HAUSDORFF MEASURE FUNCTIONS

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CONTENTS

ABSTRACT

In many works on Hausdorff Measure Theory it has been the practice to place certain restrictions on the measure functions used. These restrictions usually ensure both the monotonicity and the continuity of the functions. The aim of the first four chapters of this thesis is to find conditions under which the restrictions of continuity and monotonicity may be relaxed.

In the first chapter we deal with the monotonicity condition with respect to both measures and pre-measures. The second and third chapters are concerned with an investigation of the continuity condition with regard to measures and pre-measures, respectively. Then, having found conditions under which these restrictions may or may not be relaxed, we are able, in the fourth chapter, to generalize some known results to the case of discontinuous and non-monotonic functions.

Some of the results of the first four chapters prompted an investigation of the properties of measures corresponding to sequences of measure functions, and this is incorporated in the fifth chapter.

The main purpose of the final chapter is to determine whether or not some of the results of the earlier chapters may be extended to Hilbert space.

PREFACE

I should like to thank my supervisor, Professor H. G. Eggleston, for the advice he has given me in our many discussions on this work. The results contained in this thesis are, to the best of my knowledge, new, although many of them have been proved in less general cases; in these situations the known results are acknowledged in the introductions to the relevant chapters. Finally, I should like to express my gratitude to the Science Research Council for my studentship.

DEFINITIONS AND NOTATION

For any function $h(x)$ we define $h(x^-)$ and $h(x^+)$ as follows,

$$h(x^-) = \underline{\lim}_{y \to x} h(y)$$

$$h(x^+) = \overline{\lim}_{y \to x} h(y).$$

We note that if $h(x)$ is a monotonic increasing function we have,

$$h(x^-) = h(x-0)$$

$$h(x^+) = h(x+0)$$

If x is a point of discontinuity of $h(x)$ we shall define the size of the discontinuity at x to be $(h(x^+) - h(x^-))$.

We say that $h(x)$ is a Hausdorff measure function if it satisfies the following conditions,

$$\text{i). } \quad h(x^-) > 0 \quad \text{for} \quad x > 0$$

$$\text{ii). } \quad h(x) \to 0 \quad \text{as} \quad x \to 0$$

If $h(x)$ satisfies i). and ii). above as well as,

$$\text{iii). } \quad \underline{\lim}_{x \to 0} \frac{h(x)}{x^q} > 0,$$

for some positive integer q, we say that $h(x)$ is a q-dimensional Hausdorff measure function.

If S is any set in a metric space (X, ρ) say, we shall denote by $d(S)$ the diameter of S that is,

$$d(S) = \overline{\text{bd}}\,\{\rho(x,y) : x, y \in S\}$$

Also we denote the closure of S by $\overline{S}$.

By l^2 we shall mean the space of all real number sequences $\{x_n\}$ such that $\sum x_n^2$ is convergent. We shall denote the points of l^2 by $\underline{x}$ where $\underline{x} = \{x_n\}$. Wherever indices occur, we shall write, for example, $\underline{\alpha}^{(k)} = \{\alpha_n^{(k)}\}_{n=1,2,\dots}$. If S is a set in l^2 then we write,

$$\underline{x} + S = \{\underline{x} + \underline{y} : \underline{y} \in S\}$$

where the addition is performed component-wise. We make l^2 into a metric space by introducing the metric, ρ such that for $\underline{x}, \underline{y} \in l^2$,

$$\rho(\underline{x}, \underline{y}) = \left[\sum_{n=1}^{\infty} |x_n - y_n|^2\right]^{1/2}.$$

If X, Y are sets in l^2 then,

$$\rho(\underline{\alpha}, X) = \underline{\text{bd}}\,\{\rho(\underline{\alpha}, \underline{x}) : \underline{x} \in X\}$$

and,

$$\rho(X, Y) = \underline{\text{bd}}\,\{\rho(\underline{x}, \underline{y}) : \underline{x} \in X, \underline{y} \in Y\}.$$

A set of points S in Euclidean space or l^2 is said to be convex, if whenever two points x, y belong to S all the points of the form,

$$\lambda x + (1-\lambda) y$$

where $0 < \lambda < 1$ also belong to S. If A is any set, then by $\text{conv}\,A$ we mean the smallest convex set which contains A. The following results will be assumed wherever necessary,

a). (ℓ^2, ρ) is a complete metric space (see, for example, Sierpinski (10)),

b). $d(A) = d(\operatorname{conv} A)$ (see, for example, Eggleston (4)),

and, c). $d(\operatorname{conv} A) = d(\overline{\operatorname{conv} A})$.

For any point x and any positive real number r we shall write $S(x, r)$ for the sphere centre x, radius r. Suppose P is a set in a metric space Ω and n, d are positive real numbers. Then for every point $x \in P$ we define $N(x, n, d, P)$ to be the largest number of disjoint spheres of the form $S(p, dn^{-1})$ with $p \in P$ which can meet $S(x, d)$. We write,

$$N(n, d, P) = \sup_{x \in P} N(x, n, d, P).$$

We say that a function $h(x)$ is blanketed if for all $\alpha > 0$ there exist positive real numbers $k(\alpha, h)$ and $K(\alpha, h)$ which satisfy,

$$k(\alpha, h)\, h(t) \leq h(\alpha t) \leq K(\alpha, h)\, h(t)$$

for all $t \geq 0$. Then, if $h(x)$ is a monotonic, increasing, blanketed, Hausdorff measure function we write,

$$h(P, y) = \lim_{\tau \to 0} \overline{\lim_{\substack{d \to 0 \\ n \to \infty}}} \frac{h(dn^{-1})\, N(n, d, P \cap S(y, \tau))}{h(d)}$$

for each point y of Ω. Finally, we write,

$$h(P) = \sup_{y \in P} h(P, y).$$

A set A is said to have finite dimension if there exists a monotonic increasing, blanketed, Hausdorff measure function $h(x)$ such that, $h(A)=0$. A metric space Ω is said to be a β-space if there exist positive real numbers $\delta, \alpha\ (<\frac{1}{2})$ and $N = N(\alpha)$ such that, for all $r \leq \delta$, at most $N(\alpha)$ disjoint open spheres of radius αr can meet any given open sphere of radius r. Larman (6) has shown that a compact set A in a metric space has finite dimension if and only if A is a β-space.

If $h(x)$ is a Hausdorff measure function and S is a set in Euclidean space or l^2 then, following Hausdorff (5), we define the corresponding Hausdorff pre-measure of S denoted by $\Lambda^h_\delta(S)$ as follows,

$$\Lambda^h_\delta(S) = \underset{\bigcup U_i \supset S}{\text{bd}} \sum_i h(d(U_i)),$$

where the lower bound is taken over all coverings of S by open convex sets each of diameter strictly less than δ. We then define the Hausdorff measure $\Lambda^h(S)$ ~~af~~ of S as follows,

$$\Lambda^h(S) = \lim_{\delta \to 0} \Lambda^h_\delta(S).$$

This will be referred to as the h-measure of S. We will write $\Lambda^q_\delta(S)$ and $\Lambda^q(S)$ when $h(x) = x^q$, and $\Lambda_\delta(S)$ and $\Lambda(S)$ when $h(x) = x$. The measure $\Lambda^{h, R(q)}(S)$ is defined in a similar manner to $\Lambda^h(S)$ but here we restrict the coverings to be open q-dimensional rectangles. Similarly $\Lambda^{h, C(q)}(S)$ refers to coverings by open q-dimensional cubes. $L^h_\delta(S)$ and $L^h(S)$ will refer to coverings by closed convex sets U_i where, for $L^h_\delta(S)$ we insist that $d(U_i) \leq \delta$ for all i.

Two Hausdorff measure functions $h(x)$ and $g(x)$ will be said to be measure equivalent whenever, for all sets S, $\Lambda^h(S)$ is positive and finite if and only if $\Lambda^g(S)$ is positive and finite. Also, if $h(x)$ and

$g(x)$ are Hausdorff measure functions we write $h \prec g$ if,

$$\lim_{x \to 0} g(x)/h(x) = 0.$$

For any set S we denote the complement of S by $\mathcal{C}S$ and $X \cap \mathcal{C}Y$ by $X \setminus Y$. If x is any real number we denote the greatest integer less than or equal to x by $[x]$. Finally, we shall call $\{x_n\}$ a null sequence if $x_n \to 0$ as $n \to \infty$.

CHAPTER 1

INTRODUCTION

The first theorem of this chapter shows that in all subsequent investigations of the Hausdorff measures of sets in connected metric as well as in Euclidean spaces it is sufficient to prove theorems only for the case of monotonic Hausdorff measure functions. Theorems 2 and 3 show that this result cannot always be extended to the case of Hausdorff pre-measures. Thus throughout this work the theorems concerning Hausdorff measures will be seen to be true for both monotonic and non-monotonic measure functions. Whereas those concerning pre-measures will only be proved for the case of monotonic measure functions.

Theorem 1

Given any Hausdorff measure function $h(x)$ there exists a monotonic increasing Hausdorff measure function $H(x)$ say, such that for any set S, we have,

$$\Lambda^{H}(S) = \Lambda^{h}(S).$$

Proof

We know that $h(x) > 0$ for all positive values of x, let X be a fixed positive real number. Define $H(x)$ as follows,

$$H(x) = \inf_{y \in [x, X]} h(y) \qquad \text{and} \qquad H(x) = h(X) \qquad \text{for } x \geq X \qquad -(1)$$

1). $H(x)$ is monotonic increasing. For, if x and y are such that $x \geq y$ then,

$$H(y) = \inf_{z \in [y, X]} h(z) \leq \inf_{z \in [x, X]} h(z) = H(x).$$

that is, $H(y) \leq H(x)$.

ii). Clearly $H(x) \leq h(x)$ for all x, and so, from the definition of a Hausdorff measure function, we have,

$$H(x) \to 0 \qquad \text{as} \qquad x \to 0 \qquad -(2)$$

Also from the statement made at the beginning of the proof, we know that,

$$H(x) > 0 \qquad \text{whenever} \qquad x > 0 \qquad -(3)$$

Now let S be any set (in any metric space), we certainly have, by ii).,

$$\Lambda^H(S) \leq \Lambda^h(S). \qquad -(4)$$

Clearly if $\Lambda^H(S) = \infty$, then we have $\Lambda^h(S) = \Lambda^H(S)$. So we may assume that $\Lambda^H(S) < \infty$ for the remainder of the proof.

Given any $\varepsilon > 0$ and any $\rho > 0$ choose ρ' ($< \rho$) such that,

$$H(x) < \tfrac{1}{2} H(\rho) \qquad \text{whenever} \qquad x \leq \rho' \qquad -(5)$$

this is possible by (2) and (3).

Further, choose a covering of S by open convex sets $\{U_i^{\rho'}\}$ such that,

$$S \subset \bigcup_{i=1}^{\infty} U_i^{\rho'} \qquad -(6)$$

$$d(U_i^{\rho'}) < \rho' \qquad \text{for all } i \qquad -(7)$$

and

$$\sum_i H(d(U_i^{\rho'})) < \Lambda^H_{\rho'}(S) + \varepsilon/2. \qquad -(8)$$

We now define a new open convex covering of S by sets $\{V_i\}$ as follows;

For each i,

$$\text{if } h(d(U_i^{\rho'})) = H(d(U_i^{\rho'})) \qquad \text{we put} \qquad V_i \equiv U_i^{\rho'} \qquad -(9)$$

if $h(d(U_i^{\rho'})) \neq H(d(U_i^{\rho'}))$ then we have, from (5),

$$H(d(U_i^{\rho'})) = \inf_{y \in [d(U_i^{\rho'}), X]} h(y) < \tfrac{1}{2} \inf_{y \in [\varepsilon, X]} h(y)$$

since $d(U_i^{\rho'}) < \varepsilon'$.

Thus we can choose V_i to satisfy the following conditions,

$$d(V_i) > d(U_i^{\rho'}) \qquad -(10)$$

$$U_i^{\rho'} \subset V_i \quad \text{and } V_i \text{ open, convex} \qquad -(11)$$

$$h(d(V_i)) < H(d(U_i^{\rho'})) + \varepsilon/2^{i+1} \qquad -(12)$$

and $\qquad d(V_i) < \rho$ for all i, $\qquad -(13)$

So, $\{V_i\}$ is an open convex covering of S with

$$d(V_i) < \rho \text{ for all, } i,$$

and from (8) and (12) we have,

$$\sum_i h(d(V_i)) < \sum_i H(d(U_i^{\rho'})) + \varepsilon/2$$

$$< \Lambda^H_{\rho'}(S) + \varepsilon,$$

that is, $\qquad \Lambda^h_\rho(S) < \Lambda^H_{\rho'}(S) + \varepsilon. \qquad -(14)$

Now since $\rho' < \rho$ we know that,

$$\rho' \to 0 \quad \text{as} \quad \rho \to 0$$

Thus from (14) we have,

$$\Lambda^h(S) \leq \Lambda^H(S) + \varepsilon,$$

but the ε was an arbitrary positive real number, and so,

$$\Lambda^h(S) \leq \Lambda^H(S) \qquad -(15)$$

which, combined with (4) gives us the required result,

$$\Lambda^h(S) = \Lambda^H(S).$$

Thus we have shown that as far as Hausdorff measures are concerned we can always replace a non-monotonic function by a monotonic one which is below the original function. Theorem 2 shows us that this replacement cannot be effected by using a function which is above the original one. The result is actually a little stronger in that it is proved using pre-measures.

<u>Theorem 2</u>

There exists a Hausdorff measure function $h(x)$, such that if $H(x)$ is any monotonic Hausdorff measure function with $H(x) \geq h(x)$ for all x, then,

$$\Lambda^H_\delta(S) \geq 2\,\Lambda^h_\delta(S)$$

for arbitrarily small positive numbers δ and for all sets S on the real line.

<u>Proof</u>

Define $\qquad x_n = 1/2^n \qquad$ for $n = 0, 1, 2, \ldots$

Put, $$h(x) = x_n \quad \text{for} \quad x \in (x_{n+1}, x_n]$$

with n even or zero,

and $$h(x) = 4x_n \quad \text{for} \quad x \in (x_{n+1}, x_n]$$

with n odd.

Now let $H(x)$ be a monotonic function with $H(x) \geqslant h(x)$ for all x. Then clearly we must have,

$$H(x) \geqslant 2x \qquad \text{for all } x.$$

Thus for all δ and for all sets S we must have,

$$\Lambda^H_\delta(S) \geqslant 2\Lambda_\delta(S) \tag{16}$$

But it is easy to see that for arbitrarily small positive values of δ and for all linear sets S,

$$\Lambda^h_\delta(S) = \Lambda_\delta(S) \tag{17}$$

Thus combining (16) and (17) we have,

$$\Lambda^H_\delta(S) \geqslant 2\Lambda^h_\delta(S)$$

for arbitrarily small values of δ and for all linear sets S. Hence the theorem is proved.

Theorem 3 now shows us that as far as Hausdorff pre-measures are concerned, there exist functions which cannot be replaced by monotonic ones.

Theorem 3

There exists a Hausdorff measure function $h(x)$ and a set S such that if $H(x)$ is any monotonic Hausdorff measure function then,

$$\Lambda_{\delta}^{h}(S) \neq \Lambda_{\delta}^{H}(S),$$

for arbitrarily small positive values of δ.

Proof

We define the sequence $\{x_n\}$ of positive real numbers as follows;

Put $x_1 = 1$ and assume that $x_1, \ldots, x_{2n-1}$ have been defined, then put,

$$x_{2n} = \tfrac{1}{2}\, x_{2n-1} \qquad -(18)$$

and,

$$x_{2n+1} = \left(\frac{25}{128}\right)^2 x_{2n-1} \qquad -(19)$$

We now define the function $h(x)$ as follows,

$$h(x) = x_{2n-1}^{1/2} \quad \text{for } x \in (x_{2n}, x_{2n-1}] \qquad -(20)$$

$$= 4x_{2n-1}^{-1/2}\, x \quad \text{for } x \in \left(\frac{25}{256} x_{2n-1}, x_{2n}\right] \qquad -(21)$$

and $$h(x) = \frac{25}{64} x_{2n-1}^{1/2} \quad \text{for } x \in \left(x_{2n+1}, \frac{25}{256} x_{2n-1}\right] \qquad -(22)$$

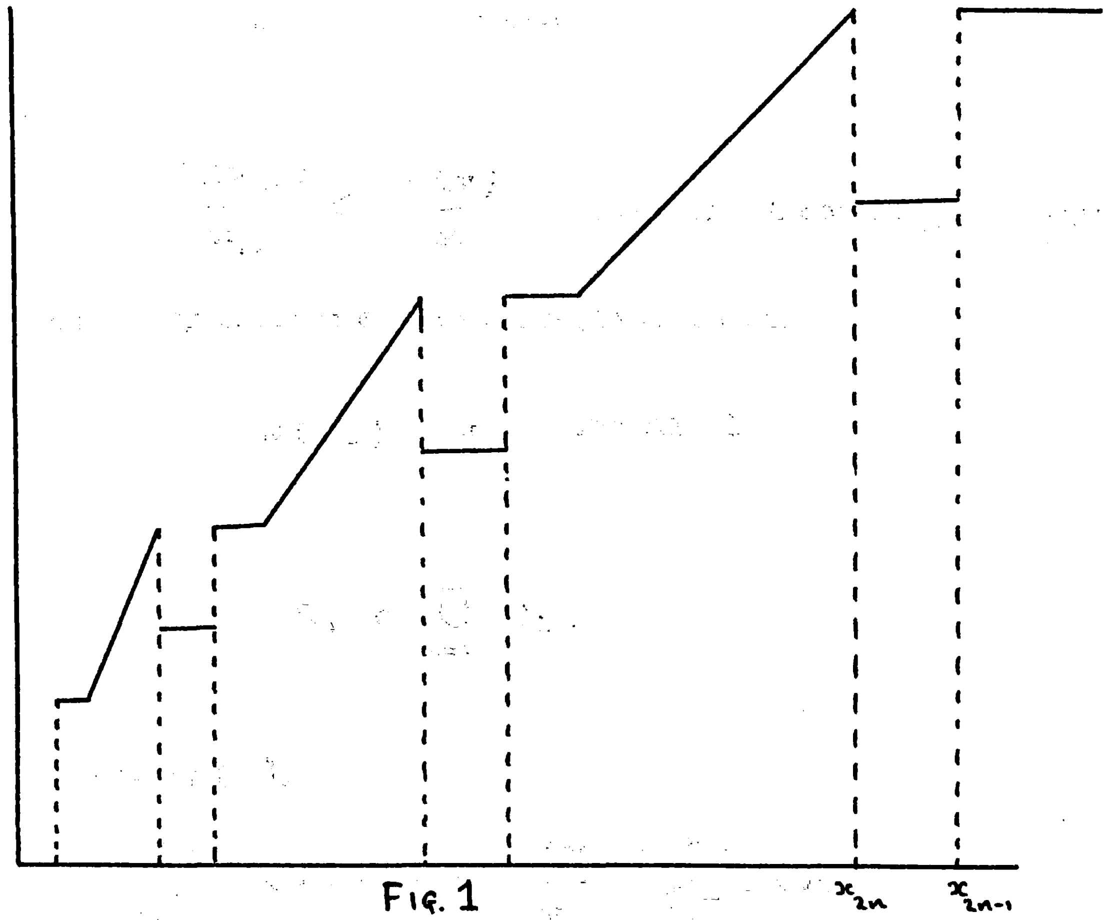

Fig. 1

Figure 1 is a rough sketch of the curve $h(x)$, and we can see that $h(x)$ is a Hausdorff measure function.

Let S_1 be the unit interval $[0, 1]$.

Now, for any integer n,

$$\frac{h(x_{2n})}{x_{2n}} = \frac{2 x_{2n-1}^{1/2}}{1/2\, x_{2n-1}} = 4 x_{2n-1}^{-1/2},$$

and, $$\frac{h(x_{2n+1})}{x_{2n+1}} = x_{2n+1}^{-1/2} = \left(\frac{128}{25}\right) x_{2n-1}^{-1/2},$$

therefore,

$$\frac{h(x_{2n})}{x_{2n}} < \frac{h(x_{2n+1})}{x_{2n+1}} \qquad -(23)$$

also,

$$\frac{h(x_{2n})}{x_{2n}} \leq \frac{h(x)}{x} \quad \text{whenever} \quad 0 < x < x_{2n} \qquad -(24)$$

Let $\{U_i\}$ be any sequence of open intervals such that,

$$d(U_i) < x_{2n} \quad \text{for all} \quad i$$

and,

$$S_1 \subset \bigcup_{i=1}^{\infty} U_i .$$

Then we have, by (24),

$$\sum_{i=1}^{\infty} h(d(U_i)) \geq \frac{h(x_{2n})}{x_{2n}} \sum_{i=1}^{\infty} d(U_i).$$

But, since $S_1 \subset \bigcup_i U_i$, we must have,

$$\sum_{i=1}^{\infty} d(U_i) \geq 1,$$

thus, combining these results,

$$\Lambda^{h}_{x_{2n}}(S_1) \geq x_{2n}^{-1}\, h(x_{2n}) \qquad -(25)$$

Now, let $H(x)$ be any monotonic increasing Hausdorff measure function.

We assume that, for all large values of n,

$$\Lambda^{H}_{x_{2n}}(S_1) = \Lambda^{h}_{x_{2n}}(S_1), \qquad -(26)$$

and show that this assumption leads to a contradiction.

Now,

$$\Lambda^{H}_{x_{2n}}(S_1) \leqslant (1 + x_{2n}^{-1})\, H(x_{2n}^{-})$$

and so, by (26) and (25),

$$x_{2n}^{-1}\, h(x_{2n}) \leqslant (1 + x_{2n}^{-1})\, H(x_{2n}^{-})$$

that is,

$$H(x_{2n}) \geqslant \frac{1}{1+x_{2n}}\, h(x_{2n}) \qquad -(27)$$

for all large values of n.

Thus, by the monotonicity of $H(x)$ we must have,

$$H(x) \geqslant \frac{2}{1+x_{2n}}\, x_{2n-1}^{1/2} \quad \text{for } x \in \left[x_{2n}, \frac{25}{256} x_{2n-3}\right] \qquad -(28)$$

Now consider any value, x, in the open interval $\left(\frac{25}{256} x_{2n-1}, x_{2n}\right)$ we have, from (23),

$$\frac{h(x)}{x} = 4\, x_{2n-1}^{-1/2} < \frac{h(x_{2n+1})}{x_{2n+1}}.$$

Thus, for any such x,

$$\Lambda_x^h(S_1) \geq 4\, x_{2n-1}^{-1/2}.$$

Again, we may assume that for all such small values of x, we have,

$$\Lambda_x^H(S_1) = \Lambda_x^h(S_1),$$

therefore,

$$4\, x_{2n-1}^{-1/2} \leq (1+x^{-1})\, H(x),$$

that is,

$$H(x) \geq \frac{1}{1+x}\, h(x).$$

Now, define the function $H'(x)$ as follows,

$$H'(x) = \frac{2}{1+x_{2n}}\, x_{2n-1}^{1/2} \quad \text{for } x \in (x_{2n}, x_{2n-1}]$$

$$= \frac{4x_{2n-1}^{-1/2}}{1+x}\, x \quad \text{for } x \in \left(\frac{25}{256} x_{2n-1}, x_{2n}\right]$$

$$= \frac{2}{1+x_{2n+2}}\, x_{2n+1}^{1/2} \quad \text{for } x \in \left(x_{2n+1}, \frac{25}{256} x_{2n-1}\right].$$

We have shown that if $H(x)$ is a monotonic function such that for any set S_1,

$$\Lambda_\delta^H(S_1) = \Lambda_\delta^h(S_1),$$

for all small values of δ, then,

$$H(x) \geq H'(x) \quad \text{for all } x \qquad -(29)$$

Also, we can see from the above definition that,

$$\lim_{x \to 0} \frac{H'(x)}{x^{1/2}} = 5/4 .$$

Thus, we have, for all sets S,

$$\Lambda^{H'}(S) \geq 5/4 \, \Lambda^{1/2}(S). \qquad -(30)$$

But we know that,

$$\lim_{x \to 0} \frac{h(x)}{x^{1/2}} = 1.$$

Now let $\{A_n\}$ be a sequence of positive integers such that $\sum A_n^{-1}$ is convergent and,

$$\prod_{n=1}^{\infty} (1 - A_n^{-1}) > 9/10 \qquad -(31)$$

Let $\{B_m\}$ be an increasing sequence of positive real numbers such that,

$$B_m \to \infty \quad \text{as} \quad m \to \infty \qquad -(32)$$

and $$B_m \geq 2 \qquad \text{for all } m. \qquad -(33)$$

Further, we can choose a null-sequence $\{x_n\}$ say, such that each x_n is a point of continuity of $h(x)$ and,

$$\frac{h(x_n)}{x_n^{1/2}} \to 1 \quad \text{as} \quad n \to \infty \qquad -(34)$$

We inductively define a sequence $\{t_n\}$ ~~x~~ of real numbers as follows, choose t_0 arbitrarily such that $t_0 \in \{x_n\}$ now assume that $t_0, \ldots, t_{m-1}$ have been defined.

Choose t_m such that,

i). $\quad 0 < t_m < \frac{1}{2} t_{m-1} \quad$ and $\quad t_m \in \{x_n\}$

ii). $\quad C_m t_m^{1/2} = t_{m-1}^{1/2}, \qquad C_m > A_m$

iii). $\quad B_m C_m t_m < t_{m-1}.$

Denote by K_m the integral part of C_m. Let S_0 be the set of points of a closed interval of length t_0 on the real line. In S_0 construct K_1 closed intervals of length t_1 equally spaced distance y_1 apart and such that there is an interval of length t_1 at each end of S_0. Denote by S_1 the K_1 closed intervals so formed. In each interval of S_1 construct K_2 closed intervals of length t_2 equally spaced distance y_2 apart with an interval of length t_2 at each end of the interval of S_1. Denote by S_2 the $K_1 K_2$ closed intervals so formed. In general S_n is a set of $K_1 \ldots K_n$ closed intervals of length t_n such that in any interval of S_{n-1} there are K_n intervals of S_n equally spaced distance y_n apart with an interval of length t_n at each end of the intervals of S_{n-1}. We write,

$$S = \bigcap_{n=0}^{\infty} S_n .$$

Then clearly,

$$\Lambda^{1/2}(S) \leq K_1 \ldots K_m t_m^{1/2} \quad \text{for all } m$$

$$\leq t_0^{1/2}. \tag{35}$$

Since S is compact we need only consider finite open coverings. Also, if

$\{V_i\}$ is a finite covering of S by means of intervals of S_n for different values of n, then we have,

$$\sum_i (d(V_i))^{1/2} \geq (c_1 - 1)\ldots(c_N - 1)\, t_N^{1/2}$$

for all large integers N.

This is because we may replace an interval of S_n by the K_{n+1} intervals of S_{n+1} which it contains. So we have, by (31),

$$\sum_i (d(V_i))^{1/2} \geq \prod_{i=1}^{N} (1 - c_i^{-1})\, t_0^{1/2} \geq {}^9\!/_{10}\, t_0^{1/2}. \qquad -(36)$$

Now let $\{U_i\}$ be any sequence of open intervals which form a covering of S. Consider a particular interval U_i of this covering. There is a least integer m, say, such that $S \cap U_i$ is contained in one interval of S_{m-1} but has points in common with at least two different intervals of S_m. Let l_i be the length of the interval U_i and r the number of intervals of S_m which intersect $S \cap U_i$. Then we have,

$$l_i \leq t_{m-1} \qquad -(37)$$

and,

$$l_i \geq (r-2)\, t_m + (r-1)\, y_m. \qquad -(38)$$

From (37) we have,

$$\frac{t_{m-1}^{1/2}}{t_{m-1}} \leq \frac{l_i^{1/2}}{l_i}. \qquad -(39)$$

Also we know that,

$$t_{m-1} = K_m t_m + (K_m - 1)\, y_m, \qquad -(40)$$

therefore, by iii). we have,

$$y_m = \frac{t_{m-1} - K_m t_m}{K_m - 1}$$

$$\geq \frac{(B_m - 1)}{K_m - 1} K_m t_m \geq (B_m - 1) t_m \qquad - (41)$$

Thus, combining (38) and (40),

$$\frac{l_i}{t_{m-1}} \geq \frac{(r-2)\, t_m + (r-1)\, y_m}{K_m t_m + (K_m - 1)\, y_m}.$$

So that,

$$r t_m^{1/2} \leq \frac{r t_{m-1}^{1/2}}{K_m} \leq \frac{r}{K_m} \left[\frac{K_m t_m + (K_m - 1)\, y_m}{(r-2) t_m + (r-1)\, y_m} \right]^{1/2} l_i^{1/2}$$

$$= \left[\frac{r^2 K_m t_m + r^2 (K_m - 1)\, y_m}{K_m^2 r t_m + K_m^2 (r-1)\, y_m - 2 K_m^2 t_m} \right]^{1/2} l_i^{1/2}$$

$$\leq \left[\frac{1}{1 - \dfrac{2}{r + (r-1)\, {}^{y_m}/_{t_m}}} \right]^{1/2} l_i^{1/2}$$

$$\leq \left[\frac{1}{1 - \dfrac{2}{2 + {}^{y_m}/_{t_m}}} \right]^{1/2} l_i^{1/2}, \quad \text{since } r \geq 2$$

$$\leq \left(\frac{B_M + 1}{B_M - 1} \right)^{1/2} l_i^{1/2}, \qquad \text{by } (41)$$

where M is the greatest integer m such that,

$$d(U_i) < y_{m-1} \qquad \text{for all } i.$$

Therefore we have, by (36),

$$\sum_i (d(U_i))^{1/2} \geq \left(\frac{B_M - 1}{B_M + 1}\right)^{1/2} \frac{9}{10} t_0^{1/2}$$

that is,

$$\Lambda^{1/2}(S) \geq 9/10 \; t_0^{1/2}, \qquad -(42)$$

since $B_m \to \infty$ as $m \to \infty$.

Also we have,

$$\Lambda^h(S) \leq K_1 \dots K_n \, h(t_n) \quad \text{for all integers } n$$

$$= (K_1 \dots K_n t_n^{1/2}) \frac{h(t_n)}{t_n^{1/2}}$$

$$\leq t_0^{1/2} . \; h(t_n)/t_n^{1/2} .$$

Therefore, by i). and (34),

$$\Lambda^h(S) \leq t_0^{1/2} \qquad -(43)$$

Thus, combining (30), (42) and (43), we have,

$$\Lambda^{H'}(S) \geq 5/4 \; \Lambda^{1/2}(S) \geq 9/8 \; t_0^{1/2} \geq 9/8 \; \Lambda^h(S) > \Lambda^h(S).$$

Hence, for all small values of δ we have,

$$\Lambda_\delta^H(S) > \Lambda_\delta^h(S) \qquad -(44)$$

as required. Thus, if we have $\Lambda_\delta^H(S_1) = \Lambda_\delta^h(S_1)$ for all small values of δ then we must also have statement (44) and hence the theorem is proved.

CHAPTER 2

INTRODUCTION

The ideas discussed in this chapter came initially, from a study of Dvoretzky's paper (2) in which he proves that, given any continuous, monotonic Hausdorff measure function, a necessary and sufficient condition for there to be a set in q-dimensional Euclidean space with the property that $\Lambda^h(S)$ is positive and finite is that $h(x)$ should be a q-dimensional Hausdorff measure function. In his remarks at the end of the paper, Dvoretzky explains how, in the one-dimensional case, this result can be extended to discontinuous functions. Following these results it seemed interesting to investigate whether discontinuous functions need any special consideration with regard to Hausdorff measures, or whether, in fact, it is sufficient to consider only continuous functions. If we alter the definition of Hausdorff measure so that we consider either, just closed convex coverings, or coverings consisting of any convex sets, then Dvoretzky's result generalises directly to discontinuous functions for sets in q-dimensional Euclidean space. In fact we see, from Theorem 4, that Dvoretzky's result does apply to discontinuous functions in q-dimensional Euclidean space, when only convex open coverings are permitted in the definition of Hausdorff measure. Theorems 5 and 6 show that when considering a particular set of finite, positive measure it is necessary only to consider continuous functions. Theorems 7 and 8 show that for any discontinuous one-dimensional Hausdorff measure function $h(x)$ there is a continuous one-dimensional Hausdorff measure function $H(x)$ such that, for all linear sets S, $\Lambda^h(S)$ is equal to $\Lambda^H(S)$. But, under certain conditions in q-dimensional space ($q > 1$) the discontinuous functions require special consideration. This latter result suggests that some of the theorems which have been established for continuous functions may not generalise to the discontinuous case.

Before proving these theorems we need to prove a lemma.

Lemma 1

For any set S in q-dimensional Euclidean space and any Hausdorff measure function $h(x)$, we have,

$$\Lambda^{h,R(q)}(S) \geq \Lambda^{h}(S) \geq ([\sqrt{q}]+1)^{-q}\,\Lambda^{h,R(q)}(S).$$

Proof.

Given any $\varepsilon > 0$ and $\delta > 0$ let $\{U_i^\delta\}$ be an open covering of S such that,

$$d(U_i^\delta) < \delta \quad \text{for all } i \qquad -(1)$$

and,
$$\sum_i h(d(U_i^\delta)) < \Lambda^h(S) + ([\sqrt{q}]+1)^{-q}\varepsilon \qquad -(2)$$

Now replace each set U_i^δ by $([\sqrt{q}]+1)^q$ open q-dimensional rectangles $R_{i_1}, \ldots, R_{i_{([\sqrt{q}]+1)^q}}$, with sides parallel to the coordinate axes, and such that,

$$d(R_{i_s}) = d(U_i^\delta) \text{ for } \quad s = 1, \ldots, ([\sqrt{q}]+1)^q \quad \text{and for each } i$$

$$U_i^\delta \subset \bigcup_{s=1}^{([\sqrt{q}]+1)^q} R_{i_s} \qquad \text{for each } i$$

In this manner we get a covering of S by open rectangles R_{i_j} such that,

$$d(R_{ij}) < \delta \qquad \text{for all } i, j$$

and $$\sum_{i,j} h(d(R_{ij})) = ([\sqrt{q}]+1)^q \sum_i h(d(U_i^\delta)).$$

Thus, by (2), andthe definition of Hausdorff measure,

$$\Lambda_\delta^{h,R(q)}(S) \le ([\sqrt{q}]+1)^q \Lambda^h(S) + \varepsilon.$$

But, this result is true for arbitrary δ and ε so we have,

$$\Lambda^{h,R(q)}(S) \le ([\sqrt{q}]+1)^q \Lambda^h(S).$$

The inequality $\Lambda^{h,R(q)}(S) \ge \Lambda^h(S)$ is trivial and thus we have the required result.

Theorem 4

Let $h(x)$ be any Hausdorff measure function. Then a necessary and sufficient condition for there to exist a set S in q-dimensional Euclidean space with $\Lambda^h(S)$ positive and finite is,

$$\varliminf_{x \to 0} \frac{h(x)}{x^q} > 0,$$

that is, $h(x)$ is a q-dimensional Hausdorff measure function.

Proof

Assume,

$$\varliminf_{x \to 0} \frac{h(x)}{x^q} = 0,$$

Let R be any q-dimensional rectangle with longest side d, say. Choose a sequence $\{x_n\}$ of real numbers such that,

$$x_n \to 0 \qquad \text{as} \quad n \to \infty$$

and,
$$\frac{h(x_n)}{x_n^q} \to 0 \qquad \text{as} \quad n \to \infty$$

For any integer n, we may cover R with $\left(\left[\frac{d\sqrt{q}}{x_n}\right]+1\right)^q$ squares of diameter x_n. Thus, for any positive number δ, we have,

$$\Lambda^h_\delta(R) \leq \left(\left[\frac{d\sqrt{q}}{x_n}\right]+1\right)^q h(x_n)$$

for all large integers n. That is,

$$\Lambda^h_\delta(R) \leq (2d\sqrt{q})^q \frac{h(x_n)}{x_n^q}$$

But, $$\frac{h(x_n)}{x_n^q} \to 0 \quad \text{as} \quad n \to \infty,$$

so $\Lambda^h_\delta(R)$ is zero for every rectangle R and for every $\delta > 0$, thus $\Lambda^{h,R(q)}(S) = 0$ for every set S in R and so by Lemma 1, $\Lambda^h(S) = 0$ for all sets S. Hence the necessity of the condition is proved.

We now prove the sufficiency of the condition.

Case 1:

$$\lim_{x \to 0} \frac{h(x)}{x^q} = \kappa,$$

where κ is a finite, positive constant.

Now, given any $\varepsilon > 0$, there exists an x_0 such that,

$$\frac{h(x)}{x^q} > \kappa - \varepsilon \qquad \text{for all} \quad x \leq x_0,$$

also we know that,

$$\frac{h(x)}{x^q} < \kappa + \varepsilon$$

for infinitely many small values of x.

Now let R denote not only a rectangle, but also its q-dimensional volume (that is, the product of the lengths of its sides). Let $\{U_i\}$ be any open covering of R with $d(U_i) < \delta < x_0$ for all i, then,

$$\Lambda^h_\delta(R) = \underset{\{U_i\}}{\text{bd}} \sum_i h(d(U_i)) \geq \underset{\{U_i\}}{\text{bd}} \sum_i (\alpha - \tau)(d(U_i))^q.$$

Each set $\{U_i\}$ can be enclosed in a q-dimensional cube of side $d(U_i)$ with sides parallel to the coordinate axes. These cubes then cover R and so we have,

$$\sum_i (d(U_i))^q \geq R$$

thus we have,

$$\Lambda^h_\delta(R) \geq (\alpha - \tau) R. \qquad -(3)$$

Now let $\{a_1, \ldots, a_q\}$ denote the side-lengths of the rectangle R then,

$$\prod_{i=1}^{q} a_i = R.$$

Also, for any $x < \delta$,

$$\Lambda^h_\delta(R) \leq \prod_{i=1}^{q} \left(\left[\frac{a_i\sqrt{q}}{x}\right] + 1\right) . h(x).$$

Now choose x so small that,

$$x < \delta$$

$$\min_{1 \leq i \leq q} \left\{\left[\frac{a_i\sqrt{q}}{x}\right]\right\} > 1/\tau,$$

and,

$$\frac{h(x)}{x^q} < \alpha + \tau.$$

Then we have,

$$\Lambda^h_\delta(R) \leq (1+\varepsilon)^q (\sqrt{q})^q \prod_{i=1}^{q} \left(\frac{a_i}{x}\right) . h(x)$$

$$= (1+\varepsilon)^q (\sqrt{q})^q R \frac{h(x)}{x^q}$$

$$< (1+\varepsilon)^q (\sqrt{q})^q (\alpha+\varepsilon) R \qquad -(4)$$

The inequalities (3) and (4) hold for all positive values δ and ε. Thus we have,

$$\alpha R \leq \Lambda^h(R) \leq \alpha (\sqrt{q})^q R.$$

Hence the sufficiency is proved for $\lim\limits_{x \to 0} \frac{h(x)}{x^q} = \alpha$ where we have, $0 < \alpha < \infty$.

Case 2:

$$\underline{\lim_{x \to 0}} \frac{h(x)}{x^q} = \infty \quad \text{and hence} \quad \lim_{x \to 0} \frac{h(x)}{x^q} = \infty \qquad -(5)$$

We shall now construct a set S in a similar manner to Dvoretzky's construction and prove that this has the property that $\Lambda^h(S)$ is positive and finite.

Since $h(x)$ may be assumed to be monotonic increasing, it has only a countable number of points of discontinuity. Let these points be denoted by $d_1, d_2, \ldots$

Thus from (5) we have,

$$\lim_{\substack{x \to 0 \\ x \neq d_i \text{ for} \\ \text{any } i}} \frac{h(x)}{x^q} = \infty \qquad -(6)$$

From (6) it follows that there exist arbitrarily small positive numbers x, satisfying,

$$x \neq d_i \quad \text{for any } i, \qquad \frac{h(x)}{x^q} < \frac{2h(t)}{t^q} \qquad -(7)$$

for all t satisfying $0 < t \leq x$ and $t \neq d_i$ for any i.

Now let $\{A_n\}$ be a sequence of positive numbers such that all the terms of the sequence are greater than two and such that the series $\sum (1/A_n)$ converges. Also, let B be a positive number greater than or equal to two. We now proceed to construct the sequence $\{x_n\}$ of positive numbers, as follows;

choose x_0 to be any positive number such that $x = x_0$ satisfies (7). Having chosen x_ν for $0 \leq \nu \leq n-1$ we choose x_n ($0 < x_n < x_{n-1}$) such that,

a). (7) holds for $x = x_n$

b). $h(x_{n-1}) = C_n^q h(x_n)$ with $C_n > A_n$

c). $B\, C_n x_n < x_{n-1}$.

It is clear that these choices are permissible from the restrictions placed on $h(x)$.

We assume that $x_n \to 0$ as $n \to \infty$.

Denote by K_n the integral part of C_n. Let S_0 be the set of points of a closed cube J_0 with sides parallel to the coordinate axes and each of length $x_0/\sqrt{q}$. Each side of this q-dimensional cube is a closed interval of length $x_0/\sqrt{q}$. From each side we remove $K_1 - 1$ open intervals each of length $y_1/\sqrt{q}$ so that there remain K_1 closed intervals each of length $x_1/\sqrt{q}$. This is possible since,

$$\frac{K_1 x_1}{\sqrt{q}} \leq \frac{C_1 x_1}{\sqrt{q}} < \frac{x_0}{\sqrt{q}} \Big/ B < \frac{x_0}{\sqrt{q}}.$$

We do this in such a way that when considering any one face of the cube, the opposite face is dissected in an exactly symmetrical fashion. We now form K_1^q closed cubes of side $x_1/\sqrt{q}$, inside J_0, by taking cartesian products of the intervals of length $x_1/\sqrt{q}$. Each one of these closed cubes is denoted by J_1 and S_1 denotes the set of points contained in the K_1^q cubes J_1. In each J_1 we construct K_2^q cubes J_2 each of side $x_2/\sqrt{q}$ in a similar manner, and denote by S_2 the set of points contained in the $K_1^q K_2^q$ closedcubes J_2. Continuing in this way we define S as the set of points contained in all S_n $(n=0,1,2,\dots)$. Then, S is a perfect nowhere dense set.

We have now to show that $\Lambda^h(S)$ is positive and finite.

Given any $\delta > 0$, there is a sufficiently large n for which $x_n < \delta$. The set S being included in S_n can be covered by the $K_1^q \dots K_n^q$ closed cubes J_n.

Now,

$$\sum_{\substack{i \\ U_i \in S_n}} h(d(U_i)) = K_1^q \dots K_n^q \, h(x_n) \qquad - (8)$$

where, the U_i are the cubes J_n which form the set S_n.

Using b). and the fact that $K_n \leq C_n$ in (8), weget,

$$K_1^q \dots K_n^q \, h(x_n) \leq K_1^q \dots K_{n-1}^q \, h(x_{n-1}) \leq \dots \leq h(x_0) \qquad -(9)$$

Since each of the points x_n is a point of continuity of $h(x)$ we can replace each of the closed sets U_i by open sets V_i containing U_i and such that $\sum_i h(d(V_i))$ differs from $\sum_i h(d(U_i))$ by an arbitrarily small amount. Combining this fact with (8) and (9) and the definition of Hausdorff measure, we have,

$$\Lambda^h_\delta(S) \leq \sum_i h(d(V_i)) < 2h(x_0) \qquad -(10)$$

and therefore, since (10) holds for all $\delta > 0$,

$$\Lambda^h(S) \leq 2h(x_0) < \infty.$$

Thus we have proved that $\Lambda^h(S)$ is finite.

Before proving $\Lambda^h(S) > 0$ we note the following. If we cover S with the $K_1^q \ldots K_n^q$ cubes I_n, then $\sum_{S_n} h(x)$ will be,

$$K_1^q \ldots K_n^q h(x_n) > (c_1-1)^q \ldots (c_n-1)^q h(x_n)$$

$$= (c_1-1)^q \ldots \frac{(c_n-1)^q}{c_n^q} h(x_{n-1}) \text{ by } b).$$

$$= \prod_{i=1}^{n} \left(\frac{c_i-1}{c_i}\right)^q h(x_0).$$

Now we know that $\sum 1/A_n$ converges and that $c_n > A_n$ so $\sum 1/c_n$ converges and hence so does $\prod(1-1/c_n)^q$. Denoting this product by P, we have, for every n,

$$\sum_{I_n} h(x) \geq P h(x_0)$$

This inequality still holds if we enclose S in a finite number of cubes $I_m, I_n, \ldots$ not necessarily bearing the same index. This follows from (9) which shows that if any J_{n-1} is replaced by the K_n^q cubes J_n included in it the contribution to $\Sigma h(x)$ cannot increase.

We now prove that $\Lambda^h(S) > 0$.

We show that if we enclose S in any finite number of open rectangles $R_1, R_2, \ldots$ and if $\Sigma h(x)$ for these rectangles is H, then there is a covering of S by cubes J_n for which $\Sigma h(x) < cH$, where c is a constant. Hence by virtue of our previous remarks, $H > 1/c\, P h(x_0)$ and consequently $\Lambda^{h,R(q)}(S) > 0$.

If each one of the rectangles R which does not contain any point of S is deleted, and every other rectangle R is replaced by the largest closed rectangle $\mathfrak{R}$ the interior of which is contained in R and such that each one of its edges contains at least one point of S then the rectangles $\mathfrak{R}$ still cover S and $\Sigma h(x)$ is only diminished by this replacement. It is sufficient to show that if $\mathfrak{R}$ is any one of the rectangles thus obtained, then it can be replaced by r^q cubes J_n with $r^q h(x_n) < c h(\ell)$ where ℓ is the diameter of $\mathfrak{R}$.

Consider $\mathfrak{R}$ and the sets $S_1, S_2, \ldots$; there must be a first set S_n such that $\mathfrak{R}$ contains points not belonging to S_n. Then $\mathfrak{R}$ is contained in one cube J_{n-1} but has points in common with at most r^q cubes J_n contained in J_{n-1} where r is the greatest number of cubes J_n met by any one edge of $\mathfrak{R}$ ($r \geq 2$). This is because every side of $\mathfrak{R}$ contains at least one point of S.

Since $\ell \leq x_{n-1}$ we have that,

$$\frac{h(x_{n-1})}{x_{n-1}^q} < \frac{2h(\ell)}{\ell^q}, \qquad \text{if } \ell \neq d_i \text{ for any } i.$$

If $\ell = d_i$ for some i, then there is an ℓ' such that $\frac{1}{2}\ell < \ell' < \ell$ and ℓ' is a point of continuity of $h(x)$. Then $\ell' < x_{n-1}$ and hence,

$$\frac{h(x_{n-1})}{x_{n-1}^q} < \frac{2h(\ell')}{\ell'^q}.$$

Also we have,

$$l \geq (r-2)\frac{x_n}{\sqrt{q}} + (r-1)\frac{y_n}{\sqrt{q}}$$

where $y_n/\sqrt{q}$ is the distance between two adjacent cubes of S_n,

and therefore,

$$l' > \frac{1}{2\sqrt{q}}\left\{(r-2)x_n + (r-1)y_n\right\}$$

Now,

$$x_{n-1} = K_n x_n + (K_n - 1)y_n \,,$$

and therefore,

$$y_n = \frac{x_{n-1} - K_n x_n}{K_n - 1} > \frac{BK_n x_n - K_n x_n}{K_n - 1} \quad \text{by c).}$$

$$> x_n .$$

Thus,

$$\frac{l'}{x_{n-1}} > \frac{\frac{1}{2\sqrt{q}}\left[(r-2)x_n + (r-1)y_n\right]}{K_n x_n + (K_n - 1)y_n}$$

$$> \frac{\frac{1}{2\sqrt{q}}(r-1)}{2K_n - 1} .$$

Therefore,

$$r^q h(x_n) \leq \frac{r^q}{K_n^q} h(x_{n-1}) < \frac{2r^q x_{n-1}^q}{K_n^q l'^q} h(l')$$

$$< \frac{2r^q}{K_n^q}\left[\frac{2K_n - 1}{\frac{1}{2\sqrt{q}}(r-1)}\right]^q h(l')$$

$$< 2(8\sqrt{q})^q h(l') = ch(l').$$

So we have,
$$\sum h(l') > 1/c\, \rho h(x_0).$$

But, in every case $l' \leq l$ and hence,

$$\sum h(l) \geq \sum h(l')$$

therefore,
$$\Lambda^{h,\, R(q)}(S) \geq 1/c\, \rho h(x_0).$$

Thus, using Lemma 1,

$$\Lambda^h(S) \geq \frac{1}{c([\sqrt{q}]+1)^q}\, \rho h(x_0) > 0.$$

Hence the theorem is proved.

We now proceed to investigate whether it is always possible to replace discontinuous functions by continuous ones.

<u>Theorem 5</u>

If $h(x)$ is any one-dimensional Hausdorff measure function and S is a set on the real line, such that $\Lambda^h(S)$ is positive and finite, then there exists a continuous one-dimensional Hausdorff measure function $H(x)$ say, such that,

$$\Lambda^h(S) = \Lambda^H(S).$$

<u>Proof</u>

Let $\bigcup_{i=1}^{\infty} X_i$ be any covering of S by open intervals X_i. Take any one of these intervals X_i; if it overlaps another interval of the cover X_j, say, then let,

$$X_i \equiv (a_i, b_i), \quad X_j \equiv (a_j, b_j) \qquad \text{and } a_j < b_i \text{ say.}$$

Then there exist points ξ_j and η_i such that,

$$a_j < \xi_j < \eta_i < b_i$$

and both $(\eta_i - a_i)$ and $(b_j - \xi_j)$ are points of continuity of $h(x)$. Replace X_i, X_j by the open intervals X_i', X_j' such that,

$$X_i' \equiv (a_i, \eta_i) \quad \text{and} \quad X_j' \equiv (\xi_j, b_j).$$

Then, these two intervals cover as much of S as the original two did, their diameters are less than the original and are both at points of continuity of $h(x)$.

If, however, X_i does not overlap any other X_j for $j \neq i$ then, we proceed as follows; given any $\tau > 0$ there exists a δ_i such that, for integral values of i,

$$h(\delta_i) < \tau/2^i$$

and,
$$0 < \delta_i < (b_i - a_i).$$

The intervals $X_i \equiv (a_i, b_i)$ and $(b_i - \frac{\delta_i}{2}, b_i + \frac{\delta_i}{2})$ overlap and cover as much of S as X_i did. We now replace these intervals, as before, by (a_i, η_i) and $(\xi_i, b_i + \frac{\delta_i}{2})$ with,

$$b_i - \frac{\delta_i}{2} < \xi_i < \eta_i < b_i,$$

such that $(\eta_i - a_i)$ and $(b_i + \frac{\delta_i}{2} - \xi_i)$ are points of continuity of $h(x)$.

In this way, we get a covering $\bigcup_{i=1}^{\infty} X_i'$ of S such that,

$$\sum_i h(d(X_i')) < \sum_i h(d(X_i)) + \varepsilon \qquad -(11)$$

for any given $\varepsilon > 0$ and with the property that $d(X_i')$ is a point of continuity of $h(x)$.

Let $x_1, x_2, \ldots$ be an enumeration of all the points of discontinuity of $h(x)$. Given any $\varepsilon > 0$ and any $\delta > 0$, there exists a covering $\bigcup_i X_i$ of S such that,

$$\Lambda_\delta^h(S) \leq \sum_{i=1}^{\infty} h(d(X_i)) < \Lambda_\delta^h(S) + \varepsilon \qquad -(12)$$

and $d(X_i) < \delta$ for all i.

Replace this covering by the corresponding covering $\bigcup_{i=1}^{\infty} X_i'$ of S.

Then, we have, by (11) and (12),

$$\Lambda_\delta^h(S) \leq \sum_i h(d(X_i')) < \Lambda_\delta^h(S) + 2\varepsilon, \qquad -(13)$$

with $d(X_i') < \delta$ for all i, and $d(X_i')$ is a point of continuity of $h(x)$ for each i.

We now choose δ such that,

$$\Lambda^h(S) \geq \Lambda_\delta^h(S) > \Lambda^h(S) - \varepsilon \qquad -(14)$$

Thus, combining (13) and (14) we see that, given any $\varepsilon > 0$ there exists $\delta' > 0$ such that for all $\delta < \delta'$ there is a covering $\bigcup_{i=1}^{\infty} X_i'$ of S such that $d(X_i') < \delta$ for all i, $d(X_i')$ is a point of continuity of $h(x)$ for each i, and,

$$\Lambda^h(S) - \varepsilon < \sum_i h(d(X_i')) < \Lambda^h(S) + 2\varepsilon \qquad -(15)$$

Now choose a sequence $\{\delta_n\}$ of positive numbers satisfying the following conditions,

$$\delta_n > \delta_{n+1} \qquad \text{for all } n \qquad -(16)$$

$$\delta_n \downarrow 0 \qquad \text{as } n \to \infty \qquad -(17)$$

$$\delta_n < \delta' \qquad \text{for all } n \qquad -(18)$$

and such that each δ_n is a point of continuity of $h(x)$. Then, as shown above, for any n, there exists a covering $\bigcup_{i=1}^{\infty} X'_{ni}$ of S such that $d(X'_{ni}) < \delta_n$ for all i, $d(X'_{ni})$ is a point of continuity of $h(x)$ for each i, and,

$$\Lambda^h(S) - \tau < \sum_i h(d(X'_{ni})) < \Lambda^h(S) + 2\tau \qquad -(19)$$

Since, for any n, $\sum_i h(d(X'_{ni}))$ is convergent we must have,

$$h(d(X'_{ni})) \to 0 \qquad \text{as } i \to \infty$$

and therefore, since $h(x) > 0$ for $x > 0$,

$$d(X'_{ni}) \to 0 \qquad \text{as } i \to \infty.$$

Thus zero is the only possible limit point of the sequence of diameters $\{d(X'_{ni})\}_{i=1}^{\infty}$. Because of this, we can now enclose each x_i in an interval $l_{1i} \equiv (\alpha_{1i}, \beta_{1i})$ such that no l_{1i} contains a $d(X'_{1i})$. If at any stage we find that a point of discontinuity x_i is already included in an l_{1j} for some $j \neq i$ we leave it alone. We, further, insist that α_{1i} and β_{1i} should be points of continuity of $h(x)$ for each i and that no l_{1i} should contain any δ_n. This last restriction is permissible as zero is the only limit point of the sequence $\{d_n\}$.

We now define the continuous function $H^{(1)}(x)$ as follows,

$$H^{(1)}(x) = h(x) \qquad \text{if } x \notin \ell_{1i} \text{ for each } i$$

$$H^{(1)}(x) = h(\beta_{1i}) \qquad \text{if } x \in [x_i, \beta_{1i})$$

$$H^{(1)}(\phi_{1i}^{(m)}) = h(\phi_{1i}^{(m-1)}+0) \qquad \text{for } m = 1, 2, \ldots$$

where $\phi_{1i}^{(m)} = \alpha_{1i} + \frac{1}{2^m}(x_i - \alpha_{1i})$ for $m = 0, 1, 2, \ldots$.

Finally define $H^{(1)}(x)$ to be linear and continuous in $[\phi_{1i}^{(m)}, \phi_{1i}^{(m-1)}]$ for $m = 2, 3, \ldots$ and in $[\phi_{1i}^{(1)}, x_i]$.

Thus, we have defined a continuous, increasing function $H^{(1)}(x)$ such that $H^{(1)}(x) \geq h(x)$ for all x and,

$$H^{(1)}(d(X'_{1i})) = h(d(X'_{1i})) \qquad \text{for all } i.$$

So we have,

$$\lambda^h(S) - \varepsilon < \sum_i H^{(1)}(d(X'_{1i})) < \lambda^h(S) + 2\varepsilon \qquad -(20)$$

and,

$$d(X'_{1i}) < \delta_1 \qquad \text{for all } i.$$

We now consider the covering $\bigcup_{i=1}^{\infty} X'_{2i}$ of S. If necessary, we shrink any of the intervals ℓ_{1i} to intervals ℓ_{2i} contained in ℓ_{1i} with $\ell_{2i} \equiv (\alpha_{2i}, \beta_{1i})$ and such that no ℓ_{2i} contains a $d(X'_{2i})$. For those values of i where $\ell_{2i} \not\equiv \ell_{1i}$, we insist that $\alpha_{2i} > \phi_{1i}^{(1)}$. Also, if any of the discontinuities of $h(x)$ becomes 'exposed' by this shrinking we similarly enclose them in a suitable ℓ_{2i} which does not contain any $\phi_{1i}^{(m)}$ unless it is the discontinuity itself. Again we define the continuous, increasing function $H^{(2)}(x)$ as follows,

$$H^{(2)}(x) = h(x) \qquad \text{if} \quad x \notin I_{2i} \text{ for each } i$$

$$H^{(2)}(x) = h(\beta_{2i}) \qquad \text{if} \quad x \in [x_i, \beta_{2i})$$

$$H^{(2)}(\phi_{2i}^{(m)}) = h(\phi_{2i}^{(m-1)}+0) \qquad \text{for} \quad m = 1, 2, \ldots$$

where $\phi_{2i}^{(m)} = \alpha_{2i} + 1/2^m (x_i - \alpha_{2i})$ for $m = 0, 1, 2, \ldots$

Finally define $H^{(2)}(x)$ to be linear and continuous in $[\phi_{2i}^{(m)}, \phi_{2i}^{(m-1)}]$ for $m = 2, 3, \ldots$ and in $[\phi_{2i}^{(1)}, x_i]$.

Thus, we see that,

$$H^{(1)}(x) \geq H^{(2)}(x) \geq h(x) \qquad \text{for all} \quad x,$$

and $$H^{(2)}(d(X'_{2i})) = h(d(X'_{2i})) \qquad \text{for all } i.$$

Again we have,

$$\Lambda^h(S) - \varepsilon < \sum_i H^{(2)}(d(X'_{2i})) < \Lambda^h(S) + 2\varepsilon \qquad -(21)$$

and $$d(X'_{2i}) < \delta_2 \qquad \text{for all } i.$$

Continuing in this manner we get continuous, increasing functions $H^{(n)}(x)$ such that,

$$H^{(n-1)}(x) \geq H^{(n)}(x) \geq h(x) \qquad \text{for all } x,$$

$$H^{(n)}(d(X'_{ni})) = h(d(X'_{ni})) \qquad \text{for all } i$$

$$\Lambda^h(S) - \varepsilon < \sum_i H^{(n)}(d(X'_{ni})) < \Lambda^h(S) + 2\varepsilon \qquad -(22)$$

and $$d(X'_{ni}) < \delta_n \qquad \text{for all } i.$$

We now define the function $H(x)$ as follows,

$$H(x) = H^{(n)}(x) \qquad \text{for } x \in [\delta_{n+1}, \delta_n).$$

We see that $H(x)$ is continuous, increasing and $H(x) \to 0$ as $x \to 0$. Also we have,

$$H(x) \geq h(x) \qquad \text{for all } x \qquad -(23)$$

and $$H(x) \leq H^{(n)}(x) \qquad \text{for } x \in (0, \delta_n). \qquad -(24)$$

From (23), we deduce that $H(x)$ is also a one-dimensional Hausdorff measure function. For any integer n, we have, using (24) and (22),

$$\Lambda^{H}_{\delta_n}(S) \leq \Lambda^{H^{(n)}}_{\delta_n}(S) \leq \sum_i H^{(n)}(d(X'_{n_i}))$$

$$< \Lambda^{h}(S) + 2\varepsilon.$$

But,

$$\lim_{n \to \infty} \Lambda^{H}_{\delta_n}(S) = \Lambda^{H}(S),$$

thus, $$\Lambda^{H}(S) \leq \Lambda^{h}(S) + 2\varepsilon.$$

From (23) we see that, $\Lambda^{H}(S) \geq \Lambda^{h}(S)$.

Hence we have shown that, given any $\varepsilon > 0$ there is a continuous one-dimensional Hausdorff measure function $H(x)$ such that,

$$\Lambda^{h}(S) \leq \Lambda^{H}(S) \leq \Lambda^{h}(S) + 2\varepsilon.$$

Thus we get the required continuous function merely by multiplying $H(x)$ by the appropriate constant, that is,

$$\Lambda^h(S) \Big/ \Lambda^H(S)$$

We see that the proof of this theorem relied very heavily on the fact that we were working with sets on the real line. The next theorem shows that the result is also true in q-dimensional Euclidean space ($q > 1$).

<u>Theorem 6</u>

If $h(x)$ is any q-dimensional Hausdorff measure function and S is a set in q-dimensional Euclidean space such that $\Lambda^h(S)$ is positive and finite, then there exists a continuous q-dimensional Hausdorff measure function $H(x)$ such that,

$$\Lambda^H(S) = \Lambda^h(S).$$

<u>Proof</u>

Let $\{\delta_n\}$ be a sequence such that $\delta_n \searrow 0$ as $n \to \infty$ and such that each δ_n is a point of continuity of $h(x)$. Given any $\varepsilon > 0$, then corresponding to each δ_n there exists a covering $\bigcup_{i=1}^{\infty} X_{n_i}$ of S such that,

$$d(X_{n_i}) < \delta_n \qquad \text{for all } i,$$

and, $$\Lambda^h_{\delta_n}(S) \leq \sum_i h(d(X_{n_i})) < \Lambda^h_{\delta_n}(S) + \varepsilon \qquad -(25)$$

We may assume that the X_{n_i} are ordered such that,

$$d(X_{n_{i+1}}) \leq d(X_{n_i}).$$

Define the function $F^{(1)}(x)$ as follows,

$$F^{(1)}(x) = h(d(X_{1_i})) \qquad \text{for } x \in (d(X_{1_{i+1}}), d(X_{1_i})]$$
$$\text{for } i = 1, 2, \ldots$$

Then $F^{(1)}(x) \geq h(x)$ for $x \in (0, d(X_{1_i}))$ and,

$$\Lambda_{\delta_1}^{F^{(1)}}(S) \leq \sum_i F^{(1)}(d(X_{1_i})) = \sum_i h(d(X_{1_i})).$$

Thus, by (25), we have,

$$\Lambda_{\delta_1}^{F^{(1)}}(S) < \Lambda_{\delta_1}^{h}(S) + \varepsilon \qquad -(26)$$

Define $F^{(2)}(x)$ as follows,

$$F^{(2)}(x) = F^{(1)}(x) \qquad \text{for } x \in (d(X_{2_1}), d(X_{1_1})]$$

$$F^{(2)}(x) = h(d(X_{2_i})) \qquad \text{if } x \in (d(X_{2_{i+1}}), d(X_{2_i})]$$
$$\text{and } F^{(1)}(x) \geq h(d(X_{2_i}))$$

and,
$$F^{(2)}(x) = F^{(1)}(x) \qquad \text{if } x \in (d(X_{2_{i+1}}), d(X_{2_i})]$$
$$\text{and } F^{(1)}(x) < h(d(X_{2_i})).$$

Then,

$$h(x) \leq F^{(2)}(x) \leq F^{(1)}(x) \quad \text{in} \quad (0, d(X_{1_1})]$$

and,
$$\Lambda_{\delta_2}^{F^{(2)}}(S) < \Lambda_{\delta_2}^{h}(S) + \varepsilon. \qquad -(27)$$

Continuing in this manner we get $F^{(n)}(x)$ satisfying,

$$h(x) \leq F^{(n)}(x) \leq F^{(n-1)}(x) \qquad -(28)$$

and,
$$\Lambda^{F^{(n)}}_{\delta_n}(S) < \Lambda^{h}_{\delta_n}(S) + \tau. \qquad -(29)$$

Now define $F(x)$ as follows,

$$F(x) = F^{(n)}(x) \qquad \text{for } x \in \left(d(X_{(n+1),1}), d(X_{n,1})\right].$$

Then, by (28),

$$F(x) \geq h(x) \qquad \text{for all } x$$

and thus $F(x)$ is a q-dimensional Hausdorff measure function. Also, for each n,

$$F(x) \leq F^{(n)}(x) \qquad \text{for } x \in \left(0, d(X_{n,1})\right].$$

So for each n, we have,

$$\Lambda^{F}_{\delta_n}(S) \leq \Lambda^{F^{(n)}}_{\delta_n}(S) < \Lambda^{h}_{\delta_n}(S) + \tau,$$

and therefore,

$$\Lambda^{h}(S) \leq \Lambda^{F}(S) < \Lambda^{h}(S) + \tau.$$

Thus, for the proof of the theorem, we need only consider those discontinuous functions which are step functions with zero as the only possible limit point of their points of discontinuity, and at any one of the discontinuities x, say, $h(x) = h(x-0)$.

Now let $h(x)$ be such a function and let S be a set such that $\Lambda^{h}(S)$ is positive and finite. We show that this can be replaced by a continuous function $H(x)$ such that $\Lambda^{h}(S) = \Lambda^{H}(S)$.

Let $\bigcup_{i=1}^{\infty} X_i$ be a covering of S by open q-dimensional rectangles with sides parallel to the coordinate axes. Then if, for some i, $d(X_i)$ is a point of discontinuity of $h(x)$ we can replace this rectangle X_i by two other rectangles Y_j, Y_k such that,

$$d(Y_j) < d(X_i) \quad \text{and} \quad d(Y_k) < d(X_i)$$

$$h(d(Y_j)) = h(d(Y_k)) = h(d(X_i))$$

and, Y_j and Y_k together cover at least as much of S as X_i did. Finally, we can also guarantee that $d(Y_j)$ and $d(Y_k)$ will be points of continuity of $h(x)$.

Thus, if $\bigcup_{i=1}^{\infty} X_i$ is any covering of S by open rectangles we can replace it by another such covering $\bigcup_{j=1}^{\infty} Y_j$ of S, such that $d(Y_j)$ is a point of continuity of $h(x)$ for each j and,

$$\sum_j h(d(Y_j)) \leq 2 \sum_i h(d(X_i)).$$

Further, if $d(X_i) < \delta$ for all i, then $d(Y_j) < \delta$ for all j. We now proceed to the construction of the continuous function $H(x)$ by a diagonal argument similar to the one used in the proof of Theorem 5.

Let $\{\delta_n\}$ be a sequence of positive numbers such that $\delta_n \searrow 0$ as $n \to \infty$ and each δ_n is a point of continuity of $h(x)$. Corresponding to each δ_n and to any given $\varepsilon > 0$, there is a covering $\bigcup_i X_{n_i}$ of S, such that,

$$\Lambda_{\delta_n}^{h, R(q)}(S) \leq \sum_i h(d(X_{n_i})) < \Lambda_{\delta_n}^{h, R(q)}(S) + \varepsilon$$

with $\qquad d(X_{n_i}) < \delta_n \qquad$ for all i.

Replace this covering by the corresponding covering $\bigcup_j Y_{n_j}$ so that, given any $\varepsilon > 0$ we have,

$$\Lambda_{\delta_n}^{h, R(q)}(S) \leq \sum_j h(d(Y_{n_j})) < 2\Lambda_{\delta_n}^{h, R(q)}(S) + 2\varepsilon \qquad -(30)$$

with $d(Y_{n_j}) < \delta_n$ for all j, and $d(Y_{n_j})$ is a point of continuiuty of $h(x)$ for each j. Let $x_1, x_2, \ldots$ be an enumeration of all the points of discontinuity of $h(x)$.

Enclose each x_i in an open interval $\ell_{1i} \equiv (\alpha_{1i}, \beta_{1i})$ such that, both α_{1i} and β_{1i} are points of continuity of $h(x)$, there are no x_j in ℓ_{1i} with $j \neq i$ and there are no $d(Y_{1j})$ nor δ_n in ℓ_{1i}. Define the continuous function $H^{(1)}(x)$ as follows,

$$H^{(1)}(x) = h(x) \qquad \text{if } x \notin \ell_{1i} \text{ for each } i$$

$$H^{(1)}(x_i) = h(x_i + 0)$$

$$H^{(1)}(x) = h(x) \qquad \text{for } x \in (x_i, \beta_{1i}]$$

and define $H^{(1)}(x)$ to be continuous, increasing and greater than or equal to $h(x)$ in the interval $[\alpha_{1i}, x_i]$.

So we have,

$$H^{(1)}(x) \geq h(x) \qquad \text{for all } x,$$

and,
$$\Lambda_{\delta_1}^{H^{(1)}, R(a)}(S) \leq \sum_i H^{(1)}(d(Y_{1i})) = \sum_i h(d(Y_{1i}))$$

therefore, by (30), we have,

$$\Lambda_{\delta_1}^{H^{(1)}, R(a)}(S) < 2\Lambda_{\delta_1}^{h, R(a)}(S) + 2\varepsilon.$$

Now consider the covering $\bigcup_i Y_{2i}$ of S. Enclose each x_i in an open interval $\ell_{2i} \equiv (\alpha_{2i}, \beta_{2i})$ contained in ℓ_{1i} such that, both α_{2i} and β_{2i} are points of continuity of $h(x)$, there are no x_j in ℓ_{2i} with $j \neq i$ and there are no $d(Y_{2j})$ in ℓ_{2i}. Define the continuous function $H^{(2)}(x)$ as follows,

$$H^{(2)}(x) = h(x) \qquad \text{if } x \notin \ell_{2i} \text{ for each } i,$$

$$H^{(2)}(x_i) = h(x_i+0)$$

$$H^{(2)}(x) = h(x) \qquad \text{for } x \in (x_i, \beta_{2i}]$$

and define $H^{(2)}(x)$ to be continuous, increasing, greater than or equal to $h(x)$ and less than or equal to $H^{(1)}(x)$ in the interval $[\alpha_{2i}, x_i]$.

So we have,

$$H^{(1)}(x) \geq H^{(2)}(x) \geq h(x) \qquad \text{for all } x,$$

and, $$\Lambda_{\delta_2}^{H^{(2)}, R(q)}(S) < 2\Lambda_{\delta_2}^{h, R(q)}(S) + 2\varepsilon.$$

Continuing in this manner we define the continuous function $H^{(n)}(x)$ such that,

$$h(x) \leq H^{(n)}(x) \leq H^{(n-1)}(x) \quad \text{for all } x,$$

and, $$\Lambda_{\delta_n}^{H^{(n)}, R(q)}(S) < 2\Lambda_{\delta_n}^{h, R(q)}(S) + 2\varepsilon.$$

Finally, we define the continuous function $H(x)$ as follows,

$$H(x) = H^{(n)}(x) \qquad \text{for } x \in [\delta_{n+1}, \delta_n)$$

Now we see that $H(x)$ is a continuous q-dimensional Hausdorff measure function with

$$h(x) \leq H(x) \leq H^{(n)}(x) \qquad \text{for all } x \text{ and for all } n$$

Thus,

$$\Lambda^h(S) \leq \Lambda^H(S).$$

Also, for each value of n,

$$\Lambda_{\delta_n}^{H,R(q)}(S) \leq \Lambda_{\delta_n}^{H^{(n)},R(q)}(S) < 2\Lambda_{\delta_n}^{h,R(q)}(S) + 2\varepsilon$$

$$\leq 2\Lambda^{h,R(q)}(S) + 2\varepsilon.$$

So we have,

$$\Lambda^{H}(S) \leq \Lambda^{H,R(q)}(S) \leq 2\Lambda^{h,R(q)}(S) + 2\varepsilon.$$

Therefore, using Lemma 1,

$$\Lambda^{h}(S) \leq \Lambda^{H}(S) \leq 2\left([\sqrt{q}]+1\right)^{q}\Lambda^{h}(S) + 2\varepsilon.$$

The remainder of the proof is trivial, since we have only to multiply $H(x)$ by a constant to get the required continuous function.

From the proofs of Theorems 5 and 6 we see that the continuous functions, $H(x)$, are dependent on the set S under consideration. We shall now see that these results can, under certain restrictions, be extended to give a continuous function which is independent of the set S.

Theorem 7

If $h(x)$ is any q-dimensional Hausdorff measure function with the property that,

$$\frac{h(x)}{x^{q-1}} \to 0 \qquad \text{as } x \to 0$$

then there exists a continuous q-dimensional Hausdorff measure function $H(x)$ such that,

$$\Lambda^{h}(S) = \Lambda^{H}(S)$$

for all sets S in q-dimensional Euclidean space.

Proof.

Let C be a convex set of diameter d in q-dimensional Euclidean space. Denote by C' the set $\{x : s(x, \mathcal{C}C) > \delta\}$ for some $\delta > 0$, where s is the metric in the space. Then we can cover the set $C \setminus C'$ with $\left[\frac{K d^{q-1}}{\delta^{q-1}}\right]$ q-dimensional cubes of diameter δ, where K is a constant dependent on q. (See Appendix 1, page 143).

Let $\{\varepsilon_i\}$ be a decreasing sequence of positive real numbers such that $\varepsilon_i \downarrow 0$ as $i \to \infty$. Let $\{\rho_n\}$ be a strictly decreasing sequence of positive numbers such that $\rho_n \downarrow 0$ as $n \to \infty$ and each point ρ_n is a point of continuity of $h(x)$. Let $x_1^{(1)}, \ldots, x_{n_1}^{(1)}$ be those points of discontinuity of size greater than $\varepsilon_1 \rho_1^q / 2^3$ with $x_i^{(1)} > \rho_1$ and $x_i^{(1)} > x_{i+1}^{(1)}$ for $i = 1, \ldots, n_1$.
In general let $x_{n_{j-1}+1}^{(j)}, \ldots, x_{n_j}^{(j)}$ be those discontinuities of $h(x)$ of size greater than $\varepsilon_j \rho_j^q / 2^{j+2}$ with $\rho_{j-1} > x_i^{(j)} > \rho_j$ and $x_i^{(j)} > x_{i+1}^{(j)}$ for $i = n_{j-1}+1, \ldots, n_j$.
Corresponding to each $x_i^{(j)}$ define $\eta_i^{(j)}$ such that ($n_{j-1}+1 \le i \le n_j$),

$$\eta_i^{(j)} \notin \{x_k^{(s)}\}_{k=n_{s-1}+1,\ s=1}^{n_s\quad \infty} \quad \text{and} \quad x_i^{(j)} - \eta_i^{(j)} > \max(\tfrac{1}{2} x_i^{(j)}, x_{i+1}^{(j)}, \rho_j) \qquad -(31)$$

$$x_i^{(j)} - \eta_i^{(j)} \text{ is a point of continuity of } h(x) \qquad -(32)$$

$$h(\eta_i^{(j)}) \Big/ (\eta_i^{(j)})^{q-1} < \varepsilon_j x_i^{(j)} \Big/ 2^{j+2} K \qquad -(33)$$

$$\eta_k^{(s)} \notin (x_i^{(j)} - \eta_i^{(j)}, x_i^{(j)}] \qquad \text{for all } s \le j \qquad -(34)$$

and $k = n_{s-1}+1, \ldots, n_s$ for $s \ne j$

$k = n_{j-1}+1, \ldots, i-1$ for $s = j$.

Define the continuous q-dimensional Hausdorff measure function $H(x)$ as follows,

1). for $x \in (\rho_j, \rho_{j-1})$ and $x \notin [x_i^{(j)} - \eta_i^{(j)}, x_i^{(j)}]$ for all $i = n_{j-1}+1, \ldots, n_j$

define $H(x)$ to be continuous, increasing, greater than or equal to $h(x)$ and such that,

$$H(x) \leq h(x) + \varepsilon_j \rho_j^q / 2^{j+2} ,$$

this is possible because of the definition of $x_i^{(j)}$;

ii). $H(x_i^{(j)} - \eta_i^{(j)}) = h(x_i^{(j)} - \eta_i^{(j)})$ for all j and for all $i = n_{j-1}+1, \ldots, n_j$;

iii). $H(\rho_j) = h(\rho_j)$ for all j ;

iv). for $x \in (x_i^{(j)} - \eta_i^{(j)}, x_i^{(j)}]$ define $H(x)$ to be increasing, greater than or equal to $h(x)$, continuous in $(x_i^{(j)} - \eta_i^{(j)}, x_i^{(j)})$ and continuous on the left at $x_i^{(j)}$ with $H(x_i^{(j)}) = h(x_i^{(j)}+0)$.

Then $H(x)$ is continuous and $H(x) \geq h(x)$ for all x, thus,

$$\Lambda^H(S) \geq \Lambda^h(S) \qquad \text{for all sets } S. \qquad -(35)$$

Now let S be a set in q-dimensional Euclidean space, contained in a q-dimensional cube of diameter $1/2$.

Let δ be a positive number such that,

$$x_{i+1}^{(j)} < \delta < x_i^{(j)} - \eta_i^{(j)} \qquad \text{for some } j \text{ and some } i, \quad n_{j-1}+1 \leq i \leq n_j .$$

Let $\{U_i^\delta\}$ be an open covering of S with,

$$d(U_i^\delta) < \delta \qquad \text{for all } i,$$

and, $$\Lambda^h(S) + \varepsilon_{J_\delta} > \sum_i h(d(U_i^\delta)) \qquad -(36)$$

where J_δ is the least integer j such that,

$$x_k^{(j)} < \delta \qquad \text{for all } k = n_{j-1}+1, \ldots, n_j ,$$

(clearly $J_\delta \to \infty$ as $\delta \to 0$).

Now, for each i,

$$d(U_i^\delta) \in [\rho_j, \rho_{j-1}) \qquad \text{for some } j.$$

If $d(U_i^\delta) \notin (x_k^{(j)} - \eta_k^{(j)}, x_k^{(j)}]$ for all $k = n_{j-1}+1, \ldots, n_j$, then by i)., ii)., and iii).,

$$H(d(U_i^\delta)) \leq h(d(U_i^\delta)) + \varepsilon_j \rho_j^q / 2^{j+2}.$$

If $d(U_i^\delta) \in (x_k^{(j)} - \eta_k^{(j)}, x_k^{(j)}]$ for some k, then we have, by (33),

$$h(d(U_i^\delta)) > h(x_k^{(j)} - \eta_k^{(j)}) > h(x_k^{(j)} - \eta_k^{(j)}) + h(\eta_k^{(j)}) - \frac{\varepsilon_j (\eta_k^{(j)})^{q-1}}{2^{j+2} K} x_k^{(j)} \qquad -(37)$$

Now, $\eta_k^{(j)} \in [\rho_s, \rho_{s-1})$ for some $s \geq j$, and,

$$\eta_k^{(j)} \notin (x_i^{(s)} - \eta_i^{(s)}, x_i^{(s)}] \qquad \text{for all } i = n_{s-1}+1, \ldots, n_s.$$

Therefore,

$$H(\eta_k^{(j)}) \leq h(\eta_k^{(j)}) + \varepsilon_s \rho_s^q / 2^{s+2} \qquad -(38)$$

Also,

$$x_k^{(j)} - \eta_k^{(j)} \in [\rho_j, \rho_{j-1})$$

and,

$$H(x_k^{(j)} - \eta_k^{(j)}) = h(x_k^{(j)} - \eta_k^{(j)}) \qquad -(39)$$

Replace each U_i^δ which satisfies $d(U_i^\delta) \in (x_k^{(j)} - \eta_k^{(j)},\ x_k^{(j)}]$ for some k, by a set of diameter $x_k^{(j)} - \eta_k^{(j)}$ together with $\left[\frac{K (x_k^{(j)})^{q-1}}{(\eta_k^{(j)})^{q-1}}\right]$ q-dimensional open cubes of diameter $\eta_k^{(j)}$, denote these replacement sets by V_i^δ. If $d(U_i^\delta) \notin (x_k^{(j)} - \eta_k^{(j)},\ x_k^{(j)}]$ for all $k = n_{j-1}+1, \dots, n_j$ we put $V_i^\delta \equiv U_i^\delta$. Thus, we get another open covering $\bigcup_i V_i^\delta$ of S with $d(V_i^\delta) < \delta$ for all i. Now, since S is contained in a cube of diameter $1/2$, we may suppose that there are less than $(x_k^{(j)})^{-q}$ values $d(U_i^\delta)$ in $(x_k^{(j)} - \eta_k^{(j)},\ x_k^{(j)}]$ and less than ρ_j^{-q} values $d(U_i^\delta)$ in $[\rho_j, \rho_{j-1})$. This is because $(x_k^{(j)})^{-q}$ such values of $d(U_i^\delta)$ or ρ_j^{-q} such values of $d(U_i^\delta)$ could arise fromx a collection of sets $\{U_i^\delta\}$ which would be sufficient to cover the whole cube. Thus, for those sets V_i^δ for which $V_i^\delta \not\equiv U_i^\delta$ we have, from (37),

$$\sum_{\substack{i \\ V_i^\delta \not\equiv U_i^\delta}} h(d(U_i^\delta)) \geq \sum_{\substack{i \\ V_i^\delta \not\equiv U_i^\delta}} H(d(V_i^\delta)) - \sum_{\substack{k,j \\ x_k^{(j)} < \delta}} (x_k^{(j)})^{-q} \cdot \frac{K (x_k^{(j)})^{q-1}}{(\eta_k^{(j)})^{q-1}} \cdot \frac{\varepsilon_j (\eta_k^{(j)})^{q-1}}{2^{j+1} K} \cdot x_k^{(j)}.$$

So we must have the following inequality, involving all the sets U_i^δ and V_i^δ,

$$\sum_i h(d(U_i^\delta)) \geq \sum_i h(d(V_i^\delta)) - \frac{\varepsilon_{J_\delta}}{2^{J_\delta + 1}}.$$

Now, by the definitions of the sets V_i^δ and (38) and (39), we have,

$$\sum_i h(d(V_i^\delta)) \geq \sum_i H(d(V_i^\delta)) - \sum_{\substack{j \\ \rho_j < \delta}} \rho_j^{-q} \left(\frac{\varepsilon_j \rho_j^q}{2^{j+2}} \right) - \sum_{\substack{k,j \\ x_k^{(j)} < \delta}} (x_k^{(j)})^{-q} \cdot \frac{K (x_k^{(j)})^{q-1}}{(\eta_k^{(j)})^{q-1}} \cdot \frac{\varepsilon_s \rho_s^q}{2^{s+2}}$$

$$\geq \sum_i H(d(V_i^\delta)) - \frac{\varepsilon_{J_\delta}}{2^{J_\delta + 1}} - \frac{K \varepsilon_{J_\delta}}{2^{J_\delta + 1}}.$$

Therefore,

$$\sum_i h(d(U_i^\delta)) \geq \sum_i H(d(V_i^\delta)) - \frac{\varepsilon_{J_\delta}}{2^{J_\delta}} - \frac{K \varepsilon_{J_\delta}}{2^{J_\delta + 1}}$$

$$\geq \Lambda^H_\delta(S) - \frac{\varepsilon_{J_\delta}}{2^{J_\delta}} - \frac{K\varepsilon_{J_\delta}}{2^{J_\delta+1}}.$$

Thus we have shown that for any δ of the prescribed form,

$$\Lambda^h(S) \geq \Lambda^H_\delta(S) - \frac{\varepsilon_{J_\delta}}{2^{J_\delta}} - \frac{K\varepsilon_{J_\delta}}{2^{J_\delta+1}} - \varepsilon_{J_\delta}.$$

But there are arbitrarily small δ of this form, and so,

$$\Lambda^h(S) \geq \Lambda^H(S)$$

since $\varepsilon_i \to 0$ as $i \to \infty$. Hence, combining this with (35) we have the required result for sets S lying in cubes of diameter $\frac{1}{4}$.

Now let S be any set in q-dimensional Euclidean space. We may divide the space up into a countable set of closed cubes $\{C_i\}$ say, each of diameter $\frac{1}{2}$. Since we have,

$$\frac{h(x)}{x^{q-1}} \to 0 \qquad \text{as} \quad x \to 0,$$

those points of S which lie on the intersection of two of these cubes form a set S_0 such that $\Lambda^h(S_0) = 0$. Hence we may write,

$$S = \bigcup_{i=0}^{\infty} S_i,$$

where $\Lambda^h(S_i) = \Lambda^H(S_i)$ for $i = 1, 2, \ldots$ and $\Lambda^h(S_0) = \Lambda^H(S_0) = 0$.

Also we have,

$$\Lambda^h(S) = \sum_{i=1}^{\infty} \Lambda^h(S_i) = \sum_{i=1}^{\infty} \Lambda^H(S_i)$$

$$= \Lambda^H(S),$$

and hence the result extends to all sets S in q-dimensional Euclidean space.

Corollary

If $h(x)$ is any one-dimensional Hausdorff measure function, then there is a continuous one-dimensional Hausdorff measure function $H(x)$ such that,

$$\Lambda^h(S) = \Lambda^H(S)$$

for all linear sets S.

The next theorem shows that it is not always possible to replace discontinuous functions by continuous ones. That is to say that the restrictions imposed in Theorem 7 cannot be relaxed.

Theorem 8

There exists a two-dimensional Hausdorff measure function $h(x)$, say, with,

$$\frac{h(x)}{x} \to \infty \qquad \text{as } x \to 0,$$

such that, for any continuous two-dimensional Hausdorff measure function $H(x)$ say, there is a set S with the property that,

$$\Lambda^h(S) \neq \Lambda^H(S).$$

Proof

Let $\qquad x_n = 1/4^n \qquad$ for $\quad n = 0, 1, 2, \ldots$

Define $h(x)$ as follows,

$$h(x) = 1 \qquad \text{for} \quad x \geq x_0$$

$$h(x_n) = \alpha^{-1} h(x_{n-1}) \qquad \text{for } n = 1, 2, \ldots$$
$$\text{where } 1 < \alpha < 2$$

and
$$h(x) = h(x_n) \qquad \text{for } x \in (x_{n+1}, x_n].$$

Then, clearly,

$$\frac{h(x)}{x} \to \infty \qquad \text{as } x \to 0.$$

Also,

$$\frac{h(x_n)}{x_n} \leq \frac{h(t)}{t} \qquad \text{for all } t \in (0, x_n]$$
$$\text{and for } n = 1, 2, \ldots$$

and
$$\frac{h(x)}{x} \leq \frac{\alpha h(t)}{t} \qquad \text{for all } t \in (0, x]$$
$$\text{and for all } x.$$

Now let $H(x)$ be any continuous Hausdorff measure function. Then for each n we have,

$$\text{either} \qquad H(x_n) \geq \alpha^{1/2} h(x_n)$$

$$\text{or} \qquad H(x_n) < \alpha^{1/2} h(x_n)$$

We, firstly assume that,

$$H(x_n) < \alpha^{1/2} h(x_n) \qquad \text{for infinitely many } n.$$

So, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that,

$$H(x_{n_i}) < \alpha^{1/2} h(x_{n_i}) \qquad \text{for all } i.$$

Because of the continuity of $H(x)$, there exist $\varepsilon_i > 0$ such that, for all i,

$$H(x) < \alpha^{-1/4} h(x) \qquad \text{for } x \in (x_{n_i}, x_{n_i} + \varepsilon_i),$$

we may, further, insist that,

$$x_{n_i} + \varepsilon_i < x_{n_i - 1}.$$

Now let $\{A_n\}$ be a sequence of positive numbers such that, $\sum A_n^{-1}$ is convergent and,

$$\prod_{n=1}^{\infty} (1 - 2/A_n) > \alpha^{-1/4}.$$

Also let $\{B_n\}$ be a sequence of positive numbers such that,

$$B_n \uparrow \infty \quad \text{as } n \uparrow \infty \qquad \text{and } B_n > 2 \text{ for all } n.$$

We now construct a sequence $\{y_n\}$ as follows:

choose y_0 arbitrarily from the open interval $(x_{n_1}, x_{n_1} + \varepsilon_1)$

Having chosen y_ν for $\nu = 1, \ldots, n-1$, choose y_n such that,

i). $0 < y_n < \frac{1}{2} y_{n-1}$

ii). if $y_{n-1} \in (x_{n_i}, x_{n_i} + \varepsilon_i)$ then $y_n \in (x_{n_j}, x_{n_j} + \varepsilon_j)$ with $j > i$,

iii). $y_{n-1} - 2y_n > x_{n_i}$ where $y_{n-1} \in (x_{n_i}, x_{n_i} + \varepsilon_i)$

iv). $C_n h(y_n) = h(y_{n-1})$ with $C_n > A_n$

v). $B_n C_n y_n < y_{n-1}$.

Write $K_n = 2\left[{}^{c_n}/_2\right]$.

We now proceed to construct the set S in two-dimensional Euclidean space. Denote by S_0 the closed circle with centre at the origin and of diameter y_0. Draw the diameters of S_0 at angles $\theta_1, 2\theta_1, \ldots, ({}^{K_1}/_2 - 1)\theta_1$ to the positive x-axis, where $\theta_1 = {}^{2\pi}/K_1$. At each end of these diameters and at the intersections of the x-axis with the perimeter of the circle we draw a closed circle of diameter y_1, inside S_0, having one point of contact with the circumference of S_0 and having centre on the diameter of S_0. Denote by S_1 the union of these K_1 closed circles, Inside each circle of S_1 we draw ${}^{K_2}/_2$ diameters at an angle θ_2 apart, where $\theta_2 = {}^{2\pi}/K_2$. At the ends of these diameters we draw closed circles of diameter y_2 in a similar manner to that described above. Thus, we have $K_1 K_2$ closed circles of diameter y_2 and these we denote by S_2. Continuing in this manner we get sets $S_1, S_2, \ldots$ with S_n consisting of $K_1 \ldots K_n$ closed circles of diameter y_n. Also we have $S_n \supset S_{n+1}$ for all n.

We define the set $S = \bigcap_{n=0}^{\infty} S_n$.

Now since each y_i is a point of continuity of $h(x)$, and since $y_n \to 0$ as $n \to \infty$ we have,

$$\Lambda^h(S) \leq K_1 \ldots K_n h(y_n) \quad \text{for all } n$$

$$\leq h(y_0). \qquad -(40)$$

We, also, note that,

$$K_1 \ldots K_n h(y_n) > (c_1 - 2) \ldots (c_n - 2) h(y_n)$$

$$= \prod_{\nu=1}^{n} (1 - {}^{2}/c_\nu) h(y_0)$$

$$> \alpha^{-1/4} h(y_0) \qquad \text{since } C_n \geq A_n. \qquad -(41)$$

Now, given any $\varepsilon > 0$, choose $\delta > 0$ such that,

$$B_n > 4/\varepsilon \qquad \text{whenever } y_n < \delta. \qquad -(42)$$

Let $\{U_i\}$ be an arbitrary open cåvering of S such that,

$$d(U_i) < \delta \qquad \text{for all } i.$$

Since S is compact, we may assume that $\{U_i\}$ consists of a finite number of open sets.

Let n be such that,

$$y_n < d(U_i) \qquad \text{for all } i.$$

If U_i is such that,

$$y_n < d(U_i) \leq y_{n-1} \qquad -(43)$$

then U_i intersects at most one circle of S_{n-1} (by v).).

If $d(U_i) \geq y_{n-1} - 2y_n$, then,

$$h(d(U_i)) = h(y_{n-1}) \qquad \text{by iii).}$$

and so we can replace U_i by the circle of S_{n-1} which it intersects, without increasing $\sum_i h(d(U_i))$ and the circle covers at least as much of S as U_i did.

Now assume $d(U_i) < y_{n-1} - 2y_n$ and that U_i intersects more than one circle of S_n. Then U_i intersects at most m circles of S_n, where,

$$m-1 = \left[\frac{2}{\theta_n}\sin^{-1}\left(\frac{d(U_i)+y_n}{y_{n-1}-y_n}\right)\right] \qquad -(44)$$

Since U_i intersects more than one of the circles of S_n,

$$d(U_i) \geq (y_{n-1}-y_n)\sin\theta_n/_2 - y_n$$

$$\geq (y_{n-1}-y_n)\frac{2}{K_n} - y_n, \qquad \text{since } \sin x \geq \frac{2x}{\pi} \quad -(45)$$

for $0 \leq x \leq \pi/_2$.

So we have,

$$mh(y_n) \leq \left(\frac{K_n}{\pi}\sin^{-1}\left\{\frac{d(U_i)+y_n}{y_{n-1}-y_n}\right\}+1\right)h(y_n) \qquad \text{by (44)}$$

$$\leq \left(\frac{K_n}{2}\left\{\frac{d(U_i)+y_n}{y_{n-1}-y_n}\right\}+1\right)h(y_n)$$

$$\leq \left(\frac{d(U_i)+y_n}{y_{n-1}-y_n}\right)K_n h(y_n) \qquad \text{by(45)}$$

$$\leq \left(\frac{B_nC_n(d(U_i)+y_n)}{B_nC_n-1}\right)\frac{h(y_{n-1})}{y_{n-1}}. \qquad \text{by iv). and v)..}$$

Now, if $d(U_i) \leq x_{n_i}/_\alpha$, where $y_{n-1} \in (x_{n_i}, x_{n_i}+\varepsilon_i)$ then,

$$h(y_{n-1})/_{y_{n-1}} \leq h(d(U_i))/_{d(U_i)}$$

thus,

$$mh(y_n) \leq \left(\frac{B_nC_n(d(U_i)+y_n)}{(B_nC_n-1)\,d(U_i)}\right)h(d(U_i)).$$

$$< \left\{1 + \frac{1}{B_n C_n - 1}\left(1 + \frac{y_{n-1}}{d(U_i)}\right)\right\} h(d(U_i))$$

$$\leqslant \left\{1 + \frac{1}{B_n C_n - 1}\left(1 + \frac{y_{n-1}}{(y_{n-1} - y_n)\frac{2}{K_n} - y_n}\right)\right\} h(d(U_i)) \quad \text{by (45)}$$

$$\leqslant \left\{1 + \frac{2}{B_n C_n} + \frac{2}{2B_n - 2/K_n - 1}\right\} h(d(U_i)) \quad \text{by v).}$$

$$< (1+\tau)\, h(d(U_i)) \quad \text{by (42) and since } K_n \geq 1. \qquad -(46)$$

So we have shown that in this case we can replace U_i by m circles of S_n causing $h(d(U_i))$ to increase by a factor of at most $(1+\tau)$ and the circles cover at least as much of S as U_i did. If U_i meets only one circle of S_n then since $d(U_i) > y_n$ we may replace it by this circle of S_n.

Now, if,

$$d(U_i) > x_{ni}/\alpha$$

then,

$$h(d(U_i)) = 1/\alpha\, h(y_{n-1}) > 1/2\, h(y_{n-1}). \qquad -(47)$$

Also, since we have assumed that $d(U_i) < y_{n-1} - 2y_n$ we know that U_i intersects less than $K_n/2$ circles of S_n.

Now, if there is a U_j with $j \neq i$ and,

$$x_{ni}/\alpha < d(U_j) < y_{n-1} - 2y_n,$$

such that U_j intersects the same circle of S_{n-1} as U_i does, then we may replace U_i and U_j by the circle of S_{n-1}, since,

$$h(d(U_i)) + h(d(U_j)) > h(y_{n-1}),$$

and U_i, U_j together intersect less than K_n circles of S_n. Also if we have such a U_i, U_{i*} say, and if the remainder of the circle of S_{n-1} which U_{i*} intersects is covered by members of $\{U_i\}$ all with,

$$y_n < d(U_i) \leq y_{n-1},$$

then we may replace all these U_i's together with U_{i*} by the circle of S_{n-1}. Thus, so far, we have replaced each U_i satisfying,

$$y_n < d(U_i) \leq y_{n-1}$$

except those such that,

$$x_{ni}/\alpha < d(U_i) < y_{n-1} - 2y_n \qquad -(48)$$

and such that there is no other U_j meeting the same circle of S_{n-1} as U_i does and satisfies (48) and (43).

We now make similar replacements with respect to those U_i for which,

$$y_{n-1} < d(U_i) \leq y_{n-2}$$

and repeat the procedure up to and including the case where,

$$y_1 < d(U_i) \leq y_0.$$

Clearly we may assume that,

$$d(U_i) \leq y_0 \qquad \text{for all } i.$$

Now assume that there is a U_i which has not been replaced, let m be such that,

$$y_m < d(U_i) \leq y_{m-1}$$

then if the circle of S_{m-1} which it intersects has been used to replace a different member of $\{U_i\}$ then we may ignore the set U_i. We now assume that there are still some U_i remaining which cannot be ignored, as explained above and which have not been replaced. Let there be such a U_i with,

$$y_n < d(U_i) \leq y_{n-1}.$$

Then if S^i_{n-1} is the circle of S_{n-1} which U_i intersects, we know that S^i_{n-1} is partially covered by a U_j with $d(U_j) > y_{n-1}$ and U_j has not been replaced. Therefore, if $y_m < d(U_j) \leq y_{m-1}$ $(m < n)$ then S^j_{m-1} is partially covered by a U_k with $d(U_k) > y_{m-1}$ and U_k has not been replaced. Thus we see that if there were to be such a U_i then given any integer t, there is a U_s belonging to $\{U_i\}$ such that, $d(U_s) > y_{t-1}$ and U_s has not been replaced. If we take $t = 1$ we see that this is a contradiction since we can assume that,

$$d(U_i) \leq y_0 \qquad \text{for all } i.$$

Thus we conclude that all U_i with

$$y_n < d(U_i) \leq y_{n-1}$$

have been replaced.

We now consider those remaining U_i for which,

$$y_{n-1} < d(U_i) \leq y_{n-2}.$$

Then S^i_{n-2} is partially covered either,

a). by U_j with $y_n < d(U_j) \leq y_{n-1}$ and which have not been replaced, or,

b). by U_k with $d(U_k) > y_{n-1}$ and such that U_k has not been replaced.

It has just been shown that a). is impossible and we can get a similar contradiction from b)..

Continuing in this manner we see that all the U_i have been replaced by a collection $\{c_i\}$ of circles of $S_0 \cup \cdots \cup S_n$ such that,

$$\bigcup_i c_i \supset S \qquad \text{and} \qquad (1+\varepsilon)\sum_i h(d(U_i)) \geq \sum_i h(d(c_i)).$$

We further note that the inequality (41) holds for any finite collection of circles of $\bigcup_{n=0}^{\infty} S_n$ which covers S - this is because we may replace any circle of S_{n-1} by the K_n circles of S_n which it contains.

Thus,

$$(1+\varepsilon)\sum_i h(d(U_i)) > \sum_i h(d(c_i)) \geq P h(y_0),$$

where $P = \prod_{\nu=1}^{\infty}(1-2/A_\nu)$.

Therefore, since $\{U_i\}$ was an arbitrary covering of S, we have,

$$\Lambda^h_\delta(S) \geq (1+\varepsilon)^{-1} P h(y_0),$$

thus,

$$\Lambda^h(S) \geq (1+\varepsilon)^{-1} P h(y_0),$$

but the ε was an arbitrary positive number and so,

$$\Lambda^h(S) \geq P h(y_0) \qquad -(49)$$

Also, we have,

$$\Lambda^H(S) \leq K_1 \dots K_n H(y_n) \qquad \text{for all } n$$
$$< \alpha^{-1/4} K_1 \dots K_n h(y_n)$$
$$\leq \alpha^{-1/4} h(y_0)$$
$$< P h(y_0). \qquad -(50)$$

Therefore,

$$\Lambda^H(S) \neq \Lambda^h(S).$$

Thus if $H(x_n) < \alpha^{1/2} h(x_n)$ for infinitely many small x_n, then we have constructed a set S such that,

$$\Lambda^H(S) \neq \Lambda^h(S).$$

Now assume that,

$$H(x_n) \geq \alpha^{1/2} h(x_n) \qquad \text{for all small } x_n.$$

So we can assume that,

$$H(x_n) \geq \alpha^{1/2} h(x_n) \qquad \text{for all } n.$$

Then there exists $\varepsilon_n > 0$ with,

$$x_n - \varepsilon_n > x_{n+1} \qquad \frac{h(x_n)}{x_n - \varepsilon_n} < \frac{h(x_{n+1})}{x_{n+1}}$$

such that,

$$H(x) > \alpha^{1/4} h(x) \qquad \text{for } x \in [x_n - \varepsilon_n, x_n]$$

Now define $H'(x)$ such that,

$$H'(x) = \alpha^{1/4} h(x_n) \qquad \text{for } x \in (x_n - \tau_n, x_{n-1} - \tau_{n-1}]$$

Then $H'(x) \leq H(x)$ and,

$$\frac{H'(x_{n-1} - \tau_{n-1})}{x_{n-1} - \tau_{n-1}} = \frac{\alpha^{-3/4} h(x_{n-1})}{x_{n-1} - \tau_{n-1}} < \frac{\alpha^{-3/4} h(x_n)}{x_n}$$

$$< \frac{\alpha^{-3/4} h(x_n)}{x_n - \tau_n} = \frac{H'(x_n - \tau_n)}{x_n - \tau_n}$$

Therefore,

$$\frac{H'(x_n - \tau_n)}{x_n - \tau_n} \leq \frac{H'(t)}{t} \qquad \text{for all } t \in (0, x_n - \tau_n]$$

Also,

$$\frac{H'(x_n - \tau_n)}{x_n - \tau_n} = \frac{\alpha^{-3/4} h(x_n)}{x_n - \tau_n} > \frac{\alpha^{-3/4} h(x_n)}{x_n},$$

therefore,

$$\frac{H'(x_n - \tau_n)}{x_n - \tau_n} \to \infty \qquad \text{as } n \to \infty$$

that is, $$H'(x)/x \to \infty \qquad \text{as } x \to 0.$$

Further, $$\alpha H'(x_n - \tau_n) = H'(x_{n-1} - \tau_{n-1}).$$

Thus, as in the previous part of the proof we can construct a sequence $\{y_n\}$ and a set S such that,

$$H'(y_0) \geq \Lambda^{H'}(S) \geq \rho H'(y_0),$$

and, $$\Lambda^h(S) < \rho H'(y_0).$$

But $H'(x) \leq H(x)$ and so we have,

$$\Lambda^{H}(S) \geq \rho H'(y_0),$$

therefore, $$\Lambda^{h}(S) \neq \Lambda^{H}(S).$$

Hence the theorem is proved.

CHAPTER 3

INTRODUCTION

We saw in the last chapter, how, under certain conditions we could replace discontinuous functions by continuous ones without altering the corresponding Hausdorff measures. In this chapter we investigate the possibility of extending these results to the case of Hausdorff pre-measures. Theorems 9, 10, 11, 12, and 13 are concerned with the extension of some results of Sion and Sjerve (11) to the case of discontinuous functions. Theorem 14 shows us some conditions under which discontinuous functions can be replaced by continuous ones. Finally, Theorem 15 shows that the replacement used in Theorem 7 of Chapter 2 cannot be used in the case of Hausdorff pre-measures.

Theorem 9

If $h(x)$ is any monotonic increasing q-dimensional Hausdorff measure function with the property that,

$$\frac{h(x)}{x^{q-1}} \to 0 \qquad \text{as } x \to 0$$

then for any $\delta > 0$ and any increasing sequence $\{S_n\}$ of sets in q-dimensional Euclidean space we have,

$$\Lambda^h_\delta\left(\bigcup_{n=1}^{\infty} S_n\right) = \lim_{n\to\infty} \Lambda^h_\delta(S_n).$$

Proof

Let C be any convex set, in q-dimensional Euclidean space, with diameter d. Write,

$$C' = \{x : \rho(x, \mathcal{E}C) > \delta\}$$

where ρ is the metric in the space. Then, as in Theorem 7 of Chapter 2, we can cover the set $C \setminus C'$ with $\left[K\left(d/\delta\right)^{q-1}\right]$ sets of diameter δ, where K is a constant.

Consider any set S and any positive number δ. Given any $\varepsilon > 0$ let, $\{U_i^\delta\}$ be a sequence of open sets such that,

$$S \subset \bigcup_i U_i^\delta$$

$$d(U_i^\delta) < \delta \qquad \text{for all } i$$

and,
$$\sum_i h(d(U_i^\delta)) < \Lambda_\delta^h(S) + \varepsilon.$$

Clearly we have,

$$S \subset \bigcup_i \overline{U_i^\delta}$$

$$d(\overline{U_i^\delta}) < \delta \qquad \text{for all } i$$

and
$$\sum_i h(d(\overline{U_i^\delta})) < \Lambda_\delta^h(S) + \varepsilon.$$

Thus,

$$L_\delta^h(S) < \Lambda_\delta^h(S) + \varepsilon.$$

Hence, since ε was arbitrary and positive, we have shown that,

$$\Lambda_\delta^h(S) \geq L_\delta^h(S) \qquad -(1)$$

Now, given any $\varepsilon > 0$, we can choose a closed covering $\{U_i^\delta\}$ of S with $d(U_i^\delta) \leq \delta$ for all i, and,

$$\sum_i h(d(U_i^\delta)) < L_\delta^h(S) + \varepsilon/2 \qquad -(2)$$

For each i, choose $\eta > 0$ such that,

$$\frac{K\{d(U_i^\delta)\}^{q-1}}{\eta^{q-1}} h(\eta) < \tau/2^{i+1}$$

and,
$$\eta < \delta.$$

Then we can replace U_i^δ by an open set of diameter $(d(U_i^\delta) - 2\eta)$ together with $\left[\frac{K\{d(U_i^\delta)\}^{q-1}}{\eta^{q-1}}\right]$ open sets of diameter η. Hence we get a new open covering $\{V_i^\delta\}$ of S with,

$$d(V_i^\delta) < \delta \qquad \text{for all } i$$

and,
$$\sum_i h(d(V_i^\delta)) \leq \sum_i h(d(U_i^\delta)) + \tau/2 \qquad -(3)$$

Thus, combining (2) and (3), we have,

$$\Lambda_\delta^h(S) < \sum_i h(d(V_i^\delta)) < L_\delta^h(S) + \tau.$$

Combining this result with (1) we see that ,

$$\Lambda_\delta^h(S) = L_\delta^h(S),$$

and so it is sufficient to prove that,

$$L_\delta^h\left(\bigcup_{n=1}^\infty S_n\right) = \lim_{n\to\infty} L_\delta^h(S_n).$$

We now define a pseudo-metric on the space of subsets of q-dimensional Euclidean space. Denote by $S(t)$ the set,

$$\{x : \rho(x, S) < t\}$$

for any set S. Define the distance between two subsets S, T by,

$$s(S,T) = \min\{t : S \subset T(t) \text{ and } T \subset S(t)\}.$$

We then write $S_n \to S$ when,

$$s(S_n, S) \to 0 \qquad \text{as} \quad n \to \infty.$$

Firstly, we suppose that $\bigcup_{n=1}^{\infty} S_n$ is bounded in q-dimensional Euclidean space.

Now, for each n, we consider a sequence $\{U_i^n\}$ of closed sets with the following properties,

i). $S_n \subset \bigcup_{i=1}^{\infty} U_i^n$

ii). $d(U_{i+1}^n) \leq d(U_i^n) \leq \delta$ for all i

iii). $\sum_i h(d(U_i^n)) \leq L_\delta^h(S_n) + 1/n$

iv). $U_i^n \to V_i$ as $n \to \infty$ (V_i compact).

We can satisfy condition iv). because of Blaschke's Selection Theorem and from the fact that we may assume that the sets U_i^n are uniformly bounded (e.g. see Eggleston (4)).

Now, since $h(x) > 0$ for all $x > 0$, we see from iii). that,

$$d(U_i^n) \to 0 \qquad \text{as} \quad i \to \infty.$$

Now let,

$$a = \varliminf_{n \to \infty} \sum_{i=1}^{\infty} h(d(U_i^n))$$

then, given any $\varepsilon > 0$ we can find a strictly increasing sequence $\{n_k\}$ of integers such that,

$$\sum_i h(d(U_i^{n_k})) < a + \varepsilon \qquad \text{for all } k. \qquad -(4)$$

By iv). we have,

$$\sum_i h(d(V_i) - 0) \leq a \qquad -(5)$$

and so,

$$d(V_i) \to 0 \qquad \text{as } i \to \infty.$$

Now define,

$$b = a - \sum_i h(d(V_i) - 0). \qquad -(6)$$

By the argument given at the beginning of the proof, using the fact that,

$$\frac{h(x)}{x^{q-1}} \to 0 \qquad \text{as } x \to 0,$$

we can find open sets W_i^j, $j = 1, \dots m_i$ such that for each i,

$$d(W_i^j) < d(V_i) \qquad \text{for } j = 1, \dots, m_i$$

$$V_i \subset \bigcup_{j=1}^{m_i} W_i^j$$

$$\sum_{j=1}^{m_i} h(d(W_i^j)) < h(d(V_i) - 0) + \varepsilon/2^i \qquad -(7)$$

and $$d\left(\bigcup_{j=1}^{m_i} W_i^j\right) \to 0 \qquad \text{as } i \to \infty.$$

Given any $\rho > 0$ choose an integer I such that for all $i \geq I$,

$$d\left(\bigcup_{j=1}^{m_i} W_i^j\right) < \rho$$

and, $$\sum_{i>I} h(d(V_i)-0) < \varepsilon \qquad -(8)$$

Further, given any integer n, choose $N = n_k\ (> n\ )$ for some k, such that for $i = 1, \ldots, I$,

$$U_i^N \subset W_i^1 \cup \cdots \cup W_i^{m_i}$$

and, $$\sum_{i=1}^{I} \{h(d(V_i)-0) - h(d(U_i^N))\} < \varepsilon. \qquad -(9)$$

Then,

$$S_n \setminus \bigcup_{i=1}^{I} \bigcup_{j=1}^{m_i} W_i^j \subset \bigcup_{i>I} U_i^N,$$

but, for $i > I$,

$$d(U_i^N) \leq d(U_I^N) \leq d\Big(\bigcup_{j=1}^{m_I} W_I^j\Big) < e.$$

So we have,

$$L_\rho^h\{S_n \setminus \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{m_i} W_i^j\} \leq \sum_{i>I} h(d(U_i^N))$$

$$= \sum_{i=1}^{\infty} h(d(U_i^N)) - \sum_{i=1}^{\infty} h(d(V_i)-0) + \sum_{i=1}^{I} h(d(V_i)-0)$$

$$- \sum_{i=1}^{I} h(d(U_i^N)) + \sum_{i>I} h(d(V_i)-0)$$

$< b + 3\varepsilon,$ using (4), (6), (9) and (8).

Thus, letting $\rho \downarrow 0$ we have,

$$L^h\{S_n \setminus \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{m_i} W_i^j\} \leq b + 3\varepsilon.$$

Therefore, we have,

$$L^h\{S_n \setminus (\bigcup_{i=1}^{I} V_i \cup \bigcup_{i>I} \bigcup_{j=1}^{m_i} W_i^j)\} \leq b+3\tau,$$

because we could choose for $\bigcup_{j=1}^{m_i} W_i^j$ a descending sequence of open sets whose intersection is V_i, for the cases $i=1,\dots,I$. (It is well known that for any ascending sequence of sets $\{E_n\}$ we have,

$$L^h(\bigcup_n E_n) = \lim_{n\to\infty} L^h(E_n) \quad).$$

So we have,

$$L^h_\rho\{S_n \setminus (\bigcup_{i=1}^{I} V_i \cup \bigcup_{i=I+1}^{\infty} \bigcup_{j=1}^{m_i} W_i^j)\} \leq b+3\tau.$$

Thus,

$$L^h_\rho\{S_n \setminus \bigcup_{i=1}^{\infty} V_i\} \leq b+3\tau + \sum_{i=I+1}^{\infty} \sum_{j=1}^{m_i} h(d(W_i^j))$$

$$< b+5\tau \qquad \text{using (7) and (8).}$$

So, letting $\rho \searrow 0$ and $\tau \searrow 0$, we have,

$$L^h\{S_n \setminus \bigcup_{i=1}^{\infty} V_i\} \leq b.$$

That is, by iii). and the fact that $L^h\{\bigcup_n S_n \setminus \bigcup_i V_i\} = \lim_{n\to\infty} L^h\{S_n \setminus \bigcup_i V_i\}$,

$$\sum_{i=1}^{\infty} h(d(V_i)-0) + L^h\{\bigcup_n S_n \setminus \bigcup_i V_i\} \leq \lim_{n\to\infty} L^h_\delta(S_n).$$

Now, we know that $d(V_i) \leq \delta$ for all i. Thus, given any $\tau > 0$, we can cover $\bigcup_i V_i$ by closed sets $\{W_i\}$ such that,

$$\bigcup_{i=1}^{\infty} V_i \subset \bigcup_{i=1}^{\infty} W_i$$

$$d(W_i) \leq \delta \qquad \text{for all } i$$

and,
$$\sum_i h(d(W_i)) < \sum_i h(d(V_i)-0) + \varepsilon.$$

Thus,
$$L^h_\delta(\bigcup_n S_n) \leq L^h_\delta(\bigcup_n S_n \setminus \bigcup_i V_i) + \sum_i h(d(W_i))$$
$$< L^h_\delta(\bigcup_n S_n \setminus \bigcup_i V_i) + \sum_i h(d(V_i)-0) + \varepsilon$$
$$\leq \lim_{n\to\infty} L^h_\delta(S_n) + \varepsilon.$$

This is true for arbitrary positive ε, and so we have,
$$L^h_\delta(\bigcup_n S_n) \leq \lim_{n\to\infty} L^h_\delta(S_n).$$

Clearly,
$$L^h_\delta(\bigcup_n S_n) \geq \lim_{n\to\infty} L^h_\delta(S_n),$$

thus we have shown that,
$$L^h_\delta(\bigcup_n S_n) = \lim_{n\to\infty} L^h_\delta(S_n).$$

Hence we have proved the theorem when $\bigcup_n S_n$ is bounded. We can, now, extend this result to the case of unbounded sets by a method of Davies (1). Suppose that $\bigcup_n S_n$ is unbounded. The result is obvious if $\lim_{n\to\infty} \Lambda^h_\delta(S_n)$ is infinite, so we must now assume that the limit is finite.

Let C be a q-dimensional cube, sides length 2δ parallel to the coordinate axes. Let C^i denote 2^q cubes of side δ into which C may be divided. Let $\{C^i_r\}$ be an enumeration of all the distinct cubes which may be obtained from C^i by translations whose components are integral multiples of 2δ.

For each i, the cubes C^i_r $(r=1,2,\dots)$ are a distance not less than δ from one another. Thus we have,

$$\Lambda^h_\delta\left(\bigcup_n S_n \cap \bigcup_r C^i_r\right) = \sum_{r=1}^{\infty} \Lambda^h_\delta\left(\bigcup_n S_n \cap C^i_r\right) \qquad -(11)$$

Suppose that the series in (11) were divergent for at least one value of i. In that case we could choose R so large that,

$$\Lambda^h_\delta\left(\bigcup_n S_n \cap \bigcup_{r=1}^{R} C^i_r\right) = \sum_{r=1}^{R} \Lambda^h_\delta\left(\bigcup_n S_n \cap C^i_r\right) > \lim_{n\to\infty} \Lambda^h_\delta(S_n). \qquad -(12)$$

Then clearly,

$$\lim_{n\to\infty} \Lambda^h_\delta(S_n) \geq \lim_{n\to\infty} \Lambda^h_\delta\left(S_n \cap \bigcup_{r=1}^{R} C^i_r\right),$$

and so, from (12),

$$\Lambda^h_\delta\left(\bigcup_n S_n \cap \bigcup_{r=1}^{R} C^i_r\right) > \lim_{n\to\infty} \Lambda^h_\delta\left(S_n \cap \bigcup_{r=1}^{R} C^i_r\right),$$

contradicting the theorem for bounded sets. Thus, for each i, the series in (11) is convergent. Given any $\varepsilon > 0$ choose a value of R such that,

$$\sum_{i=1}^{2^q} \sum_{r=R+1}^{\infty} \Lambda^h_\delta\left(\bigcup_n S_n \cap C^i_r\right) < \varepsilon.$$

Then we have,

$$\Lambda^h_\delta\left(\bigcup_n S_n\right) \leq \Lambda^h_\delta\left(\bigcup_n S_n \cap \bigcup_{i=1}^{2^q} \bigcup_{r=1}^{R} C^i_r\right) + \Lambda^h_\delta\left(\bigcup_n S_n \setminus \left\{\bigcup_n S_n \cap \bigcup_{i=1}^{2^q} \bigcup_{r=1}^{R} C^i_r\right\}\right)$$

$$< \Lambda^h_\delta\left(\bigcup_n S_n \cap \bigcup_{i=1}^{2^q} \bigcup_{r=1}^{R} C^i_r\right) + \varepsilon$$

but the bounded case of the theorem gives us,

$$\Lambda^h_\delta\left(\bigcup_n S_n \cap \bigcup_{i=1}^{2^q} \bigcup_{r=1}^{R} C^i_r\right) = \lim_{n\to\infty} \Lambda^h_\delta\left(S_n \cap \bigcup_{i=1}^{2^q} \bigcup_{r=1}^{R} C^i_r\right)$$

$$\leqslant \lim_{n\to\infty} \Lambda^h_\delta(S_n).$$

Hence,

$$\Lambda^h_\delta\left(\bigcup_n S_n\right) < \lim_{n\to\infty} \Lambda^h_\delta(S_n) + \varepsilon \qquad \text{for every } \varepsilon > 0,$$

and therefore,

$$\Lambda^h_\delta\left(\bigcup_n S_n\right) \leqslant \lim_{n\to\infty} \Lambda^h_\delta(S_n).$$

The reverse inequality is trivial and hence the theorem is proved.

Corollary

If $h(x)$ is any monotonic increasing one-dimensional Hausdorff measure function, then for any $\delta > 0$ and any increasing sequence of sets $\{S_n\}$ on the real line, we have,

$$\Lambda^h_\delta\left(\bigcup_n S_n\right) = \lim_{n\to\infty} \Lambda^h_\delta(S_n).$$

Next, instead of considering sequences of sets we look at convergent sequences of values of δ.

Theorem 10

If $h(x)$ is any monotonic increasing q-dimensional Hausdorff measure function with the property that,

$$\frac{h(x)}{x^{q-1}} \to 0 \qquad \text{as } x \to 0$$

then for any $\delta > 0$ any set S in q-dimensional Euclidean space and any sequence $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$ as $n \to \infty$ we have,

$$\Lambda^h_\delta(S) = \lim_{n\to\infty} \Lambda^h_{\delta+\varepsilon_n}(S).$$

<u>Proof</u>

Clearly we have,

$$\Lambda^h_\delta(S) \geq \Lambda^h_{\delta+\varepsilon_n}(S) \qquad \text{for all } n. \tag{13}$$

We now assume that the set S is bounded. Thus we can assume that S is contained in a q-dimensional cube of side length C, say. Given any $\varepsilon > 0$, for each integer n, let $\{U^n_i\}$ be a sequence of open sets such that,

$$S \subset \bigcup_{i=1}^{\infty} U^n_i$$

$$d(U^n_i) < \delta + \varepsilon_n \qquad \text{for all } i,$$

and,

$$\sum_i h(d(U^n_i)) < \Lambda^h_{\delta+\varepsilon_n}(S) + \varepsilon/2 \tag{14}$$

Choose n so large that,

$$\varepsilon_n < \delta \tag{15}$$

and,

$$\left(\left[\frac{C\sqrt{q}}{\delta}\right]+1\right)^q K\left(\frac{2\delta}{\varepsilon_n}\right)^{q-1} h(\varepsilon_n) < \varepsilon/2 \tag{16}$$

where K is the constant introduced in Theorem 9. Now we can replace each U^n_i with the property $d(U^n_i) \geq \delta$ by a set V^n_i with $d(V^n_i) < \delta$

together with $\left[K\left(\frac{2\delta}{\varepsilon_n}\right)^{q-1}\right]$ sets of diameter ε_n. Thus, we get a new covering of S, $\{W_i^n\}$ say, with,

$$d(W_i^n) < \delta \qquad \text{for all } i \qquad -(17)$$

There are at most $\left(\left[\frac{c\sqrt{q}}{\delta}\right]+1\right)^q$ sets U_i^n with $d(U_i^n) \geq \delta$, since, this number of such sets would be sufficient to cover S.

So we have,

$$\sum_i h(d(W_i^n)) < \sum_i' h(d(V_i^n)) + \left(\left[\frac{c\sqrt{q}}{\delta}\right]+1\right)^q K\left(\frac{2\delta}{\varepsilon_n}\right)^{q-1} h(\varepsilon_n)$$

$$\leq \sum_i' h(d(U_i^n)) + \varepsilon/2 \quad \text{by (16)}$$

$$< \Lambda^h_{\delta+\varepsilon_n}(S) + \varepsilon. \quad \text{by (14).}$$

Therefore, for all large n,

$$\Lambda^h_\delta(S) < \Lambda^h_{\delta+\varepsilon_n}(S) + \varepsilon \qquad -(18)$$

Hence we have the required result from (18), (13) and the fact that ε was an arbitrary positive number.

Thus we have proved that, for bounded sets,

$$\Lambda^h_{\delta+\varepsilon_n}(S) \to \Lambda^h_\delta(S) \qquad \text{as } n \to \infty.$$

Now let S be an arbitrary set in q-dimensional Euclidean space. Then we write,

$$S = \bigcup_{i=1}^{\infty} S_i$$

with each S_i bounded and $S_i \subset S_{i+1}$ for all i.

Then by Theorem 9,

$$\Lambda^h_\delta(S) = \lim_{i\to\infty} \Lambda^h_\delta(S_i)$$

$$= \lim_{i\to\infty} \lim_{n\to\infty} \Lambda^h_{\delta+\varepsilon_n}(S_i)$$

$$\leq \lim_{n\to\infty} \Lambda^h_{\delta+\varepsilon_n}(S). \qquad -(19)$$

Thus by (13) and (19), we have,

$$\lim_{n\to\infty} \Lambda^h_{\delta+\varepsilon_n}(S) = \Lambda^h_\delta(S)$$ as required.

Corollary

If $h(x)$ is any monotonic increasing one-dimensional Hausdorff measure function then for any $\delta > 0$ any linear set S and any sequence $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$ as $n \to \infty$ we have,

$$\Lambda^h_\delta(S) = \lim_{n\to\infty} \Lambda^h_{\delta+\varepsilon_n}(S).$$

Davies (1) shows that the result of Theorem 9 sometimes breaks down even in the case of continuous functions when we don't insist on the property,

$$\frac{h(x)}{x^{q-1}} \to 0 \qquad \text{as } x \to 0.$$

But Sion and Sjerve (11) have shown that the result is true for the pre-measure L_δ^h in the continuous case even without the above property. Theorem 11 now shows that the latter result does not extend to the discontinuous case.

<u>Theorem 11</u>

There exists a discontinuous two-dimensional Hausdorff measure function $h(x)$ say, with,

$$\frac{h(x)}{x} \not\to 0 \qquad \text{as} \quad x \to 0$$

and a positive number δ and an increasing sequence of sets $\{S_i\}$ in two-dimensional Euclidean space such that,

$$\lim_{i\to\infty} L_\delta^h(S_i) \neq L_\delta^h(\bigcup_i S_i).$$

<u>Proof</u>

Let

$$x_n = 1/2^n \qquad \text{for} \quad n = 1, 2, \ldots$$

Define $h(x)$ as follows,

$$h(x) = x_n \qquad \text{for} \quad x \in (x_{n+1}, x_n)$$

and,
$$h(x_n) = 3/2\, x_n.$$

Then, clearly, $h(x)$ is a two-dimensional Hausdorff measure function. Take $\delta = x_N$ for some positive integer N.
Denote by S_n the common part of the closed discs,

$$x^2+y^2 \leq 1/4\, \delta^2 \quad \text{and} \quad (x-2/n)^2 + y^2 \leq 1/4\, \delta^2.$$

Then we have,

$$S_n \subset S_{n+1} \qquad \text{for all } n.$$

Also we see that, $\bigcup_{n=1}^{\infty} S_n$ is the open disc $x^2+y^2 < 1/4\,\delta^2$ together with that part of the circumference which lies to the right of the y-axis. So that we have,

$$L^h_\delta(S_n) \leq x_N \qquad \text{for all } n.$$

It is clear that,

$$L^h_\delta\left(\bigcup_{n=1}^{\infty} S_n\right) \leq 3/2\, x_N,$$

since we can cover $\bigcup_n S_n$ by its own closure. We now assume that all the sets of the covering have diameter strictly less than δ. Thus, let $\{U_i\}$ be any closed covering of $\bigcup_n S_n$ such that,

$$d(U_i) < \delta \qquad \text{for all } i.$$

Clearly, we see that no set U_i can contain points on the boundary of $\bigcup_n S_n$ which are diametrically opposite. Let $\{U_{n_i}\}$ be a subsequence of $\{U_i\}$ such that each U_{n_i} has at least one point in common with the boundary of $\bigcup_n S_n$. Let the intersection of each U_{n_i} with the boundary of $\bigcup_n S_n$ subtend an angle $2\phi_i$ at the centre of $\bigcup_n S_n$.

Then,

$$\sin \phi_i \leq \frac{d(U_{n_i})}{x_N}$$

and $$0 \leq \phi_i \leq \pi/2 \qquad \text{for each } i.$$

Clearly we must have,

$$\sum_i 2\phi_i \geq 2\pi$$

in order that the sets U_i form a covering of $\bigcup_n S_n$.

Also, we know that,

$$\sin \phi_i \geq \frac{2\phi_i}{\pi}$$

Thus,

$$\sum_i d(U_i) \geq \sum_i d(U_{n_i}) \geq \frac{x_N}{\pi} \sum_i 2\phi_i \geq 2x_N.$$

Hence, since $h(x) \geq x$ for all x, we must have,

$$L^h_\delta(\bigcup_n S_n) = 3/2\, x_N.$$

Thus the theorem is proved.

We now show that we cannot always relax the conditions imposed in Theorem 10.

<u>Theorem 12</u>

There exists a discontinuous, two-dimensional Hausdorff measure function $h(x)$, say, with,

$$\frac{h(x)}{x} \not\to 0 \qquad \text{as } x \to 0,$$

a $\delta > 0$ and a set S in two-dimensional Euclidean space, such that,

$$\lim_{n\to\infty} \Lambda^h_{\delta+1/n}(S) \neq \Lambda^h_\delta(S).$$

Proof

Let,

$$x_n = \left(\tfrac{2}{3}\right)^n \qquad \text{for } n = 1, 2, \ldots$$

Define,

$$h(x) = x_n \qquad \text{for } x \in (x_{n+1}, x_n].$$

Choose $\delta = x_N$ for some positive integer N, and denote by S the open disc $x^2 + y^2 < \tfrac{1}{4}\delta^2$ together with that part of the circumference which lies to the right of the y-axis.

Then for all integers n,

$$\Lambda^h_{\delta+1/n}(S) \leq x_{N-1} = \tfrac{3}{2}x_N,$$

and, by a similar argument to that given in the previous theorem we see that,

$$\Lambda^h_\delta(S) = 2x_N.$$

Hence the theorem is proved.

Next, we extend the result of Theorem 10.

Theorem 13

If $h(x)$ is any monotonic increasing q-dimensional Hausdorff measure function with the property that,

$$\frac{h(x)}{x^{q-1}} \to 0 \qquad \text{as} \quad x \to 0$$

then for any $\delta > 0$, any set S in q-dimensional Euclidean space and any sequence $\{\varepsilon_n\}$ with $\varepsilon_n \searrow 0$ as $n \to \infty$ we have,

$$\Lambda^h_\delta(S) = \lim_{n\to\infty} \Lambda^h_{\delta-\varepsilon_n}(S).$$

Proof

Clearly we have,

$$\Lambda^h_\delta(S) \leq \Lambda^h_{\delta-\varepsilon_n}(S) \qquad \text{for all } n. \qquad -(20)$$

So that, if S is such that $\Lambda^h_\delta(S) = \infty$, then we have,

$$\lim_{n\to\infty} \Lambda^h_{\delta-\varepsilon_n}(S) = \Lambda^h_\delta(S).$$

So it is sufficient to prove the theorem for sets S such that $\Lambda^h_\delta(S)$ is finite.

Given any $\varepsilon > 0$, let $\{U_i\}$ be a sequence of open sets with the following properties,

$$S \subset \bigcup_{i=1}^{\infty} S_i \quad , \quad d(U_i) < \delta \qquad \text{for all } i$$

and,
$$\sum_{i=1}^{\infty} h(d(U_i)) < \Lambda^h_\delta(S) + \varepsilon/2 \qquad -(21)$$

From (21) and the fact that $h(x) > 0$ for all $x > 0$ we have,

$$d(U_i) \to 0 \qquad \text{as} \quad i \to \infty$$

Thus, for some constant C, there are at most C sets U_i with $d(U_i) \geq \delta - \varepsilon_1$.

Now choose N such that for all $n \geq N$,

$$\varepsilon_n < \delta/2 \qquad -(22)$$

and,

$$CK\left(\frac{\delta}{\varepsilon_n}\right)^{q-1} h(\varepsilon_n) < \varepsilon/2 \qquad -(23)$$

where K is the constant introduced in Theorem 9. For each $n \geq N$, replace each U_i with the property $d(U_i) \geq \delta - \varepsilon_n$ by an open set V_i^n with $d(V_i^n) < \delta - \varepsilon_n$ together with $\left[K\left(\delta/\varepsilon_n\right)^{q-1}\right]$ open sets of diameter ε_n. Thus we get another open covering of S by sets W_i^n, say, such that,

$$d(W_i^n) < \delta - \varepsilon_n \qquad \text{for all } i.$$

Also, since there are at most C sets U_i with $d(U_i) \geq \delta - \varepsilon_n$ we must have,

$$\begin{aligned}
\sum_i h(d(W_i^n)) &\leq \sum_i h(d(V_i^n)) + CK\left(\delta/\varepsilon_n\right)^{q-1} h(\varepsilon_n) \\
&\leq \sum_i h(d(U_i)) + CK\left(\delta/\varepsilon_n\right)^{q-1} h(\varepsilon_n) \\
&< \Lambda_\delta^h(S) + \varepsilon,
\end{aligned}$$

by (21) and (23).

Thus, for all $n \geq N$,

$$\Lambda^h_{\delta-\varepsilon_n}(S) < \Lambda^h_\delta(S) + \varepsilon \qquad -(24)$$

Therefore, using (20) and (24) we have,

$$\lim_{n \to \infty} \Lambda^h_{\delta - \varepsilon_n}(S) = \Lambda^h_\delta(S),$$

which completes the proof of the theorem.

Corollary ofTheorems 10 and 13

If $h(x)$ is any monotonic increasing q-dimensional Hausdorff measure function with the property that,

$$\frac{h(x)}{x^{q-1}} \to 0 \qquad \text{as } x \to 0$$

then for any set S in q-dimensional Euclidean space and for any sequence $\{\delta_n\}$ with $\delta_n \to \delta$ as $n \to \infty$ for some positive real number δ, we have,

$$\lim_{n \to \infty} \Lambda^h_{\delta_n}(S) = \Lambda^h_\delta(S).$$

Proof

Given any $\varepsilon > 0$, we know from Theorems 10 and 13 that there exists a positive integer N' such that for all $n \geq N'$,

$$\Lambda^h_{\delta + 1/n}(S) > \Lambda^h_\delta(S) - \varepsilon,$$

and,
$$\Lambda^h_{\delta - 1/n}(S) < \Lambda^h_\delta(S) + \varepsilon.$$

Now, since $\delta_n \to \delta$ as $n \to \infty$ there exists a positive integer N such that, for all $n \geq N$,

$$\delta - 1/N' < \delta_n < \delta + 1/N'$$

So we have, for all $n \geq N$,

$$\Lambda^h_{\delta_n}(S) \leq \Lambda^h_{\delta - 1/N'}(S) < \Lambda^h_\delta(S) + \varepsilon,$$

and, $$\Lambda^h_{\delta_n}(S) \geq \Lambda^h_{\delta + 1/N'}(S) > \Lambda^h_\delta(S) - \varepsilon.$$

Thus we have proved that,

$$\lim_{n \to \infty} \Lambda^h_{\delta_n}(S) = \Lambda^h_\delta(S).$$

We now give conditions under which it is possible to replace discontinuous functions by continuous ones without altering the corresponding Hausdorff pre-measures.

Theorem 14

Let $h(x)$ be any monotonic increasing q-dimensional Hausdorff measure function with the property that,

$$\frac{h(x)}{x^{q-1}} \to 0 \qquad \text{as} \quad x \to 0$$

and such that its points of discontinuity have zero as their only limit point. Then there is a continuous Hausdorff measure function $H(x)$, say, such that for any $\delta > 0$, and any set S in q-dimensional Euclidean space,

$$\Lambda^h_\delta(S) = \Lambda^H_\delta(S).$$

Proof

Let $\{x_i\}$ be an enumeration of all the points of discontinuity of $h(x)$.

We may assume that $x_i > x_{i+1}$ for all i, and that,

$$x_i \to 0 \qquad \text{as} \quad i \to \infty.$$

Choose τ_1 arbitrary positive such that,

$$\tau_1 \notin \{x_i\} \quad \text{and} \quad \tau_1 \notin \{\tfrac{1}{2}x_i\}.$$

Assume that we have chosen $\tau_1, \dots, \tau_{i-1}$ we then choose τ_i to be positive and such that,

$$x_i + \tau_i < x_{i-1} \qquad -(25)$$

$$\tau_i < x_i \qquad -(26)$$

$$\tau_i < x_j - (x_{j+1} + \tau_{j+1}) \qquad \text{for } j = 1, \dots, i-1 \qquad -(27)$$

$$h(x_i + \tau_i) \leq h(2x_i) \qquad -(28)$$

$$\tau_i \notin \{x_i\} \quad \text{and} \quad \tau_i \notin \{\tfrac{1}{2}x_i\} \qquad -(29)$$

and, $$h(2\eta) < \left(\frac{\eta}{2x_i}\right)^{q-1} \{h(x_i+0) - h(x_i-0)\} K^{-1} \qquad -(30)$$

for all $\eta \in (0, \tau_i]$, where K is the constant introduced in Theorem 9. Define $H(x)$ as follows,

$$H(x) = h(x) \qquad \text{for } x \in [x_i + \tau_i, x_{i-1}) \text{ for some } i$$

$$H(x_i) = h(x_i - 0),$$

in the intervals $(x_i, x_i + \tau_i)$ define $H(x)$ to be continuous and monotonic increasing so that,

$$H(x_i+\eta) = H(x_i) + K\left(\frac{2x_i}{\eta}\right)^{q-1} H_i^*(\eta) \qquad \text{for} \qquad 0 < \eta < \tau_i,$$

where $H_i^*(x)$ is a continuous increasing function with the following properties,

$$H_i^*(x) \geq h(2x) \qquad \text{for all } x \qquad -(31)$$

$$\frac{H_i^*(x)}{x^{q-1}} \to 0 \qquad \text{as } x \to 0 \qquad -(32)$$

and
$$H_i^*(\tau_i) = \frac{1}{K}\left(\frac{\tau_i}{2x_i}\right)^{q-1}\left\{h(x_i+\tau_i) - h(x_i-0)\right\} \qquad -(33)$$

This definition makes $H(x)$ continuous at x_i because of (32); and continuity at $x_i + \tau_i$ follows from (33).

Also we have,

$$H_i^*(\tau_i) \geq \frac{1}{K}\left(\frac{\tau_i}{2x_i}\right)^{q-1}\left\{h(x_i+0) - h(x_i-0)\right\}$$

$$> h(2\tau_i) \qquad \text{by (30).}$$

Hence the equations (31) and (33) are consistent. It is clear that (31) and (32) are consistent since,

$$\frac{h(2x)}{x^{q-1}} \to 0 \qquad \text{as } x \to 0$$

Finally, we need to show that we can choose such an $H_i^*(x)$ and ensure that,

$$H(x_i+\eta) \leq h(x_i+\eta) \qquad \text{for} \qquad 0 < \eta < \tau_i.$$

Now, using (30),

$$h(x_i-0)+K\left(\frac{2x_i}{\eta}\right)^{q-1}h(2\eta)<h(x_i+0)\leq h(x_i+\eta).$$

Hence we can choose such a function $H_i^*(x)$. Thus, we see that ,

$$H(x)\leq h(x) \qquad \text{for all } x,$$

and so we have,

$$\Lambda_\delta^H(S)\leq\Lambda_\delta^h(S),$$

for all sets S and positive numbers δ. Let S be a set in q-dimensional Euclidean space and δ a positive number, then given any $\tau>0$ we can choose a sequence of open sets such that,

$$S\subset\bigcup_i U_i^\delta \qquad \text{and} \qquad d(U_i^\delta)<\delta \qquad \text{for all } i,$$

and,
$$\sum_i H(d(U_i^\delta))<\Lambda_\delta^H(S)+\tau.$$

Now we assume that for some i,

$$H(d(U_i^\delta))\neq h(d(U_i^\delta)),$$

then we must have,

$$d(U_i^\delta)\in[x_j,x_j+\tau_j) \qquad \text{for some } j.$$

Let $d(U_i^\delta)=x_j+\eta$ where $0\leq\eta<\tau_j$. Then if $\eta\in[x_k,x_{k-1})$ for some k ($k>j$) then there is a λ such that,

$$\eta+\lambda\in[x_k+\tau_k,x_{k-1}),$$

with $$0 < \lambda \leq \varepsilon_k .$$

Choose open sets $\{V_{i,s}^{\delta}\}$ with $s = 1, 2, \ldots, \left[K\left(\frac{x_j+\eta}{\eta+\lambda}\right)^{q-1}\right]+1$, such that,

$$d(V_{i,1}^{\delta}) = x_j - \lambda$$

$$d(V_{i,s}^{\delta}) = \eta + \lambda \quad \text{for } s = 2, \ldots, \left[K\left(\frac{x_j+\eta}{\eta+\lambda}\right)^{q-1}\right]+1$$

and,
$$U_i^{\delta} \subset \bigcup_{s=1}^{\left[K\left(\frac{x_j+\eta}{\eta+\lambda}\right)^{q-1}\right]+1} V_{i,s}^{\delta} .$$

Then,

$$H(d(U_i^{\delta})) = H(x_j+\eta) = H(x_j) + K\left(\frac{2x_j}{\eta}\right)^{q-1} H_j^{*}(\eta)$$

$$\geq H(x_j) + K\left(\frac{x_j+\eta}{\eta+\lambda}\right)^{q-1} h(2\eta)$$

$$\geq H(x_j-\lambda) + K\left(\frac{x_j+\eta}{\eta+\lambda}\right)^{q-1} h(\eta+\lambda)$$

$$\geq H(x_j-\lambda) + K\left(\frac{x_j+\eta}{\eta+\lambda}\right)^{q-1} H(\eta+\lambda)$$

$$\geq \sum_s H(d(V_{i,s}^{\delta})).$$

Also we see that,

$$x_j - \lambda \geq x_j - \tau_k > x_{j+1} + \tau_{j+1} \qquad \text{by (27)}$$

therefore,

$$x_j - \lambda \in (x_{j+1} + \tau_{j+1}, x_j)$$

and so we have,

$$H(d(V_{i,s}^{\delta})) = h(d(V_{i,s}^{\delta})) \qquad \text{for all } i \text{ and } s.$$

Further, we note that,

$$d(V_{i,s}^{\delta}) < \delta \qquad \text{for all } i \text{ and } s.$$

Thus we have,

$$\Lambda_{\delta}^{h}(S) \leq \sum_{i} \sum_{s} h(d(V_{i,s}^{\delta})) \leq \sum_{i} H(d(U_{i}^{\delta})) < \Lambda_{\delta}^{H}(S) + \varepsilon.$$

So, since the ε was arbitrarily small we have,

$$\Lambda_{\delta}^{h}(S) = \Lambda_{\delta}^{H}(S).$$

Hence the theorem is proved.

In Theorem 7 of Chapter 2 we showed that as far as Hausdorff measures are concerned, any discontinuous one-dimensional Hausdorff measure function $h(x)$ can be replaced by a continuous function $H(x)$ with $H(x) \geq h(x)$. We now show that this is not possible for the pre-measures.

<u>Theorem 15</u>

There is a discontinuous one-dimensional Hausdorff measure function $h(x)$, say, such that if $H(x)$ is a continuous function with $H(x) \geq h(x)$ for all x, then there exists a positive number δ and a set S on the real line such that,

$$\Lambda^h_\delta(S) \neq \Lambda^H_\delta(S).$$

<u>Proof</u>

Let

$$x_n = 1/16^n \qquad \text{for } n = 1, 2, \ldots$$

Define $h(x)$ as follows,

$$h(x) = x_n^{1/2} \qquad \text{for } x \in (x_{n+1}, x_n]$$

Denote by S_n the closed interval $[0, x_n]$. Then, clearly,

$$\Lambda^h_{x_n}(S_n) = h(x_n) = x_n^{1/2} = 2^{-2n}.$$

Now let $H(x)$ be any continuous function such that,

$$H(x) \geq h(x) \qquad \text{for all } x.$$

Then we must have,

$$H(x_n) \geq \frac{1}{16^{\frac{n-1}{2}}}.$$

So, there is a positive real number δ such that,

$$H(x) \geq 3/2^{2n} \qquad \text{for } x \in [x_n - \delta, x_n].$$

$$\delta < 1/2\, x_n$$

Now let $\{U_i^{x_n}\}$ be any open covering of S_n such that,

$$d(U_i^{x_n}) < x_n \qquad \text{for all } i.$$

Then if,

$$d(U_i^{x_n}) \in [x_n - \delta, x_n) \qquad \text{for some } i,$$

we have,

$$H(d(U_i^{x_n})) \geq 3/2^{2n} = 3\Lambda^h_{x_n}(S_n).$$

Finally, we assume that, for all i,

$$d(U_i^{x_n}) < x_n - \delta.$$

Then,

$$\sum_i H(d(U_i^{x_n})) \geq \Lambda^H_{x_n-\delta}(S_n)$$

$$\geq (x_n - \delta)^{1/2} + \delta^{1/2}$$

$$\geq \frac{x_n}{(x_n-\delta)^{1/2}} = \left(\frac{x_n}{x_n-\delta}\right)^{1/2} \Lambda^h_{x_n}(S_n).$$

Thus, in either case,

$$\sum_i H(d(U_i^{x_n})) \geq c\,\Lambda^h_{x_n}(S_n),$$

where $c > 1$.

But the covering $\{U_i^{x_n}\}$ was arbitrary and so we have,

$$\Lambda^{H}_{x_n}(S_n) \geq c\,\Lambda^{h}_{x_n}(S_n),$$

that is,

$$\Lambda^{H}_{x_n}(S_n) \neq \Lambda^{h}_{x_n}(S_n).$$

Hence the theorem is proved.

CHAPTER 4

INTRODUCTION

In this chapter, rather than considering the exact values of the Hausdorff measure of certain sets, we will only be interested in whether or not the measure is positive and finite. The first theorem gives us sufficient conditions to ensure the measure equivalence of two Hausdorff measure functions. The following four theorems are concerned with an investigation into the necessity of these conditions. In the last five theorems we use the results of the first half of the chapter to extend some work of Rogers (9) and Larman (6,7,8) to the case of discontinuous functions, and to show that a result of Eggleston (3) does not remain true for discontinuous functions.

Theorem 16

Let $h(x)$ and $H(x)$ be two q-dimensional Hausdorff measure functions. If there exists a decreasing sequence $\{x_n\}$ of positive real numbers such that,

i). $x_n \downarrow 0$ as $n \to \infty$

ii). $\underline{\lim}_{n \to \infty} \dfrac{x_{n+1}}{x_n} > 0$

and iii). $\dfrac{H(x_n)}{h(x_n)} \to l$ as $n \to \infty$ where $0 < l < \infty$.

Then the functions $h(x)$ and $H(x)$ are measure equivalent, for sets in q-dimensional Euclidean space.

Proof

We know that for any set S in q-dimensional Euclidean space, $\Lambda^h(S)$

is positive and finite if and only if $\Lambda^{h,c(q)}(S)$ is positive and finite.

Now let,

$$\lim_{x \to 0} \frac{x_{n+1}}{x_n} = 2\alpha \ (>0).$$

Let S be a set in q-dimensional Euclidean space such that $\Lambda^{h,c(q)}(S)$ is positive and finite. Let ε and δ be two given positive numbers, then there exists an open covering $\{U_i^\delta\}$ of S by cubes such that,

$$\Lambda^{h,c(q)}(S) - \varepsilon < \sum_i h(d(U_i^\delta)) < \Lambda^{h,c(q)}(S) + \varepsilon \qquad -(1)$$

and $$d(U_i^\delta) < \delta \qquad \text{for all } i. \qquad -(2)$$

Now assume that δ is so small that,

$$\frac{x_{n+1}}{x_n} > \alpha \qquad \text{for all } n \text{ such that } x_{n+1} < \delta \ -(3)$$

Then, for each i, using (2) and (3), we have,

$$x_{n_i+1} < d(U_i^\delta) \leq x_{n_i} < 1/\alpha \, x_{n_i+1} \qquad \text{for some integer } n_i.$$

So we can replace each cube U_i^δ by $\left(\left[\frac{1}{\alpha}\right]+1\right)^q$ cubes V_i of diameter x_{n_i+1}. Thus there exists another open cover $\{V_i\}$ of S by cubes such that,

$$d(V_i) \in \{x_n\} \qquad \text{for all } i$$

$$d(V_i) < \delta \qquad \text{for all } i$$

and, $$\sum_i h(d(V_i)) < \left(\left[\frac{1}{\alpha}\right]+1\right)^q \sum_i h(d(U_i^\delta))$$

$$< \left(\left[\tfrac{1}{\alpha}\right]+1\right)^{q}\left(\Lambda^{h,\,c(q)}(S) + \varepsilon\right) \qquad -(4)$$

Now we know that,

$$\frac{H(x_n)}{h(x_n)} \to l \qquad \text{as } n \to \infty,$$

therefore there exists an integer N such that,

$$l - \varepsilon < \frac{H(x_n)}{h(x_n)} < l + \varepsilon \qquad \text{for all } n \geq N.$$

Also, there exists a real number $\delta' > 0$ such that for all $\delta < \delta'$ we have,

$$\frac{H(x_n)}{h(x_n)} < l + \varepsilon \qquad \text{whenever } x_n < \delta.$$

Thus, since (4) holds for arbitrarily small values of δ, we have,

$$\sum_i H(d(V_i)) < (l+\varepsilon)\sum_i h(d(V_i)) < \left(\left[\tfrac{1}{\alpha}\right]+1\right)^{q}(l+\varepsilon)\left(\Lambda^{h,c(q)}(S)+\varepsilon\right),$$

therefore,

$$\Lambda^{H,\,c(q)}(S) \leq \left(\left[\tfrac{1}{\alpha}\right]+1\right)^{q} l\,\Lambda^{h,\,c(q)}(S).$$

Hence the theorem is proved because of the symmetry of condition iii)..

Corollary

For any discontinuous Hausdorff measure function $h(x)$, there exists a continuous Hausdorff measure function $H(x)$ such that, for sets in Euclidean space, $h(x)$ and $H(x)$ are measure equivalent.

We note that the above results can easily be extended to compact finite dimensional metric spaces. We see this from the following;

If we have,

$$x_{n_i+1} < d(U_i^\delta) \leq x_{n_i} < 1/_\alpha \, x_{n_i+1},$$

for some set U_i^δ of an open covering of S, then,

$$U_i^\delta \subset S(x, x_{n_i}) \qquad \text{for some} \quad x \in U_i^\delta .$$

Now there exist at most $N(\alpha/_6)$ disjoint spheres of radius $\alpha/_6 \, x_{n_i}$ meeting $S(x, x_{n_i})$. Thus, U_i^δ is contained in $N(\alpha/_6)$ spheres of radius $\alpha/_2 \, x_{n_i}$, which in turn are contained in $N(\alpha/_6)$ sets of diameter x_{n_i+1}. The remainder of the proof is analogous to that given in Theorem 16.

We now show that Theorem 16 would not hold true if we dropped the condition ii)..

Theorem 17

For every decreasing sequence of positive numbers $\{x_n\}$ with $x_n \downarrow 0$ as $n \to \infty$ and,

$$\lim_{n \to \infty} \frac{x_{n+1}}{x_n} = 0,$$

there exist two one-dimensional Hausdorff measure functions $h(x)$ and $H(x)$ with,

$$\frac{H(x_n)}{h(x_n)} \to 1 \qquad \text{as } n \to \infty$$

and a set S such that $\Lambda^H(S)$ is positive and finite whilst $\Lambda^h(S)$ is zero.

Proof

Since,

$$\lim_{n \to \infty} \frac{x_{n+1}}{x_n} = 0,$$

we may write,

$$x_{n+1} = \phi(n)\, x_n$$

where, $\qquad \phi(i) \downarrow 0 \qquad$ as $\quad i \to \infty.$

Define,

$$
\begin{aligned}
h(x) &= (\phi(1))^{1/2} && \text{for} \quad x \geq x_3 \\
&= (\phi(1)\,\phi(2))^{1/2} && \text{for} \quad x_3 > x \geq x_4 \\
&\;\vdots && \qquad\quad \vdots \\
&= (\phi(1)\ldots\phi(m-1))^{1/2} && \text{for} \quad x_m > x \geq x_{m+1} \\
&\;\vdots && \qquad\quad \vdots
\end{aligned}
$$

and,

$$
\begin{aligned}
H(x) &= (\phi(1))^{1/2} && \text{for} \quad x > x_4 \\
&= (\phi(1)\,\phi(2))^{1/2} && \text{for} \quad x_5 < x \leq x_4 \\
&\;\vdots && \qquad\quad \vdots \\
&= (\phi(1)\ldots\phi(m-1))^{1/2} && \text{for} \quad x_{m+2} < x \leq x_{m+1} \\
&\;\vdots && \qquad\quad \vdots
\end{aligned}
$$

Clearly these functions satisfy the postulates of the theorem.

For all i, there exists $\quad y_i \in (x_{i+1}, x_i) \quad$ such that,

$$\frac{H(y_i)}{y_i} \leq \frac{H(t)}{t} \quad \text{and} \quad \frac{h(y_i)}{y_i} \leq \frac{h(t)}{t} \quad \text{for all} \quad t \in (0, y_i].$$

We now construct the set S as in Theorem 4 of Chapter 2, by means of

a sequence $\{z_n\} \subset \{y_n\}$ with respect to the function $H(x)$. We see that $\Lambda^H(S)$ is positive and finite, but $\Lambda^h(S)$ is zero.
Hence the theorem is proved.

Clearly if $h(x)$ and $H(x)$ are continuous functions then either,

i). $H(x)/h(x) \to 0$ as $x \to 0$

ii). $H(x)/h(x) \to \infty$ as $x \to 0$

or iii). there exists a sequence $\{x_n\}$ such that $x_n \to 0$ as $n \to \infty$ and,

$$\frac{H(x_n)}{h(x_n)} \to l \quad \text{as } n \to \infty \text{ where } 0 < l < \infty.$$

We now see that this is not true if the functions are discontinuous.

Thoerem 18

There exist two q-dimensional Hausdorff measure functions $H(x)$ and $h(x)$ such that,

$$\overline{\lim_{x \to 0}} \frac{H(x)}{h(x)} = \infty, \quad \text{and} \quad \underline{\lim_{x \to 0}} \frac{H(x)}{h(x)} = 0,$$

and with the property that there are no convergent sequences $\left\{ H(x_n)/h(x_n) \right\}$ with non-zero limit, where $\{x_n\}$ is a null sequence. For these two functions there are sets S_1, S_2 such that,

$$0 < \Lambda^h(S_1) < \infty \qquad \Lambda^H(S_1) = 0$$

and $$0 < \Lambda^{H}(S_2) < \infty \qquad \Lambda^{h}(S_2) = 0.$$

Proof

To prove this theorem it suffices to define two appropriate functions, the constructions of the sets S_1, S_2 can then be carried out using the methods of Theorem 4 of Chapter 2.

Define,

$$H(x) = \frac{1}{(2n-1)!} \qquad \text{for } x \in \left([(2n+1)!]^{-2/q}, [(2n)!]^{-2/q}\right]$$

$$h(x) = \frac{1}{(2n)!} \qquad \text{for } x \in \left([(2n+1)!]^{-2/q}, [(2n)!]^{-2/q}\right]$$

$$H(x) = \frac{1}{(2n-1)!} \qquad \text{for } x \in \left([(2n)!]^{-2/q}, [(2n-1)!]^{-2/q}\right]$$

$$h(x) = \frac{1}{(2n-2)!} \qquad \text{for } x \in \left([(2n)!]^{-2/q}, [(2n-1)!]^{-2/q}\right].$$

It is easy to see that these functions have the required properties.

Next, we prove that it is possible to have measure equivalence even when the conditions in Theorem 16 are contradicted.

Theorem 19

There exist two measure equivalent q-dimensional Hausdorff measure functions $h(x)$, $H(x)$ such that if $\left\{H(x_n)/h(x_n)\right\}$ is convergent for some null sequence $\{x_n\}$ then,

$$\lim_{n \to \infty} \frac{x_{n+1}}{x_n} = 0.$$

Proof

Define,

$$h(x) = 1/n! \qquad \text{for } x \in \left([(n+1)!]^{-2/q}, [n!]^{-2/q}\right]$$

$$H(x) = 2/(2n)! \qquad \text{for } x \in \left([(2n+1)!]^{-2/q}, [(2n)!]^{-2/q}\right]$$

$$= \frac{1}{2.(2n+1)!} \qquad \text{for } x \in \left([(2n+2)!]^{-2/q}, [(2n+1)!]^{-2/q}\right].$$

We see from the definition that,

$$\tfrac{1}{2} h(x) \leq H(x) \leq 2h(x) \qquad \text{for all } x,$$

hence we clearly have measure equivalence and the theorem is proved.

Clearly we can see that if,

$$\overline{\lim_{x\to 0}} \frac{H(x)}{h(x)} < \infty \quad \text{and} \quad \underline{\lim_{x\to 0}} \frac{H(x)}{h(x)} > 0,$$

then $h(x)$ and $H(x)$ are measure equivalent. But, by considering the following example we see that there exist two q-dimensional Hausdorff measure functions $h(x)$ and $H(x)$ with,

$$\overline{\lim_{x\to 0}} \frac{H(x)}{h(x)} = \infty$$

and for any set S, $\Lambda^H(S)$ is positive and finite if and only if $\Lambda^h(S)$ is positive and finite.

Define,

$$h(x) = 1/n! \qquad \text{for } x \in \left([(n+1)!]^{-2/q}, [n!]^{-2/q}\right]$$

$$H(x) = 1/n! \qquad \text{for } x \in \left[[(n+1)!]^{-2/q}, [n!]^{-2/q}\right).$$

$\dfrac{\Lambda^H(S)}{\Lambda^h(S)}$ is always positive and finite, since $H(x)$ is less than any continuous function which is greater than $h(x)$ and because for any set S we can always find a continuous function $g(x)$ greater than $h(x)$ for which,

$$\Lambda^g(S) < 2([\sqrt{q}]+1)^2 \Lambda^h(S)$$

(this fact was proved in Theorems 5 and 6 of Chapter 2).

It is interesting to investigate whether or not measure equivalence implies the existence of a null sequence on which the ratio of the functions is convergent to a non-zero limit. It is easy to see that this is the case if we are only considering continuous functions. For, if there is no null sequence with the required property we are left with only two possibilities,

i). $\lim\limits_{x \to 0} \dfrac{h(x)}{H(x)} = 0$

or, ii). $\lim\limits_{x \to 0} \dfrac{h(x)}{H(x)} = \infty.$

Clearly both these possibilities are inconsistent with measure equivalence.

The next theorem shows that the opposite result is true for discontinuous functions.

<u>Theorem 20</u>

There exist two measure equivalent one-dimensional Hausdorff measure functions $h(x)$ and $H(x)$ such that there is no convergent sequence $\left\{ h(x_n)/H(x_n) \right\}$ (where $\{x_n\}$ is a null sequence), with non-zero limit.

Proof

Define

$$x_n = 1/2^n \qquad \text{for } n = 0, 1, 2, \ldots$$

$$h(x) = \frac{1}{2^{1+\frac{1}{2}n(n+1)}} \qquad \text{for } x \in \left(x_{\frac{1}{2}(n+2)(n+3)}, x_{\frac{1}{2}n(n+1)}\right]$$

for $n = 3, 5, 7, \ldots$

$$H(x) = \frac{1}{2^{1+\frac{1}{2}(n+1)(n+2)}} \qquad \text{for } x \in \left(x_{\frac{1}{2}(n+3)(n+4)}, x_{\frac{1}{2}(n+1)(n+2)}\right]$$

Then, $n = 1, 3, 5, \ldots$

$$\frac{h(x_{\frac{1}{2}n(n+1)})}{x_{\frac{1}{2}n(n+1)}} = \frac{1}{2} \qquad \text{for all } n = 3, 5, 7, \ldots$$

and,

$$\frac{H(x_{\frac{1}{2}n(n+1)})}{x_{\frac{1}{2}n(n+1)}} = \frac{1}{2} \qquad \text{for all } n = 2, 4, 6, \ldots$$

Thus,

$$\underline{\lim}_{x \to 0} \frac{h(x)}{x} = \underline{\lim}_{x \to 0} \frac{H(x)}{x} = \frac{1}{2}.$$

So, we must have,

$$\Lambda^h(S) \geq \tfrac{1}{2}\Lambda(S) \qquad \text{for all sets } S.$$

Now let S be any set on the real line and, given any $\delta > 0$, let $\{U_i^\delta\}$ be a sequence of open intervals such that,

$$S \subset \bigcup_{i=1}^{\infty} U_i^\delta$$

$$d(U_i^\delta) < \delta \qquad \text{for all } i,$$

and, $$\Lambda_\delta(S) \leq \sum_i d(U_i^\delta) < \Lambda_\delta(S) + \varepsilon.$$

Let $\{x_n\}$ be a sequence of positive real numbers such that,

$$\frac{h(x_n)}{x_n} \to \tfrac{1}{2} \qquad \text{as } n \to \infty$$

and let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ such that, for each i,

$$d(U_i^\delta) \in (x_{n_i+1}, x_{n_i}].$$

We can replace each U_i^δ by $\left(\left[\frac{d(U_i^\delta)}{x_{n_i+1}}\right]+1\right)$ open intervals of length x_{n_i+1}. Thus, we get another open covering $\{V_i^\delta\}$ of S such that,

$$d(V_i^\delta) < \delta \qquad \text{for all } i,$$

and,

$$\sum_i (d(V_i^\delta)) = \sum_i \left(\left[\frac{d(U_i^\delta)}{x_{n_i+1}}\right]+1\right)x_{n_i+1}$$

$$\leq \sum_i (d(U_i^\delta) + x_{n_i+1})$$

$$\leq 2\sum_i d(U_i^\delta) < 2\Lambda_\delta(S) + 2\varepsilon.$$

Thus, for all $\delta > 0$, there is an open covering $\{V_i^\delta\}$ of S such that,

$$d(V_i^\delta) < \delta \qquad \text{for all } i$$

$$d(V_i^\delta) \in \{x_n\}$$

and $$\Lambda_\delta(S) \leq \sum_i d(V_i^\delta) < 2\Lambda_\delta(S) + 2\varepsilon.$$

We can choose a positive real number δ', say, such that,

$$\frac{h(d(V_i^\delta))}{d(V_i^\delta)} < 1 \qquad \text{for all } i \text{ whenever } 0 < \delta < \delta'.$$

Thus, for $0 < \delta < \delta'$ we have,

$$\Lambda_\delta^h(S) \leq \sum_i h(d(V_i^\delta)) \leq \sum_i d(V_i^\delta)$$

$$< 2\Lambda_\delta(S) + 2\varepsilon.$$

Hence we have,

$$\tfrac{1}{2}\Lambda(S) \leq \Lambda^h(S) \leq 2\Lambda(S).$$

Clearly, we can get the same result with regard to $\Lambda^H(S)$ and so we have proved that the functions $h(x)$ and $H(x)$ are measure equivalent for sets on the real line. Also it is clear that there are no convergent sequences $\left\{\frac{h(x_n)}{H(x_n)}\right\}$ with non-zero limit. Hence the theorem is proved.

Eggleston (3) has shown that given any positive number α and any function $h(x)$ satisfying,

i). $h(x)$ continuous and strictly increasing

ii). $x^\alpha/h(x)$ is an increasing function of x

and, iii). $h(0)=0$, $\lim_{x\to 0+} x^n/h(x) = 0$,

we can construct a set A in n-dimensional Euclidean space so that $\Lambda^h(A) = \alpha$. It is now easily possible to extend this result to functions satisfying,

a). $h(x) > 0$ for $x > 0$

b). $\lim_{x\to 0} h(x) = 0$

and, c). $\lim_{x\to 0} \frac{h(x)}{x^n} = \infty.$

In the same paper Eggleston defines two functions (satisfying i)., ii). and iii).) to be incomparable when,

$$\varliminf_{x\to 0+} \frac{h(x)}{H(x)} = \varliminf_{x\to 0+} \frac{H(x)}{h(x)} = 0.$$

He shows that for two incomparable functions we can construct a set A such that $\Lambda^h(A)$ is positive and finite whilst $\Lambda^H(A) = \infty$. Our next theorem shows that this result does not extend to the case of discontinuous functions.

Theorem 21

There are two incomparable q-dimensional Hausdorff measure functions $h(x)$ and $H(x)$ say, such that if S is a set in q-dimensional Euclidean space, then if $\Lambda^h(S)$ is finite we must also have $\Lambda^H(S)$ finite.

Proof

Define the decreasing sequence $\{x_n\}$ as follows,

$$x_1 = 1 \qquad \text{and} \qquad x_{n+1} = \frac{x_n}{[2(n+1)^2]^{1/q}},$$

then, clearly,

$$x_{n+1} < x_n \qquad \text{and} \quad x_n \downarrow 0 \qquad \text{as} \quad n \to \infty.$$

Define the function $g(x)$ as follows,

$$g(x) = 1 \qquad \text{for} \quad x > x_2$$

$$g(x) = \frac{g(x_{n-1})}{n^2} \qquad \text{for} \quad x \in (x_{n+1}, x_n].$$

Then $g(x)$ is a q-dimensional Hausdorff measure function.

Define $h(x)$ such that,

$$h(x) = g(x) \qquad \text{for} \quad x \notin \{x_i\}$$

$$h(x_n) = n g(x_n) \qquad \text{for all} \ n.$$

Define $H(x)$ such that,

$$H(x) = g(x) \qquad \text{for} \quad x \notin \{x_i\}$$

$$H(x_{2n}) = g(x_{2n}) \qquad \text{for all} \ n$$

$$H(x_{2n+1}) = g(x_{2n}) \qquad \text{for all} \ n.$$

Then both $h(x)$ and $H(x)$ are q-dimensional Hausdorff measure functions.

Also,

$$\frac{h(x_{2n+1})}{H(x_{2n+1})} = \frac{(2n+1)\, g(x_{2n+1})}{g(x_{2n})} = \frac{1}{2n+1},$$

and,
$$\frac{H(x_{2n})}{h(x_{2n})} = \frac{g(x_{2n})}{2n\, g(x_{2n})} = \frac{1}{2n},$$

thus,
$$\lim_{x \to 0+} \frac{h(x)}{H(x)} = \lim_{x \to 0+} \frac{H(x)}{h(x)} = 0,$$

that is, the functions $h(x)$ and $H(x)$ are incomparable.
Now let S be any set such that $\Lambda^h(S)$ is finite, then since $h(x) \geq g(x)$ we have $\Lambda^g(S)$ is finite. From Theorems 5 and 6 of Chapter 2 we see that, given any $\tau > 0$, there is a continuous Hausdorff measure function $G(x)$ say, such that $G(x) \geq g(x)$ for all x and,

$$\Lambda^g(S) \leq \Lambda^G(S) < 2([\sqrt{q}]+1)^q \Lambda^g(S) + \tau.$$

Now we have $h(x) \geq g(x)$ and $H(x) \geq g(x)$ and both $h(x)$ and $H(x)$ are less than any continuous function which is greater than $g(x)$, that is,

$$h(x) \leq G(x) \quad \text{and} \quad H(x) \leq G(x).$$

Thus,

$$\Lambda^g(S) \leq \Lambda^h(S) \leq \Lambda^G(S) < 2([\sqrt{q}]+1)^q \Lambda^g(S) + \tau,$$

and,

$$\Lambda^g(S) \leq \Lambda^H(S) \leq \Lambda^G(S) < 2([\sqrt{q}]+1)^q \Lambda^g(S) + \tau.$$

Hence the theorem is proved.

Combining this result with the corollary to Theorem 16 of this chapter, we see that Eggleston's theorem does not hold true for functions $h(x)$, satisfying,

i). $h(x)$ is continuous and strictly increasing

ii). $h(0)=0$, $\lim_{x \to 0+} \frac{x^n}{h(x)} = 0$.

Since, if it held true in the continuous case it would also be true for discontinuous functions, and we have just seen that this is not so. Thus we have a negative answer to the problem of whether we can always replace any Hausdorff measure function by another one $h(x)$, say, with the property that $h(x)/x^q$ is a decreasing function of x.

We now generalize a result of Rogers (9) to the case of discontinuous functions.

Theorem 22

Let $h(x)$ be a q-dimensional Hausdorff measure function and E a compact set of non-σ-finite h-measure in a Euclidean space (or in a compact, finite-dimensional metric space). Then there is a continuous Hausdorff measure function $g(x)$ with $h \prec g$ and such that E is of non-σ-finite g-measure.

Proof

Let $\{x_n\}$ be an enumeration of all the discontinuities of $h(x)$. Define the decreasing sequence $\{y_n\}$ as follows. Choose y_1 arbitrarily such that $y_1 \notin \{x_n\}$; having chosen $y_1, \dots, y_{n-1}$, choose y_n such that,

$$\tfrac{1}{3} y_{n-1} < y_n \leq \tfrac{1}{2} y_{n-1},$$

and,

$$y_n \notin \{x_n\}.$$

Define $H(x)$ to be continuous increasing and,

$$H(y_n) = h(y_n) \qquad \text{for all } n,$$

with, $$H(x) \leq h(x) \qquad \text{for all } x.$$

Then we have,

i). $y_n \downarrow 0$ as $n \to \infty$

ii). $\exists\, y_{n+1} > y_n$ for all n

iii). $H(y_n)/h(y_n) \to 1$ as $n \to \infty$.

Thus all the conditions of Theorem 16 are satisfied. So, for any set S, $\Lambda^h(S)$ is positive and finite if and only if $\Lambda^H(S)$ is positive and finite. Now, since E is a compact set of non-σ-finite h-measure it must be of non-σ-finite H-measure. Thus, from Rogers' result (9), there exists a continuous Hausdorff measure function $g(x)$ with $H \prec g$ such that E is of non-σ-finite g-measure. But $H(x) \leq h(x)$ for all x and thus $h \prec g$. Hence the theorem is proved.

We, next, generalize some results of Larman (6, 7).

<u>Theorem 23</u>

Let E be a finite dimensional compact metric space, and suppose that $h(x)$ is a Hausdorff measure function such that $\Lambda^h(E)$ is infinite. Then it is possible to select from E a closed subset of any given h-measure.

Proof

If α is any given positive number, it is sufficient to find a closed subset P such that $\Lambda^h(P) \geq \alpha$. Larman (7) proves this result for the case when $h(x)$ is a continuous (on the right) function. From Theorem 16 we can find a continuous function $H(x)$, with $H(x) \leq h(x)$ and such that $H(x)$ and $h(x)$ are measure equivalent.
Thus $\Lambda^H(E) = \infty$ and therefore from Larman's result we can find a closed subset P such that,

$$\Lambda^H(P) \geq \alpha.$$

But $h(x) \geq H(x)$ for all x, and therefore,

$$\Lambda^h(P) \geq \alpha.$$

This completes the proof of the theorem.

Theorem 24

Let $h(x)$ be a q-dimensional Hausdorff measure function. Then it is possible to construct in ℓ^2 a closed set A such that,

$$0 < \Lambda^h(A) < \infty.$$

Proof

Let $H(x)$ be a continuous function with $H(x) \geq h(x)$ for all x and such that $h(x)$ and $H(x)$ are measure equivalent for sets in compact finite dimensional metric spaces. Then since $H(x) \geq h(x)$ we know that H is a q-dimensional measure function. In particular, we know that,

$$\frac{H(x)}{x^{q+1}} \to \infty \qquad \text{as } x \to 0.$$

Thus, there exists a decreasing sequence $\{x_n\}$ such that,

$$x_n \downarrow 0 \qquad \text{as } n \to \infty,$$

and for each n,

$$\frac{H(x_n)}{x_n^{q+1}} \leq \frac{H(t)}{t^{q+1}} \qquad \text{for all } t \in (0, x_n],$$

since $H(x)$ is continuous.

In the closed interval $[x_{n+1}, x_n]$ define the continuous function $H'(x)$ as follows,

$$H'(x) = x^{q+1} \inf_{y \in [x_{n+1}, x_n]} \left\{ \frac{H(y)}{y^{q+1}} \right\}.$$

Then $H'(x) \leq H(x)$ for all x and $H'(x)/x^{q+1}$ increases from $H(x_n)/x_n^{q+1}$ to $H(x_{n+1})/x_{n+1}^{q+1}$ as x decreases from x_n to x_{n+1}.

Now consider two real numbers x, z such that $x > z$ then because of the continuity of $H(x)$ we have,

$$\frac{H'(x)}{x^{q+1}} = \frac{H(y_x)}{y_x^{q+1}},$$

where $H'(y_x) = H(y_x)$ and $y_x \in [x_{n+1}, x]$ for some integer n, and,

$$\frac{H'(z)}{z^{q+1}} = \frac{H(y_z)}{y_z^{q+1}},$$

where $H'(y_z) = H(y_z)$ and $y_z \in [x_{n+1}, z]$ for the same integer n.
Now, if

$$\frac{H'(x)}{x^{q+1}} = \frac{H'(z)}{z^{q+1}},$$

we must have $H'(x) \geq H'(z)$.

If,

$$\frac{H'(x)}{x^{q+1}} < \frac{H'(z)}{z^{q+1}},$$

then we must have, $y_x \in (z, x]$.

In the interval $[z, y_x]$, the function $H(x)/x^{q+1}$ takes all values from $H(y_x)/y_x^{q+1}$ to $H(z)/z^{q+1}$ which is greater than or equal to $H(y_z)/y_z^{q+1}$. Therefore, there exists $t \in [z, y_x]$ such that,

$$\frac{H(t)}{t^{q+1}} = \frac{H(y_z)}{y_z^{q+1}}.$$

Thus, we have, because of the monotonicity of $H(x)$,

$$H'(z) \leq H(t) \leq H(y_x) \leq H'(x)$$

and so we have proved that the function $H'(x)$ is monotonic increasing. Also, $H'(x)$ is continuous and $H'(x)/x^{q+1}$ increases to infinity as x decreases to zero. Larman (6) shows that for functions of this type it is possible to construct, in λ^2, a compact, perfect set A such that,

$$0 < H'(A) < \infty, \quad \text{(for definition, see p.4)}$$

and,

$$0 < \Lambda^{H'}(A) < \infty.$$

Now since $H(x) \geq H'(x)$ for all x, we have,

$$\Lambda^{H}(A) \geq \Lambda^{H'}(A) > 0.$$

Define the function $g(x)$ as follows,

$$g(x) = x\, H'(x)$$

then we have,

$$g(A) = 0,$$

and so A is a compact finite dimensional metric space.
So, if $\Lambda^{H}(A)$ is finite, then we must have,

$$0 < \Lambda^{h}(A) < \infty$$

because of the measure equivalence of $h(x)$ and $H(x)$.
Now, if $\Lambda^{H}(A) = \infty$ we can use the result of Theorem 23 to select a closed subset P of A such that,

$$0 < \Lambda^{H}(P) < \infty,$$

and again we have,

$$0 < \Lambda^{h}(P) < \infty.$$

Hence the theorem is proved.

Finally, we state a theorem of Larman (8) which can easily be generalized to the discontinuous case using the corollary to Theorem 16.

<u>Theorem 25</u>

Let $h(x)$ be a Hausdorff measure function and A an analytic set of non-σ-finite h-measure in a compact finite dimensional metric space, then we can construct $2^{\aleph_0}$ disjoint closed subsets of A which have non-σ-finite h-measure.

CHAPTER 5

INTRODUCTION

In Chapter 3 we obtained some results relating Hausdorff pre-measures and convergent sequences $\{\delta_n\}$ of positive real numbers. Following this, it seemed interesting to investigate the properties of the Hausdorff measures of a set with regard to functions $h_n(x)$ where $\{h_n(x)\}$ is a convergent sequence of functions.

Theorem 26

There exists a sequence of Hausdorff measure functions $\{h_n(x)\}$ such that,

$$h_n(x) \to h(x) \qquad \text{uniformly as } n \to \infty$$

where $h(x)$ is a Hausdorff measure function, and a set S with the property,

$$\lim_{n\to\infty} \Lambda^{h_n}(S) \neq \Lambda^{h}(S).$$

Proof

We shall in fact show that there exists such a sequence of functions $\{h_n(x)\}$ with limit $h(x)$ such that,

$$\lim_{n\to\infty} \Lambda^{h_n}(S) = \infty \qquad \text{whenever } \Lambda^{h}(S) > 0.$$

Let $\qquad x_n = 1/n^3 \qquad$ for $n = 1, 2, \ldots$

and, $\qquad \alpha_n = 1/n^2, \qquad$ for $n = 1, 2, \ldots$

Define

$$h(x) = \alpha_n \qquad \text{for } x \in (x_{n+1}, x_n].$$

We note that α_n is such that for all positive integers q,

$$\frac{\alpha_n}{x_n^q} \to \infty \qquad \text{as } n \to \infty.$$

This ensures that $h(x)$ is a q-dimensional Hausdorff measure function.
For each integer n, define $h_n(x)$ as follows,

$$h_n(x) = \left(1 + \frac{m}{n}\right)\alpha_m \qquad \text{for } x \in (x_{m+1}, x_m].$$

Then each $h_n(x)$ is a q-dimensional Hausdorff measure function.
Clearly,

$$h_n(x) \to h(x) \qquad \text{uniformly as } n \to \infty.$$

Choose any set S such that,

$$0 < \Lambda^h(S) < \infty.$$

Consider the function $h_n(x)$ for some fixed positive integer n. Then, given any real number A, there exists an integer $M = M(n)$ such that,

$$\frac{h_n(x_m)}{h(x_m)} > A \qquad \text{for all } m \geq M.$$

Now, given any $\varepsilon > 0$, choose a sequence $\{U_{n,i}^{x_m}\}$ of open sets such that,

$$S \subset \bigcup_{i=1}^{\infty} U_{n,i}^{x_m}$$

$$d(U_{n,i}^{x_m}) < x_m \qquad \text{for all } i,$$

and,

$$\Lambda^{h_n}_{x_m}(S) \leq \sum_i h_n(d(U^{x_m}_{n,i})) < \Lambda^{h_n}_{x_m}(S) + \varepsilon.$$

Then, for all $m \geq M$, we have,

$$\Lambda^{h}_{x_m}(S) \leq \sum_i h(d(U^{x_m}_{n,i})) < \frac{1}{A}\left(\sum_i h_n(d(U^{x_m}_{n,i}))\right)$$

$$< \frac{1}{A}\left(\Lambda^{h_n}_{x_m}(S) + \varepsilon\right),$$

that is,

$$\Lambda^{h_n}_{x_m}(S) > A\,\Lambda^{h}_{x_m}(S) - \varepsilon.$$

Thus, since n, A and ε were arbitrary and because,

$$0 < \Lambda^{h}(S) < \infty,$$

we have,

$$\lim_{n\to\infty} \Lambda^{h_n}(S) = \infty.$$

Hence the theorem is proved.

Corollary

There exists a sequence of Hausdorff measure functions $\{h_n(x)\}$ and a function $h(x)$ such that,

$$\frac{h_n(x)}{h(x)} \to 1 \qquad \text{as } n \to \infty$$

(the convergence being point-wise), and a set S such that,

$$\lim_{n \to \infty} \Lambda^{h_n}(S) \neq \Lambda^{h}(S).$$

Thus, we see that uniform convergence of the functions is not sufficient to ensure that the limit operation commutes with the Hausdorff measure. We, now, establish sufficient conditions for this property to hold true.

<u>Theorem 27</u>

For any Hausdorff measure function $h(x)$ and any sequence $\{h_n(x)\}$ of Hausdorff measure functions such that,

$$\frac{h_n(x)}{h(x)} \to 1 \qquad \text{uniformly as } n \to \infty$$

we have,

$$\lim_{n \to \infty} \Lambda_\delta^{h_n}(S) = \Lambda_\delta^{h}(S),$$

for all sets S and for any positive real number δ.

<u>Proof</u>

Given any $\varepsilon > 0$ we can choose a positive integer N such that,

$$1 - \varepsilon < \frac{h_n(x)}{h(x)} < 1 + \varepsilon,$$

for all $n \geq N$ and for all x.

Thus, for all $n \geq N$, any set S and any positive real number δ, we have,

$$(1-\varepsilon)\Lambda_\delta^{h}(S) \leq \Lambda_\delta^{h_n}(S) \leq (1+\varepsilon)\Lambda_\delta^{h}(S),$$

that is,

$$(1-\varepsilon)\Lambda_\delta^h(S) \leq \lim_{n\to\infty} \Lambda_\delta^{h_n}(S) \leq (1+\varepsilon)\Lambda_\delta^h(S).$$

Hence, since the ε was arbitrary, we have,

$$\lim_{n\to\infty} \Lambda_\delta^{h_n}(S) = \Lambda_\delta^h(S),$$

as required.

CHAPTER 6

INTRODUCTION

In this chapter we work in the space ℓ^2 and investigate whether or not some of the theorems of previous chapters can be extended to this Non-Euclidean space. In Theorems 4 and 24 of Chapters 2 and 4, respectively, we showed that, corresponding to any q-dimensional Hausdorff measure function, firstly there is a set in q-dimensional Euclidean space with positive, finite h-measure, and secondly there is a set in ℓ^2 with positive, finite h-measure. The first theorem of this chapter shows that there are Hausdorff measure functions such that,

$$\lim_{x \to 0} \frac{h(x)}{x^q} = 0,$$

for all positive integers q, and sets S in ℓ^2 such that $\Lambda^h(S)$ is positive and finite. Clearly, the sets S could not be embedded in any Euclidean space. The second theorem shows that there are discontinuous functions such that for sets in ℓ^2 there are no measure equivalent continuous functions. Finally, we show that Theorem 10 of Chapter 3 does not extend to the space ℓ^2.

Theorem 28

There exists a compact set S in ℓ^2 and a Hausdorff measure function $h(x)$ such that, for any positive integer q,

$$\lim_{x \to 0} \frac{h(x)}{x^q} = 0,$$

for which,

$$0 < \Lambda^h(S) < \infty.$$

Proof

Define the sequence $\{x_n\}$ of positive real numbers so that,

$$x_n = 1/2^n \qquad \text{for} \quad n = 0, 1, 2, \ldots$$

Define the function $h(x)$ as follows,

$$h(x) = x_n^n \qquad \text{for} \quad x \in (x_{n+1}, x_n],$$

then, clearly, $h(x)$ satisfies the conditions of the theorem.
Let $\{A_n\}$ be a sequence of integers such that $\sum A_n^{-1}$ is convergent and $A_n \geq 2$ for all n. We now inductively define a sequence $\{t_n\}$ of real numbers; choose t_0 to be an arbitrary positive number such that,

$$t_0 < x_1$$

and,

$$t_0 \notin \{x_n\}$$

we assume that

$$t_0 \in (x_{n_0+1}, x_{n_0})$$

for some positive integer n_0.
We now suppose that $t_0, \ldots, t_{m-1}$ have been defined and that,

$$t_{m-1} \in (x_{n_{m-1}+1}, x_{n_{m-1}})$$

for some positive integer n_{m-1}.
Choose t_m as follows,

a). $0 < t_m < 1/3\, t_{m-1}$,

b). $c_m h(t_m) = h(t_{m-1})$ with $c_m > A_m$, c). $t_{m-1} - 2t_m > x_{n_{m-1}+1}$,

and d). $t_m \notin \{x_n\}$.

Now put,

$$K_m = [c_m] \qquad \text{for } m = 1, 2, \ldots$$

Let $S'(0)$ be the collection of all points of the form,

$$(0, \ldots, 0, t_0/\sqrt{2}, 0, \ldots).$$

Put,

$$S(0) = \overline{\operatorname{conv}[\underline{\alpha}^{(0)} + S'(0)]}$$

where $\alpha_j^{(0)} = 0$ for all j.

Let $S'(1)$ be the collection of all points of the form,

$$(0, \ldots, 0, t_1/\sqrt{2}, 0, \ldots).$$

Put,

$$S(1, i_1) = \overline{\operatorname{conv}[\underline{\alpha}^{(i_1)} + S'(1)]}$$

where $\alpha_j^{(i_1)} = 0$ for $j \neq i_1$

and $$\alpha_{i_1}^{(i_1)} = \frac{t_0 - t_1}{\sqrt{2}}.$$

Then define,

$$S(1) = \bigcup_{i_1=1}^{K_1} S(1, i_1).$$

In general, let $S'(n)$ denote the set of all points of the form,

$$(0, \ldots, 0, t_n/\sqrt{2}, 0, \ldots).$$

Put,

$$S(n, i_1, \ldots, i_n) = \overline{\operatorname{conv}[\underline{\alpha}^{(i_1,\ldots,i_n)} + S'(n)]},$$

where,

$$\alpha_{i_1}^{(i_1,\ldots,i_n)} = \frac{t_0 - t_1}{\sqrt{2}},$$

$$\alpha_{K_1+i_2}^{(i_1,\ldots,i_n)} = \frac{t_1 - t_2}{\sqrt{2}},$$

$$\vdots \qquad \vdots$$

$$\alpha_{K_1+\cdots+K_{n-1}+i_n}^{(i_1,\ldots,i_n)} = \frac{t_{n-1} - t_n}{\sqrt{2}},$$

and, $\alpha_j^{(i_1,\ldots,i_n)} = 0$ for all other values of j.

Then define,

$$S(n) = \bigcup_{i_1=1}^{K_1} \cdots \bigcup_{i_n=1}^{K_n} S(n, i_1, \ldots, i_n).$$

Having defined $S(n)$ for $n = 0, 1, 2, \ldots$ we need, firstly, to show that $S(n+1) \subset S(n)$ for all integers n. To this end, it is sufficient to prove that, for each n,

$$S(n, i_1, \ldots, i_{n-1}, i_n) \subset S(n-1, i_1, \ldots, i_{n-1}),$$

for $i_n = 1, \ldots, K_n$.

Consider a point of $S(n, i_1, \ldots, i_n)$ of the form,

$$\underline{\alpha}^{(i_1,\dots,i_n)} + (0,\dots, t_n/\sqrt{2}, 0, \dots),$$

where the value $t_n/\sqrt{2}$ appears in the position p. We can write this in the form,

$$\lambda\left[\underline{\alpha}^{(i_1,\dots,i_{n-1})} + (0,\dots,0,\frac{t_{n-1}}{\sqrt{2}},0,\dots)\right] + (1-\lambda)\left[\underline{\alpha}^{(i_1,\dots,i_{n-1})} + (0,\dots,\frac{t_{n-1}}{\sqrt{2}},0,\dots)\right]$$

where the term $\frac{t_{n-1}}{\sqrt{2}}$ in the first square bracket is in position p and that in the second square bracket in position $K_1+\dots+K_{n-1}+i_n$, if we put $\lambda = \frac{t_n}{t_{n-1}}$. Thus by the convexity of $S(n-1, i_1,\dots,i_{n-1})$ we have,

$$S(n, i_1,\dots,i_{n-1}, i_n) \subset S(n-1, i_1,\dots,i_{n-1}),$$

for $i_n = 1,\dots,K_n$ as required.

Also we can see that,

$$d[S(n, i_1,\dots,i_n)] = t_n \qquad \text{for all } n.$$

Thus, since $t_n \to 0$ as $n \to \infty$ and because ℓ^2 is complete, we can define the non-empty set,

$$S = \bigcap_{n=0}^{\infty} S(n).$$

We, next, prove that S is compact. Let $\{\underline{x}_n\}$ be an infinite sequence of points of S. We now construct a convergent subsequence, using the fact that for any integer n there are only finitely many sets $S(n, i_1,\dots,i_n)$. Choose $\underline{x}_{n_1} \in \{\underline{x}_n\}$ such that,

$$\underline{x}_{n_1} \in S(1, i_1) \qquad \text{for some } i_1 \text{ with } 1 \le i_1 \le K_1,$$

and such that $S(1, i_1)$ contains infinitely many points of the sequence $\{\underline{x}_n\}$.

Now, assume that $\underline{x}_{n_{m-1}}$ has been chosen so that,

$$\underline{x}_{n_{m-1}} \in S(m-1, i_1, \ldots, i_{m-1}),$$

where $1 \leq i_j \leq K_j$ for $j = 1, \ldots, m-1$.

Choose $\underline{x}_{n_m}$ so that,

$$\underline{x}_{n_m} \in \{\underline{x}_n\}$$

$$\underline{x}_{n_m} \neq \underline{x}_{n_{m-1}}$$

and, $\underline{x}_{n_m} \in S(m, i_1, \ldots, i_{m-1}, i_m)$ for some i_m with $1 \leq i_m \leq K_m$

and such that $S(m, i_1, \ldots, i_m)$ contains infinitely many points of the sequence $\{\underline{x}_n\}$. Thus we have defined a subsequence $\{\underline{x}_{n_m}\}$ of $\{\underline{x}_n\}$.

Now, given any $\varepsilon > 0$, there exists a positive integer M such that,

$$t_m < \varepsilon \qquad \text{for all } m \geq M.$$

But, from the construction of the subsequence, there exists, for every $m \geq M$ a set $S(m, i_1, \ldots, i_m)$ such that,

$$\underline{x}_{n_j} \in S(m, i_1, \ldots, i_m) \qquad \text{for all } j \geq m.$$

Thus,

$$\rho(\underline{x}_{n_m}, \underline{x}_{n_{m+i}}) \leq t_m < \varepsilon,$$

for all $m \geq M$ and for any positive integer i. Thus $\{\underline{x}_{n_m}\}$ is a Cauchy sequence, and, by the completeness of ℓ^2 has a limit point. Also, we know that this limit point must be in S since S is, clearly, closed. Hence S is compact.

Now define,

$$T(m, i_1, \ldots, i_m) = \{x : \rho(x, S(m, i_1, \ldots, i_m)) < \tfrac{1}{4}(x_{n_m} - t_m)\}.$$

Then we have the following,

$$T(m, i_1, \ldots, i_m) \qquad \text{is open and convex}$$

$$S(m, i_1, \ldots, i_m) \subset T(m, i_1, \ldots, i_m)$$

$$d[T(m, i_1, \ldots, i_m)] = t_m + \tfrac{1}{2}(x_{n_m} - t_m) < x_{n_m},$$

and therefore,

$$h[d\{T(m, i_1, \ldots, i_m)\}] = h[d\{S(m, i_1, \ldots, i_m)\}] = h(t_m).$$

Also we have,

$$d[T(m, i_1, \ldots, i_m)] \to 0, \qquad \text{as} \quad m \to \infty,$$

and thus,

$$\Lambda^h(S) \leq K_1 \ldots K_m h[d(T(m, i_1, \ldots, i_m))] \quad \text{for all } m$$

$$\leq C_1 \ldots C_m h(t_m) = h(t_0), \qquad \text{by b)}..$$

Thus, we have shown that,

$$\Lambda^h(S) < \infty.$$

Now, let $\{U_i\}_{i=1,\ldots,N}$ be any covering of S by a sequence of open convex sets. We need only consider finite coverings because of the compactness of S.

Consider a particular set U_i, there exists a positive integer $m = m(i)$ such that all the points of $S \cap U_i$ belong to the same $S(m-1, i_1, \dots, i_{m-1})$ but to at least two different sets $S(m, i_1, \dots, i_m)$.

Thus, we may assume that,

$$d(U_i) \leq t_{m-1}.$$

Now, for $i_m \neq j_m$,

$$\rho\left[S(m, i_1, \dots, i_{m-1}, i_m), S(m, i_1, \dots, i_{m-1}, j_m)\right] = t_{m-1} - 2t_m,$$

thus, we must have,

$$d(U_i) \geq t_{m-1} - 2t_m,$$

and so,

$$h(d(U_i)) = x_{n_{m-1}}^{n_{m-1}} = h(t_{m-1}), \quad \text{by c)..}$$

Hence, we may replace the set U_i by the corresponding set $T(m-1, i_1, \dots, i_{m-1})$. Thus, since the U_i ~~x~~ was arbitrary we may assume that any covering of S consists of sets of the form $T(n, i_1, \dots, i_n)$ for finitely many values of n. Let n^* be the largest such value of n, then from b). we may assume that the covering consists of the $K_1 \dots K_{n^*}$ sets $T(n^*, i_1, \dots, i_{n^*})$.

Now,

$$K_1 \dots K_{n^*} h[d\{T(n^*, i_1, \dots, i_{n^*})\}] = K_1 \dots K_{n^*} h(t_{n^*})$$

$$> (c_1 - 1) \dots (c_{n^*} - 1) h(t_{n^*})$$

$$= \prod_{i=1}^{n^*} (1 - 1/c_i)\, h(t_0)$$

$$\geq \prod_{i=1}^{\infty} (1 - 1/A_i)\, h(t_0).$$

But $\sum A_i^{-1}$ is convergent, thus $\prod(1-1/A_i)$ is convergent with product P, say, and so we have,

$$\Lambda^h(S) \geq P h(t_0) > 0.$$

Thus, we have shown that,

$$0 < \Lambda^h(S) < \infty.$$

Also, we note that since P can be made arbitrarily close to one by appropriate choice of $\{A_n\}$ we can construct sets S in ℓ^2 of h-measure arbitrarily close to any given value.

It can be seen from the proof of Theorem 28 that the only property of $h(x)$ used in the construction of the set S is that $h(x)$ is a monotonic increasing step function. Now, if $h(x)$ is any monotonic increasing continuous function we can always find a step function $H(x)$, say, such that $h(x) \leq H(x) \leq 3h(x)$. Hence for all sets S we will have,

$$\Lambda^h(S) \leq \Lambda^H(S) \leq 3\Lambda^h(S).$$

Thus, we see that for any continuous Hausdorff measure function $h(x)$, there exists a set S in ℓ^2 such that,

$$0 < \Lambda^h(S) < \infty.$$

We have shown that, in Euclidean space and, in fact, in compact, finite dimensional metric spaces, given any discontinuous Hausdorff measure function there is a continuous measure equivalent function. The next theorem shows us that this is not the case in the space ℓ^2.

Theorem 29

There exists a discontinuous Hausdorff measure function $h(x)$ such that for sets in ℓ^2 there is no continuous measure equivalent function.

Proof

Let,

$$x_n = 1/2^n \qquad \text{for } n = 0, 1, 2, \ldots \; .$$

Define $h(x)$ as follows,

$$h(x) = 1/(n!)^2 \qquad \text{for } x \in (x_{n+1}, x_n].$$

Now, let $H'(x)$ be any continuous monotonic increasing function. Then, either,

$$\text{i).} \quad H'(x_n) > \frac{1}{(n-1)!\, n!} \qquad \text{for infinitely many values of } n,$$

$$\text{or, ii).} \quad H'(x_n) \leq \frac{1}{(n-1)!\, n!} \qquad \text{for all large values of } n.$$

Consider case i)., since $H'(x)$ is continuous, there exists a positive real number τ_n, and a subsequence $\{y_n\}$ of $\{x_n\}$ such that,

$$y_{n+1} < y_n - \tau_n ,$$

and, $$H'(x) > \frac{1}{(n-1)!\, n!} \qquad \text{for } x \in (y_n - \tau_n, y_n].$$

Define the function $H(x)$ as follows,

$$H(x) = \frac{1}{(n-1)!\, n!} \qquad \text{for } x \in (y_n - \tau_n, y_{n-1} - \tau_{n-1}].$$

Then we have,

$$H'(x) \geq H(x) \qquad \text{for all } x.$$

Let $\{A_n\}$ be a given sequence of integers such that $\sum A_n^{-1}$ is convergent. We now define a decreasing sequence $\{t_n\}$ of positive real numbers; choose t_0 arbitrarily in the open interval $(y_{n_0} - \tau_{n_0}, y_{n_0})$ for some positive integer n_0; assume that $t_0, \ldots, t_{m-1}$ have been chosen and that

$$t_{m-1} \in (y_{n_{m-1}} - \tau_{n_{m-1}}, y_{n_{m-1}}).$$

Choose t_m such that,

a). $t_m < \frac{1}{2} t_{m-1}$

b). $C_m H(t_m) = H(t_{m-1})$ with $C_m > A_m$

c). $t_{m-1} - 2t_m > y_{n_{m-1}} - \tau_{n_{m-1}}$

d). $t_m \in (y_{n_m} - \tau_{n_m}, y_{n_m})$ for some positive integer $n_m > n_{m-1}$.

Construct the set S with respect to the sequence $\{t_m\}$ just as in the proof of Theorem 28. Again, we have,

$$0 < \lambda^H(S) < \infty.$$

Now we also know that,

$$\Lambda_\delta^h(S) \leq K_1 \ldots K_m\, h[d\{T(m, i_1, \ldots, i_m)\}],$$

for arbitrary positive real numbers δ and for all large integral values of m.

Thus we have,

$$\Lambda_\delta^h(S) \leq c_1 \ldots c_m\, h(t_m)$$

$$= \frac{1}{N_m} c_1 \ldots c_m H(t_m) = \frac{1}{N_m} H(t_0),$$

where $y_{n_m} = x_{N_m}$.

But,

$$N_m \to \infty \qquad \text{as} \qquad m \to \infty,$$

and so we have,

$$\Lambda^h(S) = 0.$$

Thus, the theorem is proved for the case i).; it is easy to see that an analogous proof will deal with case ii)..

We have shown that if $h(x)$ is any monotonic q-dimensional Hausdorff measure function with the property that,

$$\frac{h(x)}{x^{q-1}} \to 0 \qquad \text{as} \qquad x \to 0,$$

then for any set S in q-dimensional Euclidean space and for any sequence $\{\delta_n\}$ with $\delta_n \to \delta$ as $n \to \infty$ for some positive real number δ, we have,

$$\lim_{n\to\infty} \Lambda_{\delta_n}^h(S) = \Lambda_\delta^h(S).$$

The next theorem shows that this result does not extend to the space l^2.

Theorem 30

There exists a Hausdorff measure function $h(x)$, a compact set S in l^2 and for arbitrarily small positive values of δ, a sequence $\{\delta_n\}$ such that $\delta_n \to \delta$ as $n \to \infty$ with the following properties,

i). $\dfrac{h(x)}{x^q} \to 0$ as $x \to 0$ for all positive integers q,

ii). $0 < \Lambda^h(S) < \infty$,

and, iii). $\lim\limits_{n \to \infty} \Lambda^h_{\delta_n}(S) \neq \Lambda^h_\delta(S)$.

Proof

Let,

$$x_n = 1/2^n \qquad \text{for } n = 0, 1, 2, \ldots .$$

Define $h(x)$ as follows,

$$h(x) = (n!)^{-2} \qquad \text{for } x \in [x_n, x_{n-1}).$$

Thus we see that,

$$h(x) > 0 \qquad \text{for } x > 0$$

$$h(x) \downarrow 0 \qquad \text{as } x \to 0$$

and,

$$\frac{h(x)}{x^q} \to 0 \qquad \text{as } x \to 0$$

for all positive integers q.

Choose t_0 to be an arbitrary positive number such that $t_0 \in \{x_n\}$. Now

assume that $t_0, \ldots, t_{m-1}$ have been chosen and that $t_{m-1} = x_{n_{m-1}} \in \{x_n\}$.
We choose t_m as follows,

i). $t_m = x_{n_m} \in \{x_n\}$ with $n_m > n_{m-1} + m$,

ii). $K_m h(t_m) = (1 + 1/m^2) h(t_{m-1})$ with K_m a positive integer,

iii). $t_{m-1} - 2t_m > x_{n_{m-1}+1}$.

Condition ii). can be satisfied since,

$$(1 + 1/m^2) \frac{h(x_{n_{m-1}})}{h(x_{n_m})} = \frac{m^2+1}{m^2} \frac{(n_m!)^2}{(n_{m-1}!)^2}$$

$$= \frac{m^2+1}{m^2} \left[(n_{m-1}+1) \ldots (n_m) \right]^2,$$

and this must be integral because of condition i)..
We now proceed to the construction of the set S.
Choose K_1 points of the form,

$$(0, \ldots, 0, t_0/\sqrt{2}, 0, \ldots)$$

where the entry $t_0/\sqrt{2}$ is in position i_1 for $i_1 = 1, \ldots, K_1$. Denote these points by $\underline{s}(\alpha_1)$ with $\alpha_1 = 1, \ldots, K_1$.
Now, choose $K_1 K_2$ points of the form,

$$\left(0, \ldots, 0, \frac{t_0^2 - t_1^2}{t_0\sqrt{2}}, 0, \ldots, 0, \left[\frac{t_1^2}{2} - \frac{t_1^4}{2t_0^2}\right]^{1/2}, 0, \ldots, 0, t_1/\sqrt{2}, 0, \ldots\right)$$

where $\frac{t_0^2 - t_1^2}{t_0\sqrt{2}}$ is in position i_1; $\left[\frac{t_1^2}{2} - \frac{t_1^4}{2t_0^2}\right]^{1/2}$ is in position $K_1 + i_1$

and $t_1/\sqrt{2}$ is in position $2K_1+i_1K_2+i_2$, for $i_1=1,\dots,K_1$ and $i_2=1,\dots,K_2$. Denote these points by $\xi(\alpha_1,\alpha_2)$ with $\alpha_1=1,\dots,K_1$ and $\alpha_2=1,\dots,K_2$. We note the following facts,

$$\rho[\xi(\alpha_1),\ \xi(\beta_1)]=t_0 \qquad \text{for} \quad \alpha_1\neq\beta_1,$$

$$\rho[\xi(\alpha_1,\alpha_2),\ \xi(\alpha_1,\beta_2)]=t_1 \qquad \text{for} \quad \alpha_2\neq\beta_2,$$

$$\rho[\xi(\alpha_1,\alpha_2),\ \xi(\beta_1,\beta_2)]=t_0 \qquad \text{for} \quad \alpha_1\neq\beta_1,$$

$$\rho[\xi(\alpha_1,\alpha_2),\ \xi(\alpha_1)]=t_1$$

and,
$$\rho[\xi(\alpha_1,\alpha_2),\ \xi(\beta_1)]=t_0 \qquad \text{for} \quad \alpha_1\neq\beta_1.$$

Now choose $K_1K_2K_3$ points of the form,

$$\left(0,\dots,0,\frac{t_0^2-t_1^2}{t_0\sqrt{2}},0,\dots,0,\left[\frac{t_1^2}{2}-\frac{t_1^4}{2t_0^2}\right]^{1/2},0,\dots,0,\frac{t_1^2-t_2^2}{t_1\sqrt{2}},0,\dots,0,\left[\frac{t_2^2}{2}-\frac{t_2^4}{2t_1^2}\right]^{1/2},0,\dots,0,\frac{t_2}{\sqrt{2}},0,\dots\right)$$

where, $\dfrac{t_0^2-t_1^2}{t_0\sqrt{2}}$ is in position i_1; $\left[\frac{t_1^2}{2}-\frac{t_1^4}{2t_0^2}\right]^{1/2}$ is in position K_1+i_1; $\dfrac{t_1^2-t_2^2}{t_1\sqrt{2}}$ is in position $2K_1+i_1K_2+i_2$; $\left[\frac{t_2^2}{2}-\frac{t_2^4}{2t_1^2}\right]^{1/2}$ is in position $2K_1+K_1K_2+K_2+i_1K_2+i_2$, and $t_2/\sqrt{2}$ is in position

$2(K_1+K_1K_2+K_2)+i_1K_3(K_2+1)+i_2K_3+i_3$, for $i_j=1,\dots,K_j$, $j=1,2,3$. Denote these points by $\xi(\alpha_1,\alpha_2,\alpha_3)$ where $\alpha_j=1,\dots,K_j$, $j=1,2,3$. We note the following facts,

$$\rho[\xi(\alpha_1,\alpha_2,\alpha_3),\ \xi(\beta_1)]=t_0 \qquad \text{for} \quad \alpha_1\neq\beta_1,$$

$$\rho[\xi(\alpha_1,\alpha_2,\alpha_3),\ \xi(\alpha_1,\beta_2)]=t_1 \qquad \text{for} \quad \alpha_2\neq\beta_2,$$

$$\rho[\xi(\alpha_1,\alpha_2,\alpha_3),\ \xi(\alpha_1,\alpha_2,\beta_3)]=t_2 \qquad \text{for} \quad \alpha_3\neq\beta_3,$$

$$\rho[\xi(\alpha_1,\alpha_2,\alpha_3),\ \xi(\beta_1,\beta_2,\beta_3)] = t_0 \quad \text{for} \quad \alpha_1 \neq \beta_1,$$

and $$\rho[\xi(\alpha_1,\alpha_2,\alpha_3),\ \xi(\alpha_1,\beta_2,\beta_3)] = t_1 \quad \text{for} \quad \alpha_2 \neq \beta_2.$$

Now assume that $K_1 \dots K_{n-1}$ points $\xi(\alpha_1,\dots,\alpha_{n-1})$ with $\alpha_i = 1,\dots,K_i$ for $i = 1,\dots,n-1$ have been defined. We define the $K_1 \dots K_n$ points $\xi(\alpha_1,\dots,\alpha_n)$ with $\alpha_i = 1,\dots,K_i$ for $i = 1,\dots,n$ in a similar manner to that described above, so that we have,

$$\rho[\xi(\alpha_1,\dots,\alpha_n),\ \xi(\alpha_1,\dots,\alpha_{n-i-1},\beta_{n-i})] = t_{n-i-1},$$

where $\alpha_{n-i} \neq \beta_{n-i}$ for $i = 0,\dots,n-1,$

and,

$$\rho[\xi(\alpha_1,\dots,\alpha_n),\ \xi(\alpha_1,\dots,\alpha_{n-i-1},\beta_{n-i},\beta_{n-i+1},\dots,\beta_n)] = t_{n-i-1},$$

where $\alpha_{n-i} \neq \beta_{n-i}$ for $i = 0,\dots,n-1.$

Now, suppose that this selection of points has been carried out for every positive integer n. Then, if $\{\alpha_i\}$ is any sequence of integers with $1 \leq \alpha_i \leq K_i$ for $i = 1,2,\dots$ the corresponding sequence of points $\{\xi(\alpha_1,\dots,\alpha_n)\}_{n=1,2,\dots}$ has the following property,

$$\rho[\xi(\alpha_1,\dots,\alpha_n),\ \xi(\alpha_1,\dots,\alpha_{n+r})] = t_n,$$

for any positive integers n and r. Thus, since we know that $t_n \to 0$ as $n \to \infty$ we see that $\{\xi(\alpha_1,\dots,\alpha_n)\}$ is a Cauchy sequence, and by the completeness of ℓ^2 must converge to a point, which we denote by $\xi(\alpha_1,\alpha_2,\dots)$. We note that,

$$\rho\left[\underline{\xi}(\alpha_1,\ldots,\alpha_n,\alpha_{n+1},\ldots),\ \underline{\xi}(\alpha_1,\ldots,\alpha_n,\beta_{n+1},\ldots)\right] = t_n,$$

if $\alpha_{n+1} \neq \beta_{n+1}$.

Now let,

$$S = \{\underline{x} : \underline{x} = \underline{\xi}(\alpha_1,\alpha_2,\ldots)\}.$$

Define,

$$S(\alpha_1,\ldots,\alpha_n) = \overline{\operatorname{conv}\{\underline{x} : \underline{x} = \underline{\xi}(\alpha_1,\ldots,\alpha_n,\gamma_{n+1},\ldots)\}}$$

where the γ_i are integers such that $1 \leq \gamma_i \leq K_i$ for $i = n+1, n+2, \ldots$.

Then,

$$d[S(\alpha_1,\ldots,\alpha_n)] = t_n.$$

Define,

$$T(\alpha_1,\ldots,\alpha_n) = \{\underline{x} : \rho(\underline{x}, S(\alpha_1,\ldots,\alpha_n)) < 1/4\, t_n\},$$

then we have,

$$h[d(T(\alpha_1,\ldots,\alpha_n))] = h[d(S(\alpha_1,\ldots,\alpha_n))].$$

We see that S can be covered by the $K_1 \ldots K_n$ open convex sets $T(\alpha_1,\ldots,\alpha_n)$ for any positive integer n.

Take an infinite sequence $\{\underline{x}^n\}$ of points $\underline{x}^n$ in S and write,

$$\underline{x}^n = \underline{\xi}(\alpha_1^n, \alpha_2^n, \ldots) \qquad \text{for } n = 1, 2, \ldots$$

then $\alpha_1^n = \alpha_1^*$ say, for infinitely many values of n.

Choose the subsequence $\{\underline{x}^{n_i}\}$ of $\{\underline{x}^n\}$ such that,

$$\underline{x}^{n_i} = \underline{c}(\alpha_1^*, \alpha_2^{n_i}, \alpha_3^{n_i}, \ldots) \qquad \text{for } i = 1, 2, \ldots,$$

then $\alpha_2^{n_i} = \alpha_2^*$ say, for infinitely many values of i. Choose the subsequence $\{\underline{x}^{n_{ij}}\}$ of $\{\underline{x}^{n_i}\}$ such that,

$$\underline{x}^{n_{ij}} = \underline{c}(\alpha_1^*, \alpha_2^*, \alpha_3^{n_{ij}}, \ldots) \qquad \text{for } j = 1, 2, \ldots .$$

In this manner we generate a convergent subsequence of $\{\underline{x}^n\}$ and hence we prove the compactness of S.

Now, let δ be an arbitrary positive number such that $\delta \notin \{t_n\}$. Let m be a positive number such that,

$$t_m < \delta < t_{m-1}.$$

Let $\{U_i\}$ be a sequence of open convex sets, such that,

$$d(U_i) < \delta \qquad \text{for all } i,$$

and,
$$S \subset \bigcup_{i=1}^{\infty} U_i .$$

We want to find the lower bound of the sum $\sum_i h(d(U_i))$ over all such sequences of sets. Firstly, because of the compactness of S we need only consider finite coverings. Now, consider a set $U_i \in \{U_i\}$ such that,

$$d(U_i) \in [t_n, t_{n-1}) \qquad \text{for some integer } n \text{ with } n \geq m.$$

Now U_i can only contain points of S which lie in the same set $T(\alpha_1, \ldots, \alpha_n)$. But, we know that $h(d(U_i))$ is greater than or equal to $h[d\{T(\alpha_1, \ldots, \alpha_n)\}]$, and so we may replace the U_i by the corresponding set $T(\alpha_1, \ldots, \alpha_n)$.

Thus we need only consider coverings consisting of sets of the form $T(\alpha_1,\dots,\alpha_n)$ for finitely many values of n. Let N be the greatest value of these integers n. Then there must be ρK_N sets $T(\alpha_1,\dots,\alpha_N)$ where ρ is an integer, assuming none of the covering sets is redundant. Now,

$$
\begin{aligned}
K_N h[d\{T(\alpha_1,\dots,\alpha_N)\}] &= K_N h[d\{S(\alpha_1,\dots,\alpha_N)\}] \\
&= K_N h(t_N) \\
&= (1+1/N^2)h(t_{N-1}) \qquad \text{by ii).,} \\
&= (1+1/N^2)h[d\{T(\alpha_1,\dots,\alpha_{N-1})\}].
\end{aligned}
$$

Thus, we should replace each block of K_N sets $T(\alpha_1,\dots,\alpha_N)$ by the single set $T(\alpha_1,\dots,\alpha_{N-1})$ which they all intersect. Continuing in this manner we eventually get,

$$
\begin{aligned}
\Lambda^h_\delta(S) &= K_1\dots K_m h[d\{T(\alpha_1,\dots,\alpha_m)\}] \\
&= K_1\dots K_m h(t_m) \\
&= \prod_{i=1}^{m}(1+1/i^2)h(t_0).
\end{aligned}
$$

Hence,

$$
\Lambda^h(S) = \lim_{m\to\infty}\prod_{i=1}^{m}(1+1/i^2)h(t_0),
$$

which is positive and finite.

Now, consider $\delta = t_N$ for some integer N, since we are only interested in sets of diameter less than t_N we get,

$$\Lambda^h_\delta(S) = K_1 \dots K_{N+1}\, h[d\{T(\alpha_1,\dots,\alpha_{N+1})\}]$$

by similar reasoning to that above.

Thus we have,

$$\Lambda^h_\delta(S) = \prod_{i=1}^{N+1}(1+1/i^2)\, h(t_0).$$

Now, let $\{\delta_n\}$ be any strictly decreasing sequence such that,

$$\delta_n \to \delta \qquad \text{as } n \to \infty.$$

Then for all large values of n we have,

$$\Lambda^h_{\delta_n}(S) = \prod_{i=1}^{N}(1+1/i^2)\, h(t_0).$$

That is,

$$\lim_{n\to\infty} \Lambda^h_{\delta_n}(S) = \prod_{i=1}^{N}(1+1/i^2)\, h(t_0) = \frac{1}{1+\frac{1}{(N+1)^2}}\Lambda^h_\delta(S) \neq \Lambda^h_\delta(S).$$

Hence the theorem is proved.

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APPENDIX 1

Let C be a convex set of diameter d in q-dimensional Euclidean space. Denote by C' the set $\{x : \rho(x, \partial C) > \delta\}$ for some small $\delta > 0$, where ρ is the metric in the space.

Assertion

We can cover $C \setminus C'$ with $K \left(d/\delta\right)^{q-1}$ cubes of diameter δ, where K is a constant dependent only on q.

Proof

Let P, Q be polytopes such that,

$$P \supset C \quad \text{and} \quad \{x : \rho(x, C) < \delta/8\} \supset P,$$

$$C' \supset Q \quad \text{and} \quad \{x : \rho(x, Q) < \delta/8\} \supset C'.$$

Then,

$$P \supset Q \supset \{x : \rho(x, \partial P) > 2\delta\}.$$

Since, if $p \in \{x : \rho(x, \partial P) > 2\delta\}$ and $x \in \partial C$, then there exists $q \in \partial P$ such that,

$$\rho(x, q) < \delta/4$$

So we have,

$$\rho(p, x) + \rho(x, q) \geq \rho(p, q) > 2\delta,$$

that is,

$$\rho(p, x) > 2\delta - \delta/4 = 7\delta/4.$$

Hence $p \in C'$.

Now, if $p \in C' \setminus Q$ then there exists $s \in C \setminus C'$ such that,

$$\rho(p, s) < \delta/4.$$

Also, there exists $x \in \mathcal{C}C$ such that $\rho(s, x) \leq \delta$ and hence,

$$\rho(p, x) < \delta/4 + \delta = 5\delta/4,$$

that is, $p \notin C' \setminus Q$ and therefore $p \in Q$.

Hence $\{x : \rho(x, \mathcal{C}P) > 2\delta\} \subset Q$ and clearly $Q \subset P$.

Now let $x \in C \setminus C'$ then there exists at least one point p on the frontier of P such that,

$$\rho(p, x) = \min\{\rho(q, x) : q \in \text{frontier of } P\}.$$

Let $s = \rho(p, x)$ then $s > 0$, since x is an interior point of P. Then from the definition of p, we have $S(x, s) \subset P$. Also, since p lies on the frontier of P there is a support hyperplane H of P through p. Clearly H must also support $S(x, s)$ and hence H is the unique support hyperplane through p. Further, we see from this argument that if we erect a right-cylinder of height 2δ on each facet of P we will have a covering of $C \setminus C'$.

Now let $\{V_i\}_{i=1}^{N}$ be a finite covering of $C \setminus C'$ by disjoint cubes V_i each of diameter δ. Then we must have,

$$\bigcup_{i=1}^{N} V_i \subset \{x : \rho(x, C) \leq \delta\} \setminus \{x : \rho(x, \mathcal{C}C') > \delta\}.$$

So, by a similar argument to the one above we have,

$$N\left(\frac{\delta}{\sqrt{q}}\right)^q \leq a_q (2d)^{q-1} 6\delta$$

where, a_q is the surface area of the unit q-dimensional sphere. Hence we may choose $K = 6a_q 2^{q-1}\sqrt{q}$ and the assertion is proved.

Correction to Pages 79 and 80.

To prove that $L^h_\delta \left(\bigcup_{n=1}^{\infty} S_n \right) = \frac{3}{2} x_N$.

It is clear that we can cover $\bigcup_n S_n$ by its own closure. We now assume that all the sets of the covering have diameter strictly less than δ. Thus, let $\{U_i\}$ be any closed covering of $\bigcup_n S_n$ such that,

$$d(U_i^\delta) < \delta \qquad \text{for all } i .$$

If there is more than one U_i such that $d(U_i) > x_{N+1}$ then clearly.

$$\sum h(d(U_i)) \geq 2x_N .$$

Now assume that there is at most one such U_i then,

$$d(U_i) < \delta - \varepsilon \text{ for all } i \text{ and for all small } \varepsilon > 0 .$$

Choose one such ε, let $\{U_{n_i}\}$ be a subsequence of $\{U_i\}$ such that each U_{n_i} has at least one point in common with the circle $x^2 + y^2 = \frac{1}{4}(\delta-\varepsilon)^2$. Clearly, no U_i can contain diametrically opposite points of this circle. Let the intersection of U_{n_i} with the circle subtend an angle $2\phi_i$ at the origin then,

$$\sin \phi_i \leq \frac{d(U_{n_i})}{x_N - \varepsilon} \text{ and } 0 \leq \phi_i < \pi/2$$

Since the circle must be covered we have,

$$\Sigma\, 2\phi_i \geq 2\pi$$

and using the fact that $\sin \phi_i \geq {}^{2\phi_i}/_{\pi}$, we get,

$$\sum_i d(U_i) \geq \sum_i d(U_{n_i}) > \frac{(x_N - \varepsilon)}{\pi}\, \Sigma 2\, \phi_i \geq 2(x_N - \varepsilon)$$

But this is true for all small values of ε, so that,

$$\sum_i d(U_i) \geq 2x_N \,.$$

Hence, since $h(x) \geq x$ for all x, we must have,

$$L_\delta^h \left(\bigcup_n S_n\right) = {}^3/_2\, x_N$$