1} \in S_1$, then $M_U \models \phi(f_0^U,\ldots,f_{n-1}^U)$ iff $\{x \in M : M \models \phi(f_0(x),\ldots,f_{n-1}(x))\}$ is a set in U (it is clearly in $\mathbb{B}$).

Further, if for each $a \in M$ we denote by $\hat{a}$ the function in S_1 with constant value a, the map $e: M \to M_U$, defined by $e(a) = \hat{a}^U$ $(\forall a \in M)$, is an elementary embedding of M into M_U.

From now on we shall identify M with its natural image (i.e. its image under the map e, above) in M_U.

<u>Theorem 6.2.3.</u>

Let id denote the identity function on M. Then $id \in S_1$ and we have :

(i) $M_U[id^U] = M_U$. Hence M_U is simple, where

U is any ultrafilter over $\mathbb{B}$.

(ii) For any simple model M^* of T, there is an ultrafilter U, over $\mathbb{B}$, s. th. $M^* \simeq M_U$.

Proof.

(i) is obvious.

For (ii) suppose $M^* = M^*[a]$, $a \in M^*$, and let $U = \{A \in \mathbb{B} : M^* \models a \in A\}$. Then U is an ultrafilter and the map taking a to id^U can clearly be extended to an isomorphism of M^* onto M_U.

□

Working towards our aim of constructing models of T with prescribed lattices of substructures we introduce the following notions similar to those used by Paris in [10] (p. 253).

For f, $g \in S_1$ and $B \in \mathbb{B}$, define

$f \trianglelefteq_B g$ iff $\mathbb{M} \models (\forall x,y \in B)(g(x) = g(y) \rightarrow f(x) = f(y))$

$f \equiv_B g$ iff $f \trianglelefteq_B g$ and $g \trianglelefteq_B f$.

If U is an ultrafilter over $\mathbb{B}$ define

$f \trianglelefteq_U g$ iff $\exists B \in U$ $f \trianglelefteq_B g$.

$f \equiv_U g$ iff $f \trianglelefteq_U g$ and $g \trianglelefteq_U f$.

$f \triangleleft_U g$ iff $f \trianglelefteq_U g$ and not $f \equiv_U g$.

The point of these definitions becomes clear with the following

Lemma 6.2.4.

Let U be any ultrafilter over $\mathbb{B}$, and f, $g \in S_1$. Then $M_U[f^U] \preccurlyeq M_U[g^U]$ iff $f \trianglelefteq_U g$.

Proof.

Suppose $M_U[f^U] \preccurlyeq M_U[g^U]$. Then $\exists h \in S_1$, s. th.

$B = \{x \in M : M \models h(g(x)) = f(x)\} \in U$.

Clearly $f \trianglelefteq_B g$, hence $f \trianglelefteq_U g$.

Now suppose $f \trianglelefteq_U g$. Then $\exists B \in U$ s. th. $f \trianglelefteq_B g$; i.e. $M \models (\forall x,y)(g(x) = g(y) \rightarrow f(x) = f(y))\ldots(1)$.

Define $h \in S_1$ by :

$$h(y) = \begin{cases} f(x), \text{ where } x = \mu t \in B : g(t) = y \\ \text{if } \exists t \in B : g(t) = y. \\ 0 \text{ otherwise.} \end{cases}$$

Then I claim $B \subset \{x : h(g(x)) = f(x)\} = A\ldots(2)$.

For. suppose $x \in B$, and let $x_0 = \mu t \in B : g(t) = g(x)$. Then x_0, $x \in B$ and $g(x_0) = g(x)$. Therefore, by (1), $f(x) = f(x_0)$. But $h(g(x)) = f(x_0)$, by the def. of h. So $h(g(x)) = f(x)$, from which (2) follows.

Now $B \subset A \Rightarrow A \in U$, since $B \in U$ by choice of B. Hence $M_U \models h(g^U) = f^U$ (from (2) and 6.2.2.). Therefore, since $h \in S_1$, we have $M_U[f^U] \preccurlyeq M_U[g^U]$ as required.

□

Now $\equiv_U$ is an equivalence relation on S_1, as is easily checked, and it is also easy to show that $\trianglelefteq_U$ induces an upper-semi lattice ordering on the equivalence classes. We denote this upper-semi lattice by L_U and have the following result, analagous to Aczel's theorem in [10] (lemma 0).

<u>Lemma 6.2.5.</u>

$\$(M_U) \simeq$ The ideals of L_U.

Proof.

It follows from lemma 6.2.4. that the map $\theta : \$(M_U) \rightarrow$ The ideals of L_U, given by $\theta(M') = \{f/\equiv_U : f^U \in M'\}$, where $f/\equiv_U$ is the $\equiv_U$ equivalence class containing f ($\in S_1$), is the required isomorphism.

□

Thus we have reduced our original problem to one of investigating certain combinatorial or partition properties of M. Before we do this however, we require a lemma which reduces the complexity of partitions we shall have to consider later, and also provides us with the negative results promised earlier.

6.3. The main lemma and some negative results.

We first require the following definition and results.

Def 6.3.1.

If $M_1, M_2 \models T$ and $M_1 \subset M_2$, we say M_1 is cofinal in M_2 or that M_2 is a cofinal extension of M_1 iff $(\forall x \in M_2)(\exists y \in M_1)\ M_2 \models y > x$.

Lemma 6.3.2.

Suppose $M_1, M_2, M^* \models T$, $M_1 \preccurlyeq M^*$ and $M_2 \preccurlyeq M^*$, and $M_1 \vee M_2$ is cofinal in M^*. Then either M_1 or M_2 is cofinal in M^*.

Lemma 6.3.3. (Paris, Gaifman, unpublished).

Suppose $M^* \models T$ and that there is a lattice embedding of C_5 (see fig. (ii)) into $\$(M^*)$ which takes the least element of C_5 onto M and the greatest element of C_5 onto M^*. Then M^* is a cofinal extension of M.

The first result is easy to prove and is left to the reader whereas the proof of 6.3.5. below is a generalisation of Paris and Gaifman's proof of 6.3.3. and we therefore omit it also. 6.3.3. shows, of course, that there is no model, M^* of $Th(N)$ s. th. $\$(M^*) \simeq C_5$.

Def. 6.3.4.

If $M_1, M_2 \models T$ we write $M_1 \preccurlyeq^m M_2$ if $M_1 \preccurlyeq M_2$, $M_1 \neq M_2$, and $\forall M'$, $M_1 \preccurlyeq M' \preccurlyeq M_2 \Rightarrow M' = M_1$ or $M' = M_2$; M_2 is then called a minimal elementary extension of M_1.

We can now prove :

Lemma 6.3.5.

Suppose $M^* \models T$ and that M^* is not a cofinal extension of M. Suppose further that $\exists M_1, M_2, M_3 \preccurlyeq M^*$ s. th.

(i) $M \preccurlyeq^m M_1 \precnsim M_2 \preccurlyeq^m M^*$.

(ii) $M_3 \vee M_1 = M^*$ and $M_3 \wedge M_2 = M$.

(iii) $\forall M' \preccurlyeq M_2$, $M' \succcurlyeq M_1$ or $M' = M$.

(iv) $M' \succcurlyeq M_1$, $M' \preccurlyeq M_2$ or $M' = M^*$.

Then $\forall M' \precnsim M^*$, $M' \preccurlyeq M_2$ or $M' = M_3$.

Proof.

We first show that $\forall M' \precnsim M^*$, $M' \preccurlyeq M_2$ or $M' \wedge M_2 = M$ and $M' \vee M_1 = M^*$ .... (1)

So suppose $M' \precnsim M^*$ and $M' \not\preccurlyeq M_2$.

Now $M' \wedge M_2 \preccurlyeq M_2$; therefore by (iii) $M' \wedge M_2 \succcurlyeq M_1$

or $M' \wedge M_2 = M$. But $M' \wedge M_2 \succcurlyeq M_1 \Rightarrow M' \succcurlyeq M_1$, and thus by (iv), $M' \preccurlyeq M_2$ or $M' = M^*$ which is contrary to our assumption above. Hence $M' \wedge M_2 = M$.

Similarly $M' \underset{\neq}{\preccurlyeq} M^*$ and $M' \not\preccurlyeq M_2 \Rightarrow M' \vee M_1 = M^*$, and (1) is thus proved.

Now let $M' \underset{\neq}{\preccurlyeq} M^*$, $M' \not\preccurlyeq M_2$. ....(2).

We now claim that $M'-M > M_2$ (cf. def.4.1.2.)..(3)

For suppose (3) false. Then $\exists a \in M'-M$ and $b \in M_2$ s. th. $a < b$. (We work in M^* throughout this proof unless otherwise stated).

Now by (1), (2) : $M' \wedge M_2 = M$. Therefore $M'[a] \wedge M_2 = M$, since $M'[a] \preccurlyeq M'$. But $M'[a] \underset{\neq}{\succcurlyeq} M$, by choice of a, so $M'[a] \not\preccurlyeq M_2$. Hence by (1) we have

both $M'[a] \wedge M_2 = M$, ...(4),

and $M'[a] \vee M_1 = M^*$, ...(5).

Now suppose $\exists c \in M_2 - M$ s. th. $c < a$ $(< b)$.(*). Then $M_2 \succcurlyeq M_2[c] \underset{\neq}{\succcurlyeq} M$. So by (iii), $M_2[c] \succcurlyeq M_1$. Using this and (5), we see that there must be some $f \in S_2$ s. th. $f(c,a) = b$. Define $F \in S_1$ by :

$$\begin{cases} F(0) = 0. \\ F(i+1) = i+1+\max.\{f(j,k) : j,k \leqslant F(i)\}. \end{cases}$$

Then F is strictly increasing. Hence we can define i_0, i_1 as follows :

$i_0 = \mu i: F(i) \geqslant b.$

$i_1 = \mu i: F(i) \geqslant a.$

Clearly $i_0 \in M_2[b] \preccurlyeq M_2$, and $i_1 \in M'[a]$. But since $c < a < b$ we have, by the def. of F, that either $i_0 = i_1$, or $i_0 = i_1 + 1$. In either case $i_0 \in M'[i_1] \preccurlyeq M'[a]$. Therefore $i_0 \in M'[a] \wedge M_2 = M$ (by (4)). Thus we have :

$F(i_0) \in M$ and $F(i_0) \geqslant b > a > c$, ...(6).

But from (5) and lemma 6.3.2. it follows that either $M'[a]$ or M_1 is cofinal in M^*. Let us first suppose that $M'[a]$ is.

Choose $d \in M'[a]$ s. th. $d > M$. (This is possible since M^* and therefore $M'[a]$ is not a cofinal extension of M by the lemma hypotheses).

Let $g \in S_1$ be s. th. $g(a) = d$. Define $g^* \in S_1$ by : $g^*(x) = \max.\{g(y) : y \leqslant F(x)\}$.

Then by (6) : $g^*(i_0) \geqslant g(a) = d > M$. But $i_0 \in M$, so $g^*(i_0) \in M$ - a contradiction.

Now suppose that M_1 is cofinal in M^*. Choose $d \in M_1$ s. th. $d > M$.

Now $M_2[c] \preccurlyeq M_2$. Therefore by (iii) $M_2[c] \succcurlyeq M_1$ or $M_2[c] = M$. In the former case, choose $g \in S_1$ s. th. $g(c) = d$ and proceed to a contradiction (using (6)) as above. The latter case is impossible by the choice of c (see (*)).

Thus we have shown (*) impossible.

Therefore $a < M_2 - M$, ...(7).

Now choose $a_1 \in M_1 - M$ and $a_2 \in M_2 - M_1$. This is possible by (i), from which it also follows that $M_1 = M_1[a_1]$.

Hence, by (5), $\exists h \in S_2$ s. th. $h(a, a_1) = a_2$. More precisely : $M^* \models h(a, a_1) = a_2$. So by (7) : $\forall d \in M_2 - M \quad M^* \models (\exists x<d)(h(x, a_1) = a_2)$.

Therefore, $\forall d \in M_2 - M \quad M_2 \models (\exists x<d)(h(x, a_1) = a_2)$....(8).

Let $x_0 = \mu x : h(x, a_1) = a_2$ (working in M^*). Then $x_0 \in M^*[a_1, a_2] \preccurlyeq M_2$. But from (8) we see that in fact $x_0 \in M = S_0$. Define g by : $g(x) = h(x_0, x)$. Then, since $x_0 \in S_0$, $g \in S_1$; and further $M^* \models g(a_1) = a_2$, - so $a_2 \in M^*[a_1] \preccurlyeq M_1$ - contradicting the choice of a_1 and a_2.

Thus the supposition that (3) is false is absurd. So $M'-M > M_2$.

We must now show that under the assumption (2), $M' = M_3$.

Now we cannot have $M' \wedge M_3 = M$ and $M' \vee M_3 = M^*$, for this would contradict lemma 6.3.3., since M^* is not a cofinal extension of M and the sublattice $<\{M, M', M_1, M_3, M^*\}, \preccurlyeq>$ of $\$(M^*)$ is isomorphic to C_5.

So say $M' \wedge M_3 = M_4 \gneqq M$ and $M' \neq M_3$. If $M_4 = M_3$, then $M' \gneqq M_3$. Also $M_1 \vee M_3 = M^*$ (from (ii)). Let $a \in M'-M_3$. Then $\exists f \in S_2$, $a_1 \in M_1$ and $b \in M_3$ s. th. $M^* \models f(a_1, b) = a$. Hence from (3) and (i) it follows that :

$\forall d \in M'-M \quad M^* \models (\exists x<d) f(x,b) = a$. Therefore :

$\forall d \in M'-M \quad M' \models (\exists x<d) f(x,b) = a$.

Arguing as before, this implies that $a \in M'[b] \preccurlyeq M_3$, contradicting the choice of a.

If $M_4 \neq M_3$, then $M \precneqq M_4 \precneqq M_3$ and we get a contradiction using (3) with "$M' = M_3$".

Using a similar method we can show that $M' \vee M_3 = M_4$ and $M_4 \precneqq M^*$ and $M' \neq M_3$ is impossible.

Hence we must have $M' = M_3$ whenever M' satisfies (2) and the proof of lemma 6.3.5. is complete.

□

Now if T is true arithmetic i.e. $T = Th(N)$, then M is N and no elementary extension of it can be a proper cofinal extension. Hence we have the following

Corollary 6.3.6.

If $N \models T$, and K is any lattice with distinct top and bottom elements and K' is any lattice with more than one element, there is no $M^* \models T$ s. th. $R \simeq \$(M^*)$; where R is the lattice represented by the diagram :

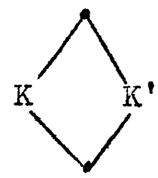
$R =$ 

fig. (iii).

In particular, there is no $M^* \models T$ s. th. $H \simeq \$(M^*)$. (See fig (ii)).

6.4. The pentagon lattice.

We now show $\exists M^* \models T$ s. th. $\$(M^*) \simeq P_5$, where T is, once again, an arbitrary complete extension of P in L.

By lemma 6.2.5. it is sufficient to find an ultrafilter U over $\mathbb{B}$ s. th. $P_5 \simeq L_U$. This, however, we do not do directly, but lemma 6.3.5. allows us to construct U with apparantly weaker properties (and also gives us some information about how we should go about it). To use lemma 6.3.5. we must first guarantee that our resulting M_U is not a cofinal extension of M, for which we need the following result.

Lemma 6.4.1.

Let U be any ultrafilter over $\mathbb{B}$. Then M_U is a cofinal extension of M iff U contains an M-finite set.

Proof.

Suppose B is M-finite and $B \in U$. Let f^U (where $f \in S_1$), be any element of M_U. Let $a = \max\{f(x): x \in B\}$ (working in M). a, of course, exists since B is M-finite, and $M_U \models f^U \leqslant a$, by theorem 6.2.2.. Hence M_U is a cofinal extension of M.

Conversely $id^U \in M_U$ and if $M_U \models id^U \leqslant a$ for some $a \in M$ (where we are identifying $\hat{a}$ with a), then $\exists B \in U$ s. th. $B = \{x \in M : M \models id(x) \leqslant a\} = \{x \in M : M \models x \leqslant a\}$, which is M-finite.

□

We now begin the construction of the required U.

First, let $\lambda x,y{:}\langle x,y\rangle \in S_2$ be a fixed pairing function and π_1, π_2 be the corresponding projection functions, i.e. $\pi_1(\langle x,y\rangle) = x$ and $\pi_2(\langle x,y\rangle) = y$.

For $B \in \mathbb{B}$ and $\langle x,y\rangle, \langle x',y'\rangle \in B$ define :

$\langle x,y\rangle \leqslant_B \langle x',y'\rangle \iff y \leqslant y' \wedge x \equiv x' \pmod{2^y}$, and

$\langle x,y\rangle \sim_B \langle x',y'\rangle \iff \langle x,y\rangle \leqslant_B \langle x',y'\rangle \wedge \langle x',y'\rangle \leqslant_B \langle x,y\rangle$.

Then $\sim_B$ is a definable (i.e. T-definable or M-definable), equivalence relation on B.

Let $\langle x,y\rangle^B = \{\langle x',y'\rangle : \langle x',y'\rangle \sim_B \langle x,y\rangle\}$, and $\mathfrak{J}_B = \{\langle x,y\rangle^B : \langle x,y\rangle \in B\}$.

$\leqslant_B$ induces a partial ordering on $\mathfrak{J}_B$ (in fact an M-binary- tree-like ordering) which we shall also denote by $\leqslant_B$. Also if $B, C \in \mathbb{B}$ and $B \subset C$, then we have $\leqslant_B = \leqslant_C \restriction B$ (in both sences of

$\leqslant_B$ and $\leqslant_C$).

We shall usually regard all sets in $\mathbb{B}$ as sets of ordered pairs. Thus we shall speak of the horizontal and vertical lines of B, for $B \in \mathbb{B}$, meaning sets of the form $\pi_2^{-1}[s] \cap B$ and $\pi_1^{-1}[s] \cap B$, for some $s \in M$, respectively.

For $A \in \mathbb{B}$, let lev(A) = the unique y s. th. $\pi_2[A] = \{y\}$, if such a unique y exists, and let lev(A) be undefined otherwise. Note that if $\emptyset \neq A \in \mathfrak{J}_B$ (for some $B \in \mathbb{B}$), then lev(A) is defined.

On setting $K = \{ \langle x,y \rangle : y \leqslant x\}$ ($\in \mathbb{B}$) we can make the following crucial

<u>Def 6.4.2.</u>

A set $B \in \mathbb{B}$ is called <u>correct</u> iff

(i) $B \subset K$.

(ii) Every set in $\mathfrak{J}_B$ is infinite.

(iii) $\mathfrak{J}_B$ has a $\leqslant_B$-least element.

(iv) Every element of $\mathfrak{J}_B$ has precisely two immediate $\leqslant_B$-successors (in $\mathfrak{J}_B$).

(v) If l, h are horizontal lines of B s. th. lev(l) $\leqslant$ lev(h), then $\pi_1[h] \subset \pi_1[l]$.

(vi) If $C, D \in \mathfrak{J}_B$ and lev(C) = lev(D), and if C', D' are immediate $\leqslant_B$-successors of C, D respectively, then lev(C') = lev(D').

We first note that if $B \in \mathbb{B}$, then each of the above conditions can be expressed by a sentence in L, and hence there is a sentence (depending on B) which is true in M iff B is a correct set.

Note K is a correct set.

Now let σ be any function in S_1 which is constant on each set in $\mathfrak{I}_K$ but takes different values on different members of $\mathfrak{I}_K$ e.g.

$$\sigma(\langle x,y\rangle) = \begin{cases} \langle rm(x,2^y),y\rangle & \text{for } \langle x,y\rangle \in K, \\ 0 & \text{otherwise,} \end{cases}$$

where $rm(s,t)$ = the remainder when s is divided by t, will suffice.

We shall now state the main combinatorial lemma concerning correct sets, and show how it implies the main theorem, as immediate justification for these rather obscure definitions.

<u>Lemma 6.4.3.</u>

Let $f \in S_1$, and B $(\in \mathbb{B})$ be any correct set. Then there is a correct set $C \subset B$, s. th. either (i) f is one-one on every horizontal line of C,

or (ii) $f \equiv_C \sigma$,

or (iii) $f \equiv_C \pi_2$,

or (iv) $f \equiv_C \hat{0}$ -i.e. f is constant on C.

<u>Lemma 6.4.4.</u>

Lemma 6.4.3. implies $\exists$an ultrafilter U, over $\mathbb{B}$, s. th. $\$(M_U) \simeq P_5$.

<u>Proof.</u>

For $A \in \mathbb{B}$, define $f_A \in S_1$ by :

$$f_A(x) = \begin{cases} 0 & \text{if } x \in A, \\ 1 & \text{if } x \notin A. \end{cases}$$

Let B be any correct set.

Apply 6.4.3. with $f = f_A$ to obtain a correct set, C, satisfying (i) or (ii) or (iii) or (iv). But f_A takes only two values, so, since C is correct we must have C satisfying (iv). Thus we have shown that if B is any correct set and $A \in \mathbb{B}$, then $\exists$a correct $C \subset B$ s. th. $C \subset A$ or $C \subset cA$ (the complement of A).

Now enumerate $S_1 X \mathbb{B}$ as follows :

$\langle f_1, B_1 \rangle$, $\langle f_2, B_2 \rangle, \ldots,$ $\langle f_n, B_n \rangle$ $, \ldots$ $n \in \omega$, $n \geq 1$.

We can now construct a sequence of sets from B, A_0, $A_1, \ldots, A_n, \ldots n \in \omega$, s. th.

(i) $A_0 = K$,

(ii) $(\forall i \in \omega)$ $A_i \supset A_{i+1}$,

(iii) $(\forall i \in \omega)$ A_i is correct,

(iv) $(\forall i \in \omega, \ i \geq 1)$ $A_i \subset B_i$ or $A_i \subset cB_i$,

(v) $(\forall i \in \omega, \ i \geq 1)$,

either (a) f_i is one-one on every horizontal line of A_i,

or (b) $f_i \equiv_{A_i} \sigma$,

or (c) $f_i \equiv_{A_i} \pi_2$,

or (d) $f_i \equiv_{A_i} \hat{0}$.

It is clear how the A_i are constructed using lemma 6.4.3. and the first part of this proof.

(ii) and (iii) now imply that $\{A_i : i \in \omega\}$ can be extended to an ultrafilter U over $\mathbb{B}$ containing no M-finite sets. (Every correct set must be M-infinite by 6.4.2.(ii)).

We claim $\$(M_U) \simeq P_5$. In fact we show the elementary substructures of M_U are arranged as follows :

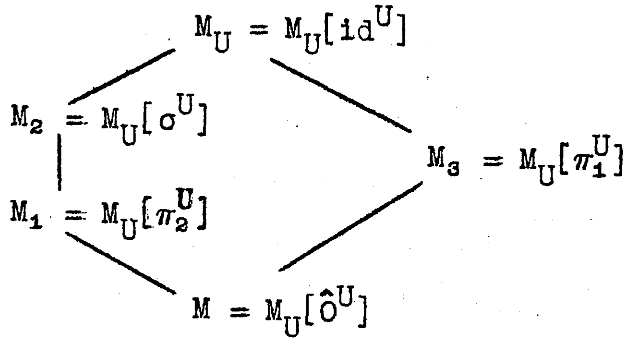


fig. (iv).

Firstly, we clearly have : $\hat{0} \trianglelefteq_K \pi_2 \trianglelefteq_K \sigma \trianglelefteq_K$ $\trianglelefteq_K id$; hence, since $K = A_0 \in U$, $M \preccurlyeq M_1 \preccurlyeq M_2 \preccurlyeq M_U$, by 6.2.4. Also $M \preccurlyeq M_3 \preccurlyeq M_U$.

Now, by construction, every set in U contains a correct set, so it follows from def. 6.4.2. that

$M \precneqq M_1 \precneqq M_2 \precneqq M_U$ ....(1),

and $M \precneqq M_3 \precneqq M_U$. ....(2).

Now suppose $M' \succcurlyeq M_1$ and $M' \succcurlyeq M_3$. Then $\pi_1^U \in M'$ and $\pi_2^U \in M'$. But the pairing function $\lambda x,y{:}\langle x,y\rangle \in S_2$. Hence $\langle \pi_1^U, \pi_2^U \rangle \in M'$; i.e. $id^U \in M'$, so $M' = M_U$.

Thus, $M_1 \vee M_3 = M_U$ ....(3).

We now show $M_2 \wedge M_3 = M$ ....(4).

Suppose $\tau \in S_1$ and $\tau^U \in M_2 \wedge M_3$. Then $\tau \trianglelefteq_U \sigma$ and $\tau \trianglelefteq_U \pi_1$ by lemma 6.2.4..

Hence $\exists B \in U$ s. th. $\tau \trianglelefteq_B \sigma$ and $\tau \trianglelefteq_B \pi_1$...(*), and we may suppose B correct by the construction of U. Let y_0 be the level of the $\leqslant_B$-least element, D, of $\mathcal{J}_B$. (This exists by 6.4.2.(iii)).

We show $\langle x,y\rangle, \langle x',y'\rangle \in B \Rightarrow \tau(\langle x,y\rangle) = \tau(\langle x',y'\rangle)$, so that $\tau \equiv_B \hat{0}$ and thus $M_2 \wedge M_3 = M$.

So suppose $<x,y>$, $<x',y'> \in B$.

Then $\pi_1(<x,y>) = \pi_1(<x,y_0>)$
and $\pi_1(<x',y'>) = \pi_1(<x',y_0>)$. ....(**).

Also $<x,y_0>$, $<x',y_0> \in B$ by 6.4.2.(v). Therefore, by the def. of D, $<x,y_0>$, $<x',y_0> \in D$, so $<x,y_0> \sim_B <x',y_0>$, which implies $\sigma(<x,y_0>) = = \sigma(<x',y_0>)$, by the def. of σ. Therefore, by (*), $\tau(<x,y_0>) = \tau(<x',y_0>)$. But by (*) and (**), $\tau(<x,y>) = \tau(<x,y_0>)$ and $\tau(<x',y'>) = \tau(x',y_0>)$. Hence $\tau(<x,y>) = \tau(<x',y'>)$, as required.

Now by the def. of U and lemma 6.2.4.,
$M' \lneq M_2 \Rightarrow M' = M_1$ or $M' = M$ ....(5).
In particular, $M \prec^m M_1$ ....(6).

Now suppose $M' \gneq M_1$. Choose $f^U \in M'-M_1$. We may suppose $\pi_2 \trianglelefteq_U f$; say $\pi_2 \trianglelefteq_B f = f_i$ and $B \in U$.

Then by the def. of U :
either (i) f_i is one-one on every horizontal line of A_i,
or (ii) $f_i \equiv_{A_i} \sigma$.

But if (i) holds we have, using $\pi_2 \trianglelefteq_B f_i$, that f_i is one-one on $B \cap A_i \in U$. Hence $f = f_i \equiv_U \equiv_U id$, and $M' = M_U$.

Suppose for no $f^U \in M'-M_1$ do we have (i) above. Then $f^U \in M'-M_1$, $f \equiv_U \sigma$. Hence $M' = M_2$.

Thus $M' \gneq M_1 \Rightarrow M' = M_2$ or $M' = M_U$ ......(7).
In particular, $M_1 \prec^m M_2 \prec^m M_U$ ......(8).

Now since U contains no M-finite sets, M_U

cannot be a cofinal extension of M, by lemma 6.4.1.. This, and (1)-(8) now imply the hypotheses of lemma 6.3.5. with M_U replacing M*.

Hence, $\forall M' \preccurlyeq\!\!\!/ M_U$, $M' \lesssim M_2$ or $M' = M_3$. From this, (3), (4), (6) and (8) we obtain $\$(M_U) \simeq P_6$, as required.

□

The proof of lemma 6.4.3. is not hard in principle ; in fact by 'drawing diagrams' it becomes fairly obvious, although the details, as we shall see, are rather messy. I should like now, however, to explain why we do not construct U directly with the required properties. For this would require a proof of lemma 6.4.3. with (i) replaced by the stronger condition :

either (ia) $f \equiv_C id$,

or (ib) $f \equiv_C \pi_1$,

and this I could not do.

However, lemma 6.3.5. tells us, essentially, that in constructing the U of 6.4.4., we only have to guarantee (i) to ensure that (ia) or (ib) must eventially occur.

Now the proof of lemma 6.4.3..

Suppose $f \in S_1$ and B is any correct set. We first construct a correct set $C' \subset B$ s. th.

$\forall A \in \mathcal{J}_{C'}$, either (i) f is constant on A, ...(*).

or (ii) f is one-one on A.

We define, by induction, sets $l_0, l_1, \ldots, l_i, \ldots$ ($i \in M$), which will be the horizontal lines of C' in ascending order of level.

Thus we will put $C' = \cup\{l_i : i \in M\}$.

We simultaneously define sets $A_0^i, \ldots, A_{2^i-1}^i$ $i \in M$, which are elements of $\mathfrak{I}_B$ and are s. th. $l_i \cap A_j^i$ for $j < 2^i$, will be all the elements of $\mathfrak{I}_{C'}$ having the same level as l_i.

We require the following induction conditions :

$(i)_i$ $l_i \subset$ some horizontal line of B, and $\mathrm{lev}(l_{i-1}) < \mathrm{lev}(l_i)$.

$(ii)_i$ $A_j^i \in \mathfrak{I}_B$ $\forall j < 2^i$, and $l_i \subset \cup\{A_j^i : j < 2^i\}$, and $l_i \cap A_j^i$ is infinite $\forall j < 2^i$, and $j \neq k \Rightarrow$ $\Rightarrow A_j^i \cap A_k^i = \emptyset$.

$(iii)_i$ Either $i = 0$ or $\forall j < 2^{i-1}$ there are precisely two numbers j_0, $j_1 < 2^i$ s. th. $\pi_1[(A_{j_0}^i \cup A_{j_1}^i) \cap l_i] \subset [A_j^{i-1} \cap l_{i-1}]$.

$(iv)_i$ $(\forall j < 2^i)$ f is either constant on $l_i \cap A_j^i$ or one-one on $l_i \cap A_j^i$.

To give the induction inertia we also require :

$(v)_i$ $\forall j < 2^i$ $\exists D_j^i \in \mathfrak{I}_B$ s. th. $\pi_1[A_j^i \cap l_i] \cap$ $\cap\, \pi_1[D']$ is infinite $\forall D' \in \mathfrak{I}_B$ s. th. $D_j^i \leqslant_B D'$.

First let $B^*(y,s)$ be a formula s. th. as y runs over M, $B_y^* = \{s \in M : M \models B^*(y,s)\}$ runs over all sets in $\mathfrak{I}_B$, and $y \neq y' \Rightarrow B_y^* \cap B_{y'}^* = \emptyset$.

<u>Def. of l_0.</u>

Let $l = \leqslant_B$-least element of $\mathfrak{I}_B$, and $t_0 = \mathrm{lev}(l)$. We define the function g on l by :

$g(0) = \langle x_0, t_0\rangle$ where $x_0 = \mu x{:}\langle x, t_0\rangle \in 1$.

$$g(y+1) = \begin{cases} \langle x', t_0\rangle \text{ where } x' = \mu x{:}\langle x,t_0\rangle \in 1 \wedge \\ \wedge\, x \in \pi_1[B^*_{y+1}] \wedge (\forall z \leq y)(x \neq \pi_1(g(z)) \wedge \\ \wedge\, f(x,t_0) \neq f(g(z))) \text{, if there} \\ \text{is such an } x. \\ = g(y), \text{ otherwise.} \end{cases}$$

If the range of g is M-infinite, let 1_0 =range(g), and $A^0_0 = 1$, whence $D^0_0 = 1$ will satisfy $(v)_0$. Conditions $(i)_0$-$(iv)_0$ areeasily checked - f being one-one on $1_0 \cap A^0_0 = 1_0$.

If the range of g is M-finite, there must be some $D \in \mathfrak{J}_B$ s. th. $f[\{\langle x,t_0\rangle : x \in \pi_1[D]\}]$ is finite. It is easy to define, in this case, a set $\bar{D} \in \mathfrak{J}_B$ s. th. $D \leq_B \bar{D}$ and a set $D'' \subset \bar{D}$ s. th. f is constant on $D^* = \{\langle x,t_0\rangle : x \in \pi_1[D'']\}$, and s. th. $\forall G \in \mathfrak{J}_B$, $\bar{D} \leq_B G \Rightarrow \pi_1[D^*] \cap \pi_1[G]$ is infinite.

We now put $1_0 = D^*$, $A^0_0 = 1$. Condition $(iv)_0$ is satisfied since f is constant on $D^* = 1_0 = 1 \cap A^0_0$, and $(v)_0$ is satisfied with $D^0_0 = \bar{D}$. The other conditions are trivial to check.

Now suppose for some i, $1_0, \ldots, 1_i$, A^i_j have been defined $(\forall j < 2^i)$ satisfying $(i)_i$-$(v)_i$. Let D^i_j $(\forall j < 2^i)$ be the sets given by $(v)_i$. We can suppose all the D^i_j have the same level and $(v)_i$ still holds.

Now consider the elements of $\mathfrak{J}_B$ which are immediate $\leq_B$-successors of the D^i_j. Each D^i_j has two such $\leq_B$-successors, say G^j_0, G^j_1 and all

G_k^j have the same level (i is fixed), say t_0. (This follows from the correctness of B).

For $k \leq 1$, $j < 2^i$ let $G_k^{j*} = \{\langle x, t_0\rangle \in G_k^j :$ $x \in \pi_1[A_j^i \cap l_i]\}$.

Now each G_k^{j*} generates a correct subset, T_k^j of B in a natural way, namely :

$T_k^j = \{\langle x,y\rangle \in B : y \geq t_0 \wedge x \in \pi_1[G_k^{j*}]\}$.

Further, G_k^{j*} is the $\leq_B$ ($= \leq_{T_k^j}$)-least element of $\mathfrak{J}_{T_k^j}$. Hence we can perform the same construction on the T_k^j as we did for B in the first part of the proof, to obtain subsets $*G_k^j$ of G_k^{j*} on which f is either one-one or constant and s. th. $(i)_{i+1}-(v)_{i+1}$ hold when we put $A_0^{i+1}, \dots, A_{2^{i+1}-1}^{i+1}$ equal to $G_0^0, G_1^0, G_0^1, G_1^1, \dots, G_0^{2^i-1}, G_1^{2^i-1}$ respectively, and $l_{i+1} = \bigcup\{*G_k^j : k \leq 1, \ j < 2^i\}$, where in $(iii)_{i+1}$ $A_{j_0}^{i+1} = G_0^j$ and $A_{j_1}^{i+1} = G_1^j$, i.e. $j_0 = 2j$, $j_1 = 2j+1$.

The induction is now complete and, putting $C' = \bigcup\{l_i : i \in M\}$, we have accomplished (*).

(Actually we have not said anything about C' being definable from B and f - but the above construction was uniform in B and f and the induction uniform in i. We conclude that C' is M-definable leaving the reader to check the details).

We now construct a correct set $C'' \subset C'$ s. th. either (i) f is constant on every set in $\mathfrak{J}_{C''}$, or (ii) f is one-one on every set in $\mathfrak{J}_{C''}$. ....(**).

First a digression.

Let $\mathfrak{T}$ be an M- full binary tree in which every element has finite level (this ordering is definable in P). By a strict subtree $\mathfrak{T}'$ of $\mathfrak{T}$, we mean a subtree of $\mathfrak{T}$ which is a full binary tree (we drop the prefix M- from now on), and s. th. if $a, b \in \mathfrak{T}'$ and a and b have the same level in $\mathfrak{T}'$, then they have the same level in $\mathfrak{T}$.

Now the existence of a C'' satisfying (**) is clearly equvalent to the following <u>claim</u> :

If every node of $\mathfrak{T}$ is coloured either red or blue, then $\mathfrak{T}$ has a monochromatic strict subtree.

To prove the claim, suppose $\mathfrak{T}$ is coloured as stated. Then one of the following must occur : either (i) $\forall a \in \mathfrak{T}$, $\exists x$, $x \geqslant$ height of a, s. th. every level of $\mathfrak{T}$ of height $\geqslant x$ contains at least two red nodes, b and c, s. th. $a \ll b$ and $a \ll c$. (Where $\ll$ is the tree ordering).

or (ii) $\exists a \in \mathfrak{T}$ s. th. there are infinitely many levels, l, above a, s. th. all nodes (except possibly one) in l which are $\gg a$, are coloured blue.

It is easy to check that in case (i) there is a red strict subtree of $\mathfrak{T}$, and in case (ii) a blue one. Hence we can construct C'' satisfying (**).

Suppose C'' satisfies (**)(ii). I claim we can find a correct set $C \subset C''$ s. th. (i) of lemma 6.4.3. holds.

Let $l_0, l_1, \ldots, l_i, \ldots$ $i \in M$, be the horizontal lines of C'' in increasing order of level.

We define $l'_0, l'_1, \ldots, l'_i, \ldots$ $i \in M$ s. th. $\forall i$:

$(i)_i$ $l'_i \subset l_i$ and $\pi_1[l'_i] \subset \pi_1[l'_{i-1}]$ (or $i = 0$),

$(ii)_i$ $D \in \mathfrak{J}_{C''}$, $D \subset l_i \Rightarrow D \cap l'_i$ is infinite,

$(iii)_i$ f is one-one on l'_i,

$(iv)_i$ $D \in \mathfrak{J}_{C''}$, $D \subset l_i \Rightarrow \pi_1[D \cap l'_i] \cap \pi_1[D']$ is infinite $\forall D' \in \mathfrak{J}_{C''}$ s. th. $D \leqslant_{C''} D'$.

Let $l'_0 = l_0$.

Suppose $l'_0, \ldots, l'_i$ have been constructed for some $i \geqslant 0$, satisfying $(i)_j$-$(iv)_j$ $\forall j \leqslant i$.

Let $lev(l_{i+1}) = t_0$.

Define $\quad G(y) \Leftrightarrow lev(C''_y{*}) \geqslant t_0$. (Where the $*$ operator is defined as on p. 70 six lines from the bottom).

Define g as follows :

$$\begin{cases} g(0) = & \mu x : x \in \pi_1[l_{i+1}] \cap \pi_1[l'_i] \\ g(y+1) = & \mu x : (x \in \pi_1[l'_i] \cap \pi_1[C''_z{*}] \text{ where} \end{cases}$$

$z = (y+1)$st. element, t, satisfying $G(t)) \wedge ((\forall p \leqslant y)$ $(f(<x,t_0>) \neq f(<g(p),t_0>)))$.

By the induction hypotheses $(i)_i$-$(iv)_i$, $g(y)$ is always defined and $range(g) \subset \pi_1[l_{i+1}]$ since $G(z) \wedge x \in \pi_1[C''_z{*}] \Rightarrow x \in \pi_1[l_{i+1}]$, by the correctness of C''.

Let $l'_{i+1} = \{<x,t_0> : x \in range(g)\}$.

$(i)_{i+1}$-$(iv)_{i+1}$ can now be verified.

Put $C = \cup\{l'_i : i \in M\}$. That C is correct and that f is one-one on every horizontal line of C (i.e. on l_i' $\forall i$) follows from the construction. Hence we have (i) of lemma 6.4.3. if C" satisfies (**)(ii).

It remains to show that if C" satisfies (**)(i) then there is a correct $C \subset C''$ s. th. (ii) or (iii) or (iv) of lemma 6.4.3. holds.

This is again equivalent to a partition theorem on trees, namely :

If $\mathcal{J}$ is any tree as described on p. (73) and $\mathcal{J}$ is coloured in any way whatsoever (possibly using infinitely many colours) then it has a strict subtree $\mathcal{J}'$ s. th.
either (i) every node of $\mathcal{J}'$ has a different colour,

or (ii) nodes of $\mathcal{J}'$ of the same level have the same colour, but nodes of different levels have different colours,

or (iii) every node of $\mathcal{J}'$ has the same colour.

To prove this, suppose $\mathcal{J}$ is coloured in any way. Suppose first that the following holds :
(+) $\forall z \in M$, $\forall x \in \mathcal{J}$, $\exists$level, l of $\mathcal{J}$ above x, s. th. $\forall$levels l' above l, $l' \cap \{y \in \mathcal{J} : y \gg x\}$ is at least z-coloured (i.e. there are z colours appearing in this set).

We define $\mathcal{J}'$ to satisfy (i), by constructing its levels $l_0, l_1, \ldots$ by induction as follows.

Let $l_0 = \{$least element of $\mathcal{J}\}$.

Suppose $l_0, \ldots, l_i$ have been constructed s. th.
$(i)_i$ every element of $\cup\{l_j : j \leq i\}$ has a

different colour,

$(2)_i$ $(\forall j \leq i)$ $l_j \subset$ some level of $\mathcal{T}$.

$(3)_i$ $(\forall j \leq i)$ l_j contains 2^j elements.

$(4)_i$ $\langle \cup\{l_j : j \leq i\}, \ll \rangle$ is a binary tree of height i.

($\ll$ once again denotes the ordering of $\mathcal{T}$, and we use the same symbol for its restriction to subsets of $\mathcal{T}$).

We construct l_{i+1} s. th. $l_0, \ldots, l_{i+1}$ satisfy $(1)_{i+1}-(4)_{i+1}$.

Take $z = 2^{i+2}$ in (+) and find a level, l of $\mathcal{T}$ s. th. $l \cap \{y \in \mathcal{T} : y \gg x\}$ is at least 2^{i+2}-coloured $\forall x \in l_i$. This is possible from (+) since l_i is finite and $\mathcal{T}$ has infinitely many levels. Suppose $l_i = \{x_0, \ldots, x_{2^i-1}\}$, and let $A_j = \{y \in \mathcal{T} : y \gg x_j\} \cap l$ $(\forall j < 2^i)$.

Then since $\cup\{l_j : j \leq i\}$ has $2^{i+1}-1$ elements, we may pick two elements, y_j^0 and y_j^1, from each A_j s. th. every element of $\cup\{l_j : j \leq i\} \cup$ $\cup \{y_0^0, y_0^1, y_1^0, y_1^1, \ldots, y_{2^i-1}^0, y_{2^i-1}^1\}$ has a different colour. Putting $l_{i+1} = \{y_0^0, y_0^1, y_1^0, y_1^1, \ldots, y_{2^i-1}^0, y_{2^i-1}^1\}$ completes the induction. $\mathcal{T}' = \{l_i : i \in N\}$ now satisfies (i) above.

If (+) is false, then using the same method as that on p. (73) we can construct a strict subtree $\mathcal{T}''$ of $\mathcal{T}$ s. th. every level of $\mathcal{T}''$ has the same colour. It is then a triviality to construct a strict subtree $\mathcal{T}'$ of $\mathcal{T}''$ (and therefore $\mathcal{T}'$ is a strict subtree of $\mathcal{T}$) s. th. either (i) or (ii) holds.

The proof of lemma 6.4.3. is now complete.

□

Lemmas 6.4.3. and 6.4.4. now give the main

<u>Theorem 6.4.5.</u>

$\exists M^* \models T$ s. th. $\$(M^*) \simeq P_5$.

<u>6.5. Cofinal extensions of models of P.</u>

A complete answer to the problem posed on p. 52 still seems a long way off - even for finite lattices. To obtain results for the simplest modular non-distributive lattices, however, lemma 6.3.3. tells us that elementary cofinal extensions of models of P must be investigated, and in this section we look at minimal cofinal extensions.

We first extend some of our previous definitions concerning simple extensions.

<u>Def. 6.5.1.</u>

If M is any model of P, M* is called a <u>simple extension of M</u> if $M^* \succ M$, and $\exists a \in M^*$ s. th. $M^* = M^*[M \cup \{a\}]$.

Now if we let $\mathbb{B}_M$, for $M \models P$, be the Boolean algebra of M-definable (i.e. definable using parameters from M) subsets of M, and U be any ultrafilter over $\mathbb{B}_M$, we can construct M_U in a similar way as in section 6.2. where the elements of M_U are now M-definable total functions (from M to M) factored modulo U. Theorems analagous to 6.2.2. and 6.2.3. can now be proved, as can one analagous to 6.2.4. when the obvious modification of the definition of $\triangleleft_U$ is introduced. We leave the details to the reader.

The point of doing all this is the following

Lemma 6.5.2.

If M is any model of P and U any ultrafilter over $\mathbb{B}_M$, then M_U is a minimal elementary extension of M, i.e. $M \preccurlyeq^m M_U$, iff every M-definable one-place function is either constant or one-one on a set in U, and U contains no singleton sets. Further, M_U is a cofinal extension of M iff U contains an M-finite set.

Proof.

All is clear from the modified 6.2.4. and 6.4.1..

□

Now Gaifman has shown [6] that given any $M \models P$, there is an ultrafilter U over $\mathbb{B}_M$, containing no M-finite sets s. th. every M-definable one-place function is either constant or one-one on some set in U. Upon observing the fairly trivial fact that $M \preccurlyeq^m M^*$ implies M^* is either a cofinal or an end extension of M, we see that M_U is a minimal elementary end extension of M.

We should like to prove an analagous result for cofinal extensions but can, unfortunately, only prove the following special cases :

Theorem 6.5.3.

Suppose M is a non-standard model of P

satisfying one of the following conditions :

either (i) $\exists a \in M$ s. th. $\{x \in M : M \models x \leqslant a\} = \aleph_0$,

or (ii) M is saturated.

Then M has a minimal cofinal elementary extension.

Proof.

Suppose M satisfies (i). Let F be the function in S_1 s. th. $P \vdash (\forall x)(F(x)$ is the number of partitions of (i.e. equivalence relations on) $\{y : y \leqslant x\})$.

By a modified over- spill arguement we can find a non-standard $b \in M$ s. th. $M \models F(b) \leqslant a$. Using (i) this implies there are only countably many distinct M-definable partitions of the set $\{y \in M : M \models y \leqslant b\}$. Thus there is a sequence $f_0, f_1, \ldots, f_n, \ldots$ $n \in \omega$, of M-definable one-place functions s. th. given any M-definable one-place function g, we can find $n \in \omega$ s. th.:

$$M \models (\forall x, y \leqslant b)(f_n(x) = f_n(y) \leftrightarrow g(x) = g(y)).$$

Thus we shall be finished if we can construct an ultrafilter U over $\mathbb{B}_M$ s. th. $\forall n \in \omega$ $\exists A \in U$ s. th. f_n is either one-one or constant on A, and $\{x \in M : M \models x \leqslant b\} \in U$.

We do this by constructing a sequence $A_0, A_1, \ldots, A_n, \ldots$ $n \in \omega$ of sets in $\mathbb{B}_M$ s. th. $\forall n \in \omega$

(i)$_n$ $\{y \in M : M| \ y \leqslant b\} \supset A_{n-1} \supset A_n$,

(ii)$_n$ $A_n \cap \omega$ is infinite,

(iii)$_n$ f_n is one-one or constant on A_n.

First, let $b = b_0, b_1, \ldots, b_n, \ldots$ $n \in \omega$ be a decreasing sequence of infinite elements of M

s. th. for all infinite $c \in M$, $\exists n \in \omega$ s. th. $M \models b_n \leqslant c$. Such a sequence exists by (i) of the theorem hypotheses.

Now suppose $A_0, \ldots, A_n$ have been constructed to satisfy $(i)_n$-$(iii)_n$. Suppose, firstly, that f_{n+1} is constant on some infinite subset of $A_n \cap \omega$ - taking the value c, say.

Let $A_{n+1} = \{x \in M : M \models x \leqslant b_{n+1} \wedge x \in A_n \wedge f_{n+1}(x)=c\}$.

If f_{n+1} is constant on no infinite subset of $A_n \cap \omega$, it must be one-one on some infinite subset of $A_n \cap \omega$. Define the function G by :

$$\begin{cases} G(0) = & \mu x : x \in A_n, \\ G(y+1) = & \mu x : x \in A_n \wedge x > G(y) \wedge (\forall z \leqslant y) \\ & (f_{n+1}(x) \neq f_{n+1}(G(z))), \text{ if} \\ & \text{such an } x \text{ exists,} \\ & G(y) \text{ otherwise.} \end{cases}$$

Then G is an M-definable function. Let $A_{n+1} = \text{range}(G) \cap \{x \in M : M \models x \leqslant b_{n+1}\}$.

In either case $(i)_{n+1}$-$(iii)_{n+1}$ are easily verified.

Now extending $\{A_n : n \in \omega\}$ to a non-principal ultrafilter over $\mathbb{B}_M$ completes the proof.

Note that if c is an infinite element of M, then $\exists A \in U$ s. th. $M \models (\forall x)(x \in A \rightarrow x \leqslant c)$. Hence $\{x \in M : M \models \text{id}(x) \leqslant c\} \in U$. i.e. $M_U \models \text{id}^U \leqslant c$. Thus id^U is an infinite element of M_U which is smaller than every infinite element of M.

Now suppose M satisfies (ii) of the theorem hypotheses and $\overline{\overline{M}} = \kappa$. Then there are κ one-place M-definable functions ; say $f_0, f_1, \ldots, f_\alpha, \ldots\ \alpha < \kappa$.

is a κ-enumeration of them.

We define elements of $\mathbb{B}_M$ $c_0 \supset c_1 \supset \cdots \supset c_\alpha \supset \cdots$ $(\alpha < \kappa)$ s. th. each c_α is an M-finite infinite set and f_α is one-one or constant on c_α $(\forall \alpha < \kappa)$.

Suppose $c_0, \ldots, c_\alpha, \ldots$ $(\alpha < \beta < \kappa)$ have been so defined. If $\beta = \gamma + 1$, let the number of elements in c_γ be a. (i.e. there is an M-definable one-one map from c_γ onto $\{x \in M : M \models x < a\}$). Then f_β must either take one value at least $[\sqrt{a}]$ times on c_γ or must take at least $[\sqrt{a}]$ values on c_γ. ($[\sqrt{x}]$ = the integer part of $\sqrt{x}$ - this is an M-definable function). It is now easy to define a subset c_β of c_γ, on which f_β is one-one or constant, having 'M-cardinality' $[\sqrt{a}]$. But a must be infinite, by our inductive hypotheses, hence so is $[\sqrt{a}]$ and thus c_β is M-finite but infinite.

Now if β is a limit ordinal, consider the set τ of formulae :

{"x codes a finite set having at least n elements" : $n \in \omega$} $\cup$ {"f is one-one or constant on the set coded by x"} $\cup$ {"the set coded by $x \subset c_\alpha$": $\alpha < \beta$}.

A similar arguement to that used above shows that τ is finitely satisfiable in M. Certainly $< \kappa$ parameters from M are mentioned in τ, and so, since M is saturated, τ is realised in M by c say. Setting c_β = the set coded by c completes our induction, and theorem 6.5.3. follows.

□

Def. 6.5.4.

For M, $M^* \models P$, M non-standard, we say that M^* is a normal extension of M if $M^* \succ M$ and $\exists a \in M^*$ s. th. $\omega < \{a\} < M-\omega$ in M^*.

Thus we proved above, in fact, that every model of P satisfying 6.5.3.(i) (in particular every non-syandard countable model) has a minimal normal extension. We now ask the same question for models satisfying (ii). It should be fairly clear that the proof we used for saturated models above can be adapted for models satisfying (i), but would not, in general, give us a normal extension. Thus we are essentially asking if the proof we used for (i) can be adapted for saturated models. We first make the following definition due to Choquet [2].

Def. 6.5.5.

A non-principal ultrafilter U over ω (i.e. on the full power set of ω) is called Ramsey if given any partition $\{a_i : i \in \omega\}$ of ω, either (i) $\exists i \in \omega$ s. th. $a_i \in U$, or (ii) $\exists a \in U$ s. th. $\overline{\overline{a \cap a_i}} \leq 1$ $\forall i \in \omega$.

We can now prove :

Theorem 6.5.6.

If M is an ω_1-saturated model of P, the following are equivalent :

(i) M has a minimal normal extension,

(ii) There exists a Ramsey ultrafilter over ω.

Proof.

Suppose U is a Ramsey ultrafilter over ω. We show that $M^* = M^{\omega}/U$ (the usual ultrapower of M over U - see [1]) is the required extension.

Certainly $M^* \succcurlyeq M$ and $\omega < id^U < M\text{-}\omega$ in M^* (where id is here the map taking each $n \in \omega$ to its copy in M), so M^* is a normal extension of M.

Now suppose $f \in M^{\omega}$. Then by considering the partition $\{f^{-1}[a] : a \in M\}$ of ω and using the fact that U is Ramsey, we see that $\exists A \in U$ s. th. f is constant on A, in which case $f^U \in M$, or f is one-one on A. In this latter case we proceed as follows.

Let τ be the following set of formulae :

$\{$'x is a finite set of ordered pairs'$\} \cup$

$\cup \{(\forall z)((\exists t)(\langle z,t\rangle \in x) \rightarrow (\exists! t)(\langle z,t\rangle \in x))\} \cup$

$\cup \{(\forall z,t,t')(\langle t,z\rangle \in x \wedge \langle t',z\rangle \in x \rightarrow t = t')\} \cup$

$\cup \{\langle n,a\rangle \in x : n \in A$ s. th. $f(n) = a$ and $a \in M\} \cup$

$\cup \{\neg(\exists y)(\langle n,y\rangle \in x : n \notin A\}$.

Clearly τ can be written properly as a set of formulae of L using parameters from M, and uses only countably many parameters from M. It is also finitely satisfiable, and so, since M is ω_1-saturated there is an element c of M realising τ.

Now c codes a one-one function with M-finite domain, which agrees with f on A. Let F be the M-definable total function defined by :

$$F(x) = \begin{cases} b, & \text{if } \langle b,x\rangle \in c, \\ 0, & \text{otherwise.} \end{cases}$$

Then $A \subset \{n \in \omega : M \models F(f(n)) = n\}$, and so $M^* \models F(f^U) = id^U$, by the usual form of Los' theorem, [1]. Thus any elementary substructure of M^* containing M and f^U, must contain id^U. But it is easy to show $M^* = M^*[id^U]$, from which it follows that M^* is a minimal elementary extension of M.

Now to show (i) implies (ii), suppose M^* is a minimal normal extension of M.

Let $a \in M^*$ be s. th. $\omega < a < M-\omega$. Then $a \in M^*-M$ and since M^* is a minimal extension of M we must have $M^* = M^*[M \cup \{a\}]$. Letting $U = \{A \in B_M : M^* \models a \in A\}$ we see that $M^* \simeq M_U$, and it follows from lemma 6.5.2. that every M-definable one-place function is either constant or one-one on some set in U.

Let $U' = \{A \cap \omega : A \in U\}$.

Then it follows from the choice of a and the fact that M is ω_1-saturated that U' is a non-principal ultrafilter over ω.

Suppose $\{a_i : i \in \omega\}$ is a partition of ω. Define $f:\omega \to \omega$ by $f(n) = \mu i \in \omega : n \in a_i$. Again using the fact that M is saturated we can find an M-definable function F, s. th. $F(i) = f(i)$ for all $i \in \omega$, using a simple types arguement similar to those above. Since F is one-one or constant on some set $A \in U$, we must have that f is one-one or constant on $A \cap \omega \in U'$. It follows that U' is the required Ramsey ultrafilter.

□

Now the existence of a Ramsey ultrafilter over ω is implied by the continuum hypothesis ([2]), but cannot be proved from the axioms of Zermelo-Fraenkel set theory with choice (ZFC) (a result of Kunen - unpublished). Hence, although we can prove (in ZFC) that every countable non-standard model of P has a minimal normal extension, it follows from theorem 6.5.6. (and the existence of ω_1-saturated models) that we cannot prove in ZFC that every non-standard model of P has such an extension.

Whether the latter comment holds when we replace 'normal' by just 'cofinal elementary', or whether every non-standard model of P does have a minimal cofinal elementary extension, we do not know.

Chapter 7. Some Open Problems.

For the most part, theorems in this thesis apply to all models of P - that is we have never exhibited model theoretic properties which distinguish different complete extensions of P. Thus our methods are not delicate enough to construct models which give, say, informative independence results in Peano arithmetic. We therefore pose the problem - 'Find a property for which there are complete extensions T_1, T_2, of P s. th. every (or some) model of T_1 has this property, but no model of T_2 has it'.

H. Friedman has suggested the property of having a certain order type of cardinality ω_1. (All countable non-standard models of P are order isomorphic).

Chapter 3 suggests the question - 'does every non-standard model of P have an elementary non-$\preccurlyeq_1$ end extension. It would be curious if this were false but I can think of no reasonable way of attacking the problem. One might think a generalisation of Friedman's theorem would help. However, we can construct an elementary extension of N, of cardinality ω_1, which is not only non-isomorphic, but non-$L_{\omega_1,\omega}$-elementarily equivalent, to all its proper initial segments. ($L_{\omega_1,\omega}$ is the language allowing conjunction and disjunction over any countable set of formulae involving only finitely many free variables.).

Finally, problems already raised implicitly are - 'Is every countable non-standard model of P isomorphic to $2^{\aleph_0}$ initial segments of itself ?', and - 'Does every non-standard model of P have a minimal cofinal elementary extension ?'.

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