ULTRAFILTERS AND ULTRAPRODUCTS.

By

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Abstract

The topics of this thesis are properties that distinguish between the $2^{2^{\aleph_0}}$ isomorphism-classes (called <u>types</u>) of non-principal ultrafilters on ω. In particular we investigate various orders on ultrafilters.

The Rudin-Frolik order is a topologically invariant order on types; it had been shewn that there are types with $2^{\aleph_0}$ predecessors in this order, and that, assuming the C.H., for every $n \in \omega$ there are types with n predecessors. We shew that, assuming the C.H., there is a type with $\aleph_0$ predecessors.

The next two main results can be phrased in terms of the minimal elements of these orders. Both assume the C.H. We find an ultrafilter that is a p-point (minimal in M.E.Rudin's "essentially greater than" order) that is not above any Ramsey ultrafilter (minimal in the Rudin-Keisler order). We also find an ultrafilter minimal in Blass' "initial segment" order that is not a p-point. These ultrafilters generate ultrapowers with interesting model-theoretic properties.

We then investigate the classification of ultrafilters when the C.H. is no longer assumed. We find various properties of ultrafilters, sometimes by assuming some substitute for the C.H. such as

Martin's Axiom, and sometimes without assuming any additional axiom of set-theory at all. Finally we relate the structure of ultrapowers to the existence of special sorts of ultrafilters.

Contents

Chapter 1. Introduction.

1·1 This thesis is about the properties of non-principal ultrafilters over N, the set of natural numbers. It is known that there are $2^{2^{\aleph_0}}$ different isomorphism-types of such ultrafilters, and an obvious and important problem is to find properties that distinguish between them.

If one assumes the Continuum Hypothesis the method of induction up to ω_1 is a very powerful tool for constructing ultrafilters with distinguishing properties, and so the classification of ultrafilters is fairly straightforward. In Chapters 3 and 4 we give an account of the model-theoretic and topological properties of ultrafilters under the assumption of the Continuum Hypothesis.

Without it, the situation is much more difficult. The most natural approach is to try and classify ultrafilter types without using any special axiom, apart from the usual axioms of set-theory and the Axiom of Choice. In Chapter 5 we define a certain property and prove from the Axioms Z.F.C. alone that some but not all ultrafilter-types possess this property, but the property is not a particularly natural one, and cannot be used for any interesting classification of ultrafilter types. We also present some theorems obtained by using some substitute for the Continuum Hypothesis.

1·2 Contents

Chapter 2 is mostly introduction; it consists of the set-theoretic terminology in which this thesis is phrased, the definitions of ultrafilters and their topology in βN and of ultraproducts. A few basic Lemmas are proved. Various special sorts of ultrafilters are defined, several examples of non-principal filters are given and results are stated on how they relate to the special sorts of ultrafilters.

Chapter 3 discusses the topology of βN. The customary classification of points in βN is by their position with respect to a certain order, called the Rudin-Frolik order. It had been proved that there are ultrafilters with $2^{\aleph_0}$ predecessors in this order, and, assuming the Continuum Hypothesis, for every $n \in \omega$ there are ultrafilters with n predecessors. We extend this classification by constructing, (again assuming the C.H.) an ultrafilter with precisely $\aleph_0$ predecessors.

In Chapter 4 we turn to the model-theory of ultrapowers. Puritz' [11] convenient notation is used. He defines the Skies and Constellations of an ultrafilter p so that (heuristically) if $f, g \in {}^{\omega}\omega$, and for no $n \in \omega$ does $f^{-1}[n] \in p$ or $g^{-1}[n] \in p$, they are in the same constellation of p if they define the same partition of the integers, modulo a set in p, and they are in the same sky of p if in the ultrapower of ω with respect to p they are in elements of roughly the same magnitude.

The sky and constellation configuration of an ultrafilter p gives a very good picture of the model-theoretic structure of the ultrapower of ω with respect to p (in terms of initial segments, cofinal extensions and the like). Also, the particular sorts of ultrafilter defined in Chapter 2 have special sky and constellation sets. The two main results of the chapter can be phrased as:-

1) An ultrafilter can have one sky but no bottom constellation. (This answers a question of A.R.D.Mathias)

2) There is an ultrafilter with more than one sky but whose ultrapower of ω has no initial segments that are ultrapowers.

So far in the literature four orderings have been introduced. 2) gives an example of an ultrafilter that is minimal in two of these orderings (the Rudin-Frolik ordering mentioned above and A.Blass' "initial-segment" ordering but not in a third (M.E.Rudin's "essentially-greater-than" ordering). At the end of the chapter we discuss the possibility of finding other classifications of ultrafilters. The simplest case is to find two Ramsey ultrafilters which do not have the same properties. The only way I have been able to find such a distinction is by assuming some additional axion such as $V = L$ or Martin's Axiom + $2^{\aleph_0} > \aleph_1$. In fact, I doubt whether any such classification is possible in general, and this conjecture is extended to all ultrafilters on ω.

Chapter consists of a very incomplete exposition of the properties of ultrafilters when the C.H. is no longer assumed. As mentioned above, a property is found which is shared by some but not all ultrafilters on ω. Then we proceed to a discussion of the possible order -type of ω^ω/p, and some results are proved relating the possible order-types to other properties of ultrafilters. The gaps in this account are stated at the end of the chapter.

1·3 The main original parts of this thesis are sections 3·3, 4·3, 4·4, 4·5 and Chapter 5. As for the other theorems, some are due to other authors, and some are basic lemmas that have been proved by many people who have worked in this field. I have given a proof of someone else's theorem when its brevity and importance for the later development seemed to justify it. When there was doubt as to who first proved a basic lemma I have not tried to credit it to anybody.

In this thesis only ultrafilters over ω and ultrapowers of the natural numbers have been considered; generalization of the theory to higher cardinals and different structures is possible, but as the methods of proof and the flavour of the results are the same I did not feel that the extra generality justified the loss of clarity and precision it would entail.

Many of the proofs here are extremely complicated; it is unfortunate that the theory of ultrafilters often utilises very involved combinatorics. Frequently it seems likely that a neat positive theorem will be true,but on further examination a very complicated counterexample can be found. The blame lies between me, for not finding the right theorems to prove, and a Providence which does not always arrange that the Truth is Beautiful.

Finally, my thanks are due to the S.R.C. for three years financial support, and to the staff of Bedford College, especially my supervisor, Mr J.C. Fernau, for their help and encouragement.

Chapter 2 Notation and Some Basic Lemmas.

2·1 We work in Z.F. Set-Theory with the Axiom of Choice. When we assume further axioms (which will frequently happen) we will state them. Our notation is fairly standard. The following is a guide, which we will keep to as far as possible, for which symbols (with or without subscripts, superscripts etc) will be used for which entities:-

m,n,i etc for natural numbers.
α,β,γ etc for ordinals.
κ,λ etc for cardinals.
a,b,c etc for sets of natural numbers.
p,q,r etc for ultrafilters.
F,G,H etc for filters.
f,g,h etc for functions.

ϕ is the empty set, N or ω the set of all natural numbers, ω_1 the set of all countable ordinals. If A is a set, $|A|$ is its cardinality, $S_\omega(A)$ is the set of all finite subsets of A, $P(A)$ is the power set of A, the set of all subsets of A. If $A \subseteq I$, $C_I(A)$ is the complement of A, i.e. $C_I(A) = \{x \in I: x \notin A\}$. The subscript will be omitted when no confusion can arise. For A and B sets, ${}^A B$ or B^A is the set of all maps from A to B. If f is a function, $\mathrm{dom}(f)$ is its domain and $\mathrm{ran}(f)$ is its range. If $a \subseteq \mathrm{dom}(f)$, $f[a] = \{f(x): x \in a\}$ and if $a \subseteq \mathrm{ran}(f)$, $f^{-1}[a] = \{x: f(x) \in a\}$. If $a \subseteq \mathrm{dom}(f)$, the function obtained by restricting f to a is written $f|a$. The function $f \in {}^{\omega}\omega$ such that $f(n) = n$ for all n is called id.

2·2

Now let I be a set.

Def 2·21 for $F \subseteq P(I)$ we say F is a Filter if the following conditions hold:

1) $a,b \in F$ implies $a \cap b \in F$.
2) $a \in F$, $a \subseteq b \subseteq I$ imply $b \in F$.

Def 2·22 We say a filter F is proper if $\phi \notin F$. Henceforth all filters are assumed to be proper.

Def 2·23 We say a filter p over I is an ultrafilter if it is maximal. Equivalently, p is an ultrafilter iff for all $a \subseteq I$, either $a \in p$ or $C_I(a) \in p$.

Def 2·24. A filter F is principal if $\cap F \in F$. Equivalently, F is principal if for some $b \in F$, $F = \{a \subseteq I: b \subseteq a\}$. In particular, an ultrafilter p over I is principal if for some $x \in I$, $p = \{a \subseteq I: x \in a\}$. If a filter is not principal it is called non-principal, or free.

Def 2·25 The dual to a filter is called an Ideal. For F a filter, the corresponding ideal is $Q = \{a: C_I(a) \in F\}$. Much of the literature speaks in terms of ideals rather than filters.

Def 2·26 We say $A \subseteq P(I)$ has the finite intersection property (henceforth abbreviated to f.i.p.) if A is contained in a proper filter.

Def 2·27 If A has the f.i.p. the least proper filter containing A (this always exists) is said to be generated by A.

Then, assuming the Axiom of Choice, (or the strictly weaker hypothesis, the Boolean Prime Ideal Theorem), any set with the f.i.p. can be extended to an ultrafilter. In particular, let $Fr = \{a \subseteq \omega : \omega - a$ is finite$\}$. Fr can be extended to an ultrafilter, in fact to $2^{2^{\aleph_0}}$ ultrafilters. See [1] for details. as $\cap Fr = \phi$, these ultrafilters are all non-principal, and all non-principal ultrafilters on ω contain Fr. Our attention in this thesis will be confined to these, the non-principal ultrafilters on ω, henceforth abbreviated to f.u.f.

2·3 Ultrafilters on ω can be regarded as the points of the Stone-Čech Compactification of the Integers, βN. See [7] for details. N is embedded in βN by the natural map ψ which takes $n \in N$ to the principal ultrafilter generated by $\{n\}$. When discussing βN we will identify $n \in \omega$ with its image under ψ, if no confusion can arise.

βN has the topology generated by sets of the form $W(a) = \{q \in \beta N: a \in q\}$, for each $a \subseteq N$. These are clopen sets, ($W(C_N(a) = \beta N - W(a))$ and the singleton $\{\psi(n)\}$ is an open set, for each $n \in N$. $\{\psi(n)\} = W(\{n\})$. βN is compact, (this is equivalent to the statement that every filter can be extended to an ultrafilter) and hence so is $N^* = \beta N - N$. In the restriction topology on N^*, $W(b) \subseteq W(a)$ iff $b - a$ is finite, and $W(a) = W(b)$ iff $(a - b) \cup (b - a)$ is finite.

2·4

Suppose that $\{\mathfrak{A}_n\}_{n \in \omega}$ is an indexed family of structures with the same similarity type, which for simplicity we will assume to consist of the single binary relation R. The generalization to another similarity type is straightforward. The domain of each $\mathfrak{A}_n$ is written A_n.

Def 2·41 $\prod_{n \in \omega} A_n$ is the Cartesian product of the domains, i.e. it is the set of all functions f such that $\mathrm{dom}(f) = \omega$ and $f(n) \in A_n$ for every n. Let p be a f.u.f.

Def 2·42 For $f, g \in \prod_{n \in \omega} A_n$, write $f \sim_p g$ iff $\{n: f(n) = g(n)\} \in p$. This is an equivalence relation.

Def 2·43 Write $f^\sim$ for $\{g: g \sim_p f\}$

Def 2·44 Define $R^\sim$ by $f^\sim R^\sim g^\sim$ iff $\{n: f(n) R g(n)\} \in p$. It is easy to check that $R^\sim$ is well-defined. (Not dependent on the choice of $f \in f^\sim$, $g \in g^\sim$.)

Def 2·45 Define $\prod_{n \in \omega} \mathfrak{A}_n / p$ to be that structure whose domain is $\{f^\sim: f \in \prod_{n \in \omega} A_n / p\}$ and with the single binary relation $R^{\sim}$. This structure is called the <u>ultraproduct</u> of $\{\mathfrak{A}_n\}_{n\in\omega}$ with respect to p.

The fundamental theorem of ultraproducts is as follows. (see [1] for a proof).

<u>Theorem 2·46</u> (Łoś)

If $\phi(v_1,\ldots v_n)$ is a formula in the language of $\{\mathfrak{A}_n\}_{n\in\omega}$, (we assume that they have the same language), and the free variables of ϕ are among $v_1,\ldots v_n$, then

$\prod_{n\in\omega}\mathfrak{A}_n/p \models \phi[f_1^{\sim},\ldots f_n^{\sim}]$ iff $\{n: \mathfrak{A}_n \models \phi[f_1(n),\ldots f_n(n)]\}$ is in p.

If p is a principal ultrafilter the ultraproduct is trivial. If p is generated by $\{n\}$, then $\prod_{n\in\omega}\mathfrak{A}_n/p$ is isomorphic to $\mathfrak{A}_n$.

A special case of the ultraproduct construction occurs when all the $\mathfrak{A}_n$ are the same.

<u>Def 2·47</u> If $\mathfrak{A}_n = \mathfrak{A}$ for all n, write $\prod_{n\in\omega}\mathfrak{A}_n/p$ as $\mathfrak{A}^{\omega}/p$. This is called the ultrapower of $\mathfrak{A}$ with respect to p. The special case of Łoś' theorem relevant to ultrapowers is:-

<u>Theorem 2·48</u> If $\phi(v_1,\ldots v_n)$ is a formula in the language of $\mathfrak{A}$ with free variables among $v_1,\ldots v_n$, then $\mathfrak{A}^{\omega}/p \models \phi[f_1^{\sim},\ldots f_n^{\sim}]$ iff $\{n: \mathfrak{A} \models \phi[f_1(n),\ldots f_n(n)]\}$ is in p.

In particular, define an embedding $e:\mathfrak{A} \to \mathfrak{A}^{\omega}/p$ by

$e(x) = f_x^{\sim}$, where $f_x(n) = x$ for all $n \in \omega$.
Then $\mathfrak{A}^{\omega}/p \models \phi[f_{x_1}^{\sim}, \ldots f_{x_n}^{\sim}]$ iff $\mathfrak{A} \models \phi[x_1, \ldots x_n]$
i.e. the embedding e is elementary.

Def 2·49 If $f^{\sim} \in \mathfrak{A}^{\omega}/p$ is of the form $f_x^{\sim}$ for some $x \in \mathrm{dom}(\mathfrak{A})$, we say $f^{\sim}$ is standard. Otherwise we say $f^{\sim}$ is non-standard, or infinite.

2·5

If p is an ultrafilter on ω, and $f \in {}^{\omega}\omega$, write $f(p) = \{ a \subseteq \omega : f^{-1}[a] \in p \}$.
Then $f(p)$ is an ultrafilter, and $f(p)$ is principal iff f is constant on some set in p.

Theorem 2·51 (W. Rudin, [15])
For p and q ultrafilters over ω, p and q are isomorphic (that is, there is a bijection ψ from p to q which preserves inclusion) iff for some permutation of the integers π, $\pi(p) = q$.

Def 2·52 If there is such a permutation π, we write $p \equiv q$. This is obviously an equivalence relation, and the equivalence classes are called types.
Write $p^{\sim} = \{q : p \equiv q\}$. $p^{\sim}$ is the type of p.

Def 2·53 Write $p^{\sim} \leq_{RK} q^{\sim}$ if for some $f \in {}^{\omega}\omega$, $f(q) = p$.
We shew that $\leq_{RK}$ is a partial order. It is called the Rudin-Keisler order.

Theorem 2·54 (Various)
If $f(p) = p$, then $\{n : f(n) = n\} \in p$; i.e., $f \sim_p \mathrm{id}$.

<u>Proof</u> Let $b_1 = \{n: f(n) = n\}$, $b_2 = \{n: f(n) < n\}$, and $b_3 = \{n: f(n) > n\}$. We shew that $b_1 \in p$.

If $b_2 \in p$, let $a_n = \{m: n$ is the first number such that $f^n(m) \notin b_2\}$. (Here f^n is the n^{th} iterate of f). $\bigcup_{n\geq 1} a_n = b_2 \in p$.

Precisely one of $\bigcup_{n\geq 1} a_{2n}$ and $\bigcup_{n\geq 1} a_{2n+1}$ is in p. But $\bigcup_{n\geq 1} a_{2n} \in p$ iff $f[\bigcup_{n\geq 1} a_{2n}] \in p$ iff $\bigcup_{n\geq 1} a_{2n+1} \in p$. this is impossible.

If $b_3 \in p$, again let $c_n = \{m: n$ is the first number such that $f^n(m) \notin b_3\}$
Similarly, $\bigcup_{n\geq 1} c_n \notin p$. Let $d = b - \bigcup_{n\geq 1} c_n \in p$.

Let $d_0 = \{n \in d: n \notin f[d]\}$
Let $d_n = \{m \in d: n$ is the least number s.t. $m \in f^n[d_0]\}$
Then either $\bigcup_{n\geq 0} d_{2n}$ or $\bigcup_{n\geq 0} d_{2n+1}$ is in p.
But $\bigcup_{n\geq 0} d_{2n} \in p$ iff $f[\bigcup_{n\geq 0} d_{2n}] \in p$ iff $\bigcup_{n\geq 0} d_{2n+1} \in p$.
This is impossible, so b_1 is in p.

<u>Corollary 2·55</u>

$\leq_{RK}$ is a partial order.

<u>Proof</u>. If $\tilde{p} \leq_{RK} \tilde{q} \leq_{RK} \tilde{p}$, then $f(p) = q$ and $g(q) = p$ for some $f,g \in {}^{\omega}\omega$. So $fg(p) = p$. fg is the identity on some set $a \in p$, and so g is one-to one on a. We can split a into two infinite halves b and b', and define g' so that g' is a permutation and $n \in b \in p$ implies that $g(n) = g'(n)$. So $\tilde{q} = \tilde{p}$.

<u>Def 2·56</u> If p and q are ultrafilters on ω, write

$$p \times q = \{a \subseteq \omega \times \omega : \{m : \{n : \langle m,n \rangle \in a\} \in p\} \in q\}.$$

Then $p \times q$ is an ultrafilter over $\omega \times \omega$, and if π_1 and π_2 denote the projections to the first and second axes respectively,

$$\pi_1(p \times q) = p, \quad \pi_2(p \times q) = q.$$

2·6

We now define some special sorts of ultrafilters, due to Choquet [4,5] and W. Rudin [15].

Def 2·61 A non-principal filter q is p-point if whenever $\langle a_n : n \in \omega \rangle$ is a partition of ω such that $a_n \notin q$ for any n there is $a \in q$ so that

$$|a \cap a_n| < \omega \text{ for all } n.$$

Def 2·62 A non-principal filter q is rare if whenever $\langle a_n : n \in \omega \rangle$ is a partition of ω into finite sets there is $a \in q$ so that

$$|a \cap a_n| = 1 \text{ for all } n.$$

Def 2·63 A non-principal filter is Ramsey if it is both rare and a p-point.

Remark 2·64 The following are equivalent:

1) q is a p-point.

2) for every $f \in {}^{\omega}\omega$, either f is constant on some set in q, or else f is finite-to-one on some set in q.

3) if A is a countable subset of q, there is b in q, $|b - a| < \omega$ for all $a \in A$.

Remark 2·65 If q is a p-point, q is an ultrafilter.

Proof. If $b \notin q$, $C_{\omega}(b)$ is infinite, as q is non-principal, so let $\langle a_n : n \in \omega \rangle$ be a partition of $C(b)$. Either $a_n \in q$ for some n, or else there is $a \in q$, $|a \cap a_n| < \omega$ for all n and $|a \cap b| < \omega$. As q is non principal, in either case $C(b) \in q$.

Remark 2·66 A rare filter is not necessarily an ultrafilter. One can construct, for example, assuming the C.H., a rare filter q such that every $a \in q$ contains infinitely many even numbers and infinitely many odd numbers.

Remark. 2·67 A filter q is rare iff it is non-principal and whenever f is a finite-to-one function in ${}^{\omega}\omega$ there is $a \in q$ such that $f|a$ is one-to-one.

Remark 2·68 An ultrafilter q is Ramsey iff whenever $\langle a_n : n \in \omega \rangle$ is a partition of ω, either $a_n \in q$ for some n or else there is $a \in q$, $|a \cap a_n| = 1$ for all n. Equivalently, for every $f \in {}^{\omega}\omega$, there is $a \in q$ so that $f|a$ is either constant or one-to-one.

Now the existence theorem.

Theorem 2·69 (Choquet) The C.H. implies

1) There are Ramsey ultrafilters.
2) There are rare ultrafilters that are not p-points.
3) There are p-points that are not rare.
4) There are ultrafilters that are neither rare nor a p-point.

Proof

Examples of 2), 3) and 4) will be given later. We give a construction of 1).

Enumerate (C.H.) ${}^{\omega}\omega$ as $\langle f_\alpha : \alpha < \omega_1 \rangle$.

For each $\alpha < \omega_1$ we will add a set d_α so that $f_\alpha | d_\alpha$ is either constant or one-to-one. Each d_α is infinite, and $\alpha > \beta$ implies that $|d_\alpha - d_\beta| < \omega$, so the collection $\{d_\alpha : \alpha < \omega_1\}$ generates a proper filter.

Add in $Fr = \{a : \omega - a \text{ is finite}\}$

Stage 0 Assume $f_0 = id$, and let $d_0 = \omega$.

Stage α We have added $\{d_\beta : \beta < \alpha\}$. α is countable, so the filter constructed so far is generated by countably many sets. Let them be $\{e_n : n \in \omega\}$.

Construct $d \subseteq \omega$ as follows:-

Let $n_1 \in e_1$.

Let $n_2 \in e_1 \cap e_2$, $n_1 \neq n_2$.

....

Let $n_i \in e_1 \cap e_2 \cap \ldots \cap e_i$, $n_j \neq n_i$ for $j < i$.

....

Let $d = \{n_1, n_2, \ldots n_i, \ldots\}$

d is infinite, and $|d - e_n| < \omega$ for all n.

Let d_α be an infinite subset of d such that $f_\alpha | d_\alpha$ is constant or one- to-one. Certainly $|d_\alpha - d_\beta| < \omega$ for all $\beta < \alpha$.

Finally let q be generated by $\{d_\alpha : \alpha < \omega_1\}$. q is a Ramsey ultrafilter.

Remark 2·610 At each stage α we could have added one of at least 2 disjoint candidates for d_α. Different choices of d_α would engender different q's. Hence we can construct $2^{\aleph_1} = 2^{2^{\aleph_0}}$ different Ramsey ultrafilters.

2·7

This section some examples of non-principal filters, and their relations to the special sorts of ultrafilter defined in 2·6.

Example 2·71 Let $\langle a_n : n \in \omega \rangle$ be a partition of ω into finite sets so that $|a_n|$ is unbounded. Let $F = \{\omega - a : |a \cap a_n| = 1$ for all $n\}$. F generates a proper non-principal filter that can (C.H.) be extended to a p-point but not to a rare filter. In fact, an ultrafilter q is non-rare iff it contains such a filter as F.

Example 2·72 Let $\langle a_n : n \in \omega \rangle$ be a partition of ω into infinite sets. Let $F = \{C_\omega(a_n) : n \in \omega\} \cup$ $\cup \{\omega - a : |a \cap a_n| < \omega$ for all $n\}$. Then F generates a proper filter that can (C.H) be extended to a rare filter but not to a p-point. In fact, an ultrafilter q is not a p-point iff it contains such a filter as F.

Example 2·73 Let $F = \{\omega - a :$ for some n, a contains no arithmetic progression of length $n\}$; Van der Waerden's theorem on arithmetic progressions implies that F is a non-principal filter, and F can be extended (C.H.) to a p-point but not to a rare filter.

Example 2·74 For $a \subseteq \omega$, define $d(a,n) = \left|\frac{a \cap \{m: m \leq n\}}{n}\right|$

Let $\rho(a) = \lim_{n\to\infty} d(a,n)$ where this exists.

Let $F = \{a: \rho(a) = 1\}$ F is a non-principal filter that cannot be extended to either a rare filter or a p-point. This filter appeared in [13].

Example 2·75 Let $F = \{\omega - (a \cup \{0\})$: for all $n,m \in a$, $n + m \notin a\}$. An application of Ramsey's theorem shews that F generates a non-principal filter over $\omega - \{0\}$. In [8] it is shewn by a non-standard argument that F cannot be extended to a Ramsey ultrafilter. We shew that F cannot be extended to a rare ultrafilter.

Proof. Let $\langle a_n : n \in \omega\rangle$ partition $\omega - \{0\}$ so that

1) $a_n < a_{n+1}$ (i.e. $x \in a_n$ and $y \in a_{n+1}$ imply $x < y$)

2) $|a_n| = 2^n$.

Suppose p were a rare ultrafilter extending F. Either $\bigcup_{n\in\omega} a_{2n}$ or $\bigcup_{n\in\omega} a_{2n+1}$ is in p, suppose $\bigcup_{n\in\omega} a_{2n}$ is in p. Let b be a choice set for $\langle a_n : n \in \omega\rangle$, and let $a = b \cap \bigcup_{n\in\omega} a_{2n} \in p$.

But if $x,y \in a$, say $x \in a_{2m}$ and $y \in a_{2r}$ where $r \geq m$.

Then $x + y \in a_{2r}$ or $x + y \in a_{2r+1}$.
In either case $x + y \notin a$, so $a \notin p$, a contradiction.

The moral of all these results seems to be:-

"Simple - to - describe filters cannot be extended to Ramsey ultrafilters."

This can be made precise as follows:-

We say a set of subsets of ω A is Σ_1^1 if $x \in A$ iff $\exists y \phi[x,y,c]$

Where c is a constant set of natural numbers and the only quantifiers in ϕ range over natural numbers.

<u>Theorem 2·76</u> (A.R.D.Mathias, unpublished)

If A is a Σ_1^1 set of subsets of ω, and q is a Ramsey ultrafilter, there is $a \in p$ such that either
1) Every infinite subset of a is in A, or
2) Every infinite subset of a is outside A.

<u>Corollary 2·77</u> If A is a Σ_1^1 filter, (and all those mentioned above are) either A is contained in some countably generated filter or else A cannot be extended to a Ramsey ultrafilter.

Mathias' result is essentially maximal, for if $V = L$ there is a Δ_2^1 well-ordering of the subsets of ω which can be used to define a Ramsey ultrafilter.

Chapter 3 Topology of βN

3·1

In section 2·3 we defined the space βN, the Stone - Cech Compactification of N with the discrete topology. The following are some trivial results on the topology on $N^* = \beta N - N$.

3·11 1) $W(a) \cap W(b) = W(a \cap b)$

2) $W(a) \cup W(b) = W(a \cup b)$

3) $W(a) = \phi$ iff a is finite.

4) $\bigcup_{n\in\omega} W(a_n) \subseteq W(\bigcup_{n\in\omega} a_n)$ and in general they are not equal.

5) $\bigcap_{n\in\omega} W(a_n) \supseteq W(\bigcap_{n\in\omega} a_n)$ and in general they are not equal.

If $X \subseteq \beta N$, we write the closure of X as $\bar{X}$. Then $q \in \bar{X}$ iff $\forall a \epsilon q\ \exists x \epsilon X,\ a \in x$.
P-points have a special topological significance. In fact, the term p-point is derived from topology.

Theorem 3·12 A f.u.f. q is a p-point iff the intersection of a countable collection of neighbourhoods of q is itself a neighbourhood of q.

Proof Let $\{U_n\}_{n\epsilon\omega}$ be such a collection. We can assume that $U_n = W(E_n)$ where $E_n \in q$. Then there is $E \in q$, $|E - E_n| < \omega$ for all n. Hence $W(E) \subseteq \bigcap_{n\epsilon\omega} W(E_n)$ is the neighbourhood of q required.

Conversely, suppose $E_n \in q$ for every n. Then

$\bigcap_{n\in\omega} W(E_n)$ is a neighbourhood of q. Let $W(E) \subseteq \bigcap_{n\in\omega} W(E_n)$ where $E \in q$. Then $|E - E_n| < \omega$ for all n, hence q is a p-point.

Corollary 3.13 If q is a p-point, q is not in the closure of any countable subset X of N^* unless $q \in X$.

Proof For every $x \in X$, let U_x be a neighbourhood of q not containing x. Then $\bigcap_{x\in X} U_x$ is a neighbourhood of q disjoint from X.

3·2

In [15] W.Rudin used the existence of p-points (assuming the C.H) to prove that N^* is not homogenous (i.e. there are two points p,q in N^* such that no auto-homeomorphism maps p to q. By the Compactness of N^* there is some $q \in N^*$ which is in the closure of a countable subset of N^*, and no homeomorphism can map q to a p-point.)

In [6] Z.Frolik proved the inhomogeneity of N^* without the C.H. by using the following ideas:-

Def 3·21 If X is a countable indexed subset of N^*, $X = \{X_n : n \in \omega\}$, X is said to be discrete iff there are sets $\{c_n : n \in \omega\}$ such that $c_n \in X_n$ for all n, and $n \neq m$ implies that $c_n \cap c_m = \phi$. Topologically, X is discrete if whenever $x \in X$, $x \notin \overline{X - \{x\}}$.

(Note; we will use X,Y,Z etc to denote countable indexed subsets of N^*, sometimes with superscripts, e.g. X^α or X^n. The n^{th} member of X in the enumeration is written X_n.)

<u>Def 3·22</u> If the conditions of 3·21 are satisfied, we say X is <u>made discrete</u> by $\{c_n: n \in \omega\}$.

<u>Lemma 3·23</u> (M.E.Rudin) Suppose Z is a countable indexed discrete (henceforth abbreviated to c.i.d.) subset of N^*, $X \subseteq Z$ and $Y \subseteq Z$. Then if $q \in \bar{X} \cap \bar{Y}$, $q \in \overline{X \cap Y}$.

<u>Proof</u>. Let Z be made discrete by $\{c_n: n \in \omega\}$.
Let $d = \cup\{c_n: Z_n \in X \cap Y\}$.
Then as $q \in \bar{X} \cap \bar{Y}$, $d \in q$. Let $a \in q$. $a \cap d \in q$, so $a \cap d \in z \in X \cap Y$. Hence $q \in \overline{X \cap Y}$.

<u>Def 3·24</u> If X is a c.i.d. subset of N^*, and $p \in N^*$, we write:-

$$\Sigma[X,p] = \{a \subseteq \omega: \{n: a \in X_n\} \in p\}$$

If $q \in \bar{X} - X$, we write:-

$$\Omega[X,q] = \{a \subseteq \omega: \forall b \in q, \exists n \in a,\ b \in X_n\}$$

Then we have:

<u>Theorem 3·25</u> 1) $\Sigma[X,p]$ and $\Omega[X,q]$ are ultrafilters.
2) $\Sigma[X,\Omega[X,q]] = q$ and $\Omega[X,\Sigma[X,p]] = p$, i.e. the

operations Σ and Ω are inverse.

Proof. All the parts involve merely untangling the definitions, apart from shewing that $\Omega[X,q]$ has the f.i.p. This follows however from Lemma 3·23.

Def 3·26 If $p,q \in N^*$, we say $\tilde{p} <_{RF} \tilde{q}$ iff there is a c.i.d. subset X of N^* such that $q = \Sigma[X,p]$ or equivalently $p = \Omega[X,q]$.

This is called the Rudin-Frolik ordering. That it is an ordering will follow from later Lemmas. The definition is well defined; e.g. if $p' \in \tilde{p}$ a different enumeration of X, say X', will give $q = \Sigma[X',p']$.

A less combinatorial definition of the ordering is as follows:

$\tilde{p} <_{RF} \tilde{q}$ iff there is some homeomorphism ψ of βN into N^* such that $\psi(p) = q$.

In fact $q = \Sigma[X,p]$ where $X_n = \psi(n)$, $\psi[\beta N] = \bar{X}$.

Similarly one can shew that if ϕ is an auto--homeomorphism of N^*, $\phi(q) = r$, and $\tilde{p} <_{RF} \tilde{q}$, then $\tilde{p} <_{RF} \tilde{r}$. So the property of having $\tilde{p}$ as a $<_{RF}$ predecessor is a topological invariant.

This ordering is weaker than the Rudin-Keisler ordering as follows:-

Theorem 3·27 $\tilde{p} <_{RF} \tilde{q}$ then $\tilde{p} \leq_{RK} \tilde{q}$.

Proof Suppose $q = \Sigma[X,p]$ and that X is made discrete by $\{a_n : n \in \omega\}$. Then if we define $f \in {}^{\omega}\omega$ so that $f^{-1}[n] = a_n$ for all n, it is easy to shew that $f(q) = p$.

<u>Corollary 3·28</u> For any $q \in N^*$, $q^\sim$ has at most $2^{\aleph_0}$ predecessors in the $<_{RF}$ ordering.

So for some $p, q \in N^*$, $p^\sim$ is not a $<_{RF}$ predecessor of $q^\sim$. So this proves, (without the C.H.) that N^* is not homogenous.

<u>Corollary 3·29</u> If $p^\sim <_{RF} q^\sim$, $p^\sim \nmid q^\sim$. So $p^\sim <_{RK} q^\sim$.

<u>Proof</u> If $a \in q$, $a \in X_n$ for some n. As $X \subseteq N^*$, X_n contains no finite set. So $a \cap a_n$ is infinite. For no $a \in q$, is $f|a$ a one-to-one function. From Theorem 2·54, $p^\sim \nmid q^\sim$.

The following gives another criterion for $p^\sim <_{RF} q^\sim$.

<u>Lemma 3·210</u> $p^\sim <_{RF} q^\sim$ iff there are countable discrete sets X and Y and $r \in N^*$ so that

1) $Y \subseteq \bar{X} - X$.
2) $r = \Sigma[X,q] = \Sigma[Y,p]$

<u>Proof</u> Suppose first that $q = \Sigma[Z,p]$ for some c.i.d. set Z. Let X be any c.i.d. set, and let $r = \Sigma[X,q]$. Define Y by $Y_n = \Sigma[X, Z_n]$. Y is a countable indexed set, $Y \subseteq \bar{X} - X$ and Y is discrete.

Then $a \in \Sigma[Y,p]$ iff $\{n : a \in Y_n\} \in p$

iff $\{n: \{m: a \in X_m\} \in Z_n\} \in p$

iff $\{m: a \in X_m\} \in q$ iff $a \in r$.

So $r = \Sigma[X,q] = \Sigma[Y,p]$

Conversely suppose the conditions hold.

Define $Z_n = \Omega[X,Y_n]$. Z is a c.i.d. set.

$a \in \Sigma[Z,p]$ iff $\{n: a \in Z_n\} \in p$

iff $\{n: \forall b \in Y_n\ \exists m \in a,\ b \in X_m\} \in p$

iff $\forall b \in r,\ \exists m \in a,\ b \in X_m$

iff $a \in q$.

So $q = \Sigma[Z,p]$, and $q^\sim >_{RF} p^\sim$.

<u>Theorem 3·211</u> If q is a f.u.f., the $<_{RF}$ predecessors of $q^\sim$ are linearly ordered.

<u>Proof</u> Suppose that $q = \Sigma[X,p]$ and $q = \Sigma[Y,r]$

<u>Case 1</u> Let $X' = \{x \in X: x \in \bar{Y} - Y\}$.

If $q \in \bar{X}'$ by 3·210, $r^\sim >_{RF} p^\sim$.

<u>Case 2</u>. Let $Y' = \{y \in Y: y \in \bar{X} - X\}$

If $q \in \bar{Y}'$, by 3·210 $p^\sim >_{RF} r^\sim$.

<u>Case 3</u> Otherwise.

Then let $X^* = X - X'$, $Y^* = Y - Y'$.

Then $X^* \cup Y^*$ is discrete, and $q \in \overline{X^*} \cap \overline{Y^*}$.

By lemma 3·23, $q \in \overline{X^* \cap Y^*}$.

So $p^\sim = r^\sim$.

The following Lemma will be needed later:-

Lemma 3·212 If $\tilde{p} <_{RF} \tilde{q}$, say $q = \Sigma[X,p]$, then $\tilde{q}$ is $<_{RF}$-minimal above $\tilde{p}$ iff $\{n: \tilde{X}_n$ is $<_{RF}$-minimal$\} \in p$.

Proof Suppose first that $X_n = \Sigma[Y^n, r_n]$ where each Y^n is a c.i.d. set, and if X is made discrete by $\{c_n : n \in \omega\}$, then $c_n \in Y^n_m$ for all n and m.

Then $Y = \bigcup_{n \in \omega} Y^n$ is a countable discrete set. $X \subseteq \bar{Y} - Y$, so in particular $q \in \bar{Y} - Y$. So if we let $r = \Omega[Y,q]$,

$\tilde{q}\ {}_{RF}{>}\ \tilde{r}\ {}_{RF}{>}\ \tilde{p}$.

Conversely, suppose $\tilde{q}\ {}_{RF}{>}\ \tilde{r}\ {}_{RF}{>}\ \tilde{p}$, where $q = \Sigma[X,p]$ and $q = \Sigma[Y,r]$. We can assume without loss of generality that $X \subseteq \bar{Y} - Y$, so if we let $Z_n = \Omega[Y,X_n]$, then $\tilde{X}_n\ {}_{RF}{>}\ \tilde{Z}_n$ for all n.

3·3

Many results have been found on the possible order types embeddable in this ordering. See e.g. [3].

Assuming the C.H. there are ultrafilter types minimal in this ordering (for example p-points), and by a re-iteration of Lemma 3·212, for every $n \in \omega$ we can construct an ultrafilter q such that $\tilde{q}$ has precisely n $<_{RF}$-predecessors. In [17] A.K. and E.F. Steiner construct an ultrafilter type with $2^{\aleph_0}$ predecessors. (This does not need the C.H.). They state at the end of the paper that they do not know whether there is a type with precisely $\aleph_0$ predecessors. We construct one such, assuming the C.H.

Firstly we discuss what possible countable order types can occur . Let q be a f.u.f. and let S be the set of $<_{RF}$ predecessors of $\tilde{q}$, ordered by $<_{RF}$.

<u>Lemma 3·31</u> If S is countable, we can assume that if we define c.i.d. sets X^p for every $\tilde{p} \in S$, where $q = \Sigma[X^p, p]$, then $\tilde{p} \;{}_{RF}{>}\; \tilde{r} \to X^r \subseteq \bar{X}^p - X^p$.

<u>Proof</u> Re-iteration of Lemma 3·210.

Now, any infinite order type must have either an infinite ascending subset or an infinite descending subset. (Or both). Henceforth we assume that S is countable.

<u>Case 1. 3·32</u> S has an infinite ascending sequence S'.

<u>Subcase 1a. 3·321</u> S' has a least upper bound. We shew that this is impossible.

Without loss of generality we assume that the least upper bound is $\tilde{q}$, and S' is the sequence

$$\tilde{p}_0 <_{RF} \tilde{p}_1 <_{RF} \cdots\cdots <_{RF} \tilde{p}_n <_{RF} \cdots\cdots <_{RF} \tilde{q}.$$

Say $q = \Sigma[X^n, p_n]$, where $X^n \subseteq \overline{X^{n+1}} - X^{n+1}$

Suppose X^o is made discrete by $\{c_m : m \in \omega\}$.

Then let $Y = \{X^n_m : c_n \in X^n_m\}$.

Y is discrete, as each X^n is. Let $a \in q$, then

$a \in X_m^o$ for some m.
$X_m^o \in \bar{X}^m - X^m$, so $a \in X_r^m$ for some r, where $c_r \in X_r^m$.
Hence $q \in \bar{Y}$.

Let $p' = \Omega[Y,q]$.
Fix $n \in \omega$. Let $Z = \{X_m^n : m \leq n\}$.
Then $q \in X^n - Z$, and $X^n - Z \subseteq \bar{Y} - Y$.
So $\tilde{p_n} <_{RF} \tilde{p'} <_{RF} \tilde{q}$ for all n.
This contradicts our assumption that $\tilde{q}$ was the least upper bound.

<u>Subcase 1b 3·322</u> S' has no least upper bound. Then we can assume that S' is of the form:

$$\tilde{p_0} <_{RF} \cdots\cdots <_{RF} \tilde{p_n} <_{RF} \cdots <_{RF} \tilde{q_m} <_{RF} \cdots <_{RF} \tilde{q_0} = \tilde{q}.$$

And there is no p' such that $\tilde{p_n} <_{RF} \tilde{p'} <_{RF} \tilde{q_m}$ for all m and n. We shew that this is impossible.

Suppose that $q = \Sigma[X^n, p_n] = \Sigma[Y^m, q_m]$ where
$X^n \subseteq \bar{Y}^m - Y^m$, $X^n \subseteq \bar{X}^{n+1} - X^{n+1}$, and $Y^{n+1} \subseteq \bar{Y}^n - Y^n$.
Let X^o be made discrete by $\{c_n : n \in \omega\}$.
Define $Z = \{Y_m^n : c_n \in Y_m^n\}$
Then Z is a countable discrete sequence, and $q \in \bar{Z}$.
Furthermore, $X^n \subseteq \bar{Z} - Z$ for all n.
For all n, let $Z' = Z - \{Y_n^m : m \leq n\}$
Then $q \in \bar{Z}'$, and $Z' \subseteq \bar{Y}^n - Y^n$.
So $\tilde{p_n} <_{RF} \tilde{p'} <_{RF} \tilde{q_m}$ for all m and n, a contradiction.

<u>Case 2 3·33</u> S has no infinite ascending sequence.
Then it has an infinite descending sequence.

Subcase 2a 3·331 S is bounded below. As case 1 did not occur, S must have a biggest lower bound.

Say $\tilde{q} \ {}_{RF}\!> \cdots {}_{RF}\!> \tilde{q_n} \ {}_{RF}\!> \cdots {}_{RF}\!> \tilde{p}$.
Where $q = \Sigma[X^n, q_n] = \Sigma[Y, p]$, and
$Y \subseteq \bar{X}^n - X^n$, $\quad X^{n+1} \subseteq \bar{X}^n - X^n$.
But this situation cannot in fact occur. We can prove, by a method similar to the construction in Subcase 1b,

Lemma 3·332 If the situation described in subcase 2a occurs, there is $p' \in N^*$ so that

$\tilde{q_n} \ {}_{RF}\!> \tilde{p'}$ for all n, $\tilde{p'} \ {}_{RF}\!> \tilde{p}$ and
$q = \Sigma[Z, p']$ where $Z \subseteq \bigcup_{n\in\omega} X^n$.

This leaves us with Subcase 2b, in which S has an infinite descending sequence not bounded below. But assuming the C.H., this case can actually happen.

Theorem 3·34 (C.H.) There is an ultrafilter q such that $\tilde{q}$ has precisely $\aleph_0$ $<_{RF}$-predecessors.

Proof Let $\{a^n_m : n, m \in \omega\}$ be infinite subsets of such that:

1) $a^n_m \cap a^n_{m'} = \phi$ if $m \neq m'$.
2) $\bigcup_{n\in\omega} a^m_n = \omega$ for all m.

3) $a^{n+1}_m = \bigcup_{r\in b_{nm}} a^n_r$ where each b_{nm} is an infinite subset of ω.

(i.e., each $\langle a_m^n : m \in \omega \rangle$ is a partition of ω into infinite sets, and $\langle a_m^{n+1} : m \in \omega \rangle$ is coarser than $\langle a_m^n : m \in \omega \rangle$)

Now let $\{X_m^o\}$ be p-points so that $a_m^o \in X_m^o$ for all m. X_m^o is a c.i.d. subset of N^*. We will define c.i.d. sets X^n for every $n \in \omega$.

Suppose we have defined X^n.

Let Y_m^n be p-points such that $b_{nm} \in Y_m^n$ and let $X_m^{n+1} = \Sigma[X^n, Y_m^n]$

Thus we can define X_m^n for all n and m. From the construction it is not hard to shew that $a_m^n \in X_m^n$ and X^n is a c.i.d. set, and $X^{n+1} \subseteq \bar{X}^n - X^n$

We wil construct an ultrafilter q such that $q \in \bigcap_{n \in \omega} \bar{X}^n$.

If $p_n = \Omega[X^n, q]$, we will require that the only $<_{RF}$-predecessors of $\tilde{q}$ are $\{\tilde{p}_n : n \in \omega\}$.
The following are some facts about this construction that we shall need.

<u>Facts</u> 1) $\tilde{q} = \tilde{p}_o \ {}_{RF}\!> \cdots {}_{RF}\!> \tilde{p}_n \ {}_{RF}\!> \cdots$

2) $p_n = \Sigma[Y^n, p_{n+1}]$

3) If $\tilde{p}_n \ {}_{RF}\!> \tilde{p} \ {}_{RF}\!> \tilde{p}_{n+1}$ then either $\tilde{p} = \tilde{p}_n$ or else $\tilde{p} = \tilde{p}_{n+1}$.

4) If $a \in X_m^{n+1}$ for some n and m then $\{r : a \in X_r^n\}$

is infinite.

5) If $\exists p \in N^*$, $p_n^\sim \;_{RF}> p^\sim$ for all n, then there is p', $p_n^\sim \;_{RF}> p'^\sim$ for all n, and $q = \Sigma[X', p']$, where X' is a countable discrete subset of $\bigcup_{n \in \omega} X^n$.

Proofs 1) is from Lemma 3·210, 2) is just calculation, 3) is from Lemma 3·212, 4) is because Y_m^n is non-principal, and 5) is Lemma 3·332.

From Facts 3) and 5), to ensure that the only $<_{RF}$-predecessors of $q^\sim$ are $\{p_n^\sim : n \in \omega\}$, it suffices to shew the following:-

If X is a countable discrete subset of $\bigcup_{n \in \omega} X^n$, and $q \in \bar{X}$, then if $p = \Omega[X, q]$, $p^\sim = p_n^\sim$ for some n. To ensure that $p^\sim = p_n^\sim$ we need only ensure that $q \in \overline{X \cap X^n}$.

So enumerate (C.H.) the countable discrete subsets of $\bigcup_{n \in \omega} X^n$ as $\langle X^\alpha : \alpha < \omega_1 \rangle$. For every α we will add a set d_α to q, such that either $d_\alpha \notin X_m^\alpha$ for any m, or else $d_\alpha = \bigcup_{m \in \omega} \{a_m^n : X_m^n \in X^\alpha\}$ for some fixed n.

Induction Hypothesis

At every stage α we have a countably generated filter F_α, so that if $a \in F_\alpha$, for every n,

$\{m: a \in X^n_m\}$ is infinite.

<u>Stage 0</u> Let $d_0 = \omega$, $F_0 = Fr$.

<u>Stage α</u> Let F be generated by $\bigcup_{\beta<\alpha} F$. As α is countable, F is countably generated. Let its generators be $\{e_n: n \in \omega\}$, and assume without loss of generality that $e_n \supseteq e_{n+1}$ for all n.

For each n, write $h_n = \bigcup_{m\in\omega} \{a^n_m: X^n_m \in X^\alpha\}$.

<u>Case 1</u> The filter generated by $F \cup \{h_n\}$ obeys the induction hypothesis, for some n. Then let $d_\alpha = h_n$, and let F_α be generated by $F \cup \{d_\alpha\}$.

<u>Case 2</u> Otherwise. We construct sets $\{a_n: n \in \omega\}$ as follows:-

<u>Stage 0</u> The filter generated by $F \cup \{h_0\}$ does not obey the induction hypothesis. Certainly for some n_0, $e_0 \cap a^0_{n_0} \in X^0_{n_0}$ and $X^0_{n_0} \notin X^\alpha$. Let $a_0 = a^0_{n_0} \cap e_0$.

<u>Stage j</u> Suppose we have defined a_i for $i < j$. The filter generated by $\{h_0 \cup \ldots \cup h_j\} \cup F$ does not obey the induction hypothesis. So for some n_j,

$e_j \cap a^j_{n_j} - (h_o \cup \dots \cup h_j) \in X^j_{n_j}$.

In particular $X^j_{n_j} \notin X^\alpha$.

Let $a_j = e_j \cap a^j_{n_j} - (h_o \cup \dots \cup h_j)$.

Let $d_\alpha = \bigcup_{j \in \omega} a_j$.

Claim 1 If $x \in X^\alpha$, $d_\alpha \notin x$.

Proof Say $x = X^n_m$ for some $n, m \in \omega$.

If $n = 0$, $d_\alpha \cap a^n_m = a^o_{n_o} \cap e_o \cap a^o_m = \phi$.

If $n > 0$, $d_\alpha \cap a^n_m \subseteq \bigcup_{r<n} a^r_{n_r}$ by the construction of d_α.

But by fact 4), if $d_\alpha \cap a^n_m \in X^n_m$,

$\{r: d_\alpha \cap a^n_m \in X^{n-1}_m\}$ is infinite.

So $d_\alpha \cap a^n_m \notin X^n_m$.

Claim 2 The filter generated by $F \cup \{d_\alpha\}$ obeys the induction hypothesis.

Proof A typical member of this filter contains $d_\alpha \cap e_n$ for some n. Fix m. Let $r = \max\{n.m\} + 1$. Then $d_\alpha \cap e_n \in X^r_{n_r}$, so $\{k: d_\alpha \cap e_n \in X^{r-1}_k\}$ is infinite. Certainly, $\{k: d_\alpha \cap e_n \in X^m_k\}$ is infinite. The induction Hypothesis is still true.

So let F_α be the filter generated by $F_\alpha \cup \{d_\alpha\}$.

Finally let G be generated by $\bigcup\{F_\alpha : \alpha < \omega_1\}$. G is not necessarily an ultrafilter. But let f be the map such that $f^{-1}[n] = a_n^o$ for every n. As every infinite subset of X^o has occured in our enumeration, f(G) is an ultrafilter.

Define $q = \Sigma[X^o, f(G)]$.
Then $q \in \bigcap_{n\in\omega} \bar{X}^n$, and by our construction, the $<_{RF}$-predecessors of $\tilde{q}$ are precisely $\{\tilde{p}_n : n \in \omega\}$.

<u>Remark 3·35</u> The existence of $<_{RF}$-minimal ultrafilters is necessary in this proof. In a model of set-theory in which there are no $<_{RF}$-minimal ultrafilters, every type has at least $\aleph_1$ predecessors. For take $\tilde{q}$, find $\tilde{p} <_{RF} \tilde{q}$. Re-iterate Lemma 3·212 to obtain a sequence $\tilde{p} <_{RF} \cdots <_{RF} \tilde{p}_n <_{RF} \cdots <_{RF} \tilde{q}$. Then re-iterate subcase 1a and subcase 1b to obtain $\aleph_1$ $<_{RF}$ predecessors.

Chapter 4 Model - theory of Ultrapowers

4·1

The following set of results, due to Keisler, clears up the problem of the structure of ultrapowers of a countable structure with a countable language over a countable set, assuming the C.H. Proofs are in [1].

Def 4·11 Let $\mathfrak{A}$ be a structure, $A \subseteq \mathrm{dom}(\mathfrak{A})$. A set Φ of formulae in the language of $\langle \mathfrak{A}, a \rangle_{a \in A}$ with x appearing as the only free variable, is said to be finitely satisfiable in $\langle \mathfrak{A}, a \rangle_{a \in A}$ if for every finite subset $\{\phi_1, \ldots \phi_n\} \subseteq \Phi$,

$$\langle \mathfrak{A}, a \rangle_{a \in A} \models \exists x(\phi_1(x) \wedge \ldots \wedge \phi_n(x)).$$

Def 4·12 $\mathfrak{A}$ is said to be κ-Saturated if whenever $A \subseteq \mathrm{dom}(\mathfrak{A})$, $|A| < \kappa$, and Φ is a set of formulae, finitely satisfiable in $\langle \mathfrak{A}, a \rangle_{a \in A}$, then there is $b \in \mathrm{dom}(\mathfrak{A})$, $\mathfrak{A} \models \phi[b]$ for all $\phi \in \Phi$.

Def 4·13 $\mathfrak{A}$ is said to be Saturated if it is $|\mathfrak{A}|$-saturated.

Certain sorts of ultrafilter give rise to saturated ultrapowers, as follows:-

<u>Def 4·14</u> An ultrafilter p is said to be <u>κ-good</u> if whenever $\lambda < \kappa$, and f is a map from $S_\omega(\lambda)$ to p, then there is a map g from $S_\omega(\lambda)$ to p, $g(a) \subseteq f(a)$ for all $a \in S_\omega(\lambda)$, and $g(a) \cap g(b) = g(a \cap b)$ for all $a,b \in S_\omega(\lambda)$.

<u>Def 4·15</u> An ultrafilter p is said to be <u>ω-incomplete</u> if there is a countable subset X of p such that $\cap X = \phi$.

<u>Theorem 4·16</u> If p is a κ-good, ω-incomplete ultrafilter on λ, and the cardinality of the language of $\mathfrak{A}$ is less than κ, $\mathfrak{A}^\lambda/p$ is κ-saturated.
In particular, if p is $|\mathfrak{A}^\lambda|$-good, $\mathfrak{A}^\lambda/p$ is saturated.

It is quite easy to check that all non-principal ultrafilters on ω are $\aleph_1$-good and ω-incomplete. By a back and forth argument, any two elementary equivalent structures of the same cardinality that are saturated are isomorphic. In particular,

<u>Theorem 4·17</u> (C.H) If $\mathfrak{A}$ is a countable structure with a countable language, and p and q are f.u.f.s,

then $\mathfrak{A}^{\omega}/p$ is isomorphic to $\mathfrak{A}^{\omega}/q$.

4·2

In view of the results of section 4·1, to obtain any results about ultrapowers of countable structures we shall have to consider a larger language.

Def 4·21 The full structure on ω, written ω^+, is the structure whose domain is ω and with all possible relations on ω.

$$\omega^+ = \langle \omega, R_\alpha \rangle_{\alpha < 2^{\aleph_0}}.$$

We now find that the structure of the ultrapower $\omega^{+\omega}/p$ depends very much on the combinatorial properties of the ultrafilter p.

The following is the result that connects the model-theoretic properties of $\omega^{+\omega}/p$ to the $\leq_{RK}$ ordering mentioned in 2·5. It appears in [8] and [12].

Theorem 4·22 $\omega^{+\omega}/p$ can be embedded as a elementary substructure of $\omega^{+\omega}/q$, (written $e: \omega^{+\omega}/p \leq \omega^{+\omega}/q$, where e. is the embedding), iff there is $f \in {}^{\omega}\omega$ such that $f(q) = p$, (i.e. $q \geq_{RK} p$).

<u>Proof</u> Suppose that $f(q) = p$. Define $\phi: {}^{\omega}\omega \to {}^{\omega}\omega$ by $(\phi(g))(n) = g(f(n))$.

Then $g \sim_p g'$ iff there is $a \in p$ such that $n \in a$ iff $g(n) = g'(n)$. Let $b = f^{-1}[a] \in q$.

For $m \in b$, $g(f(m)) = g'(f(m))$. Hence $\phi(g) \sim_q \phi(g')$.

So there is a well-defined map

$e: {\omega^+}^{\omega}/p \to {\omega^+}^{\omega}/q$, which by similar arguments is elementary.

Suppose $e: {\omega^+}^{\omega}/p \leqslant {\omega^+}^{\omega}/q$.

Let $f^{\sim} = e(\mathrm{id}^{\sim})$. For all $a \in p$, ${\omega^+}^{\omega}/p \models \mathrm{id}^{\sim} \in a$.

$({\omega^+}^{\omega}/p \models \mathrm{id}^{\sim} \in a$ iff $\{n: \mathrm{id}(n) \in a\} \in p$ iff $a \in p)$.

e is elementary so ${\omega^+}^{\omega}/q \models f^{\sim} \in a$.

So $\{n: f(n) \in a\} \in q$, i.e. $f^{-1}[a] \in q$.

Hence $f(q) = p$.

<u>Remark</u> From the proof of Theorem 4·22 it is clear that $x \in e[{\omega^+}^{\omega}/p]$, where $f(q) = p$ and e is the induced embedding, iff there is $g \in x$, $|g[f^{-1}[n]]| = 1$ for all n.

A convenient notation to describe such ultrapowers was invented by Puritz in [12]. The following definitions are his.

<u>Def 4·23</u> For $f, g \in {}^{\omega}\omega$, p a f.u.f. and $f^{\sim}$ and $g^{\sim}$ non-standard members of $\omega^{+\omega}/p$, we write $f \leqslant_p g$ iff $\exists h \in {}^{\omega}\omega$, $\{n: h(g(n)) \geqslant f(n)\} \in p$.

<u>Def 4·24</u> Write $f \equiv_p g$ iff $f \leqslant_p g$ and $f \;_p\!\geqslant g$. The equivalence classes of $\equiv_p$ are called the skies of p.

$$Sk_p(f) = \{g: f \equiv_p g\}$$

<u>Def 4·25</u> For $f, g \in {}^{\omega}\omega$, p a f.u.f. and $f^{\sim}$ and $g^{\sim}$ non-standard members of $\omega^{+\omega}/p$, we write $f \leqslant^p g$ iff $\exists h \in {}^{\omega}\omega$, $\{n: h(g(n)) = f(n)\} \in p$.

<u>Def 4·26</u>. Write $f \equiv^p g$ iff $f \leqslant^p g$ and $f \;^p\!\geqslant g$. The equivalence classes of $\equiv^p$ are called the constellations of p. $Con_p(f) = \{g: f \equiv^p g\}$

We order Skies and Constellations by extending $\leqslant_p$ and $\leqslant^p$. $Sk_p(f) \leqslant Sk_p(g)$ iff $f \leqslant_p g$, $Con_p(f) \leqslant Con_p(g)$ iff $f \leqslant^p g$. It is easy to shew that these are well-defined orders. The following gives criteria for $f \leqslant_p g$ and $f \leqslant^p g$.

<u>Lemma 4·27</u> 1) $f \leqslant_p g$ iff $\exists a \in p$, $|f[g^{-1}[n] \cap a]| < \omega$ for all n.

2) $f \leqslant^p g$ iff $\exists a \in p$, $|f[g^{-1}[n] \cap a]| = 1$ for all n.

Proof 1) If $f \leq_p g$, let h be as given in the definition, and let $a = \{n: h(g(n)) \geq f(n)\} \in p$. Then $|f[g^{-1}[n] \cap a]| \leq h(n) < \omega$.

Conversely, if the condition holds, define h so $h(m) = \max\{f[g^{-1}[m] \cap a]\}$.

Then $\{n: h(g(n)) \geq f(n)\} \supseteq a \in p$.

2) If $f \leq^p g$, let h be as given in the definition, and let $a = \{n: h(g(n)) = f(n)\} \in p$. Then $f[g^{-1}[m] \cap a] = \{h(m)\}$, so $|f[g^{-1}[m] \cap a]| = 1$ for all m.

Conversely, if the condition holds, define h so $\{h(m)\} = f[g^{-1}[m] \cap a]$.

Then $\{n: h(g(n)) = f(n)\} \supseteq a \in p$.

Corollary 4·28 (Puritz) 1) q is a p-point iff q has only one sky.

2) q is rare iff the top sky of q has only one constellation.

3) q is Ramsey iff it has only one constellation.

Proof : 1) q is a p-point iff every non-standard function f is equivalent mod q to a finite-to-one function. Hence there is $a \in q$, $|id[f^{-1}[n] \cap a]| < \omega$ for all n, so $Sk_q(f) \geq Sk_q(id)$, and id is always in the top sky of q.

2) f is in the top sky of q iff f is equivalent mod q to a finite-to-one function. But q is rare iff every finite-to-one function is equivalent mod q to a one-to-one function. Hence q is rare iff every f in the top sky of q is in the same constellation as id.

3) This follows from 1) and 2).

Corollary 4·29 1) q is a p-point iff whenever $e: {}^{\omega}\omega^{+}/p \leq {}^{\omega}\omega^{+}/q$, $e[{}^{\omega}\omega^{+}/p]$ is cofinal in ${}^{\omega}\omega^{+}/q$.
2) q is rare iff whenever $e: {}^{\omega}\omega^{+}/p \leq {}^{\omega}\omega^{+}/q$ either $e[{}^{\omega}\omega^{+}/p] = {}^{\omega}\omega^{+}/q$ or $e[{}^{\omega}\omega^{+}/p]$ is not cofinal in ${}^{\omega}\omega^{+}/q$.
3) q is Ramsey iff whenever $e: {}^{\omega}\omega^{+}/p \leq {}^{\omega}\omega^{+}/q$, then $e[{}^{\omega}\omega^{+}/p] = {}^{\omega}\omega^{+}/q$.

Proof These follow from 4·28 and the fact that if $f \in Sk_q(id)$, ${}^{\omega}\omega^{+}/f(q)$ is embeddable as a cofinal substructure of ${}^{\omega}\omega^{+}/q$.

The following facts will be useful.

Lemma 4·210 1) Skies are totally ordered.
2) Skies are made up of whole constellations.
3) If f is in the bottom sky of p, f(p) is a

p-point.

4) The converse to 3) is not true in general.

5) $f(p)$ is rare iff f is in a minimal constellation of $Sk_p(f)$.

6) Constellations are not necessarily totally ordered.

7) If f is in a minimal constellation of $Sk_p(f)$, it is in the minimum constellation of $Sk_p(f)$.

<u>Proofs</u> 1) 2) and 3) are in [12]. An example of 4) is in [14]. The proof of 5) is similar to that of 4·27. We prove 6) and 7).

6) Let $\langle a_n : n \in \omega \rangle$ be a partition of ω so that $|a_n| = n^2$, and we imagine each a_n as a nxn block. Define f and g so that f is constant on each row in each a_n and g is constant on each column in each a_n. Then f and g are finite to one, and for all $m,n \in \omega$, $|f^{-1}[n] \cap g^{-1}[m]| \leq 1$.

Let F be generated by:

$$\{\omega - a: \forall n |f^{-1}[n] \cap a| = 1\} \cup \{\omega - a: \forall m |g^{-1}[m] \cap a| = 1\}$$

Then F is a proper filter, and if $c \subseteq \omega$, $|f[g^{-1}[n] \cap c]| = 1$ for all n, then $|g^{-1}[n] \cap c| = 1$ for all n, so $C_\omega(c) \in F$. If q is any ultrafilter extending F, $Con_q(f) \nleq Con_q(g)$, and by similar arguments $Con_q(g) \not\leq Con_q(f)$.

7) Let f be in a minimal constellation of $Sk_p(f)$. Then $f(p)$ is rare. Let $g \in Sk_p(f)$. We can assume that $|f[g^{-1}[m]]| < \omega$ for all m. We construct disjoint finite sets $\{a_n : n \in \omega\}$ so that the following holds;

If $g^{-1}[m] \cap f^{-1}[a_n] \neq \phi$ and $g^{-1}[m] \cap f^{-1}[a_r] \neq \phi$, Then $n = r$ or $n = r+1$ or $r = n+1$.

Let $a_0 = \{0\}$.

Suppose we have defined a_n, and a_n is finite.

Let $a_{n+1} = \{m: \exists r,\ f^{-1}[m] \cap g^{-1}[r] \neq \phi$ and $f^{-1}[a_n] \cap g^{-1}[r] \neq \phi\} \cup \{(n+1) - \bigcup_{r \leq n} a_r\}$.

a_{n+1} is finite, as a_n is finite and $|f[g^{-1}[s]]| < \omega$ for all s.

Then $\bigcup_{n\in\omega} a_n = \omega$, and if $n \neq m$ then $a_n \cap a_m = \phi$.

As $f(p)$ is rare, let $a \in f(p)$ be such that $|a \cap a_n| = 1$ for all n.

Also either $\bigcup_{n\in\omega} a_{2n}$ or $\bigcup_{n\in\omega} a_{2n+1}$ is in $f(p)$. Say $\bigcup_{n\in\omega} a_{2n} \in f(p)$. Let $b = a \cap \bigcup_{n\in\omega} a_{2n}$.

Let $c = f^{-1}[b] \in p$.

<u>Claim</u> $|f[g^{-1}[n] \cap c]| = 1$ for all n.

Proof. If $g^{-1}[n] \cap f^{-1}[m] \neq \phi$ and $g^{-1}[n] \cap f^{-1}[m'] \neq \phi$, Say $m \in a_{2r}$. Then $m' \in a_{2r}$ or $m' \in a_{2r+1}$ or $m' \in a_{2r-1}$. The latter two are impossible, by our choice of c. Furthermore, as $|a \cap a_{2r}| = 1$, $m = m'$. So $|f[g^{-1}[n] \cap c]| = 1$. So $f \leqslant^p g$. f is in the minimum constellation of $Sk_p(f)$.

Digression 4·211 It is reasonable to ask whether all the elementary substructures of ${\omega^+}^\omega/q$ are of the form $e[{\omega^+}^\omega/p]$. The following gives a criterion for this to occur.

Def 4·212 For $f \in {}^\omega\omega$, $eq(f) = \{\langle i,j\rangle: f(i) = f(j)\}$. Then if G is a filter over $\omega x \omega$, define

$$ {\omega^+}^\omega/q|G = \{f^\sim \in {\omega^+}^\omega/q: \exists g \in f^\sim, \ eq(g) \in G\} $$

It is shewn in [9] that ${\omega^+}^\omega/q|G$ is an elementary substructure of ${\omega^+}^\omega/q$, and that all the elementary substructures of ${\omega^+}^\omega/q$ are of this form. It is further shewn that for any G,

${\omega^+}^\omega/q|G$ is isomorphic to ${\omega^+}^\omega/q|G'$, for some G' which is a filter over $\omega x \omega$ generated by equivalence relations on ω.

Now, for $f^{\sim}, g^{\sim} \in \omega^{+\omega}/q$, $\{eq(f) : f \in f^{\sim}\} = \{eq(g) : g \in g^{\sim}\}$ iff $Con_q(f) = Con_q(g)$. So $\omega^{+\omega}/q|G'$ is made up of whole constellations. Then,

Theorem 4·213 $\omega^{+\omega}/q|G \leqslant \omega^{+\omega}/q$ is itself an ultrapower iff the set of constellations included in $\omega^{+\omega}/q|G$ has a greatest element included in $\omega^{+\omega}/q|G$.

In particular, if q has only finitely many constellations, every elementary substructure is itself an ultrapower. Later an example of an ultrapower with an elementary substructure that is not an ultrapower will be presented.

4·3

In [12] Puritz constructs ultrafilters with various sky and constellation systems. For example, he shews that (assuming the C.H.) for every $n \in \omega$ there are ultrafilters with n skies. (The process is identical to that mentioned at the beginning of 3·3). A question that he asks is:-

"Does every ultrafilter have a bottom sky?"

This is related to a question posed by Choquet in [4].

"Is there an ultrafilter such that for no $f \in {}^{\omega}\omega$ is $f(q)$ a p-point?"

Mathias [11] answers both questions by proving:-

Theorem 4·31 (C.H.) There is an ultrafilter q such that for no $f \in {}^{\omega}\omega$ is $f(q)$ a p-point.
(This answers Puritz' question because of 4·210 part 3)
R.A.Pitt improved this to:-

Theorem 4·32 (C.H.) There is an ultrafilter q such that for no $f \in {}^{\omega}\omega$ is $f(q)$ either rare or a p-point.

At the 1971 Logic Conference at Cambridge, Mathias asked further if there is a p-point q such that for no $f \in {}^{\omega}\omega$ is $f(q)$ rare. Below we present a construction of one such, assuming the C.H. In Puritzian terms this ultrafilter has one sky but no bottom constellation.

Theorem 4·33* (C.H) There is an ultrafilter q such that for no $f \in {}^{\omega}\omega$ is $f(q)$ rare.

*Mathias and Pitt have also proved this result.

Remark As promised in 2·6, this is an example of a non-rare p-point.

Proof Let $F = \{f \in {}^{\omega}\omega : f$ is finite-to-one$\}$

Enumerate (C.H.) F as $\langle f_{\alpha} : \alpha < \omega_1 \rangle$.

Enumerate (C.H.) $P(\omega)$ as $\langle S_{\alpha} : \alpha < \omega_1 \rangle$.

As q is a p-point, for every non-standard $x \in \omega^{+\omega}/q$, there is $f \in F$, $f \in x$.

Induction Assumption.

For each $\alpha < \omega_1$ we will construct d_{α}, h_{α}, J_{α} so that:-

1) $d_{\alpha} \subseteq \omega$, and $\alpha > \beta$ implies $|d_{\alpha} - d_{\beta}| < \omega$.

2) $d_{\alpha} \subseteq S_{\alpha}$ or $d_{\alpha} \cap S_{\alpha} = \phi$.

3) h_{α} is a function from d_{α} to ω, such that h_{α} is finite to one; and if $\alpha > \beta$ h_{α} is coarser than h_{β} except on a finite set. That is, there is a finite set c such that $n,m \in d_{\alpha} - c$ implies that if $h_{\beta}(n) = h_{\beta}(m)$ then $h_{\alpha}(n) = h_{\alpha}(m)$.

4) J_{α} consists of at most countably many sets of subsets of ω; write $J_{\alpha} = \{J_{\alpha}^{n} : n \in \omega\}$. $\alpha > \beta$ implies $J_{\beta} \subseteq J_{\alpha}$.

5) For any α, and any finite subset of ω, $S = \{n_1, \ldots n_i\}$, there is $n \in \omega$ such that if $a_1 \in J_{\alpha}^{n_1}$, $\ldots$ $a_i \in J_{\alpha}^{n_i}$, $m \geq n$, $a_1 \cap \ldots \cap a_i \cap d_{\alpha} \cap h_{\alpha}^{-1}[m] \neq \phi$.

6) If q extends $\{d_\alpha\} \cup \bigcup J_\alpha$, $f_\alpha(q)$ is not rare.

<u>Remark</u> Conditions 1) and 2) imply that q is a p-point. For if K is a countable subset of q,

$K = \{S_{\alpha_i} : i \in \omega\}$, then $d_{\alpha_i} \subseteq S_{\alpha_i}$ for all i.

Take α greater than α_i for all i, then $d_\alpha \in q$ and $|d_\alpha - d_{\alpha_i}| < \omega$ for all i.

Certainly, $|d_\alpha - b| < \omega$ for all $b \in K$.

Now we proceed with the induction.

<u>Stage 0</u> Let $h_0 = id$, $d_0 = \omega$, $J_0 = \phi$.

<u>Stage $\alpha = \beta+1$</u> Suppose we have constructed d_β, h_β, J_β. Define h as follows:-

Let $h^{-1}[1] = \bigcup\{f_\alpha^{-1}[m] : f_\alpha^{-1}[m] \cap h_\beta^{-1}[1] \neq \phi\}$

$h^{-1}[1]$ is a finite set. Let $n_1 = 1$.

Suppose we have defined $h^{-1}[i]$ for all $i < j$, and each $h^{-1}[i]$ is a finite set. Let n_j be the first number such that $h_\beta^{-1}[n_j] \cap h^{-1}[i] = \phi$ for all $i < j$.

Let $h^{-1}[j] = \bigcup\{f_\alpha^{-1}[m] : f_\alpha^{-1}[m] \cap h_\beta^{-1}[n_j] \neq \phi\}$

Then $h^{-1}[j]$ is a finite set.

Let $d = dom(h) \cap d_\beta$. Then h is a well-defined finite-to-one function on d.

Let k be a function such that $dom(k) = d$ and

1) $h(n) = h(m)$ implies $k(n) = k(m)$ for $n, m \in d$.

2) $|h[k^{-1}[n]]|$ is finite but increasing.

i.e. k is finite-to-one but coarser than h.

Define $H_m = \{\omega - a: |h[k^{-1}[n] \cap a]| \leq m$ for all $n\}$.

<u>Claim 1</u> If q is a f.u.f. that contains d and $\bigcup_{m\in\omega} H_m$, $f_\alpha(q)$ is not rare.

<u>Proof</u> h is coarser than f_α, so $con_q(h) \leq con_q(f_\alpha)$.
k is coarser than h, so $con_q(k) \leq con_q(h)$.
But for no set $a \in q$, is $|h[k^{-1}[n] \cap a]| = 1$ for all n. So the constellation of k is strictly less than the constellation of f_α. By Lemma 4·210 part 5 $f_\alpha(q)$ is not rare.

<u>Claim 2</u> There is an infinite $I \subseteq \omega$, and $d_\alpha \subseteq d$, so that either $d_\alpha \subseteq S_\alpha$ or $d_\alpha \cap S_\alpha = \phi$ and the following holds:-

If $S = \{n_1, \ldots n_i\}$ is a finite subset of ω and $r \in \omega$ there is $n \in I$ such that whenever $a_1 \in J_\beta^{n_1}, \ldots$ .... $a_i \in J_\beta^{n_i}$, $b \in H_r$, and $m \in I$, $m \geq n$,

$$a_1 \cap \ldots \cap a_i \cap b \cap d_\alpha \cap k^{-1}[m] \neq \phi.$$

Proof If not. Then there is an infinite $I_1 \subseteq \omega$ and $m \in \omega$ and $n_1, \ldots n_i \in \omega$ and $a_1 \in J_\beta^{n_1}, \ldots a_i \in J_\beta^{n_i}$ and $a \in H_m$ so that for all $n \in I_1$,

$$a \cap a_1 \cap \ldots \cap a_i \cap d \cap S_\alpha \cap k^{-1}[n] = \phi.$$

Then there is an infinite $I_2 \subseteq I_1$ and $r \in \omega$ and $m_1, \ldots m_j \in \omega$ and $b \in H_r$ and $b_1 \in J_\beta^{m_1}, \ldots b_j \in J_\beta^{m_j}$, such that for all $n \in I_2$,

$$b \cap b_1 \cap \ldots \cap b_j \cap d \cap C_\omega(S_\alpha) \cap k^{-1}[n] = \phi.$$

So for every $n \in I_2$,

$$a \cap b \cap b_1 \cap \ldots \cap b_j \cap a_1 \cap \ldots \cap a_i \cap d \cap k^{-1}[n] = \phi.$$

We shew this is impossible.

Take s so big that if $s' \geq s$ and if $h_\beta^{-1}[t] \cap k^{-1}[s'] \neq \phi$ then $b_1 \cap \ldots \cap b_j \cap a_1 \cap \ldots \cap a_i \cap d_\beta \cap h_\beta^{-1}[t] \neq \phi$, and $|h[k^{-1}[s']]| > m + r$. Then as I_2 is infinite, there is $s' \geq s$, $s' \in I_2$, so that

$$a \cap b \cap b_1 \cap \ldots \cap b_j \cap a_1 \cap \ldots \cap a_i \cap d \cap k^{-1}[s'] \neq \phi.$$

A contradiction.

So let d_α be as in the claim. Let ψ be a map from I to ω which is one-to-one and onto.

Let $h_\alpha(n) = \psi k(n)$.

Then h_α is a map from d_α to ω which is coarser than h_β. Let $J_\alpha = J_\beta \cup \{H_m : m \in \omega\}$. Claims 1) and 2) imply that that the induction assumption still holds.

<u>Stage α, a limit ordinal.</u>

Let $\alpha = \{\gamma_n : n \in \omega\}$, and let $\{\alpha_n : n \in \omega\}$ be an increasing subset of α.

Let $K = \bigcup_{n\in\omega} J_{\gamma_n}$ and enumerate K as $K = \{K_n : n \in \omega\}$

Define h as follows:-

Let $n_1 \in d_{\alpha_1}$ be the first number such that:

1) If $K_1 \in J_{\alpha_1}$, $a \in K_1$, and $m \geqslant n_1$, $a \cap d_{\alpha_1} \cap h_{\alpha_1}^{-1}[m] \neq \phi$.

2) If $\gamma_1 \leqslant \alpha_1$, $m, m' \in d_{\alpha_1}$, $m, m' \geqslant n_1$, then $m, m' \in d_{\alpha_1}$ and $h_{\gamma_1}(m) = h_{\gamma_1}(m')$ implies that $h_{\alpha_1}(m) = h_{\alpha_1}(m')$.

Then let $h^{-1}[1] = h_{\alpha_1}^{-1}[n_1]$. This is a finite set.

Suppose we have defined $h^{-1}[i]$ for all $i < j$, and each $h^{-1}[i]$ is a finite set. Let $n_j \in d_{\alpha_j}$ be the first number such that:-

1) Let those K_i, $i \leqslant j$, which are in J_{α_j} be K_{i_1}, $\ldots K_{i_k}$. Then if $a_1 \in K_{i_1}, \ldots a_k \in K_{i_k}$, and $m \geqslant n_j$, $a_1 \cap \cdots \cap a_k \cap d_{\alpha_j} \cap h_{\alpha_j}^{-1}[m] \neq \phi$.

2) Let those γ_i, $i \leqslant j$which are less than or equal to α_j be $\gamma_{i_1}, \ldots \gamma_{i_k}$. Then if $m, m' \in d_{\alpha_j}$, $m, m' \geqslant n_j$, $m, m' \in d_{\gamma_{i_r}}$, for $1 \leqslant r \leqslant k$ and $h_{\gamma_{i_r}}(m) = h_{\gamma_{i_r}}(m')$ implies $h_{\alpha_j}(m) = h_{\alpha_j}(m')$ for $1 \leqslant r \leqslant k$.

3) $h_{\alpha_j}^{-1}[m] \cap h^{-1}[i] = \phi$ for $m \geqslant n_j$, $i < j$.

(note: in the induction assumption clauses 1), 3) and 5) say that d_α, h_α, J_α behave regularly except on a finite set. In the definition of n_j we are taking n_j so big that all these finite sets have been exhausted in $\bigcup_{m<n_j} h_{\alpha_j}^{-1}[m]$.)

Let $h^{-1}[j] = h_{\alpha_j}^{-1}[n_j]$. This is a finite set, and if we let $d = \mathrm{dom}(h)$ it is clear that h is a well-defined function on d. By our construction it is also clear that:

1) $|d - d_\beta| < \omega$ for $\beta < \alpha$. In fact, $d - d_{\gamma_k}$ is included in $\bigcup_{i \leqslant j} h^{-1}[i]$, where j is the first number such that $\alpha_j > \gamma_k$ and $k < j$

2) For every $\beta < \alpha$ there is a finite set c such that if $m, n \in d - c$, $h_\beta(m) = h_\beta(n)$ implies $h(n) = h(m)$. In fact such a c is $\bigcup_{i \leqslant j} h^{-1}[i]$, where j is the

first number such that $\alpha_j > \gamma_k$ and $j > k$, $(\beta = \gamma_k)$.

3) If $n_1, \ldots n_i$ is a finite subset of ω, there is $n \in \omega$ so that whenever $a_1 \in K_{n_1}, \ldots a_i \in K_{n_i}$ and $m \geq n$, $a_1 \cap \ldots \cap a_i \cap d \cap h^{-1}[m] \neq \phi$. In fact, such an n is the first number such that $K_{n_1}, \ldots K_{n_i} \in J_{\alpha_n}$, and $n_1, \ldots n_i < n$.

So we can proceed to construct h_α, d_α, J_α, exactly as in the successor ordinal case.

Finally let q be generated by $\{d_\alpha : \alpha < \omega_1\} \cup \cup \bigcup\{J_\alpha : \alpha < \omega_1\}$. q is a p-point such that for no $f \in {}^\omega\omega$ is f(q) rare.

<u>4·4</u>

We now consider two other orderings on ultrafilters, weaker than the Rudin-Keisler order but stronger than the Rudin-Frolik order.

Suppose now that p,q are f.u.f.s and that $f(q) = p$ for some $f \in {}^\omega\omega$.

<u>Def 4·41</u> We say $q \,_{EG}\!> p$ if for no $a \in q$ is $|f^{-1}[n] \cap a| < \omega$ for all n. This ordering is due to M.E.Rudin in [14].

Def 4·42 We say $q \;_{IS}\!\!> p$ if the canonical embedding $e: \omega^{+^{\omega}}/p \leqslant \omega^{+^{\omega}}/q$ is such that $e[\omega^{+^{\omega}}/p]$ is an initial segment of $\omega^{+^{\omega}}/q$. This definition is due to Blass in [2].

The chain of implication is:-

$\tilde{q} \;_{RF}\!\!> \tilde{p} \rightarrow q \;_{IS}\!\!> p \rightarrow q \;_{EG}\!\!> p \rightarrow q \;_{RK}\!\!> p$; none of the reverse implications hold. Most of the proofs and counterexamples are trivial. First here is a criterion for $q \;_{IS}\!\!> p$.

Lemma 4·43 Suppose $f(q)= p$. Then $q \;_{IS}\!\!> p$ iff whenever $h \in {}^{\omega}\omega$ is such that $|h[f^{-1}[n]]| < \omega$ for all n, there is $a \in q$, $|h[f^{-1}[n] \cap a]| = 1$ for all n.

Proof Suppose $e[\omega^{+^{\omega}}/p]$ is an initial segment of $\omega^{+^{\omega}}/q$. Let h be a function satisfying the condition of the Lemma. Define f' so that $f'|f^{-1}[n]$ is constant with value greater than $\max\{h[f^{-1}[n]]\}$. Then $\tilde{f'} \in e[\omega^{+^{\omega}}/p]$, and $\tilde{f'} > \tilde{h}$. So $\tilde{h} \in e[\omega^{+^{\omega}}/p]$, and so there is $h' \in \tilde{h}$, $h'|f^{-1}[n]$ is constant for all n. So if we let $a = \{m: h(m) = h'(m)\}$, $a \in p$ and $h[f^{-1}[n] \cap a] = 1$ for all n.

Conversely, suppose the condition holds. Let $\tilde{h} \leqslant \tilde{g} \in e[\omega^{+^{\omega}}/p]$. Then we can assume $h(n) \leqslant g(n)$ for all n. But $|g[f^{-1}[n]]| = 1$ for all n, so $|h[f^{-1}[n]]| < \omega$ for all n.
Let a be the set such that $|h[f^{-1}[n] \cap a]| = 1$ for all n. Define h' so that $h'|f^{-1}[n]$ is constant

with the same value as $h|f^{-1}[n] \cap a$.
Then $h' \in h^{\sim}$, and $h'^{\sim} \in e[\omega^{+^{\omega}}/p]$.
So $e[\omega^{+^{\omega}}/p]$ is an initial segment of $\omega^{+^{\omega}}/q$
Using this we can shew:-

<u>Theorem 4·44</u> $p^{\sim} <_{RF} q^{\sim}$ implies $p <_{IS} q$.

<u>Proof</u> Let $q = \Sigma[X,p]$, where X is made discrete by $\{a_n : n \in \omega\}$. Then $f(q) = p$, where $f[a_n] = \{n\}$.

Suppose h is a function such that $|h[a_n]| < \omega$ for all n. Then for all n, there is $b_n \in X_n$, $b_n \subseteq a_n$, such that $|h[b_n]| = 1$.

If we let $b = \bigcup_{n \in \omega} b_n$, $\{n: b \in X_n\} = \omega \in p$, So $b \in q$. By the lemma, $p <_{IS} q$.

The only hard part in the chain of implication is to shew that the converse of theorem 4·44 does not hold. Proofs are in [2] and [14].

We will now consider the minimal elements in these four orderings. It is not hard to shew that:

1) A $<_{RK}$-minimal ultrafilter is Ramsey.
2) A $<_{EG}$-minimal ultrafilter is a p-point.

In [13] M.E.Rudin asked the following two questions:-

1) Is there a $<_{RF}$-minimal ultrafilter that is

not a p-point?

2) If the answer to 1) is yes, is there an ultrafilter that is not in the closure of any countable discrete set, but is in the closure of some countable set?

Kunen found examples for both these conjectures, assuming the Continuum Hypothesis. His results are announced in [10]. They are:-

1) There is a f.u.f. p, not a p-point, such that p is not in the closure of any countable set.

2) There is a countable subset X of N^*, such that if $x \in X$, x is not in the closure of any countable discrete set, yet $x \in \overline{X - \{x\}}$.

An answer to question 1) would be found by exhibiting an ultrafilter that is $<_{RF}$-minimal but not $<_{EG}$-minimal. Rudin and Blass both construct an ultrafilter that is $<_{RF}$-minimal but not $<_{IS}$-minimal. Here we construct, assuming the C.H., an ultrafilter that is $<_{IS}$-minimal but not $<_{EG}$-minimal.

<u>Theorem 4·45</u> (C.H.) There is an ultrafilter q, not a p-point, such that for no $p \in N^*$ is $\omega^{+\omega}/p$ embeddable as a proper initial segment of $\omega^{+\omega}/q$.

<u>Proof</u> Let f be any function such that $|f^{-1}[n]| = \omega$ for all n. Let the filter F_0 be generated by: $\{\omega - a: |a \cap f^{-1}[n]| < \omega \text{ for all } n\} \cup \{\bigcup_{m \geq n} f^{-1}[m]: n \in \omega\}$.

Then if q extends F_0 q is not a p-point.

Enumerate ${}^{\omega}\omega$ as $\langle f_\alpha : \alpha < \omega_1 \rangle$. For each α we will ensure that q contains sets so that $\omega^{+\omega}/f_\alpha(q)$ is not embeddable as a proper initial segment of $\omega^{+\omega}/q$.

<u>Def</u> Let $H = \{h \in {}^{\omega}\omega : |h[f^{-1}[n]]| < \omega$ for all $n\}$.

<u>Def</u> If $h, j \in H$, a <u>concatenation</u> of h and j is a function $k \in H$ such that for all n, m
$k(n) = k(m)$ iff $h(n) = h(m)$ and $j(n) = j(m)$.
(i.e. k is a finer function than both h and j).

<u>Def</u> If $L = \{h_n : n \in \omega\} \subseteq H$, a concatenation of L is a function $k \in H$ such that if $i, j \in f^{-1}[n]$,
$k(i) = k(j)$ iff $h_m(i) = h_m(j)$ for all $m \leqslant n$.
(i.e. for each n, k is finer than h_n on $\bigcup_{m>n} f^{-1}[m]$, which is a set in F_0.)

<u>Induction Assumption</u>

For every α we will define d_α, h_α, J_α and F_α such that:-

1) F_α is a proper filter generated by
$F_0 \cup \{d_\beta : \beta < \alpha\} \cup \bigcup\{J_\beta : \beta < \alpha\}$.

2) $h_\alpha \in H$, and if $\beta < \alpha$, there is $m \in \omega$, h_α is finer than h_β on $\bigcup_{n \geqslant m} f^{-1}[n]$.

3) If $\beta \leqslant \alpha$, $J_\beta \subseteq J_\alpha$, and $a \in J_\beta$, there is $n \in \omega$, for all $m \geqslant n$, then if $h_\alpha^{-1}[m] \cap a \neq \phi$, then $h_\alpha^{-1}[m] \subseteq a$.

<u>Stage 0</u> We have constructed F_0. Let $d_0 = \omega$, $h_0 = f$,

$J_0 = \phi$.

Stage $\alpha > 0$ Let F be generated by $\bigcup_{\beta<\alpha} F_\beta$, and let $J = \bigcup_{\beta<\alpha} J_\beta$. Let h be a concatenation of $\{h_\beta : \beta < \alpha\}$. Relabel $\{d_\beta : \beta < \alpha\}$ as $\{e_n : n \in \omega\}$, and assume without loss of generality that $e_n \supseteq e_{n+1}$ for all n.

Let $A_n = \{m: |f_\alpha[h^{-1}[m] \cap e_n]| = \omega\}$.
Let $B_n = \bigcup_{m \in A_n} h^{-1}[m] \cap e_n$.

Case 1 For some n, $F \cup \{f_\alpha^{-1}[n]\}$ has the f.i.p. Let $d_\alpha = f_\alpha^{-1}[n]$, $J_\alpha = J$, $h_\alpha = h$. The induction hypothesis still holds.

Case 2 $F \cup \{B_n : n \in \omega\}$ still has the f.i.p. Then we can find a set d_α so that $f_\alpha | d_\alpha$ is one-to-one, and $d_\alpha \cap e_n \cap h^{-1}[m]$ is infinite for $m \in A_n$. Let $J_\alpha = J$, $h_\alpha = h$. The induction hypothesis still holds.

Case 3 Neither case 1 nor case 2 hold.
So for some $n \in \omega$, B_n cannot be added to F.
Certainly $d_\alpha = e_n \cap C_\omega(B_n)$ is already in F.
Without loss of generality we can assume that

$|f_\alpha[h^{-1}[m]]| < \omega$ for all m.

Let k be a concatenation of f_α and h. Define h_α as follows:-

k is finer than f_α.
So if $k^{-1}[n] \cap f_\alpha^{-1}[m] \neq \phi$, $k^{-1}[n] \subseteq f_\alpha^{-1}[m]$.
Let l be a function such that:

$|l[f_\alpha^{-1}[n]]| = n$ for all n.
If $k^{-1}[m] \cap e_n \cap f_\alpha^{-1}[n]$ is infinite, then $|l[k^{-1}[m] \cap e_n]| = n$ and $l^{-1}[r] \cap k^{-1}[m] \cap e_n$ is either void or infinite.

Now let h_α be the concatenation of k and l.
Let $J_\alpha = \{\omega - a: \exists m |h_\alpha[k^{-1}[n] \cap a]| < m$ for all $n\} \cup J$.

<u>Claim 1</u> $F \cup \{d_\alpha\} \cup \bigcup J_\alpha$ has the f.i.p.

<u>Proof</u> Say $\omega - a$ is such that $|h_\alpha[k^{-1}[n] \cap a]| < m$ for all n, Let $n \in \omega$, and let $b \in J$.

Case 1 did not occur. Hence we can find $m' > m$, so that $e_n \cap f_\alpha^{-1}[m'] \cap k^{-1}[n']$ is infinite, for some n' so that $k^{-1}[n'] \subseteq b$. Then certainly $|h_\alpha[k^{-1}[n'] \cap e_n \cap f_\alpha^{-1}[m'] \cap (\omega - a)]| \geq m' - m$. Certainly $b \cap e_n \cap d_\alpha \cap (\omega - a) \neq \phi$.

It is easy to check that the induction hypothesis is still true.

Finally, let q be an ultrafilter extending $\bigcup_{\beta<\omega_1} F_\beta$. Firstly q is not a p-point. Let $g \in {}^\omega\omega$. $g = f_\alpha$ for some α.

<u>Claim 2</u> ${\omega^+}^\omega/f_\alpha(q)$ is not embeddable as a proper initial segment of ${\omega^+}^\omega/q$.

<u>Proof</u> Suppose Case 1 occured. Then $f_\alpha(q)$ is not a f.u.f.
Suppose Case 2 occured, Then f_α is one-to-one on

a set d_α in q. So $\omega^{+^\omega}/f_\alpha(q)$ is isomorphic to ω^{+^ω}/q.

Suppose Case 3 occured. l is a function such that $|l[f_\alpha^{-1}[n]]| < \omega$ for all n, yet for no $a \in q$ is $|l[f_\alpha^{-1}[n] \cap a]| = 1$ for all n.

By Lemma 4·43, $\omega^{+^\omega}/f_\alpha(q)$ is not embeddable as an initial segment of ω^{+^ω}/q.

q is $<_{IS}$-minimal but not $<_{EG}$-minimal.

Remark 4·46 q has only one constellation in its top sky, so q is an example of a rare ultrafilter that is not a p-point.

Remark 4·47 Though $\omega^{+\omega}/q$ has no proper initial segment that is an ultrapower, it has as initial segment that is a limit ultrapower.

viz $\omega^{+^\omega}/q|G$, where G is the filter generated by $\{eq(h): h \in H\}$

This is an example of an elementary substructure of an ultrapower that is not an ultrapower, as promised in 4·2.

4·5

So far we have classified ultrafilters by their topological properties and by their sky and constellation sets. The question now arises: how complete is this classification?

Firstly note that neither collection of properties is sufficient by itself to categorize all the properties of ultrafilters. In [14] an example is given of two ultrafilters with the same sky and constellation configuration yet with different topological properties, and in [15] it is shewn that any two p-points have the same topological properties, though one may be rare and the other not.

<u>Problem</u> If p and q are f.u.f.s with isomorphic sky and constellation sets and with an auto-homeomorphism of N^* mapping p to q, (so that p and q have the same topological properties), find a property Φ possessed by p but not by q.

Of course, we wish to exclude the cases when Φ is of the form "$a \in p$" for some $a \subseteq \omega$. So we require that Φ is invariant under permutations, that is, if $\Phi(p)$ holds, and π is a permutation of the integers, $\Phi(\pi(p))$ holds.

The simplest case is to find some permutation invariant property possessed by some but not all Ramsey ultrafilters. I have not been successful in looking for such a property. In fact, I would conjecture that:

1) There is a model of Z.F.C. + C.H. in which every Ramsey ultrafilter has the same permutation invariant properties.

2) There is a model of Z.F.C. + C.H. in which the following holds: whenever p and q are f.u.f.s

with isomorphic sky and constellation sets and with an auto-homeomorphism of N^* mapping p to q then p and q possess the same permutation invariant properties.

<u>Remark</u> It cannot be true that in every model of Z.F.C. + C.H. every Ramsey ultrafilter has the same permutation invariant properties, for if V = L, there is a definable well-ordering of the subsets of ω which can be used to define a Ramsey ultrafilter p_0. So if we take Φ to be "p is isomorphic to p_0", some but not all Ramsey ultrafilters possess this property.

We can find a property shared by some but not all Ramsey ultrafilters if we assume Martin's Axiom + $2^{\aleph_0} > \aleph_1$.

<u>Def 4·51</u> If P is a partially ordered set, we say $D \subseteq P$ is <u>dense</u> iff $\forall x \in P, \exists y \in D, \ y \leqslant x$.

<u>Def 4·52</u> If $x, y \in P$, we say x and y are <u>compatible</u> if there is $z \in P$, $z \leqslant x$ and $z \leqslant y$.

Martin's Axiom is the following statement:-

<u>4·53</u> Whenever P is a partially ordered set, and S is a collection of dense subsets of P, and $|P| < 2^{\aleph_0}$, and $|S| < 2^{\aleph_0}$, and every set of mutually incompatible elements is at worst countable, then there is a set $G \subseteq P$ such that every two members of G are compatible and $G \cap D \neq \emptyset$ for every $D \in S$.

We abbreviate this to M.A. The set G found is said to be generic for S.

It can be shewn that C.H. implies M.A., yet it is consistent that M.A. and $2^{\aleph_0} > \aleph_1$. See [16]

<u>Def 4·54</u> For q a f.u.f., we say q is <u>Super-Ramsey</u> if it is Ramsey and whenever $S \subseteq q$, $|S| < 2^{\aleph_0}$, there is $a \in q$, $|a - b| < \omega$ for all $b \in S$.

<u>Theorem 4·55</u>

1) M.A. implies there are Super-Ramsey ultrafilters.

2) M.A. + $2^{\aleph_0} > \aleph_1$ implies that there are Ramsey ultrafilters that are not Super-Ramsey.

<u>Proof</u> 1) is due to Booth [3]. It follows from the next lemma by using the construction of 2·69.

<u>Lemma 4·56</u> M.A. implies that if F is a non-principal filter generated by $\kappa < 2^{\aleph_0}$ sets, then there is an infinite $a \subseteq \omega$, $|a - b| < \omega$ for all $b \in F$.

<u>Proof of 2)</u> Suppose $2^{\aleph_0} = \lambda > \aleph_1$. Let $\langle a_\alpha : \alpha < \omega_1 \rangle$ be a sequence of sets such that

1) $|a_\alpha - a_\beta| < \omega$ for $\alpha > \beta$.

2) $|a_\beta - a_\alpha| = \omega$ for $\alpha > \beta$.

We will construct a Ramsey ultrafilter p such that $a_\alpha \in p$ for all $\alpha < \omega_1$, yet for no $a \in p$, $|a - a_\alpha| < \omega$ for all $\alpha < \omega_1$.

Enumerate ${}^\omega\omega$ as $\langle f_\beta : \omega_1 \leqslant \beta < \lambda \rangle$.

For every β, $\omega_1 \leq \beta < \lambda$ we will add a set d_β such that f_β is one-to-one or constant on d_β, and if e is a member of the filter generated by $\{d_\beta : \beta < \lambda\}$, then $|e - a_\alpha| = \omega$ for some $\alpha < \omega_1$. Certainly $|e - a_\delta| = \omega$ for all $\delta \geq \alpha$.
For convenience let $d_\beta = a_\beta$ for $\beta < \omega_1$.
Suppose we have found d_γ for all $\gamma < \beta$, $\beta \geq \omega_1$.
Let $|\beta| = \kappa < 2^{\aleph_0}$. Let F be generated by $\{d_\gamma : \gamma < \beta\}$.
Let $\{e_\gamma : \gamma < \kappa\}$ be a base for F; we can assume that this base is closed under finite intersection.

<u>Induction Assumption</u> For every γ there is $\alpha < \omega_1$, $|e_\gamma - a_\alpha| = \omega$.

Consider f_β. First we try to make f_β constant on d_β.

<u>Case 1</u> For some $n \in \omega$, for all γ there is α, $|e_\gamma \cap f_\beta^{-1}[n] - a_\alpha| = \omega$. Then let $d_\beta = f_\beta^{-1}[n]$.

<u>Case 2</u> Case 1) did not occur. We will make f_β one-to-one on d_β.

<u>Claim</u> For all γ there is $\alpha_\gamma < \omega_1$ such that $|f_\beta[e_\gamma - a_{\alpha_\gamma}]| = \omega$.

<u>Proof</u> Fix γ. Suppose the claim does not hold at γ. So $|f_\beta[e_\gamma - a_\alpha]| < \omega$ for all $\alpha < \omega_1$.

Let $A_\alpha = \{n : |f_\beta^{-1}[n] \cap (e_\gamma - a_\alpha)| = \omega\}$. Then A_α is finite for all α, and as $\alpha > \beta$ implies that

$|a_\alpha - a_\beta| < \omega$, $\alpha > \beta$ implies $A_\alpha \supseteq A_\beta$.

So for some α^*, A_α must remain fixed for $\alpha \geq \alpha^*$. Case 1 did not hold. So for all $n \in \omega$, there is γ_n, so that for all α,
$|e_{\gamma_n} \cap f_\beta^{-1}[n] - a_\alpha| < \omega$. Let $e = \bigcap_{n \in A_{\alpha^*}} e_{\gamma_n}$.
Then $|e \cap f_\beta^{-1}[n] - a_\alpha| < \omega$ for all $n \in A_{\alpha^*}$, all α.
Hence $|e \cap e_\gamma - a_\alpha| < \omega$ for all α, contradicting the induction hypothesis for $e \cap e_\gamma$.

Define a partially ordered set P as follows:
The elements of P are of the form $<s,t>$, where $s = <<n_1,m_1>,\ldots<n_i,m_i>>$ for $f_\beta(n_j) = m_j$, $1 \leq j \leq i$, and $n_j = n_k$ iff $m_j = m_k$, $1 \leq j,k \leq i$.
t is a finite subset of κ.

We say $<s',t'> \leq <s,t>$ iff
1) s' extends s.
2) t' includes t.
3) if $<s',t'> \neq <s,t>$, then for every $\gamma \in t$ there is $<n,m> \in s' - s$, $n \in e_\gamma - a_{\alpha_\gamma}$.
Now, $|P| = \kappa < \lambda$.
$<s,t>$ and $<s',t'>$ are compatible if $s = s'$, and so every set of mutually incompatible elements is at worst countable.

Let $A_\gamma = \{<s,t>: \gamma \in t\}$ for all $\gamma < \kappa$.
Let $B_n = \{<s,t>: |s| \geq n\}$ for all $n \in \omega$.

By the claim, each A_γ and B_n is dense. So let G be a generic set meeting them.
Let $d_\beta = \{n:$ for some $<s,t> \in G$, $n \in dom(s)\}$

Then d_β is an infinite set, as G meets every B_n.

If $<n,m> \in s$ where $<s,t> \in G$, and $<n',m> \in s'$ where $<s',t'> \in G$, then as $<s,t>$ and $<s',t'>$ are compatible, $n' = n$. So $f_\beta | d_\beta$ is one-to-one.

Also G meets every A_γ. Hence for every γ and every n d_β will contain at least n members of $e_\gamma - a_{\alpha_\gamma}$. So $|d_\beta \cap e_\gamma - a_{\alpha_\gamma}| = \omega$.

The filter generated by $F \cup \{d_\beta\}$ is proper and obeys the induction hypothesis.

Finally let q be generated by $\{d_\beta : \beta < \lambda\}$. q is a Ramsey ultrafilter that is not Super-Ramsey.

Chapter 5 Ultrafilters without the Continuum Hypothesis.

5·1

Classification of ultrafilters becomes very difficult when the C.H. is no longer assumed. The special sorts of ultrafilters discussed previously do not necessarily exist in all models of set theory. For example:

Theorem 5·11 (Kunen, unpublished)

If M is a model of Z.F.C. obtained by adding $\aleph_2$ random reals to L, there is no Ramsey ultrafilter.

But we noted in 4·5 that M.A. implies that there are Ramsey ultrafilters.

In fact, in his thesis [2], Blass even conjectured that it is consistent with Z.F.C. that there are no special sorts of ultrafilter at all; that is, for every permutation invariant formula Φ there is a model of Z.F.C, in which either every f.u.f. possesses this property or no f.u.f. posses this property. We produce a counterexample to this conjecture. Firstly we need a result of Kunen [10].

Theorem 5·12 There is an ultrafilter which is not generated by less than $2^{\aleph_0}$ sets. (Though it is consistent with Z.F.C. that $2^{\aleph_0} > \aleph_1$ and there is an ultrafilter generated by $\aleph_1$ sets.)

Recall that if q is a f.u.f., $q \times q = \{a \subseteq \omega \times \omega : \{n : \{m : \langle m,n\rangle \in a\} \in q\} \in q\}$. This is then a non-principal ultrafilter over $\omega \times \omega$.

Our sentence Φ is:-

$\Phi(p)$ iff "there is an ultrafilter generated by less than $2^{\aleph_0}$ sets and p is one such or else every ultrafilter is generated by at least $2^{\aleph_0}$ sets and p is isomorphic to an ultrafilter of the form $q \times q$, for some f.u.f. q."

<u>Theorem 5·13</u> Φ is permutation invariant and some but not all f.u.f.s have property Φ.

<u>Proof</u> The only non-trivial part is to shew that if no ultrafilter is generated by less than $2^{\aleph_0}$ sets then there is an ultrafilter p not isomorphic to $q \times q$ for some q. We assume that no ultrafilter is generated by less than $2^{\aleph_0}$ sets and construct p by induction.

Enumerate the bijections from ω to $\omega \times \omega$ as $\langle f_\alpha : \alpha < 2^{\aleph_0}\rangle$. For every $\alpha < 2^{\aleph_0}$ we will construct a filter F_α such that $f_\alpha(F_\alpha)$ cannot be extended to $q \times q$ for any f.u.f. q.

<u>Induction Hypothesis:-</u>

1) $\alpha > \beta \rightarrow F_\alpha \supseteq F_\beta$.

2) F_α is generated by at most $|\alpha| + \omega$ sets.

Suppose we have constructed F_β for all $\beta < \alpha$. Let F be generated by $\bigcup_{\beta<\alpha} F_\beta$. F is generated by at most $|\alpha| + \omega$ sets.

Let $G = f_\alpha(F)$. Let π_1 and π_2 denote the projections of $\omega x \omega$ onto the first and second co-ordinates respectively.

<u>Case 1</u> For some i, if we let $a = \{i\} \times \omega$, then $G \cup \{a\}$ has the f.i.p. Suppose qxq extends $G \cup \{a\}$.

Then $\{m: \langle m,n\rangle \in a\} = \{i\} \in q$. q is principal. Let F_α be generated by $\{f_\alpha^{-1}[b]: b \in G \cup \{a\}\}$. F_α is still generated by $|\alpha| + \omega$ sets.

<u>Case 2</u> For some j, if we let $a = \omega \times \{j\}$, then $G \cup \{a\}$ still has the f.i.p. Suppose qxq extends $G \cup \{a\}$. Then $\{n: \{m: \langle m,n\rangle \in a\} \in q\} = \{j\} \in q$. So q is principal. Let F_α be generated by $\{f_\alpha^{-1}[b]: b \in G \cup \{a\}\}$. F_α is still generated by $|\alpha| + \omega$ sets.

<u>Case 3</u> Neither case 1 nor case 2 occur.
So neither $\pi_1(G)$ nor $\pi_2(G)$ can be extended to a principal ultrafilter. But $\pi_1(G)$ is generated by less than $2^{\aleph_0}$ sets, and so cannot be an ultrafilter.

Let $a_1 \subseteq \omega$ be such that both $\pi_1(G) \cup \{a_1\}$ and $\pi_1(G) \cup \{\omega - a_1\}$ possess the f.i.p.

The filter generated by $\pi_1(G) \cup \{a_1\}$ is still generated by less than $2^{\aleph_0}$ sets, so let $a_2 \subseteq a_1$

be such that both $\pi_1(G) \cup \{a_2\}$ and $\pi_1(G) \cup \{a_1 - a_2\}$ possess the f.i.p.

Re-iterate this process to obtain a sequence of sets $a_1 \supset a_2 \supset \ldots \supset a_n \supset \ldots$ such that $\pi_1(G) \cup \{a_n\}$ and $\pi_1(G) \cup \{a_n - a_{n+1}\}$ possess the f.i.p., for every n. Let $b = \{<m,n>: m \notin a_n\}$.
Let G' be generated by $G \cup \{b\} \cup \{a_n \times \omega: n \in \omega\}$.

<u>Claim 1</u> G' is a proper filter.

<u>Proof</u> Let $a_{n_1} \times \omega, \ldots a_{n_i} \times \omega$ be a finite subset of $\{a_n \times \omega: n \in \omega\}$.
Take $r > \max\{n_1, \ldots n_i\}$. Then $a_r \times \omega \subseteq a_{n_j} \times \omega$, $1 \leq j \leq i$.
Let $c = \omega \times \{s: s > r\}$. Then $c \in G$ already, as case 2 did not occur.

Let $d \in G$. We shew that $d \cap b \cap (a_r \times \omega) \neq \phi$.
$\pi_1(G) \cup \{a_r - a_{r+1}\}$ possesses the f.i.p. So $G \cup \{(a_r - a_{r+1}) \times \omega\}$ possesses the f.i.p. In particular, $d \cap c \cap (a_r - a_{r+1}) \times \omega \neq \phi$
Let $<m,n> \in d \cap c \cap (a_r - a_{r+1}) \times \omega$.
Then $n > r$ as $<m,n> \in c$. $m \notin a_{r+1}$ but $m \in a_r$.
Certainly $m \notin a_n$.
So $<m,n> \in b \cap d \cap a_r \times \omega$.

<u>Claim 2</u> G' cannot be extended to an ultrafilter of the form qxq, for q a f.u.f.

<u>Proof</u> Suppose not. Let $q \times q \supseteq G'$. Certainly

$\pi_1(G') \subseteq \pi_1(qxq) = q$.

In particular $a_n \in q$ for every n.

Hence $\{m: \langle m,n\rangle \in C_{\omega x\omega}(b)\} = a_n \in q$ for every n.

$\{n: \{m: \langle m,n\rangle \in C_{\omega x\omega}(b)\} \in q\} = \omega \in q$.

So $C_{\omega x\omega}(b) \in qxq$, contradicting the fact that $b \in qxq$.

Let F_α be generated by $\{f_\alpha^{-1}[d]: d \in G'\}$. F_α is generated by $|\alpha| + \omega + 1 = |\alpha| + \omega$ sets.

Finally let p extend $\cup\{F_\alpha: \alpha < 2^{\aleph_0}\}$. p is never isomorphic to qxq, for q a f.u.f.

Remark 5·14 As noted in 3·2 the property of having p as a ${}_{RF}$>-predecessor is a topological invariant. Also an ultrafilter is generated by less than $2^{\aleph_0}$ sets iff in N^* it has a neighbourhood base of power less than $2^{\aleph_0}$. This is also a topologically invariant property.

So if we define Φ' by:-

Φ'(p) iff "there is a point of N^* with a neighbourhood base of power less than $2^{\aleph_0}$ and p is one such or else no point of N^* has a neighbourhood base of power less than $2^{\aleph_0}$ and p has a ${}_{RF}$>-predecessor isomorphic to qxq, for some $q \in N^*$".

Then a modification of Theorem 5·13 will shew that some but not all ultrafilters have the property Φ', and that Φ' is a topologically invariant property.

Remark 5·15 These properties Φ and Φ' are not very natural or significant, and it is doubtful whether they can be used for some interesting classification of ultrafilters.

5·2

As mentioned at the beginning of Chapter 4, if the C.H. holds, and $\mathfrak{A}$ is a countable model with a countable language, and p is a f.u.f., $\mathfrak{A}^{\omega}/p$ is saturated.

This is not necessarily true if the C.H. is no longer assumed. Let us consider the order type of ω^{ω}/p.

Def 5·21 An order type S is said to be an η_{α}-set if whenever $A, B \subseteq S$, $0 \leq |A|, |B| < \aleph_{\alpha}$, and $A < B$, (that is, if $a \in A$ and $b \in B$ $a < b$) then there is $c \in S$, $A < c < B$.

η_{α}-sets are $\aleph_{\alpha}$-saturated order types. If X and Y are η_{α}-sets of cardinality $\aleph_{\alpha}$ they are isomorphic. As all f.u.f.s are $\aleph_1$-good, the order type of ω^{ω}/p is $\omega + (\omega^{*} + \omega)\eta$ where η is an η_1-set.

Let the order type of ω^{ω}/p be denoted by $\omega + (\omega^{*} + \omega)\eta_p$. First note that if $2^{\aleph_0} = \aleph_{\alpha} > \aleph_1$ it does not necessarily follow that η_p is not a

η_α-set, for every f.u.f p. In fact,

<u>Theorem 5·22</u> M.A. implies that there is a f.u.f. p such that η_p is a η_α-set, where $2^{\aleph_0} = \aleph_\alpha$.

<u>Remark 5·23</u> Solovay, Silver and Rucker (unpublished) have proved a stronger result, that M.A. implies that there is an ultrafilter p such that for every countable model $\mathfrak{A}$ with a countable language, $\mathfrak{A}^\omega/p$ is saturated. The proof of this result is by a generalization of the proof of 5·22; we will give a sketch proof of 5·22.

<u>Proof</u> We will consider all the possible pairs <A,B> in ω^ω/p such that A < B. We construct p by induction; suppose at stage γ we have a filter generated by S, $|S| < 2^{\aleph_0}$, and have to consider the $\gamma^{\underline{th}}$ pair <A,B>.

Define a partially ordered set P by: an element of P is of the form <r,s,u,v> where r is a function from a finite subset of ω to ω, $s \in S_\omega(S)$, $u \in S_\omega(A)$, $v \in S_\omega(B)$.
We say $\langle r',s',u',v'\rangle \leqslant \langle r,s,u,v\rangle$ whenever

1) r' extends r, $s' \supseteq s$, $u' \supseteq u$, $v' \supseteq v$.

2) If $\langle n,m\rangle \in r' - r$, then $n \in d$ for all $d \in s$, $f(n) < m$ for all $f \in u$, $m < g(n)$ for all $g \in v$.

Then P has no uncountable set of mutually incomparable elements, and $|P| < 2^{\aleph_0}$.

Define dense sets as follows:

$$A_b = \{<r,s,u,v>: b \in s\} \text{ for each } b \in S.$$
$$B_f = \{<r,s,u,v>: f \in u\} \text{ for each } f^\sim \in A.$$
$$C_g = \{<r,s,u,v>: g \in v\} \text{ for each } g^\sim \in B.$$
$$D_n = \{<r,s,u,v>: |r| \geq n\} \text{ for each } n \in \omega.$$

Let G be a generic set meeting them all. Define a partial function h by

$$h(n) = m \text{ iff } \exists <r,s,u,v> \in G, \ <n,m> \in r.$$

Let $d = \mathrm{dom}(h)$. Then we can add d to the filter, and if q is a f.u.f. extending it, in ω^ω/q,

$$A < h^\sim < B.$$

<u>5·3</u>

We now introduce the notion of a scale.

<u>Def 5·31</u> If $f,g \in {}^\omega\omega$, we write $f \;{}_s\!> g$ iff there is $k \in \omega$, for all $n \geq k$, $f(n) > g(n)$. ${}_s\!>$ is a partial order, and a <u>Scale</u> is a subset S of ${}^\omega\omega$, cofinal in ${}^\omega\omega$ under ${}_s\!>$, (i.e. for all $g \in {}^\omega\omega$ there is $f \in S$, $f \;{}_s\!> g$), which is totally ordered by ${}_s\!>$.

If the C.H. holds, it is easy to construct a scale. But they do not necessarily exist. In fact, it has been pointed out by various people (nowhere published, however) that it is consistent with Z.F.C. + $2^{\aleph_0} > \aleph_1$ that

1) There is no scale.

2) There is a scale of cardinality less than $2^{\aleph_0}$.

3) There is a scale of cardinality $2^{\aleph_0}$.

M.A. implies 3), which will be shewn later.

Def 5·32 For S an ordered set, the Upward Cofinality of S is the least cardinal of a set $S' \subseteq S$ such that $\forall x \in S, \exists y \in S', x < y$.

The downward cofinality of S is the least cardinal of a set $S' \subseteq S$ such that $\forall x \in S, \exists y \in S', y < x$.

Then obviously, if there is a scale of cardinality κ, for any f.u.f. p, the upward cofinality of η_p is κ. Also,

Theorem 5·33 If there is a scale of cardinality κ, and q has a least sky, then the downward cofinality of η_q is also κ.

Proof If f is in the bottom sky of q, f(q) is a p-point. Without loss of generality, we can assume that q itself is a p-point. So for every $g \in {}^{\omega}\omega$, we can assume that g is finite-to-one.

Firstly suppose that $S \subseteq {}^{\omega}\omega$, $|S| < \kappa$. We can find $\psi: {}^{\omega}\omega \rightarrow {}^{\omega}\omega$ which "inverts the axes", that is $f \;{}_{s}\!> g$ iff $\psi(g) \;{}_{s}\!> \psi(f)$. Find $h \in {}^{\omega}\omega$ so that $h \;{}_{s}\!> \psi(f)$ for every $f \in S$. We can re-invert the

axes, finding a function h' that is non-decreasing and $f_s > h'$ for all $f \in S$.
So the set $\{f^{\sim}: f \in S\}$ is bounded below in $\omega^{\omega}/q - \omega$.

Conversely, let the scale be S. Invert the axes by ψ to find $S' = \psi[S]$, then for any non-decreasing function h, there is $g \in S'$, $h_s > g$. So the downward cofinality of η_q is precisely κ.

In [2], Blass uses the following hypothesis as a substitute for C.H.

<u>Def 5·34</u> FRH(ω) iff "Any filter generated by less than $2^{\aleph_0}$ sets is contained in a filter generated by at most $\aleph_0$ sets."

FRH(ω) is equivalent to:

If F is a non-principal filter generated by less than $2^{\aleph_0}$ sets, then there is an infinite $a \subseteq \omega$, $|a - b| < \omega$ for all $b \in F$.

It was stated in Chapter 4 that M.A. implies FRH(ω). We now shew:-

<u>Theorem 5·35</u> FRH(ω) implies that there is a scale of cardinality $2^{\aleph_0}$.

<u>Proof</u> This follows from the following lemma by induction up to $2^{\aleph_0}$.

<u>Lemma</u> FRH(ω) implies that if $S \subseteq {}^{\omega}\omega$, $|S| < 2^{\aleph_0}$, there

is $f \in {}^{\omega}\omega$, $f \underset{s}{\geq} g$ for all $g \in S$.

<u>Proof</u> Without loss of generality we can assume that every $g \in S$ is non decreasing. Let $\langle a_n : n \in \omega \rangle$ partition ω into infinite sets.

For each $g \in S$ define $a_g \subseteq \omega$ so that $a_g \cap a_n = \{m \in a_n : m \geq \text{the } g(n)^{\underline{th}} \text{ member of } a_n\}$. Then $|a_g \cap a_n| = \omega$ for all n.

Let F be the filter generated by $\{C_\omega(a_n) : n \in \omega\} \cup \{a_g : g \in S\}$. F is a proper non-principal filter generated by less than $2^{\aleph_0}$ sets.

Use $FRH(\omega)$ to find a set $a \subseteq \omega$, $|a - b| < \omega$ for all $b \in F$. $|a - C_\omega(a_n)| < \omega$ for all n, and a is infinite, so $\{n : a \cap a_n \neq \phi\} = T$ is infinite. Enumerate T as $\{n_i : i \in \omega\}$. Define f as follows:-

If $n_i < n \leq n_{i+1}$, $f(n) = m$, where if r is the first member of $a \cap a_{n_{i+1}}$, r is the $m^{\underline{th}}$ member of $a_{n_{i+1}}$.

<u>Claim</u> For $g \in S$, $f \underset{s}{\geq} g$

<u>Proof</u> $|a - a_g| < \omega$. Take i_0 so great that $a - a_g \subseteq \bigcup_{n<i_0} a_n$. This is possible. Then if $n \geq i_0$, say $n_i < n \leq n_{i+1}$, and $f(n) = m$, the $m^{\underline{th}}$ member of $a_{n_{i+1}}$ is certainly in $a_g \cap a_{n_{i+1}}$.
Hence $f(n) = f(n_{i+1}) \geq g(n_{i+1}) \geq g(n)$. This proves the

Claim and the lemma.

We can obtain cofinal subsets of ${}^{\omega}\omega$ if we have rare filters.

Theorem 5·36 If there is a rare filter generated by S, $|S| = \kappa$, then there is a cofinal subset of ${}^{\omega}\omega$ (under ${}_{s}>$) of power κ.

Proof For each $b \in S$, define f_b by $f_b(n)$ = the $n+1^{\underline{th}}$ member of b. Suppose $f \in {}^{\omega}\omega$. Without loss of generality we can assume that f is strictly increasing.

Define a partition of ω by $a_n = \{m : f(n-1) < m \leqslant f(n)\}$. Then as S generates a rare filter, there is $b \in S$, $|b \cap a_n| \leqslant 1$ for all n.

Then certainly the $n^{\underline{th}}$ member of b is greater than $f(n-1)$. So $f_b(n) > f(n)$ for all n.

Corollary 5·37 If there is a rare filter generated by $\aleph_1$ sets, there is a scale of cardinality $\aleph_1$.

5·4

Now we connect scales with other properties of ultrafilters.

Def 5·41 Abbreviate the hypothesis "there is a scale of cardinality $2^{\aleph_0}$" to C.S.

C.S. is quite a powerful hypothesis.

<u>Theorem 5·42</u> C.S. implies that no ultrafilter is generated by less than $2^{\aleph_0}$ sets.

<u>Proof</u> Let F be a filter generated by S, where S is closed under finite intersections and $|S| < 2^{\aleph_0}$.

For $a \subseteq \omega$, a infinite, define $f_a \in {}^{\omega}\omega$ by, $f_a(n) =$ the n^{th} member of a.

Then we can use C.S. to find $f \in {}^{\omega}\omega$, $f \;{}_{S}\!> f_a$ for every $a \in S$.

We define two sequences $\langle a_n : n \in \omega \rangle$ and $\langle b_n : n \in \omega \rangle$ of finite sets as follows:-

Let $a_1 = \{$the first $f(1)$ members of $\omega.\}$
If we have defined $a_1, \ldots a_n$, let $|a_1 \cup \ldots \cup a_n| = m$ and let $r = \max\{a_1 \cup \ldots \cup a_n\}$.

Then let $b_n = \{i : r < i \leqslant f(m+1)\}$ if this is non-empty, and $b_n = \{r+1\}$ otherwise.

If we have defined $b_1, \ldots b_n$, let $|b_1 \cup \ldots \cup b_n| = m$ and let $r = \max\{b_1 \cup \ldots \cup b_n\}$.

Then let $a_{n+1} = \{i : r < i \leqslant f(m+1)\}$ if this is non-empty, and let $a_{n+1} = \{r+1\}$ otherwise.

Let $a = \bigcup_{n \geqslant 1} a_n$, and $b = \bigcup_{n \geqslant 1} b_n$. Then $a \cup b = \omega$.

Suppose $a \in F$. Then for some $c \in S$, $a \supseteq c$. Certainly $f_a(n) \leqslant f_c(n)$ for all n.

But by the construction of a, for infinitely many m's, the $(m+1)^{\underline{th}}$ member of a occurs after $f(m+1)$. So $f_a(m+1) > f(m+1)$. This contradicts the fact that $f_s > f_c$. So $a \notin F$ and by similar arguments $b \notin F$. F is therefore not an ultrafilter.

<u>Theorem 5·43</u> C.S. implies that there are p-points.

<u>Proof</u> Enumerate ${}^{\omega}\omega$ as $<f_\alpha : \alpha < 2^{\aleph_0}>$. At each step α we will add a set a_α so that f_α is either constant or finite-to-one on a_α. The filter generated at stage α is F_α.

<u>Stage 0</u> Let $F_0 = Fr$.

<u>Stage α</u> Suppose we have constructed F_β for all $\beta < \alpha$. Let F be generated by $\bigcup_{\beta<\alpha} F_\beta$. F has less than $2^{\aleph_0}$ generators, so let them be S. Assume that S is closed under finite intersection.

<u>Case 1</u> For some $n \in \omega$, $f_\alpha^{-1}[n] \cup G$ has the f.i.p. Let $a_\alpha = f_\alpha^{-1}[n]$.

<u>Case 2</u> Otherwise. Then for all $b \in S$, $\{n : b \cap f_\alpha^{-1}[n] \neq \phi\}$ is infinite. For each $b \in S$, define g_b as follows:

If $b \cap f_\alpha^{-1}[n] = \phi$, then $g_b(n) = 0$.

If r is the first element of $f_\alpha^{-1}[n] \cap b$, and r is the $m^{\underline{th}}$ element of $f_\alpha^{-1}[n]$, then $g_b(n) = m$.

Here we have $|S| < 2^{\aleph_0}$ functions. Let $f \in {}^\omega\omega$ be such that $f >_s g_b$ for all $b \in S$.

Define $a_\alpha \subseteq \omega$ to be such that $a_\alpha \cap f_\alpha^{-1}[n] = \{$the first $f(n)$ members of $f_\alpha^{-1}[n]\}$.

Then $|a_\alpha \cap f_\alpha^{-1}[n]| < \omega$ for all n, so $f_\alpha | a_\alpha$ is finite-to-one. We shew $b \cap a_\alpha \neq \phi$ for all $b \in S$.

Fix b. Let k be so great that $m \geqslant k$ implies $f(m) > g_b(m)$. For some $n \geqslant k$, $b \cap f_\alpha^{-1}[n] \neq \phi$. Then if $r \in b \cap f_\alpha^{-1}[n]$, r is among the first $g_b(n)$ elements of $f_\alpha^{-1}[n]$, so it is certainly among the first $f(n)$ elements of $f_\alpha^{-1}[n]$. $r \in b \cap a_\alpha$. Let F_α be generated by $F \cup \{a_\alpha\}$.

Finally let q be generated by $\cup\{F_\alpha : \alpha < 2^{\aleph_0}\}$. By our construction, q is a p-point.

6·5

Conclusion

This chapter has been a very incomplete exposition of the properties of ultrafilters without using the C.H. Let us list some of the questions that have been raised implicitly.

1) Does $2^{\aleph_0} > \aleph_1$ imply there is a f.u.f. p such that η_p is not a η_α-set, where $2^{\aleph_0} = \aleph_\alpha$? In particular, does M.A. imply this?

2) If there is a scale, and p does not have a bottom sky, what is the downward cofinality of η_p?

3) If there is no scale, can one find f.u.f.'s p and q so that the upward cofinalities of η_p and η_q are different?

4) Does C.S. imply FRH(ω)?

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