DERIVATIONS IN FREE POWER SERIES RINGS AND FREE ASSOCIATIVE ALGEBRAS

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by

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ABSTRACT.

A derivation d in any associative ring R is a linear mapping such that $(ab)^d = a^d b + ab^d$, any $a,b \in R$. The kernel of d is a subring of R which can sometimes be a ring of the same type as R. In particular, if R is a free power series ring, $F\langle\langle x_1,\ldots,x_q \rangle\rangle$, over a commutative field of characteristic zero, we find conditions under which Ker d is again a free power series ring. This happens e.g. if all the nonzero elements of the set $\{x_i^d;\ i = 1,\ldots,q\}$ are homogeneous of the same order, or if at least one element in this set has a nonzero constant term.

For every derivation d in a complete inversely filtered F-algebra S satisfying the [n-term] inverse weak algorithm it is at least true that Ker d is [an n-fir] a semifir, i.e. Ker d is then again a ring in which every finitely generated [by at most n generators] right ideal is a free right S-module of unique rank. This is also true for the fixed rings of suitably chosen automorphisms of S, for if α is an automorphism which maps every element onto itself plus an element of higher order, then $\log\alpha$ is a derivation such that $\mathrm{Fix}\,\alpha = \mathrm{Ker}\,(\log\alpha)$.

In a free associative algebra $F\langle X \rangle$, X a countable set, the kernel of any derivation d such that the nonzero elements of the set $\{x^d;\ x \in X\}$ are homogeneous of the same degree, is also a free associative algebra over F. In particular, the kernel of the derivation $\frac{\partial}{\partial x}$ has a free generating set consisting of $\{y \in X;\ y \neq x\}$ together with the set of all commutators of the form $[..[[y,x],..,x]$. This makes it possible to regard $F\langle X \rangle$ as a skew polynomial ring in x over $\mathrm{Ker}\,\frac{\partial}{\partial x}$, a fact which characterizes x up to a "constant" in $\mathrm{Ker}\,\frac{\partial}{\partial x}$.

aan Ena

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INTRODUCTION.

If the development of general (noncommutative) associative ring theory is superficially divided into rings with descending chain conditions, rings with ascending chain conditions and rings without chain conditions; then free rings, in particular free algebras and free power series rings over a field, fall under the last heading. In other words they belong to a part of ring theory which is still relatively little explored and as such present a fruitful area for research.

It is still fairly recently that general methods for their study became available. Among these tools are the weak algorithm (which is a generalization of the classical Euclidean algorithm) in free associative algebras and its counterpart, the inverse weak algorithm, in free power series rings. A survey which indicates the techniques presented by the weak algorithm can be found in [17] and this paper also contains references to relevant literature in this field. The inverse weak algorithm was first defined in [10].

Whenever an algebraic structure, e.g. a free ring, is being studied it is important to know as much as possible about the endomorphisms, in particular the automorphisms, and other related mappings, such as derivations, in this structure. Free rings have the property that every (suitable) subset in them which can be taken as the image set of the set of free generators, determines both an endomorphism and a derivation in the ring. However, it can be quite difficult to decide whether a given endomorphism is an automorphism. In a free associative algebra (of rank > 2) it is not even known yet if every automorphism is "tame" in the sense that it may be regarded as a product of certain "elementary" automorphisms (see page 122).

Derivations in free associative algebras have not before been studied in any depth, and the research for this thesis started with an attempt to use derivations to characterize sets of free generators in such an algebra. Any derivation in a free algebra, $F\langle X\rangle$, can only marginally decrease the order of an element to which it is applied. This suggests that it might be worthwhile to study the effect of derivations on δ-dependence relations ($\delta(a)$ = the order of $a \in F\langle X\rangle$), something which can be done more advantageously in free power series rings, i.e. the topological completions of free associative algebras relative to the topology induced by the order-function. In this way we obtained information about the kernels of derivations in complete inversely filtered rings, and in particular free power series rings.

It is well-known in mathematics that elements which belong to the kernel of a derivation (or differentiation) are often also fixed elements of some automorphism, and vice versa. We investigated this connection between derivations and automorphisms in a complete inversely filtered ring, and have shown how it enables us to extend results on the kernels of derivations to fixed rings of automorphisms.

Additional information on the endomorphisms, automorphisms and derivations of a free power series ring can be found in [1], but even so, all that has yet been said in this respect touches only a small part of the vast field of investigation presented by derivations in free rings.

ABBREVIATIONS AND SPECIAL NOTATION.

(The numbers on the left refer to the pages on which additional information can be found.)

14	WA_n	= n-term weak algorithm
15	WA	= weak algorithm
15	IWA_n	= n-term inverse weak algorithm
16	IWA	= inverse weak algorithm
16	gr(R)	= graded ring associated to a filtered or inversely filtered ring
16	$gr_n(R)$	= group of (homogeneous) elements of degree n in gr(R)
18	UFD	= unique factorization domain
22	Ker d	= kernel of the derivation d
	$Fix\alpha (\subseteq A)$	= the subalgebra consisting of all elements $a \in A$ such that $a^\alpha = a$; α an automorphism of A.
11	v	= filtration, or degree function in a free associative algebra
12	ϑ	= inverse filtration
21	δ	= order function in a free associative algebra or free power series ring
	$(a_i)^T$	= transpose of the row vector (a_i)
46	$\bar{a}$	= least homogeneous component of a
51, 104	$\frac{\partial}{\partial x_i}$	= the (continuous) derivation in $F\langle\langle X\rangle\rangle$ or $F\langle X\rangle$ which sends $x_i \mapsto \delta_{ij}$

32	$f\frac{\partial}{\partial x_i}$	= the derivation in $F< X >$ which sends $x_i \mapsto f\,\delta_{ij}$
	$[a,b]$	= the commutator $ab - ba$
52	$[u,x_1{}^{[k]}]$	= $[\ldots[[u,\underbrace{x_1],x_1],\ldots,x_1}_{k \text{ times}}]$
	$\mathbb{C}$	= the field of all complex numbers
	$\mathbb{Q}$	= the field of all rational numbers
	$\mathbb{Z}$	= the ring of all integers
	$\mathbb{Z}^+$	= the semigroup of all positive integers
	//	= the end (or absence) of a proof

CHAPTER 1.

DEFINITIONS AND PRELIMINARIES.

The definitions and basic properties of all the concepts used and discussed in the thesis form the subject matter for this chapter. We have proved the assertions only if it was inconvenient or impossible to provide a reference to a satisfactory proof in the relevant literature.

1. Filtered and inversely filtered rings.

We are primarily concerned with free associative algebras and free power series rings, but some of the basic notions which will be our tools for studying derivations in these rings belong naturally to a more general context. We take this as a starting point for listing the necessary definitions. All rings are taken to have unit-elements which are also the unit-elements of all their subrings, and all homomorphisms map the unit-elements onto themselves.

A <u>filtered ring</u> R is a ring with a non-negative integer-valued function v defined on its subset of nonzero elements and satisfying the conditions:

i) $v(x) \geq 0$ for $x \neq 0$, $v(1) = 0$;

ii) $v(x - y) \leq \max\{v(x), v(y)\}$;

iii) $v(xy) \leq v(x) + v(y)$.

1.1

Extend v to the whole of R by taking $v(0) = -\infty$. An equivalent way to get R to be a filtered ring is to require that there must be a sequence (R_n) of subgroups of the additive group of R satisfying the conditions

a) $0 = R_{-\infty} \subseteq R_0 \subseteq R_1 \subseteq R_2 \subseteq \dots \quad , 1 \in R_0$;

b) $\bigcup R_t = R$;

c) $R_i R_j \subseteq R_{i+j}$.

The equivalence between these two sets of conditions stems from the fact that a) - c) follows from i) - iii) if we take

$$R_n = \{ x \in R \mid v(x) \leq n \} ;$$

and conversely i) - iii) follows from a) - c) if we define v on R by

$$v(x) = \min \{ n \mid x \in R_n \} .$$

For any a ($\neq 0$) $\in$ R the integer v(a) is called the <u>degree</u> of a.

On the other hand, if we have a non-negative integer-valued function ϑ defined on the set of nonzero elements of the ring R and satisfying the conditions

1) $\vartheta(x) \geq 0$ for $x \neq 0$, $\vartheta(1) = 0$;

2) $\vartheta(x-y) \geq \min \{ \vartheta(x), \vartheta(y) \}$;

3) $\vartheta(xy) \geq \vartheta(x) + \vartheta(y)$;

and if we extend ϑ to the whole of R by taking $\vartheta(0) = \infty$, we say that R is <u>inversely filtered</u> by ϑ.

1.1

Now put

$$R_t = \{ x \in R \mid \vartheta(x) \geq t \} ,$$

then (R_t) is a descending sequence of subgroups of the additive group of R

a') $R = R_0 \supseteq R_1 \supseteq R_2 \supseteq \cdots\cdots$ such that

b') $\bigcap R_t = 0$,

c') $R_i R_j \subseteq R_{i+j}$.

Conversely, if a') - c') are given in R, we can get a ϑ satisfying 1) - 3) by taking

$$\vartheta(x) = \begin{cases} \min \{ t \mid x \notin R_{t+1} \} & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 . \end{cases}$$

For any $a(\neq 0) \in R$ the integer $\vartheta(a)$ is called the <u>order</u> of a.

If R is an inversely filtered ring it may be topologized by taking the subgroups R_t to be a neighbourhood base of zero. The completion $\hat{R}$ of this topological ring is also of interest to us, because by b') R is Hausdorff and it is therefore a subring of its completion $\hat{R}$. (See e.g. [6 , ch. III] . In the next two chapters we shall always take the inversely filtered rings to be complete, so that $R = \hat{R}$ will hold.

Next we introduce the concepts of dependence and independence relative to the filtration in filtered and inversely filtered rings.

1.1

Definition: i) Let R be any filtered ring with filtration v. A family (a_i) of elements of R is right v-dependent if there exist elements $b_i \in R$, almost all zero, such that

$$v(\sum_i a_i b_i) < \max_i \{ v(a_i) + v(b_i) \} ,$$

or if some $a_i = 0$. Otherwise the family (a_i) is right v-independent.

ii) An element $a \in R$ is right v-dependent on a family (a_i) if $a = 0$ or if there exist elements $b_i \in R$, almost all zero, such that

$$v(a - \sum a_i b_i) < v(a) \quad , \quad \text{while}$$
$$v(a_i) + v(b_i) \leq v(a) \quad \text{for all } i.$$

The corresponding notions for left v-dependence are defined analogously.

If one member of a family is right v-dependent on the rest, the family is necessarily right v-dependent, but the converse is not generally true. In fact the converse just constitutes the "weak algorithm" as expressed in the following

Definition: A ring R with a filtration v is said to satisfy the n-term weak algorithm (WA_n) relative to v, if, given any right v-dependent family $a_1, \ldots, a_m$ $(m \leq n)$ such that $v(a_1) \leq \ldots \leq v(a_m)$, some a_k is right v-dependent on $a_1, \ldots, a_{k-1}$,

1.1

If R satisfies WA_n for all n, we say that R satisfies the weak algorithm (WA) relative to v.

For inversely filtered rings the corresponding definitions are as follows:

Definition: i) Let R be any inversely filtered ring with inverse filtration ϑ. A family (a_i) of elements of R is right ϑ-dependent if there exist elements b_i of R, almost all zero, such that

$$\vartheta(\sum a_i b_i) > \min_i \{ \hat{v}(a_i) + \hat{v}(b_i) \} ,$$

or if some $a_i = 0$. Otherwise the family is right ϑ-independent.

ii) An element $a \in R$ is right ϑ-dependent on a family (a_i) if a = 0 or if there exist elements $b_i \in R$, almost all zero, such that

$$\vartheta(a - \sum a_i b_i) > \vartheta(a) \quad , \text{ while}$$
$$\vartheta(a_i) + \vartheta(b_i) \geq \vartheta(a) \quad \text{for all } i.$$

Definition: A ring R with an inverse filtration ϑ is said to satisfy the n-term inverse weak algorithm (IWA_n) relative to ϑ, if, given any right ϑ-dependent family $a_1, \ldots, a_m$ $(m \leq n)$ such that $\vartheta(a_1) \leq \ldots \leq \vartheta(a_m)$, some a_k is right ϑ-dependent on $a_1, \ldots, a_{k-1}$.

1.1

If R satisfies IWA_n for all n we say that it satisfies the <u>inverse weak algorithm</u> (IWA) relative to ϑ.

Note that both the definitions of WA and IWA refer only to right dependence. This is so because the concepts are left-right symmetric in both cases. For WA_n (and WA) a proof of this fact can be found in [17] and for IWA_n (or IWA) [2] can be consulted, but the argument in the latter case is basically the same as in the former. It utilizes the notion of graded rings associated to the filtered and inversely filtered rings respectively.

For a filtered ring R the <u>associated graded ring</u>, gr(R), is obtained by considering the union of disjoint additive groups $R_n/R_{n-1} = gr_n(R)$ (say), and defining multiplication in the natural way such that $gr_i(R)\, gr_j(R) \subseteq gr_{i+j}(R)$. For an inversely filtered ring the associated graded ring is obtained in an exactly analogous way, the difference being that we now have to consider the union of the disjoint additive groups R_t/R_{t+1} . (see e.g. [5 ,Ch.3] for a discussion of the parallel commutative case.)

It is important to be aware of the fact that every dependence relation relative to the filtration in a filtered or inversely filtered ring, manifests itself by a <u>linear</u> dependence relation in the associated graded ring.

Consequently, R satisfies WA_n (resp. IWA_n) if and only if the associated graded ring, gr(R), satisfies the condition: Given any right linearly dependent family $a_1, \ldots, a_m$ ($m \leq n$) of elements in gr(R) such that $a_j \in gr_{i_j}(R)$ and $i_1 \leq \ldots \leq i_m$, then for some k, $1 < k \leq m$, a_k is right linearly dependent on $a_1, \ldots, a_{k-1}$.

2. Free ideal rings, and unique factorization domains.

A right free ideal ring (or right fir) R is a ring in which every right ideal is a free right R-module of unique rank. The definition for a left fir is exactly analogous. If a ring satisfies this property only for finitely generated right ideals, it is called a semifir. (Here it is no longer necessary to distinguish between right and left semifirs, because the notion becomes left-right symmetric.) Rings characterized by these properties were first defined by Cohn [13]. He later [16] introduced a refinement by considering rings in which all right (or left) ideals generated by n elements, are free of unique rank. Such rings are called n-firs. Bergman [2] developed these ideas further; in particular he gave a number of equivalent characterizations of such rings, one of which will be important to us in Chapter 2 (Cor. 2.3):

A ring R is an <u>n-fir</u> if and only if, for every set of m ($m \leq n$) elements $a_1, \ldots, a_m$, which are left linearly dependent over R, it is possible to find an invertible $m \times m$ matrix μ over R such that the vector $\mu(a_i)^T$ has at least one component equal to zero.

P.M.Cohn [12] generalized the notion of unique factorization to noncommutative integral domains. We recall only the basic definitions.

Let R be an integral domain. An element in R is called an <u>atom</u> if it is a non-unit which is not a product of two non-units. Two elements a,b in R are said to be <u>similar</u> if $R/aR \cong R/bR$ as right R-modules. (It is sufficient to state this only for operations on the right, because the condition is equivalent to its left-right analogue. See [12] for a detailed discussion.) If

$$a = u_1 u_2 \ldots u_r \quad , \quad b = w_1 w_2 \ldots w_s$$

are any two factorizations of a and b respectively, these factorizations are said to be <u>isomorphic</u> if r = s and there is a permutation π of $(1, \ldots, r)$ such that u_i is similar to $w_{i\pi}$. R itself is called a <u>unique factorization domain</u> (UFD) if every nonzero non-unit of R has a factorization into atoms, and any two atomic factorizations of a given element are isomorphic.

Definition: A unique factorization domain R is said to be rigid if, for any two prime factorizations of an element

$a \in R$: $a = b_1 b_2 \ldots b_r = c_1 c_2 \ldots c_r$,

there exist units $u_o, u_1, \ldots, u_r$ $(u_o = u_r = 1)$

such that $c_i = u_{i-1}^{-1} b_i u_i$, $(i = 1, \ldots, r)$.

3. Free associative algebras and free power series rings.

Let K be a commutative ring, and X a set of noncommuting indeterminates which commute without restriction with any element of K. There exists a number of equivalent ways of defining the free associative algebra, $K < X >$, generated by X over K. We give the definition which describes the elements of $K < X >$ directly in a normal form: Let S_X be the free semigroup on the set $X = \{ x_i \}$, indexed by I. This consists of all products (words)

$$x_{i_1} x_{i_2} \ldots x_{i_n} \tag{1}$$

where $(i_1, i_2, \ldots, i_n)$ runs over all finite sequences of suffixes in I (including the empty sequence which gives the unit-element 1). The free associative algebra $K < X >$ is then the semigroup-algebra of S_X over K, i.e. it is the K-algebra consisting of all elements of the form

$$\sum x_{i_1} x_{i_2} \dots x_{i_n} \lambda_{i_1 \dots i_n} , \qquad (2)$$

where the coefficients $\lambda_{i_1 \dots i_n} \in K$ are almost all zero. Two elements of $K\langle X \rangle$, written in the form (2), can be equal only if they are identical.

In $R = K\langle X \rangle$ the free generating set X is not unique, but any two free generating sets have the same cardinality, which is called the rank of R (see [17 , p.4]).

One very important general property of such a free algebra (which can actually be used to characterize it, [8;IV.2]) is the universal mapping property: For every given mapping $\varphi : X \to A$ from X into another K-algebra A, there exists a unique K-algebra homomorphism $\eta : K\langle X \rangle \to A$ such that $\varphi = i \cdot \eta$, where i is the natural injection $i: X \to K\langle X \rangle$.

Let the length of the word (1) be n, i.e. equal to the number of factors x_i appearing in it, and define a (natural) filtration on $R = K\langle X \rangle$ by taking the degree v(a) of each nonzero element $a \in K\langle X \rangle$ to be equal to the maximum of the lengths of the words appearing in a, when it is expressed in the form (2). It is straightforward to check that this definition does indeed make $K\langle X \rangle$ a filtered ring as defined in section 1).

In the following chapters we will be mainly interested in the case where K is a field F, and if that is so, it is well known [9;p.28] that $F\langle X \rangle$ satisfies WA with respect to the filtration v just defined.

1.3

It is then also true that for arbitrary $a, b \in F\langle X \rangle$

$$v(ab) = v(a) + v(b),$$

and hence the natural filtration is in fact a valuation or, as it is commonly called, a degree function.

It is also possible to define an inverse filtration, or more precisely, an order function on $F\langle X \rangle$. Take $\delta(a)$ to be the minimum of the lengths of the words appearing in the nonzero element $a \in F\langle X \rangle$ when it is expressed in the form (2), and say $\delta(0) = \infty$. Then

1') $\delta(a) \geq 0$ for $a \neq 0$, $\delta(1) = 0$;

2') $\delta(a - b) \geq \min\{\delta(a), \delta(b)\}$;

3') $\delta(ab) = \delta(a) + \delta(b)$.

Consequently $F\langle X \rangle$ may also be regarded as a topological ring, and as such it has a completion which we denote by $F\langle\langle X \rangle\rangle$. This F-algebra, $F\langle\langle X \rangle\rangle$, contains $F\langle X \rangle$ as a subalgebra. It is in fact the free power series ring in X over F [10 ,p.458]. In other words every element of $F\langle\langle X \rangle\rangle$ can be uniquely expressed in the form

$$\sum_{\text{infinite}} x_{i_1} x_{i_2} \cdots x_{i_n} \lambda_{i_1 \cdots i_n}, \tag{3}$$

where for each n, the coefficients $\lambda_{i_1 \cdots i_n} \in F$ are almost all zero. This also makes it clear how the order-function can be extended to $F\langle\langle X \rangle\rangle$.

Later it will be necessary to require the field F to be of characteristic zero at some crucial stages of the discussion, such as in lemma 2.15, where we prove that $F\langle X\rangle$ lies in the images of some derivations in $F\langle\langle X\rangle\rangle$, and therefore we make the convention that whenever free algebras over F, or free power series rings over F, are being discussed, F will always be taken to be of characteristic zero.

4. Derivations.

Derivations from a subalgebra into an algebra are mappings which are of considerable importance in algebra, and which are still finding an ever widening scope of applicability.

Definition: If A is a subalgebra of an algebra B, a derivation d of A into B is a linear mapping of A into B such that

$$(ab)^d = a^d b + ab^d , \qquad a,b \in A.$$

If A = B we say that d is a derivation in A.

The general properties of such derivations are well known. (See e.g. [20] or [4].) We list a number of these properties which are of importance for our own work:

i) The kernel, Ker d, of every derivation d: $A \to B$ is a subalgebra of A.

1.4

ii) Any two derivations which coincide on a generating set of A are identical on A.

iii) The set $\mathcal{L}$ of all derivations in A forms a Lie algebra relative to the operations of addition and multiplication given by

$(d_1,d_2) \mapsto d_1+d_2$, where $a^{d_1+d_2} = a^{d_1} + a^{d_2}$, any $a \in A$

$(d_1,d_2) \mapsto [d_1,d_2]$, where $a^{[d_1,d_2]} = (a^{d_1})^{d_2} - (a^{d_2})^{d_1}$, any $a \in A$.

iv) The Leibniz-formula for the k-th power of a derivation holds:

$$(ab)^{d^k} = \sum_{i=0}^{k} \binom{k}{i} a^{d^i} b^{d^{k-i}} \quad , \text{ any } a,b \in A.$$

v) For every element $b \in A$ the mapping defined in A by $a \mapsto ab - ba$ (any $a \in A$) is a derivation which is called the <u>inner derivation</u> determined by b.

vi) If $A = K\langle\langle X \rangle\rangle$ is a free power series ring over a commutative ring K and if d is a continuous derivation in A, then the kernel of d is a <u>closed</u> subalgebra of A; since it is the inverse image of the set $\{0\}$ which is closed in the "inverse filtration topology" on A.

The definition of a derivation can be generalized in the following way:

<u>Definition</u>: a) Let α and β be any two homomorphisms of the subalgebra A into the algebra B, then the linear mapping

1.4

$D : A \rightarrow B$ such that

$$(ab)^D = a^D b^\alpha + a^\beta b^D \quad , \quad \text{any } a,b \in A,$$

is called an (α,β)-derivation of A into B.

b) In particular an $(\alpha, 1)$-derivation is called an α-derivation.

vii) For every given β-derivation D in an integral domain K, where β is an injective endomorphism of K, there exists a ring whose elements can be uniquely expressed as polynomials

$$a_0 + xa_1 + x^2 a_2 + \dots + x^n a_n \tag{4}$$

in an indeterminate x over K, with componentwise addition and multiplication induced by the commutation rule

$$ax = xa^\beta + a^D \quad , \quad a \in K. \tag{5}$$

Conversely, if R is a ring which contains the integral domain K as a subring and which is isomorphic as right K-module to the ring of polynomials K[x] (with elements like (4)), then there exists an injective endomorphism β and a β-derivation D of K such that the multiplication in R is determined by that of K together with the rule (5).

Such rings are called skew polynomial rings, they were first studied by O.Ore [21], and they are usually denoted by $K[x;\beta,D]$.

1.4

Property (ii′) above is of special importance in free associative algebras, because it implies that in these algebras it is sufficient to know what any particular derivation does to the free generating set in order to define it explicitly and unambiguously. On the other hand, every given mapping $d : X \to K\langle X \rangle$ (K a commutative ring) extends to a unique derivation in $R = K\langle X \rangle$. This can be seen by noting that the mapping of X into M (say), the ring of 2×2 matrices over R, given by

$$x \mapsto \begin{bmatrix} x & x^d \\ 0 & x \end{bmatrix} \quad , \text{ every } x \in X,$$

extends by the universal mapping property of R to a unique homomorphism of R into M. It is then only necessary to look at the co-ordinates in the upper right hand corners of the matrices in M to see that d has been extended to a derivation in R.

For free power series rings $F\langle\langle X \rangle\rangle$ (F a field) the situation is different, mainly because the set X is <u>not</u> a generating set of $F\langle\langle X \rangle\rangle$ in the full algebraical sense of the terminology (the elements of $F\langle\langle X \rangle\rangle$ are not necessarily <u>finite</u> linear combinations of monomials in X). However the analogy is restored if we limit consideration to <u>continuous</u> derivations in $F\langle\langle X \rangle\rangle$.

Proposition 1.1: Let $\hat{R} = F\langle\langle x_1,\dots,x_q\rangle\rangle$ and let $\mathcal{D}(R)$ denote the F-space of continuous derivations in $\hat{R}$, then $\mathcal{D}(R) \cong \hat{R}^q$.

Proof: We have to prove that every continuous derivation in $\hat{R}$ is completely determined by its values on $X = \{x_1,\dots,x_q\}$ and that for every q-tuple $(u_1,\dots,u_q)$ of elements in $\hat{R}$ the mapping $d : X \to \hat{R}$ given by $x_i^d = u_i$ extends to a derivation in $\hat{R}$.

Let d be a continuous derivation in $\hat{R}$. Take any element $g \in \hat{R}$ and say $g^d = h$. For every positive integer n the element h can be written as

$$h = h_n + h_n'$$

where h_n is a polynomial of degree $\leq$ n-1 and h_n' is a power series of order $\geq$ n. Since d is continuous, there exists a positive integer m (which we take to be minimal) such that $\hat{R}_m^d \subseteq \hat{R}_n$. Now write $g = g_m + g_m'$, where g_m is a polynomial of degree $\leq$ m-1 and $g_m' \in \hat{R}_m$. Then $(g_m')^d \in \hat{R}_n$, and consequently

$$h_n \equiv g_m^d \pmod{\hat{R}_n} . \qquad (6)$$

By property (ii) above, g_m^d is uniquely determined by the values of d on the elements in X. Hence by (6) the same is true of h_n, and since this fact holds for all n, it follows that h is uniquely determined by the values of d on X. In other words if $x_i^d = u_i$ $(i = 1,\dots,q)$ then the F-linear mapping of $\mathcal{D}(\hat{R})$ into $\hat{R}^q$ given by

$$d \mapsto (u_1, \ldots, u_q) \tag{7}$$

is injective, and it only remains to show that it is also surjective.

[Note that only after this fact (7) has been established, can we use the knowledge that the continuous derivation d is determined by its values on X to see that if f is an arbitrary element of order r in $\hat{R}$, then $\delta(f^d) \geq r-1$.]

Let $(u_1, \ldots, u_q)$ be any element of $\hat{R}^q$ and define a mapping $d : X \to \hat{R}$ by sending $x_i \mapsto u_i$. The argument preceding this proposition can now be used again (in an appropriately modified form) to extend d to a unique derivation of the free algebra $F\langle X \rangle$ into $\hat{R}$. Denote this derivation also by d. Now if $f = \sum_j f_j$ (sum of homogeneous components) is an arbitrary element in $\hat{R}$, each f_j lies in $F\langle X \rangle$ and since d is uniquely determined by its values on X, it follows that $\delta(f_j{}^d) \geq j-1$. Hence the sequence $(f_j{}^d)$ of elements in $\hat{R}$ is summable (by Cauchy's criterion). Consequently, by writing

$$f^{d'} = \sum_{j=0}^{\infty} f_j{}^d$$

we get an F-linear mapping $d' : \hat{R} \to \hat{R}$ which extends d. It is straightforward to check that d' is indeed a derivation in $\hat{R}$. (See also [4 ,p.61]) This establishes that the mapping (7) is also surjective. //

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Remark: We would like to emphasize that up to now it has generally been accepted in ring theory that all derivations in a free power series ring $K \ll x_1,\ldots,x_q \gg$ are continuous. This has led N.Bourbaki [4 ;p.61] to make a mistake when he considered a power series ring $K[[x_1,\ldots,x_q]]$ in commuting indeterminates, and tried to prove that a derivation d_0 which is zero on every polynomial in this ring, is necessarily the zero derivation. The mistake was made when he applied a lemma to a power series, although it had only been proved for polynomials, and in doing so he tacitly assumed that the derivation was continuous.

We now indicate in a special case how it is possible to obtain discontinuous derivations in a free power series ring.

Example of a discontinuous derivation: Consider the free power series ring $k[[x]]$ in one indeterminate over a field of characteristic zero. Pass from the inclusion $k[x] \subseteq k[[x]]$ to the corresponding ~~localizations~~ rings of fractions of these rings, and consider $k(x) \subseteq k((x))$ i.e. consider the field of rational fractions in one indeterminate, x, over k, lying within the field of Laurent-fractions in x over k. (See [4 ;p.60]). It is well-known that $k((x))$ is a transcendental extension field of $k(x)$, and that it has infinite transcendency degree over $k(x)$. (See e.g. [4 ;p.107(Ex.13) and p.100(Prop.8)]). Hence, by the theorem of Steinitz [4 ;p.98], $k((x))$ can be obtained from $k(x)$ by a pure transcendental extension followed by an algebraic extension. Let L be the pure transcendental extension of $k(x)$ with basis $\{y_i\}$, $i \in I$. Then for every family (u_i), $i \in I$, of elements of L there exists a unique nonzero derivation $d : L \to k((x))$ which extends the zero derivation $d_0 : k(x) \to k((x))$, and which is such that $y_i^d = u_i$, all $i \in I$. [4 ;p.136,Prop.4.] Since we have taken k to be of characteristic zero, this derivation d can also be extended to a unique nonzero derivation $\hat{d} : k((x)) \to k((x))$. [4 ;p.136,Prop.5.] Thus for any family (u_i), $i \in I$, of elements (not all of them zero) in L there exists a uniquely determined nonzero derivation $\hat{d}$ in $k((x))$ which is discontinuous in $k((x))$, because it has the property that $\operatorname{Ker} \hat{d} \supseteq k(x)$. //

Let $R = K\langle\langle X \rangle\rangle$, $X = \{x_1,\dots,x_q\}$, be a power series ring over a commutative ring.

Definition: If d is a nonzero continuous derivation in R and if $n = \min_i \{\delta(x_i^d)\}$, then d is said to be of order n relative to X.

Definition: A continuous derivation d in R is called homogeneous relative to the generating set X if all the nonzero elements of the set

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$$U = \{u_i \in R \mid u_i = x_i^{\,d}\}$$

are homogeneous, and there exists a non-negative integer n such that $\delta(u_i) = n$ for every nonzero $u_i \in U$.

It should be noted that each homogeneous derivation in R is by this definition also a continuous derivation.

Now, if d is a given continuous derivation in $\hat{R} = F\langle\langle x_1,\ldots,x_q \rangle\rangle$, we know that it is completely determined by its action on the elements of X, say

$x_i^{\,d} = u_{i0} + u_{i1} + u_{i2} + \ldots\ldots$ (sum of homogeneous components)

Proposition 1.1 shows that each of the mappings (from X into $\hat{R}$) in the sequence (d_j), $j = 0,1,2,\ldots\ldots$, given by

$$x_i^{\,d_j} = u_{ij} \qquad i = 1,\ldots,q \quad ,$$

extends to a unique continuous derivation in $\hat{R}$, and according to the definition above, each d_j $(j = 0,1,2,\ldots)$ is a homogeneous derivation. Furthermore if f is an arbitrary element in $\hat{R}$, we have for every $j = 1,2,\ldots\ldots$ that $\delta(f^{\,d_j}) \geq \delta(f) + j-1$, and hence the sequence $(f^{\,d_j};\ j=0,1,2\ldots)$ is summable in $\hat{R}$. Write its sum as

$$f^{\,d_0 + d_1 + d_2 + \ldots\ldots} .$$

This implies that $d_0 + d_1 + d_2 + \ldots\ldots$ is a proper derivation in $\hat{R}$ and since it coincides with d on X, we have

$$d = d_0 + d_1 + d_2 + \ldots\ldots$$

For ease of reference we state the conclusion as

Proposition 1.2: Every continuous derivation d in $\hat{R} = F\langle\langle x_1, x_2, \ldots, x_q \rangle\rangle$, given by

$x_i^{d} = \sum_{j=0}^{\infty} u_{ij}$, (sum of homogeneous components),

$i = 1, \ldots, q$, can be regarded as the sum $d_0 + d_1 + d_2 + \ldots$ of homogeneous derivations in $\hat{R}$, where each d_j is given by

$x_i^{d_j} = u_{ij}$, $i = 1, \ldots, q$. //

Remark: It is easy to see how the preceding definition and discussion can be adapted to the case of derivations in the free associative algebra $F\langle Y \rangle$, where Y can now also be an infinite set. In this case a derivation is called homogeneous if the images of the free generating elements $y \in Y$ are either zero or homogeneous elements of the same degree, and if Y is finite every given derivation in $F\langle Y \rangle$ can be regarded as a finite sum of homogeneous derivations. The degree of such a derivation d is $\max\{\deg(y^d) : y \in Y\}$.

5. The Lie algebra of derivations in a free associative algebra.

In the previous section we recalled that the set $\mathcal{L}$ of all derivations in an algebra A forms a Lie algebra. We now want to show that if A is taken to be a free associative algebra of finite rank over a field of characteristic zero, it is possible to specify a nontrivial generating set of the

Lie algebra $\mathcal{L}$. We do not claim that the generating set given below is minimal, and the main reason for the inclusion of this discussion in the thesis, is that it displays the factors which have to be taken into account in the search for such generating sets.

Let $R = F\langle x_1,\dots,x_q\rangle$ and let $u\frac{\partial}{\partial x_i}$ denote the derivation in R which sends the free generating element x_i to $u \in R$ and all the other free generating elements to zero.

Proposition 1.3: Let $G_1 = \left\{ \frac{\partial}{\partial x_i} \; ; \; i = 1,\dots,q \right\}$,
$G_2 = \left\{ (x_i x_{i+1}) \frac{\partial}{\partial x_i} \; ; \; i = 1,\dots,q \; , \; q+1 = 1 \right\} \cup \left\{ x_q{}^2 \frac{\partial}{\partial x_2} \right\}$
$G_3 = \Big\{ h \frac{\partial}{\partial x_2}$; where h runs through the set of monomials of degree ≥ 3 which are either of the form $x_q{}^n$ $(n \geq 3)$, or of the form $h'x_j$ $(j \neq q)$ $\Big\}$,
then $G = G_1 \cup G_2 \cup G_3$ is a generating set for the Lie algebra, $\mathcal{L}$, of derivations in R.

Proof: Let $\mathcal{G}$ be the Lie algebra (over F) generated by G. Then $\mathcal{G} \subseteq \mathcal{L}$. In order to see that $\mathcal{G} = \mathcal{L}$ it is by linearity sufficient to show that all derivations of the form $g\frac{\partial}{\partial x_i}$, where g is a monomial, lie in $\mathcal{G}$. We do the verification in three steps and in order to facilitate the exposition we change the notation by taking

$$f\frac{\partial}{\partial x_i} = (x_i \,\hat{}\, f) ,$$

composing mappings on the right, and using q + k to represent the index k $(k = 1,\dots,q-1)$, whenever it is convenient to do so.

i) Monomials of degree 1 : All derivations of the form $(x_i \hat{}\ x_j)$ $i,j = 1,\dots,q$, lie in $\mathcal{G}$ since

a) $(x_i \hat{}\ x_i) = [(x_i \hat{}\ x_i x_{i+1}),(x_{i+1} \hat{}\ 1)] \in \mathcal{G}$, $i = 1,..,q$;

b) $(x_i \hat{}\ x_{i+1}) = [(x_i \hat{}\ x_i x_{i+1}),(x_i \hat{}\ 1)] \in \mathcal{G}$, $i = 1,..,q$;

c) if $i = 1,..,q$ and $t = 2,3,..,q-1$

$$(x_i \hat{}\ x_{i+t}) = [..[[(x_i \hat{}\ x_{i+1}),(x_{i+1} \hat{}\ x_{i+2})],..,(x_{i+t-1} \hat{}x_{i+t})] \in \mathcal{G}$$

ii) Monomials of degree 2 : All derivations of the form $(x_i \hat{}\ x_j x_k)$ $i,j,k = 1,\dots,q$ lie in $\mathcal{G}$, since

a) $(x_q \hat{}\ x_1 x_q) = [(x_q \hat{}\ x_q^2),(x_q \hat{}\ x_1)] - (x_q \hat{}\ x_q x_1) \in \mathcal{G}$

b) for $j \neq q$

$$(x_q \hat{}\ x_1 x_j) = [(x_q \hat{}\ x_1 x_q),(x_q \hat{}\ x_j)] \in \mathcal{G}$$
$$(x_q \hat{}\ x_j x_1) = [(x_q \hat{}\ x_q x_1),(x_q \hat{}\ x_j)] \in \mathcal{G}$$

c) for $i \neq 1,q$

$$(x_q \hat{}\ x_i x_q) = [(x_q \hat{}\ x_1 x_q),(x_1 \hat{}\ x_i)] \in \mathcal{G}$$
$$(x_q \hat{}\ x_q x_i) = [(x_q \hat{}\ x_q x_1),(x_1 \hat{}\ x_i)] \in \mathcal{G}$$

d) for $j,k = 2,..,q-1$

$$(x_q \hat{}\ x_j x_k) = [[(x_q \hat{}\ x_q x_1),(x_q \hat{}\ x_j)],(x_1 \hat{}\ x_k)] \in \mathcal{G}$$

e) for $i = 1,..,q-1$; $j = 1,..,q$; $j \neq i$

$$(x_i \hat{}\ x_i x_j) = [(x_i \hat{}\ x_q),(x_q \hat{}\ x_i x_j)] + (x_q \hat{}\ x_q x_j) \in \mathcal{G}$$
$$(x_i \hat{}\ x_j x_i) = [(x_i \hat{}\ x_q),(x_q \hat{}\ x_j x_i)] + (x_q \hat{}\ x_j x_q) \in \mathcal{G}$$

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f) for $i = 1,..,q-1$; $j,k = 1,..,q$; $j,k \neq i$

$$(x_i\hat{}\ x_j x_k) = [(x_i\hat{}\ x_q),(x_q\hat{}\ x_j x_k)] \in \mathcal{G}$$

g) for $i = 1,..,q-1$

$$(x_i\hat{}\ x_i^2) = [(x_i\hat{}\ x_q),(x_q\hat{}\ x_i^2)] + (x_q\hat{}\ (x_i x_q + x_q x_i)) \in \mathcal{G}$$

iii) Monomials of degree ≥ 3 : Order the homogeneous elements of degree n ($n \geq 1$) lexicographically (on $x_1 < \ldots < x_q$) by taking the smallest word in such an element to be the leading term. We show that for every $n \geq 3$, and every monomial g of degree n, the derivations $(x_i\hat{}\ g)$, $i = 1,..,q$, lie in $\mathcal{G}$.

Let $h = h'x_j$ be a monomial of degree ≥ 3, then

a) If $j \neq q$ we have for each $i = 1,...,q-1$ that $(x_i\hat{}\ h) \in \mathcal{G}$:

Note that

$$x_i^{(x_i\hat{}\ x_q x_j)(x_q\hat{}\ h')} = h'x_j \quad \text{and}$$

$$x_q^{(x_q\hat{}\ h')(x_i\hat{}\ x_q x_j)} = f \quad \text{(say)}$$

where f is either zero, or it is a homogeneous element of degree n, obtained from h' by replacing each factor x_i in turn by $x_q x_j$ and taking the sum of the monomials obtained in this way. This shows that

$$[(x_i\hat{}\ x_q x_j),(x_q\hat{}\ h')] + (x_q\hat{}\ f) = (x_i\hat{}\ h'x_j).$$

Now, by induction $(x_q\hat{}\ h') \in \mathcal{G}$ and by (2) below $(x_q\hat{}\ f) \in \mathcal{G}$. Hence $(x_i\hat{}\ h) \in \mathcal{G}$.

b) If $j = q$ and $i = q$ we have

$$\left.\begin{aligned} x_q^{(x_q\,\hat{}\,x_q^2)(x_q\,\hat{}\,h')} &= h'x_q + x_q h' \quad \text{and} \\ x_q^{(x_q\,\hat{}\,h')(x_q\,\hat{}\,x_q^2)} &= f' \quad \text{(say)} \end{aligned}\right\} \qquad (1)$$

where f' is either zero, or it is obtained from h' by squaring each factor x_q in turn and taking the sum of the monomials thus obtained. Consequently, the element $(f'-x_q h')$ follows h in the ordering and therefore (by induction) $(x_q\,\hat{}\,(f'-x_q h')) \in \mathcal{G}$. Now, using (1), it can be checked

that $(x_q\,\hat{}\,h'x_q) = [(x_q\,\hat{}\,x_q^2),(x_q\,\hat{}\,h')] + (x_q\,\hat{}\,(f'-x_q h'))$.
Hence $(x_q\,\hat{}\,h) \in \mathcal{G}$. This fact, together with the choice of G_3 as part of the generating set G, implies by linearity that

$$(x_q\,\hat{}\,g) \in \mathcal{G}, \quad \text{any } g \in R,\ \partial(g) \geq 3. \qquad (2)$$

c) Finally, let $j = q$ and $1 \leq i \leq q-1$:
If $h = x_q^n$ we have $(x_i\,\hat{}\,h) \in \mathcal{G}$, since

$$(x_i\,\hat{}\,x_q^n) = \tfrac{1}{2}\,[(x_i\,\hat{}\,x_q^2),(x_q\,\hat{}\,x_q^{n-1})] \in \mathcal{G}$$

So, assume h is not of the form x_q^n, then in general $h = h''x_r x_q^s$, where $r \neq q$ and $s \geq 1$. We proceed by induction on s: Note that

$$x_i^{(x_i \wedge x_q^2)(x_q \wedge h'' x_r x_q^{s-1})} = h'' x_r x_q^{s} + x_q h'' x_r x_q^{s-1}$$

$$x_q^{(x_q \wedge h'' x_r x_q^{s-1})(x_i \wedge x_q^2)} = f'' \quad \text{(say)}$$

where f'' is the image of $h'' x_r x_q^{s-1}$ under the derivation $(x_i \wedge x_q^2)$. This shows that

$$(x_i \wedge h) = [(x_i \wedge x_q^2),(x_q \wedge h'' x_r x_q^{s-1})]$$
$$-(x_i \wedge x_q h'' x_r x_q^{s-1}) + (x_q \wedge f''). \qquad (3)$$

Now take $s = 1$. By a) and (2) above we know that $(x_i \wedge x_q h'' x_r),(x_q \wedge h'' x_r)$, and $(x_q \wedge f'')$ lie in $\mathcal{G}$. Hence $(x_i \wedge h) \in \mathcal{G}$. This provides a starting point for an induction argument which can be completed with the aid of (3). Hence $(x_i \wedge h' x_q) \in \mathcal{G}$ for every $i = 1,\ldots,q-1$ and every monomial h' of degree ≥ 2.

The choice of the generating set G, together with the arguments given in a) - c) above, establish the claim that every derivation of the form $(x_i \wedge h)$, where h is any monomial of degree ≥ 3 in R, lies in $\mathcal{G}$. //

CHAPTER 2.

KERNELS OF DERIVATIONS IN FREE POWER SERIES RINGS.

In this chapter we study continuous derivations in free power series rings over a commutative field of characteristic zero in an attempt to find a fairly extensive class of such derivations with the property that their kernels are also free power series rings over the same field. Our main tool for doing this is the inverse weak algorithm, which means that we will have to consider ϑ-dependent families of elements in the kernels under consideration. (We invariably take ϑ to be the natural order function determined by the free generating set of the power series ring.) However, we start off by looking at linear dependence relations in the kernels; something which can be done equally well in the more general setting of complete inversely filtered rings, and therefore the first section is devoted almost entirely to derivations in such rings.

1. Complete inversely filtered rings.

We take a result of G.M. Bergman [2] about complete inversely filtered rings satisfying n-term inverse weak algorithm as a stepping stone to our first proposition. Since his result is not yet readily available in print we also recall the main features of the proof given in his thesis.

Proposition 2.1: Let S be a complete inversely filtered ring (with filtration ϑ) satisfying the n-term inverse weak algorithm. Then for any family A of $r \leq n$ elements of S there exists an ordering of A as $a_1, \ldots, a_r$ and a special upper triangular matrix ν such that $(a_i)\nu = (a_i')$ consists of a sequence of right ϑ-independent elements, followed by a sequence of zeros.

[A special upper triangular matrix is one which has 1's down the diagonal and zeros below it.]

Proof: For every set of m+1 elements in S, say $a_1, \ldots, a_m$ and a, there exist elements $b_1, b_2, \ldots, b_m \in S$ with $\vartheta(b_i) \geq \vartheta(a) - \vartheta(a_i)$ such that $a - \sum a_i b_i$ is either non right ϑ-dependent on $a_1, \ldots, a_m$, or else it is zero, because a sequence of elements $b_i^{(k)}$ can be found such that

$$\vartheta(a - \sum a_i b_i^{(k)}) \geq \vartheta(a) + k$$

for every k, and the completeness of S ensures that this sequence produces the said elements b_i, $i = 1, \ldots, m$.

Now let A be the finite family of elements of S mentioned in the proposition and let $a_1 = a_1'$ be any element of A of minimal order. By using the fact mentioned above, modify all other members of A by multiples of a_1 so that they are either zero or non right ϑ-dependent on a_1'. This will not decrease any orders so $\vartheta(a_1')$ will still be minimal in the resulting set. Let a_2' be of minimal order among the resulting elements other than a_1'. Again apply the same fact mentioned, this time making all remaining elements zero or non right ϑ-dependent on $\{a_1', a_2'\}$. Continuation of this process gives a sequence $a_1', \ldots, a_r'$ which can also be seen to be the image under a special triangular matrix of a certain ordering of A. After checking that no a_i' is right ϑ-dependent on the rest, it only remains to apply IWA_n to see that a sequence of ϑ-independent terms (followed by a sequence of zeros) has been obtained. //

Now we can prove

<u>Proposition 2.2</u>: Let S be a complete inversely filtered ring satisfying the n-term inverse weak algorithm and let d be any derivation in S. If $A \subseteq \operatorname{Ker} d$ is a set of m $(m \leq n)$ elements which are left linearly dependent over S, there exists an ordering $a_1, \ldots, a_m$ of A and a special upper triangular matrix μ <u>over Ker d</u> such that $\mu(a_i)^T = (a_i')^T$ is a sequence of elements which contains at least one zero.

<u>Proof</u>: Write $A = \{a_1, \ldots, a_m\}$, where the ordering is still arbitrary, and say

$$\sum_{i=1}^{m} f_i a_i = 0 , \qquad (1)$$

where the $f_i \in S$ and at least one of them is not zero. If $m = 1$ it is immediately clear that $a_1 = 0$ and then the assertion holds with $\mu = [1]$, hence we can take $m \geq 2$. Apply proposition 2.1 to the set $B = \{f_i;\ i = 1, \ldots, m\}$ and rearrange (if necessary) the indexing in (1) in such a way that the permutation called for in this proposition becomes the identity permutation. Let ν be the special upper triangular matrix given by Prop.2.1, put $(f_i')=(f_i)\nu$ and let $\nu^{-1} = (g_{jk})$. Rewrite (1) as

$$(f_i)\,\nu\nu^{-1}(a_i)^T = 0$$

which is

$$\sum_{i=1}^{m} f_i'(a_i + \sum_{k=i+1}^{m} g_{ik}\, a_k) = 0 \qquad (2)$$

Since $f_1' \neq 0$ and the f_i' are right linearly independent over S, (2) implies that

$$a_1 + \sum_{k=2}^{m} g_{1k}\, a_k = 0 \qquad (3)$$

Now if $g_{12},\ldots,g_{1m} \in$ Ker d, (3) shows that the proposition holds with

$$\mu = \begin{bmatrix} 1 & g_{12} & \cdots & g_{1m} \\ & 1 & 0 \,\cdots & 0 \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}$$

Alternatively, if $g_{1k} \notin$ Ker d for at least one k we apply d to (3), then

$$\sum_{k=2}^{m} g_{1k}{}^{d}\, a_k = 0 \qquad (4)$$

If m = 2 we see immediately in (4) that $g_{12}{}^{d} = 0$ and hence by (3) the required matrix is then

$$\begin{bmatrix} 1 & g_{12} \\ 0 & 1 \end{bmatrix}$$

Assume inductively that the proposition holds for sets of cardinality m-1. By (4) the set $\{a_2,\ldots,a_m\}$ is a set of elements in Ker d, left linearly dependent over S, hence there exists an $(m-1) \times (m-1)$ special upper triangular matrix μ_1 over Ker d such that the sequence

$$\mu_1(a_2,\ldots,a_m)^T = (a_2',\ldots,a_m')^T$$

contains at least one zero. The $m \times m$ matrix

$$\mu = \begin{bmatrix} 1 & 0 \; \ldots\ldots 0 \\ 0 & \\ & \begin{bmatrix} & & \\ & \mu_1 & \\ & & \end{bmatrix} \\ 0 & \end{bmatrix}$$

then satisfies the claim of the proposition. //

<u>Corollary 2.3</u>: The kernel of any derivation d in a complete inversely filtered ring S satisfying IWA_n is an n-fir.

<u>Proof</u>: This follows immediately from the proposition and the characterization of n-firs mentioned on page 18 . //

<u>Corollary 2.4</u>: Take S and d as before. If $a_1,\ldots,a_m$ $(m \leq n)$ is a set of elements in Ker d left linearly independent over Ker d, then for any $f_1,\ldots,f_m \in S$ the equation

$$\sum_{i=1}^{m} f_i a_i = 0$$

implies that $f_i = 0$, $i = 1,\ldots,m$.

<u>Proof</u>: If this is not so, a straightforward application of the proposition will give a left linear dependence of $a_1,\ldots,a_m$ over Ker d, contrary to the assumption. //

For the next corollary and ensuing proof we need the notion of flatness and other terminology from homological algebra. The reader is referred to some standard reference work like [22] for the definitions of these concepts which we use without introducing them beforehand.

Corollary 2.5: Let S be a complete inversely filtered ring satisfying the full IWA and let d be any derivation in S. Then S is a flat right Ker d-module.

Proof: This follows directly from the preceding two corollaries and the following criterion for flatness in semifirs, due to P.M. Cohn:

> If S is a semifir and H a right S-module, then H is flat if and only if for any finite family $\{b_1,\ldots,b_m\} \subseteq S$ of elements, left linearly independent over S, and any $h_1,\ldots,h_m \in H$, $\sum_{i=1}^{m} h_i b_i = 0$ implies that $h_i = 0$ $(i = 1,\ldots,m)$. (5)

The proof of this criterion is not conveniently available in reference material and it is repeated here:
Let $\alpha : S^n_{(\ell)} \longrightarrow S$ be the linear map defined by $(x_i) \mapsto \sum x_i b_i$. Since the b_i are left linearly independent over S, the sequence

$$0 \longrightarrow S^n_{(\ell)} \xrightarrow{\alpha} S_{(\ell)}$$

is exact, and (5) states that the induced sequence

$$0 \longrightarrow H \otimes S^n = H^n \longrightarrow H \otimes S = H \qquad (6)$$

is exact. Writing C = coker α , we see that C is a finitely related cyclic S-module, and all such modules arise in this way, because S is a semifir. Now the exactness of (6) means that $\mathrm{Tor}^1_S(H,C) = 0$. It follows from the properties of Tor that this holds for all modules C, whence H is flat. Conversely, when H is flat, $\mathrm{Tor}^1_S(H,C) = 0$, hence (6) is then exact and (5) holds. //

Turning now to free power series rings over a commutative field F of characteristic zero, we say that it is clear that all the preceding results also apply to these rings, but since there is more information available, we can also say something more about the kernels of derivations. E.g. , the fact that a free power series ring is a rigid UFD (see [10]) is inherited by the kernels of derivations in these rings. In order to prove this we need to know the following fact about local rings.

Proposition 2.6: The kernel of any derivation d in a local ring T is again a local ring.

Proof: We show that for any non-unit b $\in$ Ker d the element 1 + b is a unit in Ker d. b Must also be a non-unit in T, for if there exists an element c $\in$ T such that bc = 1, we see immediately on applying d to this expression that $bc^d = 0$, and hence $c \in$ Ker d, contradicting the fact that b is a non-unit in Ker d. Now, using the same argument, we see that 1 + b, which is a unit in T, must actually be a unit in Ker d. Hence, Ker d is a local ring. //

The situation for free power series rings is then as follows.

Theorem 2.7: Let $\hat{R} = F\langle\langle x_1,\ldots,x_q\rangle\rangle$ and let d be any derivation (continuous or not) in $\hat{R}$, then

i) Ker d is a semifir.

ii) Ker d is a rigid UFD.

iii) $\hat{R}$ is a flat right Ker d-module.

Proof: i) This follows from corollary 2.3 since $\hat{R}$ satisfies n-term IWA for each n and therefore Ker d is an n-fir for each n.

ii) Note that Ker d is atomic in the sense that every (non-zero) non-unit in it can be written as a product of atoms (i.e. non-units which cannot be written as products of two non-units). This is so because $\hat{R}$ is a rigid UFD [10, p.462], which means that every non-unit $b \in$ Ker d can be factorised uniquely (up to units) in $\hat{R}$ as $b = b_1b_2\ldots b_t$. Now if it is not possible to find an index r, $1 \leq r \leq t$, such that $b_1b_2\ldots b_r \in$ Ker d, b is certainly an atom in Ker d.

On the other hand, if such an r can be found, both the factors $b_1 b_2 \dots b_r$ and $b_{r+1} \dots b_t$ lie in Ker d and can therefore be treated in the same way. The assertion follows then from proposition 2.6 and the following theorem of P.M.Cohn [15]: An atomic semifir is a rigid UFD if and only if it is a local ring.

iii) See corollary 2.5. //

Part iii) of the theorem immediately raises the question whether $\hat{R}$ can be a <u>free</u> right Ker d-module. We do not know if this is so for arbitrary d or not. Looking at particular cases, however, we see that if d is e.g. the derivation $\frac{\partial}{\partial x_1}$ studied in example 2.12 (i), then $\hat{R}$ is indeed a free right Ker d-module with basis $\{1, x_1, x_1^2, x_1^3, \dots\}$. This is so because we then have the following commutation:

$$f x_1 = x_1 f \pmod{\text{Ker } d}, \quad \text{any } f \in \text{Ker } d.$$

A similar situation arises when primitive derivations in free associative algebras are being studied and we will leave the detailed discussion of this fact till then.

<u>Remark</u>: Theorem 2.7 as well as the results still to be discussed in the following sections are only nontrivial for continuous derivations if $\hat{R}$ has more than one generator, for if q = 1 then Ker d = F irrespective of the choice of d:

Let d be the continuous derivation in $F\langle\langle x \rangle\rangle$ given by $x \longmapsto u = \sum_{i=r}^{\infty} \nu_i x^i$, $\nu_r \neq 0$, $r \geq 0$, and suppose that

$$f = \sum_{j=0}^{\infty} \lambda_j x^j$$

is an element of Ker d. For every index $k \geq r$ the coefficient of x^k in f^d is

$$\gamma_k = \lambda_1 \nu_k + 2\lambda_2 \nu_{k-1} + \ldots + (k-r+1)\lambda_{k-r+1} \nu_r$$

and since $f^d = 0$ we have $\gamma_k = 0$, all $k \geq r$.
Putting $k = r$ gives $\lambda_1 = 0$ and for $k \geq r+1$ we have the sequence of recursion formulae

$$\lambda_{(k-r)+1} = \frac{1}{(k-r+1)\nu_r} (\lambda_1 \nu_k + 2\lambda_2 \nu_{k-1} + \ldots + (k-r)\lambda_{k-r} \nu_{r+1})$$

giving $\lambda_j = 0$ for all $j \geq 2$. Hence $f = \lambda_0 \in F$.

2. Homogeneous derivations in free power series rings.

Recall that a continuous derivation d in $\hat{R} = F\langle\langle x_1,..,x_q\rangle\rangle$ has been called homogeneous with respect to $X = \{x_1,...,x_q\}$ if $x_i^d = u_i$ $(i = 1, ...,q)$ where all the nonzero u_i's are homogeneous of the same degree in X and that any continuous derivation in $\hat{R}$ can be regarded as a "sum" of homogeneous derivations. Throughout this chapter we denote the least homogeneous component of any element a in $\hat{R}$ by $\bar{a}$. Before restricting attention to homogeneous derivations only, we state a lemma which is slightly more general than needed here, but which will be used again later.

<u>Lemma 2.8</u>: Let $d = d_r + d_{r+1} + \ldots$ (sum of homogeneous derivations) be an arbitrary continuous derivation in $\hat{R}$

with $d_r \neq 0$. Then for any $f \in \hat{R}$

$$f \in \text{Ker } d \Rightarrow \bar{f} \in \text{Ker } d_r.$$

Proof: The least homogeneous component of f^d is $\bar{f}^{d_r}$. //

The auxiliary information needed for the following proposition about homogeneous derivations is contained in

Lemma 2.9: Let d be any homogeneous derivation in $\hat{R}$, then

i) $f \in \text{Ker } d \Rightarrow \bar{f} \in \text{Ker } d$.

ii) An δ-dependence relation in Ker d

$$\delta(\sum a_i b_i) > \min_i \{ \delta(a_i) + \delta(b_i) \} \qquad (1)$$

is the shortest possible for the set $B = \{ b_1, \ldots, b_m \} \subseteq \text{Ker } d$ if and only if any proper subset of $\bar{B} = \{ \bar{b}_1, \ldots, \bar{b}_m \}$ is left linearly independent over Ker d.

Proof: i) This is just a special case of 2.8.

ii) Any left δ-dependence relation shorter than (1) satisfied by elements of B in Ker d, leads directly on taking least homogeneous components of the relevant elements to a left linear dependence over Ker d involving a proper subset of $\bar{B}$, and vice versa. //

Proposition 2.10: The kernel of any homogeneous derivation d in $\hat{R}$ satisfies the inverse weak algorithm with respect to the order function δ.

Proof: Any verification of the existence of IWA can be carried out by considering either right or left dependence relations. We find it convenient here to take the relations on the left. Let $A = \{a_i\} \subseteq \text{Ker } d$ be a set of elements which are left δ-dependent in Ker d, and consider in particular the relation

$$\delta\left(\sum_{i=1}^{m} b_i a_i\right) > \min_i \{\delta(b_i) + \delta(a_i)\} \tag{2}$$

which we take to be the shortest possible relation satisfied by members of A in Ker d. Arrange the indices in (2) in such a way that

$$\delta(a_1) \leq \delta(a_2) \leq \cdots \leq \delta(a_m).$$

By IWA in $\hat{R}$ some a_k ($k \leq m$) is left δ-dependent on $a_1, \ldots, a_{k-1}$ in $\hat{R}$, i.e. there exist elements $c_1, \ldots, c_{k-1} \in \hat{R}$ such that

$$\left.\begin{aligned} &\delta\left(a_k - \sum_{i=1}^{k-1} c_i a_i\right) > \delta(a_k) \\ &\delta(c_i) + \delta(a_i) \geq \delta(a_k) \quad ; \ i = 1, \ldots, k-1. \end{aligned}\right\} \tag{3}$$

By omitting terms if necessary we may assume in (3) that $\delta(c_i) + \delta(a_i) = \delta(a_k)$; $i = 1, \ldots, k-1$. When we look at the least homogeneous component of the left hand side of (3) we see that

$$\bar{a}_k - \sum_{i=1}^{k-1} \bar{c}_i \bar{a}_i = 0 \tag{4}$$

Now apply d to (4) and keep lemma 2.9 (i) in mind, then

$$\sum_{i=1}^{k-1} \bar{c}_i^{\,d}\, \bar{a}_i = 0 \tag{5}$$

By lemma 2.9(ii) the set $\{\bar{a}_1, \ldots, \bar{a}_{k-1}\}$ is left linearly independent over Ker d and hence by corollary 2.4 applied to (5) the elements $\bar{c}_i^{\,d}$ must all be zero. It only remains to replace every c_i in (3) by $\bar{c}_i$ to transform that relation into a left δ-dependence of a_k on $a_1,\ldots,a_{k-1}$ in Ker d. //

Theorem 2.11: The kernel of any homogeneous derivation d in $\hat{R}$ is again a free power series ring over F.

Proof: Ker d, being a closed subalgebra of $\hat{R}$, (see section 1.4, general property vi) is complete. It is also connected in the sense that Ker d = F + K_1 , where K_1 is the two-sided ideal in Ker d consisting of all elements of order ≥ 1. Now using proposition 2.10 we complete this proof with an application of the following theorem of P.M. Cohn [10, p.459]

> Let S be a complete valuated (by ϑ) connected algebra over a commutative field F. Then S is a power series ring (on a ϑ-independent almost generating set of S_1) over F if and only if S posesses an inverse weak algorithm.

[$S_1 = \{a \in S \mid \vartheta(a) \geq 1\}$. An almost generating set of S_1 is a generating set B of a right ideal I which is dense in S_1.] //

This quoted theorem gives an indication of what is to be expected of the free generating set of Ker d. We now describe such a set by constructing an appropriate modification of the "weak" algebra basis" of a filtered F-algebra introduced by G.M.Bergman [2, p.34] (see also [17, p.12])

For this we use the graded ring $H = Gr(\text{Ker } d)$ associated to the filtration

$$\text{Ker } d = K_0 \geq K_1 \geq K_2 \geq \; .....$$

which is induced in Ker d by δ. H consists of the family of disjoint abelian groups $H_i = K_i/K_{i+1}$, $i = 0,1,2,...$ where in particular $H_0 \cong F$.

For each $i \geq 1$ let $H_i' = K_i'/K_{i+1}$ be the F-subspace of H_i spanned by the elements ab where $a \in H_{j_1}$, $b \in H_{j_2}$, any $j_1 (<i)$ and $j_2 (<i)$ such that $j_1 + j_2 = i$, and let B_i be a set of representatives for a basis of the F-space H_i/H_i'. The set $B = \bigcup_{i \geq 1} B_i$ consists of homogeneous elements of Ker d none of which is right δ-dependent on the rest, for if there exists $b \in B$ such that

$$b \equiv \sum b_j g_j \pmod{K_{t+1}} \qquad (6)$$

where the $b_j \in B$, the $g_j \in K_0$ and $t = \delta(b)$, then any term $b_j g_j$ with $\delta(b_j) \neq t$ represents an element in $K_t'/K_{t+1} = H_t'$. So (6) can be written as

$$b \equiv \sum b_k \lambda_k \pmod{H_t'}$$

where $\lambda_j \in F$ and $\delta(b_k) = t$ whenever $\lambda_k \neq 0$. But this contradicts the choice of B_t as a set of representatives of a basis of the F-space H_t/H_t'. Since Ker d satisfies IWA with respect to δ, B is actually an δ-independent set.

Now let I be the right ideal in Ker d generated by B. We show that I is dense in K_1, and do this by way of a proof adapted from the work of P.M.Cohn [10, p.457]. Firstly we point out that since $H_1' = 0$ and hence $H_1 \subseteq I$, it can easily

be seen by induction that for every $i \geq 1$, $H_i \subseteq I$.
Hence if we take $a\ (\neq 0) \in K_1$ with, say $\delta(a) = n$ and write $a = a_{(n)} + p_1$ where $p_1 \in K_{n+1}$, then $a - p_1 = a_{(n)} \in I$. By induction on n we obtain a sequence $p_0 = a, p_1, p_2, \ldots$ of elements of strictly increasing order, such that $p_i - p_{i+1} \in I$. It follows that the orders $\delta(p_i - p_{i+1})$ also increase strictly, and hence

$$a = (p_0 - p_1) + (p_1 - p_2) + \ldots$$

is convergent. Thus a lies in the closure of I and the assertion follows.

Referring back to the quoted theorem, it can now be stated that the power series ring Ker d has the set B as a free generating set. Note that even though $\hat{R}$ has only finitely many generators, this set B is in general infinite.

<u>2.12 Examples</u>: The insight obtained in the theoretical discussion above is now put to use for studying the kernels of two special homogeneous derivations, singled out because they are of importance to the succeeding sections and because they reveal typical properties of such derivation - kernels.

1) Take $\hat{R} = F\langle\langle x_1, \ldots, x_q \rangle\rangle$ and $d = \frac{\partial}{\partial x_1}$ i.e. take d to be the continuous derivation which sends $x_i \longrightarrow \delta_{1i}$ (Kronecker delta) $i = 1, \ldots, q$. We use the same notation as before. Consider any homogeneous element $g \in H_n$, $n \geq 1$, and note

that g may also be taken to be homogeneous in x_1, say it is of degree $n - r$ in x_1. This means that each monomial in g has r factors x_{i_j}, $i_j \neq 1$, $j = 1,2,..,r$. Furthermore, there is at least one monomial in g of the form

$$u' = x_{i_1} x_1^{s_{i_1}} x_{i_2} x_1^{s_{i_2}} \dots x_{i_r} x_1^{s_{i_r}} , \qquad (7)$$

where $s_{i_1} + s_{i_2} + \dots + s_{i_r} = n - r$, $r < n$

If (7) is not true, g will have x_1 as left factor, say $g = x_1^t f$ where f does not have the left factor x_1. Now if we apply $d = \frac{\partial}{\partial x_1}$ to this, we get

$$0 = t x_1^{t-1} f + x_1^t f^d$$

$$\text{i.e. } tf = -x_1 f^d$$

contradicting the choice of f.

We can therefore write

$$g = \sum_i \lambda_i (x_{i_1} x_1^{s_{i_1}} \dots x_{i_r} x_1^{s_{i_r}}) + x_1 g' \qquad (8)$$

Notation: Use the symbol

$$[u, x_1^{[k]}]$$

to denote the commutator $[\dots[[u,\underbrace{x_1],x_1],\dots,x_1}_{k \text{ times}}]$, u arbitrary in $\hat{R}$.

An easy induction on k shows that $[u,x_1^{[k]}]$ is of the form

$$[u,x_1^{[k]}] = u x_1^k + x_1 f_u^{(k)} \qquad (9)$$

where $f_u^{(k)}$ is an element generated by u and x_1 in $R = F< x_1,\dots,x_q >$. We use the formula (9) repeatedly to write (7) as

2.2 (Example 1)

$$u' = [x_{i_1}, x_1^{[s_{i_1}]}](x_{i_2} \dots x_1^{s_{i_r}}) - x_1 f_{x_{i_1}}^{(s_{i_1})}(x_{i_2} \dots x_1^{s_{i_r}})$$

$$= [[x_{i_1}, x_1^{[s_{i_1}]}]x_{i_2}, x_1^{[s_{i_2}]}](x_{i_3} \dots x_1^{s_{i_r}}) - x_1 f_{\#}^{(s_{i_2})}(x_{i_3} \dots x_1^{s_{i_r}})$$

$$- x_1 f_{x_{i_1}}^{(s_{i_1})}(x_{i_2} \dots x_1^{s_{i_r}}), \quad \text{where } \# = [x_{i_1}, x_1^{[s_{i_1}]}]x_{i_2}$$

$$= \dots\dots$$

$$= [\dots[[x_{i_1}, x_1^{[s_{i_1}]}]x_{i_2}, x_1^{[s_{i_2}]}]x_{i_3}, \dots, x_1^{[s_{i_r}]}] + h_i \qquad (10)$$

where h_i represents in general such a long and involved expression that we do not endeavour to write it down explicitly. It is only important to note that h_i has the left factor x_1, say $h_i = x_1 h_i'$.

With the aid of (10) we can rewrite the expression (8) for g in the following form

$$g = \sum_i \lambda_i [\dots[x_{i_1}, x_1^{[s_{i_1}]}]x_{i_2}, \dots, x_1^{[s_{i_r}]}] + x_1(\sum_i h_i' + g') \qquad (11)$$

Since the sum of commutators in (11) lies in Ker d, it follows that

$$x_1(\sum_i h_i' + g') \in \text{Ker } d.$$

However, as we have seen above, the fact that there is a left factor x_1 will only allow this to be true if $\sum_i h_i' + g' = 0$.

2.2 (Example 1)

Let $c_i = [\ldots[x_{i_1}, x_1^{[s_{i_1}]}]x_{i_2}, \ldots, x_1^{[s_{i_r}]}]$.

Now for any r, if $s_{i_r} = 0$, c_i is of the form $[a, x_j]$, where $a, x_j \in \operatorname{Ker} d$ and hence $c_i \in H_n'$. Furthermore if $r > 1$ and $s_{i_r} \neq 0$, then c_i is of the form $[(ab), x_1^{[s_{i_r}]}]$ where $a \in H_{j_1}$, $b \in H_{j_2}$ for some j_1 and j_2 (≥ 1) such that $j_1 + j_2 = n - s_{i_r}$. Hence, by a straightforward induction starting from

$$[ab, x_1] = a[b, x_1] + [a, x_1]b$$

it follows that $c_i \in H_n'$. The outcome of this discussion is then that

$$g = \sum_{j=2}^{q} \lambda_j [x_j, x_1^{[n-1]}] \pmod{H_n'} .$$

Now we have made it clear how the set $B = \bigcup_i B_i$ of free generators should be constructed: just take $B_1 = \{ x_j \mid 2 \leq j \leq q \}$, and for every $n \geq 2$ take $B_n = \{ [x_j, x_1^{[n-1]}] \mid 2 \leq j \leq q \}$. Then the elements of B_1 are clearly F-linearly independent and by an easy induction the same is true of B_n for every $n \geq 2$. Hence by the argument on page 50 the set B consists of δ-independent elements which form an almost generating set of K_1.

So, summarizing the preceding discussion, we get

Proposition 2.13: In $\hat{R} = F \ll x_1, \ldots, x_q \gg$ the kernel of the continuous derivation $\frac{\partial}{\partial x_1}$ is a free power series ring over F with free generating set $B = \bigcup_{i=1}^{\infty} B_i$, where $B_1 = \{x_j \mid 2 \leq j \leq q\}$, and for every $i \geq 2$
$B_i = \{[x_j, x_1^{[i-1]}] \mid 2 \leq j \leq q\}$. //

Example 2: In $\hat{R} = F \ll x,y \gg$, let d be the continuous derivation which sends $x \mapsto 0$, $y \mapsto x$.

The way in which we describe a set B of free generators for the kernel of this derivation will enable us to show that it is the least closed subalgebra N of $\hat{R}$ containing x and satisfying the property

$$g \in N \Rightarrow (xgy - ygx) \in N \tag{13}$$

By lemma 3.7 in the next chapter, N will then also be the fixed ring of the automorphism

$$\exp d = 1 + d + \frac{1}{2!}d^2 + \frac{1}{3!}d^3 + \ldots..$$

which in this case turns out to be the elementary automorphism in $F \ll x,y \gg$ given by

$$\begin{cases} x \mapsto x \\ y \mapsto y + x . \end{cases}$$

Use again the same notation as before by taking H_n to represent the set of homogeneous elements of degree n in Kerd. Verify that H_1 is the F-space generated by x, and that H_2 is the F-space generated by $\{x^2, [x,y]\}$, so that we can take

2.2 (Example 2)

$B_1 = \{x\}$ and $B_2 = \{[x,y]\}$.

In general, i.e. for $r \geq 3$ it is sufficient to consider elements which are also homogeneous in y, therefore we define H_{rk} $(r \geq 2,\ 0 \leq k \leq r)$ to be the F-subspace of H_r consisting of the elements homogeneous of degree k in y.

For homogeneous elements $f \in \bar{R}$ which are also homogeneous in x and y separately (and in particular for monomials) we use the symbols $\partial_x(f)$ and $\partial_y(f)$ to indicate the degrees in x and y respectively.

For each H_{rk} $(r \geq 3,\ 0 \leq k \leq r)$ define two subspaces H_{rk}' and H_{rk}'' by taking

i) H_{rk}' = (the subspace spanned by the products ab, where a and b are homogeneous elements in Ker d such that deg a + deg b = r, $\partial_y(a) + \partial_y(b) = k$);

ii) H_{rk}'' = $\begin{cases} \text{(the zero subspace), if } k = 0,1\ ; \\ \text{(the subspace spanned by the elements} \\ xfy - yfx, \text{ with } f \in H_{r-2,k-1}), \text{ if } 2 \leq k \leq r. \end{cases}$

our object now is to prove for every $r \geq 3$, $0 \leq k \leq r$ that

$$H_{rk} = H_{rk}' \oplus H_{rk}'' \tag{14}$$

If k = 0, this is immediately clear. If k = 1, H_{rk} is generated by elements of the form $x^i y x^j - x^s y x^t$, where $i + j + 1 = s + t + 1 = r$. These elements clearly lie in H_{r1}' whenever either $j \geq 1$ or $s \geq 1$ (or both); and the elements of the form $[x^{r-1}, y]$ also lie in H_{r1}' because

2.2 (Example 2)

$$[x^{r-1},y] = x^{r-2}[x,y] + [x^{r-2},y]x \ .$$

If $r \geq 3$, $2 \leq k \leq r$, then $H_{rk}' \cap H_{rk}'' = 0$ because no non-zero element of the form $xfy - yfx$, where f is a homogeneous element in Ker d, can be written as a product of two elements in Ker d. It remains to show that for $r \geq 3$, $2 \leq k \leq r$ we have $H_{rk} \subseteq H_{rk}' + H_{rk}'' = M_{rk}$ (say).

We introduce a classification of the monomials in $\hat{R}$ in the following way:

Let h be an arbitrary monomial in H_{rk} $(r \geq 2,\ 1 \leq k \leq r)$ Distinguish between the y's appearing in h by numbering them from left to right so as to create k different factorizations of h into the product of a y and two other monomials, viz. $h_1'y_{(1)}h_1''$, $h_2'y_{(2)}h_2''$, ..., $h_k'y_{(k)}h_k''$.

We now say that <u>h is of type 'L'</u> (for leading term) if and only if it does not have left factor y, and $\partial_x(h_i') > \partial_y(h_i')$ for every $i = 1,2,\ldots,k$. The monomials $x^r \in H_{ro}$, $(r \geq 1)$ are also said to be of type 'L' .

E.g. x^2yxy is of type 'L', whilst xy^2x^2 is not.

Order the monomials in $\hat{R}$ lexicographically (with $x < y$) and extend this to an ordering of all elements of $\hat{R}$ by taking the <u>smallest term</u> of any $f \in \hat{R}$ to be its leading term.

The reason for introducing monomials of type 'L' becomes clear as soon as we note that every monomial h of type 'L' is the leading term of an element in the space $H_{rk}' + H_{rk}'' = M_{rk}$ where $r = \deg h$ and $k = \partial_y(h)$.

2.2 (Example 2)

This can be seen by noting first of all that the only monomials of degree 2 which are of type 'L' are x^2 and xy, and both of them are leading terms of elements in H_{20} and H_{21} respectively. Furthermore if h is a monomial of type 'L' (with $\deg h = r$ (>2), $\partial_y(h) = s$) we have either $h = h'x$ or $h = xh''y$. If $h = h'x$, h' is also of type 'L' and is by induction the leading term of an element $g' \in M_{r-1,s}$, so h is the leading term of $g'x \in M_{rs}$. If $h = xh''y$, h'' is also of type 'L' and is by induction the leading term of an element $g'' \in M_{r-2,s-1}$, so h is the leading term of $(xg''y - yg''x) \in M_{rs}$.

The next step is to prove that every element in Ker d has at least one term which is of type 'L'. Consider a homogeneous element $g \in \text{Ker } d$ and write it (as is generally possible) in the form

$$g = xg_1y + yg_2x + xg_3x + yg_4y \qquad (15)$$

Applying the derivation d to (15) gives

$$0 = g^d = x(g_1^d + g_4)y + y(g_2^d + g_4)x + x(g_1 + g_2 + g_3^d)x + yg_4^dy$$

from which we get

$$\left.\begin{aligned} g_1^d + g_4 &= 0 \\ g_2^d + g_4 &= 0 \\ g_1 + g_2 + g_3^d &= 0 \\ g_4^d &= 0 \end{aligned}\right\} \qquad (16)$$

By (16) clearly $(g_1 - g_2)^d = 0$. We may assume (inductively) that $g_1 - g_2$ has at least one term of type 'L'. If this term appears in g_1 the assertion about g follows immediately, so we say that none of the terms in g_1 is of type 'L' and that g_2 has at least one term of this type. Now since $g_3^d = -(g_1 + g_2)$ (by (16)) we can also assume $g_3^d \neq 0$, for $g_3^d = 0$ will imply $g_1 = -g_2$, whence g_1 has a term of type 'L' , contradiction. The assumption about g_2 implies that $g_1 + g_2 = -g_3^d \neq 0$ has a term of type 'L' and then recalling the action of d on monomials, we see that g_3 either has a term of type 'L' , or if not, it has at least one term of the form $h'yh''$ where h' and h'' are both of type 'L' and $h'y$ is not ($h'' = 1$ is also possible). In both these cases xg_3x has a term of type 'L' and this establishes the assertion about g.

Return to (14); if it does not hold there will exist $r \geq 3$ and $2 \leq s < r$ such that $M_{rs} \subsetneq H_{rs}$. Consider then $\tilde{g} \in H_{rs}/M_{rs}$ ($\tilde{g} \neq 0$) and let g be any representative of $\tilde{g}$. Write this representative in the form

$$g = \lambda_1 h_1 + \dots + \lambda_m h_m$$

where the h_i are monomials indexed in such a way that $h_i < h_{i+1}$ ($i = 1,\dots,m-1$) in the lexicographical ordering. Let t be the least index such that h_t is of type 'L' (such a t exists because $g \in \operatorname{Ker} d$). We know that h_t is the leading term of an element, say h_t^*, in M_{rs} and can therefore write

2.2 (Example 2)

$$g = \sum_{i=1}^{t-1} \lambda_i h_i + \lambda_t h_t^{*} - \lambda_t (h_t^{*} - h_t) + \sum_{j=t+1}^{m} \lambda_j h_j .$$

Hence $\tilde{g}$ can also be represented by the element

$$g' = \sum_{i=1}^{t-1} \lambda_i h_i - \lambda_t (h_t^{*} - h_t) + \sum_{j=t+1}^{m} \lambda_j h_j$$

in which the smallest term of type 'L' is larger than h_t. This obviously indicates a process of elimination of terms of type 'L' from the chosen representatives of $\tilde{g}$, and this process then ends in a representative which does not have any term of type 'L' , contradicting the fact that every element in Ker d contains at least one such term. Hence $H_{rk} = M_{rk}$ for all $r \geq 3$ and $2 \leq k \leq r$.

This discussion shows that in order to find a free generating set B for Ker d it is sufficient to confine attention to the subspaces H_{rk}'' , in fact we can take B_r ($r \geq 3$) to be a basis of the F-space $\sum_{k=0}^{r} H_{rk}''$. Hence $B_1 = \{x\}$, $B_2 = \{[x,y]\}$, $B_3 = \emptyset$, and for $r \geq 4$

$B_r = \{ xby - ybx \mid b \neq x^{r-2}, \& b \in Y_{r-2}; Y_{r-2}$ being a basis of the F-space $H_{r-2} \}$.

It is now also clear that Ker d is contained in every subalgebra satisfying (13) in $\hat{R}$, and therefore Ker d = N. So

<u>Proposition 2.14</u>: Let N be the least subalgebra of F<<x,y>> containing x and satisfying the property $g \in N \Rightarrow (xgy-ygx) \in N$, then i) N is the kernel of the continuous derivation in F <<x,y >> given by $x \mapsto 0$, $y \mapsto x$.

ii) N is the fixed ring of the elementary automorphism in $F \ll x,y \gg$ given by $x \mapsto x$, $y \mapsto x + y$.

iii) N is a free power series ring over F with free generating set $B = \bigcup_{i=1}^{\infty} B_i$, where $B_1 = \{x\}$, $B_2 = \{[x,y]\}$, $B_3 = \emptyset$ and for every $i \geq 4$ $B_i = \{ xby-ybx \mid b \neq x^{i-2}, \ \& \ b \in Y_{i-2} \}$.

Y_{i-2} is a basis of the F-subspace of N consisting of all the homogeneous elements of degree i-2. //

3. Derivations of order zero.

When the kernel of a continuous derivation in $\hat{R} = F \ll x_1,..x_q \gg$ is being studied in connection with the inverse weak algorithm the least homogeneous components of the elements appearing in the dependence relations play a constantly recurring part. This came to light in the previous section and it will do so again in our treatment of continuous derivations of order zero because for such derivations the said components determine the whole situation. Here we can strengthen theorem 2.7 (asserting that Ker d is a semifir and rigid UFD) by showing that the kernels of continuous derivations of order zero have IWA and hence are also free power series rings. The door is opened by

Lemma 2.15: If d_o is a homogeneous derivation of order zero

2.3

in $\hat{R}$, the free algebra $R = F < x_1, \ldots, x_q >$ lies in Im d_o, and in fact every homogeneous element of degree r in R is the image (under d_o) of a homogeneous element of degree r+1.

<u>Proof</u>: Let d_o be the continuous derivation given by $x_i^{d_o} = \lambda_i$ ($\lambda_i \in F$) and say $\lambda_1 \neq 0$. Take $Y = \{y_1, \ldots, y_q\}$ to be the free generating set of $\hat{R}$ obtained from $X = \{x_1 \ldots, x_q\}$ by the linear transformation: $x_1 \mapsto \frac{1}{\lambda_1} x_1$, $x_i \mapsto x_i - \frac{\lambda_i}{\lambda_1} x_1$, $2 \leq i \leq q$. We are then in fact considering the derivation $\frac{\partial}{\partial y_1}$ in $\hat{R} = F \ll y_1, \ldots, y_q \gg$.

It is sufficient to show that every monomial h in Y is the image under $d_o = \frac{\partial}{\partial y_1}$ of a homogeneous element of degree (deg h)+1 . Write

$$h = y_{i_1}^{s_1} y_{i_2}^{s_2} \ldots y_{i_m}^{s_m} \tag{1}$$

where $i_{\nu+1} \neq i_\nu$, $\nu = 1, \ldots, m-1$; and let $s = s_1 + \ldots + s_m$. If $i_\nu \neq 1$, all ν , we can write

$$h = (hy_1)^{d_o} \quad .$$

If, on the other hand, $i_\nu = 1$ for at least one $\nu \in \underline{m} = \{1, 2, \ldots, m\}$ we order the monomials in Y lexicographically (based on $y_1 < \ldots < y_q$) and extend this ordering to all elements of $R = F < y_1, \ldots, y_q >$ by taking the <u>smallest</u> term in any element of R to be its leading term. Then we choose an $f \in R$ such that $h = f^{d_o} - g$, where g follows h in the chosen ordering.

In fact, let k be the largest member of the set $\underline{m}$ such that $i_k = 1$, and take

$$f = y_{i_1}^{s_1} \dots y_{i_{k-1}}^{s_{k-1}} y_{i_k}^{s_k+1} y_{i_{k+1}}^{s_{k+1}} \dots y_{i_m}^{s_m} ,$$

then

$$h = \frac{1}{s_k+1} f^{d_o} - g \quad , \quad \text{where}$$

$$g = \begin{cases} 0 & \text{if k is also the smallest member of } \underline{m} \text{ such that } i_k = 1 ; \\ \frac{1}{s_k+1}\Big[\sum\limits_{\substack{i_j=1 \\ j< k}} s_j (y_{i_1}^{s_1} \dots y_{i_j}^{s_j-1} \dots y_{i_k}^{s_k+1} \dots y_{i_m}^{s_m})\Big] & \text{otherwise.} \end{cases}$$

Each of the monomials in g can now be treated in the same way, and since we are dealing with homogeneous elements on a finite generating set, the process must terminate. Hence it is possible to find a homogeneous element a of degree s+1 such that $a^{d_o} = h$. //

Lemma 2.16: If d is a continuous derivation of order zero in $\hat{R}$, say $d = d_o + d_1 + d_2 + \dots$ where $d_o \neq 0$, it is possible to find for any given homogeneous element $c \in \operatorname{Ker} d_o$ an element $g \in \operatorname{Ker} d$ such that $c = \bar{g}$ ($\bar{g}$ being the least homogeneous component of g).

Proof: Let $f = f_r + f_{r+1} + f_{r+2} + \dots$ (sum of homogeneous components) be an arbitrary element of order r in $\hat{R}$, then $f \in \operatorname{Ker} d$ if and only if all the homogeneous components of

f^d are zero, i.e. if and only if

$$\sum_{j=0}^{k} f_{r+j}{}^{d_{k-j}} = 0 \qquad (k = 0,1,2,\ldots)$$

(This is the homogeneous component of degree r+k-1 of f^d).

If the given element c has order r, write $c = g_r$, then we know that $g_r{}^{d_o} = 0$. Assume inductively that homogeneous elements $g_{r+1},\ldots,g_{r+k}$ exist such that

$$g_r{}^{d_1} + g_{r+1}{}^{d_o} = 0$$

$$g_r{}^{d_2} + g_{r+1}{}^{d_1} + g_{r+2}{}^{d_o} = 0$$

$$\ldots\ldots\ldots$$

$$g_r{}^{d_k} + g_{r+1}{}^{d_{k-1}} + \ldots + g_{r+k}{}^{d_o} = 0$$

and consider the element

$$g_r{}^{d_{k+1}} + g_{r+1}{}^{d_k} + \ldots + g_{r+k}{}^{d_1} = a \quad \text{(say).}$$

This is an element in R, homogeneous of degree r+k.
By lemma 2.15 there exists a homogeneous element of degree r + k + 1, say g_{r+k+1}, such that

$$a + g_{r+k+1}{}^{d_o} = 0.$$

Hence it is possible to build up a sequence (g_{r+i}; i=0,1,2,..) of elements which satisfy the system of equations

$$\sum_{j=0}^{k} g_{r+j}{}^{d_{k-j}} = 0\ , \quad k = 0,1,2,\ldots\ .$$

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Hence $g = \sum_{i=0}^{\infty} g_{r+i}$ is an element in Ker d such that $\bar{g} = c$. //

We say that a set A of elements in $\hat{R}$ is a minimal left δ-dependent set if no proper subset of A is left δ-dependent.

Before we can get the following proposition we need yet another lemma.

Lemma 2.17: Let $d = d_0 + d_1 + d_2 + \dots$ be a continuous derivation of order zero in $\hat{R}$. If $A = \{a_1,\dots,a_m\} \subseteq \text{Ker } d$ is a set, minimal left δ-dependent (in Ker d), then any proper subset of $\bar{A} = \{\bar{a}_1,\dots,\bar{a}_m\} \subseteq \text{Ker } d_0$ is left linearly independent over Ker d_0 .

Proof: Suppose $\{\bar{a}_{j_1},\dots,\bar{a}_{j_k}\}$ is a proper subset of $\bar{A}$ which is left Ker d_0-dependent and therefore satisfies a relation $\sum_{r=1}^{k} g_{j_r}\bar{a}_{j_r} = 0$, where every g_{j_r} is a homogeneous element of Ker d_0 . By lemma 2.16 it is then possible to find elements $f_{j_r} \in \text{Ker } d$ such that $g_{j_r} = \bar{f}_{j_r}$, $r = 1,2,\dots,k$. Hence

$$\delta\Big(\sum_{r=1}^{k} f_{j_r} a_{j_r}\Big) > \min_r \{\delta(f_{j_r}) + \delta(a_{j_r})\}$$

which is a left δ-dependence relation in Ker d featuring a proper subset of A, and hence contradicting the assumption about A. //

Proposition 2.18: The kernel of every continuous derivation of order zero in $\hat{R} = F \ll x_1, x_2, \ldots, x_q \gg$ satisfies the inverse weak algorithm with respect to the order filtration.

Proof: Consider the derivation $d = d_0 + d_1 + d_2 + \ldots$ where $d_0 \neq 0$. Let $A \subseteq \text{Ker } d$ be a finite set of left δ-dependent elements and take the relation

$$\delta\left(\sum_{i=1}^{m} b_i a_i\right) > \min_i \{\delta(b_i) + \delta(a_i)\} \qquad (2)$$

($a_i \in A$, $b_i \in \text{Ker } d$) to be the shortest satisfied in Ker d by elements of the set A. Arrange the indices in (2) in such a way that $\delta(a_1) \leq \ldots \leq \delta(a_m)$. (2) is also an δ-dependence relation in $\hat{R}$, so that by IWA in $\hat{R}$ some a_k is left δ-dependent on $a_1, \ldots, a_{k-1}$, i.e. there exist elements $c_1, \ldots, c_{k-1} \in \hat{R}$ such that

$$\left.\begin{aligned} &\delta\left(a_k - \sum_{i=1}^{k-1} c_i a_i\right) > \delta(a_k) \\ &\delta(c_i) + \delta(a_i) \geq \delta(a_k) \quad , \quad i = 1, \ldots, m-1 \end{aligned}\right\} \qquad (3)$$

By omitting terms if necessary, we may assume in (3) that $\delta(c_i) + \delta(a_i) = \delta(a_k)$, $(i = 1, \ldots, k-1)$.
Looking only at least homogeneous components in (3), we get

$$\bar{a}_k - \sum_{i=1}^{k-1} \bar{c}_i \bar{a}_i = 0 \qquad (4)$$

Now apply d_0 to (4) and recall lemma 2.8 . This gives

$$\sum_{i=1}^{k-1} \bar{c}_i^{\,d_0}\, \bar{a}_i = 0 \qquad (5)$$

Since (2) had been taken to be the shortest left δ-dependence relation satisfied by elements of the set A, lemma 2.17 says that the elements $\bar{a}_1,\ldots,\bar{a}_{k-1}$ are left linearly independent over Ker d_o. Hence corollary 2.4 can be applied to (5), and this gives $\bar{c}_i^{\,d_o} = 0$, for $i = 1,\ldots,k-1$. Now by lemma 2.16 there exist elements $g_1,\ldots,g_{k-1}$ in Ker d such that $\bar{g}_i = \bar{c}_i$ $(i = 1,\ldots,k-1)$. All that remains to be done to get a_k left δ-dependent (in Ker d) on the preceding a_i's, is to replace every c_i in (3) by the corresponding $g_i \in$ Ker d. //

Theorem 2.19: The kernel of any continuous derivation of order zero in $\hat{R} = F \ll x_1,\ldots,x_q \gg$ is again a power series ring over F.

Proof: Identical to the proof of theorem 2.11 except that proposition 2.18 must be used here. //

In this case the description of a free generating set for Ker d is facilitated by the information already obtained for homogeneous derivations and in particular the derivation $\frac{\partial}{\partial x_1}$ (see proposition 2.13). As was shown in the proof of lemma 2.15 we may take the derivation d_o in $d = d_o + d_1 + d_2 + \ldots$ to be $\frac{\partial}{\partial x_1}$ by employing a suitably chosen linear change of generating set.

2.3

Now Ker $\frac{\partial}{\partial x_1} = F \ll y_1, y_2, y_3, \ldots\ldots \gg$, where

$$y_i = \begin{cases} x_{i+1} & \text{if } 1 \leq i \leq q-1 \\ [x_{i-q+2}, x_1] & \text{if } q \leq i \leq 2(q-1) \\ \ldots & \\ [x_{i+n(1-q)+1}, x_1^{[n]}] & \text{if } n(q-1)+1 \leq i \\ & i \leq (n+1)(q-1) \\ \ldots\ldots\ldots\ldots & \end{cases}$$

We know by lemma 2.16 that there exists for every y_i an element $z_i \in$ Ker d such that $\bar{z}_i = y_i$ and it requires only a short argument (Which we now give) to verify that $Z = \{z_i \; ; \; i = 1,2,\ldots\}$ is a free generating set of Ker d:

The fact that $Y = \{y_i \; ; \; i = 1,2,\ldots\}$ is a right δ-independent set in Ker d_o clearly implies that Z is a right δ-independent set in Ker d. Furthermore , if K_1 is the ideal of Ker d consisting of all elements of order ≥ 1, we claim that the right ideal J generated by Z in Ker d is dense in K_1. The least homogeneous component $\bar{f}$ of any element $f \in K_1$ is an element of Ker d_o and is therefore expressible in terms of the generators Y , say

$$\bar{f} = \sum_{\text{finite}} y_{i_1} y_{i_2} \ldots y_{i_{n_i}} \lambda_{i_1 \ldots i_{n_i}} .$$

Now let g_1 be the corresponding element

$$\sum_{\text{finite}} z_{i_1} z_{i_2} \ldots z_{i_{n_i}} \lambda_{i_1 \ldots i_{n_i}} \in J,$$

then the element $f - g_1 \in K_1$ has $\delta(f - g_1) > \delta(f)$.

Again $\overline{f - g_1} \in \text{Ker } d_o$. In the same way as before we obtain an element $g_2 \in J$ such that $f - g_1 - g_2 \in K_1$ and $\delta(f-g_1-g_2) > \delta(f-g_1)$. This process can be continued to give a sequence

$$g_1, g_2, g_3, \ldots \qquad (g_i \in J, \text{ all } i)$$

of elements of ascending order with the property that for any $n \in Z^+$ there exists an $m \in Z^+$ such that

$$\delta(f - \sum_{j=1}^{m} g_j) > n.$$

Hence f lies in the closure of J as we wanted to show.

Conclusion: In order to find a free generating set for the free power series ring $\text{Ker } d \subseteq \hat{R}$ where $d = d_o + d_1 + d_2 + \ldots$ is a continuous derivation of order zero in $\hat{R} = F\langle\langle x_1, \ldots, x_q \rangle\rangle$, it is sufficient to take a free generating set $\{y_1, y_2, y_3, \ldots\}$ for the kernel of the homogeneous derivation d_o and then find a set of elements $Z = \{z_1, z_2, z_3, \ldots\} \subseteq \text{Ker } d$ such that the least homogeneous component of z_i equals y_i, every $i \geq 1$.

4. Derivations of order one.

The successful establishment of IWA in the kernels of all continuous derivations of order zero in $\hat{R}$ can be attributed chiefly to the ease with which inverse images can be found under the homogeneous derivations of order zero (see lemma 2.15).

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In the case of continuous derivations of order ≥ 1 this is not possible any more, and any attempt to prove directly that the kernel of such a derivation satisfies IWA is therefore made much more complicated, if not totally impossible. E.g. if $d = d_1 + d_2 + d_3 + \ldots$ is a continuous derivation of order 1 in $\hat{R}$ where d_1 is a "diagonal" derivation with respect to X, i.e.

$$x_i^{d_1} = \lambda_i x_i \ , \qquad \lambda_i \in F \ , \quad i = 1,\ldots,q \ ,$$

it is easy enough to say whether a given element lies in Im d_1 or not because d_1 sends every monomial in X into a scalar multiple of itself (see lemma 3.6, proof) and therefore $\hat{R} = \text{Ker } d_1 \oplus \text{Im } d_1$.

But if (say) $d_2 \neq 0$ and c is a given homogeneous element in Ker d_1, it is by no means certain that it will be possible to find an element $g \in \hat{R}$ with $\delta(g) > \delta(c)$ such that

$$c + g \in \text{Ker } d \ ,$$

because we must then in particular have that

$$c^{d_2} + \bar{g}^{d_1} = 0$$

($\bar{g}$ = least homogeneous component of g) i.e.

$$c^{d_2} \in \text{Im } d_1 \ .$$

This, however, will depend entirely on d_2 which has not been restricted in any way, and hence the whole argument may grind to a halt.

2.4

Another property which facilitated the arguments in the case of continuous derivations of order zero, was the smoothness with which the order-filtration (as determined by the free generating set) could be used, but even this changes when non-homogeneous continuous derivations of order one are being considered. See for example the following example (2.20) of a continuous derivation whose kernel does not satisfy IWA with respect to the natural order function. The significance of the existence of such derivations is then that any further investigation along the same lines will be hampered by the additional task of finding a suitable inverse filtration for the kernel of the derivation under consideration.

<u>2.20 Example</u>: In $F \ll x_1, x_2, \ldots, x_5 \gg$, consider the continuous derivations d_1 and d_2 given by

$$
d_1 : \begin{array}{l} x_1 \mapsto -x_1 \\ x_2 \mapsto x_2 \\ x_3 \mapsto x_3 + x_2 \\ x_4 \mapsto 0 \\ x_5 \mapsto 0 \end{array}
\qquad
d_2 : \begin{array}{l} x_1 \mapsto x_1 x_4 \\ x_2 \mapsto -x_4 x_2 \\ x_3 \mapsto -x_4 x_3 \\ x_4 \mapsto x_4^2 \\ x_5 \mapsto -x_4 x_5 \end{array}
$$

We show that the kernel of the derivation $d = d_1 + d_2$ (of order 1) does <u>not</u> satisfy IWA with respect to the natural order-function:

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Ker d contains the four elements

$$a_1 = x_1x_2$$
$$a_2 = x_1x_2x_5 + x_1x_3x_4x_5$$
$$b_1 = x_5x_1x_2 + x_4x_5x_1x_3$$
$$b_2 = -x_1x_2.$$

Furthermore, a_1 and a_2 are right δ-dependent in Ker d, in fact

$$\delta(a_1b_1 + a_2b_2) > \delta(a_1b_1) = \delta(a_2b_2)$$

since

$$(x_1x_2)(x_5x_1x_2) + (x_1x_2x_5)(-x_1x_2) = 0.$$

Now $x_1x_2x_5 = (x_1x_2)x_5$,

i.e. $\bar{a}_2 = \bar{a}_1x_5$,

where $\bar{a}_1$, $\bar{a}_2$ are respectively the least homogeneous components of a_1 and a_2. But x_5 can never be the least homogeneous component of an element in Ker d, for if an element

$$g = g_2 + g_3 + \dots \quad , \quad \delta(g) \geq 2,$$

could be found such that

$$x_5 + g \in \text{Ker } d$$

it will follow in particular for the homogeneous component of degree 2 of $(x_5 + g)^d$ that

$$x_5^{d_2} + g_2^{d_1} = 0 . \qquad (1)$$

(1) Shows that $x_5^{d_2}$ must then lie in Im d_1 , but

$$x_5^{d_2} = -x_4x_5 \notin \text{Im } d_1 .$$

Hence it is impossible to find $c \in$ Ker d such that

$$\delta(a_2 - a_1c) > \delta(a_2) ,$$

and this establishes the claim.

2.4

Remark: The existence of continuous derivations (in free power series rings) whose kernels do not satisfy IWA relative to the natural order function, becomes even more interesting if it is recalled from section 1 (Thm. 2.7) that each of these kernels is a semifir.

An attempt was made to determine the (very involved) kernel of the derivation in the example completely, in the hope that this might turn out to be non-free, but the indications were that another filtration could be found relative to which the kernel does satisfy IWA.

CHAPTER 3.

DERIVATIONS AND AUTOMORPHISMS IN COMPLETE INVERSELY FILTERED F-ALGEBRAS.

We can use the information, obtained in Chapter 2, about the kernels of derivations in complete inversely rings satisfying IWA to prove corresponding results for the fixed rings of certain automorphisms in complete inversely filtered F-algebras, where F is a field of characteristic zero. The theorems will then of course apply directly to free power series rings $F\langle\langle x_1,\dots,x_q\rangle\rangle$; where we find for example that a continuous automorphism α which sends the free generators x_i to $x_i^{\alpha} = x_i + g_i$, $\delta(g_i) \geq 2$, has a fixed ring which is both a semifir and a rigid UFD.

First of all an appropriate connection between automorphisms and derivations in such an inversely filtered algebra S is needed. Consider the mapping formally given by

$$\exp d = 1 + d + \frac{1}{2!}d^2 + \frac{1}{3!}d^3 + \dots\dots \qquad (1)$$

where d is a derivation in S. Initially, take

$$S = \hat{R} = F\langle\langle x_1,\dots,x_q\rangle\rangle .$$

We cannot expect (1) to represent a well-defined mapping in $\hat{R}$ for any arbitrary ~~chosen~~ derivation d of order zero in $\hat{R}$. E.g. $d = \frac{\partial}{\partial x_1}$, then exp d will be undefined in some elements of $\hat{R}$, such as $f = \sum_{i=1}^{\infty} x_1^{\,i}$, because the elements $\frac{1}{n!}f^{d^n}$ $(n \geq 1)$

all have constant term equal to 1 and therefore the formal expression $\sum_{n=0}^{\infty} \frac{1}{n!} f^{d^n}$ does <u>not</u> represent an element of $\hat{R}$.

For derivations of order 1 in $\hat{R}$ we must also move about with care and take account of the fact that even trivial continuous derivations, like the one which sends every x_i to itself, make exp d undefined in (e.g.) the points x_i ; but here the situation can be rectified by taking the field F to be suitably restricted, as we shall do in section 2.

Everything becomes much more tractable if we consider only continuous derivations of order ≥ 2 in $\hat{R}$, because such a derivation d increases the order of every element to which it is applied. Consequently, for every $f \in \hat{R}$ the sequence $(\frac{1}{k!} f^{d^k}$; $k = 0,1,2,\ldots..)$ is summable in the order filtration topology (by Cauchy's criterion), because for every $n \geq 0$ there exists an $m \geq 0$ such that $\delta(f^{d^k}) > n$ for all $k > m$. In fact this is true in any complete inversely filtered F-algebra - see proposition 3.2 below.

1. Order-increasing derivations.

Let S be a complete inversely filtered F-algebra with filtration $\hat{v}$. We call a derivation d in S <u>order-increasing</u>, if $\hat{v}(f^d) \geq \hat{v}(f) + 1$ for every $f \in S$.

Note that every order-increasing derivation is necessarily continuous because it maps every neighbourhood of zero into itself.

The following fact about exp d, which is partly a consequence of the Leibniz-formula for the powers of a derivation, is well known.

Lemma 3.1: If d is a derivation in an F-algebra R such that exp d is defined throughout R, then exp d is an automorphism of R.

Proof: Exp d is an F-linear mapping, because for each $n \geq 0$ the mapping $\frac{1}{n!}d^n$ is F-linear. It is an endomorphism, because for arbitrary $a, b \in R$

$$\begin{aligned}(ab)^{\exp d} &= \sum_{i=0}^{\infty} \frac{1}{i!}(ab)^{d^i} \\ &= \sum_{i=0}^{\infty} \frac{1}{i!}\Big(\sum_{j=0}^{i} \binom{i}{j} a^{d^j} b^{d^{i-j}}\Big) \\ &= \sum_{i=0}^{\infty} \sum_{r+s=i} \Big(\frac{1}{r!}a^{d^r}\Big)\Big(\frac{1}{s!}b^{d^s}\Big) \\ &= \Big(\sum_{r=0}^{\infty} \frac{1}{r!}a^{d^r}\Big)\Big(\sum_{s=0}^{\infty} \frac{1}{s!}b^{d^s}\Big) \\ &= a^{\exp d}\, b^{\exp d} \quad .\end{aligned}$$

Finally, exp d has an inverse, viz. exp(-d), as can be verified by a straightforward calculation. Hence exp d is an automorphism of R. //

<u>Proposition 3.2</u>: Let S be a complete inversely filtered F-algebra. If d is an order-increasing derivation in S, then exp d is a continuous automorphism which maps every nonzero element of S onto itself plus an element of higher order.

<u>Proof</u>: For every $a \in S$ the sequence $(\frac{1}{n!}a^{d^n})$ is summable, with sum $a^{\exp d}$, because the fact that d is order-increasing implies for every $m \geq 0$ that there exists an integer $t \geq 0$ such that $\hat{v}(\frac{1}{k!}a^{d^k}) > m$ for all $k > t$. Hence exp d is defined throughout S, and by lemma 3.1 it is an automorphism. Clearly it maps every nonzero element of S onto itself plus an element of higher order. //

If the derivation d is order-increasing and b is an arbitrary element in S it is not only true that $\hat{v}(b^d) \leq \hat{v}(b^{d^2}) \leq \ldots\ldots$, where equality holds for any $m \geq 1$ only if $b^{d^m} = 0$, but we also have for any sequence $\{\mu_i\}$ of nonzero scalars that

$$\hat{v}(\mu_1 b^d) \leq \hat{v}(\mu_2 b^{d^2}) \leq \ldots\ldots \qquad (2)$$

where it is again true that the equality holds for any $m \geq 1$ only if $b^{d^m} = 0$. This will allow us to determine the fixed ring of exp d .

Lemma 3.3: Let S be a complete inversely filtered F-algebra, and let d be an order-increasing derivation in S. If $a \in S$ is such that the sequence $(\lambda_n a^{d^n})_{n \geq 1}$ is summable, with sum zero, for some sequence $\{\lambda_n\}$ of nonzero scalars, then $a \in \operatorname{Ker} d$.

Proof: If $a \notin \operatorname{Ker} d$, then $\hat{v}(a^d) = k < \infty$, and hence also $\hat{v}(\lambda_1 a^d) = k$. Then by (2)

$$\hat{v}\left(\sum_{n=1}^{m} \lambda_n a^{d^n}\right) = k \quad \text{for all } m \geq 2 . \tag{3}$$

On the other hand, since the sequence $(\lambda_n a^{d^n})$ is summable, with sum zero, there exists an index $t > 1$ such that $\hat{v}\left(\sum_{n=1}^{m} \lambda_n a^{d^n}\right) > k$ for all $m > t$, which contradicts (3). Hence $a \in \operatorname{Ker} d$. //

Proposition 3.4: If d is an order-increasing derivation in a complete inversely filtered F-algebra S, then the fixed ring, Fix α, of the automorphism $\alpha = \exp d$ equals the kernel of d.

Proof: Clearly $\operatorname{Ker} d \subseteq \operatorname{Fix} \alpha$. If $a \in \operatorname{Fix} \alpha$, then the summable sequence $(\frac{1}{n!} a^{d^n} ;\ n \geq 0)$ has sum a, and therefore the subsequence $(\frac{1}{n!} a^{d^n} ;\ n \geq 1)$ has sum zero. Hence by lemma 3.3 $a \in \operatorname{Ker} d$. Hence $\operatorname{Fix}(\exp d) = \operatorname{Ker} d$. //

Definition: Call an automorphism α in S exponentially dependent on a derivation if there exists a derivation d in S such that $\alpha = \exp d$.

Now some of our results on kernels of derivations in S can be applied to the fixed rings of automorphisms in S:

Theorem 3.5: a) In a complete inversely filtered F-algebra S satisfying IWA_n , the fixed ring of any automorphism α which is exponentially dependent on an order-increasing derivation, is an n-fir. Moreover if S is a local ring, then so is Fix α .

b) Take S and α as in a). If S satisfies IWA, it is a flat right Fix α -module.

Proof: This follows immediately from proposition 3.4, corollaries 2.3 & 2.5 and proposition 2.6. //

2. Homogeneous derivations of order one in $\mathbb{C} \langle\langle x_1, \ldots, x_q \rangle\rangle$.

On page 75 we mentioned an example of a derivation d of order 1 for which exp d could only be a proper mapping in the free power series ring if the base field had the appropriate properties. So to ensure that this will be the case, the base field is now taken to be the field of complex numbers, $\mathbb{C}$.

We discuss only homogeneous derivations of order 1. The situation for arbitrary continuous derivations of order 1 is more difficult, and it will not be pursued here.

If h is any monomial in $\hat{R} = \mathbb{C} \ll x_1,\dots,x_q \gg$, let $\partial_{x_i}(h)$ denote the degree in x_i of h .

Lemma 3.6: If d is the homogeneous derivation in $\hat{R} = \mathbb{C} \ll x_1,\dots,x_q \gg$ given by

$$x_i^{\,d} = \lambda_i x_i \quad , \qquad \lambda_i \in \mathbb{C} \quad , \qquad i = 1,\dots,q \; ,$$

then the mapping exp d is an automorphism in $\hat{R}$, and Ker d = Fix(exp d).

Proof: This derivation d has the property that its action on any monomial h amounts only to a change of the coefficient, in fact

$$h^d = \Big(\sum_{i=1}^{q} \partial_{x_i}(h)\, \lambda_i \Big) h$$

$$= \sigma_h h \quad \text{(say)},$$

and hence for every $n \geq 1$

$$h^{d^n} = \sigma_h^{\,n}\, h \; ,$$

whence

$$h^{\exp d} = \Big(\sum_{k=0}^{\infty} \frac{1}{k!}\, \sigma_h^{\,k} \Big) h = (e^{\sigma_h}\, h) \in \hat{R} \, .$$

By linearity $f^{\exp d} \in \hat{R}$, for every $f \in \hat{R}$, and by lemma 3.1 exp d is then an automorphism of $\hat{R}$. Since an arbitrary element $a \in \hat{R}$, written as a sum of monomials

$$a = \mu_0 + \sum_{i=1}^{\infty} \sum_{j=1}^{m_i} \mu_{ij} h_{ij}$$

is mapped by exp d to

$$a^{\exp d} = \sum_{i=1}^{\infty} \sum_{j=1}^{m_i} \mu_{ij} e^{\sigma_{ij}} h_{ij} + \mu_0 \qquad (1)$$

where $\sigma_{ij} = \sigma_{h_{ij}}$, we have that

$$a \in \mathrm{Fix}(\exp d) \implies \sigma_{ij} = 0$$

for every i,j in (1) , and then also

$$a^d = \sum_{i=1}^{\infty} \sum_{j=1}^{m_i} \mu_{ij}\, \sigma_{ij} h_{ij} = 0.$$

Hence Ker d = Fix(exp d). //

The following lemma has been used for establishing part ii) of proposition 2.14 in which we gave an explicit description of the fixed ring of an elementary automorphism in $F\langle\langle x,y \rangle\rangle$. For this application the base field can be F, because the derivation is already given in the form (2) (see below).

Call a derivation d in an algebra A <u>locally nilpotent</u> if there exists for every $a \in A$ an $n \geq 0$ such that $a^{d^n} = 0$.

3.2

<u>Lemma 3.7</u>: Let $\hat{R} = \mathbb{C}\langle\langle X \rangle\rangle$ and $R = \mathbb{C}\langle X \rangle$, where $X = \{x_1, \ldots, x_q\}$. If d, any homogeneous derivation of order 1 in $\hat{R}$, is locally nilpotent on R, then the mapping exp d is an automorphism in $\hat{R}$, and Ker d = Fix(exp d).

<u>Proof</u>: The homogeneous derivation d in $\hat{R}$ is by definition continuous, and hence uniquely determined by its values on $\{x_1, \ldots, x_q\}$. Consequently, d is determined by a $q \times q$ matrix over $\mathbb{C}$ which is an algebraically closed field, and we may therefore assume this matrix to be in the (<u>lower</u> triangular) Jordan canonical form. Furthermore, since d is locally nilpotent on R, the matrix has only zeros along its main diagonal. Write

$$x_i^{\,d} = \begin{cases} 0 & \text{if } i = 1 \\ \lambda_{i1}x_1 + \ldots + \lambda_{i,i-1}x_{i-1} & \text{if } i = 2,3,\ldots,q \end{cases} \tag{2}$$

On the other hand, every homogeneous derivation of order 1 of this form is locally nilpotent on R. Now, if g is any homogeneous element in $\hat{R}$ the formal sum $\sum_{i=1}^{\infty} \frac{1}{i!} g^{d^i}$ (if it is nonzero) has almost all its terms equal to zero, and it is a homogeneous element of degree equal to deg g. By linearity it follows that exp d is a well-defined mapping in $\hat{R}$, and by lemma 3.1 it is then an automorphism in $\hat{R}$.

Finally, Fix(exp d) = Ker d : It is enough to consider homogeneous elements, so let a be a homogeneous element of degree k, and let $m(a) + 1 = \min\{ n \in \mathbb{Z}^+ \mid a^{d^n} = 0 \}$. Taking $a \in \mathrm{Fix}(\exp d)$ implies that

$$a^d + \frac{1}{2!} a^{d^2} + \ldots + \frac{1}{m(a)!} a^{d^{m(a)}} = 0 . \qquad (3)$$

Order all homogeneous elements of degree k lexicographically (based on $x_1 < \ldots < x_q$) by taking the <u>largest</u> term in each homogeneous element to be the leading term. Note that for each homogeneous element of degree k we have then $g^d < g$ in the chosen ordering. This implies that the leading term of a^d in (3) is zero, which is only possible if $a^d = 0$. Hence $a \in \mathrm{Ker}\, d$, and the assertion follows. //

<u>Proposition 3.8</u>: For every homogeneous derivation d of order 1 in $\hat{R} = \mathbb{C} \langle\langle x_1, \ldots, x_q \rangle\rangle$ the mapping exp d is an automorphism, and Ker d = Fix(exp d).

<u>Proof</u>: As in the preceding lemma it is sufficient to consider a d (completely determined by its values on X) which is given by a lower triangular matrix in the Jordan canonical form. We write it simply as

$$x_i^d = \begin{cases} \lambda_{ii} x_i , & \text{if } i = 1 \\ \lambda_{i,i-1} x_{i-1} + \lambda_{ii} x_i , & \text{if } i = 2, \ldots, q. \end{cases}$$

Decompose d into d = d' + d", where d' and d" are respectively the linear derivations given by

3.2

$$x_i^{d'} = \lambda_{ii} x_i \quad ; \quad i = 1,\dots,q \quad , \text{ and}$$

$$x_i^{d''} = \begin{cases} 0, \text{ if } i = 1 \\ \lambda_{i,i-1} x_{i-1} \quad , \quad \text{if } i = 2,\dots,q \;. \end{cases}$$

By the preceding two lemmas both exp d' and exp d" are automorphisms in $\hat{R}$. Furthermore it is a consequence of the special nature of the Jordan canonical form that $d'd'' - d''d' = 0$. Hence exp d' exp d" = exp(d'+d"), and therefore exp(d'+d") is an automorphism.

Clearly Ker d $\subseteq$ Fix(exp d). For the converse, note that if we can show that

Fix(exp d) = Fix(exp d') $\cap$ Fix(exp d") , then

Fix(exp d) = Ker d' $\cap$ Ker d" $\subseteq$ Ker d ,

and the proof will be completed.

An endomorphism β in an F-vector space V is said to be semi-simple if every subspace of V which is mapped into itself by β is a direct summand of V [7;p.66]. Let V_n be the finite dimensional F-space consisting of the homogeneous elements of degree n in $\hat{R}$, and note that exp d' , exp d", and exp d restrict to (vector space) automorphisms of V_n . Denote them respectively by α_n' , α_n'' , and α_n . Then we have for each n that

$$\alpha_n = \alpha_n' \alpha_n'' = \alpha_n'' \alpha_n' \quad , \qquad (4)$$

where α_n' is semi-simple and $\alpha_n'' - 1$ is nilpotent, and hence we can apply theorem 18 of [7;Ch.2] to see that both α_n'

and α_n'' are polynomials in α_n with coefficients in F. This implies that every element of V_n which is fixed by α_n is also fixed by both α_n' and α_n''. Conversely, (4) shows that every element which is fixed by both α_n' and α_n'' is also fixed by α_n. Since this is true for each n, we get that

$$\mathrm{Fix}(\exp d) = \mathrm{Fix}(\exp d') \cap \mathrm{Fix}(\exp d''). \qquad //$$

3. Automorphisms exponentially dependent on order-increasing derivations.

Returning to an arbitrary complete inversely filtered F-algebra S, we describe a class of continuous automorphisms which is exponentially dependent on order-increasing derivations. Formally speaking, it is reasonable to think that if $\alpha = \exp d$, then $d = \log \alpha$, i.e.

$$d = \beta - \tfrac{1}{2}\beta^2 + \tfrac{1}{3}\beta^3 - \tfrac{1}{4}\beta^4 + \ldots\ldots$$

where $\beta = \alpha - 1$. So if we can prove for some given automorphism α that the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (\alpha - 1)^n$$

represents a derivation in S, we will have found a derivation d' with the property that $\alpha = \exp d'$. Since d' will be order-increasing if $\beta = \alpha - 1$ is an order-increasing mapping, we restrict consideration to continuous automorphisms of S which map every nonzero element of S onto itself plus an element of higher order.

For any given automorphism α in S, $\beta = \alpha - 1$ is an α-derivation (i.e. $(1,\alpha)$-derivation), because the difference $\alpha - \gamma$, of any two automorphisms in an F-algebra is a (γ,α)-derivation; as can be seen by noting that it is F-linear, and that for any $a,b \in S$

$$\begin{aligned}(ab)^{\alpha-\gamma} &= a^{\alpha} b^{\alpha} - a^{\gamma} b^{\gamma} \\ &= a^{\alpha}(b^{\alpha} - b^{\gamma}) + (a^{\alpha} - a^{\gamma})b^{\gamma} \\ &= a^{\alpha-\gamma} b^{\gamma} + a^{\alpha} b^{\alpha-\gamma} \qquad (1)\end{aligned}$$

In (1), take $\gamma = 1$, then we get

$$\begin{aligned}(ab)^{\beta} &= a^{\alpha} b^{\beta} + a^{\beta} b \\ &= a^{\beta+1} b^{\beta} + a^{\beta} b \\ &= a^{\beta} b + ab^{\beta} + a^{\beta} b^{\beta} \quad . \qquad (2)\end{aligned}$$

We extend (2) by induction to higher powers of β :

Lemma 3.9: Let T be any F-algebra. If β is an F-linear mapping in T such that (2) holds, then for all $n \geq 1$ and any $a,b \in T$

$$(ab)^{\beta^n} = \sum_{r=0}^{n} \sum_{i=r}^{n} \binom{n}{r}\binom{n-r}{i-r} a^{\beta^{n+r-i}} b^{\beta^i} \quad . \qquad (3)$$

3.3

<u>Proof</u>: Consider $(ab)^{\beta^{n+1}} = [(ab)^{\beta^{n}}]^{\beta}$ and use (3) & (2) to write this as

$$(ab)\beta^{n+1} = \sum_{r=0}^{n}\sum_{i=r}^{n}\binom{n}{r}\binom{n-r}{i-r}[a\beta^{n+r-i+1}b\beta^{i} + a\beta^{n+r-i}b\beta^{i+1} + a\beta^{n+r-i+1}b\beta^{i+1}]$$

$$= [\sum_{r=0}^{n}\sum_{i=r}^{n}\binom{n}{r}\binom{n-r}{i-r} + \sum_{r=0}^{n}\sum_{i=r+1}^{n+1}\binom{n}{r}\binom{n-r}{i-r-1} + \sum_{r=1}^{n+1}\sum_{i=r}^{n+1}\binom{n}{r-1}\binom{n+1-r}{i-r}]\, a\beta^{n+r-i+1}\, b\beta^{i}$$

$$= \binom{n}{0}a\beta^{n+1}b + \sum_{i=1}^{n}\binom{n}{i}a\beta^{n-i+1}b\beta^{i} + \sum_{r=1}^{n}\binom{n}{r}a\beta^{n+1}b\beta^{r}$$
$$+ \sum_{i=1}^{n}\binom{n}{i-1}a\beta^{n-i+1}b\beta^{i} + \binom{n}{n}a\, b\beta^{n+1} + \sum_{r=1}^{n}\binom{n}{r-1}a\beta^{n+1}b\beta^{r}$$
$$+ \sum_{r=1}^{n}\sum_{i=r+1}^{n}[\binom{n}{r}\binom{n-r}{i-r} + \binom{n}{r}\binom{n-r}{i-r-1} + \binom{n}{r-1}\binom{n-r+1}{i-r}]\, a\beta^{n+r-i+1}\, b\beta^{i}$$
$$+ \sum_{r=1}^{n}\binom{n}{r}a\beta^{r}\, b\beta^{n+1} + \sum_{r=1}^{n}\binom{n}{r-1}a\beta^{r}\, b\beta^{n+1}$$
$$+ a\beta^{n+1}\, b\beta^{n+1}$$

Rewriting this with the aid of the formula $\binom{m}{p}+\binom{m}{p-1}=\binom{m+1}{p}$ gives

$$(ab)\beta^{n+1} = \sum_{i=0}^{n+1}\binom{n+1}{0}\binom{n+1}{i}a\beta^{n+1-i}\, b\beta^{i}$$
$$+ \sum_{r=1}^{n}\sum_{i=r}^{n+1}\binom{n+1}{r}\binom{n+1-r}{i-r}a\beta^{n+1+r-i}\, b\beta^{i}$$
$$+ \binom{n+1}{n+1}\binom{0}{0}a\beta^{n+1}\, b\beta^{n+1}$$

$$= \sum_{r=0}^{n+1}\sum_{i=r}^{n+1}\binom{n+1}{r}\binom{n+1-r}{i-r}a\beta^{n+1+r-i}\, b\beta^{i}$$

Hence (3) holds for all $n \geq 1$. //

3.3

The proof of lemma 3.11 below requires some identities which we establish beforehand:

Lemma 3.10: The following identities hold in the ring of integers : For every $s \geq 1$ let $t = \frac{s}{2}$ if s is even, and $t = \frac{s-1}{2}$ if s is uneven;

and let $< s,i,k > = \frac{(-1)^{k+1}}{k} \binom{k}{s-k} \binom{2k-s}{i-s+k}$.

If $0 \leq i \leq t$, then $\sum_{k=s-i}^{s} < s,i,k > = \begin{cases} (-1)^{s+1}(\frac{1}{s}) & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$

and if $t+1 \leq i \leq s$, $\sum_{k=i}^{s} < s,i,k > = \begin{cases} (-1)^{s+1}(\frac{1}{s}) & \text{if } i = s \\ 0 & \text{otherwise.} \end{cases}$

Proof: Consider the free commutative power series ring $\mathbb{Z}[[x,y]]$. If we apply the logarithmic mapping to the element $1 + x + y + xy = (1+x)(1+y)$ we get

$$\log(1+x+y+xy) = \log(1+x) + \log(1+y)$$

which is just a convenient way of stating that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (x+y+xy)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (x^k+y^k) \qquad (4)$$

By an induction proof, which is exactly analogous to the one given in lemma 3.9 we see that for every $k \geq 1$

$$(x+y+xy)^k = \sum_{r=0}^{k} \sum_{i=r}^{k} \binom{k}{r}\binom{k-r}{i-r} x^{k+r-i} y^i .$$

Substitute this in (4) , then

$$\sum_{k=1}^{\infty}\sum_{r=0}^{k}\sum_{i=r}^{k}\frac{(-1)^{k+1}}{k}\binom{k}{r}\binom{k-r}{i-r}x^{k+r-i}y^{i}$$

$$=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}(x^k+y^k) \qquad (5)$$

In (5), equate homogeneous components of degree s, then

$$\sum_{k=s-t}^{s}\sum_{i=s-k}^{k}\frac{(-1)^{k+1}}{k}\binom{k}{s-k}\binom{2k-s}{i-s+k}x^{s-i}y^{i}=\frac{(-1)^{s+1}}{s}(x^s+y^s)$$

i.e. $\left[\sum_{i=0}^{t}\left[\sum_{k=s-i}^{s}\frac{(-1)^{k+1}}{k}\binom{k}{s-k}\binom{2k-s}{i-s+k}\right]x^{s-i}y^{i}\right]$

$$+\sum_{i=t+1}^{s}\left[\sum_{k=i}^{s}\frac{(-1)^{k+1}}{k}\binom{k}{s-k}\binom{2k-s}{i-s+k}\right]x^{s-i}y^{i}=\frac{(-1)^{s+1}}{s}(x^s+y^s).$$

The lemma follows immediately on equating coefficients of $x^m\ y^i$ in this last identity. //

<u>Lemma 3.11</u>: Let S be a complete inversely filtered F-algebra. If β is an F-linear mapping in S such that both

$$(ab)^{\beta}=a^{\beta}b+ab^{\beta}+a^{\beta}b^{\beta} \quad , \text{ any } a,b\in S \qquad (6)$$

and the mapping $d=\beta-\frac{1}{2}\beta^2+\frac{1}{3}\beta^3\ .....$ is defined everywhere in S, then this d is a derivation in S.

<u>Proof</u>: By lemma 3.9 it is true for arbitrary $a,b\in S$ that

$$(ab)^d=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}(ab)^{\beta^k}$$

$$=\sum_{k=1}^{\infty}\sum_{r=0}^{k}\sum_{i=r}^{k}\frac{(-1)^{k+1}}{k}\binom{k}{r}\binom{k-r}{i-r}a^{\beta^{k+r-i}}b^{\beta^i} \qquad (7)$$

Now use the knowledge gained in rewriting the summations in the proof of lemma 3.10 to rewrite (7) in an exactly analogous way as

3.3

$$(ab)^d = \sum_{s=1}^{\infty} \Big[\sum_{i=0}^{t} \Big(\sum_{k=s-i}^{s} < s,i,k > \Big) a\beta^{s-i} b\beta^{i} + \sum_{i=t+1}^{s} \Big(\sum_{k=i}^{s} < s,i,k > \Big) a\beta^{s-i} b\beta^{i} \Big]$$

where again $< s,i,k > = \frac{(-1)^{k+1}}{k} \binom{k}{s-k} \binom{2k-s}{i-s+k}$.

Then by lemma 3.10

$$(ab)^d = \sum_{s=1}^{\infty} \frac{(-1)^{s+1}}{s} [a^{\beta^s} b + ab^{\beta^s}] = a^d b + ab^d$$

Hence d is a derivation in S. //

Proposition 3.12: Let S be a complete inversely filtered F-algebra. If the continuous automorphism α maps every element of S onto itself plus an element of higher order, then $d = \log \alpha$ is a derivation in S such that $\alpha = \exp d$.

Proof: Put $\alpha = 1 + \beta$, then β satisfies (6) (See the discussion leading up to (2) above.) Also,

$$d = \log \alpha = \sum_{k=1}^{\infty} (-1)^{k+1} (\tfrac{1}{k}) \beta^k$$

is a well defined mapping of S, because β is an order-increasing mapping and therefore the sequence $(\frac{(-1)^{k+1}}{k} a^{\beta^k})$ is summable for every $a \in S$. Hence by lemma 3.11 the mapping d is a derivation which is also order-increasing. Furthermore $\alpha = \exp d$, as can be seen by a direct calculation. //

Propositions 3.2 and 3.12 combine to give a characterization of continuous automorphisms which are exponentially dependent on order-increasing derivations.

Theorem 3.13: Let S be a complete inversely filtered F-algebra. A continuous automorphism α in S is exponentially dependent on an order-increasing derivation d in S, if and only if α maps every nonzero element of S onto itself plus an element of higher order. //

We also get the following corollary to theorem 3.5(a)

Corollary 3.14: If S is a complete inversely filtered F-algebra satisfying IWA_n , then the fixed ring of any continuous automorphism which maps every element of S onto itself plus an element of higher order, is also an n-fir. //

It is interesting to compare this corollary with a recent result of G.M.Bergman [3] on the fixed rings of endomorphisms in filtered rings with IWA_2 :

> If R is a ring satisfying IWA_2 with respect to some filtration, then the fixed ring R' of any semigroup of ring endomorphisms of R is still a 2-fir, and two elements of R' right commensurable in R are right commensurable in R' .

[Two elements $a,b \in R$ are right commensurable if $aR \cap bR \neq 0$.]

Bergman could not extend the proof of this theorem to higher n-fir conditions because the elementary operations by which linearly dependent n-tuples are brought to standard forms are not uniquely determined in these cases, but note that the theorem covers all endomorphisms of the rings under consideration.

In the case of complete inversely filtered rings it is possible to handle higher n-fir or even semifir conditions, as we have done above, but we could only prove the corresponding result for a restricted class of endomorphisms. It should be pointed out, however, that this restriction was brought about by the particular "exponential" connection which we established between derivations and automorphisms, and not by properties of the endomorphisms themselves.

For free power series rings S.Andreadakis [1] established another connection between continuous endomorphisms and continuous derivations by showing that for every given continuous endomorphism β of $\hat{Q} = \mathbb{Q} \ll x_1, \ldots, x_q \gg$ which sends the free generators to $x_i + f_i$, where each $f_i \in \hat{Q}$ has order ≥ 1, it is possible to define an infinite sequence $D_o = 1, D_1, D_2, \ldots$ of Q-linear mappings in $\hat{Q}$ in such a way that

$$\beta = \sum_{n=0}^{\infty} \frac{1}{n!} D_n \qquad (8)$$

D_1 is in particular the continuous derivation in $\hat{Q}$ given by $x_i \mapsto f_i$, $i = 1,2,\ldots,q$, and the other D_n $(n \geq 2)$ are differential operators of higher order with such intricate definitions that it is not worthwhile to describe them here in greater detail, because (8) does not connect β with a single derivation in such a way that we could have used it to prove e.g. that the fixed ring of β is a semifir.

Now let $\hat{R} = F\langle\langle x_1,\ldots,x_q \rangle\rangle$ and let α be a continuous automorphism which maps every nonzero element of $\hat{R}$ onto itself plus an element of higher order. Then, in particular $x_i^{\alpha} = x_i + f_i$, where $\delta(f_i) \geq 2$ $(i = 1,\ldots,q)$, and α is completely determined once the elements f_i are known. On the other hand every continuous automorphism α in $\hat{R}$ whose values on $X = \{x_1,\ldots,x_q\}$ are of the form $x_i^{\alpha} = x_i + g_i$, where $\delta(g_i) \geq 2$ $(i = 1,\ldots,q)$, maps each element of $\hat{R}$ onto itself plus an element of higher order, and hence it is exponentially dependent on an order-increasing derivation. By proposition 3.4 we then get the following corollary to theorem 2.7 :

<u>Theorem 3.15</u>: Let $\hat{R} = F\langle\langle X \rangle\rangle$, $X = \{x_1,\ldots,x_q\}$, and let α be a continuous automorphism whose values on X are $x_i^{\alpha} = x_i + f_i\,(x_1,\ldots,x_q)$, where $\delta(f_i) \geq 2$, $i = 1,\ldots,q$, then

i) Fix α is a semifir

ii) Fix α is a rigid UFD

iii) $\hat{R}$ is a flat right Fix α -module. //

4. Linear automorphisms in $\mathbb{C}\langle\langle x_1,\ldots,x_q \rangle\rangle$.

A continuous automorphism α in a free power series ring $F\langle\langle X \rangle\rangle$ is called linear if each x^α is an F-linear combination of a finite number of x's in X.

For the free power series ring $\mathbb{C}\langle\langle x_1,\ldots,x_q \rangle\rangle$ in particular we can show that the class of automorphisms which are exponentially dependent on derivations, includes not only those automorphisms which map every x_i onto itself plus an element of higher order, but also all linear automorphisms. This implies that the fixed ring of every linear automorphism in this ring is again a free power series ring over $\mathbb{C}$ (Cor.3.19)

Lemma 3.16: If α is the continuous automorphism in $\hat{R} = \mathbb{C}\langle\langle x_1,\ldots,x_q \rangle\rangle$ given by

$$x_i^\alpha = \lambda_i x_i \quad , \quad \lambda_i \in \mathbb{C} \quad , \quad i = 1,\ldots,q \quad ,$$

then the mapping $d = \log\alpha$ is a derivation in $\hat{R}$.

Proof: Let $\alpha = 1 + \gamma$, then $x_i^\gamma = (\lambda_i - 1)x_i = \nu_i x_i$ (say), and γ is a $\mathbb{C}$-linear mapping which satisfies (6) of section 3) above. If $h = x_{i_1}\cdot\ldots\cdot x_{i_n}$ is an arbitrary monomial in $\hat{R}$, let $I = \{1,\ldots,n\}$, and note that if we extend the action of γ on products of elements (as given in (6)) by induction on n, we find

3.4

$$h^{\gamma} = \sum_{r=1}^{n} \sum_{\substack{(j1,..,jr)\in I^r \\ j1<\ldots<jr}} x_{i_1}\ldots x_{i_{j_1}}^{\gamma}\ldots x_{i_{j_r}}^{\gamma}\ldots x_{i_n} \quad (1)$$

$$= [\sum_{r=1}^{n} \sum_{\substack{(j1,\ldots,jr)\in I^r \\ j1<\ldots<jr}} \nu_{i_{j1}} \ldots \nu_{i_{jr}}]\, h$$

$$= \tau_h \, h \quad \text{(say)}$$

and $h^{\gamma^k} = \tau_h^k \, h$, each $k \geq 1$.

Hence $h^d = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} h^{\gamma^k}$

$$= (\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \tau_h^k) \, h$$

$$= [(\log \tau_h)\, h] \in \hat{R}.$$

By linearity d is a well-defined mapping of $\hat{R}$, and hence by lemma 3.11 it is a derivation. //

<u>Lemma 3.17</u>: If α is the continuous automorphism in $\hat{R}$ given by

$$x_i^{\alpha} = \begin{cases} x_i & \text{if } i = 1 \\ \lambda_{i1}x_1 + \ldots + \lambda_{i,i-1}x_{i-1} + x_i & \text{if } i = 2,\ldots,q, \end{cases}$$

then the mapping $d = \log\alpha$ is a derivation in $\hat{R}$.

<u>Proof</u>: Let $\alpha = 1 + \gamma$, then

$$x_i^{\gamma} = \begin{cases} 0 & \text{if } i = 1 \\ \lambda_{i1}x_1 + \ldots + \lambda_{i,i-1}x_{i-1} , & \text{if } i = 2,\ldots,q. \end{cases} \quad (2)$$

Order the homogeneous elements of degree n in $\hat{R}$ lexicographically (on $x_1 < \dots < x_q$) by taking the <u>largest</u> term in each element to be its leading term. For every monomial $h = x_{i_1} \dots x_{i_n}$ in $\hat{R}$ we have as in (1) that

$$h^{\gamma} = \sum_{r=1}^{n} \sum_{\substack{(j1,..,jr)\in I^r \\ j1< \dots < jr}} x_{i_1} \dots x_{i_{j1}}^{\gamma} \dots x_{i_{jr}}^{\gamma} \dots x_{i_n} ,$$

$$I = \{1,\dots,n\} .$$

Then by (2) h^{γ} is either zero or homogeneous of degree n and if it is nonzero it precedes h in the lexicographical ordering. since $(x_1{}^n)^{\gamma} = 0$, this implies that an integer $m \geq 1$ can be found such that $h^{\gamma^m} = 0$. Consequently, γ is locally nilpotent on $\mathbb{C} < x_1,\dots,x_q >$. Hence if f is an arbitrary element in $\hat{R}$ and $t \geq 1$ any given integer, it is possible to find an index m such that $\delta(f^{\gamma^r}) > t$ for all $r \geq m$. This shows that the sequence $(\frac{(-1)^{k+1}}{k} f^{\gamma^k}; k = 1,2,\dots)$ is summable, with sum $f^{\log\alpha}$, for every $f \in \hat{R}$. Thus d is a well defined mapping which, by lemma 3.11, is a derivation in $\hat{R}$. //

<u>Proposition 3.18</u>: Every linear automorphism (see p. 94) α in $\hat{R} = \mathbb{C} \ll x_1,\dots,x_q \gg$ is exponentially dependent on a derivation d.

Proof: The automorphism α is completely determined by its values on $x_1, \ldots, x_q$, and each of these values is a $\mathbb{C}$-linear combination of $x_1, \ldots, x_q$. Let A be the coefficient matrix of the x_i^{α}. Since $\mathbb{C}$ is algebraically closed we may assume (by a linear change of the free generating set, if necessary) that A is a lower triangular matrix in the Jordan canonical form. Let D be a diagonal matrix with its diagonal identical to that of A, and let N = A-D. Then N is a lower triangular matrix with zeros along its diagonal. Furthermore the fact that A is in the Jordan canonical form implies that ND = DN, and hence also $D(I+D^{-1}N)=(I+D^{-1}N)D$. Let α' and α'' be respectively the linear automorphisms in $\hat{R}$ with defining matrices D and $I+D^{-1}N$ over $\mathbb{C}$, then $\alpha = \alpha'\alpha'' = \alpha''\alpha'$, and hence $\log\alpha = \log\alpha' + \log\alpha''$. By lemmas 3.16 and 3.17 we know that both $\log\alpha'$ and $\log\alpha''$ are derivations in $\hat{R}$. Hence $\log\alpha = d$ (say) is a derivation. A straightforward calculation shows that $\alpha = \exp d$. //

Corollary 3.19: The fixed ring of any linear automorphism α in $\mathbb{C}\langle\langle x_1, \ldots, x_q \rangle\rangle$ is again a free power series ring over $\mathbb{C}$.

Proof: By the proposition $\alpha = \exp d$ where the derivation $d = \log\alpha$ is also homogeneous of order 1. Hence by prop. 3.8, $\mathrm{Fix}\,\alpha = \mathrm{Ker}\, d$, and the corollary follows from thm.2.11. //

CHAPTER 4.

DERIVATIONS IN FREE ASSOCIATIVE ALGEBRAS.

In Chapter 2 we used the excellent divisibility properties available in complete inversely filtered rings satisfying IWA_n, when we started off with the fact that in such a ring any m-tuple ($m \leq n$) of elements, taken in a suitable order, can be reduced by a <u>special</u> upper triangular matrix to a sequence of right $\hat{v}$-independent elements followed by zeros. For filtered rings R with WA_n, the corresponding reduction of an m-tuple ($m \leq n$) of elements to a sequence of right v-independent elements followed by zeros, can be carried out by a matrix which lies in GE_m (R) (see [2] or [16]) This means that the matrix can be written as a product of elementary matrices, i.e. invertible matrices with not more than one nonzero off-diagonal term; but then we cannot be sure that the matrix is such that its inverse has a 1 in its first row (cf. proof of proposition 1.2). Consequently, we cannot use an analogous argument to obtain corresponding general results for derivations in filtered rings with WA_n.

In free associative algebras we can use techniques which are similar to those employed for studying derivations and their kernels in free power series rings. Consider the derivation $d = d_0 + d_1 + \dots + d_\rho$ (sum of homogeneous derivations)

4.0

in $R = F\langle x_1,\ldots,x_q\rangle$, and let $a = a_r + a_{r+1} + \ldots + a_s$ $(s \geq r)$ (sum of homogeneous components) be an element in Ker d. Since each homogeneous component of a^d equals zero, we have the following set of equations:

$$\begin{aligned}
&a_s^{d_p} = 0\\
&a_s^{d_{p-1}} + a_{s-1}^{d_p} = 0\\
&a_s^{d_{p-2}} + a_{s-1}^{d_{p-1}} + a_{s-2}^{d_p} = 0\\
&\ldots\ldots\ldots\ldots\\
&a_{r+2}^{d_o} + a_{r+1}^{d_1} + a_r^{d_2} = 0\\
&a_{r+1}^{d_o} + a_r^{d_1} = 0\\
&a_r^{d_o} = 0
\end{aligned} \tag{1}$$

Conversely, if one wants to find an element in Ker d starting from a homogeneous element $a_s \in \text{Ker } d_p$, (1) shows that a finite sequence of homogeneous elements $a_{s-1}, a_{s-2}, \ldots\ldots$ has to be found such that $a_s^{d_{p-1}} = -a_{s-1}^{d_p}$, $a_s^{d_{p-2}} + a_{s-1}^{d_{p-1}} = -a_{s-2}^{d_p}$, etc. However, finding appropriate inverse images under an arbitrary homogeneous derivation is very difficult if not impossible, and this reasoning will only lead somewhere if we keep to the simplest case in which $d = d_p$ is a homogeneous derivation.

4.1

1. Kernels of homogeneous derivations in $F<X>$.

Let $R = F<X>$. Throughout this section and the next, F is a commutative field of characteristic zero and $X=\{x_1,x_2,..\}$ is a finite or countable free generating set, unless it is stated explicitly that X must be taken to be finite. R is filtered by the natural degree function v on X, and we consider derivations which are v-homogeneous, i.e. they are homogeneous with respect to X in the sense that they map the free generators to elements which are either zero or homogeneous of the same degree in X. We show that the kernels of such derivations satisfy the weak algorithm with respect to v, and consequently they are also free algebras over F.

Lemma 4.1: Let d be a v-homogeneous derivation in $R = F<X>$. If the nonzero homogeneous elements $a_1,a_2,\ldots,a_m$ in Ker d are right linearly dependent over R, and if $v(a_1)\leq \ldots\leq v(a_m)$, then some a_k $(k \leq m)$ is right linearly dependent on $a_1,a_2,\ldots,a_{k-1}$ over Ker d.

Proof: Let $f_1,f_2,\ldots,f_m$ be elements in R such that

$$\sum_{i=1}^{m} a_i f_i = 0 \qquad (2)$$

It is clear that we may take the f_i to be homogeneous.

4.1

By the WA in Gr R, the graded ring associated to R w.r.t. v (see section 1.1), there exist homogeneous elements $g_1, .., g_{r-1}$ $(r \leq m)$ in R such that

$$a_r = \sum_{j=1}^{r-1} a_j \, g_j \qquad (3)$$

If all the elements g_j lie in Ker d there is nothing more to prove, and this is indeed the case if m = 2, for if we apply d to the expression

$$a_2 = a_1 \, g_1$$

we get $\quad 0 = a_1 \, g_1^{\,d}$, whence $g_1^{\,d} = 0$.

Hence, we may assume inductively that the assertion holds for all relations (2) of length less than m. Now apply d to (3), where we assume that $g_j \notin$ Ker d for at least one j, then

$$0 = \sum_{j=1}^{r-1} a_j \, g_j^{\,d} \qquad (4)$$

By the induction assumption some a_k , $k \leq r-1$, is right linearly dependent on $a_1, \ldots, a_{k-1}$ over Ker d. This proves the lemma. //

<u>Proposition 4.2</u>: Let d be a v-homogeneous derivation in $R = F\langle X \rangle$. If the finite set of nonzero elements $b_1, b_2, \ldots, b_m$ in Ker d is right v-dependent <u>over R</u> , then some b_k , $k \leq m$, is right v-dependent on $b_1, \ldots, b_{k-1}$ <u>over Ker d</u>.

Proof: Let the given v-dependence be

$$v\left(\sum_{i=1}^{m} b_i \; f_i \right) < \max_i \{ v(b_i) + v(f_i) \} = n \tag{5}$$

where the f_i's are elements of R. If

$$R_{n-1} = \{ f \in R \mid v(f) \leq n-1 \}$$

and we take (5) modulo R_{n-1} we get a linear dependence

$$\sum_{j=1}^{r} \bar{b}_{i_j} \; \bar{f}_{i_j} = 0$$

where we have denoted the representative of an arbitrary element f modulo R_{n-1} by $\bar{f}$. Now, since d is a homogeneous derivation, each $\bar{b}_{i_j}$ lies in Ker d; hence by lemma 4.1 there exist homogeneous elements $c_1,\ldots,c_{k-1}$ $(k \leq r)$ in Ker d such that

$$\bar{b}_{i_k} = \sum_{j=1}^{k-1} \bar{b}_{i_j} \; c_j$$

from which it follows that

$$\left.\begin{aligned} & v\left(b_{i_k} - \sum_{j=1}^{k-1} b_{i_j} \; c_j\right) < v(b_{i_k}) \\ & \text{and} \\ & v(b_{i_j}) + v(c_j) = v(b_{i_k}), \quad j = 1,\ldots,k-1 \end{aligned}\right\} \tag{6}$$

In (6) we have obtained a right v-dependence of b_{i_k} on $b_{i_1},\ldots,b_{i_{k-1}}$ over Ker d. //

The promised result follows as a corollary to this proposition.

Corollary 4.3: If d is a v-homogeneous derivation in $R = F\langle X \rangle$, then Ker d satisfies WA relative to v. //

Theorem 4.4: The kernel of any homogeneous derivation d in $R = F\langle X \rangle$ is also a free associative algebra over F.

Proof: According to our definition of a homogeneous derivation there exists a free generating set Y and a natural degree function v' relative to Y such that d is v'-homogeneous. By the preceding corollary we can apply the following theorem of P.M.Cohn [9] to establish our result:

> Let A be an algebra over a commutative field F, with degree function v such that for any non-zero $a \in A$, $v(a) = 0$ if and only if $a \in F$. Then A is a free associative algebra over F with a right v-independent free generating set if and only if WA holds in A. //

Remark: In order to find a free generating set for Ker d we can construct a "weak algebra basis" for this algebra in the way described by P.M.Cohn in [17]. We do not reproduce that construction here, but we use it in the next section to describe a free generating set for the kernel of a rather special homogeneous derivation.

4.2

2. Primitive derivations in $F\langle X \rangle$.

We say that an element $y \in R = F\langle X \rangle$ is a primitive element in R if there exists a free generating set Y of R with $y \in Y$. In a free generating set Y of R each primitive element y uniquely determines a derivation in R; just take the derivation in $R = F\langle Y \rangle$ given by the mapping from Y into R which sends $y \mapsto 1$ and $z \mapsto 0$, each $z \in Y - \{y\}$. We call such a derivation in R a primitive derivation and denote it by $\frac{\partial}{\partial y}$.

So, if it is said that a given derivation d in $R = F\langle X \rangle$ is a primitive derivation, it means that for some free generating set Y of R there exists a $y \in Y$ such that $d = \frac{\partial}{\partial y}$. This d is then a homogeneous derivation of degree zero with respect to Y, and hence, by theorem 4.4, Ker d is also a free associative algebra over F. We now describe a free generating set for this algebra.

Let d be the primitive derivation $\frac{\partial}{\partial x_1}$ in $R = F\langle X \rangle$, where $X = \{x_1, x_2, \ldots\ldots\}$. Denote the degree-function relative to X by v, and let $N_n = \text{Ker } d \cap R_n$, $R_n = \{f \in R \mid v(f) \leq n\}$, then the filtration induced in Ker d by v is

$$N_0 = F \subseteq N_1 \subseteq N_2 \subseteq \ldots\ldots\ldots\ldots$$

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Define, for each $n \geq 1$, an F-subspace N_n' of N_n by taking N_n' to be the space spanned by all products ab with $a,b \in R_{n-1}$ $v(a) + v(b) \leq n$. Let B_n be a set of representatives for a basis of the F-space N_n/N_n'. In order to describe such a set B_n explicitly we have to find the homogeneous elements of degree n which lie in Ker d but not in N_n'. Denote commutators of the form $[\ldots[[f,\underbrace{x_1],x_1],\ldots,x_1}_{k\text{-times}}]$ by $[f,x_1^{[k]}]$ and repeat the argument given in the discussion of example 2.12(1) in section 2.2. This shows that every homogeneous element of degree n which lies in Ker d can be written as

$$g = \sum_{i>1} \lambda_i [x_i, x_1^{[n-1]}] \pmod{N_n'},$$

where almost all the terms in the sum are equal to zero. Furthermore, since for each $n > 1$ the commutators $[x_i, x_1^{[n-1]}]$ $(i > 1)$ form a basis of the F-space generated by them, we can take

$$B_1 = \{x_i : i \geq 2\} \text{ and for each } j \geq 2$$

$$B_j = \{[x_i, x_1^{[j-1]}] : i \geq 2\}.$$

Then $B = \bigcup_{j=1}^{\infty} B_j$ is a free generating set of the F-algebra Ker $\frac{\partial}{\partial x_1}$ (see P.M.Cohn [17 ,p.12]) We state the conclusion as proposition 4.5.

Proposition 4.5: If d is the primitive derivation determined by the primitive element $y \in Y$ in $F\langle Y \rangle$, then Ker d is a free subalgebra of $F\langle Y \rangle$, with free generating set $C = \bigcup_{i=1}^{\infty} C_i$, where

$$C_1 = \{z \in Y \mid z \neq y\} \text{ and for every } i \geq 2$$
$$C_i = \{[z, y^{[i-1]}] \mid z \in Y,\ z \neq y\} \text{ .} \qquad //$$

Next we indicate how Ker $\frac{\partial}{\partial x_1}$ lies embedded in $F\langle X \rangle$ by proving that $F\langle X \rangle$ may be regarded as a skew polynomial ring over Ker $\frac{\partial}{\partial x_1}$; but before we can do that we have to introduce another valuation on $F\langle X \rangle$.

Proposition 4.6: If d is any locally nilpotent derivation in an integral domain S (with 1), then the $\mathbb{Z} \cup \{-\infty\}$- valued function w defined on S by

$$w(a) = \begin{cases} k-1 & \text{if } k = \min\{j \in \mathbb{Z}^+ \mid a^{d^j} = 0,\quad a \neq 0\} \\ -\infty & \text{if } a = 0 \end{cases}$$

is a valuation.

Proof: $w(1) = 0$ since $1^d = 0$. Also, for any $a, b \in S$

$$\begin{aligned} w(a-b) &= \min\{j \in \mathbb{Z}^+ \mid (a-b)^{d^j} = 0\} -1 \\ &\leq \min\{\min\{i \in \mathbb{Z}^+ \mid a^{d^i} = 0\} , \\ &\qquad \min\{j \in \mathbb{Z}^+ \mid b^{d^j} = 0\}\} -1 \\ &= \min\{\min\{i \in \mathbb{Z}^+ \mid a^{d^i} = 0\} -1 , \\ &\qquad \min\{j \in \mathbb{Z}^+ \mid b^{d^j} = 0\} -1\} \\ &= \min\{w(a), w(b)\} \quad , \end{aligned}$$

4.2

and $w(ab) = \min\{k \in \mathbb{Z}^+ \mid (ab)^{d^k} = 0\} - 1$, so that by the lemma below we have

$$w(ab) = (\min\{i \in \mathbb{Z}^+ \mid a^{d^i} = 0\} - 1) + (\min\{j \in \mathbb{Z}^+ \mid b^{d^j} = 0\} - 1) = w(a) + w(b) .$$

<u>Lemma</u>: Let S and d be as in the proposition.

If $r = \min\{i \in \mathbb{Z}^+ \mid a^{d^i} = 0\}$

$s = \min\{j \in \mathbb{Z}^+ \mid b^{d^j} = 0\}$

then $\min\{k \in \mathbb{Z}^+ \mid (ab)^{d^k} = 0\} = r + s - 1$.

<u>Proof of lemma</u>: Put $r + s - 1 = t$ and consider $(ab)^{d^t}$.

By the Leibniz-formula

$$(ab)^{d^t} = \sum_{i=0}^{t} \binom{t}{i} a^{d^i} b^{d^{t-i}} ,$$

and since $b^{d^{t-i}} = 0$ if $0 \le i \le r-1$, $a^{d^i} = 0$ if $r \le i \le t$, we have $(ab)^{d^t} = 0$. Now if also $(ab)^{d^{t-1}} = 0$, i.e. if

$$\sum_{j=0}^{t-1} \binom{t-1}{j} a^{d^j} b^{d^{t-j-1}} = 0 \qquad (1)$$

and if we omit all terms in (1) which are zero by virtue of the fact that either

$a^{d^j} = 0$ (which occurs when $r \le j \le t-1$)

or $b^{d^{t-j-1}} = 0$ (which occurs when $0 \le j \le r-2$) ,

then (1) reduces to

$$\binom{t-1}{r-1} a^{d^{r-1}} b^{d^{s-1}} = 0 .$$

Hence either $a^{d^{r-1}} = 0$ or $b^{d^{s-1}} = 0$ which constitutes a contradiction against the minimality of either r or s. Hence $t = \min\{ k \in \mathbb{Z}^+ \mid (ab)^{d^k} = 0 \}$. //

<u>Theorem 4.7</u>: If d is the primitive derivation $\frac{\partial}{\partial x_1}$ in $R = F\langle X \rangle$, $X = \{x_1, x_2, \ldots\}$, then the inner derivation given by $\Delta :\ a \to [a, x_1]$ on R restricts to a derivation Δ_{x_1} in Ker d, and R may be regarded as the skew polynomial ring Ker d $[x_1; 1, \Delta_{x_1}]$.

<u>Proof</u>: To see that Δ_{x_1} is a derivation in Ker d it is sufficient to know that Ker d is closed with respect to the mapping Δ, and this is clear, because for any $b \in$ Ker d we have

$$[b, x_1]^d = b - b = 0 .$$

For the rest of the theorem we start off by showing that R is a free right Ker d-module with basis

$$\{1, x_1, x_1^2, x_1^3, \ldots\} :$$

Since $d = \frac{\partial}{\partial x_1}$ is clearly a locally nilpotent derivation on R we can introduce a valuation w on R in the same way as in proposition 4.6, and use this valuation in an induction argument to show that every element $a \in R$ with $w(a) = n$ can be written uniquely in the form

$$a = a_0 + x_1 a_1 + x_1^2 a_2 + \ldots + x_1^n a_n \qquad (2)$$

where $a_i \in \text{Ker } d$, $i = 0,1,\ldots,n$. This is trivially true for all $a \in R$ such that $w(a) = 0$ (or $-\infty$), i.e. all $a \in \text{Ker } d$. Assume (2) for all $a \in R$ such that $w(a) < n$ and consider an element $b \in R$ with $w(b) = n$. By the definition of w we have $w(b^d) = n-1$, and hence, by the assumption, there exist uniquely determined elements $c_o, c_1, \ldots, c_{n-1} \in \text{Ker } d$ such that

$$b^d = c_o + x_1c_1 + x_1^2c_2 + \ldots\ldots + x_1^{n-1}c_{n-1} .$$

Let $b^* = x_1c_o + \frac{1}{2}x_1^2c_1 + \ldots\ldots + \frac{1}{n}x_1^nc_{n-1}$, then

$$(b^*)^d = c_o + x_1c_1 + \ldots\ldots + x_1^{n-1}c_{n-1} = b^d .$$

Hence $(b-b^*)^d = 0$. Now let $b_o = b-b^*$, $b_1 = c_o$, $b_2 = \frac{1}{2}c_1$, ..., $b_n = \frac{1}{n}c_{n-1}$;

then rewriting $b = b_o + b^*$ we get

$$b = b_o + x_1b_1 + x_1^2b_2 + \ldots\ldots + x_1^n b_n$$

where the b_i's are uniquely determined elements of Ker d. This establishes the claim.

Finally, R may be regarded as the skew polynomial ring $\text{Ker } d\,[x_1; 1, \Delta_{x_1}]$, because each of its elements can be written uniquely in the form (2), and because the multiplication of such elements is completely governed by the commutation rule $gx_1 = x_1g + [g,x_1]$, any $g \in \text{Ker } d$. //

4.2

Now that we have seen how every primitive element $y \in F\langle X \rangle = R$ determines a derivation d such that R may be regarded as a skew polynomial ring over Ker d, it will be interesting to find out how far this property goes towards characterizing y as a primitive element. We attempt to answer this question in the case where R is free of finite rank, i.e. the free generating set X is a finite set, (see p.20) and we find that the property only determines y up to an arbitrary "constant" in Ker d.

<u>Theorem 4.8</u>: Let $R = F\langle X \rangle$, where $X = \{x_1, \ldots, x_q\}$ and $q \geq 2$. There exist a derivation d and an element $z \in R$ such that

i) the mapping $\Delta_z : g \rightarrow [g,z]$ defines a derivation in Ker d , and

ii) R may be regarded as the skew polynomial ring Ker $d[z;1,\Delta_z]$

if and only if $z = y+b$, where y is a primitive element in R, $b \in$ Ker d, and $d = \frac{\partial}{\partial y}$.

<u>Proof</u>: $\Longrightarrow$: Under assumption ii) clearly $z \notin$ Ker d, but for every $g \in$ Ker d, $[g,z^d] = [g,z]^d$, and by i) this is zero. Hence Ker d is a subring of the centralizer C of z^d in R. We want to show that $z^d \in F$. Now if we assume $z^d \notin F$ it follows that this centralizer is a commutative subring of R

(see [11 ,p.349]), and then by ii) R will be a skew polynomial ring over a commutative ring, which implies that R is a right and left Ore domain. This, however, is impossible since R was taken to be free on more than one generator; and therefore $z^d \in F$. Say $z^d = \lambda$ ($\neq 0$).

Furthermore, if we use v to denote the degree function determined by X, we can also see that the derivation d reduces the v-degree of every element to which it is applied. This is immediately clear for all the nonzero elements in Ker d, and since any element $a \notin$ Ker d can be written uniquely as

$$a = a_0 + za_1 + \ldots + z^k a_k \ , \ a_i \in \text{Ker } d \ , \ i=0,1,.,k \qquad (3)$$

we know that

$$v(a) \geq \max_i \{ v(z^i a_i) \ ; \ i = 1,2,\ldots,k \} \ .$$

Now apply d to (3) , then

$$a^d = \lambda a_1 + 2\lambda z a_2 + \ldots + k\lambda z^{k-1} a_k \ ,$$

and hence

$$\begin{aligned} v(a^d) &= \max_i \{ v(z^{i-1} a_i) \ ; \quad i = 1,2,\ldots,k \} \\ &< \max_i \{ v(z^i a_i) \ ; \quad i = 1,2,\ldots,k \} \\ &\leq v(a) \ . \end{aligned}$$

If we apply this knowledge to the free generators, x_j, we see that for each $j = 1,2,\ldots,q$, $v(x_j{}^d) \leq 0$ and hence $x_j{}^d = \eta_j \in F$, where we may assume that (say) $\eta_1 \neq 0$.

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The next step is to change the free generating set linearly to a more suitable one. Consider the elements

$$y_i = \begin{cases} \frac{1}{\eta_1} x_1 & \text{if } i = 1 \\ x_i - \frac{\eta_i}{\eta_1} x_1 & \text{if } i = 2,3,\dots,q \end{cases},$$

and verify that $Y = y_1,\dots,y_q$ is also a free generating set of R. Note that $y_i^d = \delta_{1i}$ (Kronecker delta), $i = 1,\dots,q$.

This makes it abundantly clear that the given d must be a primitive derivation in R. We also know by theorem 4.7 that $R = \text{Ker } d\,[y_1;1, \Delta_{y_1}]$, hence there exist uniquely determined elements $b_o,b_1,\dots,b_r \in \text{Ker } d$ ($b_r \neq 0$, $r \geq 1$) such that

$$z = b_o + y_1 b_1 + \dots + y_1^r b_r . \tag{4}$$

Apply d to (4), then

$$\lambda = b_1 + 2y_1 b_2 + \dots + ry_1^{r-1} b_r ,$$

From which it is clear that $r = 1$ and that $b_1 = \lambda$. Hence by (4) $z = \lambda y_1 + b_o$, and since λy_1 is also a primitive element in R, this proves the necessity of the conditions i) and ii).

$\Longleftarrow$: Consider an arbitrary primitive element y in R, and let d be the derivation $\frac{\partial}{\partial y}$, then we know by theorem 4.7 that $R = \text{Ker } d\,[y;1, \Delta_y]$, where Δ_y is the derivation in Ker d given by $g \mapsto [g,y]$, all $g \in$ Ker d. We now show that if a is an arbitrary element in Ker d, and $z = y+a$, then the mapping Δ_z defined on Ker d by $g \mapsto [g,z]$ is a derivation in Ker d, and R may also be regarded as the skew polynomial ring Ker d $[z;1, \Delta_z]$. Ker d is closed under the mapping Δ_z since (for every $g \in$ Ker d), $[g,z] = [g,y] + [g,a]$, where $[g,y]$ and $[g,a]$ both lie in Ker d. Consequently, Δ_z is a derivation in Ker d, because it is the restriction to Ker d of an inner derivation in R.

The rest of the assertion follows from the fact that the skew polynomial rings

$$M_1 = \text{Ker d } [y;1, \Delta_y] \quad \text{and}$$

$$M_2 = \text{Ker d } [z;1, \Delta_z]$$

are isomorphic as right Ker d-modules : Take $\varphi: M_1 \to M_2$ to be the mapping which sends

$$b = b_o + yb_1 + \ldots\ldots + y^m b_m \in M_1 \quad (b_m \neq 0)$$

to

$$b^\varphi = b_o + (z-a)b_1 + \ldots\ldots + ((z-a)^m)b_m \in M_2 \quad .$$

By repeatedly using the identity

$$az = za + [a,z]$$

it is possible to find for every $i = 1,\ldots,m$ uniquely determined elements $c_{io}, c_{i1}, \ldots, c_{ii} \in$ Ker d such that $(z-a)^i = c_{io} + zc_{i1} + \ldots\ldots + z^i c_{ii}$. (In particular each $c_{ii} = 1$)

4.2

Hence $$b^{\varphi} = \sum_{i=0}^{m} \Big(\sum_{j=0}^{i} z^j c_{ij} \Big) b_i$$
$$= \sum_{j=0}^{m} z^j \Big(\sum_{i=j}^{m} c_{ij} b_i \Big) \,. \qquad (5)$$

It is straightforward to check that φ is a Ker d-module homomorphism. Furthermore Ker $\varphi = 0$, for if $b^{\varphi} = 0$ we see from (5) that for every $j = o,1,\ldots,m$

$$\sum_{i=j}^{m} c_{ij} b_i = 0 \quad . \qquad (6)$$

Taking $j = m$ in (6) gives $c_{mm} b_m = 0$, i.e. $b_m = 0$, which is a contradiction. φ is also a mapping <u>onto</u> M_2 because every element

$$g_o + zg_1 + \ldots\ldots + z^k g_k \in M_2$$

is the image under φ of the element

$$g_o + (y+a)g_1 + \ldots\ldots + (y+a)^k g_k \in M_1 . \qquad //$$

<u>Remark</u>: The case of F-algebras which are free on <u>one</u> generator is completely trivial, because the kernel of every nonzero F-linear derivation d on F[X] is equal to F: Let b be an arbitrary element of Ker d and say

$$b = \lambda_o + \lambda_1 x + \ldots\ldots \lambda_m x^m \,.$$

Then since R is commutative

$$0 = b^d = \lambda_1 x^d + 2\lambda_2 x x^d + \ldots + m\lambda_m x^{m-1} x^d$$
$$= (\lambda_1 + 2\lambda_2 x + \ldots + m\lambda_m x^{m-1}) x^d$$

and since $x^d \neq 0$ we get

$$\lambda_1 = 2\lambda_2 = \ldots\ldots = m\lambda_m = 0.$$

F has characteristic zero, hence $\lambda_1 = \lambda_2 = \ldots = \lambda_m = 0$, and therefore $b = \lambda_0 \in F$. //

The knowledge obtained in theorems 4.5 and 4.7 can of course be applied again to the free algebra which constitutes the kernel of the primitive derivation $\frac{\partial}{\partial x_1}$, givinga repetitive process which will allow us to describe a descending chain of free subalgebras of $R = F\langle X \rangle$, each with the property that R is a free module over it.

Denote the kernels of the primitive derivations $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots$ in $F\langle X \rangle$, $X = \{x_1, x_2, \ldots\}$, respectively by $N_1, N_2, N_3, \ldots$, and let $N^{(n)} = \bigcap_{i=1}^{n} N_i$. Extend the notation $[g, x_i^{[r]}]$, which we introduced at the beginning of this section, to include also the case where $r = 0$ by taking $[g, x_i^{[0]}] = g$.

<u>Theorem 4.9</u>: For every $n \geq 1$ the subalgebra $N^{(n)}$ of $R = F\langle X \rangle$, $X = \{x_1, x_2, \ldots\}$, is a free algebra over F, with a free generating set consisting of the elements x_j $(j \geq n+1)$ and all the left normed commutators of the form $[\ldots[[x_k, x_1^{[r_1]}], x_2^{[r_2]}], \ldots, x_n^{[r_n]}]$, where $k > \min_i \{ r_i \neq 0 \}$ and each $r_i \geq 0$, (at least one $r_i \neq 0$). Furthermore, when R is regarded as a right $N^{(n)}$-module it is free with basis the set of all products of the form $x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$ (each $i_j \geq 0$).

Proof: By proposition 4.5 and theorem 4.7 this theorem is valid if $n = 1$, and hence we may assume inductively that it holds for $n - 1$. Note that when the primitive derivation $\frac{\partial}{\partial x_n}$ in R is restricted to $N^{(n-1)}$ we get exactly the primitive derivation (say ∂_n) determined by the free generator x_n of $N^{(n-1)}$, because $\frac{\partial}{\partial x_n}$ sends

$$x_n \mapsto 1,$$

$$x_j \mapsto 0 \quad (\text{all } j > n)\text{ , and}$$

$$[..[[x_k,x_1{}^{[r_1]}],x_2{}^{[r_2]}],\dots,x_{n-1}{}^{[r_{n-1}]}] \mapsto 0 \ (\text{all } r_i \geq 0).$$

All these commutators go to zero because in the cases where this is not trivially true, i.e. when $k = n$, it follows from the fact that $(x_n)\frac{\partial}{\partial x_n} = 1$ and $(x_t)\frac{\partial}{\partial x_n} = 0$ if $1 \leq t \leq n-1$. Now by proposition 4.5 applied to the free algebra $N^{(n-1)}$ we see that Ker ∂_n, which is the same as $N^{(n-1)} \cap N_n = N^{(n)}$, is a free algebra with a free generating set as described in the formulation of this theorem.

Furthermore, by theorem 4.7, we have then that $N^{(n-1)}$ is a free right $N^{(n)}$-module with basis $\{1,x_n,x_n{}^2,x_n{}^3,\dots\}$, and we know by induction that when R is regarded as right $N^{(n-1)}$-module, it is free on the basis consisting of all products of the form $(x_1{}^{i1}\ x_2{}^{i2}\ \dots\ x_{n-1}{}^{in-1})$. Hence it follows that when R is regarded as right $N^{(n)}$-module, it is free on the basis consisting of all products of the form

$(x_1{}^{i_1}\ x_2{}^{i_2}\ \dots\ x_n{}^{i_n})$. //

4.2

Theorem 4.9 leads directly to the results of G.Falk [19] on the intersection of the kernels of the derivations $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_2}$,........ in a free associative algebra F< X >. It is only necessary to allow n to run through the whole index set of $X = \{x_1, x_2, \ldots\}$ in this theorem to get

Corollary 4.10:(G.Falk): The subalgebra $N = \bigcap_{i=1}^{\infty} \mathrm{Ker}\, \frac{\partial}{\partial x_i}$ of the free associative algebra $F< x_1, x_2, \ldots >$ (where F is a commutative field of characteristic zero) is also a free algebra with free generating set equal to the set of all left normed commutators of the form

$$[\ldots[[x_k, x_1^{[r_1]}], x_2^{[r_2]}], \ldots, x_t^{[r_t]}],$$

where $k > \min_i \{ r_i \neq 0 \}$ and each $r_i \geq 0$, $r_t \neq 0$. Furthermore, when R is regarded as right N-module, it is free on a basis consisting of all the products of the form

$$x_1^{i_1} x_2^{i_2} \ldots x_t^{i_t} \quad \text{where } i_t \neq 0 \text{ unless } t = 1,$$

in which case it can possibly be zero. //

Remark: Corollary 4.10 actually improves Falk's results, because it not only states that N is generated by the left normed commutators mentioned above, but that it is generated by them as a *free* algebra over F.

CHAPTER 5.

THE TRACE OF A DERIVATION IN F< X >.

Up to now we have concerned ourselves almost entirely with the kernels of derivations in free associative algebras which we could take to be of countable rank; however, when we turn to discuss the trace of a derivation, the free algebra has to be of finite rank.

Initially we define the trace of a given derivation relative to one free generating set, say X, and then investigate to what extent this definition is bound to the particular generating set chosen.

1. Defining the trace of a derivation in $F< x_1,\ldots,x_q >$.

Let $R = F < X >$, where F is a commutative field of characteristic zero and $X = \{ x_1,\ldots,x_q \}$. R, regarded as a graded ring, is a direct sum of the finite dimensional F-vector spaces $gr_n(R)$, and an arbitrary derivation d in R induces (for each $n \geq 0$) an F-linear transformation δ_n in $gr_n(R)$. This δ_n corresponds bi-uniquely to a square matrix over F, and as usual we define the trace of δ_n to equal the trace of this matrix.

5.1

Consider d in the same way as before (section 1.4) as a sum of homogeneous derivations $d = d_o + d_1 + \dots + d_p$: The fact that a homogeneous derivation of degree i, when restricted to $gr_n(R)$, can ~~only~~ map $gr_n(R)$ into itself only if $i = 1$, implies that for each $n \geq 1$

$$\delta_n = d_1 \mid gr_n(R) \ . \tag{1}$$

Denote the trace of δ_n by $\tau_x(\delta_n)$.

<u>Proposition 5.1</u>: If d_1 is a homogeneous derivation of degree 1 in $R = F< X >$, then for every $n \geq 1$,

$$\tau_x(d_1 \mid gr_n(R)) = nq^{n-1} \tau_x(d_1 \mid gr_1(R)) \tag{2}$$

<u>Proof</u>: Recall that if $gr_1(R) = V$, then for every $n \geq 1$, $gr_n(R) \cong V \otimes \dots \otimes V$ (n factors), where the tensor products are taken over F. Suppose $\alpha_1, \dots, \alpha_n$ are linear transformations in V. It is well-known that

$$\tau_x(\alpha_1 \otimes \dots \otimes \alpha_n) = \tau_x(\alpha_1) \dots \tau_x(\alpha_n), \tag{3}$$

but for completeness we sketch the proof: Let each α_i be given (relative to the basis $\{x_1, \dots, x_q\}$ of V) by

$x_j^{\alpha_i} = \sum_{r=1}^{q} x_r \lambda_{ijr}$, $j = 1, \dots, q$, then the linear transformation $\alpha_1 \otimes \alpha_2$ in $V \otimes V$ is given by

$$(x_j \otimes x_k)^{\alpha_1 \otimes \alpha_2} = x_j^{\alpha_1} \otimes x_k^{\alpha_2} = \sum_{r,s=1}^{q} (x_r \otimes x_s) \lambda_{1jr} \lambda_{2ks} ,$$

all $j,k = 1, \dots, q$.

5.1

Hence $\tau_x(\alpha_1 \otimes \alpha_2) = \sum_{j,k=1}^{q} \lambda_{1jj} \lambda_{2kk}$

$$= (\sum_{j=1}^{q} \lambda_{1jj})(\sum_{k=1}^{q} \lambda_{2kk})$$

$$= \tau_x(\alpha_1)\ \tau_x(\alpha_2)\ .$$

By induction this extends to (3).

Note that the action of the derivation d_1 on $gr_n(R)$ is mirrored exactly in the action of the linear transformation

$$(\delta_1 \otimes 1 \otimes \ldots \otimes 1) + (1 \otimes \delta_1 \otimes \ldots \otimes 1) + \ldots + (1 \otimes \ldots \otimes 1 \otimes \delta_1) \quad (4)$$

on the F-space $V \otimes \ldots \otimes V$ (n factors).

Let $\rho_i = (1 \otimes \ldots \otimes 1 \otimes \underset{\text{i th position}}{\delta_1} \otimes 1 \otimes \ldots \otimes 1)$, then

since $\tau_x(1_V) = q$, we have by (3) that

$$\tau_x(\rho_i) = q^{n-1}\ \tau_x(\delta_1)\ , \quad i = 1,\ldots,n.$$

Hence $\tau_x(\sum_{i=1}^{n} \rho_i) = nq^{n-1}\ \tau_x(\delta_1)$, and this implies

that $\tau_x(d_1 | gr_n(R)) = nq^{n-1}\ \tau_x(d_1 | gr_1(R))$ //

This proposition shows that the trace of each δ_n is just an integral multiple of the trace of δ_1 , and therefore we can define the trace of the derivation d as follows

Definition: If d is a derivation in $R = F\langle X \rangle$, $X = \{x_1,\ldots,x_q\}$, and if d_1 is the derivation in R given by $x_i^{d_1} = u_i$, where u_i is the homogeneous component of degree one of x_i^d $(i = 1,\ldots,q)$; define the trace of d relative to X by

$$\mathrm{trace}_X(d) = \mathrm{trace}_X(d_1 \mid gr_1(R)) \qquad (5)$$

$\mathrm{Trace}_X(\ )$ then assigns to every derivation d a uniquely determined scalar in F in such a way that it gives an additive mapping from the F-space of derivations of R into F.

2. Effect of a change in the free generating set on the trace of a derivation.

Next we check whether the trace function defined in (5) is independent of the choice of free generating set. Simple examples show that this is not so.

In $F\langle x_1,x_2 \rangle$, let d be the derivation given by

$$x_1^d = x_1 + x_1^2\mu \qquad (\mu \neq 0)$$

$$x_2^d = x_2 + x_2^2\nu \qquad (\nu \neq 0),$$

and let α be the translation given by

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$$x_1^{\alpha} = y_1 = x_1 + \lambda_1$$

$$x_2^{\alpha} = y_2 = x_2 + \lambda_2 \quad , \text{ where } (\lambda_1\mu + \lambda_2\nu) \neq 0.$$

Then $y_1^{d} = x_1^{d} = (y_1 - \lambda_1) + (y_1 - \lambda_1)^2\mu$

$$= (\lambda_1^2\mu - \lambda_1) + y_1(1-2\lambda_1\mu) + y_1^2\mu \quad ,$$

and $y_2^{d} = x_2^{d} = (y_2 - \lambda_2) + (y_2 - \lambda_2)^2\nu$

$$= (\lambda_2^2\nu - \lambda_2) + y_2(1-2\lambda_2\nu) + y_2^2\nu \quad .$$

Hence by (5) $\text{trace}_X(d) = 2$, and

$$\text{trace}_Y(d) = 2-2(\lambda_1\mu + \lambda_2\nu) \neq \text{trace}_X(d).$$

This suggests that we limit ourselves to augmentation preserving automorphisms of $R = F< x_1,\ldots,x_q >$, i.e. automorphisms α such that x_i^{α} has zero constant term for each $i = 1,\ldots,q$.

An augmentation preserving automorphism in R is said to be tame if it can be expressed as a product of elementary automorphisms, i.e. automorphisms in which

i) an element $x \in X$ is replaced by λx (λ a nonzero element in F) and the rest remain unchanged;

ii) the elements of X are permuted in any way;

iii) an element $x \in X$ is replaced by $x + f(x_1,\ldots,x_q)$, where f is an expression in the elements of X which are distinct from x; and the elements $\neq$ x in X remain unchanged. (See e.g. [17 ,p.33]

Proposition 5.2: Let d be a derivation in $R = F\langle X \rangle$, where $X = \{x_1,\ldots,x_q\}$, and let $Y = \{y_1,\ldots,y_q\}$ be another free generating set which is the image of X under a tame automorphism of R, then $\mathrm{trace}_Y(d) = \mathrm{trace}_X(d)$.

Proof: It is sufficient to prove that $\mathrm{trace}_Y(d) = \mathrm{trace}_X(d)$ for every Y which is the image of X under an elementary automorphism of R. Let d_1 be the derivation given by

$$x_i^{d_1} = u_i = \mu_{i1}x_1 + \ldots + \mu_{iq}x_q \quad , \quad i = 1,\ldots,q$$

where each u_i is the homogeneous component of degree 1 of x_i^d , then by (5) in section 5.1 $\mathrm{trace}_X(d) = \sum_{j=1}^{q} \mu_{jj}$.

Any elementary automorphism given in i) or ii) above induces a vector space automorphism in the q-dimensional space $gr_1(R)$, and it is well-known that such an automorphism does not change the trace of the linear transformation d_1 in this vector space. The same is true of any elementary automorphism given in iii) if $\deg f(x_1,\ldots,x_q) = 1$. Hence it only remains to consider an elementary automorphism α which is given on X by

$$x_k^{\alpha} = y_k = x_k \quad \text{if } k \neq i \text{ , and}$$

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$$x_i^{\alpha} = y_i = x_i + f(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_q) = x_i + f_x \text{ (say)},$$

where $\text{order}_X f_x \geq 2$.

Note that $y_i^{\alpha^{-1}} = x_i = y_i - f(y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_q)$

$$= y_i - f_y \quad \text{(say)} \quad ,$$

where $\text{order}_Y f_y \geq 2$.

The derivation d_1 is defined relative to Y by

$$y_k^{d_1} = x_k^{d_1} \qquad \text{if } k \neq i$$

$$= \sum_{j=1}^{\ell} \mu_{kj} x_j$$

$$= \sum_{j\neq i} \mu_{kj} y_j + \mu_{ki} y_i - \mu_{ki} f_y$$

and $\quad y_i^{d_1} = x_i^{d_1} + f_x^{d_1}$

$$= \sum_{j=1}^{\ell} \mu_{ij} x_j + f_x^{d_1}$$

$$= \Big(\sum_{j\neq i} \mu_{ij} y_j \Big) + \mu_{ii} y_i - \mu_{ii} f_y + (f_x^{d_1})^{\alpha^{-1}}$$

where $\text{order}_Y (f_x^{d_1 \alpha^{-1}} - \mu_{ii} f_y) \geq 2$.

Hence $\text{trace}_Y(d) = \sum_{k=1}^{\ell} \mu_{kk} = \text{trace}_X(d)$. //

There exists at the present time a conjecture that <u>all</u> augmentation preserving automorphisms in a free associative algebra of finite rank are tame. If this is true the preceding proposition will show that the trace of a derivation is independent of any change of free generating set caused

5.2

by an augmentation preserving automorphism.

Remark: Anastasia Czerniakiewicz [18] proved that in a free associative algebra of rank 2, F< x,y > , any automorphism which preserves the commutator xy-yx is tame. Moreover, she recently announced that she is now able to prove that all automorphisms of F< x,y > are tame.

--ooOoo--

REFERENCES.

[1] S.Andreadakis : On the automorphisms of free groups and free nilpotent groups. Proc. London Math. Soc., (3) 15 (1965) 239-268.

[2] G.M.Bergman : Commuting elements in free algebras and related topics in ring theory. (Thesis, Harvard University, 1967)

[3] G.M.Bergman : Groups acting on hereditary rings. Proc. London Math. Soc. (To appear)

[4] N.Bourbaki : Éléments de mathématique, Algèbre, Ch IV & V 2nd Ed (Paris, 1959).

[5] N.Bourbaki : Éléments de mathématique, Algèbre Commutative, Ch III & IV (Paris, 1961).

[6] N.Bourbaki : Elements of mathematics, General Topology Part 1 (Paris, 1966).

[7] C.Chevalley : Théorie des Groupes de Lie (Paris,1968).

[8] P.M.Cohn : Universal Algebra (New York, 1965).

[9] P.M.Cohn : On a generalization of the Euclidean algorithm. Proc. Cambridge Philos. Soc. 57 (1961) 18-30.

[10] P.M.Cohn : Factorization in noncommutative power series rings. Proc. Cambridge Philos. Soc. 58 (1962) 452-464.

[11] P.M.Cohn : Rings with a weak algorithm. Trans. Amer. Math. Soc. 109 (1963) 332-356.

[12] P.M.Cohn : Noncommutative unique factorization domains. Trans. Amer. Math. Soc. 109 (1963) 313-331

[13] P.M.Cohn : Free ideal rings. J. Algebra 1 (1964) 47-69.

[14] P.M.Cohn : Subalgebras of free associative algebras. Proc. London Math. Soc. (3) 14 (1964) 618-632.

[15] P.M.Cohn : Hereditary local rings. Nagoya Math. J. 27 (1966) 223-230.

[16] P.M.Cohn : Dependence in rings II. The dependence number. Trans. Amer. Math. Soc. 135 (1969) 267-279.

[17] P.M.Cohn : Free associative algebras. Bull. London Math. Soc. 1 (1969) 1-39.

[18] A.J.Czerniakiewicz : Endomorphisms of a free associative algebra of rank 2. Trans. Amer. Math. Soc. (to appear).

[19] G.Falk : Konstanzelemente in Ringen mit Differentiation. Math. Annalen, 124 (1952) 182-186.

[20] N.Jacobson : Lectures in Abstract Algebra, Vol III (Princeton, 1964).

[21] O.Ore : Theory of noncommutative polynomials. Ann. of Math. (2) 34 (1933) 480-508.

[22] S.MacLane : Homology (Berlin, 1963).